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Research article

A progressive approach to solving a generalized CEV-type model by applying symmetry-invariant surface conditions

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Abstract: In this paper, we examine a type of constant elasticity of variance model that is subject to its terminal condition. We prove that certain transformations may be applied to obtain a simpler equation that has known solution processes. Four cases are obtained that play a role in specifying the many unknown parameters of the model. The corresponding terminal condition is transformed into an initial condition, and we then demonstrate how to solve this Cauchy problem by using Lie symmetries and Poisson's formula. Finally, we examine the behaviour of the obtained solutions.

Keywords: constant elasticity of variance; Lie symmetries; convolution; boundary conditions
Mathematics Subject Classification: 35K05, 35K15, 37C79

1. Introduction

In the theory of partial differential equations (PDEs) of a financial nature, the Black-Scholes model [1] is arguably the most researched equation in mathematical finance. Most importantly, in [1], the equation was reduced to the classical heat equation. Other mathematical interest in such equations is attributable to stock prices often being linked to Brownian motion [2]. However, more complicated models exist that may not be reduced to simple models with known solutions.

A constant elasticity of variance (CEV) model [3] is commonly used to investigate option and asset pricing. Consider a type of CEV model that has been modified for a portfolio optimisation problem, and that contains unknown constants [4], viz.,

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= -\left(\frac{\alpha\lambda-\delta}{1-\lambda}+\frac{\lambda x(\beta-\alpha)^2}{K^2\left(2(\lambda-1)^2\right)}\right)u(x,t)-\left(\frac{2\gamma\lambda(\beta-\alpha)}{\lambda-1}-2\beta\gamma\right)x\frac{\partial}{\partial x}u(x,t) \\ &\quad -\gamma(2\gamma+1)K^2\frac{\partial u(x,t)}{\partial x}-2\gamma^2K^2x\frac{\partial^2}{\partial x\,\partial x}u(x,t), \quad x>0,\ t\in[0,T] \end{aligned} \tag{1.1}$$

where x is the stock/asset price at current time t and $u(x, t)$ is the value function for the portfolio optimisation problem. The constants $\alpha, \beta, \gamma, \delta, \lambda$ and K represent the interest rate, the expected instantaneous return rate of stock, the elasticity parameter, a subjective discount, a parameter linked to a utility function assumed by the model for optimal investment and consumption and a constant related to the instantaneous volatility of the stock, respectively.

This model must satisfy a terminal or boundary condition, namely [4],

$$u(x, T) = 1 \tag{1.2}$$

where T is the time of expiry. This approach will work for other types of terminal conditions.

Equation (1.1) is a linear parabolic equation, which may be studied from various perspectives [5–7], but finding a solution is nontrivial.

Within the remarkable theory of Lie's classification of linear second-order PDEs [8] is a nontrivial and rarely used mechanism to transform certain models into a heat transfer equation. This is our approach in the present study. We find new transformations that convert the above model into the classical heat equation. This provides us with selection criteria for the numerous free parameters of the model, resulting in four cases of interest.

The solution of Cauchy problems for parabolic PDEs, such as (1.1), can be manipulated into a particular solution with specific singularities or so-called fundamental solutions [9]. In some cases, the fundamental solution is an invariant solution [10]. We shall prove this property for the model under study.

As a major achievement in this study, the invariant surface condition is exploited to find a symmetry generator that will solve the model that is subject to the terminal condition. This is also a nontrivial task, and it is the first time that such solutions are pursued. As we shall demonstrate, these new symmetries are extremely involved; but nonetheless, invariant solutions are possible.

The technique of Lie symmetries is rarely applied to evaluate PDEs that are subject to initial or boundary conditions. In most of the literature, initial conditions are ignored under symmetry analysis; however, in this study, we highlight the need for symmetries when tackling such problems.

The structure of the paper follows. In Section 2, we prove the transformations of the model that we require. Section 3 describes the initial conditions and the theory used to obtain the solutions of the model. Finally, we conclude in Section 4.

2. The construction of transformations

In what follows, we use the heat equation as a tool to solve a Cauchy problem. Hence, we first need to transform the PDE (1.1) and its terminal condition. It turns out that this model is transformable under specific restrictions on its arbitrary parameters. Hence, we prove the following result.

Theorem 1. *The model (1.1) is reducible to the 1+1 heat equation*

$$\omega_\tau - \omega_{yy} = 0 \tag{2.1}$$

under the conditions of the following cases:

(a) Case I: $\alpha = \sqrt{2},\ \lambda = \frac{1}{2},\ \beta = 1,\ K = \frac{1}{\gamma},\ \gamma = -1$ *and* δ *is arbitrary.*

The transformations are

$$\tau = T - t, \tag{2.2}$$

$$y = \sqrt{2x}. \tag{2.3}$$

(b) *Case II:* $\alpha = \sqrt{\lambda + 1},\ \lambda = 2,\ \beta = \sqrt{3\lambda},\ K = -1,\ \gamma = 1$ *and* δ *is arbitrary.*

The transformations are

$$\tau = T - t, \tag{2.4}$$

$$y = \sqrt{2x}. \tag{2.5}$$

(c) *Case III:* $\alpha = \frac{1}{\beta},\ \lambda = 4,\ \beta = \sqrt{2},\ \gamma = -1$ *and* $K,\ \delta$ *are arbitrary.*

The transformations are

$$\tau = T - t, \tag{2.6}$$

$$y = \frac{\sqrt{2x}}{K}. \tag{2.7}$$

(d) *Case IV:* $\beta = \alpha\sqrt{\lambda},\ \gamma = 1$ *and* $\alpha,\ \lambda,\ K,\ \delta$ *are arbitrary.*

The transformations are

$$\tau = T - t, \tag{2.8}$$

$$y = \frac{\sqrt{2x}}{\gamma K}. \tag{2.9}$$

Proof of Theorem 1. Consider Case I. An invertible change of the independent variables of the form y and τ reduces (1.1) to

$$a(y,\tau)u_y + c(y,\tau)u + u_\tau - u_{yy} = 0, \tag{2.10}$$

where

$$u(y,\tau), a(y,\tau) = \left(\sqrt{2} - 2\right)y \text{ and } c(y,\tau) = 2\delta + \left(\sqrt{2} - \frac{3}{2}\right)y^2 - \sqrt{2}.$$

Thereafter, under another transformation, the dependent variable $u(y,\tau)$ undergoes a transformation, viz.,

$$u(y,\tau) = \omega(y,\tau)e^{-\phi(y,\tau)}, \tag{2.11}$$

where

$$\phi(y,\tau) = 2\delta\tau + \tau - \frac{3\tau}{\sqrt{2}} - \frac{1}{4}\left(\sqrt{2} - 2\right)y^2. \tag{2.12}$$

This transformation procedure converts (2.10) to the heat Eq (2.1).

The proofs for Case II–IV are analogous, but where

$$a(y,\tau) = -\frac{2\left(\frac{1}{2}\left(\sqrt{6} - 2\sqrt{3}\right)y^2 + 1\right)}{y} \quad \text{and} \quad c(y,\tau) = \frac{1}{2}\left(2\left(2\sqrt{3} - \delta\right) - \left(\sqrt{3} - \sqrt{6}\right)^2 y^2\right)$$

for Case II,

$$a(y,\tau) = -\frac{1}{3}\left(\sqrt{2}y\right) \quad \text{and} \quad c(y,\tau) = \frac{1}{3}\left(2\sqrt{2} - \delta\right) - \frac{1}{9}\left(\frac{1}{\sqrt{2}} - \sqrt{2}\right)^2 y^2$$

for Case III, and

$$a(y,\tau) = \frac{\sqrt{\lambda}\left(\alpha y^2 - 2\right) - 2}{\left(\sqrt{\lambda}+1\right)y} \quad \text{and} \quad c(y,\tau) = \frac{\alpha\lambda - \delta}{\lambda - 1} - \frac{\alpha^2 \lambda y^2}{4\left(\sqrt{\lambda}+1\right)^2}$$

for Case IV.

Similarly, we define the transformation (2.11)

$$\phi(y,\tau) = -\delta\tau - \sqrt{3}\tau + 3\sqrt{\frac{3}{2}}\tau + \frac{1}{4}\left(\sqrt{2}-2\right)\sqrt{3}y^2 + \log(y) \tag{2.13}$$

for Case II,

$$\phi(y,\tau) = -\frac{\delta\tau}{3} + \frac{5\tau}{3\sqrt{2}} + \frac{y^2}{6\sqrt{2}} \tag{2.14}$$

for Case III, and

$$\phi(y,\tau) = \frac{3\alpha\sqrt{\lambda}\tau}{2(\lambda-1)} - \frac{\alpha\lambda\tau}{2(\lambda-1)} - \frac{\delta\tau}{\lambda-1} - \frac{\alpha\sqrt{\lambda}y^2}{4\left(\sqrt{\lambda}+1\right)} + \log(y) \tag{2.15}$$

for Case IV. □

All of the restrictions on the above parameters are mathematical requirements, but they do not violate any financial interpretations of the model. The elasticity parameter γ is often negative, but special situations do exist for positive elasticity. It follows that the same transformation is applicable to the terminal condition; hence, we have the following lemma.

Lemma 1. *The terminal condition (1.2) is transformed via the transformations given by (2.2)–(2.15), such that*

$$\omega(y,\tau) = e^{2\delta\tau+\tau-\frac{3\tau}{\sqrt{2}}-\frac{1}{4}\left(\sqrt{2}-2\right)y^2} \tag{2.16}$$

for Case I,

$$\omega(y,\tau) = e^{-\delta\tau-\sqrt{3}\tau+3\sqrt{\frac{3}{2}}\tau+\frac{1}{4}\left(\sqrt{2}-2\right)\sqrt{3}y^2+\log(y)} \tag{2.17}$$

for Case II,

$$\omega(y,\tau) = e^{-\frac{\delta\tau}{3}+\frac{5\tau}{3\sqrt{2}}+\frac{y^2}{6\sqrt{2}}} \tag{2.18}$$

for Case III, and

$$\omega(y,\tau) = e^{\frac{3\alpha\sqrt{\lambda}\tau}{2(\lambda-1)}-\frac{\alpha\lambda\tau}{2(\lambda-1)}-\frac{\delta\tau}{\lambda-1}-\frac{\alpha\sqrt{\lambda}y^2}{4\left(\sqrt{\lambda}+1\right)}+\log(y)} \tag{2.19}$$

for Case IV.

The proof is rudimentary and therefore left to the reader.

3. The initial problem

It is well known that the heat Eq (2.1) is an important PDE that has fuelled many analytical studies of the past [11–15]. Significantly, it admits a fundamental solution, which is pivotal to our solution process.

From the lemma, let $\omega(y,0) = F(y)$; next, the Lie point symmetries of (2.1) are of the form

$$\begin{aligned} X &= \xi(y,\tau,\omega)\frac{\partial}{\partial y} + \eta(y,\tau,\omega)\frac{\partial}{\partial \tau} + \phi(y,\tau,\omega)\frac{\partial}{\partial \omega} \\ &= \left(4c_1\tau^2 + 2c_2\tau + c_6\right)\frac{\partial}{\partial \tau} + \left(4c_1 y\tau + c_2 y + 2c_3\tau + c_4\right)\frac{\partial}{\partial y} \\ &\quad \left(\left(c_1(-2\tau - y^2) - c_3 y + c_5\right)\omega + \alpha(y,\tau)\right)\frac{\partial}{\partial \omega}, \end{aligned} \tag{3.1}$$

where $\alpha(y,\tau)$ is an arbitrary solution of the heat equation, $\alpha_\tau = \alpha_{yy}$. Hence, the invariant surface condition

$$\xi(y,\tau,\omega)\omega_y + \eta(y,\tau,\omega)\omega_\tau - \bar{\phi}(y,\tau,\omega) = 0 \tag{3.2}$$

holds at the boundary where $\omega(y,0)$, such that [16]

$$\frac{\bar{\phi}(y,0,F(y)) - \xi(y,0,F(y))F'(y)}{\eta(y,0,F(y))} = \omega_{yy}. \tag{3.3}$$

From condition (3.3), we have that

$$\alpha(y,0) = \left(c_1 y^2 + c_3 y - c_5\right)F(y) + \left(c_5 y + c_4\right)F'(y) + c_6 F''(y). \tag{3.4}$$

Consider the following initial value problem for the heat Eq (2.1):

$$\begin{cases} \alpha_\tau - \alpha_{yy} = 0, & \text{in } \mathbb{R}\times(0,\infty), \\ \alpha(y,0), & \text{on } \mathbb{R}\times\{\tau = 0\}. \end{cases} \tag{3.5}$$

The Cauchy problem (3.5) is well-posed. For the uniqueness of the solution, see Tikhonov [17].

Now, the fundamental solution of (2.1) is given by

$$A(y,\tau) := \begin{cases} \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{y^2}{4\tau}}, & (y\in\mathbb{R}, \tau > 0), \\ 0, & (y\in\mathbb{R}, \tau < 0), \end{cases} \tag{3.6}$$

which is singular at the point (0,0). The function $A(y-m,\tau)$ is also a solution for each fixed $y \in \mathbb{R}$; then, consequently, the convolution [18]

$$\alpha(y,\tau) = \int_{\mathbb{R}} A(y-m,\tau)\alpha(m,0)dm, \quad (y\in\mathbb{R}, \tau > 0) \tag{3.7}$$

is a solution to the one-dimensional heat Eq (2.1). This integral is also called Poisson's formula.

As mentioned before, we may now apply the above theoretical considerations to solve the heat Eq (2.1) as subject to its initial values. Subsequently, we reverse all of the transformations and ultimately solve the original PDE, i.e., the CEV model. By virtue of the above formulation, these solutions will satisfy the terminal condition (1.2). Note that these solutions are lengthy and we have not provided an exhaustive list of obtainable solutions, but, rather, we have depicted selected solutions for each of the four cases. More solutions do exist.

Suppose that we begin with Case I, and let

$$c_4 = 1,\ c_1 = c_2 = c_3 = c_5 = c_6 = 0$$

for simplicity. Therefore,

$$\alpha(y,0) = F'(y),$$

and the resultant symmetry that incorporates the initial condition is given by

$$X = \frac{\partial}{\partial y} + \alpha(y,\tau)\frac{\partial}{\partial \omega}. \tag{3.8}$$

Hence, with the condition (3.2), we find that the solution to (2.1) is given by

$$\omega(y,\tau) = \int \alpha(y,\tau)dy + H(\tau).$$

Without loss of generality, we let $H(\tau) = 0$, and we must then determine $\alpha(y,\tau)$. Since the α function is a solution to the heat Eq (2.1), it will also satisfy the convolution (3.7). Thus, by integration via Mathematica, we obtain

$$\alpha(y,\tau) = -\frac{\left(\sqrt{2}-2\right) y e^{-\frac{\left(\sqrt{2}-2\right)y^2}{4\left(\sqrt{2}-2\right)\tau+4}}}{2\sqrt{\frac{1}{\tau}+\sqrt{2}-2}\,\sqrt{\tau}\left(\left(\sqrt{2}-2\right)\tau+1\right)}, \tag{3.9}$$

and we can integrate (3.9) to find $\omega(y,\tau)$, which is a solution to the heat Eq (2.1), viz.,

$$\omega(y,\tau) = \frac{e^{-\frac{\left(\sqrt{2}-2\right)y^2}{4\left(\sqrt{2}-2\right)\tau+4}}}{\sqrt{\frac{1}{\tau}+\sqrt{2}-2}\,\sqrt{\tau}}. \tag{3.10}$$

Now, the transformations from Theorem 1 are used to reverse the substitutions and yield the solution to the model (1.1), viz., $u(x,t)$;

$$u(x,t) = \frac{\exp\left(2\delta(t-T) - \frac{2\left(\sqrt{2}-2\right)x}{4\left(\sqrt{2}-2\right)(T-t)+4} + \frac{3(T-t)}{\sqrt{2}} + t - T + \frac{1}{2}\left(\sqrt{2}-2\right)x\right)}{\sqrt{\left(\sqrt{2}-2\right)(T-t)+1}}. \tag{3.11}$$

If we let $t = T$ be the time of expiry, we see that (3.11) simplifies to

$$u(x,T) = 1,$$

which shows that the solution is subject to the terminal condition. All solutions below satisfy this relation.

Similarly, for Case II, the same procedure is followed and we obtain the following function

$$\alpha(y,\tau)=\sqrt{\frac{1}{\tau}+2\sqrt{3}-\sqrt{6}}\,\sqrt{\tau}e^{\frac{\sqrt{3}\left(\sqrt{2}-2\right)y^2}{4-4\sqrt{3}\left(\sqrt{2}-2\right)\tau}}\times\frac{\left(\left(36-24\sqrt{2}\right)\tau^2-2\tau\left(\left(9-6\sqrt{2}\right)y^2+2\left(\sqrt{2}-2\right)\sqrt{3}\right)+\left(\sqrt{2}-2\right)\sqrt{3}y^2+2\right)}{2\left(\sqrt{3}\left(\sqrt{2}-2\right)\tau-1\right)\left(6\left(7\sqrt{2}-10\right)\sqrt{3}\tau^3+18\left(2\sqrt{2}-3\right)\tau^2+3\left(\sqrt{2}-2\right)\sqrt{3}\tau-1\right)}. \tag{3.12}$$

We then integrate (3.12) to find

$$\omega(y,\tau)=-\frac{\sqrt{\frac{1}{\tau}+2\sqrt{3}-\sqrt{6}}\,\sqrt{\tau}\left(6\left(2\sqrt{2}-3\right)\tau+\left(\sqrt{2}-2\right)\sqrt{3}\right)ye^{\frac{\sqrt{3}\left(\sqrt{2}-2\right)y^2}{4-4\sqrt{3}\left(\sqrt{2}-2\right)\tau}}}{\sqrt{3}\left(\sqrt{2}-2\right)\left(6\left(7\sqrt{2}-10\right)\sqrt{3}\tau^3+18\left(2\sqrt{2}-3\right)\tau^2+3\left(\sqrt{2}-2\right)\sqrt{3}\tau-1\right)}, \tag{3.13}$$

and it follows that the solution to the original PDE is given by

$$\begin{aligned}u(x,t)=&-\exp\left(\delta(T-t)+\frac{2\left(\sqrt{2}-2\right)\sqrt{3}x}{4\left(\sqrt{2}-2\right)\sqrt{3}(t-T)+4}\right)\\&\times\exp\left(\sqrt{3}(T-t)+3\sqrt{\frac{3}{2}}(t-T)-\frac{1}{2}\sqrt{3}\left(\sqrt{2}-2\right)x\right)\\&\times\frac{\sqrt{\left(2\sqrt{3}-\sqrt{6}\right)(T-t)+1}\left(6\left(2\sqrt{2}-3\right)(T-t)+\left(\sqrt{2}-2\right)\sqrt{3}\right)}{\sqrt{3}\left(\sqrt{2}-2\right)}\\&\times\frac{1}{\left(6\left(7\sqrt{2}-10\right)\sqrt{3}(T-t)^3+3\left(\sqrt{2}-2\right)\sqrt{3}(T-t)+18\left(2\sqrt{2}-3\right)(t-T)^2-1\right)}.\end{aligned} \tag{3.14}$$

For Case III, we have the following functions:

$$\alpha(y,\tau)=\frac{\sqrt{\frac{3}{2}}ye^{\frac{y^2}{6\sqrt{2}-4\tau}}}{\left(3-\sqrt{2}\tau\right)^{3/2}} \tag{3.15}$$

and

$$\omega(y,\tau)=\frac{\sqrt{\frac{3}{2}}\left(3\sqrt{2}-2\tau\right)e^{\frac{y^2}{6\sqrt{2}-4\tau}}}{\left(3-\sqrt{2}\tau\right)^{3/2}}, \tag{3.16}$$

and it follows that the solution to (1.1) is

$$u(x,t)=\frac{\sqrt{\frac{3}{2}}\left(2t-2T+3\sqrt{2}\right)\exp\left(\frac{1}{6}\left(-2\delta(t-T)+\frac{6x}{K^2\left(2t-2T+3\sqrt{2}\right)}+5\sqrt{2}(t-T)-\frac{\sqrt{2}x}{K^2}\right)\right)}{\left(\sqrt{2}t-\sqrt{2}T+3\right)^{3/2}}. \tag{3.17}$$

For Case IV, we have the following functions:

$$\alpha(y,\tau)=\frac{\left(\sqrt{\lambda}+1\right)^2\sqrt{\tau}\sqrt{-\frac{\alpha}{\sqrt{\lambda}+1}+\alpha+\frac{1}{\tau}}e^{-\frac{\alpha\sqrt{\lambda}y^2}{4\sqrt{\lambda}(\alpha\tau+1)+4}}\left(\sqrt{\lambda}\left(2\alpha\tau-\alpha y^2+2\right)+2\right)}{2\left(\sqrt{\lambda}(\alpha\tau+1)+1\right)^3} \tag{3.18}$$

and

$$\omega(y,\tau)=\frac{\left(\sqrt{\lambda}+1\right)^2\sqrt{\tau}y\sqrt{-\frac{\alpha}{\sqrt{\lambda}+1}+\alpha+\frac{1}{\tau}}e^{-\frac{\alpha\sqrt{\lambda}y^2}{4\sqrt{\lambda}(\alpha\tau+1)+4}}}{\left(\sqrt{\lambda}(\alpha\tau+1)+1\right)^2}, \tag{3.19}$$

and it follows that another solution to the original Eq (1.1) is given by

$$\begin{aligned} u(x,t)=&\exp\left(\frac{1}{2}\left(-\frac{t\left(\alpha\lambda-3\alpha\sqrt{\lambda}+2\delta\right)}{\lambda-1}+\frac{\alpha\sqrt{\lambda}x\left(\frac{1}{\sqrt{\lambda}+1}+\frac{1}{\sqrt{\lambda}(\alpha t-\alpha T-1)-1}\right)}{K^2}\right)\right)\\ &\exp\left(\frac{1}{2}\left(\frac{T\left(\alpha\lambda-3\alpha\sqrt{\lambda}+2\delta\right)}{\lambda-1}\right)\right)\frac{\left(\sqrt{\lambda}+1\right)^2\sqrt{\left(\alpha-\frac{\alpha}{\sqrt{\lambda}+1}\right)(T-t)+1}}{\left(\sqrt{\lambda}(\alpha t-\alpha T-1)-1\right)^2}. \end{aligned} \tag{3.20}$$

We have recorded four unique solutions for (1.1), as seen above; they are obtainable through convolution and invariance, as well as restrictions on the parameters $K,\alpha,\beta,\gamma,\delta$ and λ. Theorem 1 highlights how these parameters were selected.

Table 1 shows the arbitrary parameters for various solutions, as per Figure 1.

Table 1. Values of the arbitrary constants across all four solutions.

Parameter	Solution 1 (3.11)	Solution 2 (3.14)	Solution 3 (3.17)	Solution 4 (3.20)
T	2	2	2	2
δ	0.5	0.5	0.5	0.5
K			2	2
α				1.5
λ				0.75

Below are the graphical illustrations of the analytical solutions (3.11), (3.14), (3.17) and (3.20), respectively.

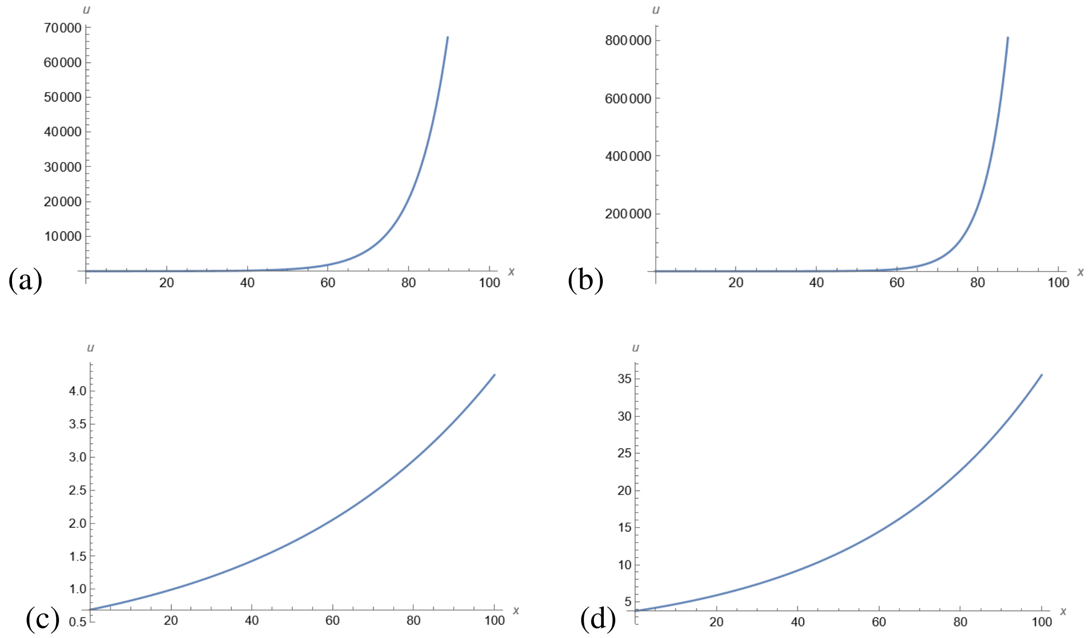

Figure 1. Graphical illustrations of the four analytical solutions ((a)–(d): (3.11), (3.14), (3.17) and (3.20)), showing the relationship between $u(x, t)$ and the asset price x. The respective t value is $t = 1.5$ in (a)–(d).

4. Conclusions

Investment and consumption problems constitute an active area of study, although not many findings report exact solutions that are subject to the terminal condition. In this work, we have successfully derived such solutions for several parameter values. This approach may be applied to similar types of models; however, the convolution integrals may sometimes be challenging to solve.

We have derived new symmetry generators, viz., (3.8) for each case with the α coefficients respectively given by (3.9), (3.12), (3.15) and (3.18). These symmetries with the α coefficients are extremely involved, but they are essential in the sense that they incorporate the important initial conditions. Hence, these are novel solutions and we have presented a substantial advancement in the application of symmetry groups to models in the mathematics of finance.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Political Eco.*, **81** (1973), 637–654. https://doi.org/10.1086/260062
2. L. Bachelier, Théorie de la spéculation, *An. Sci. Ecole Norm. Super.*, **17** (1900), 21–86. https://doi.org/10.24033/asens.476
3. J. C. Cox, The constant elasticity of variance option pricing model, *J. Portfolio Manage.*, **22** (1996), 15–17. https://doi.org/10.3905/jpm.1996.015
4. H. Chang, X. Rong, H. Zhao, C. Zhang, Optimal investment and consumption decisions under the constant elasticity of variance model, *Math. Probl. Eng.*, **2013** (2013), 974098. https://doi.org/10.1155/2013/974098
5. Y. L. Hsu, T. I. Lin, C. F. Lee, Constant elasticity of variance (CEV) option pricing model: integration and detailed derivation, *Math. Comput. Simul.*, **79** (2008), 60–71. https://doi.org/10.1016/j.matcom.2007.09.012
6. T. Motsepa, T. Aziz, A. Fatima, C. M. Khalique, Algebraic aspects of evolution partial differential equation arising in the study of constant elasticity of variance model from financial mathematics, *Open Phys.*, **16** (2018), 31–36. https://doi.org/10.1515/phys-2018-0006
7. A. Bakkaloglu, T. Aziz, A. Fatima, F. M. Mahomed, C. M. Khalique, Invariant approach to optimal investment-consumption problem: the constant elasticity of variance (CEV) model, *Math. Methods Appl. Sci.*, **40** (2017), 1382–1395. https://doi.org/10.1002/mma.4060
8. S. Lie, On integration of a class of linear partial differential equations by means of definite integrals, *Arch. Math.*, **6** (1881), 328–368.
9. J. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, Yale University Press, 1923.
10. N. H. Ibragimov, *Primer on group analysis*, Moscow: Znanie, 1989.
11. R. Maphanga, S. Jamal, A terminal condition in linear bond-pricing under symmetry invariance, *J. Nonlinear Math. Phys.*, **30** (2023), 1295–1304. https://doi.org/10.1007/s44198-023-00132-6
12. S. Jamal, Imaging noise suppression: fourth-order partial differential equations and travelling wave, *Mathematics*, **8** (2020), 2019. https://doi.org/10.3390/math8112019
13. U. Obaidullah, S. Jamal, A computational procedure for exact solutions of Burgers' hierarchy of nonlinear partial differential equations, *J. Appl. Math. Comput.*, **65** (2021), 541–551. https://doi.org/10.1007/s12190-020-01403-x
14. U. Obaidullah, S. Jamal, On the formulaic solution of a $(n + 1)$th order differential equation, *Int. J. Appl. Comput. Math.*, **7** (2021), 58. https://doi.org/10.1007/s40819-021-01010-9
15. F. M. Mahomed, Complete invariant characterization of scalar linear (1+1) parabolic equations, *J. Nonlinear Math. Phys.*, **15** (2008), 112–123. https://doi.org/10.2991/jnmp.2008.15.s1.10

16. J. Goard, Noninvariant boundary conditions, *Appl. Anal.*, **82** (2003), 473–481. https://doi.org/10.1080/0003681031000109639

17. A. N. Tikhonov, Uniqueness theorems for the heat equation, *Dokl. Akad. Nauk SSSR*, **6** (1935), 294–300.

18. L. C. Evans, *Partial differential equations*, 2 Eds., Rhode Island: American Mathematical Society, 2010. https://doi.org/10.1090/gsm/019