Schur's 1926 Partition Theorem and Related Identities

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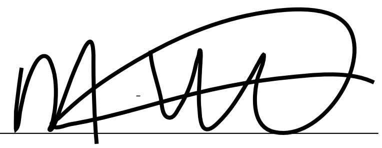
Declaration

I declare that this dissertation is my own, unaided work. It is submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted prior for any degree or examination at any other University.

Signature:  **Date:** 05 April 2022

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Abstract

In this paper we study Schur's 1926 Partition Theorem rigorously and in depth. The fundamental partition theorem by Schur asserts the equality of the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ and the number of partitions of n into parts that differ by at least 3 with multiples of 3 differing by at least 6. We aim to throw more light on the proof of the aforementioned theorem from its origins through to its generalizations. Consequently, from our desire to tackle the theorem from its foundation, we also examine related identities that led to the discovery of the theorem which includes Euler's Theorem together with its extensions, Euler Pairs and Glaisher's Theorem. We will especially showcase two different proofs of the best known generalization of Schur's Theorem, and make the proofs more comprehensive.

Contents

Chapter 1

Introduction

1.1 Background and Literature Review

The theory of partitions of a positive integer n is an interesting topic in number theory. The formal study of partitions was pioneered by the prominent mathematician Leonhard Euler in the 18th century. The famous “Euler identity" was named after him, which states that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts. French mathematician Adrien-Marie Legendre was actually the one who rephrased the identity purely in terms of partitions [16, p.24]. Numerous ways have been developed over the years to prove it including the Glaisher bijection. However, in proving the Euler identity, the bijection treats it as a base case for the proof of a more general identity called Glaisher’s partition identity.

In the 19th century, British mathematician James Sylvester (1814-1897) advanced the study of partitions in a paper entitled, "A Constructive Theory of Partitions, Arranged in Three Acts, an Interact, and an Oxodion“. The aforementioned is where he introduced a Ferrers graph into partition theory, an idea invented by Norman Macleod Ferrers (1829-1903) who never employed it on paper. Sylvester developed a bijection technique involving Ferrers graphs which also can be used to prove Euler’s identity [18]. Sylvester’s bijection ultimately implied an effective refinement of Euler’s identity.

As decades progressed, an illustrious pair of mathematicians, namely G.H. Hardy and S. Ramanujan, whose paths crossed in the early 20th century, would join forces. During World War I, the two discussed mathematics daily and jointly produced a number of monumental papers on various topics in number theory [19]. Their collaboration came about when Ramanujan, who had no formal education, wrote to Hardy who was then a reputable mathematics professor in England. Ramanujan’s letters contained a long list of discoveries he had made at the time. Together, they subsequently made contributions to numerous topics in mathematics in general, most notably in number theory.

With aid of the work of Percy A. MacMahon, Hardy and Ramanujan went on to make arguably necessary contribution in partition theory [16, p.51]. From the approximation formula for the number of partitions of any positive integer n, denoted by $p(n)$, to certain partition identities such as the Rogers-Ramanujan identities, their work would become a crucial stepping stone to all mathematicians studying partition theory till date.

Partition identities can be proved by constructing a bijection or using generating functions, or both where plausible. Upon proving an identity, it is wise to consider which route is more efficient or possible by inspection. The latter usually requires a lot of manipulation and application of combinatorial functions. When opting for the generating functions route, it might be necessary to be knowledgeable about certain special functions. In our case, we will require hypergeometric functions which we shall briefly discuss in the next chapter.

Separated from the developments in British mathematics by the first world war, Russian mathematician Issai Schur (1875 - 1941) sat in Germany and undertook independent groundbreaking researches on partitions. He managed to rediscover the Rogers-Ramanujan Identities independently and formulated a bijection to prove them. Schur observed that the Euler identity and the first Rogers-Ramanujan identity have a similar shape which initially led him to an incorrect conjectured generalization of both. However, in 1926, Schur found the correct way of modifying the failed conjecture which led to the identity now known as "Schur's Partition Theorem" [16, p.35].

In the late 20th century, American mathematician David Bressoud (1950 -) developed a bijective proof of Schur's theorem which involves the manipulation of Young diagrams of partitions. We shall examine in detail Bressoud's bijection and how it operates, and elucidate why it is such an elegant proving method. Furthermore, with co-author K.A. Post, Bressoud found a combinatorial proof of a generalization of Schur's theorem [22], [28]. As this is part of our main objectives, we shall patiently go through Bressoud's proof with the aim of filling any omitted gaps and generally improving it.

American mathematician George. E. Andrews (1938 -) is presently the most famous number theorist studying partitions. Having found "Ramanujan's Lost Notebook", he utilized the revolutionary knowledge it contained and vastly improved the field of partitions. His work on the subject has been published in numerous books and consequently providing important resources for many mathematicians studying partitions [17]. We will make use of some of his results in certain parts of this dissertation. The theory of partitions has been studied and discussed by many other prominent mathematicians such as Gauss, Jacobi,. . . , etc, but the joint work of Hardy and Ramanujan made a revolutionary change in the field [16, p. ix].

1.2 Overview

In Chapter 2 we give basic definitions which are essential in understanding any further work. This includes the definitions of an integer partition, a Young diagram and generating functions with the focus briefly on hypergeometric functions which shall be useful in completing one of our main objectives. In Chapter 3 we introduce and illustrate Glaisher's bijection. We then introduce Sylvester's bijection which is an alternative way to prove Euler's identity. From there, we introduce Sylvester's refinement of Euler's identity and include a bijective proof using Young diagrams, then one using generating functions. The proof makes use of

Young diagrams in an intelligent way and as such serves as an ideal way of illustrating the utmost importance of their role in partition theory. From here, we define Euler Pairs, which we will use in proving the first part of the special case of Schur's theorem. We conclude the chapter by introducing the first of the partition bijections produced by Bressoud.

The succeeding chapters will be of utmost importance in the project. In Chapter 4 we shall briefly discuss how Schur's theorem initially emerged. Then we will introduce a special case of Schur's theorem. From there, we shall employ more of Bressoud's work by illustrating another of his bijections which serves as an elegant way of proving the special case of Schur's theorem. Thereafter we give a proof using generating functions. In Chapter 5 we discuss Schur's theorem in full generality and provide two best known proofs. The final chapter will comprise some of the new developments in the theory and some advanced results for further explorations in future.

Chapter 2

Partitions, Young Diagrams and Generating Functions

2.1 Partitions

An *integer partition* or a *partition* λ of n is a representation of n as an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$, $\lambda_i > 0$ for each i. We call the integers λ_i *parts* of the partition λ, and call n the weight of λ.
We will write $\lambda \vdash n$ or $|\lambda| = n$ to denote that λ is a partition of n.

For partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, define the sum $\lambda + \mu$ to be a partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$, where the shorter partition may be padded with zeros. Similarly, define the union $\lambda \cup \mu$ to be a partition with all the parts $\{\lambda_i, \mu_j\}$ arranged in non-increasing order, with repetitions, [27, p.8].

2.2 Young Diagrams

A Young diagram $[\lambda]$ of a partition $\lambda \vdash n$ is a collection of n squares (i, j) on a square grid $\mathbb{Z}^2$, with $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$, where $\ell(\lambda)$ is the number of parts of λ. The first co-ordinate i increases downwards while the second coordinate j increases from left to right. Figures 2.1 and 2.2 show Young diagrams of the sum and union of $\beta = (5, 4, 4, 2, 1)$ and $\eta = (4, 3, 1, 1)$ respectively, [27, p.8].

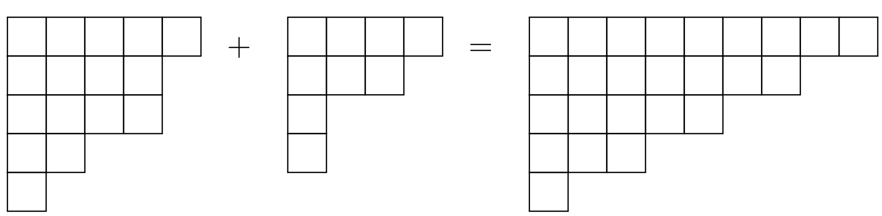


Figure 2.1: Young diagram of $\beta + \eta = (9, 7, 5, 3, 1)$.

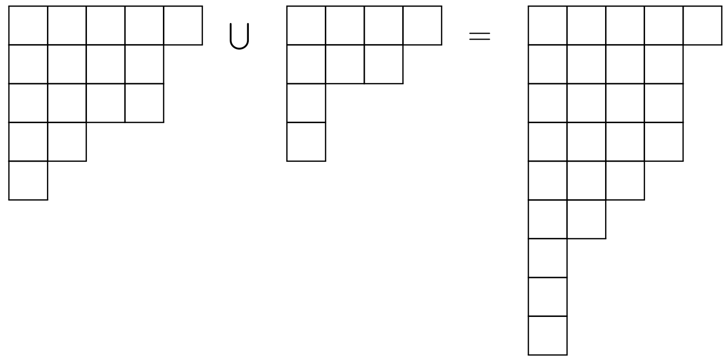


Figure 2.2: Young diagram of $\beta \cup \eta = (5, 4, 4, 4, 3, 2, 1, 1, 1)$.

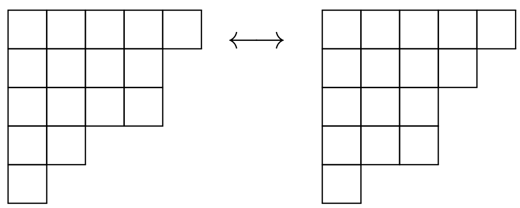


Figure 2.3: Young diagram showing the transformation of β to β' by conjugation.

Suppose we have a Young diagram of some partition λ. Swapping the rows and columns of λ gives a new partition. The resultant partition is called the *conjugate partition*, denoted by λ'. The process of swapping rows and columns is called conjugation and it is illustrated in Figure **2.3**.

2.2.1 Modular Diagram

We define a 2-*modular diagram* $[\mu]_2$ to be a Young diagram with the integers 1 or 2 written in squares such that 1 can only appear in the last square of a row, and no 2 can appear below 1, [27, p.9]. There exists a natural bijection between Young diagrams and 2-modular diagrams $[\mu]_2$ obtained by collapsing two consecutive squares into one 2-square as illustrated in Figure 2.4. We shall use this bijection in Chapter 3 of this dissertation.

In general, an m-modular diagram $[\mu]_m$ is defined by having an integer m written in all squares of $[\mu]$ which are not the last of a row; any integer i such that $1 \leq i \leq m$ can be written in the last square a row.

2.3 Generating Functions

A *Generating Function* is a way of encoding an infinite sequence of numbers (a_n) by treating them as coefficients of a formal series. The series is called the generating function of the sequence. An elementary example in partition theory is the generating function for the

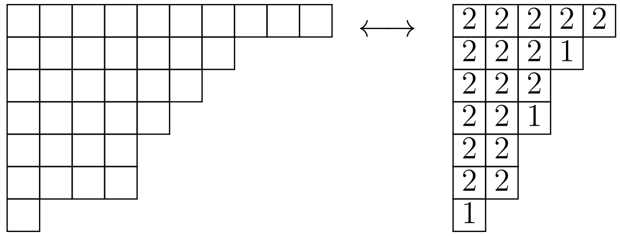


Figure 2.4: Young diagram of $[10, 7, 6, 5, 4, 4, 1]$ and the corresponding 2-modular diagram.

number of partitions of n which is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where the coefficient of each q^i represents the number of partitions of i and $|q| < 1$. As the generating function for unrestricted partitions is unrestrained, every positive integer is incorporated and repetition is allowed and unbounded. As such, the generating function for unrestricted partitions emerges as follows:

$$\sum_{n=0}^{\infty} p(n)q^n = (1 + q^1 + q^{1+1} + \cdots)(1 + q^2 + q^{2+2} + \cdots)(1 + q^3 + q^{3+3} + \cdots)\cdots$$
$$= \Big(\frac{1}{1-q}\Big)\Big(\frac{1}{1-q^2}\Big)\Big(\frac{1}{1-q^3}\Big)\cdots$$

since $|q| < 1$. Which simplifies to

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

2.3.1 Hypergeometric Functions [23]

Let $\{a_i\}_{i=0}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $b_j \neq -n$ with $n \in \mathbb{N}_0$ for $j = 1, 2, \ldots, s$. Then the *ordinary hypergeometric series* with variable z is defined as

$${}_{1+r}F_s \left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \ldots (a_r)_n}{n!(b_1)_n \ldots (b_s)_n} z^n \tag{2.1}$$

where $(c)_0 := 1$ and $(c)_n = c(c+1)\cdots(c+n-1)$ for $n = 1, 2, \ldots$.

The *basic hypergeometric series* with variable z is defined by

$${}_{1+r}\phi_s \left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_0; q)_n (a_1; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \{(-1)^n q^{\frac{n(n-1)}{2}}\}^{s-r} z^n. \tag{2.2}$$

where the notation $(a;q)_n$ denotes the *q-Pochhammer symbol* defined by

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{i=0}^{n-1}(1-aq^i) \tag{2.3}$$

for strictly non-negative n.

The q-Pochhammer symbol can be extended to an infinite product as follows:

$$(a;q)_\infty := (1-a)(1-aq)\cdots(1-aq^{n-1})\cdots = \prod_{i=0}^{\infty}(1-aq^i).$$

The binomial theorem is commonly applied in a lot of problems especially in combinatorics. As such, it serves as an ideal elementary way to illustrate the relationship between equations (2.1) and (2.2). The latter is commonly explained to be the q-analog of the former. In terms of hypergeometric series, the famous binomial theorem is as follows:

$${}_1F_0\left[\begin{matrix} c \\ - \end{matrix};z\right] = \sum_{n=0}^{\infty}\frac{(c)_n}{n!}z^n = \frac{1}{(1-z)^c}, \qquad (|z|<1). \tag{2.4}$$

The q-analog is given as follows:

$${}_1\phi_0\left[\begin{matrix} c \\ - \end{matrix};q,z\right] = \frac{(cz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(c;q)_n}{(q;q)_n}z^n, \qquad (|z|<1). \tag{2.5}$$

Setting $c := 0$ gives a special case which subsequently yields the definition of $1/(z;q)_\infty$. Another special case which is the inversion for the aforementioned case is given as follows:

$$(z;q)_\infty = {}_1\phi_1\left[\begin{matrix} 0 \\ - \end{matrix};q,z\right] = \sum_{n=0}^{\infty}\frac{(-z)^n}{(q;q)_n}q^{\binom{n}{2}}. \tag{2.6}$$

The case for when $r=s=1$ on (2.1) gives

$${}_2F_1\left[\begin{matrix} a_0,\ a_1 \\ b_1 \end{matrix};z\right] = \frac{(cz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(a_0)_n(a_1)_n}{n!(b_1)_n}z^n. \tag{2.7}$$

The q-analog of equation (2.6) is given by

$${}_2\phi_1\left[\begin{matrix} a_0 & a_1 \\ & b_1 \end{matrix};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_0;q)_n(a_1;q)_n}{(q;q)_n(b_1;q)_n}z^n. \tag{2.8}$$

In the mid 1800s, E. Heine studied the q-analog and discovered the following identities,

$${}_2\phi_1\left[\begin{matrix} a & b \\ & c \end{matrix};q,z\right] = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b & z \\ & az \end{matrix};q,b\right] \tag{2.9}$$

$$ {}_2\phi_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; q, z\right] = \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} abz/c & b \\ & bz \end{matrix} ; q, \frac{c}{b}\right] \tag{2.10} $$

$$ {}_2\phi_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; q, z\right] = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} c/a & c/b \\ & c \end{matrix} ; q, \frac{abz}{c}\right] \tag{2.11} $$

which are now known as Heine's q-Euler transformations.

From equation (2.9), ${}_2\phi_1(abz/c, b, bz; q, c/b)$ on the RHS can be re-expressed as follows:

$$ {}_2\phi_1\left[\begin{matrix} abz/c & b \\ & bz \end{matrix} ; q, \frac{c}{b}\right] = \sum_{n=0}^{\infty} \frac{(abz/c;q)_n(b;q)_n}{(q;q)_n(bz;q)_n}\left(\frac{c}{b}\right)^n $$

$$ = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n(bz;q)_n}\left(1-\frac{abz}{c}\right)\left(1-\frac{abz}{c}q\right)\dots\left(1-\frac{abz}{c}q^{n-1}\right)\left(\frac{c}{b}\right)^n $$

$$ = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n(bz;q)_n}\left(\frac{c}{b}-az\right)\left(\frac{c}{b}-azq\right)\dots\left(\frac{c}{b}-azq^{n-1}\right). \tag{2.12} $$

We introduce the following result which we are going to apply in the next chapter. We will not prove it as the proof assumes a number of prior results and this is not directly linked to our main objective.

Corollary 2.4. For $|q| < 1$,

$$ 1 + \sum_{n=1}^{\infty} \frac{(a;q)_n q^{n(n+1)/2}}{(q;q)_n} = \prod_{m=1}^{\infty}(1-aq^{2m-1})(1+q^m). $$

Generating functions are powerful and faultless in proving partition identities and we shall use them later.

Chapter 3

Euler's Partition Theorem and Some Extensions

Euler's identity served as a pathway for the emergence of many partition identities. In this chapter, we discuss some of its many proofs, including the influential proofs provided by J. J. Sylvester (1814 - 1897). These are followed by a discussion of one of its several refinements that have been discovered over the years.

Euler Pairs is also a concept that came as a result of Euler's identity. We also discuss them in this chapter together with Bressoud's bijective proof for relating partitions into distinct parts and partitions into super distinct parts (any pair of super-distinct parts differ by at least 2).

Firstly, we discuss Glaisher's generalization of Euler's identity.

3.1 Glaisher's Bijection

Firstly, we recall the Euler Identity.

Theorem 3.1. *Let $\mathcal{O}(n)$ be the set of partitions of n into odd parts and $\mathcal{D}(n)$ the set of partitions of n into distinct parts. Then Euler's identity states that:*

$$|\mathcal{O}(n)| = |\mathcal{D}(n)|.$$

To introduce Glaisher's bijection $\psi : \mathcal{O}(n) \mapsto \mathcal{D}(n)$, we use it to prove Theorem 3.1 which will serve as a base case of the bijection.

Proof of Theorem 3.1. Let $\lambda = (1^{m_1}, 3^{m_2}, \ldots) \in \mathcal{O}(n)$. For every odd part i, let $\psi(\lambda)$ contain a part $i \cdot 2^r$ if and only if the integer m_i when written in binary has a 1 at the r-th position from right to left. Conversely, let $\phi : \mathcal{D}(n) \mapsto \mathcal{O}(n)$ be defined as an iterative procedure. Let $\lambda^* \in \mathcal{D}(n)$. Suppose $\lambda^* = (\lambda_1, \lambda_2, \ldots)$. Replace every even part (λ_i) with two parts $\frac{\lambda_i}{2}$. Repeat until the resulting partition μ has no even parts, and set $\phi(\lambda^*) := \mu$. $\square$

Example 3.1. Let $\lambda = (1^4, 3, 7^3, 11^2, 13^5) \in \mathcal{O}(115)$. The m_i's when represented in binary give $4 = (100)_2,\ 1 = (1)_2,\ 3 = (11)_2,\ 2 = (10)_2$ and $5 = (101)_2$. Thus we have that

$$\psi(\lambda) = (1 \cdot 2^2, 3 \cdot 2^0, 7 \cdot 2^0, 7 \cdot 2^1, 11 \cdot 2^1, 13 \cdot 2^0, 13 \cdot 2^2),$$

which gives

$$\psi(\lambda) = (3, 4, 7, 13, 14, 22, 52).$$

Conversely

$$\phi : \lambda^* = (3, 4, 7, 13, 14, 22, 52) \mapsto (2^2, 3, 7, 7^2, 11^2, 13, 26^2) \mapsto (1^4, 3, 7^3, 11^2, 13^5).$$

Note: Equivalently, $\mathcal{O}(n)$ and $\mathcal{D}(n)$ are the sets of partitions of n into parts not divisible by 2 and parts repeated less than 2 times respectively.

Generally, we have the following theorem from Andrews [14]:

Theorem 3.2 (Glaisher's Theorem)**.** *Let $\mathcal{O}_k(n)$ be the set of partitions of n into parts not divisible by k and $\mathcal{D}_k(n)$ the set of partitions of n into parts repeated less than k times. Then*

$$|\mathcal{O}_k(n)| = |\mathcal{D}_k(n)|.$$

Proof. Let Glaisher's bijection $\psi : \mathcal{O}_k(n) \mapsto \mathcal{D}_k(n)$ be defined as follows. Suppose $\lambda \in \mathcal{O}_k(n)$. Then $\lambda = (1^{d_1}, 2^{d_2}, \ldots, (k-1)^{d_{k-1}}, (k+1)^{d_k}, \ldots)$. For each part, the image $\psi(\lambda)$ will contain parts $(a_i \cdot k^r)^s$ if and only if the integer d_i represented in base k has $s \neq 0$ at the r-th position. It remains to show that $s < k$. Each d_i when represented in base k has coefficients $\leq k-1$. Thus giving that $s < k$. Since $a \cdot b^r = p \cdot b^s \iff a = p$ and $r = s$, $\forall\ a, p \nmid b$ and $b \nmid a, p$, this gives the forward implication.

Conversely, assume that $\lambda^* = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{D}_k(n)$. If λ_i is a multiple of k, then $\lambda_i = a \cdot k^b$, $k \nmid a$. Define $\phi : \mathcal{D}_k(n) \mapsto \mathcal{O}_k(n)$ by replacing every such part λ_i by parts a, k^b times. This gives the complete bijection. □

Example 3.2. Let $\beta = (1^4, 2^7, 4, 5^2, 7^6, 8^3, 10^{10}) \in \mathcal{O}_3(198)$. The number of times in which each part is repeated when represented in ternary gives $4 = (11)_3,\ 6 = (20)_3,\ 2 = (2)_3,\ 3 = (10)_3,\ 7 = (21)_3$ and $10 = (101)_3$. β is transformed as follows:

$$\psi(\beta) = ((1 \cdot 3^0)^1, (1 \cdot 3^1)^1, (2 \cdot 3^0)^1, (2 \cdot 3^1)^2, (4 \cdot 3^0)^1, (5 \cdot 3^0)^2, (7 \cdot 3^1)^2, (8 \cdot 3^1)^1, (10 \cdot 3^0)^1, (10 \cdot 3^2)^1)$$

which gives

$$\psi(\beta) = (1, 2, 3, 4, 5^2, 6^2, 10, 21^2, 24, 90) \in \mathcal{D}_3(198).$$

Conversely, given $\beta^* = (1, 2, 3, 4, 5^2, 6^2, 10, 21^2, 24, 90) \in \mathcal{D}_3(198)$, we note that

$3, 6, 21, 24, 90 \equiv 0 \pmod 3$. Now, we factorise each of these parts to a product of some integer k that is not divisible by 3 and an exponent of 3 as follows:

$$3 = 1 \cdot 3^1,\ 6 = 2 \cdot 3^1,\ 21 = 7 \cdot 3^1,\ 24 = 8 \cdot 3^1,\ 90 = 10 \cdot 3^2.$$

The converse $\phi = \psi^{-1} : \mathcal{D}_3(n) \mapsto \mathcal{O}_3(n)$, is given as follows:

$$\phi(\beta^*) = (1, 1^{3^1}, 2, (2^{3^1})^2, 4, 5^2, (7^{3^1})^2, 8^{3^1}, 10, 10^{3^2}),$$

giving

$$\phi(\beta^*) = (1^4, 2^7, 4, 5^2, 7^6, 8^3, 10^{10}),$$

which gives the bijection.

3.2 Sylvester's Refinement and Bijective Proof of Euler's Theorem

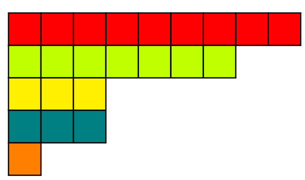
In this section we give alternative bijective proofs of Theorem 3.1 by Sylvester. In fact, we will illustrate three different bijections that give the same correspondence. Following these we introduce Sylvester's refinement of Euler's identity.

We now give the three versions of Sylvester's bijection retrieved from [27, p.21-22].

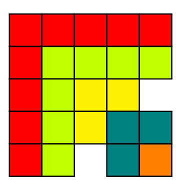
3.2.1 Sylvester's Proofs of Euler's Identity

First Bijection

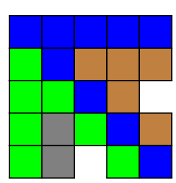
We start by defining the first version of Sylvester's proof. Let the bijection $\pi : \mathcal{O}(n) \to \mathcal{D}(n)$ be given as follows. Consider the partition $(9, 7, 3, 3, 1)$ given by the following Young diagram:



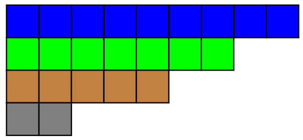
Firstly, arrange all odd parts symmetrically, folding them as hooks, as follows:



Now read the blocks diagonally as follows:



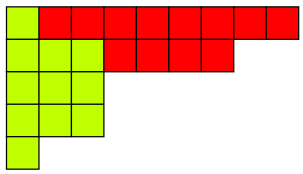
which gives



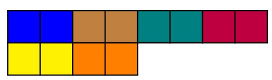
the Young diagram for $(9, 7, 5, 2)$.

Second Bijection

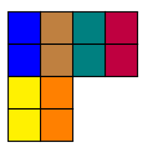
For the second bijection $\theta : \mathcal{O}(n) \to \mathcal{D}(n)$, we divide the diagram $[\lambda]$ into two parts, along the line $j = 1 + 2i$.



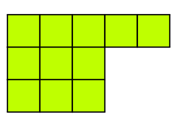
Read the parts above and below as diagrams of partitions α and β respectively. In our example, α will be the partition $(8, 4)$ and $\beta = (1, 3, 3, 3, 1)$. Because of the nature of the transformation, we represent $\alpha = (8, 4)$ such that adjacent parts act as a unit block, as follows:



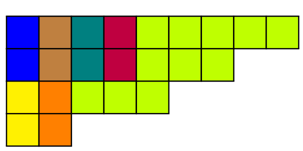
Now, let $\theta(\lambda) = (2\cdot(\alpha/2)')' + \beta'$. To obtain the diagram described by the first term, conjugate each pair of the above diagram for α. Thus giving the diagrams:



and



for $(2 \cdot (\alpha/2)')'$ and β'. Merging the diagrams gives:



a diagram for $\theta(\lambda) = (9, 7, 5, 2)$.

For demonstration, consider $\lambda = (9, 7, 3, 3, 1) \in \mathcal{O}(23)$ as given with $\alpha = (8, 4)$ and $\beta = (1, 3, 3, 3, 1)$. The first term of $\theta(\lambda)$ is evaluated as follows:

$$(2 \cdot (\alpha/2)')' = (2 \cdot (4, 2)')' = (2 \cdot (2, 2, 1, 1))' = (4, 4, 2, 2)' = (4, 4, 2, 2).$$

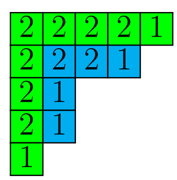
The second term $\beta' = (1, 3, 3, 3, 1)' = (5, 3, 3)$. Thus giving that $\theta(\lambda) = (4, 4, 2, 2)+(5, 3, 3) = (9, 7, 5, 2)$. Note that β' is evaluated when β is written in descending order.

Third Bijection

The third bijection is the same as the one Sylvester used to prove his refinement for Euler's theorem, which we will discuss in the upcoming section. Define $\zeta : \mathcal{O}(n) \to \mathcal{D}(n)$ as follows. Draw a 2-modular diagram $[\lambda]_2$ corresponding to λ. The diagram for $\lambda = (9, 7, 3, 3, 1)$ is given by:

2	2	2	2	1
2	2	2	1	
2	1			
2	1			
1				

Draw successive hooks $H_1, H_2, \ldots$ each indicated by a unique color as follows:



Let μ_1 be the number of squares in H_1 and let t_1 be the number of $2's$ in H_1, let μ_2 be the number of squares in H_2 and let t_2 be the number of $2's$ in H_2, etc. Define $\zeta(\lambda) = (\mu_1, t_1, \mu_2, t_2, \ldots) := \mu$.

Clearly, H_1 has 9 squares and 7 2's, while H_2 has 5 squares and 2 2's. Thus giving $\zeta(\lambda) = (9, 7, 5, 2)$.

Remark: The maps $\pi, \theta, \zeta : \mathcal{O}(n) \to \mathcal{D}(n)$ are bijections with identical correspondences, that is, $\pi = \theta = \zeta$.

3.2.2 Sylvester's Refinement of Euler's Theorem

We state the theorem, and then discuss two proofs.

Theorem 3.3. *Let $\mathcal{A}_k(n)$ denote the number of partitions of n into odd parts (each of which may be repeated) such that exactly k different parts occur and let $\mathcal{B}_k(n)$ denote the number* *of partitions of n into k separate sequences of one or more consecutive integers (with parts distinct). Then $\mathcal{A}_k(n) = \mathcal{B}_k(n)$.*

Remark: In general, combining all the cases for k, that is, letting k be indefinite, gives Euler's identity.
We will use the notations $\{\mathcal{A}_k(n)\}$ and $\{\mathcal{B}_k(n)\}$ to denote the corresponding sets enumerated by $\mathcal{A}_k(n)$ and $\mathcal{B}_k(n)$, respectively.

For example, when $n = 13$ and $k = 3$, the four partitions in $\{\mathcal{A}_3(13)\}$ are

$$(9, 3, 1),\ (7, 5, 1),\ (7, 3, 1^3)\ (5, 3, 3, 1^2),\ (5, 3, 1^4),$$

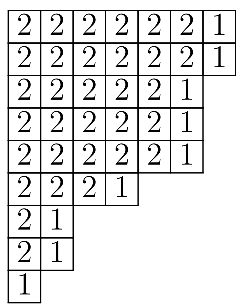
and the four partitions in $\{\mathcal{B}_3(10)\}$ are

$$(9, 3, 1),\ (8, 4, 1),\ (7, 5, 1),\ (7, 4, 2),\ (6, 4, 2, 1).$$

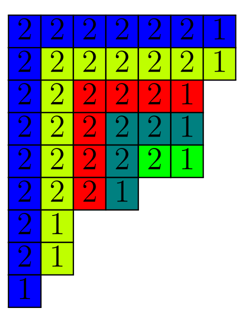
Bijective Proof of Theorem 3.3

We prove this using a technique Sylvester used to prove Euler's theorem, the third of the three bijections discussed in the previous section.

Define $\zeta : \{\mathcal{A}_k(n)\} \mapsto \{\mathcal{B}_k(n)\}$ as follows. For every $\lambda \in \{\mathcal{A}_k(n)\}$ draw the 2-modular diagram $[\lambda]_2$. For example, consider $\lambda = (13^2, 11^3, 7, 3^2, 1) \in \{\mathcal{A}_5(73)\}$. Then the 2-modular diagram $[\lambda]_2$ is given as follows:



We read off successive hooks $H_1, H_2, \ldots, H_5$, each indicated by a distinct color, starting with the largest hook H_1 (in blue color) to the smallest, innermost hook H_5 (in green color).



Assume that the image of λ is given by $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \ldots) \in \{\mathcal{B}_5(73)\}$.

Define μ_1 to be the number of squares in H_1 and μ_2 to be the number of $2's$ in H_1; define μ_3 to be the number of squares in H_2 and μ_4 to be the number of $2's$ in H_2; etc. Thus each μ_{2r-1} counts the number of squares in H_r and μ_{2r} counts the number of 2's in H_r, for $r \geq 1$. In this case, we obtain $\zeta(\lambda) = \mu = (15, 13, 12, 9, 7, 6, 5, 3, 2, 1)$, a partition with 5 sequences of consecutive integers, that is, $(15), (13, 12), (9), (7, 6, 5), (3, 2, 1)$.

Conversely, we mostly elucidate Bessenrodt's article [20], and reason as follows. Without loss of generality we consider $\mu = (15, 13, 12, 9, 7, 6, 5, 3, 2, 1) \in \{\mathcal{B}_5(73)\}$, and construct the hooks $H_1, \ldots, H_5$ in reverse order, while positioning them together successively to eventually produce $\zeta^{-1}(\mu) = \lambda$. We will use the parts of μ labelled as follows:

$$\mu_1 = 15,\ \mu_2 = 13,\ \mu_3 = 12,\ \mu_4 = 9,\ \mu_5 = 7,\ \mu_6 = 6,\ \mu_7 = 5,\ \mu_8 = 3,\ \mu_9 = 2,\ \mu_{10} = 1.$$

Notice that $(\mu_9, \mu_{10}) = (2, 1)$ gives the innermost hook H_5 of $\zeta^{-1}(\mu)$. H_5 will have leg length $\mu_9 - \mu_{10} - 1 = 1 - 1 = 0$ and arm length $\mu_{10} = 1$.

$$H_5 : \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$$

Then H_4 will have leg length $(\mu_7 - \mu_8) + (\mu_9 - \mu_{10}) - 1 = 2 + 1 - 1 = 2$ and arm length $\mu_8 - (\mu_9 - \mu_{10}) = 3 - 1 = 2$. This next hook is positioned such that its lowest 2 is put to the left of the lowest 1 of the diagram already constructed.

$$H_4 + (H_5) : \quad \begin{array}{ccc} 2 & 2 & 1 \\ 2 & & \\ 1 & & \end{array} \quad + \quad \begin{array}{cc} 2 & 1 \end{array} \quad = \quad \begin{array}{ccc} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & & \end{array}$$

Similarly H_3 has leg length $(\mu_5 - \mu_6) + (\mu_7 - \mu_8) + (\mu_9 - \mu_{10}) - 1 = 1 + 2 + 1 - 1 = 3$ and arm length $\mu_6 - (\mu_7 - \mu_8) - (\mu_9 - \mu_{10}) = 6 - 2 - 1 = 3$. Therefore, we have

$$H_3 + (H_4 + H_5) : \quad \begin{array}{cccc} 2 & 2 & 2 & 1 \\ 2 & & & \\ 2 & & & \\ 2 & & & \end{array} \quad + \quad \begin{array}{ccc} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & & \end{array} \quad = \quad \begin{array}{cccc} 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & & \end{array}$$

We continue like this and obtain H_2 and H_1, each of which is attached to the preceding diagram as described above, until the full $\zeta^{-1}(\mu) = \lambda$ emerges again. The following computations may be easily verified:

H_2: leg length $= (\mu_3 - \mu_4) + (\mu_5 - \mu_6) + (\mu_7 - \mu_8) + (\mu_9 - \mu_{10}) - 1 = 3 + 1 + 2 + 1 - 1 = 6$;
arm length $= \mu_4 - (\mu_5 - \mu_6) - (\mu_7 - \mu_8) - (\mu_9 - \mu_{10}) = 9 - 1 - 2 - 1 = 5$.

H_1: leg length $= (\mu_1 - \mu_2) + (\mu_3 - \mu_4) + \cdots + (\mu_9 - \mu_{10}) - 1 = 2 + 3 + 1 + 2 + 1 - 1 = 8$;
arm length $= \mu_2 - (\mu_3 - \mu_4) - (\mu_5 - \mu_6) - (\mu_7 - \mu_8) - (\mu_9 - \mu_{10}) = 13 - 3 - 1 - 2 - 1 = 6$.

This completes our proof.

Generating Functions Proof of Sylvester's Refinement, Theorem 3.3

Before we introduce the proof by generating functions, we introduce an identity and a lemma which will be essential in understanding some parts of the generating function proof.

Proposition 3.3. The number of partitions of n into m parts that are distinct is equal to the number of partitions of n in which each of the integers $1, 2, \dots, m$ appear at least once.

Proof. We prove this by conjugation. Consider $\upsilon = (7, 5, 1)$ and $\rho = (7, 4, 2)$, partitions of 13 into 3 parts that are distinct. Clearly the conjugate $\upsilon' = (3, 2^4, 1^2)$, and $\rho' = (3^2, 2^2, 1^3)$, which are both partitions in which each integer in $\{1, 2, 3\}$ appears at least once.
Generally, suppose we have a partition, $P = (a_1, a_2, \dots, a_m)$ with $a_1 > a_2 > \cdots > a_m$. Then

$$P' = (m^{a_m}, (m-1)^{a_{m-1}-a_m}, \dots, 1^{a_1-a_2}), \tag{3.1}$$

which is a partition described by the second class of partitions in the proposition. The converse is also deduced by conjugation which completes the proof. □

Lemma 3.1. The generating function for partitions enumerated by $\mathcal{B}_k(n)$ is given by

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{B}_k(n) z^k q^n = 1 + \sum_{m=1}^{\infty} \frac{zq^m}{1-q^m} \prod_{i=1}^{m-1} q^i \Big(1 + \frac{zq^i}{1-q^i}\Big).$$

Proof. In order to find a two-variable generating function for $\mathcal{B}_k(n)$ we proceed in an indirect manner (see Andrews [14]). We start by examining the conjugates of the partitions enumerated by $\mathcal{B}_k(n)$. To simplify formulating a generating function for these conjugates, we divide them into two classes.
Let $\mathcal{B}'_k(n)$ denote the number of the conjugates of all partitions enumerated by $\mathcal{B}_k(n)$.
Consider a partition $P = (12, 11, 8, 7, 6, 3, 1) \in \{\mathcal{B}_4(48)\}$, i.e. the last part $a_7 = 1$. By formula 3.1, we have that $P' = (7, 6^2, 5^3, 4, 3, 2^3, 1)$. Clearly, we can see that the largest part of P', i.e. 7 appears exactly once. Also, we observe that exactly $k - 1 = 4 - 1 = 3$ of the integers $1, 2, \dots, 6$ in P' appear more than once.

Generally given a partition $P = (a_1, a_2, \dots, a_m) \in \{\mathcal{B}_k(n)\}$ with $a_m = 1$, then P' is a partition into parts $1, 2, \dots, m$, where each of the integers $1, 2, \dots, m$ appear, the largest part m appears exactly once and exactly $k - 1$ of the integers $1, 2, \dots m - 1$ appear more than once. This realization can be best seen by aid of a Young diagram, or by formula 3.1. This gives us the first class of partitions described by $\mathcal{B}'_k(n)$, $n > 0$. Using the variable z to mark distinct parts, we thus obtain the following generating function for this class of partitions:

$$\sum_{m=1}^{\infty} zq^m(q^1 + zq^{1+1} + \cdots)(q^2 + zq^{2+2} + \cdots)\cdots(q^{m-1} + zq^{(m-1)+(m-1)} + \cdots) \tag{3.2}$$

Using a similar argument for $P \in \{\mathcal{B}_k(n)\}$, with $a_m > 1$, we obtain that P' comprises of m as a part appearing a_m times and each of the integers $1, 2, \dots m-1$ appearing at least once. Moreover, exactly $k-1$ of the integers $1, 2, \dots, m-1$ appear more than once. So ultimately exactly k parts appear more than once. This gives us the second class of partitions described by $\mathcal{B}'_k(n)$, $n > 0$. The corresponding generating function is given by

$$\sum_{m=1}^{\infty}(zq^{m+m} + zq^{m+m+m} + \cdots)(q^1 + zq^{1+1} + \cdots)(q^2 + zq^{2+2} + \cdots)\cdots(q^{m-1} + zq^{(m-1)+(m-1)} + \cdots). \tag{3.3}$$

Hence addition of the expressions (3.2) and (3.3) gives the generating function for $\mathcal{B}'_k(n)$ as follows (where we conventionally add 1 for the case $n = 0$):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{B}'_k(n)z^kq^n = 1+\sum_{m=1}^{\infty} zq^m(q^1+zq^{1+1}+\cdots)(q^2+zq^{2+2}+\cdots)\cdots(q^{m-1}+zq^{(m-1)+(m-1)}+\cdots)$$

$$+\sum_{m=1}^{\infty}(zq^{m+m}+zq^{m+m+m}+\cdots)(q^1+zq^{1+1}+\cdots)(q^2+zq^{2+2}+\cdots)\cdots(q^{m-1}+zq^{(m-1)+(m-1)}+\cdots)$$

$$= 1+\sum_{m=1}^{\infty} zq^m\prod_{i=1}^{m-1}(q^i+zq^{2i}+\cdots)+\sum_{m=1}^{\infty}(zq^{2m}+zq^{3m}+\cdots)\prod_{i=1}^{m-1}(q^i+zq^{2i}+zq^{3i}+\cdots)$$

$$= 1+\sum_{m=1}^{\infty} zq^m\prod_{i=1}^{m-1}q^i\Big(1+\frac{zq^i}{1-q^i}\Big)+\sum_{m=1}^{\infty}\frac{zq^{2m}}{1-q^m}\prod_{i=1}^{m-1}q^i\Big(1+\frac{zq^i}{1-q^i}\Big)$$

$$= 1+\sum_{m=1}^{\infty} zq^m\prod_{i=1}^{m-1}q^i\Big(1+\frac{zq^i}{1-q^i}\Big)\Big(1+\frac{q^m}{1-q^m}\Big)$$

$$= 1+\sum_{m=1}^{\infty}\frac{zq^m}{1-q^m}\prod_{i=1}^{m-1}q^i\Big(1+\frac{zq^i}{1-q^i}\Big).$$

Since conjugation is an involution on any set of partitions of n, it follows that $\mathcal{B}_k(n) = \mathcal{B}'_k(n)$ which implies that

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{B}_k(n)z^kq^n = 1+\sum_{m=1}^{\infty}\frac{zq^m}{1-q^m}\prod_{i=1}^{m-1}q^i\Big(1+\frac{zq^i}{1-q^i}\Big).$$

□

We are now ready to give a complete proof of Theorem 3.3 using generating functions.

Proof of Theorem 3.3. The generating function for $\mathcal{A}_k(n)$ is as follows:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{A}_k(n)z^kq^n = (1+zq^1+zq^{1+1}+\cdots)(1+zq^3+zq^{3+3}+\cdots)\cdots$$

$$= \Big(1+\frac{zq}{1-q}\Big)\Big(1+\frac{zq^3}{1-q^3}\Big)\Big(1+\frac{zq^5}{1-q^5}\Big)\cdots$$

$$= \prod_{i=1}^{\infty}\Big(1+\frac{zq^{2i-1}}{1-q^{2i-1}}\Big)$$

$$= \prod_{i=1}^{\infty} \Big(\frac{1-(1-z)q^{2i-1}}{1-q^{2i-1}} \Big)$$

$$= \frac{((1-z)q;q^2)_\infty}{(q;q^2)_\infty} \qquad \text{(using the notation (2.3.1))}$$

$$= ((1-z)q;q^2)_\infty(-q;q)_\infty,$$

where the last equality follows from the analytic Euler-Theorem transformation; in fact $1/(q;q^2)_\infty$ is equivalent to

$$\frac{(q;q)_\infty}{(q;q^2)_\infty(q;q)_\infty} = \frac{(q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{(q;q)_\infty(-q;q)_\infty}{(q;q)_\infty} = (-q;q)_\infty.$$

Now, by Lemma 3.1, we have that the generating function for $\mathcal{B}_k(n)$ is given by:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{B}_k(n) z^k q^n = 1 + \sum_{m=1}^{\infty} \frac{zq^m}{1-q^m} \prod_{i=1}^{m-1} q^i \Big(1 + \frac{zq^i}{1-q^i}\Big),$$

which we can further simplify as follows:

$$= 1 + \sum_{m=1}^{\infty} \frac{zq^m}{1-q^m} \prod_{i=1}^{m-1} q^i \Big(\frac{1-(1-z)q^i}{1-q^i}\Big)$$

$$= 1 + \sum_{m=1}^{\infty} \frac{zq^m \prod_{i=1}^{m-1} q^i(1-(1-z)q^i)}{(q;q)_m}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{\prod_{i=0}^{m-1} q^i(1-(1-z)q^i) \prod_{i=1}^{m} q^i)}{(q;q)_m}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{((1-z);q)_m q^{m(m+1)/2}}{(q;q)_m}$$

$$= \sum_{m=0}^{\infty} \frac{((1-z);q)_m q^{m(m+1)/2}}{(q;q)_m}.$$

By Corollary 2.4, this simplifies to

$$= \prod_{t=1}^{\infty} (1-(1-z)q^{2t-1})(1+q^t)$$

$$= ((1-z)q;q^2)_\infty(-q;q)_\infty$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{A}_k(n) z^k q^n.$$

This completes the proof. $\square$

3.4 Euler Pairs

The concept of *Euler Pairs* arises from Euler's Identity. The idea was originally found by I. Schur who never published his work on it. In fact, it first appeared in full in a paper of Andrews [15]. An Euler Pair is any pair of sets M and N such that:

- for each $m_i \in M$, there do not exist $m_j \in M$ such that $m_i = 2^k \cdot m_j$, k an integer.
- N contains all the elements of M together with their multiples of powers of 2.

Equivalently, if M and N satisfy $2N \subseteq N$ and $M = N - 2N$.

The theorem on Euler pairs is as follows:

Theorem 3.4. *The number of partitions of n into parts in M is equal to the number of partitions of n into distinct parts in N, where M and N are Euler Pairs as previously described.*

The proof can be found in [14].

Example 3.3. Consider $\mathcal{M} = \{x \in \mathbb{Z}^+ | x \equiv 1, 5 \text{ or } 6 \pmod 8\}$ and $\mathcal{N} = \{x \in \mathbb{Z}^+ | \ x \not\equiv 3 \pmod 4\}$. $\mathcal{M}$ and $\mathcal{N}$ are Euler Pairs and so the number of partitions of n into parts in $\mathcal{M}$ is equal to the number of partitions of n into distinct parts in $\mathcal{N}$.

Proof. The proof to this is identical to that for Theorem 3.1. Let $\lambda = (1^{a_1}, 5^{a_2}, 6^{a_3}, \ldots)$ be a partition into parts in $\mathcal{M}$. Then we express each a_i as a sum of distinct exponents of two (base 2), i.e.

$$a_i = \mu_0 \cdot 2^q + \mu_1 \cdot 2^{q-1} + \mu_2 \cdot 2^{q-2} + \cdots + \mu_{q-1} \cdot 2 + \mu_q$$

where $\mu_j = 0$ or 1. So then $a_i = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_s}$ with $p_i > p_{i+1}$ and $s \leq q$.

Each part is then mapped as follows:

$$(8k+1)^{a_i} \mapsto (8k+1) \cdot 2^{p_0} + (8k+1) \cdot 2^{p_1} + \cdots + (8k+1) \cdot 2^{p_s}. \tag{3.4}$$

The parts are distinct and congruent to $0, 1, 2 \pmod 4$ and congruent to $1 \pmod 4$ if and only if $p_s = 0$.

Note: $(8k+1) \cdot 2^p$ can be uniquely written in exactly one of the forms $4t, 4t+1$ or $4t+2$ with corresponding restrictions on p, but not in the form $4k+3$. That is,

$$(8k+1) \cdot 2^p = \begin{cases} 4(2k)+1 \equiv \ 1 \pmod 4 & \text{if } p = 0, \\ 4(4k)+2 \equiv 2 \pmod 4 & \text{if } p = 1, \\ 4(8k+1) \cdot 2^{p-2} \equiv \ 0 \pmod 4 & \text{if } p \geq 2. \end{cases}$$

Clearly no integer can be expressed in more than one of the forms above.

The same idea holds for parts congruent to $5, 6 \pmod 8$, which are mapped in the same way as demonstrated by Theorem (3.1). For parts congruent to $5 \pmod 8$, these uniquely map to parts congruent to $0, 1$ or $2 \pmod 4$ as given below:

$$(8k+5) \cdot 2^p = \begin{cases} 4(2k+1)+1 \equiv \ 1 \pmod 4 & \text{if } p = 0, \\ 4(4k+2)+2 \equiv \ 2 \pmod 4 & \text{if } p = 1, \\ 4(8k+5) \cdot 2^{p-2} \equiv \ 0 \pmod 4 & \text{if } p \geq 2. \end{cases}$$

For parts congruent to $6 \pmod 8$, we have just two cases as follows:

$$(8k+6) \cdot 2^p = \begin{cases} 4(2k+1)+2 \equiv \ 2 \pmod 4 & \text{if } p = 0, \\ 4(4k+3) \cdot 2^{p-1} \equiv \ 0 \pmod 4 & \text{if } p \geq 1. \end{cases}$$

Clearly, there do not exist integers k, p such that either of the image parts (as given in the three piecewise functions) are equal. For instance, $4(2k)+1 = 4(2k+1)+1$ and $4(8k+5) \cdot 2^{p-2} = 4(4k+3) \cdot 2^{p-1}$ both give no solution. Doing this and seeking k, p for each of the forms that are equivalent modulo 4 yields no solution on k and p. Thus giving that the parts in the image are indeed distinct. This gives the forward implication.

Conversely, let $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$, where each $\beta_i \in \mathcal{N}$ and $\beta_i \neq \beta_j$. If β_i is a multiple of 2 that is not contained in $\mathcal{M}$, define the converse as an iterative process as follows: split β_i into two parts $\frac{\beta_i}{2}$. If each of the $\frac{\beta_i}{2}$'s are multiples of 2 not contained in $\mathcal{M}$, repeat the process until each of the parts are integers contained in $\mathcal{M}$. The piecewise functions defined in the forward implication guarantee that this process will eventually yield parts in $\mathcal{M}$ as required. This completes the bijective proof. □

Example 3.4. For a more practical example, consider $\beta = (17, 13^3, 6^7, 5^2, 1^{10})$, a partition of 118 into parts in $\mathcal{M}$. β is mapped as follows:

- $17 \mapsto 17$
- $13^3 \mapsto 13 \cdot 2, 13 = 26, 13$
- $6^7 \mapsto 6 \cdot 2^2, 6 \cdot 2, 6 = 24, 12, 6$
- $5^2 \mapsto 5 \cdot 2 = 10$
- $1^{10} \mapsto 1 \cdot 2^3, 1 \cdot 2 = 8, 2$

giving that $\beta \mapsto (17) \cup (26, 13) \cup (24, 12, 6) \cup (10) \cup (8, 2) = (26, 24, 17, 13, 12, 10, 8, 6, 2)$, a partition into distinct parts in $\mathcal{N}$.

Conversely, the image $\beta = (26, 24, 17, 13, 12, 10, 8, 6, 2)$ is mapped as follows:

$26 \mapsto 13^2$, $24 \mapsto 12^2$. But $12 \notin \mathcal{M}$. So we split 12^2 again to obtain, $12^2 \mapsto 6^4$, $6 \in \mathcal{M}$ as required. The rest are mapped as $17 \mapsto 17$, $13 \mapsto 13$, $12 \mapsto 6^2$, $10 \mapsto 5^2$, $6 \mapsto 6$ and $2 \mapsto 1^2$.

Thus we have $\beta \mapsto (17, 13^3, 6^7, 5^2, 1^{10})$.

Examples of Euler Pairs

1. $\mathcal{M}_1 = \{n \in \mathbb{N} | \, n \equiv \ 1 \pmod 2\}$ and $\mathcal{N}_1 = \mathbb{N}$, i.e. the set of odd numbers and natural numbers.
2. $\mathcal{M}_2 = \{n \in \mathbb{N} | \, n \equiv \ \pm 1 \pmod 6\}$ and $\mathcal{N}_2 = \{n \in \mathbb{N} | \, n \equiv \ 1 \pmod 2\}$.
3. $\mathcal{M}_3 = \{n \in \mathbb{N} | \, n \equiv \ \pm 1 \pmod 3\}$ and $\mathcal{N}_3 = \{n \in \mathbb{N} | \, n \not\equiv \ 2, 3 \pmod 6\}$.
4. $\mathcal{M}_4 = \{n \in \mathbb{N} | \, n \equiv \ 2, 5, 11 \pmod{12}\}$ and $\mathcal{N}_4 = \{n \in \mathbb{N} | \, n \equiv \ \pm 2, 5 \pmod 6\}$.

3.5 Bressoud's Theorem for Partitions into Distinct and Super-Distinct Parts

We say that $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is a partition into distinct parts if and only if $|\mu_i - \mu_j| \geq 1$ $\forall \ \ i \neq j$. Similarly, we say that μ is a partition into *super-distinct* parts if and only if the pairwise difference between any two parts is at least 2. The following theorem by Bressoud relates the two aforementioned classes of partitions.

Theorem 3.5. *[16, p.22-23] The number of partitions of n into super-distinct parts is equal to the number of partitions of n into distinct parts with each even part greater than 2 times the number of odd parts.*

Proof. The bijective proof goes as follows:

- Take the Young diagram of the partition into super-distinct parts and split it into a sum of two Young diagrams.
- The first is such that, starting from the bottom, the first row has 1 square, the second has 3, the third has 5, ...etc. This will be a representation of a partition into k distinct consecutive odd parts, where k is the number of parts of μ. So the largest part will be $2k - 1$ and the smallest will be 1.
- The second Young diagram represents some regular partition. Order the rows of this diagram in descending order starting with odd parts, then even parts.
- Merge the two resulting Young diagrams. This gives the forward implication.

  See Figure 3.1 for illustration.

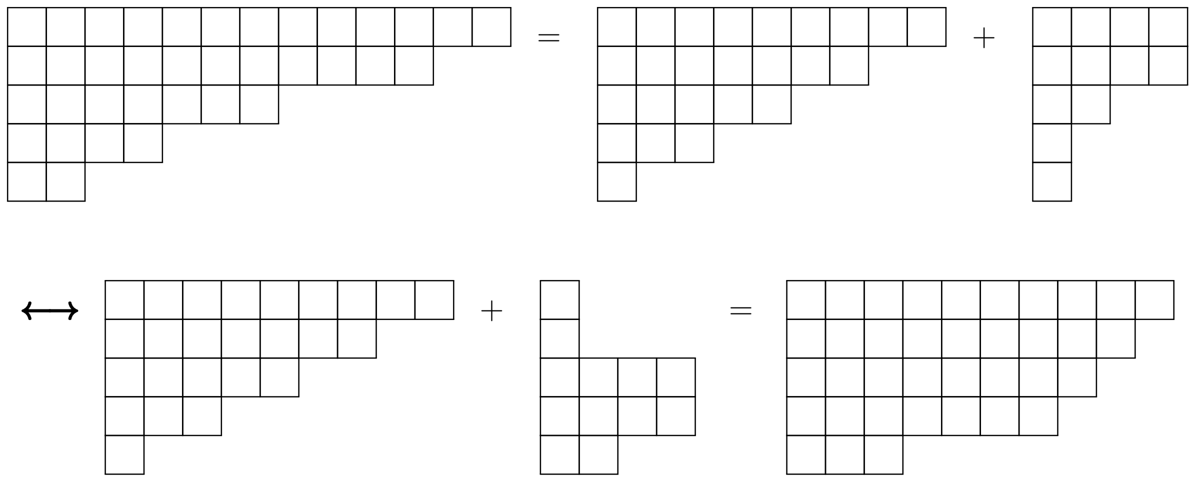


Figure 3.1: Bressoud's Bijection: $(13, 11, 7, 4, 2) \longrightarrow (10, 9, 8, 7, 3)$.

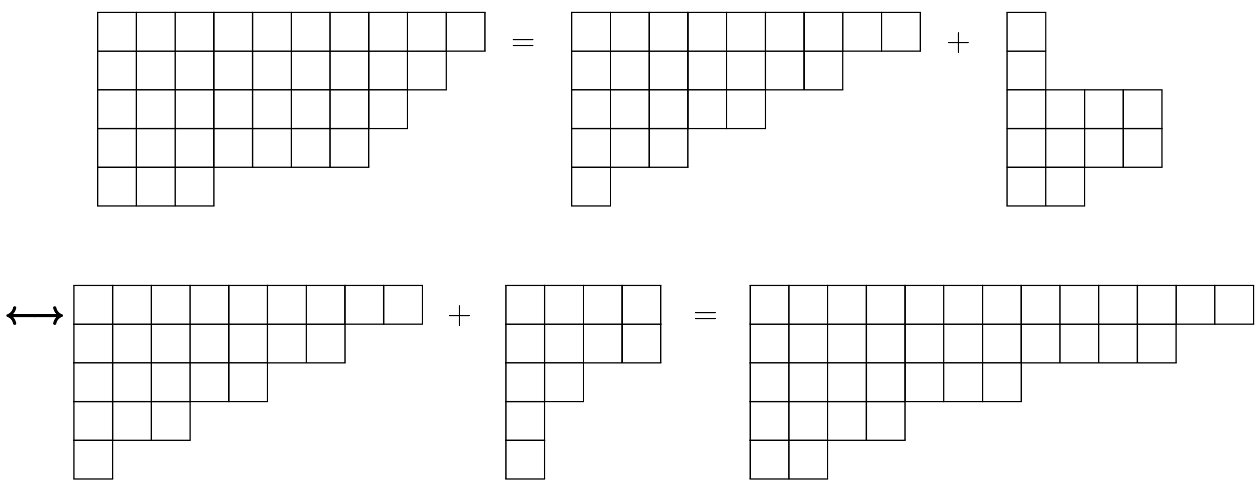


Figure 3.2: Bressoud's Bijection: $(10, 9, 8, 7, 3) \longrightarrow (13, 11, 7, 4, 2)$.

Conversely, suppose we have a partition λ into distinct parts such that each even part is greater than two times the number of odd parts. We order the parts in descending order, starting with even parts, then odd parts. Then split the Young diagram of the partition into two as described in the forward implication. Order the second Young diagram in descending order starting with even parts then odd. Merging the resultant Young diagrams gives a partition into super-distinct parts. See Figure 3.2 for illustration. □

Remark: The resultant partition satisfies the desired condition. Suppose initially we have a partition P into super-distinct parts with exactly k odd parts. If we disjoin the Young diagram of P into two as described by the bijection, the second Young diagram will have exactly k even parts. If we order it in descending order starting with odd rows and then even, we have then that the last k rows will be even.

Now, if we merge the first Young diagram which is made up of only odd rows and the resultant second Young diagram to get a Young diagram of the image P^*, we then have that the last k rows of P^* are odd. So the largest odd part will be as a result of merging a part $2k-1$ and the largest even part of the second Young diagram. Hence the part above which is the smallest even part of P^* will be as a result of merging a part $2k+1$ and the smallest odd part from the second diagram. Thus giving that the smallest even part of P^* is $\geq 2k+2$.

Chapter 4

Schur's Partition Theorem and its Relatives

German mathematician Issai Schur is one of the major contributors in partition theory. Some of his works include the renowned 'Schur Partition Theorem' which resulted from an incorrect conjecture deduced from the Euler and the first Rogers-Ramanujan identity. In Section 3.1, we discussed two ways of expressing Euler's identity. Another way of rewriting the identity is as follows:

$$\mathrm{p}(n \mid 1\text{-distinct parts }) = \mathrm{p}(n \mid \text{parts} \equiv \pm 1 \pmod 4).$$

The first Rogers-Ramanujan identity is as follows:

$$\mathrm{p}(n \mid 2\text{-distinct parts }) = \mathrm{p}(n \mid \text{parts} \equiv \pm 1 \pmod 5).$$

With closer inspection of these identities, one realises they have a very similar shape which led to the following conjecture:

$$\mathrm{p}(n \mid m\text{-distinct parts }) = \mathrm{p}(n \mid \text{parts} \equiv \pm 1 \pmod{m+3}).$$

This conjecture however fails when $m = 3$. To see this, we resort to the classical tabular way of finding partition identities to examine whether

$$\mathrm{p}(n \mid 3\text{-distinct parts }) = \mathrm{p}(n \mid \text{parts in } \mathcal{S}),$$

where $\mathcal{S} = \{1, 5, 7, 11, 13, \ldots\}$; see for example [6].

n	$\lambda \vdash n$ 3-distinct parts	$\mathrm{p}(n \mid 3\text{-dist parts })$	$\lambda \vdash n$ parts in $\mathcal{S}$	$\mathrm{p}(n \mid \text{parts in } \mathcal{S})$
1	(1)	1	(1)	1
2	(2)	1	$(1, 1)$	1
3	(3)	1	(1^3)	1
4	(4)	1	(1^4)	1
5	$(5), (4, 1)$	2	$(5), (1^5)$	2
6	$(6), (5, 1)$	2	$(5, 1), (1^6)$	2
7	$(7), (6, 1), (5, 2)$	3	$(7), (5, 1, 1), (1^7)$	3
8	$(8), (7, 1), (6, 2)$	3	$(7, 1), (5, 1^3), (1^8)$	3
9	$(9), (8, 1), (7, 2), (6, 3)$	4	$(7, 1^2), (5, 1^4), (1^9)$	3

The above table shows that the conjecture fails when $n = 9$, where we have four partitions of 9 into 3-distinct parts and three partitions of 9 into parts in $\mathcal{S}$.

Schur correctly modified the identity which led to Schur's Partition Theorem given below.

4.1 Schur's Partition Theorem

Theorem 4.1. (I. Schur [11, p.56-58])

1. *Let $\mathcal{A}(n)$ denote the set of partitions of n into parts congruent to $\pm 1 \pmod 6$.*
2. *Let $\mathcal{B}(n)$ denote the set of partitions of n into 3-distinct parts where no consecutive multiples of 3 appear (i.e., minimal difference between multiples of 3 is 6).*
3. *Let $\mathcal{C}(n)$ denote the set of partitions of n into distinct parts congruent to $\pm 1 \pmod 3$.*
4. *Let $\mathcal{D}(n)$ denote the set of partitions of n into odd parts appearing at most twice.*
5. *Consider the set of partitions of n in which no part appears more than twice, odd parts appear only once, the difference between two parts is always greater than 1 and can be 2 only if both are odd. Define $E(n)$ to be the weighted count of these partitions with weight $(-1)^e$ for each partition that has exactly e different parts that appear twice.*

Then

$$|\mathcal{A}(n)| = |\mathcal{B}(n)| = |\mathcal{C}(n)| = |\mathcal{D}(n)| = E(n).$$

The first equality is the modification by Schur and the formulation he originally introduced. To demonstrate the function $E(n)$, consider the case for $n = 10$. $E(10)$ gives a weight count of the seven partitions $(10), (9,1), (8,2), (7,3), (6,3,1)$ and $(6,2,2)$. The last is the only partition that gives weight -1, the remaining give $+1$. Thus giving $E(10) = 4$. $|\mathcal{C}(10)| = 4$ counts $(10), (8,2), (7,2,1)$ and $(5,4,1)$. Thus $|\mathcal{C}(n)| = E(n)$ holds for $n = 10$. The cardinalities of $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$ also give 4 for $n = 10$. We will not provide the proof linking the weight count function $E(n)$ with the rest of the functions (for a proof, see [32]).

We note that $\mathcal{A}(n)$ and $\mathcal{C}(n)$ are Euler pairs. However, we shall provide the generating function proofs for the equality of $|\mathcal{A}(n)|$, $|\mathcal{C}(n)|$ and $|\mathcal{D}(n)|$ in the next section. We commence by proving the second equality by illustrating a bijection provided by Bressoud.

4.1.1 Bressoud's Proof of Schur's Theorem

Let $\pi : \mathcal{C}(n) \mapsto \mathcal{B}(n)$ or $\pi^{-1} := \tau : \mathcal{B}(n) \mapsto \mathcal{C}(n)$ be defined iteratively as follows. Suppose we have a partition in $\mathcal{C}(n)$ whose parts are arranged in descending order in a column. Add consecutive parts that are not 3-distinct starting from the bottom. For example, for $\xi = (19, 17, 14, 11, 10, 5, 4, 2) \in \mathcal{C}(82)$, the first step is as illustrated in Figure 4.1: $\xi \longrightarrow ((19+17), 14, (11+10), 5, (4+2)) = (36, 14, 21, 5, 6)$.

$$\xi = \xi_0 : \begin{array}{c} 19 \\ 17 \\ 14 \\ 11 \\ 10 \\ 5 \\ 4 \\ 2 \end{array} \longrightarrow \xi_1 : \begin{array}{c} 36 \\ 14 \\ 21 \\ 5 \\ 6 \end{array}$$

Figure 4.1 : First step π of Bressoud on ξ.

We observe that a part $3k + 1$ can only merge with parts $3k - 1$ or $3k + 2$ and vice versa. Hence the resultant merges yield multiples of 3. Moreover, these multiples of 3 will have a minimal difference of $(2 + i)3$ if i parts lie between them in the column. As such, the multiples of 3 are sufficiently distinct as required.

For the second step, separate ξ_1 into a sum of two partitions. The first partition is the result of subtracting consecutive non-negative multiples of 3 from bottom upward, and the second consists of multiples of 3 increasing upward, as shown in Figure 4.2.

$$\xi_1 : \begin{array}{c} 36 \\ 14 \\ 21 \\ 5 \\ 6 \end{array} \longrightarrow \xi_2 : \begin{array}{cc} 24 & 12 \\ 5 & 9 \\ 15 & 6 \\ 2 & 3 \\ 6 & 0 \end{array}$$

Figure 4.2 : Second step π of Bressoud on ξ.

Reorder the parts of the first partition in ξ_2 in descending order to give ξ_3. Thereafter combine the two partitions in ξ_3 by adding up corresponding parts to obtain ξ_4.
Therefore ultimately $\xi = (19, 17, 14, 11, 10, 5, 4, 2) \in \mathcal{C}(82)$ is mapped to $\xi_4 = (36, 24, 12, 8, 2) \in \mathcal{B}(82)$, thus giving the resultant partition into 3-distinct parts with no consecutive multiples of 3 as desired.

$$\xi_2 : \begin{array}{cc} 24 & 12 \\ 5 & 9 \\ 15 & 6 \\ 2 & 3 \\ 6 & 0 \end{array} \longrightarrow \xi_3 : \begin{array}{cc} 24 & 12 \\ 15 & 9 \\ 6 & 6 \\ 5 & 3 \\ 2 & 0 \end{array} \longrightarrow \xi_4 := \zeta : \begin{array}{c} 36 \\ 24 \\ 12 \\ 8 \\ 2 \end{array}$$

Figure 4.3 : The third and final step π of Bressoud on ξ.

Remark. Suppose we have two arbitrary parts a and b, each positioned at $q + 1$ and $p + 1$ respectively, with $p < q$. Also, suppose after the subtraction of $3q$ and $3p$, i.e., the corresponding multiples of 3, and reordering, $a - 3q$ and $b - 3p$ correspond to some $3m$ and $3k$ respectively, with k, m, p, q all different and notably $k < m$. Then the resultant partition will consist of $r := a + 3(m - q)$ and $s := b + 3(k - p)$ as desired. This is because:

1. If a and b are both multiples of 3, and $|a-b| \geq 6$ as required, then $|r-s| \geq 6$.
2. If either or none is a multiple of 3, then $|r-s| \geq 3$ as $p < q$.

Conversely, given a partition ξ_4 into 3-distinct parts with no consecutive multiples of 3, it is clear to see that the map from ξ_4 to ξ_3 is reversible (that is, subtract $3(j-1)$ from the jth level from bottom upward, and store each multiple of 3 at the same level, in a second column). We next need to find a unique way to reorder the entries in the first column of ξ_3 such that it gives ξ_2. Define a map by rewriting each multiple of 3 in the first column as a sum of three parts $u+v+w$, $u > v > w$, to obtain ζ, where u is a 'remainder' equal to the corresponding multiple of 3 in the second summand, while v and w are non-multiples of 3 that differ by 1 or 2 (these numbers can be negative). This is shown in the second mapping given in Figure 4.4.

$$\xi_4: \begin{array}{c} 36 \\ 24 \\ 12 \\ 8 \\ 2 \end{array} \longrightarrow \xi_3: \begin{array}{cc} 24 & 12 \\ 15 & 9 \\ 6 & 6 \\ 5 & 3 \\ 2 & 0 \end{array} \longrightarrow \zeta: \begin{array}{cc} 12+7+5 & 12 \\ 9+4+2 & 9 \\ 6+1+(-1) & 6 \\ 5 & 3 \\ 2 & 0 \end{array}$$

Figure 4.4 : The first two steps of τ on ξ_4, the image of ξ.

Remark. Since $v+w$ is a multiple of 3, the pair v, w always exists uniquely. If the multiple of 3 is even, say $3k = 3(2k')$, the unique split yields $3k'+1$ and $3k'-1$ which differ by 2. If the multiples of 3 is odd, say $3k = 3(2k'+1)$, the unique split gives $3k'+2$ and $3k'+1$ with absolute difference 1.

Now reorder the first column of ζ as follows: search from bottom upward for a 'remainder plus pair' (that is, an instance of $u+v+w = u+(v+w)$) lying immediately above a single part such that the smallest part is strictly less than the single part. If there is such an occurrence, interchange the single part with the remainder plus pair, then subtract 3 from the remainder (so that it equals the corresponding multiple of 3 in the second column), and add 3 to the smallest member. Repeat the process until the smallest member of each decomposition is greater than or equal to the single part lying below it, as shown below. (Note that $3+1+2$ is treated again as $3+2+1$ below).

$$\zeta: \begin{array}{cc} 12+7+5 & 12 \\ 9+4+2 & 9 \\ 6+1+(-1) & 6 \\ 5 & 3 \\ 2 & 0 \end{array} \longrightarrow \zeta_1: \begin{array}{cc} 12+7+5 & 12 \\ 9+4+2 & 9 \\ 5 & 6 \\ 3+1+2 & 3 \\ 2 & 0 \end{array} \longrightarrow \zeta_2: \begin{array}{c} 12+7+5 \\ 9+4+2 \\ 5 \\ 2 \\ 0+4+2 \end{array}$$

Figure 4.5.1 : Reordering the remainder plus pair column.

$$\zeta_2 : \begin{array}{c} 12+7+5 \\ 9+4+2 \\ 5 \\ 2 \\ 0+4+2 \end{array} \longrightarrow \zeta_3 : \begin{array}{cc} 12+7+5 & 12 \\ 5 & 9 \\ 6+4+5 & 6 \\ 2 & 3 \\ 0+4+2 & 0 \end{array}$$

Figure 4.5.2 : Reordering the remainder plus pair column.

Finally sum up each single part and the corresponding multiple of 3 in the second column. For each 'remainder plus pair', add the remainder (u) to one of the parts of the pair (say v) and the other part of the pair (say w) to the corresponding entry in the second column. Since the sums involve two parts, one that is a multiple of three and the other not divisible by 3, the resulting parts are not multiples of 3.

$$\zeta_3 : \begin{array}{cc} 12+7+5 & 12 \\ 5 & 9 \\ 6+4+5 & 6 \\ 2 & 3 \\ 0+4+2 & 0 \end{array} \longrightarrow \zeta_4 : \begin{array}{c} 19+17 \\ 14 \\ 10+11 \\ 5 \\ 4+2 \end{array}$$

Figure 4.6 : Final summation of parts.

Therefore, the desired image ξ emerges as the collection of all the parts of ζ_4, sorted in decreasing order, $\zeta_4 \to \xi = (19, 17, 14, 11, 10, 5, 4, 2)$.

4.2 Proof for Schur's Partition Theorem using generating functions

In this section, our primary objective is to show that

$$\sum_{n=0}^{\infty} |\mathcal{A}(n)|q^n = \sum_{n=0}^{\infty} |\mathcal{B}(n)|q^n. \tag{4.1}$$

We would once again use the set $\mathcal{C}(n)$ by showing that it has the same generating function as those for $\mathcal{A}(n)$ and $\mathcal{B}(n)$, i.e

$$\sum_{n=0}^{\infty} |\mathcal{A}(n)|q^n = \sum_{n=0}^{\infty} |\mathcal{C}(n)|q^n = \sum_{n=0}^{\infty} |\mathcal{B}(n)|q^n. \tag{4.2}$$

Firstly, we prove the first equality in equation (4.2).

Proof. [11, p.54–56],[6, p.191-195] The first equality is shown using algebraic manipulation of infinite products as shown below.

$$\begin{aligned}\sum_{n=0}^{\infty}|\mathcal{C}(n)|q^n &= \prod_{n=1}^{\infty}(1+q^{3n-1})(1+q^{3n+1}) \qquad (4.3)\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{6n-4})(1-q^{6n-2})}{(1-q^{3n-1})(1-q^{3n+1})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{6n-4})(1-q^{6n-2})}{(1-q^{6n-1})(1-q^{6n+2})(1-q^{6n-2})(1-q^{6n+1})}\\ &= \prod_{n=1}^{\infty}\frac{1}{(1-q^{6n-1})(1-q^{6n+1})}\\ &= \sum_{n=0}^{\infty}|\mathcal{A}(n)|q^n.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{n=0}^{\infty}|\mathcal{D}(n)|q^n &= \prod_{n=0}^{\infty}(1+q^{2n-1}+q^{2(2n-1)})\\ &= \prod_{n=0}^{\infty}\frac{1-q^{6n-3}}{1-q^{2n-1}}\\ &= \prod_{n=0}^{\infty}\frac{1-q^{6n-3}}{(1-q^{6n-5})(1-q^{6n-3})(1-q^{6n-1})}\\ &= \prod_{n=0}^{\infty}\frac{1}{(1-q^{6n-5})(1-q^{6n-1})}\\ &= \sum_{n=0}^{\infty}|\mathcal{A}(n)|q^n.\end{aligned}$$

Thus giving the generating function proofs for the pairwise equality of $|\mathcal{A}(n)|$, $|\mathcal{C}(n)|$ and $|\mathcal{D}(n)|$.

To prove the second equality in equation (4.2), we introduce a refinement of $|\mathcal{B}(n)|$. Let $\gamma_j(m,n)$ denote the number of partitions of n into m parts, each $> j$, that are 3-distinct and the difference between multiples of 3 is at least 6. It follows that

$$|\mathcal{B}(n)| = \sum_{n\geq 0}\gamma_0(m,n). \qquad (4.4)$$

We begin by noting the following identities for $\gamma_j(m,n)$:

$$\gamma_0(m,n)-\gamma_1(m,n) = \gamma_0(m-1,n-3m+2), \qquad (4.5)$$

$$\gamma_1(m,n) - \gamma_2(m,n) = \gamma_1(m-1, n-3m+1), \tag{4.6}$$

$$\gamma_2(m,n) - \gamma_3(m,n) = \gamma_3(m-1, n-3m), \tag{4.7}$$

$$\gamma_3(m,n) = \gamma_0(m, n-3m). \tag{4.8}$$

The proof of equation (4.5) runs as follows. The expression $\gamma_0(m,n) - \gamma_1(m,n)$ enumerates partitions counted by $\gamma_0(m,n)$ with the condition that 1 is compulsory as a part. Now, delete 1 as a summand and subtract 1 from each of the remaining $m-1$ summands. (We now have a partition of $n-m$ with the minimum summand greater than or equal to 3). Now subtract 2 from each of the remaining parts to obtain the enumeration $\gamma_0(m-1, n-3m+2)$.

Conversely, consider partitions enumerated by $\gamma_0(m-1, n-3m+2)$, i.e., partitions of $n-3m+2$ into $m-1$ distinct parts that differ by at least 3 and multiples of 3 differ by at least 6. Add 3 on each part for each part to be greater than 3. Insert 1 in each of the partitions to make 1 compulsory. Thus giving the enumeration $\gamma_0(m,n) - \gamma_1(m,n)$, i.e. the count of partitions of n into m parts that differ by at least 3 with multiples of 3 differing by at least 6, and 1 compulsory as a part. This transformation therefore establishes a bijection between the partitions enumerated by $\gamma_0(m,n) - \gamma_1(m,n)$ and those enumerated by $\gamma_0(m-1, n-3m+2)$. Hence equation (4.5) is established.

The other three identities are treated similarly. The fact that multiples of 3 differ by at least 6 suffices to prove equation (4.7).

We now translate equations (4.5)-(4.8) into generating functions identities. Let

$$G_i(x) = G_i(x,q) = 1 + \sum_{n,m\geq 1} \gamma_i(m,n) x^m q^n. \tag{4.9}$$

Then the identities (4.5)-(4.8) are equivalent to the following respectively:

$$G_0(x) - G_1(x) = xqG_0(xq^3), \tag{4.10}$$

$$G_1(x) - G_2(x) = xq^2G_1(xq^3), \tag{4.11}$$

$$G_2(x) - G_3(x) = xq^3G_3(xq^3) = xq^3G_0(xq^6), \tag{4.12}$$

$$G_3(x) = G_0(xq^3). \tag{4.13}$$

These emerge as a result of comparing coefficients. Equation (4.10) emerges as follows:

$$G_0(x) - G_1(x) = \sum_{m,n\geq 1} (\gamma_0(m,n) - \gamma_1(m,n)) x^m q^n \tag{4.14}$$

$$= \sum_{m,n\geq 1} \gamma_0(m-1, n-3m+2)x^m q^n$$
$$= xq + xq \sum_{m,n\geq 2} \gamma_0(m-1, (n-1)-3(m-1))x^{m-1}q^{n-1}$$
$$= xq + xq \sum_{m,n\geq 1} \gamma_0(m, n-3m)x^m q^n$$
$$= xq + xq \sum_{m,n\geq 1} \gamma_0(m, n)x^m q^{n+3m}$$
$$= xq + xq \sum_{m,n\geq 1} \gamma_0(m, n)(xq^3)^m q^n = xqG_0(xq^3).$$

The rest are also transformed to generating functions identities in the same way.

Substituting equations (4.10) and (4.12) with (4.13), into equation (4.11), we obtain

$$G_0(x) - xqG_0(xq^3) - G_0(xq^3) - xq^3G_0(xq^6) = xq^2(G_0(x) - xqG_0(xq^3)). \tag{4.15}$$

Since x and q are both dummy variables we use for bookkeeping, we can alter them on both sides of the equation without affecting any significant result. As such, for the term on the RHS inside the bracket, we replace x with xq^3. So equation (4.15) then becomes

$$G_0(x) - xqG_0(xq^3) - G_0(xq^3) - xq^3G_0(xq^6) = xq^2(G_0(xq^3) - xq^4G_0(xq^6)), \tag{4.16}$$

making $G_0(x)$ the subject gives

$$G_0(x) = (1 + xq + xq^2)G_0(xq^3) + xq^3(1 - xq^3)G_0(xq^6). \tag{4.17}$$

Set

$$g(x) := \frac{G_0(x)}{\prod_{n=0}^{\infty}(1 - xq^{3n})}. \tag{4.18}$$

Then

$$(1-x)g(x) = \frac{(1 + xq + xq^2)G_0(xq^3)}{\prod_{n=1}^{\infty}(1 - xq^{3n})} + \frac{xq^3(1 - xq^3)G_0(xq^6)}{\prod_{n=1}^{\infty}(1 - xq^{3n})}, \tag{4.19}$$

so

$$(1-x)g(x) = \frac{(1 + xq + xq^2)G_0(xq^3)}{\prod_{n=1}^{\infty}(1 - (xq^3)q^{3(n-1)})} + \frac{xq^3(1 - xq^3)G_0(xq^6)}{(1 - xq^3)\prod_{n=2}^{\infty}(1 - (xq^6)q^{3(n-2)})},$$

which yields

$$(1-x)g(x) = (1 + xq + xq^2)g(xq^3) + xq^3g(xq^6).$$

The first term for equation (4.9) and the term corresponding to $n = 0$ for the expansion of $g(x)$ are both 1. Thus if g is the power series $\sum_{n\geq 0} a_n(q)x^n$, then $a_0(q) = g(0) = 1$ and for $n > 0$,

$$a_n(q) - a_{n-1}(q) = q^{3n}a_n(q) + q^{3n-2}a_{n-1}(q) + q^{3n-1}a_{n-1}(q) + q^{6n-3}a_{n-1}(q). \tag{4.20}$$

Making $a_n(q)$ the subject yields

$$a_n(q) = \frac{(1+q^{3n-1})(1+q^{3n-2})}{1-q^{3n}} a_{n-1}(q). \tag{4.21}$$

The solution to this linear recurrence is

$$a_n(q) = \frac{(1+q)(1+q^4)\cdots(1+q^{3n-2})(1+q^2)(1+q^5)\cdots(1+q^{3n-1})}{(1-q^3)(1-q^6)\cdots(1-q^{3n})}, \tag{4.22}$$

which by equation (2.3) can be rewritten as

$$a_n(q) = \frac{(-q;q^3)_n(-q^2;q^3)_n}{(q^3;q^3)_n}.$$

So, by equation (4.18) and the definition of $g(x)$, we have

$$G_0(x) = (x;q^3)_\infty \sum_{n=0}^{\infty} \frac{(-q;q^3)_n(-q^2;q^3)_n}{(q^3;q^3)_n} x^n, \tag{4.23}$$

which by equation (2.7) is the basic hypergeometric series

$$= (x;q^3)_\infty \, {}_2\phi_1 \left[\begin{matrix} -q & -q^2 \\ & 0 \end{matrix} ; q^3, x \right], \tag{4.24}$$

where q is replaced by q^3 and $a_0 = -q$, $a_1 = -q^2$, $b_1 = 0$, $z = x$.

By equations (2.9) and (2.11), substituting for the above values gives an alternative way of expressing equation (4.24) which is given by

$$G_0(x) = (-q^2x;q^3)_\infty \sum_{n=0}^{\infty} \frac{(-q^2;q^3)_n}{(q^3;q^3)_n(-q^2x;q^3)_n} (qx)(q^4x)\dots(q^{3n-2}x), \tag{4.25}$$

which simplifies to

$$G_0(x) = (-q^2x;q^3)_\infty \sum_{n=0}^{\infty} \frac{(-q^2;q^3)_n}{(q^3;q^3)_n(-q^2x;q^3)_n} q^{n(3n-1)/2} x^n. \tag{4.26}$$

Hence

$$G_0(1) = (-q^2;q^3)_\infty \sum_{n=0}^{\infty} \frac{1}{(q^3;q^3)_n} q^{n(3n-1)/2}, \tag{4.27}$$

so

$$G_0(1) = (-q^2;q^3)_\infty \sum_{n=0}^{\infty} \frac{1}{(q^3;q^3)_n} q^{\frac{3n(n-1)}{2}+n},$$

giving

$$G_0(1) = (-q^2; q^3)_\infty \sum_{n=0}^{\infty} \frac{1}{(q^3; q^3)_n} q^{3n(n-1)/2} q^n.$$

Using the identity given by equation (2.5), where q is replaced by q^3 and $z = -q$, this simplifies to

$$G_0(1) = (-q^2; q^3)_\infty (-q; q^3)_\infty = \sum_{n=0}^{\infty} |\mathcal{C}(n)| q^n, \tag{4.28}$$

by equation (4.3).

By equations (4.4) and (4.9), we see that

$$\sum_{n \geq 0} |\mathcal{B}(n)| q^n = \sum_{m,n \geq 0} \gamma_0(m,n) q^n = G_0(1). \tag{4.29}$$

By equations (4.28) and (4.29), we see that $|\mathcal{C}(n)| = |\mathcal{B}(n)|$ for all n and this completes the proof. □

Chapter 5

Schur's Theorem in Full Generality

Schur's alteration of the initially failed conjecture became a success for $m = 3$. But why stop at $m = 3$? Could the alteration also be true for $m = 4$? The answer to the question turned out to be "No". In fact, no further alteration could give such an identity which extends to all $m \geq 3$.

However, it emerged to be possible to formulate an identity for $m \geq 3$ when relating distinct parts and m-distinct parts with a further restriction imposed on the parts used on the latter class. The theorem is as follows:

Theorem 5.1. *[22],[28] Suppose r and m are positive integers such that $r < \frac{m}{2}$.
Let $C_{r,m}(n)$ be the number of partitions of n into distinct parts congruent to $\pm r \pmod m$, and let $D_{r,m}(n)$ be the number of partitions of n into m-distinct parts congruent to $0, \pm r \pmod m$, with minimal difference $2m$ between multiples of m.
Then $C_{r,m}(n) = D_{r,m}(n)$ for all n.*

The bijection given in Section 4.1 as a one-to-one correspondence between $\mathcal{C}(n)$ and $\mathcal{B}(n)$ is the case for when $r = 1$ and $m = 3$. Clearly $C_{1,3} = |\mathcal{C}(n)|$, the number of partitions of n into distinct parts congruent to $\pm 1 \pmod 3$, and $D_{1,3} = |\mathcal{B}(n)|$, the number of partitions of n into 3-distinct parts with minimal difference 6 between multiples of 3. We note that the set of positive integers congruent to $0, \pm 1 \pmod 3$ is just $\mathbb{N}$.

We provide two combinatorial proofs of Theorem 5.1.

5.1 Proof of Schur's Theorem in Full Generality by Bressoud

We start by giving a one-to-one correspondence for Schur's general partition theorem and we do so by introducing new concepts so as to make the proof clear and convenient to follow. In this section, the parts of partitions will be written in weakly increasing order.

Definition 5.1. Let r, m, k and n be positive integers such that $r < \frac{m}{2}$, $n - \frac{mk(k-1)}{2} \geq rk$. An r, m *underlying partition* for n and k, or an underlying partition, is a partition of $n - \frac{mk(k-1)}{2}$ into exactly k parts, each of which is congruent to $0, \pm r \pmod{m}$ and such that multiples of m are distinct.

For example, given $r = 1, m = 5$, $n = 533$ and $k = 11$, then
$\lambda = (4, 6, 6, 11, 15, 20, 34, 36, 40, 41, 45)$ is a 1, 5 underlying partition for 533 and 11. It may be verified that the weight of λ is given by $n - \frac{mk(k-1)}{2} = 533 - \frac{5 \cdot 11 \cdot 10}{2} = 258$.

Lemma 5.1. There is a one-to-one correspondence between r, m underlying partitions for n and k and the partitions counted by $D_{r,m}(n)$ which have exactly k parts.

Proof. Let $(a_1, a_2, \ldots, a_k)$, $a_1 \leq a_2 \leq \cdots \leq a_k$, be an r, m underlying partition for n and k. For $1 \leq i \leq k$, define $b_i = a_i + m(i-1)$. (Note that $\sum_{i=1}^{k} (i-1) = \frac{k(k-1)}{2}$). Then $(b_1, b_2, \ldots, b_k)$ with $b_1 < b_2 < \cdots < b_k$ is a partition counted by $D_{r,m}(n)$, and this correspondence is uniquely reversible. □

Example 5.1. The 1,5 underlying partition for 533 and 11 given by

$$(4, 6, 6, 11, 15, 20, 34, 36, 40, 41, 45)$$

corresponds to

$$(4, 11, 16, 26, 35, 45, 64, 71, 80, 86, 95) \in D_{1,5}(533).$$

Definition 5.2. An ordering of the parts in an r, m underlying partition for n and k, say $a_1, a_2, \ldots, a_k$, is called an Ω-*ordering* if the following inequalities are satisfied for any i and j, $1 \leq i < j \leq k$.

1. $a_i \equiv a_j \equiv 0 \pmod{m} \implies a_i < a_j$;

2. $a_i \equiv \pm a_j \equiv r, m - r \pmod{m} \implies a_i \leq a_j$, $r > 0$;

3. $$\left.\begin{array}{l} a_i \equiv r, m-r \ (mod\ m) \\ a_j \equiv 0 \ (mod\ m) \end{array}\right\} \implies a_i + mi \leq \left\lfloor \frac{a_j + mi}{2} \right\rfloor_r;$$

4. $$\left.\begin{array}{l} a_i \equiv 0 \ (mod\ m) \\ a_j \equiv r, m-r \ (mod\ m) \end{array}\right\} \implies \left\lfloor \frac{a_i + mi}{2} \right\rfloor_r < a_j + mi;$$

   or equivalently,
   $$\left\lceil \frac{a_i + m(i-1)}{2} \right\rceil_r < a_j + mi$$
   where $\lfloor x \rfloor_r$ is the greatest integer less than or equal to x and congruent to $\pm r$ (mod m) and $\lceil x \rceil_r$ is the least integer greater than or equal to x and congruent to $\pm r$ (mod m).

5.1.1 Establishing an Ω-ordering

Suppose we have an r, m underlying partition for n and k, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_i \leq \lambda_j \; \forall \, i, j$. We seek to reorder the λ_i's such that λ is of Ω-ordering. We note that since λ is an ascending sequence, then conditions (1) and (2) are clearly satisfied (So firstly, we need to ensure the sequence is in ascending order before proceeding). Hence it suffices to reorder λ such that condition 3 is satisfied. To achieve that, we do as follows:

Search for the largest λ_j divisible by m and the smallest λ_i congruent to $\pm r \pmod m$ such that condition (3) does not hold. If no such parts are found, then λ is already ordered as required. If such parts are found, take all the parts greater than or equal to λ_i that are congruent to $\pm r \pmod m$ and place them adjacently to the right of λ_j. (This ensures that all λ_i's that violate condition (3) w.r.t. the given λ_j are relocated to rid of the relevance of condition (3). Also, this ensures parts of equal residues remain in ascending order so as to not infringe the first two conditions). Repeat the process until no such λ_i and λ_j are found. At convergence, the process yields λ that is of Ω-ordering. (Note that the new order does not violate condition (4). For condition (4) not to hold, a_i has to be significantly larger than a_j, and ordering using condition (3) does not allow for such a case).

Example 5.2. Consider $\lambda = (4, 6, 6, 11, 15, 20, 34, 36, 40, 41, 45)$, a 1, 5 underlying partition of 533. The largest λ_j congruent to (mod 5) and smallest λ_i congruent to $1, 4 \pmod 5$ such that condition (3) does not hold are 45 and 34 respectively. i.e.

$$69 = 34 + 5(7) \not\leq \left\lfloor \frac{45 + 5(7)}{2} \right\rfloor_1 = 39.$$

So the new sequence is $4, 6, 6, 11, 15, 20, 40, 45, 34, 36, 41$.

The next such parts are 40 and 11 which give the contradiction $31 \leq 29$ as per condition (3), i.e.,

$$31 = 11 + 5(4) \not\leq \left\lfloor \frac{40 + 5(4)}{2} \right\rfloor_1 = 29.$$

The new sequence therefore becomes $4, 6, 6, 15, 20, 40, 11, 45, 34, 36, 41$.

45 and 11 are the next such parts which give that

$$46 = 11 + 5(7) \not\leq \left\lfloor \frac{45 + 5(7)}{2} \right\rfloor_1 = 39,$$

resulting in the new sequence of $4, 6, 6, 15, 20, 40, 45, 11, 34, 36, 41$.

The next and final such parts are 20 and 6 in position $i = 2$ which give

$$16 = 6 + 5(2) \not\leq \left\lfloor \frac{20 + 5(2)}{2} \right\rfloor_1 = 14.$$

This gives the final ordering $4, 15, 20, 6, 6, 40, 45, 11, 34, 36, 41$ which is of Ω-ordering.

5.1.2 Uniqueness of an Ω-ordering

Lemma 5.2. Every underlying partition has a unique Ω-ordering.

Proof. Given an r,m underlying partition for n and k, let $b_1, b_2, \ldots, b_\beta$ with $b_1 \le b_2 \le \cdots \le b_\beta$ be the parts congruent to $r, m-r \pmod m$ and $c_1, c_2, \ldots, c_\gamma$ with $c_1 < c_2 < \cdots < c_\gamma$ be the parts divisible by m. Suppose $(a_1, a_2, \ldots, a_k)$ is an Ω-ordering of this partition. Then a_1 equals either b_1 or c_1. By Conditions (3) and (4) of an Ω-ordering, if $b_1 + m \le \left\lfloor \frac{c_1+m}{2} \right\rfloor_r$, then $a_1 = b_1$. If $b_1 + m > \left\lfloor \frac{c_1+m}{2} \right\rfloor_r$, then $a_1 = c_1$. Thus a_1 is uniquely determined. We proceed inductively and assume that $a_1, a_2, \ldots, a_{j-1}$ have been uniquely determined, and that we have used parts $b_1, b_2, \ldots, b_{s-1}, c_1, c_2, \ldots, c_{t-1}$. Thus a_j equals either b_s or c_t. Once again, by properties (3) and (4), if $b_s + mj \le \left\lfloor \frac{c_t+mj}{2} \right\rfloor_r$, then $a_j = b_s$. If $b_s + mj > \left\lfloor \frac{c_t+mj}{2} \right\rfloor_r$, then $a_j = c_t$. Thus a_j is uniquely determined. This completes the proof. ◻

Note that the proof also gives an alternative way of reordering an underlying partition such that it is of Ω-ordering.

Definition 5.3. Given a partition $(a_1, a_2, \ldots, a_p)$ with $a_1 < a_2 < \cdots < a_p$, counted by $C_{r,m}(n)$, we subdivide it, from left to right, into blocks of at most two parts such that no two elements with difference greater than or equal to m ever occupy the same block. The *order* of the partition is the number of such blocks.

Example 5.3. The partition $(4, 9, 11, 14, 16, 21, 26, 31, 34, 36, 39, 46, 74, 81, 91)$ counted by $C_{1,5}(533)$ is decomposed into blocks as $(4|9, 11|14, 16|21|26|31, 34|36, 39|46|74|81|91)$ and so has order 11.

Lemma 5.3. There is a one-to-one correspondence between r,m underlying partitions for n and k and the partitions counted by $C_{r,m}(n)$ with order k.

Proof. Let $(a_1, a_2, \ldots, a_k)$ be an r,m underlying partition for n and k of Ω-ordering. Define a partition $(b_1, b_2, \ldots, b_k)$ of n by setting $b_i := a_i + m(i-1)$ for each $i = 1, 2, \ldots, k$. Each b_i that is divisible by m is replaced by the pair $\left\lfloor \frac{b_i}{2} \right\rfloor_r, \left\lceil \frac{b_i}{2} \right\rceil_r$. This resultant partition will have distinct parts. To show this, we take advantage of the properties of an Ω-ordering for $(a_1, a_2, \ldots, a_k)$ as follows:

If $i < j$ $(i \le j-1)$, $a_i \le a_j$ and each a_i is congruent to $r, m-r \pmod m$, then

$$a_i + m(i-1) \le a_j + m(j-2) \implies a_i + m(i-1) < a_j + m(j-1) \implies b_i < b_j.$$

which gives that the parts not divisible by m are distinct.

Setting $b_i := a_i + m(i-1)$ gives that $b_i + m = a_i + mi$, and by property (3) of an Ω-ordering, we have that if $i < j \implies i \leq j-1$, $a_i \equiv r, m-r \pmod{m}$ and $a_j \equiv 0 \pmod{m}$, then

$$b_i + m = a_i + mi \leq \left\lfloor \frac{a_j + mi}{2} \right\rfloor_r \leq \left\lfloor \frac{a_j + (j-1)}{2} \right\rfloor_r = \left\lfloor \frac{b_j}{2} \right\rfloor_r,$$

i.e.

$$b_i + m \leq \left\lfloor \frac{b_j}{2} \right\rfloor_r \implies b_i < \left\lfloor \frac{b_j}{2} \right\rfloor_r.$$

By property (4), we have that if $i < j \implies i \leq j-1$, $b_i \pmod{m}$ and $b_j \equiv \pm r \pmod{m}$, then $\left\lceil \frac{b_i}{2} \right\rceil_r < b_j$ using an argument similar to the above. Therefore giving that each pair $\left\lfloor \frac{b_i}{2} \right\rfloor_r, \left\lceil \frac{b_i}{2} \right\rceil_r$ will be distinct from the other parts. Thus giving that all the parts will be distinct. This procedure is uniquely reversed if parts sharing the same block are first added together, and then $m(i-1)$ is subtracted from the ith part as read from the left. ☐

Example 5.4. The $1, 5$ underlying partition for 533 and 11, given by

$$(4, 15, 20, 6, 6, 40, 45, 11, 34, 36, 41),$$

becomes

$$(4, 20, 30, 21, 26, 65, 75, 46, 74, 81, 91),$$

which becomes

$$(4, 9, 11, 14, 16, 21, 26, 31, 34, 36, 39, 46, 74, 81, 91).$$

Lemma 5.1 and Lemma 5.3, combined, give the desired correspondence which proves Schur's theorem in full generality.

5.2 An Alternative Proof of Theorem 5.1

In this section, we provide a simple and elegant alternative proof of Schur's Theorem in full generality, that is, Theorem 5.1. The proof is due to Raghavendra et al [30].
For convenience we return to the notation of a partition as a list of decreasing positive integers.

Proof. We construct a mapping from partitions enumerated by $C_{r,m}(n)$ to those enumerated by $D_{r,m}(n)$. Let $\eta = (c_1, c_2, \ldots, c_s)$ denote a partition enumerated by $C_{r,m}(n)$.

Suppose every pair c_i, c_{i+1} is such that $c_i - c_{i+1} \geq m$, then η is also enumerated by $D_{r,m}(n)$, that is, η is a fixed point of the mapping.

The forward implication in our construction then begins as follows.

Step S_1 : List the parts of η in a column in descending order. Let η_1 denote this partition.

Step S_2 : From the top, look for the first i, say β, for which $c_\beta - c_{\beta+1} < m$.

Since the parts of η belong to two residue classes modulo m, namely, r and $-r \equiv m - r \pmod{m}$, it follows that c_β and $c_{\beta+1}$ satisfy

(i) $c_\beta = m(k+1) - r$ and $c_{\beta+1} = mk + r$ or

(ii) $c_\beta = mk + r$ and $c_{\beta+1} = mk - r$.

In either case, replace c_β and $c_{\beta+1}$ with their sum $d_\beta = c_\beta + c_{\beta+1}$. Note that d_β will always be divisible by m: in case (i), $d_\beta = m(2k+1)$ and in case (ii), $d_\beta = m(2k)$.

For example, when $m = 5$ and $r = 2$, we have

$$(i) \quad \begin{matrix} 3 \\ 2 \end{matrix} \longrightarrow 5 \qquad\qquad (ii) \quad \begin{matrix} 7 \\ 3 \end{matrix} \longrightarrow 10.$$

Since the new part d_β may be smaller than the previous part $c_{\beta-1}$, we obtain the two possibilities:

Case 1. $c_{\beta-1} - d_\beta < m$,

Case 2. $c_{\beta-1} - d_\beta > m$.

Note that $m \nmid (c_{\beta-1} - d_\beta)$ since $m \nmid c_{\beta-1}$ but $m \mid d_\beta$.

For Case 1 we apply the transformation

$$\begin{pmatrix} c_{\beta-1} \\ d_\beta \end{pmatrix} \longmapsto \begin{pmatrix} d_\beta + m \\ c_{\beta-1} - m \end{pmatrix}.$$

For example, when $m = 8$ and $r = 1$, we have

$$(i) \quad \begin{matrix} 23 \\ 9 \\ 1 \end{matrix} \longrightarrow \begin{matrix} 23 \\ 10 \end{matrix} \qquad\qquad (ii) \quad \begin{matrix} 25 \\ 15 \\ 7 \end{matrix} \longrightarrow \begin{matrix} 25 \\ 22 \end{matrix} \longrightarrow \begin{matrix} 30 \\ 17 \end{matrix}$$

Case 1 in turn leads to two possibilities:

Case 1.1. $c_\beta - (d_\beta + m) < m$,

Case 1.2. $c_\beta - (d_\beta + m) > m$.

In Case 1.1., we repeat the procedure we applied for Case 1. We continue this procedure until we meet Case 1.2. or the part $(d_\beta + tm)$ reaches the top.

In Case 2 we search from the top for the next i, say ν such that $c_\nu - c_{\nu+1} < m$ and repeat the procedure S_2 until we meet Case 1.2. for ν.

The requirement that every pair of multiples of m in the resulting partition is at least $2m$ is fulfilled for the following reason.

Let $\eta = (\ldots, a_p, a_{p+1}, \ldots, a_s, a_{s+1}, \ldots)$ where (a_p, a_{p+1}) and (a_s, a_{s+1}) are pairs that qualify to be adjusted by Step S_2. Evidently $a_p + a_{p+1} \geq a_s + a_{s+1} + (\omega+2)m$ where ω is the number of parts between a_2 and a_s. Each application of this step may lift up the parts (a_p, a_{p+1}) and (a_s, a_{s+1}). For every lifting each part is increased by m; but there is no way the part induced by (a_s, a_{s+1}) can be lifted above the part induced by (a_p, a_{p+1}). Thus the final two parts must have minimal difference $2m$.

Following repeated applications of Step S_2 (a finite number of times) we reach a stage where the desired difference conditions are fulfilled for all parts of η. We associate this resulting partition η_4 which is enumerated by $D_{r,m}(n)$ to η. This gives the forward implication.

Example 5.5. Let $r = 1$ and $m = 5$, and consider $\eta = (36, 31, 29, 26, 19, 14, 11, 6, 4)$ which is counted by $C_{1,5}(176)$.

Then the mapping $C_{1,5}(176) \mapsto D_{1,5}(176)$ is as follows.

$$\begin{array}{c}36\\ \left.\begin{array}{c}31\\29\end{array}\right\}\\26\\19\\14\\11\\6\\4\end{array} \longrightarrow \begin{array}{c}\left.\begin{array}{c}36\\60\end{array}\right\}\\26\\19\\14\\11\\6\\4\end{array} \longrightarrow \begin{array}{c}65\\31\\26\\19\\ \left.\begin{array}{c}14\\11\end{array}\right\}\\6\\4\end{array} \longrightarrow \begin{array}{c}65\\31\\26\\ \left.\begin{array}{c}19\\25\end{array}\right\}\\6\\4\end{array} \longrightarrow \begin{array}{c}65\\31\\ \left.\begin{array}{c}26\\30\end{array}\right\}\\14\\6\\4\end{array} \longrightarrow \begin{array}{c}65\\ \left.\begin{array}{c}31\\35\end{array}\right\}\\21\\14\\6\\4\end{array} \longrightarrow \begin{array}{c}65\\40\\26\\21\\14\\6\\4\end{array}$$

$$\begin{array}{c}65\\40\\26\\21\\14\\ \left.\begin{array}{c}6\\4\end{array}\right\}\end{array} \longrightarrow \begin{array}{c}65\\40\\26\\21\\ \left.\begin{array}{c}14\\10\end{array}\right\}\end{array} \longrightarrow \begin{array}{c}65\\40\\26\\21\\15\\9\end{array}$$

The last object $(65, 40, 26, 21, 15, 9)$ is the partition enumerated by $D_{1,5}(176)$ that corresponds to η.

Conversely, let μ be a partition enumerated by $D_{r,m}(n)$. If μ contains no multiple of m as a part, then μ is also enumerated by $C_{r,m}(n)$. If μ contains multiples of m, then we transform it as follows to map it to $C_{r,m}(n)$:

Step T_1: Arrange the parts of μ in a column in decreasing order. Denote this partition by μ_1.

Step T_2: From the bottom upward search for the first multiple of m, say x. Now split x into (x_1, x_2) tentatively, as follows:

$$\begin{aligned} x = m \cdot (2k) &\longrightarrow (mk + r, mk - r), \\ x = m \cdot (2k + 1) &\longrightarrow (m(k + 1) - r, mk + r). \end{aligned} \tag{5.1}$$

Suppose there is a part y below x. If $y < x_2$, then the tentative splitting gives the result we desire, that is,

$$x \longrightarrow \begin{pmatrix} mk + r \\ mk - r \end{pmatrix} \quad \text{or} \quad x \longrightarrow \begin{pmatrix} m(k + 1) - r \\ mk + r \end{pmatrix}.$$

But if $y \geq x_2$, then we replace

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} y+m \\ x-m \end{pmatrix}.$$

Now split $x - m$ into (x_1, x_2) as before, and apply the same transformation to $x - m$ and the part below it. Repeat the process until the part below $x - tm$ is sufficiently small and then finalise the split. Do this for all multiples of m.

Clearly, Step $\mathcal{S}_2$ will not form new multiples of m. If $y < x_2$, the substitute entries x_1 and x_2 certainly are not multiples of m. If $y \geq x_2$, we note that we add m and subtract m from y and x respectively which does not alter the parities of the pair (mod m).

We apply step $\mathcal{S}_2$ until all multiples of m in μ are split into parts congruent to $\pm\, r (\text{mod } m)$.

The resulting partition will be a partition enumerated by $C_{r,m}(n)$.

Remark: The splitting caused by x will never go beneath or between the ones resulting from y.

Example 5.6. Let $m = 5$ and $r = 1$. Consider the image partition obtained earlier, that is, $\mu = (65, 40, 26, 21, 15, 9)$ which is counted by $D_{1,5}(176)$. We now recover the pre-image partition counted by $C_{1,5}(176)$ as follows:

$$\begin{array}{c}65\\40\\26\\21\\\left.\begin{array}{c}15\\9\end{array}\right\}\\ \\ \\ \end{array}\longrightarrow\begin{array}{c}65\\40\\26\\21\\14\\\left.10\right\}\\ \\ \\ \end{array}\longrightarrow\begin{array}{c}65\\\left.\begin{array}{c}40\\26\end{array}\right\}\\21\\14\\6\\4\\ \\ \end{array}\longrightarrow\begin{array}{c}65\\31\\\left.\begin{array}{c}35\\21\end{array}\right\}\\14\\6\\4\\ \\ \end{array}\longrightarrow\begin{array}{c}65\\31\\26\\\left.\begin{array}{c}30\\14\end{array}\right\}\\6\\4\\ \\ \end{array}\longrightarrow\begin{array}{c}65\\31\\26\\19\\\left.25\right\}\\6\\4\\ \\ \end{array}\longrightarrow\begin{array}{c}65\\31\\26\\19\\14\\11\\6\\4\end{array}$$

$$\begin{array}{c}\left.\begin{array}{c}65\\31\end{array}\right\}\\26\\19\\14\\11\\6\\4\\ \\ \end{array}\longrightarrow\begin{array}{c}36\\\left.60\right\}\\26\\19\\14\\11\\6\\4\\ \\ \end{array}\longrightarrow\begin{array}{c}36\\31\\29\\26\\19\\14\\11\\6\\4\end{array}$$

Thus we arrive again at the partition $(36, 31, 29, 26, 19, 14, 11, 6, 4)$ counted by $C_{1,5}(176)$, as expected.
Evidently, the two mappings $C_{r,m}(n) \longrightarrow D_{r,m}(n)$ and $D_{r,m}(n) \longrightarrow C_{r,m}(n)$ are inverses to each other.
This completes our proof. $\square$

Chapter 6

Discussion and Conclusion

6.1 Further Work

Profound discoveries are hardly built upon foundations of misconceptions. Schur's Partition Theorem is one of such seldom encountered cases. Throughout the paper, we have covered the topic through different lenses. All the same, an extended generalization of Schur's partition theorem was studied by Alladi and Gordon, and we will mention it shortly. Schur's theorem can also be studied within the domain of a broader class of partitions called **overpartitions**. In fact, a corresponding analogue of Schur's theorem exists and we shall present the summary below.

We state the full generalization of Schur's Partition Theorem by Alladi and Gordon.

6.1.1 Alladi-Gordon's Generalization

Theorem 6.1. *[2] Let $M \geq 3$ and $0 < \alpha < \beta < M \leq \alpha + \beta$. Let $A(n;k)$ denote the number of partitions of n into k distinct parts congruent to α or β (mod M). Let $B(n;k)$ denote the number of partitions of n into k parts congruent to α, β or $\alpha + \beta$ (mod M) such that:*

(i) the difference between any two parts is $\geq M$;

(ii) the difference between parts congruent to $(\alpha + \beta)$ (mod M) is $> M$;

(iii) the parts congruent to $(\alpha + \beta)$ (mod M) are counted twice.

Then

$$A(n;k) = B(n;k).$$

This reduces to the fundamental theorem of Schur when $\alpha = 1$, $\beta = 2$ and $M = 3$ (see Theorem 4.1, parts 2 and 3).

6.1.2 Overpartitions and Schur's Theorem for Overpartitions

An **overpartition** of a positive integer n is a non-increasing sequence of natural numbers that sum up to n in which the first occurrence of a number may be overlined. For example, we have that $\overline{p}(4) = 14$ and set $\overline{p}(0) := 1$. The 14 overpartitions of 4 are $4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1$ and $\overline{1}+1+1+1$.

An Analogue of Schur's Theorem on Overpartitions

Theorem 6.2. *[25] Let $A(k,n)$ denote the number of overpartitions of n into parts congruent to $\pm 1 \pmod 3$ with k non-overlined parts. Let $B(k,n)$ denote the number of overpartitions of n with k non-overlined parts, where parts differ by at least 3 if the smaller is overlined **or** both parts are divisible by 3, and parts differ by at least 6 if the smaller is overlined **and** both parts are divisible by 3. Then $A(k,n) = B(k,n)$.*

The case $k = 0$ gives Schur's theorem for regular partitions.

The theorem can be proved by aid of generating functions in three ways. The most undemanding of the three is of a similar nature to the one for regular partitions. We start by defining a function $b_j(k,m,n)$, that counts the partitions counted by $B(k,n)$ having m parts such that the smallest is greater than j. From there, formulate and solve a system of four recurrences.

The details may be found in [25].

6.1.3 An Analogue of Bressoud's Generalization on Overpartitions

Theorem 6.3. *[29] Given positive integers k, r and m such that $r < \frac{m}{2}$, let $C_{k,r,m}(n)$ denote the number of overpartitions into parts congruent to $\pm r \pmod m$ with k non-overlined parts. Let $D_{k,r,m}(n)$ denote the number of partitions of n into parts congruent to $0, \pm r \pmod m$ where parts differ by at least $2m$ if the smaller is overlined and both parts are divisible by m **and** there are k non-overlined parts. Then*

$$C_{k,r,m}(n) = D_{k,r,m}(n) \quad \forall \;\; n.$$

The proof is analogous to that for Schur's Theorem in full generality discussed in Section 5.2. On the forward implication, we instead search for the i's that are such that $b_i - b_{i+1} < m$ and b_{i+1} is overlined. All such parts are once again replaced with their sum, with all cases considered rigorously.

Conversely, from the bottom, search for the first multiple of m and split it into (x_1, x_2) **if** there is no part below it. From there, the proof follows a way similar to that in Section 5.2. The reader is referred to [29] for the details.

6.2 Conclusion

Following the elapse of almost two years and a roller coaster of brainstorming sessions, intense studying and careful scrutiny of the subject at hand, the research has resulted into a remarkable success. Indeed we did not cover the subject together with its extensions in entirety. That of course is nothing short of what we hoped to have pursued but unfortunately, due to time constraint, and the extensions of the topic being over the scope of what is required for my dissertation and the qualification, that could not be translated into reality. Else, the dissertation has been an enormous success. I hope to soon resume my quest on studying Schur's theorem in its entirety.

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