The Role of Convection in Stellar Models

by

Sashin Moonsamy

Thesis presented in fulfilment of the requirements for the degree of

Master of Science in Physics

at the University of the Witwatersrand

School of Physics,
University of the Witwatersrand,
PO Box Wits, Johannesburg, 2050, South Africa

Supervisor: Dr. F.A.M Frescura

May 2011

Abstract

The Role of Convection in Stellar Models

S. Moonsamy

School of Physics,

University of the Witwatersrand,

PO Box Wits, Johannesburg, 2050, South Africa

Thesis: MSc (Physics)

May 2011

This thesis reviews the essential ingredients of local, time-independent mixing-length theory, the convective stability of fluid elements in the stellar interior, the origins and influence of convection zones in stars, and the numerical implementation of convection in some popular stellar evolution codes. The thesis concludes with a brief discussion of the future role of mixing-length theory in stellar modelling.

Contents

Chapter 1

Introduction

"It remains to call attention to the chief outstanding difficulty of our subject." H. Lamb.

A key problem in stellar astrophysics is understanding the structure and evolution of stars. Such understanding cannot be achieved through observation alone, but is based on the correlation of model predictions with real observations. The mathematical models are highly complex. They involve coupled systems of partial differential equations which do not admit analytic solution in terms of elementary functions. To explore the properties of these models, we must resort to numerical solution through computer codes.

Computer models of stellar structure have been severely hampered by several key problems. These include the description of stellar convection, the inclusion of non-linear effects in stellar pulsation, and the effect of rapid rotation on the pulsation modes [96]. To date, these problems have been inadequately addressed by replacing the full theory by highly simplified versions thereof. In particular, current models of stellar structure do not incorporate in a realistic way one of the key features of real stars: turbulent convection.

Until recently, stellar models have relied almost exclusively on a semi phenomenological theory of stellar convection, known as mixing-length theory (MLT) [5]. Recent work in the field has suggested that this theory underestimates hugely the rate of energy transport through the stellar material [10]. This has far reaching consequences since the predicted rate of energy transport impacts severely on the evolution of stars, their ages and lifetimes and consequently also on our understanding of the evolution and structure of galaxies and of the cosmos.

The modelling and theoretical understanding of this phenomenon has proven to be a considerable challenge to both theorists and observers. However, during the last two decades, rapid advances in computer technology coupled with observational data of an unprecedented resolution and accuracy has permitted a new look at the field.

The purpose of this thesis is to review the essential ingredients of mixing length theory and consider its role in our present understanding of the structure and evolution of stars.

Physically, convection involves mass motions: hot material moves upward as cooler, denser material sinks. This often chaotic, overturning motion that occurs in fluids under special circumstances is an important mode of heat transfer in nature.

For example, the large-scale movement of air in the atmosphere is a convective phenomenon and is responsible for the distribution of thermal energy on the Earth's surface. In the oceans, thermohaline circulation triggered by differences in the temperature and salinity of water plays an important role in determining the climate of the Earth. Convection also plays an essential role in the cores of planets where a dynamo like effect converts mechanical energy into magnetic energy and is responsible for the planetary magnetic field.

In stars, convection is one of the most important physical processes that occur and has a strong influence on several stages of stellar evolution. It is the dominant mode of energy transfer in the cores of moderate to high mass stars (about F0V or earlier) during the main-sequence phase. This has important consequences, since the size of a star's convective core affects its luminosity, effective temperature and the lifetime of the corresponding evolutionary phase.

The envelopes of smaller, cooler, main-sequence stars are also dominated by convection. The extent of these outer convection zones may have an effect on the surface abundances of different chemical species. In general, the depth of the convective envelope in stars cooler than about F0-2V increases with decreasing mass, and stars with masses less than about $0.35M_\odot$ are thought to be fully convective. This fully convective state also extends to other low mass objects in the universe, including brown dwarfs and even giant planets [61].

Convection also influences the late stages of stellar evolution since nearly all stars develop significant convective envelopes or shells once they leave the main-sequence. The size of the intermediate convective shells in massive stars is thought to affect the extension of blue loops in the HR diagram [99] and convection is even believed to be important in the thin outer atmospheres of white dwarfs [27].

Convection is an intrinsically non-local, time dependent and 3-dimensional phenomenon. No truly satisfactory model yet exists to describe it adequately in stellar environments. This is mostly due to our inability to solve analytically the Navier-Stokes equations or to produce a full direct numerical simulation of the behaviour that they predict, and has resulted in the development of convection theories which approximate the full equations in some way. The most widely used of these is the mixing-length theory, which has dominated the treatment of convective heat transport in the stellar interior for several decades now.

While having served as a useful phenomenological model of the convective process, MLT is however not without its flaws. The theory is littered with seemingly arbitrary parameters which have been the source of some disagreement in the literature [60]. One of the major sources of uncertainty in MLT is the value to be used for the mixing-length itself. However, MLT does provide a qualitatively reasonable description of convection. Since no truly satisfactory prescription currently exists, physicists have been using it for want of a better theory.

The purpose of this thesis is to examine in detail the elements of mixing-length theory and its role in stellar models. I begin by discussing the stability of a fluid element against perturbations of its initial state and derive the criteria needed to determine whether convection will occur. We then discuss the formation of convection zones in stars and their effects on stellar evolution. I then consider the history, assumptions and major results of mixing-length theory. A brief discussion on the role of turbulence follows. I also give a short summary of the numerical implementation of convection in some popular stellar evolution codes, and conclude with some remarks on possible future developments in the field.

Chapter 2

Convective Stability

> *"...let us say that stability depends on the ability of the gas - particles and photons - at any given point to sustain the weight of the overlying layers by means of the pressure it exerts, so as to maintain exact balance despite possible perturbations"* D. Prialnik.

The question of whether or not convection occurs in a certain region of a star is really a question of stability. This is because the occurence of convection at a particular level depends on whether or not a given mass element will be stable against local perturbations of its thermodynamic state at that level. If it is not, then these fluctuations may give rise to the large macroscopic motions of the stellar material which we identify as convection. In this chapter we will consider the motion of such elements, and derive criteria for the convective stability of a given region of the star.

2.1 Dynamical Stability

The dynamical stability of a displaced mass element is based on the assumption that the motion of the element is adiabatic. Consider a mass element situated a distance r from the stellar centre. Suppose that it is slightly hotter than its surroundings. One might then also expect the element to have a pressure excess over its surroundings but were this so, the element would immediately expand at the local speed of sound until a state of pressure equilibrium is achieved. Since this expansion occurs much more rapidly than any other motion of the element, we will assume that it maintains the same pressure as its immediate surroundings throughout its lifetime. We will also assume that both the element and its surroundings obey the ideal gas law. Thus, the element's temperature excess implies that it should be lighter than its surroundings and hence be driven radially outwards in the star under the action of buoyancy forces.

Consider an element at position r that is initially in complete equilibrium with its surroundings. Suppose that this element is now placed at position $r + \Delta r$. Denote by D_ρ the density difference between the element and its surroundings. Then D_ρ at $r + \Delta r$ will be given by,

$$D_\rho = \left[\left(\frac{d\rho}{dr} \right)_e - \left(\frac{d\rho}{dr} \right)_s \right] \Delta r \tag{2.1.1}$$

where we have used the subscripts e and s to distinguish between quantities which refer to the element and its surroundings respectively.

The value of D_ρ will determine the effect of the buoyancy force and whether the element is stable in its new position. There are clearly two situations of interest here. When $D_\rho < 0$ the element is lighter than its surroundings and the buoyancy force $B = -gD_\rho > 0$. This is an unstable situation since the element is driven further from its equilibrium position and will thus continue to rise. When $D_\rho > 0$, $B < 0$ and the element, which is now heavier than its new surroundings, falls back to its original position. In this case, the layer is said to be stable.

The condition for stability is therefore,

$$\left(\frac{d\rho}{dr} \right)_e > \left(\frac{d\rho}{dr} \right)_s \tag{2.1.2}$$

In practice, it is inconvenient to use the criterion in this form since the structure equations do not contain explicity any term involving density gradients. It would be preferable therefore to express them in terms of something that does appear explicitly in the equations. The natural choice is the temperature gradient. To do this, we use the equation of state $\rho = \rho(P, T, \mu)$ in the following differential form:

$$\frac{d\rho}{\rho} = \alpha\frac{dP}{P} - \delta\frac{dT}{T} + \varphi\frac{d\mu}{\mu} \tag{2.1.3}$$

where

$$\alpha \equiv \left(\frac{\partial \ln \rho}{\partial \ln P}\right)_T \tag{2.1.4}$$

$$\delta \equiv -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_\mu \tag{2.1.5}$$

$$\varphi \equiv \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_P \tag{2.1.6}$$

Here the subscripts refer to the fact that the derivatives are being taken at constant T, μ and P.

We ignore the effects of ionization and assume that the mean molecular weight μ varies only with change of chemical composition, so that for an element that moves without change, we have $d\mu = 0$. The composition of the surroundings however, may not be constant since in general the composition may be stratified in layers.

Using (2.1.3) and applying the condition for pressure equilibrium, we can rewrite (2.1.2) in the form

$$\left(\frac{\delta}{T}\frac{dT}{dr}\right)_e - \left(\frac{\delta}{T}\frac{dT}{dr}\right)_s + \left(\frac{\varphi}{\mu}\frac{d\mu}{dr}\right)_s < 0 \tag{2.1.7}$$

It is customary at this point to multiply the terms in this inequality by the pressure scale height, H_p, which is defined to be

$$H_p = -P\frac{dr}{dP} \tag{2.1.8}$$

in order to convert the gradients with respect to distance into gradients with respect to pressure. This is preferred since pressure is a state variable and is thus a more direct measure of the thermodynamic state of the material in any given layer of the star. Also, since pressure decreases monotonically with distance away from the core, it may be used as a measure of position in the stellar interior. Doing this yields

$$\left(\frac{d\ln T}{d\ln P}\right)_s < \left(\frac{d\ln T}{d\ln P}\right)_e + \frac{\varphi}{\delta}\left(\frac{d\ln \mu}{d\ln P}\right)_s \tag{2.1.9}$$

We now define the following three derivatives:

$$\nabla = \left(\frac{d\ln T}{d\ln P}\right)_s, \qquad \nabla_e = \left(\frac{d\ln T}{d\ln P}\right)_e, \qquad \nabla_\mu = \left(\frac{d\ln \mu}{d\ln P}\right)_s \tag{2.1.10}$$

With these definitions (2.1.9) becomes

$$\nabla < \nabla_e + \frac{\varphi}{\delta}\nabla_\mu \tag{2.1.11}$$

Consider a layer in the star where all the energy is transported via radiation so that $\nabla = \nabla_{rad}$. According to (2.1.11) such a layer is stable if

$$\nabla_{rad} < \nabla_{ad} + \frac{\varphi}{\delta}\nabla_\mu \tag{2.1.12}$$

(assuming that the convecting elements move adiabatically[1]). Equation (2.1.12) is known as the *Ledoux criterion* for dynamical stability [54]. If it is violated, the layer is said to be dynamically unstable and small perturbations will tend to grow until convection occurs. This could be caused by one of two things: either ∇_{rad} has become too high (which can happen if there is a large flux through the medium, or if the stellar material increases in opacity) or ∇_{ad} has become too low.

In a region of homogenous chemical composition, $\nabla_\mu = 0$ and (2.1.12) reduces to

$$\nabla_{rad} < \nabla_{ad} \tag{2.1.13}$$

which is known as the *Schwarzschild criterion* [76].

Note that both (2.1.12) and (2.1.13) are strictly local criteria. In certain situations, these criteria may not be sufficient to determine the stability of a

[1]See Appendix (A)

given layer since they only incorporate the effect of local forces on the convective process. However, convective motions may be coupled (via momentum transfer, inertia and the equation of continuity) to the neighbouring layers of the star. In extreme cases, such as those involving semi-convection [2] and the determination of the boundary of a convective zone, one should also take into account the reaction of nearby layers to a local perturbation.

The above criteria are in fact equivalent to the statement that the fluid will be convectively unstable in those regions where the entropy decreases outward ($dS/dr < 0$). In other words, convection does not take place in hydrostatic stars where the entropy increases outward. It will be shown that in regions where the convection is very efficient, ∇ is only slightly greater than ∇_{ad}, and the entropy is nearly constant with height.

A rigorous study of the conditions for stability against convection, based on a detailed linear stability analysis, has been published by Lebovitz [52]. This analysis yields the same criterion for convective instability as does the Schwarzschild criterion, whose validity in the case of a general relativistic fluid has been established by Chandrasekhar [14] and Thorne [89]

2.1.1 The Relationship Between Stability and Entropy

From the first and second laws of thermodynamics we have that for infinitesimal, reversible changes,

$$TdS = dE - \frac{P}{\rho^2} d\rho \tag{2.1.14}$$

We now express E and ρ in terms of P and T. The differentials may then be expanded into partials with respect to P and T and transformed using standard thermodynamic rules [3] to obtain

$$\frac{dS}{dr} = c_p \left(\nabla - \nabla_{ad}\right) \frac{d \ln P}{dr} \tag{2.1.15}$$

[2] See Section (2.3)

[3] See for example, [49]

where we have made use of the fact that [4]

$$\nabla_{ad} = \frac{P\delta}{T\rho c_p} \tag{2.1.16}$$

It is the presence of $(\nabla - \nabla_{ad})$ in (2.1.15) that is of interest here. Since hydrostatic equilibrium requires that $d\ln P/dr \leqslant 0$, the following must be true:

- If the star is locally radiative so that $\nabla < \nabla_{ad}$, then $dS/dr > 0$ and the entropy increases radially outwards.
- If the star is convectively unstable so that $\nabla > \nabla_{ad}$, then $dS/dr < 0$ and the entropy decreases outward. In the special case of highly efficient convection, ∇ exceeds ∇_{ad} by a negligible amount and we may thus set $\nabla = \nabla_{ad}$. For this situation, S is effectively constant throughout the convective region.

2.1.2 The Brunt-Väisälä Frequency

In a dynamically stable layer, a displaced mass element will be drawn back to its original position by buoyancy and will in general overshoot this point as a result of the excess momentum gained during its motion. This results in an oscillation around an equilibrium level defined by its original position. The frequency of this oscillation is known as the Brunt-Väisälä frequency and we now proceed to derive an expression for it.

Consider a mass element displaced radially from its equilibrium position by an amount Δr. The excess density D_ρ possessed by the element over its surroundings at this position is given by (2.1.1). Assuming that the element undergoes no changes in its composition and maintains a state of pressure equilibrium, we can use (2.1.3) to show that

$$D_\rho = \rho\left[-\left(\frac{\delta}{T}\frac{dT}{dr}\right)_e + \left(\frac{\delta}{T}\frac{dT}{dr}\right)_s - \left(\frac{\varphi}{\mu}\frac{d\mu}{dr}\right)_s\right]\Delta r \tag{2.1.17}$$

[4] See also Appendix (A)

Using (2.1.8) and (2.1.10) we then have the following

$$D_\rho = \frac{\rho\delta}{H_p}\left[\left(\frac{d\ln T}{d\ln P}\right)_e - \left(\frac{d\ln T}{d\ln P}\right)_s + \frac{\varphi}{\delta}\left(\frac{d\ln \mu}{d\ln P}\right)_s\right]\Delta r \quad (2.1.18)$$

$$= \frac{\rho\delta}{H_p}\left[\nabla_{ad} - \nabla + \nabla_\mu\right]\Delta r \quad (2.1.19)$$

Note that we have assumed that the element moves adiabatically throughout its motion and hence $\nabla_e = \nabla_{ad}$ in the equation above. The equation of motion of the element is then given by

$$\frac{\partial^2(\Delta r)}{\partial t^2} = -\frac{g\delta}{H_p}\left[\nabla_{ad} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right]\Delta r \quad (2.1.20)$$

The solution of (2.1.20) is of the form

$$\Delta r = r_0 e^{i\omega t} \quad (2.1.21)$$

and the frequency ω of this oscillation is the Brunt-Väisälä frequency mentioned previously. Clearly

$$\omega = \sqrt{\frac{g\delta}{H_p}\left[\nabla_{ad} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right]} \quad (2.1.22)$$

In an unstable layer, the Ledoux criterion is violated and we see from (2.1.22) that this results in a non-real value of ω. This implies that, in such a situation, the displaced mass element moves away exponentially instead of oscillating.

2.2 Vibrational Stability

We now consider the effects of deviations from adiabaticity on the motion of a mass element. We will assume that these deviations are small enough that the thermal adjustment time of the element is much larger than its period of oscillation.

The temperature excess of the element over its surroundings is given by

$$DT = \left[\left(\frac{dT}{dr}\right)_e - \left(\frac{dT}{dr}\right)_s\right]\Delta r \quad (2.2.1)$$

$$= -\frac{T}{H_p}(\nabla_e - \nabla)\,\Delta r \qquad (2.2.2)$$

If $DT > 0$, the element will lose heat to its surroundings via radiation. If $DT < 0$ it will gain heat. For a dynamically stable layer that is also chemically homogenous, we have from (2.1.11) that $\nabla_e - \nabla > 0$. This then implies that $DT < 0$ for $\Delta r > 0$ in (2.2.2). The element therefore gains heat from its surroundings by radiation. This has the effect of reducing $\nabla_e - \nabla$, D_ρ and the restoring force, with the net result that the element oscillates with a slowly decreasing amplitude.

If the layer is dynamically stable but chemically inhomogeneous, we could have a situation in which ∇_μ is so large that $\nabla_e - \nabla < 0$ in spite of (2.1.11). In this case $DT > 0$ for $\Delta r > 0$ in (2.2.2), and the displaced element loses heat to its surroundings. This has the effect of increasing $\nabla_e - \nabla$, D_ρ and the restoring force, with the result that the element oscillates with a slowly increasing amplitude. This vibrational instability may lead to a chemical mixing of elements within the star, resulting in the overall reduction or perhaps even destruction of the gradient ∇_μ. It is unclear at present, whether a local analysis suffices in such critical situations. It may be that the reaction of other layers in the star may provide enough damping to suppress this type of instability [1].

We must therefore distinguish between *dynamical stability* and *vibrational stability*. The former refers to the stability of a displaced mass which is assumed to be moving adiabatically, while the latter takes into account heat exchanges between the element and its surroundings. It is possible to have a layer in the star that is dynamically stable and yet vibrationally unstable. This occurs whenever the actual temperature gradient for that region is such that the Ledoux criterion is satisfied but the Schwarzschild criterion is not i.e.

$$\nabla_{ad} < \nabla < \nabla_{ad} + \frac{\varphi}{\delta}\nabla_\mu \qquad (2.2.3)$$

The unique effects that arise in this situation are the source of debate in the literature and will be discussed below.

2.3 Semi-convection

In 1958, Schwarzschild and Härm [77] found that during central hydrogen burning, the convective cores of massive ($M > 10M_{\odot}$) stars leave behind a certain hydrogen profile as they retreat. This results in a region of outwardly increasing hydrogen content such that,

$$\nabla_{\mu} > 0 \tag{2.3.1}$$

and the layer is dynamically stable. Inside the core, we expect ∇ to be very close to ∇_{ad} and outside the core we must have $\nabla = \nabla_{rad}$. By considering the stability criteria presented in section (2.1), it can be shown that

$$\nabla_{ad} < \nabla_{rad} < \nabla_{ad} + \frac{\varphi}{\delta}\nabla_{\mu} \tag{2.3.2}$$

which implies that the layer is vibrationally unstable. Thus, a mass element which has been slighly perturbed will tend to oscillate with increasing amplitude, and will penetrate deeper and deeper into regions of different chemical composition. The mixing that occurs as a result of this process is known as semi-convection. As the central convection zone evolves, it creates a composition discontinuity at its boundary which, once sufficiently large, overcomes the effects of viscosity and causes the compositional changes to propogate as waves of chemical discontinuity from the convective core into the region where the gas still has its original composition.

An accurate physical model of semi-convection requires an understanding of non-linear hydrodynamic stability and the effects of turbulence near the boundary of the convective instability. Since these processes are very poorly understood at present, modern stellar evolution models have had to resort to a series of approximations in their treatment of semi-convection [80], [22], [85]. Langer et al. [53] have modelled this phenomenon as a diffusion process with a diffusion coefficient, D_{sc}, given by

$$D_{sc} = \frac{a_{sc}}{6} D_r \frac{\nabla - \nabla_{ad}}{\nabla_{ad} + \nabla_{\mu} - \nabla} \tag{2.3.3}$$

where D_r is the diffusion coefficient corresponding to energy transport by radiation, ie.

$$D_r = \frac{4}{3}\frac{c}{\kappa\rho}\frac{aT^3}{c_\rho} \tag{2.3.4}$$

and a is the radiation density constant. The quantity, a_{sc}, in equation (2.3.3) is an adjustable free parameter. They have suggested that values around 0.1 are suitable for massive stars and have also found that the resulting chemical profile depends on the ratio of the diffusion time-scale to the adjustment time of the star (ie. the timescale on which the stellar properties change).

Several other prescriptions for the treatment of semi-convection also exist, such as those suggested by Robertson and Faulkner [73], Castellani et al. [13], and Spruit [84]. Whichever method is used, the outcome is the same, namely, the smoothing of the composition gradients at the interface between the radiative and convective zones.

Caloi and Mazitelli [9] have also shown that, in the case of core helium burning, an appropriate implementation of overshooting can mimic semi-convection reasonably well, and is better suited to numerical simulations. On the other hand, Mowlavi and Forestini [66] find that for stars of mass $10M_\odot \leqslant M \leqslant 20M_\odot$, both overshooting *and* semi-convection considerably modify the structural evolution of the star after core hydrogen burning has occured. By making a comparison between their calculated evolutionary tracks and observational data, they conclude that semi-convection is in fact more important.

Detailed discussions of these issues can be found in the papers by Chiosi [19], Trimble [90] and Simpson [80].

2.4 Secular Stability

Consider a mass element situated in a region of different but homogenous chemical composition such that $D\mu \neq 0$ but $\nabla_\mu = 0$. The element is assumed to be in mechanical equilibrium with its surroundings. From (2.1.3) we must then have that

$$\frac{DT}{T} = \frac{\varphi}{\delta}\frac{D\mu}{\mu} \tag{2.4.1}$$

For $D\mu > 0$, the element radiates energy into its surroundings thereby increasing its density. It then sinks (or rises for $D\mu < 0$) with a velocity v_μ such that DT always remains constant according to (2.4.1).

The temperature of the element changes as a result of radiation at a rate of $-DT/\tau_{adj}$ where τ_{adj} is its thermal adjustment time. It also changes as a result of the adiabatic compression (or expansion) that the element undergoes in order to maintain pressure equilibrium with its surroundings. We therefore have that

$$\frac{1}{T}\frac{\partial}{\partial t}DT = -\frac{1}{T}\frac{DT}{\tau_{adj}} + (\nabla_{ad} - \nabla)\frac{\partial \ln P}{\partial t} \tag{2.4.2}$$

We now deduce an expression for v_μ by noting that the rate of change of pressure experienced by the element can be expressed in terms of its velocity and the pressure scale height as follows,

$$\frac{\partial \ln P}{\partial t} = -\frac{v_\mu}{H_p} \tag{2.4.3}$$

Note also that

$$\frac{\partial}{\partial t}DT = 0 \tag{2.4.4}$$

from (2.4.1), since $D\mu$ does not vary in a chemically homogeneous region. With this in mind we can solve equations (2.4.2) and (2.4.3) simultaneously to obtain

$$v_\mu = -\frac{H_p}{(\nabla_{ad} - \nabla)\,\tau_{adj}}\frac{\varphi}{\delta}\frac{D\mu}{\mu} \tag{2.4.5}$$

Thus for $D\mu > 0$ and $\nabla_{ad} > \nabla$, the element sinks through a dynamically stable surrounding at a velocity v_μ which depends on the thermal adjustment time for radiative losses.

Secular instabilities of the kind discussed here can occur, for example, in stars of roughly one solar mass [46]. Once the hydrogen in the core of such stars has been converted into helium, the central region is cooled by the outward flux of neutrinos which carry away energy without interacting with the stellar matter. The temperature in these stars is therefore highest somewhere off-centre. If helium burning is ignited in the region of maximum temperature, a carbon shell will eventually form around the core. Carbon "fingers" or plumes will grow and sink inwards towards the stellar centre since the carbon shell will have a higher molecular weight than the regions below it. This phenomenon can of course also occur with elements heavier than carbon in the later stages of stellar evolution [32].

2.5 Reasons for Convective Instabilities

2.5.1 Instabilities due to a large energy flux F

According to (2.1.13), convection will occur in a chemically homogeneous layer whenever

$$\nabla_{rad} > \nabla_{ad} \tag{2.5.1}$$

As mentioned previously, this could be the result of either ∇_{rad} becoming very large or ∇_{ad} becoming very small. Let us begin by considering the conditions under which the former is true.

The local value of ∇_{rad} can be expressed in terms of the total flux, F, at some point r in the star as [5]

$$\nabla_{rad} = \frac{3F\kappa P}{4acT^4 g} \tag{2.5.2}$$

where κ is the opacity of the stellar material at that point and c is the speed of light. Thus, one way in which the radiative gradient can become large relative to ∇_{ad} is if F becomes very large. This can happen close to the core of massive stars where the central temperatures are so high that the CNO cycle serves as the primary source of energy [23]. Due to the extreme temperature sensitivity of this process, the energy generation will be strongly concentrated towards

[5] See section (4.2)

the stellar centre. This implies that a large energy flux should exist there since the total luminosity will be generated within a sphere of extremely small radius relative to the total stellar radius, R. This phenomenon gives rise to the convective instability responsible for convection in the central zone of massive stars.

2.5.2 Instabilities due to a steep increase in κ

From (2.5.2) we see that convective instabilities can also occur if κ becomes large while P also remains large. This will happen if opacity increases steeply with depth. An example of this occurs in the temperature region of stars in which hydrogen begins to ionize. Because hydrogen is abundant, even $0,1\%$ ionization increases the number of free electrons by a factor of 10. The increased electron density leads to a rise in the H^- absorption coefficient, resulting in a very steep increase in κ. The coefficient also increases, as a result of the increasing excitation of the second and third energy levels of hydrogen. ∇_{rad} therefore becomes extremely large, giving rise to a convective instability in this region of the star.

Hydrogen convection zones are usually found to occur next to helium convection zones. In fact, in most stars, these zones eventually merge [51]. While on the main sequence, it is only in early F stars that the two zones remain distinct [6]. Below these zones, we find layers that are in radiative equilibrium down to the core, except in the case of very cool main sequence stars. In these, the hydrogen and helium convection zones can be so extended that they may even approach the stellar centre [28]. It is interesting to note that, in general, convection sets in at lower optical depths for stars with higher effective temperatures [6].

[6]See section (3.1)

2.5.3 Instabilities due to small values of ∇_{ad}

The adiabatic temperature gradient can become very small in temperature regions where an abundant element like hydrogen or helium ionizes. As discussed in section (2.5.2), these ionization zones correspond to large values of κ and hence of ∇_{rad}. These are thus regions of convective instability. Large values of ∇_{rad} and small values of ∇_{ad} therefore occur simultaneously in the same temperature range and are hence jointly responsible for the occurence of these zones. The small values of ∇_{ad} in these areas increases the extent of the convection zone, which ceases to exist once the ionization is complete.

Small values of ∇_{ad} are also found in the surface regions of M stars, where convection zones arise due to the dissociation of hydrogen molecules and the large specific heat of the stellar material there [6]. These unstable zones are separated from the hydrogen convection zones by regions that are in radiative equilibrium.

Chapter 3

Convection Zones

> *"The surface layers of most stars are influenced by convection. This may be considered unfortunate from the point of view of the theory of stellar structure and evolution, where the lack of a proper description of convection has, for many years, represented a major uncertainty."* A. Nordlund.

In general, the atmospheres of stars of spectral types A and earlier are in radiative equilibrium, while convection becomes important in the middle F stars, and dominates in later types. The existence of convection zones in stars is a result of the temperature and density dependence of the opacity as well as the reduction of the adiabatic temperature gradient caused by the partial ionization of hydrogen. The importance of these mechanisms and the existence of extensive hydrogen convection zones in stellar envelopes was first recognized by Unsöld [92], who found that the amount of H^- present in the hydrogen ionization zone depends on the electron number density, which increases very rapidly with increasing optical depth in cool stars. Thus, opacity must also increase rapidly due to the presence of H^- ions. As will be shown below, this causes convection.

3.1 The Onset of Convection in Stellar Envelopes

3.1.1 Constant opacity

A reason that convection is favoured in ionization zones is that the temperature gradient needed for convection[1] is not very steep there. We will show that this is due to the fact that the adiabatic index, γ, is close to unity in these regions of the star. The larger the number of degrees of freedom available to the gas particles, the smaller the value of γ. Thus, if a gas particle can absorb heat by exciting internal degrees of freedom such as vibration or rotation, γ is smaller and the gradient becomes less steep. This is also the case if heat can be absorbed by the dissociation of molecules or the ionization of atoms. This means that a rising bubble of gas does not cool very rapidly and is more likely to remain bouyant if electron recombination can provide some of the energy needed for it to expand.

The Eddington relation between the temperature and optical depth states that

$$T^4 = \frac{T_{eff}^4}{2}\left(1 + \frac{3}{2}\tau\right) \tag{3.1.1}$$

where T_{eff} is the effective temperature and τ is the optical depth. This can be rewritten as

$$4\ln T = \ln\left(\frac{1}{2}T_{eff}^4\right) + \ln\left(1 + \frac{3}{2}\tau\right) \tag{3.1.2}$$

from which we have that

$$\frac{dT}{d\tau} = \frac{3T}{8 + 12\tau} \tag{3.1.3}$$

For convection to occur, the radiative gradient must exceed the adiabatic gradient, which is given by

$$\left(\frac{dT}{d\tau}\right)_{ad} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{d\tau} \tag{3.1.4}$$

[1]See appendix (A)

Note that both the temperature and pressure gradients in this equation are negative. Convection requires that the temperature decrease rapidly with height. This is determined by the value of the adiabatic index γ and the fall-off in pressure. We now deduce the value of γ necessary for convection to occur.

The condition that the surface layer be in hydrostatic equilibrium implies that

$$\frac{dP}{d\tau} = \frac{\rho g}{\kappa} \tag{3.1.5}$$

Integration of (3.1.5) between $\tau = 0$ and τ (assuming that κ is independent of τ) yields

$$P = \frac{\rho g \tau}{\kappa} \tag{3.1.6}$$

which implies that

$$\frac{1}{P}\frac{dP}{d\tau} = \frac{1}{\tau} \tag{3.1.7}$$

Substituting this result into (3.1.4) gives us

$$\left(\frac{dT}{d\tau}\right)_{ad} = \frac{\gamma - 1}{\gamma}\frac{T}{\tau} \tag{3.1.8}$$

Equations (3.1.3) and (3.1.7) can now be used to express the Schwarzschild criterion for convective instability as

$$\frac{3\tau}{8 + 12\tau} > \frac{\gamma - 1}{\gamma} \tag{3.1.9}$$

The maximum value of the left hand side of this inequality occurs when $\tau \to \infty$. It follows then that when κ is independent of T and ρ (or P), that the condition for convection to occur is

$$\gamma < \frac{4}{3} \tag{3.1.10}$$

3.1.2 Variable opacity

In cool stars, the opacity tends to decrease rather steeply as the surface is approached. We show that if the opacity increases with increasing pressure, then convection occurs at relatively small optical depths.

κ is known to obey a power law of the form:

$$\kappa = \kappa_0 P^x \tag{3.1.11}$$

where κ_0 and x are positive integers. Equation (3.1.5) can thus be written as

$$(\kappa_0 P^x) \frac{dP}{d\tau} = \rho g \tag{3.1.12}$$

Integrating this equation between $\tau = 0$ and τ and solving for P yields,

$$P = \frac{\rho g \tau (x+1)}{\kappa_0 P^x} \tag{3.1.13}$$

Equations (3.1.12) and (3.1.13) imply that

$$\frac{1}{P}\frac{dP}{d\tau} = \frac{1}{(x+1)\,\tau} \tag{3.1.14}$$

which we now use to determine the adiabatic temperature gradient in (3.1.4) to be

$$\left(\frac{dT}{d\tau}\right)_{ad} = \frac{\gamma - 1}{\gamma}\frac{T}{(x+1)\,\tau} \tag{3.1.15}$$

Equations (3.1.3) and (3.1.15) can now be used to express the Schwarzschild criterion for convective instability as

$$\frac{3\tau}{8 + 12\tau} > \frac{\gamma - 1}{(x+1)\,\gamma} \tag{3.1.16}$$

Since the maximum value of the left hand side of this inequality is 1/4, it follows that convection occurs whenever

$$\frac{x+1}{4} > \frac{\gamma - 1}{\gamma} \tag{3.1.17}$$

This implies that if $\gamma = 5/3$ (as is the case for a non-relativistic, ideal monatomic gas), the condition for convection is

$$x > \frac{3}{5} \tag{3.1.18}$$

It is interesting to note that since dynamical stability requires $\gamma > 4/3$ and since γ is at most 5/3, a star in hydrostatic equilibrium must satisfy

$$\frac{4}{3} < \gamma \leqslant \frac{5}{3} \tag{3.1.19}$$

which means that if the configuration of a star is to be approximated by a polytrope, the polytropic index, n, may only vary between 1.5 and 3, due to the fact that

$$n = \frac{1}{\gamma - 1} \tag{3.1.20}$$

3.2 The Depth of the Outer Convection Zones

The envelopes in early stars are generally in a state of radiative equilibrium due to the complete ionization of hydrogen (thin, weak convection zones associated with He^0 and He^+ ionization do exist, but only transport an extremely small percentage of the flux). In A stars, thin hydrogen convection zones begin to develop at shallow depths ($\tau = 0.2$), while in F stars the convection zone starts somewhat deeper, and is thicker. By types $F2$ to $F5$ convection will transport essentially all of the flux within the zone and for later types, the zone extends deeper into the star as the convection becomes more efficient. In M stars the convective envelope is so extensive that it determines the structure of the star as a whole [56].

Knowledge of the extent of the convective regions in a star is thus of extreme importance for our understanding of its chemical evolution. Determining the depth of the outer convection zones, however, is not a trivial matter since the depth of these regions depends very sensitively on the efficiency of the convective energy transport there. The difference in depth obtained for different efficiencies is not small. This is evidenced by the fact that Unsöld [92] calculated a depth of 2000km for the outer convection zone of the Sun when assuming a temperature stratification corresponding to radiative equilibrium, whereas Biermann [2], who assumed an adiabatic temperature stratification, determined the depth to be 20000km. We can at least qualitatively say, that the lower boundary of these zones must occur where the ionization of the most abundant elements is nearly complete, since not only does κ decrease appreciably once ionization ceases, but we also have that the average κ decreases for deeper layers as $\kappa \propto T^{-3.5}$ from Kramers' Law.

3.2.1 Dependence of the convection zone depth on T_{eff}

From (3.1.1) we see that stars with lower effective temperatures have actual temperatures that increase more slowly with τ. Since the absorption coefficients are smaller at lower temperatures, a given value of τ and T will occur at a greater depth. Thus, the onset of hydrogen ionization occurs deeper within the stellar interior for stars with lower effective temperatures. This implies that the resulting convection zones should have higher pressures, smaller temperature gradients, and therefore more efficient convection. Also, the depth of the zone increases since the ionization is completed deeper within the star, where the temperatures and pressures are higher. It may even be possible that for very cool main-sequence stars, the outer convection zones extend down to the core.

Similarly, for stars with larger effective temperatures, we expect the ionization zones to start higher up in the atmosphere. For early F stars the convectively unstable region may even start at optical depths as high as $\tau = 0.2$, which must mean that the convection that occurs there is inefficient (because of the extremely low pressures). For $T_{eff} \simeq 7600$K at spectral type $A9$ or $F0$, the temperature stratification no longer differs from the radiative equilibrium stratification and the convection zones become very thin [7].

3.2.2 The Lithium problem

Since the efficiency of the convective energy transport depends on the characteristic length, l, of the moving elements, so too must the depth of the convection zone. In many parts of the Hertzsprung-Russel diagram (HRD), the computed stellar strucuture depends very sensitively the characteristic length, making it an important quantity to determine. In principle, we should be able to determine l if we can somehow measure the depth of the convection zone.

Convection zones tend to become thoroughly mixed in relatively short periods of time due to the turbulent motions that occur there. We can therefore assume that the composition in a convective region always remains homogenous. This has important consequences for many stages of stellar evolution. For example, remnants of earlier nuclear burnings may be brought to the surface, fresh fuel may be carried into a zone of nuclear burning, or discontinuities can be produced that drastically influence the later evolution of the star.

We generally find a certain amount of overshoot at the boundaries of these zones due to the excess momentum carried by convecting particles. This effect becomes particularly important when trying to determine the abundance of lithium in the stellar atmosphere. For temperatures above 2×10^6K, the Li^6 nuclei are destroyed in reactions with protons,

$$Li^6 + H^1 \rightarrow He^4 + He^3 \tag{3.2.1}$$

and Li^7 starts to burn at temperatures in excess of 2.4×10^6K according to the following reaction

$$Li^7 + H^1 \rightarrow 2He^4 \tag{3.2.2}$$

The fact that we do not observe any Li^6 content in the solar spectrum suggests that outer convection zone of the Sun must extend down to regions where the temperature is in excess of 2×10^6 K. If this is the case, then Li^6 would be mixed down to layers which are hot enough to destroy it. On the other hand, we do observe a weak Li^7 line, but the solar abundance of this isotope is about 100 times smaller than that observed in the most lithium rich stars, the young T Tauri. It appears that the solar convection zone does not quite reach down to layers with temperatures that are in excess of 2.4×10^6 K, however, some mixing does still occur at those depths as a result of overshoot and this reduces the Li^7 content on a timescale of 10^9 years [7].

By assuming that $l = H_p$, one can calculate that the solar convection zone should reach down to a temperature of approximately 2×10^6K. This agrees with the destruction of Li^6 and the slow overshoot mixing that destoys Li^7. For $l = 2H_p$ it would extend as far as 3 million kelvin, which would be incompatible with the presence of any Li^7 in the solar atmosphere.

Li^6 has not been detected in any of the main-sequence stars, but Li^7 has been observed in many of them [15]. Note that, in general, we expect the timescale on which the element is destroyed to be smaller for cooler stars due to them having deeper convection zones. Difficulties however arise when trying to understand the structure of F stars. Calculations show that the convection zones in these stars do not reach down to layers of 2×10^6 K, nor do we expect any overshoot mixing down to layers with temperatures of 2.4×10^6 K. Convective mixing cannot be the explanation for the destruction of lithium in these stars. Furthermore, observations by Boesgaard and Tropicco [4] show that in some young cluster stars, the depletion of lithium in the surface regions is largest for spectral types around $F5$, a feature which cannot be accurately explained. It may be that rotation induced mixing and diffusion are important phenomena in these stars.

3.3 Convective Cores

Convection can also be an important effect in the central part of a star where thermonuclear energy is generated in a small region near the stellar centre. As discussed in section (2.5.1), convection generally occurs in the cores of massive main-sequence stars, where hydrogen burning takes place via the extremely temperature sensitive CNO cycle. The convective core increases in extent the larger the star and can cover as much as 70% of the stellar mass in stars of $50M_\odot$ or more. As the total stellar mass, M, increases further, the convective core will eventually approach the surface, resulting in stars that are fully convective.

In less massive stars, like the Sun, the central energy generating regions are much larger due to the burning of hydrogen via the p-p chain (which is less temperature dependent). ∇_{rad} is therefore much smaller near the core and convection is less likely to occur.

The above observations can be described quantitatively by noting that the core of a star will become convective if the power generated per unit mass within it exceeds a certain critical value. To find this critical value, we equate the radiative and adiabatic temperature gradients, (2.5.2) and (3.1.4) respectively, and make use of the condition for hydrostatic equilibrium, (3.1.5), to obtain

$$\frac{3\kappa\rho}{4acT^4}\frac{L(r)}{4\pi r^2} = \frac{(\gamma-1)}{\gamma}\frac{T}{P}\frac{\rho G m(r)}{r^2} \tag{3.3.1}$$

where $L(r)$ and $m(r)$ are respectively the luminosity and mass at a distance r from the centre of the star and G is the gravitational constant. Using the fact that acT^4 is equivalent to the radiation pressure, P_r, we find that the value of $L(r)/m(r)$ needed for convection is

$$\frac{L(r)}{m(r)} = \frac{(\gamma-1)}{\gamma}\frac{16\pi Gc}{\kappa}\frac{P_r}{P} \tag{3.3.2}$$

Thus, a convective core of radius r will be produced if the power generated per unit mass within r exceeds this limit. If $L(r)/m(r)$ is smaller than this value, energy can be transported from the core by radiative diffusion without inducing convection.

3.4 The Influence of Convection Zones on Stellar Structure

There are two important ways in which convection can change the structure of a star:

- The radius of the star becomes smaller

- There is an increase in the amount of energy generation

If the outward energy transport due to convection is increased then the star should tend to lose more energy than is generated and hence cool off. However, this does not actually happen, as the resulting reduction in internal gas pressure would cause the gravitational pull to exceed the pressure force. The star instead contracts, increasing its internal temperature and hence the rate of energy generation, ϵ. This allows it to balance the increase in energy loss and thus return to a state of thermal equilibrium. In doing so its radius decreases and luminosity increases, implying that its effective temperature becomes larger.

3.4.1 The Hayashi line

The Hayashi line (HL) is defined as the locus on the HRD of fully convective stars which correspond to a given mass and chemical composition. In deriving some properties of the HL we will assume that the temperature throughout the star is stratified adiabatically and shall neglect the depression of ∇_{ad} that occurs near the surface due to the partial ionization of H and He. ∇_{ad} will therefore be taken as being constant throughout the star's interior. These simplifications will certainly introduce errors in the P-T stratification, however, they will be nearly the same for neighbouring models and one can hope to obtain at least the correct differential behaviour. We now proceed to model the star as a simple polytrope with

$$P = K^{'}T^{1+n} \tag{3.4.1}$$

where the polytropic index, n, is defined by

$$n = \frac{1-\nabla_{ad}}{\nabla_{ad}} \tag{3.4.2}$$

and $K^{'}$ is some constant. Note that for an ideal gas, $\nabla_{ad} = 0.4$, in which case $n = 3/2$.

In writing down (3.4.1) we note that in the extreme case when convection continues down to the centre of the star, the constant $K^{'}$ cannot be arbitrary because (3.4.1) must have solutions corresponding to a complete polytropic model with the appropriate central boundary conditions. In other words, given M and R, $K^{'}$ must satisfy the following well known relation for ideal gas polytropes,

$$K^{'} = K^{-n}\left(\frac{N_A k}{\mu}\right)^{1+n} \tag{3.4.3}$$

where N_A and k are Avogadro's number and the gas constant respectively, and

$$K = \left[\frac{4\pi}{\xi_n^{n+1}\left(-\theta_n^{'}\right)^{n-1}}\right]_{\xi_1}^{1/n} \frac{G}{n+1} M^{1-1/n} R^{-1+3/n} \tag{3.4.4}$$

Recall that θ and ξ are the dimensionless density and radial coordinates respectively. Note that ξ_1 denotes the surface value of ξ.

One way to ensure this is to follow the prescription of Schwarzschild [78], and rewrite the pressure and temperature in terms of the following dimensionless variables:

$$p = \frac{4\pi}{G}\frac{R^4}{M^2}P \tag{3.4.5}$$

$$t = \frac{N_A k}{G}\frac{R}{\mu M}T \tag{3.4.6}$$

Equation (3.4.1), then becomes

$$p = E_0 t^{5/2} \tag{3.4.7}$$

where

$$E_0 = K^{'} 4\pi \left(\frac{\mu}{N_A k}\right)^{5/2} G^{3/2} M^{1/2} R^{3/2} \tag{3.4.8}$$

But according to (3.4.3) and (3.4.4)

$$K^{'} = \frac{2.5^{3/2}}{4\pi}\left[\xi_{3/2}^{5/2}\left(-\theta_{3/2}^{'}\right)^{1/2}\right]_{\xi_1}\left(\frac{N_A k}{\mu}\right)^{5/2}\frac{1}{G^{3/2}M^{1/2}R^{3/2}} \tag{3.4.9}$$

for an ideal gas. Substituting this expression into (3.4.8) gives us

$$E_0 = \left(\frac{-125}{8}\xi_{3/2}^{5}\theta_{3/2}^{'}\right)_{\xi_1}^{1/2} = 45.48 \tag{3.4.10}$$

This shows that E_0 does not depend on any of the physical parameters of the model (such as mass, radius or composition) but rather contains only the surface values of the polytropic variables, making it a well defined constant for a given class of polytrope.

For luminosities $L \leqslant L_\odot$, Hayashi [34] has derived the following analytical relation between the stellar parameters:

$$\log\left(\frac{L}{L_0}\right) = 0.272 - 1.835\log\left(\frac{M}{M_0}\right) + 9.17\left(\log T_{eff} - 3.70\right) + 2.27\log\left(\frac{E_0}{40}\right) + 0.699\left(\log\kappa_0(Z) + 15.58\right) \quad (3.4.11)$$

where Z is the star's metal abundance. This equation defines an almost vertical line, in the HRD, on which fully convective stars of a given mass (ie. stars with $E_0 = 45.48$) are found. It is also known as the Hayashi Line.

Note the strong dependence of L on T_{eff}. Changing T_{eff} by only 4% changes L by a factor of approximately 10. Since stars with zones that are in radiative equilibrium have a higher T_{eff} than fully convective stars, there can be no stars with T_{eff} lower than for completely convective stars. The HL therefore gives a lower limit for the T_{eff} of stars in hydrostatic equilibrium. In other words, the coolest stars are those which are fully convective and hence lie on the HL.

A direct application of this discussion is to the evolution of pre-main sequence stars which form from the contraction of protostellar clouds and are initially fully convective. They follow a near vertical path in the HRD that is qualitatively similar to that given by (3.4.11). These paths are appropriately known as *Hayashi tracks* and are almost parallel to the ascending giant branch but covered in the reverse direction. As these stars continue to contract, their luminosity may decrease to the point where the deep interior ultimately becomes radiative and hydrogen burning is ignited. It is at this point that the main-sequence stage of evolution begins. These tracks correspond to the radii and effective temperatures of the very young T Tauri stars [33].

Chapter 4

Mixing Length Theory

> *"Stellar turbulent convection is going through an interesting phase. For some forty years there was an intellectual stagnation since most of the data did not require more than the MLT, a model that in spite of its rudimentary physical assumptions, proved to be a very useful tool."* V. M. Canuto.

The mixing length theory was originally formulated by G.I. Taylor [88] and L. Prandtl [72] to describe incompressible, terrestrial convection. Although Taylor was the first to introduce the idea of a mixing length, credit is generally given to Prandtl, who developed the theory in complete analogy to molecular heat transfer while working in Göttingen during the 1920's. It was later adapted to stars by Biermann [3], Vitense [95] and Bohm-Vitense [5], and has since been extended and modified in so many ways that there now exists several "versions" of the theory. Despite several key shortcomings, the basic formalism of MLT is still the most widely used model of convection in present day stellar evolution codes.

4.1 The Assumptions of Mixing Length Theory

In MLT, the convection zone is treated as if it consisted of groups of "average" convective elements. This replaces the conceptually difficult situation, involving eddies of different shapes, sizes, velocities and lifetimes, that actually occurs inside these zones since each of the "average" convective elements are assumed to have the same physical properties at a given radial distance r from the stellar centre.

Convective elements are assumed to form as a result of instabilities or perturbations in the stellar fluid. Once formed, they will rise or fall under the action of bouyancy forces through a characteristic distance known as the mixing length, before losing their identity and merging with the surrounding fluid. In this sense, the mixing length for a given convective element is its "mean free path" in the convective motion. The elements may of course also exchange heat with their surroundings via radiation as a result of temperature imbalances.

According to Gough [31]: *"It is as a result of ignoring different combinations of these processes, approximating the remaining ones in slightly different ways, and making different assumptions about the geometry of the flow that different mixing length models have emerged."* The differences in these models have not been stressed in the literature since in almost all cases they result in formulae for the advection of heat through the stellar interior that are essentially the same up to factors of order unity. They do however become apparent in situations where the convectively unstable region is not static, such as in studies of stellar pulsation.

Some authors ([71], [68]) have proposed a different picture of convection based on cells consisting of rising and falling columns of fluid surrounded by a rising or falling cylinder of fluid. It has however been shown that this amounts to little more than thinking of pairs of rising and falling elements as a single entity in the MLT picture [31]. With this in mind, we may state the basic assumptions of MLT formally as follows:

A 1: On average, each convective element is assumed to travel through a distance λ, the mixing length, before dissipating into the surroundings. Note that in general $\lambda = \lambda(r)$.

A 2: The mixing length is assumed to be much shorter than any scale length associated with the structure of the star.

A 3: The precise shape of the elements is not specified, but they are all assumed to possess the same characteristic dimension. This characteristic dimension will be taken to be of the same order as the mixing length.

A 4: All convective elements at a given radial distance r from the centre of the star are assumed to convect at the same *average* speed $\overline{v}$, which is taken to be the speed of upward and downward moving elements averaged over both the mixing length and the surface defined by r. Note that $\overline{v} = \overline{v}(r)$.

A 5: Each element is assumed to maintain a state of complete pressure equilibrium with its surroundings as it rises or falls. This means that each material element at a given distance r from the stellar centre is at exactly the same value of pressure. This assumption implies that $\overline{v}$ should never exceed the local sound speed, v_s, in the mixing length approach. Were this to happen, the assumption of complete pressure equilibration would not be very realistic, since the mechanical adjustment time t_p is of the order λ/v_s whereas the mean eddy lifetime t is of the order of $\lambda/\overline{v}$. Hence $t_p/t \sim \overline{v}/v_s$ which means that $\overline{v} < v_s$ in order for $t_p < t$.

A 6: The star is assumed to be in a steady state. The amount of matter rising at each level is thus, at each time, the same as the amount sinking. This assumption is equivalent to that of hydrostatic equilibrium throughout the star.

A 7: The effects of magnetic fields and rotation are ignored as well as all acoustic phenomena.

A 8: Temperature and density differences between the element and its surroundings are assumed to be small.

A 9: The effects of turbulent pressure on the the convective process are neglected. It will be shown that this is a reasonable assumption to make provided that the convection is subsonic.

The above assumptions represent an extreme simplification of the convective processes that occur in the stellar interior. The constraints represented by these assumptions are all derived from laboratory based simulations of convection and may not be applicable to the much more complicated situations encountered in the interiors of stars. In fact, it will be shown that in practice, MLT ends up violating one of its own assumptions. This failure is traceable to the fact that the MLT assumptions are essentially equivalent to what is known as the Boussinesq Approximation. This approximation assumes that the fluid is almost incompressible, and that variations in its temperature and density may be ignored except insofar as they give rise to the buoyancy forces that drive convection. While the Boussinesq Approximation has been shown to work very well in laboratory situations where the size of the fluid system is much smaller than any of the associated scale heights, it turns out, that when applied to stars, reasonable results are obtained only if the mixing length is chosen to be approximately equal to the pressure (or some other) scale height characteristic of the star.

To make matters worse, the question as to which scale height is the most physically significant with regard to the mixing length theory is still unresolved. It is customary to set $\lambda = \alpha H$ where H is usually taken to be either the pressure or density scale height and α is a free parameter. Use of the density scale height may however lead to difficulties when studying the stellar surface since it becomes negative in the case of density inversion. It has also been suggested by Faulkner et al. [26] that the temperature scale height be used in certain applications. Most authors tend to adopt the pressure scale height for the sake of convenience.

One generally uses values of α that lie between 1 and 4, but choosing a value between 1 and 2 seems to work best. The reason for this is as follows. Convecting elements which are much smaller than a scale height in size will radiate away considerable amounts of energy during their motion before dissolving and will consequently be poor carriers of convective flux. This is due both to their small size and the short distance (of the order of λ) through which they move. Most of the convective flux will then presumably be carried by the larger elements. However, due to the turbulent nature of these elements, they are not likely to retain their identity as they move through a distance of much more than a few scale heights since they will be travelling through regions whose physical conditions differ significantly from those in which they originated. For example, it can be shown that the volume of a turbulent element will increase roughly by a factor of 2 for every scale height through which it moves.

There is however another troublesome feature present in the assumptions. Using the virial theorem, one can easily show that the average value of the pressure scale height in a star is of the same order of magnitude as the stellar radius, R, so that $\lambda \simeq H_p \simeq R$. Now if the size of the element is assumed to be of the same order as the mixing length, then this suggests that it does not get very far before mixing.

Furthermore, if the cross-sections of rising and falling gas colums were originally equal in a convective layer, then the rising gas must have expanded by a factor of e after moving through a pressure scale height, in order to maintain a state of pressure equilibrium with the surroundings. This means that after a distance comparable to H_p there is very little room left for the falling gas plumes. But we require that the amount of material falling be equal to the amount which is rising. The only way in which this can happen is if some of the rising gas elements are dragged down by the falling material. Thus, for any given layer, a large percentage of the falling matter must have been taken out of the rising columns. There is however, no satisfactory method for computing these percentages.

Lastly, our entire picture of convective heat transfer would have to be drastically modified in order to account properly for situations involving supersonic convection. In the absence of a theory which can accurately account for these effects, we will force v_s to be the upper limit of $\overline{v}$, so that should the equations of MLT ever yield a value of $\overline{v} > v_s$ at some point in a convection zone we would simply set $\overline{v}/v_s = 1$. This assumption is arbitrary and ad hoc. The results obtained using it are likely to be incorrect, both due to fundamental crudity of MLT and the fact that the theory may not even be applicable at all in such regions.

With these considerations in mind, we proceed to develop the essential results of the theory.

4.2 The Energy Flux in a Star

The total energy flux $F(r)$ through a surface of given radius r in a star is related to the total luminosity L at that radius by

$$F(r) = \frac{L}{4\pi r^2}$$

In general, $F(r)$ consists of the sum of the radiative, conductive and convective fluxes. The radiative and conductive fluxes, F_{rad} and F_{cond} respectively, can be combined into a single formula as

$$F_{cond+rad} = F_{cond} + F_{rad} = \frac{4ac}{3}\frac{T^4}{\kappa\rho}\frac{1}{H_p}\nabla$$
$$= \frac{4ac}{3}\frac{T^4 g}{\kappa P}\nabla \qquad (4.2.1)$$

where κ is the effective opacity defined by

$$\frac{1}{\kappa} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}$$

It is customary to use F_{rad} as an abbreviation for the symbol $F_{cond+rad}$. Note also that F_{rad} has been averaged over all the matter at radial distance r

Now define the radiative gradient ∇_{rad} by the relation,

$$F(r) = F_{rad} + F_{con} \equiv \frac{4ac}{3}\frac{T^4 g}{\kappa P}\nabla_{rad} \qquad (4.2.2)$$

where F_{con} is the convective flux at radius r. Note that ∇_{rad} is a fictitious temp gradient. It is the gradient which would exist at a given point in the star if *all* the energy were transported by radiation at that radius.

4.3 The Convective Flux

The *average* energy delivered per unit area, per unit time, by *upward* moving elements is

$$F_{con} = \frac{1}{2}\rho c_p \overline{v}\overline{\Delta T} \qquad (4.3.1)$$
$$= \frac{1}{2}\rho c_p \overline{v}\left(\nabla - \nabla_e\right)\frac{\lambda}{2}\frac{T}{H_p} \qquad (4.3.2)$$

where c_p is used because of assumption (A5).

Note that this equation represents only half of the total flux, since approximately one half of the matter is rising while the other half is falling at any given level.

To determine the convective flux at a given radius in the star, we first need to obtain expressions for $\overline{v}$ and ΔT at that radius.

4.3.1 The Convective Velocity

The equation of motion:

Begin by considering the equation of motion of a convecting element in a non-viscous medium,

$$\frac{d^2r}{dt^2} = -g\frac{\Delta\rho}{\rho}$$

The net force per unit mass acting on the element is therefore

$$F = -g\frac{\Delta\rho}{\rho} \tag{4.3.3}$$

Using $\rho = \rho(\mu, P, T)$ we can relate $\Delta\rho$ to ΔT to obtain

$$\Delta \ln \rho = -Q\Delta \ln T \tag{4.3.4}$$

where

$$\begin{aligned} -Q &= \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_{P,T}\left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{\mu,P} \\ &= \varphi\left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P + \delta \end{aligned}$$

Note that Q is taken at constant pressure due to (A5). Now, for an ideal gas, the equation of state is $P = \left(\frac{\Re}{\mu}\right)\rho T$, which implies that

$$Q = 1 - \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P$$

Clearly $Q = 1$ if the element is chemically homogenous (ie. μ is constant). If in addition one includes the effects of radiation pressure, then it can be shown that [20]

$$Q = \frac{4-3\beta}{\beta} - \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P$$

where β is the ratio of gas pressure to total pressure.

Using (4.3.4) we can rewrite (4.3.3) as

$$F = \frac{Qg}{T}\Delta T \tag{4.3.5}$$

and using (2.2.2) we can substitute for $\frac{\Delta T}{T}$ to get

$$F = \frac{Qg}{H_p}(\nabla - \nabla_e)\Delta r \tag{4.3.6}$$

The work function:

The work done on the element will in general be a function of Δr. We must now average $W(\Delta r)$ over all possible values of Δr. This is generally done by multiplying $W(\Delta r)$ by a numerical factor and replacing Δr by its average value. Most authors tend to choose the numerical factor in such a way as to obtain agreement with the work of Böhm-Vitense [5]. For our purposes, this will mean setting it equal to 1/2. It will be demonstrated that the final result obtained for the work function is in fact, up to an order of magnitude, insensitive to this seemingly arbitrary value introduced here.

As for the average value of Δr: consider a level in the star situated a distance r from its centre. The elements crossing this level will in general have different values of v and ΔT since they could have each started their motion at anything up to a mixing length away from r. We will therefore assume that the average distance travelled by one of these elements is half of its mixing length, so that $\Delta r = \frac{\lambda}{2}$ in (4.3.6). The *average* work done on the element over this distance is therefore,

$$\overline{W} = \frac{Qg\lambda^2}{8H_p}\left(\nabla - \nabla_e\right) \tag{4.3.7}$$

In the absence of dissipative forces, all of $\overline{W}$ would be transformed into the kinetic energy of the element. However, in real stars, we know that this is not the case. All real fluids are viscous and energy will, in general, be lost by the element as it collides with the stellar material it encounters on its journey through the star. We will therefore assume that only about 1/2 of this work appears as the kinetic energy of the moving element while the remainder is lost to the surroundings, which have to be "pushed aside" as the element convects. Therefore,

$$\overline{v}^2 = \frac{Qg\lambda^2}{8H_p}\left(\nabla - \nabla_e\right)$$

and hence,

$$\overline{v} = \frac{g\lambda}{2}\sqrt{\frac{\rho Q\left(\nabla - \nabla_e\right)}{2P}} \tag{4.3.8}$$

which may also be expressed in terms of the adiabatic sound velocity, v_s, as

$$\frac{\overline{v}}{v_s} = \left[\frac{\lambda}{2H_p}\sqrt{\frac{Q}{2\Gamma_1}}\right](\nabla - \nabla_e)^{\frac{1}{2}} \qquad (4.3.9)$$

Note that it can be shown that the factor multiplying $(\nabla - \nabla_e)^{\frac{1}{2}}$ in (4.3.9) is of order unity when $\lambda \sim H_p$ [20]. This suggests that the numerical factors introduced in the derivation of $\overline{v}$ are insignificant, at least from a qualitive point of view.

As noted previously, we will take v_s to be an upper limit for the physically possible values of $\overline{v}$. This means that if we ever have a sitution where $\overline{v} > v_s$ during the course of an MLT calculation, we will have to replace $\overline{v}$ by v_s. Not only does this prescription ensure that our assumption of pressure equilibrium is not violated, but it also lends itself to the assertion that the disspation of energy through shock formation may very well prevent $\overline{v}$ from exceeding v_s by a significant amount in a real star. This constraint also implies that the outermost layers of a star, which are characterized by very low densities, must be in radiative equilibrium since, at subsonic velocities, convective energy transport in these regions becomes highly inefficient [1]. This result is in agreement with observation.

Note that $\overline{v}$ is the velocity of the element *averaged* over λ. However, in practice, one generally uses the *local* value of the velocity (and other variables) when performing calculations. This is done for the sake of computational convenience and is one of the reasons that the MLT is referred to as a *local theory of convection.* Non-local mixing length calculations have also been carried out [83]. Hofmeister and Weigert [41] have performed such an analysis and conclude that the overall structure of the convection zone obtained using their methods, does not differ significantly from that obtained using a local model. Nevertheless, there are differences in the calculated density profiles for the outermost layers of these zones in red giants.

[1]See section (4.7)

Having solved for $\overline{v}$ we are now in a position to find the convective flux in this version of MLT. Equation (4.3.2) becomes

$$F_{con} = \sigma_{con} \left(\nabla - \nabla_e\right)^{\frac{3}{2}} \tag{4.3.10}$$

where

$$\sigma_{con} \equiv \frac{g^2 \lambda^2 Q^{1/2} \rho^{5/2} c_p T}{4\sqrt{2}\, P^{3/2}} \left(\nabla - \nabla_e\right)^{\frac{3}{2}} \tag{4.3.11}$$

4.4 The Net Flux

We are now in a position to solve for the net energy flux at a given distance r from the stellar centre. Recall from (4.2.2) that

$$F(r) = F_{rad} + F_{con} \tag{4.4.1}$$

Using (4.2.1) and (4.3.10) we can rewrite this as

$$F(r) = \sigma_{rad} \nabla + \sigma_{con} \left(\nabla - \nabla_e\right)^{\frac{3}{2}} \tag{4.4.2}$$

where

$$\sigma_{rad} \equiv \frac{4acT^4 g}{3\kappa P} \tag{4.4.3}$$

and σ_{con} is defined (4.3.11). If we now express F(r) in terms of the fictitious radiative gradient ∇_{rad} defined in (4.2.2) we get

$$\nabla_{rad} = \nabla + \frac{9}{4} A \left(\nabla - \nabla_e\right)^{\frac{3}{2}} \tag{4.4.4}$$

where

$$A \equiv \frac{4}{9}\left(\frac{\sigma_{con}}{\sigma_{rad}}\right) = \frac{c_p \kappa g Q^{1/2} \rho^{5/2} \lambda^2}{12\sqrt{2}\, ac P^{1/2} T^3} \tag{4.4.5}$$

Now σ_{rad} and σ_{con} in (4.4.2) each have the dimensions of an energy flux and may be regarded as a type of radiative and convective "conductivity". This implies that the dimensionless quantity, A, is a measure of the ability of the stellar material to transport energy by convection.

4.4.1 The Properties of A

Using (4.3.9) we see that

$$A = \frac{4}{9}\left[\frac{3\sqrt{Q}\,\rho v_s c_p \kappa P \alpha^2}{16\sqrt{2\Gamma_1}\,acT^3 g}\right] \tag{4.4.6}$$

If in addition, the ideal gas law, $\rho = (\mu/\Re)\,P/T$, is applicable, we have that

$$A = \left[\frac{Q^{1/2}\,(\mu/\Re)^{1/2}\,c_p\kappa\alpha^2}{12\sqrt{2}\,acg}\right]\frac{P^2}{T^{3.5}} \tag{4.4.7}$$

We now consider the variation of A with depth in a stellar envelope. Below the region of hydrogen ionization, for an envelope in hydrostatic equilibrium,

$$P = KT^{n+1} \tag{4.4.8}$$

where K is a constant that depends on the luminosity L, mass M, radius R, and chemical composition of the star. We also almost always have that $2.5 \leqslant n+1 \leqslant 5$.

Substituting (4.4.8) into (4.4.7) gives

$$A = \left[\frac{Q^{1/2}\,(\mu/\Re)^{1/2}\,c_p\kappa\alpha^2 K^2}{12\sqrt{2}\,acg}\right]T^{2(n+1)-3.5} \tag{4.4.9}$$

MLT ignores any variation with depth in the coefficient of T in (4.4.9) and treats it as a constant. Since the exponent of T in this equation is practically always greater than unity it follows that, in general, A increases with depth in such envelopes.

Interestingly, it can be shown that in the deep interior of a star, we have

$$A \simeq \left(\frac{t_{ff}}{t_k}\right)^{-1} \tag{4.4.10}$$

where t_{ff} and t_k are the free-fall and Kelvin times respectively. This gives a value of $A \simeq 10^{12}$ for a solar type star.

Now equation (4.4.4) relates the three gradients ∇_{rad}, ∇ and ∇_e to each other in terms of the local values of the physical variables contained in A. We assume that these variables and the local value of $\nabla_e = \nabla_{ad}$ are known. If in addition, we regard the value of ∇ to be known at a given point, then (4.4.4) can be used to calculate ∇_{rad} at that point. This can then be used to calculate the corresponding values of $\overline{v}$, F_{con}, and F_{rad} via equations (4.3.8), (4.3.10) and (4.2.2) respectively. Alternatively, if the value of ∇_{rad} is regarded as known at r, then (4.4.4) must first be solved for ∇, and the remaining quantities of interest can then be calculated as before.

If $\nabla_e = \nabla_{ad}$ is not a valid approximation, another expression relating the four gradients must be obtained. This expression, along with (4.4.4), would constitute 2 equations amongst 4 unknowns. Now, ∇_{ad} can always be calculated at a given point if we know the local values of the physical variables contained in A. Consequently, given the value of any one of the remaining three gradients, we can use these relations to determine the other two. Usually either ∇_{rad} or ∇ is known. Which of the two is known depends on the nature of the problem. The other gradient of the pair (∇_{rad}, ∇) is then regarded as a function of the known one. We shall develop this expression in the section that follows.

4.5 Convective Efficiencies

A convecting element which is hotter than its surroundings will lose heat via radiation. It may also gain heat if it possesses nuclear energy sources. These losses and gains do not contribute *directly* to the star's outward heat flux and are said to be "horizontal" since they are taken relative to the instantaneous surroundings of the element, averaged over a spherical shell. On average, there are as many cool elements as hot ones at any given level in the stellar interior. There is therefore no net vertical contribution to the heat transport in the star as a result of these gains and losses. They do however, have an effect on the efficiency of convection and so contribute indirectly to the outward radial transport of heat.

4.5.1 The definition of Γ

The convective efficiency, Γ, is defined as:

$$\Gamma = \frac{\textit{Excess heat content just before mixing}}{\textit{Energy radiated during lifetime}} \tag{4.5.1}$$

Large values of Γ are typical for very dense matter where radiation losses are relatively unimportant when compared with the convective flux. In regions where matter is not very dense, the losses due to radiation can be so large that the convecting elements lose nearly all of their excess heat content by radiation and cool to approximately the temperature of their surroundings. In such situations, even extremely violent movements of the stellar material are ineffective in transporting energy by convection and Γ is very small[2].

The excess heat content:

The excess heat content posessed by the element over its surroundings just before it dissolves is given by

$$\rho V c_p \Delta T_{end} \tag{4.5.2}$$

where T_{end} is the element's temperature excess at the end of its motion.

It is customary to set $\Delta T_{end} = 2\Delta T$ in accordance with [5], where ΔT is defined as the temperature difference between the centre of the element and its surface, averaged over its lifetime. Since the characteristic size of a convective element is taken to be roughly of the same order as the mixing length, we will assume that the distance over which ΔT occurs is $\lambda/2$.

[2]See section (4.7.1.3)

The energy radiated:

In order to complete the expression for Γ, we need to determine the energy radiated by element during its lifetime. If we assume that the element is optically thick [3], then the outward flux of energy from it is given by

$$\frac{4acT^3}{3\kappa\rho}\frac{\Delta T}{\lambda/2} \qquad \text{for} \qquad \kappa\rho\frac{\lambda}{2} \gg 1$$

The *net* energy radiated by the element to the surroundings during its lifetime is then,

$$\left(\frac{4acT^3}{3\kappa\rho}\frac{\Delta T}{\lambda/2}\right)\frac{\lambda}{\overline{v}}\mathscr{A} \tag{4.5.3}$$

where $\mathscr{A}$ and $\lambda/\overline{v}$ are its surface area and lifetime respectively.

An expression for Γ:

Using (4.5.2) and (4.5.3) in (4.5.1) we get

$$\Gamma = \frac{3c_p\kappa\rho^2\overline{v}}{4acT^3}\frac{V}{\mathscr{A}} \tag{4.5.4}$$

Equation (4.5.4) contains a *form factor* $V/\mathscr{A}$, which depends on the geometry of the element. A sphere of diameter λ, a cube of side λ and a cylinder of diameter and height λ, all have $V/\mathscr{A} = \lambda/6$. However, it is customary in the literature to set $V/\mathscr{A} = (2/9)\lambda$ in order to obtain numerical agreement with [5]. This gives

$$\Gamma = \frac{c_p\kappa\rho^2\overline{v}\lambda}{6acT^3} \tag{4.5.5}$$

Substituting for $\overline{v}$ using (4.3.8), we get,

$$\Gamma = A\left(\nabla - \nabla_e\right)^{\frac{1}{2}} \tag{4.5.6}$$

Finally, equation (4.5.6) can be used to rewrite (4.4.4) in terms of Γ as follows:

$$\nabla_{rad} = \nabla + \frac{9}{4}\Gamma\left(\nabla - \nabla_e\right) \tag{4.5.7}$$

[3] This is a reasonable assumption due to the large sizes of the convective elements (of the order of the pressure scale height) and the relatively high opacity of stellar material.

4.5.2 A second expression for Γ

We now deduce a second expression for Γ. For polytropic changes we have,

$$\nabla_e = \nabla_{ad} \frac{1}{1-(c/c_p)} \tag{4.5.8}$$

where

$$c = \frac{dQ}{dT_e}$$

is a generalized specific heat per unit mass and T_e is the temperature of the element during its motion.

It can be shown that

$$\frac{c}{c_p} = \frac{-(1-\eta)/\Gamma}{1-[\nabla/(\nabla-\nabla_e)]} \tag{4.5.9}$$

where

$$\eta = \frac{\Delta\varepsilon}{\Delta(\nabla\cdot F/\rho)} \tag{4.5.10}$$

$\Delta\varepsilon$ and $\Delta(\nabla\cdot F/\rho)$ are, respectively, the excess rate of energy generation within the element and the excess rate of energy loss due to radiation by the element, both per unit mass.

Subtituting (4.5.9) into (4.5.8) gives

$$\Gamma = (1-\eta)\frac{\nabla-\nabla_e}{\nabla_e-\nabla_{ad}} \tag{4.5.11}$$

The factor $(1-\eta)$ on the right hand side of this equation is clearly due to the effect of energy sources on convection.

If we assume that the local values of ∇_{ad} and the physical variables contained in A are known, then equations (4.5.6), (4.5.7) and (4.5.11) form a set of 3 equations in 4 unknowns, namely, Γ, ∇, ∇_e and ∇_{rad}. Clearly only one of these is independent. The values of the remaining three are obtained by solving the equations simultaneously.

4.5.3 Interpretation of the equations

There are two special cases of interest as regards Γ. These correspond to situations in which the convection is highly efficient ($\Gamma\to\infty$), and in which it is highly inefficient ($\Gamma \to 0$). In both instances we would like to know the fraction of the total flux that is carried by convection. Begin by noting that from (4.2.1) and (4.2.2),

$$\frac{F_{rad}}{F} = \frac{\nabla}{\nabla_{rad}} \tag{4.5.12}$$

Therefore,

$$\begin{aligned} \frac{F_{con}}{F} &= \frac{F - F_{rad}}{F} \\ &= \frac{\nabla_{rad} - \nabla}{\nabla_{rad}} \end{aligned} \tag{4.5.13}$$

From (4.5.7) we see that, as $\Gamma \to 0$, $\nabla \to \nabla_{rad}$ and hence $F_{con}/F \to 0$. This means that convection will not significantly effect the structure of a region of the star in which Γ is small since it carries only a small amount of the total flux and causes ∇ to deviate only slightly from ∇_{rad}.

Now consider the case where convection is highly efficient. As $\Gamma \to \infty$ we have from (4.5.7) and (4.5.11) respectively, that $\nabla \to \nabla_e$ and $\nabla_e \to \nabla_{ad}$ implying that $\nabla \to \nabla_{ad}$. Therefore,

$$\frac{F_{con}}{F} \simeq \frac{\nabla_{rad} - \nabla_{ad}}{\nabla_{rad}} \tag{4.5.14}$$

Thus if $\nabla_{rad} \gg \nabla_{ad}$ then $F_{con}/F \to 1$, but we could also have a situation, such as that which occurs near the boundary of a convection zone, where $\nabla_{rad} \to \nabla_{ad}$ in which case $F_{con}/F \to 0$. Therefore a high convective efficiency does not necessarily mean that convection will carry most of the flux.

4.6 Order of Magnitude Estimates

We now derive order of magnitude estimates for the convective velocity, timescale and efficiency as described by the MLT. We will then use the expressions obtained to estimate these parameters for the sun.

The velocity $\overline{v}$:

Using (4.3.5) and the work energy theorem, we have that

$$\begin{aligned}\frac{1}{2}\rho\overline{v}^2 &= g\lambda\frac{\Delta T}{T} \\ &\simeq \frac{GM}{R}\frac{\Delta T}{T}\end{aligned} \quad (4.6.1)$$

where we have set $g = GM/R^2$ and $\lambda = R$ in accordance with the discussion at the end of section (4.1). From (4.6.1) we have that

$$\overline{v} \simeq \sqrt{\frac{GM}{R}}\sqrt{\frac{\Delta T}{T}} \quad (4.6.2)$$

The sound velocity for the whole star can be evaluated using the virial theorem to give, to an order of magnitude,

$$\overline{v}_s \simeq \sqrt{\frac{GM}{R}}$$

We therefore have that

$$\frac{\overline{v}}{\overline{v}_s} \simeq \sqrt{\frac{\Delta T}{T}} \quad (4.6.3)$$

The timescale t:

The mean lifetime of a convecting element is given by

$$t = \frac{\lambda}{\overline{v}} \simeq \frac{R}{\overline{v}}$$

Substituting for $\overline{v}$ in the above expression using (4.6.2) and noting that the free-fall time t_{ff} is approximately $(R^3/GM)^{1/2}$ gives

$$t \simeq t_{ff}\left(\frac{\Delta T}{T}\right)^{-1/2} \quad (4.6.4)$$

From equations (4.6.3) and (4.6.4) we note the following: Were $\Delta T/T$ unity, the element would be accelerated with the full gravitational acceleration and, after having moved through a distance of $\lambda \simeq R$, would be travelling at approximately the free fall velocity (which is of the same order of magnitude as $\overline{v}_s$). Equation (4.6.3) shows that $\overline{v}$ is smaller than $\overline{v}_s$ by a factor of approximately $(\Delta T/T)^{1/2}$, which implies that $(\Delta T/T)$ is the fraction of the total graviational acceleration that the element actually experiences during its motion. Note also that (4.6.4) implies that the mean lifetime should be larger than the free fall time by a factor of approximately $(\Delta T/T)^{-1/2}$.

The efficiency Γ:

Let us begin by noting that since the surface area of the element is of the same order of magnitude as that of the star, the luminosity of the element differs from the radiative luminosity of star (which is assumed to be of the same order as the total stellar luminosity) only because the temperature gradient inside the element is different from that of its surroundings. Inside the element, the temperature gradient is of the order $\Delta T/\lambda \simeq \Delta T/R$ whereas in the rest of the star it is of order T/R. The luminosity of the element is therefore of the order $\Delta T/T$ which suggests that the energy it radiates during its lifetime is of order

$$t\,(\Delta T/T) \simeq t_k\,(\Delta T/T)^2 \tag{4.6.5}$$

where $t_k \simeq GM^2/LR$ is the Kelvin-Helmholtz timescale. To complete our estimate for Γ we take the mass M of a typical element to be of the same order of magnitude as the star and note that the excess heat content posessed by the element over its surroundings is of order $Mc_pT\,(\Delta T/T)$. We therefore have from the definition of Γ in (4.5.1) that, correct to an order of magnitude,

$$\Gamma \simeq \left(\frac{\Delta T}{T}\right)^{-1} \tag{4.6.6}$$

An estimate for $(\Delta T/T)$:

All of the order of magnitude estimates given above depend on the value of $(\Delta T/T)$. To approximate this quantity, note from (4.6.5) that

$$t \simeq t_k \,(\Delta T/T) \tag{4.6.7}$$

Substituting this expression into (4.6.4) gives

$$\frac{\Delta T}{T} \simeq \left(\frac{t_{ff}}{t_k}\right)^{2/3} \tag{4.6.8}$$

For a solar type star, we have

$$\frac{t_{ff}}{t_k} \simeq \frac{LR^{5/2}}{G^{3/2}M^{5/2}} \simeq 10^{-12} \tag{4.6.9}$$

and hence $\Delta T/T \simeq 10^{-8}$. This implies that, in a convection zone situated in the deep interior of a star, the temperature gradient is superadiabatic by only a negligible amount. This is due to the fact that the fractional excess of the actual temperature gradient over the adiabatic temperature gradient is of the same order of magnitude as $\Delta T/T$.

Finally we have from (4.6.3), (4.6.4) that

$$\frac{\overline{v}}{\overline{v}_s} \simeq \left(\frac{t_{ff}}{t_k}\right)^{1/3} \simeq 10^{-4} \tag{4.6.10}$$

$$t \simeq t_{ff} \left(\frac{t_{ff}}{t_k}\right)^{1/3} \simeq t_{ff} 10^4 \tag{4.6.11}$$

$$\Gamma \simeq \left(\frac{t_{ff}}{t_k}\right)^{-2/3} \simeq 10^8 \tag{4.6.12}$$

Taking $\overline{v}_s \simeq 10^5$ m/s, $T \simeq 10^7$ K, $t_{ff} \simeq 10^3$ s and $t_k \simeq 10^{15}$ s (which are the appropriate order of magnitude values for the sun) we obtain that $\overline{v} \simeq 10$ m/s, $\Delta T \simeq 10^{-1}$ K and $t \simeq 10^7$ s for a solar type star. These values should be compared with those corresponding to the stellar surface where $\overline{v} \simeq 10^3$ m/s, $\Delta T \simeq 10^2$ K and $t \simeq 10^2$ s. Clearly convection is more efficient at transferring heat in the outer layers of such a star than it is in the deep interior. This phenomenon can be attributed to the differences in the values of ρ, λ, and to a smaller extent T, between the core and surface regions. The density drops by a factor of approximately 10^8 while the mixing length and temperature drop by approximately 10^3. Note that the result obtained by this calculation agrees with the fact that solar mass stars have radiative cores and convective envelopes.

4.7 Solving the MLT Equations

4.7.1 Solution when ∇_{rad} is specified

We now solve the three MLT equations: (4.5.6), (4.5.7) and (4.5.11), for ∇, ∇_e and Γ, at a given point r in a convection zone assuming that the total flux $F(r)$ is known at that point. This implies by way of (4.2.2) that we know the value of ∇_{rad} at r. We will also assume, in this section and the next, that the effects of energy sources on the convective process may be ignored and that the local values of ∇_{ad} and the physical variables contained in A are known.

We state again, for completeness, the equations we would like to solve:

$$\Gamma = A\left(\nabla - \nabla_e\right)^{1/2} \tag{4.7.1}$$

$$\nabla_{rad} - \nabla = a_0 A\left(\nabla - \nabla_e\right)^{3/2} \tag{4.7.2}$$

$$\Gamma = \frac{\nabla - \nabla_e}{\nabla_e - \nabla_{ad}} \tag{4.7.3}$$

Note that the numerical factor of 9/4 in (4.5.7) has been replaced by the parameter a_0 in (4.7.2). This is done for the sake of generality since different versions of the MLT use different values for this constant.

Define

$$\xi \equiv \frac{\nabla_{rad} - \nabla}{\nabla_{rad} - \nabla_{ad}} \tag{4.7.4}$$

We now show that the MLT equations can be reduced to a single cubic equation in ξ. Once the value of ξ has been determined, we can solve for the actual gradient ∇ which, from (4.7.4), is given by

$$\nabla = (1 - \xi)\,\nabla_{rad} + \xi \nabla_{ad} \tag{4.7.5}$$

From (4.7.1) and (4.7.2) we have

$$\nabla_{rad} - \nabla = a_0 \Gamma \,(\nabla - \nabla_e) \tag{4.7.6}$$

Hence

$$\nabla = \frac{\nabla_{rad} + a_0 \Gamma \nabla_e}{1 + a_0 \Gamma} \tag{4.7.7}$$

from which

$$\nabla_{rad} - \nabla = \frac{a_0 \Gamma\,(\nabla_{rad} - \nabla_e)}{1 + a_0 \Gamma} \tag{4.7.8}$$

We now solve equations (4.7.3) and (4.7.7) simultaneously for ∇_e and write the resulting equation as follows:

$$\nabla_{rad} - \nabla_e = \frac{\Gamma\,(1 + a_0 \Gamma)\,(\nabla_{rad} - \nabla_{ad})}{1 + \Gamma\,(1 + a_0 \Gamma)} \tag{4.7.9}$$

Substituting this equation into (4.7.8) and dividing the resulting expression by $\nabla_{rad} - \nabla_{ad}$ gives

$$\xi = \frac{a_0 \Gamma^2}{1 + \Gamma\,(1 + a_0 \Gamma)} \tag{4.7.10}$$

This suggests that ξ may be regarded as a measure of the convective efficiency since it is a function of Γ alone. From (4.7.10) and (4.7.5) we see that

- When convection is inefficient: $\Gamma \to 0$, $\xi \to 0$ and $\nabla \to \nabla_{rad}$
- When convection is efficient: $\Gamma \to \infty$, $\xi \to 1$ and $\nabla \to \nabla_{ad}$
- In the transition region between inefficient and efficient convection: $\Gamma \simeq 1$, $\xi \simeq 0.5$ and $\nabla \simeq \frac{1}{2}\,(\nabla_{rad} + \nabla_{ad})$

Using (4.7.1), (4.7.6) and (4.7.10) we can write Γ in terms of ξ as follows

$$\Gamma = B\xi^{1/3} \tag{4.7.11}$$

Where

$$B \equiv \left[\frac{A^2}{a_0}\left(\nabla_{rad} - \nabla_{ad}\right)\right]^{1/3} \tag{4.7.12}$$

Clearly B is also a measure of the convective efficiency since ξ is a function of Γ alone.

- When convection is inefficient: $\Gamma \to 0$ and $B \to 0$
- When convection is efficient: $\Gamma \to \infty$ and $B \to \Gamma$

Hence $B \gg 1$ implies a high convective efficiency whereas $B \ll 1$ implies the opposite. However, if (A5) is to be satisfied, then $B \gg 1$ is alone an insufficient condition for highly efficient convection. It will be demonstrated later that if $A \leqslant 1$, then $B \gg 1$ implies $\overline{v} > v_s$. Thus for highly efficient, subsonic convection we require in addition to $B \gg 1$ that we also have $A \gg 1$.

Substituting (4.7.11) into (4.7.10) gives the desired cubic equation in ξ:

$$\xi^{1/3} + B\xi^{2/3} + a_0 B^2 \xi - a_0 B^2 = 0 \tag{4.7.13}$$

Note that this equation has only one real root for $0 \leqslant \xi \leqslant 1$. We also have $\xi \to 0$ as $B \to 0$ and $\xi \to 1$ as $B \to \infty$, which is in agreement with the discussion above.

4.7.1.1 Iterative Solutions of (4.7.13):

The iterative solutions of (4.7.13) are now presented since these are particularly well suited to numerical simulations of convection.

Case 1: $B < 1$

We start with $\left(a_0 B^2\right)^3$ as the initial trial value of ξ. Carrying out the iterative procedure analytically then results in the following expansion for ξ:

$$\xi = \left(a_0 B^2\right)^3 \left\{1 - 3B\left(a_0 B^2\right) - \left[3 - (9/a_0)\right]\left(a_0 B^2\right)^3 + \ldots\right\} \tag{4.7.14}$$

Case 2: $B \geqslant 1$

Begin by re-writing (4.7.13) in the form

$$1 - \xi = \frac{\left(\xi^{1/3} + B\xi^{2/3}\right)}{a_0 B^2} \tag{4.7.15}$$

and take

$$\xi = 1 - \left[\frac{(1+B)}{a_0 B^2}\right] \tag{4.7.16}$$

as the initial trial value of ξ. Carrying out the iterative procedure analytically then results in the following expansion for $1 - \xi$:

$$1-\xi = \frac{1+B}{a_0 B^2}\left[1 - \frac{1+2B}{3\,(1+B)}\left(\frac{1+B}{a_0 B^2}\right) + \frac{1}{9}\left(\left(\frac{1+2B}{1+B}\right)^2 - 1\right)\left(\frac{1+B}{a_0 B^2}\right)^2 + ...\right] \tag{4.7.17}$$

4.7.1.2 Determining the Convective Quantities:

Once the value of ξ is known, we can easily determine the remaining quantities of interest:

- The convective efficiency Γ can be determined from (4.7.11)
- The actual temperature gradient ∇ can be determined from (4.7.5)
- The difference between the actual temperature gradient and that of the element $\nabla - \nabla_e$, can be determined from (4.7.1)
- The fraction of the total flux carried by convection can be determined from:

  $$\frac{F_{con}}{F} = \left[1 - \left(\frac{\nabla_{ad}}{\nabla_{rad}}\right)\right]\xi \tag{4.7.18}$$

  which is obtained from (4.5.13) and (4.7.4).

- The average velocity of a convecting element $\overline{v}$, can be determined from:

  $$\overline{v} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\,\Gamma_1^{1/2}}\left[\left(\frac{\nabla_{rad} - \nabla_{ad}}{a_0 A}\right)\xi\right]^{1/3} v_s \tag{4.7.19}$$

  which is obtained from (4.3.9) and (4.7.1)

- The degree to which the actual gradient is superadiabatic $\nabla - \nabla_{ad}$, can be determined from:

$$\nabla - \nabla_{ad} = \frac{1+\Gamma}{a_0\Gamma^2}\left[\xi\left(\nabla_{rad} - \nabla_{ad}\right)\right] \tag{4.7.20}$$

which is obtained from (4.7.5) and (4.7.10)

4.7.1.3 The Limiting Cases

Highly Inefficient Convection:

We derive expressions for some of the quantities of interest in the limit of $B \ll 1$ convection.

Starting from (4.7.14) we have that

$$\xi \simeq a_0^3 B^6 \tag{4.7.21}$$

$$= a_0 A^4 \left(\nabla_{rad} - \nabla_{ad}\right)^2 \tag{4.7.22}$$

$$\ll 1 \tag{4.7.23}$$

We can now use this to deduce the following:

- The convective efficiency:

$$\Gamma = B\xi^{1/3} \tag{4.7.24}$$

$$\simeq A^2 \left(\nabla_{rad} - \nabla_{ad}\right) \tag{4.7.25}$$

$$\ll 1 \tag{4.7.26}$$

- The fraction of the total flux carried by convection:

$$\frac{F_{con}}{F} = \left(\frac{\nabla_{rad} - \nabla_{ad}}{\nabla_{rad}}\right)\xi \tag{4.7.27}$$

$$\simeq \frac{a_0 A^4 \left(\nabla_{rad} - \nabla_{ad}\right)^3}{\nabla_{rad}} \tag{4.7.28}$$

$$\ll 1 \tag{4.7.29}$$

- The average convective velocity:

$$\overline{v} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}\frac{\Gamma}{A}v_s \quad (4.7.30)$$

$$\simeq \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}A\left(\nabla_{rad} - \nabla_{ad}\right)v_s \quad (4.7.31)$$

Note from (4.7.31) that for $B \ll 1$ we do not necessarily have $\overline{v}/v_s \ll 1$. Such a situation could occur when energy is being transported by convection despite the relative "inability" of the stellar material to do so. To see this, consider the case when $\overline{v}$ is comparable to v_s. From (4.7.30) we must then have that $A \leqslant \Gamma \ll 1$, since the coefficient of Γ/A is normally of order unity. Hence from (4.7.31), $(\nabla_{rad} - \nabla_{ad}) \gg 1$. Thus in this case, the "driving force" for convection (represented by $\nabla_{rad} - \nabla_{ad}$) is very strong but the ability of the material to convect efficiently (represented by A) is very small. Large convective velocities are therefore necessary in order to transport the required amount of energy by convection.

- The superadiabaticity of the actual gradient:

$$\nabla - \nabla_{ad} = (1 - \xi)\left(\nabla_{rad} - \nabla_{ad}\right) \quad (4.7.32)$$

$$\simeq \nabla_{rad} - \nabla_{ad} \quad (4.7.33)$$

Hence $\nabla \simeq \nabla_{rad}$ as expected.

Highly Efficient Convection:

We now consider the limit of $B \gg 1$ convection. In this case, we may set $\xi = 1$. The following results are then obtained:

- The convective efficiency:

$$\Gamma = B\xi^{1/3} \quad (4.7.34)$$

$$\simeq B \quad (4.7.35)$$

$$= \left[\left(\frac{A^2}{a_0}\right)(\nabla_{rad} - \nabla_{ad})\right]^{1/3} \tag{4.7.36}$$

$$\gg 1 \tag{4.7.37}$$

- The fraction of the total flux carried by convection:

$$\frac{F_{con}}{F} = \left(\frac{\nabla_{rad} - \nabla_{ad}}{\nabla_{rad}}\right)\xi \tag{4.7.38}$$

$$\simeq 1 - \left(\frac{\nabla_{ad}}{\nabla_{rad}}\right) \tag{4.7.39}$$

$$\tag{4.7.40}$$

Note that if $\nabla_{rad} \gg \nabla_{ad}$ we have that $F_{con}/F = 1$

- The average convective velocity:

$$\overline{v} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}\frac{\Gamma}{A}v_s \tag{4.7.41}$$

$$\simeq \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}\left[\frac{a_0(\nabla_{rad} - \nabla_{ad})}{A}\right]^{1/3} v_s \tag{4.7.42}$$

For $B \gg 1$ we must have that $A \gg 1$ if the average convective velocity is to be subsonic. This is expected since the better the material's "capacity" for efficient convection, the smaller the velocity required for energy to be transported by convection.

- The superadiabaticity of the actual gradient:

$$\nabla - \nabla_{ad} = \frac{1+\Gamma}{a_0\Gamma^2}\left[\xi(\nabla_{rad} - \nabla_{ad})\right] \tag{4.7.43}$$

$$\simeq \frac{(\nabla_{rad} - \nabla_{ad})}{a_0\Gamma} \tag{4.7.44}$$

$$\ll \nabla_{rad} - \nabla_{ad} \tag{4.7.45}$$

Hence $\nabla \to \nabla_{ad}$ in this limit, as expected.

Of course, even for the most efficient convection, the actual temperature gradient can never be *exactly* adiabatic (or less than than adiabatic), since this would cause the layer to become stable against convection. Were this to happen, the convective energy transport would decay and the temperature gradient would increase until convection could set in again.

Lastly, since $A \propto \lambda^2$, we see from the above discussions that Γ, F_{con}/F and $\overline{v}/v_s$ are more sensitive to the value of λ for small convective efficiencies than they are for large ones.

4.7.2 Solution when ∇ is specified

In this section we solve for ∇_{rad}, ∇_e and Γ assuming that the local value of ∇ is known. Instead of reducing (4.7.1), (4.7.2) and (4.7.3) to a cubic equation in ξ, we choose for convenience to work with the equivalent set of equations (4.7.1), (4.7.6) and (4.7.20), which we now reduce to a quadratic equation in Γ.

Begin by eliminiating ξ from (4.7.20) using (4.7.4). Substite (4.7.6) into the resulting expression to obtain

$$\nabla - \nabla_{ad} = \frac{1+\Gamma}{\Gamma}\left(\nabla - \nabla_e\right) \tag{4.7.46}$$

Using (4.7.1), eliminate $(\nabla - \nabla_e)$ to obtain the desired equation:

$$\Gamma^2 + \Gamma - A^2\left(\nabla - \nabla_{ad}\right) = 0 \tag{4.7.47}$$

The solution of (4.7.47) which is of physical interest is

$$\Gamma = \frac{1}{2}\left[\sqrt{1 + 4A^2\left(\nabla - \nabla_{ad}\right)} - 1\right] \tag{4.7.48}$$

From (4.7.47) we see that the quantity $A^2\left(\nabla - \nabla_{ad}\right)$ may be regarded as a measure of the convective efficiency since it is a function of Γ alone. Note also from (4.7.48) that for

- inefficient convection: $\Gamma \ll 1$; $A^2\left(\nabla - \nabla_{ad}\right) \ll 1$
- efficient convection: $\Gamma \gg 1$; $A^2\left(\nabla - \nabla_{ad}\right) \gg 1$

4.7.2.1 Determining the Convective Quantities

- The radiative temperature gradient can be determined from

$$\nabla_{rad} = \nabla + \left(\frac{a_0}{8A^2}\right)\left[\sqrt{1+4A^2\left(\nabla-\nabla_{ad}\right)}-1\right]^3 \tag{4.7.49}$$

  which is obtained by eliminating $(\nabla - \nabla_e)$ from (4.7.6) using (4.7.1) and substituting (4.7.48) for Γ into the resulting equation.

- The convective flux can be determined begin by noting that:

  from (4.4.4) we have

$$\left(\nabla-\nabla_e\right)^{\frac{3}{2}} = \frac{\sigma_{rad}}{\sigma_{con}}\left(\nabla_{rad}-\nabla\right) \tag{4.7.50}$$

  Substituting (4.7.50) into (4.3.10) and using (4.7.49) to eliminate $(\nabla_{rad} - \nabla)$ from the resulting expression gives

$$F_{con} = \sigma_{rad}\left(\frac{a_0}{8A^2}\right)\left[\sqrt{1+4A^2\left(\nabla-\nabla_{ad}\right)}-1\right]^3 \tag{4.7.51}$$

- The fraction of the total flux carried by convection is given as before by (4.5.13)

- The average convective velocity can be determined from

$$\overline{v} = \frac{Q^{1/2}\alpha}{4\sqrt{2}\,\Gamma_1^{1/2}A}\left[\sqrt{1+4A^2\left(\nabla-\nabla_{ad}\right)}-1\right]v_s \tag{4.7.52}$$

  Which is obtained by subtituting (4.7.48) into (4.7.30)

4.7.2.2 Taking Radiative Heat Losses into Account

We now define the quantity

$$f \equiv \frac{\nabla-\nabla_e}{\nabla-\nabla_{ad}} \tag{4.7.53}$$

which from (4.7.1) and (4.7.48) may also be written as

$$f = \frac{\left[\sqrt{1+4A^2\left(\nabla-\nabla_{ad}\right)}-1\right]^2}{4A^2\left(\nabla-\nabla_{ad}\right)} \tag{4.7.54}$$

In the basic equations (4.3.9) and (4.3.10), one may choose to neglect the effects of heat losses due to radiation from the convecting elements by replacing $(\nabla - \nabla_e)$ in these equations with the approximate factor $(\nabla - \nabla_{ad})$. Such a situation may arise if we do not, for example, have knowledge of ∇_e. One can, however, easily take into account radiative losses by multiplying the uncorrected values of $\overline{v}$ and F_{con} by $f^{1/2}$ and $f^{3/2}$ respectively.

Note that

- for highly efficient convection $A^2\,(\nabla - \nabla_{ad}) \gg 1$ and hence $f \simeq 1$ by (4.7.54). From (4.7.53) we then have that $(\nabla - \nabla_e) \rightarrow (\nabla - \nabla_{ad})$ in this limit.

- for highly inefficient convection $A^2\,(\nabla - \nabla_{ad}) \ll 1$ and hence $f \ll 1$ by (4.7.54). Hence at small convective efficiencies, use of $(\nabla - \nabla_{ad})$ in the basic equations for $\overline{v}$ and F_{con} can lead to large overestimates for these quantities.

Chapter 5

Turbulent Convection

"Although the turbulent motion has been extensively discussed in the literature from different points of view, the very essence of this phenomenon is still lacking sufficient clearness." L.D. Landau.

The picture of convection that we have developed in the previous chapter represents a gross simplification of the actual behaviour of the stellar fluid. The reality of the situation is clearly much more complicated.

Convective flows in stars are highly turbulent and consist of an intricate heirarchy of eddies and bubbles moving and interacting in an extremely complicated way. The description of this phenomenon poses many physical and mathematical problems of great complexity. A theory of convection that can accurately account for turbulent processes does not yet exist. The principal difficulty lies in closing the system of equations that describe the flow. All closure models invoke additional heuristic or ad hoc hypotheses to close the system of equations at some chosen order. They either relate statistical quantities to each other, or propose relationships between the mean flow and turbulence. Existent models can be classified into four categories:

- *Algebraic models.* This includes the mixing-length approach of Prandtl [72].

- *One-equation models*, which use a modified turbulent kinetic energy equation along with a prescribed mixing length [74], [57].
- *Two-equation models*, the most famous of which is Kolmogorov's k-ϵ theory [43].
- *Reynolds stress models*, which use transport equations for the Reynolds shear stresses [75], [50].

In this chapter we start from the hydrodynamic equations for a vertically stratified medium and derive an expression for the convective heat flux within the MLT regime.

5.1 Mixing Length Theory Revisited

Begin by considering a plane parallel fluid layer of infinite horizontal extent. As usual, we ignore the effects of magnetic fields, rotation and nuclear energy generation. The fluid equations for such a system may be written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{5.1.1}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{5.1.2}$$

$$\rho\left[\frac{\partial}{\partial t}\left(U + \frac{u^2}{2}\right) + u_i \frac{\partial}{\partial x_i}\left(U + \frac{u^2}{2}\right)\right] = \rho u_i g_i - \frac{\partial}{\partial x_i}\left(F_{r,i} + \rho u_i - u_i \tau_{ij}\right) \tag{5.1.3}$$

where u_i is the velocity, p is the gas plus radiation pressure, U is the internal energy per unit mass, $F_{r,i}$ is the heat flux carried by both radiation and conduction and τ_{ij} is the viscous stress tensor.

All flow variables are assumed to be decomposable into a sum of the form

$$f_i = \overline{f_i} + f_i^{'} \tag{5.1.4}$$

where $\overline{f_i}$ is the ensemble average of the f_i and $f_i^{'}$ represents the fluctuation of f_i about the average state. Applying this decomposition to the continuity equation results in the following expressions for the mean and fluctuating parts,

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\overline{u_i} + \overline{\rho^{'} u_i^{'}}\right) = 0 \tag{5.1.5}$$

$$\frac{\partial \rho^{'}}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho^{'}\overline{u_i} + \overline{\rho}u_i^{'} + \rho^{'} u_i^{'}\right) = 0 \tag{5.1.6}$$

We now linearize the equations for the fluctuating parts and ignore pressure fluctuations, except when they occur in the momentum equation. This is known as the anelastic approximation [29], which amounts to filtering out high-frequency phenomena such as acoustic waves. In making this approximation we will clearly end up overestimating the convective flux since, in reality, kinetic energy from the turbulent flow will be converted to acoustic radiation via the generation of sound waves in the fluid. With this in mind (5.1.5) and (5.1.6) become

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\overline{u_i}\right) = 0 \tag{5.1.7}$$

$$\frac{\partial}{\partial x_i}\left(\overline{\rho}u_i^{'}\right) = 0 \tag{5.1.8}$$

respectively, and the mean momentum and energy equations may be expressed as

$$\frac{D\overline{u_i}}{Dt} = g_i - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial}{\partial x_j}\overline{u_i^{'}u_j^{'}} + \frac{1}{\overline{\rho}}\frac{\partial \overline{\tau_{ij}}}{\partial x_j} \tag{5.1.9}$$

and

$$\rho\frac{D}{Dt}\left(\overline{U} + \frac{1}{2}\overline{u^{'2}}\right) = -\frac{\partial \overline{F_{r,i}}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\overline{\left(\overline{\rho}h\right)^{'} u_i^{'}} + \overline{\frac{1}{2}\left(\overline{\rho}u^{'2}\right)^{'} u_i^{'}} - \overline{u_i^{'}\tau_{ij}}\right) - p\frac{\partial \overline{u_i}}{\partial x_i} + \overline{\left(\tau_{ij} - \overline{\rho}u_i^{'}u_j^{'}\right)}\frac{\partial \overline{u_i}}{\partial x_j} \tag{5.1.10}$$

respectively, where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{u_i}\frac{\partial}{\partial x_i} \tag{5.1.11}$$

and h is the specific enthalpy. The Reynolds number in stars has been estimated to be of the order 10^{10} [82] which implies that the molecular viscosity of the stellar fluid is very small. We can thus ignore the presence of $\overline{u_i'\tau_{ij}}$ and τ_{ij} in equation (5.1.10) since these will be small relative to the turbulent energy flux and Reynolds stress respectively.

An equation for the mean specific turbulent energy, appearing in (5.1.10), can be derived by averaging the equation obtained by taking the scalar product of (5.1.9) with u_i'. The resulting expression contains a term, ϵ, which describes the rate of dissipation of turbulent kinetic energy and may be defined as the trace of

$$\epsilon_{ij} = 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_j'}{\partial x_k}\right)} \tag{5.1.12}$$

In the standard MLT model of convection, one usually ignores ϵ and all other terms that contain ν. However, not only does this violate the conservation of energy, but it also leads to an overestimation of the turbulent heat and momentum fluxes [12]. The reason for this stems from the fact that ϵ remains finite even when $\nu \to 0$, since the mean-square vorticity diverges in this limit. A decrease in ν decreases the scale at which viscous dissipation occurs but not the rate of dissipation. This is consistent with the idea of a spectral energy cascade in which energy is transferred from large to small scale eddies in a manner which depends on the *inviscid* large scale dynamics of the turbulent field.

In one-equation models, ϵ is often estimated as

$$\epsilon \equiv \frac{\left(\overline{u'^2}\right)^{3/2}}{l_\epsilon} \tag{5.1.13}$$

where l_ϵ defines the length scale over which energy dissipation occurs. In the context of stellar convection, several authors (eg. [79], [98]) have assumed that $l_\epsilon = \lambda/D$ where λ is the mixing length and D is a drag coefficient of order unity.

For an incompressible fluid, the above assumptions constitute the Boussinesq approximation, which is used in almost all treatments of stellar convection that take the mixing-length approach. Within this approximation the fluid equations become:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{5.1.14}$$

$$\frac{\partial u_i^{'}}{\partial x_i} = 0 \tag{5.1.15}$$

$$\frac{D\overline{u_i}}{Dt} = g_i - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial}{\partial x_j}\left(\overline{u_i^{'}u_j^{'}}\right) \tag{5.1.16}$$

$$\frac{Du_i^{'}}{Dt} = \frac{\rho^{'}}{\overline{\rho}}g_i - \frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x_i} - \frac{\partial}{\partial x_j}\left(u_i^{'}u_j^{'} - \overline{u_i^{'}u_j^{'}}\right) - u_j^{'}\frac{\partial \overline{u_i}}{\partial x_j} + \nu\frac{\partial^2 u_i^{'}}{\partial x_j^2} \tag{5.1.17}$$

$$\frac{D\overline{T}}{Dt} = \frac{\overline{\delta}}{\overline{\rho c_p}}\frac{D\overline{p}}{Dt} - \frac{1}{\overline{\rho c_p}}\frac{\partial}{\partial x_i}\left(\overline{F_{r,i}} + \overline{\rho c_p}\overline{u_i^{'}T^{'}}\right) + \frac{\epsilon}{c_p} \tag{5.1.18}$$

$$\frac{DT^{'}}{Dt} = -\frac{\partial}{\partial x_i}\left(u_i^{'}T^{'} - \overline{u_i^{'}T^{'}}\right) + u_i^{'}\beta - \frac{1}{\overline{\rho c_p}}\frac{\partial F_{r,i}^{'}}{\partial x_i} + \frac{\nu}{c_p}\left[\left(\frac{\partial u_i^{'}}{\partial x_j}\right)^2 - \overline{\left(\frac{\partial u_i^{'}}{\partial x_j}\right)^2}\right] \tag{5.1.19}$$

where the superadiabatic temperature gradient, β, is defined as

$$\beta = -\frac{1}{\overline{c_p}}\left(\frac{d\overline{h}}{dx_3} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dx_3}\right) \simeq -\left(\frac{d\overline{T}}{dx_3} - \frac{\overline{\delta}}{\overline{\rho c_p}}\frac{d\overline{p}}{dx_3}\right) \tag{5.1.20}$$

and we have ignored fluctuations in δ and c_p.

Note that the terms describing the dissipation of momentum and energy in equations (5.1.17) and (5.1.19) respectively, also remain finite in the limit $\nu \to 0$ for the same reason as before.

We now use the fact that the mean horizontal velocity of our system is zero ($\overline{u_i} = 0$) and assume that the mean quantities in the fluid equations are stationary. (5.1.16) and (5.1.18) can then be simplified to,

$$\frac{d}{dz}\left(\overline{p_g} + \overline{p_t}\right) = -g\overline{\rho} \tag{5.1.21}$$

and

$$\frac{d}{dz}\left(\overline{F_{r,z}} + \overline{F_{c,z}}\right) = 0 \tag{5.1.22}$$

respectively, where z represents the vertical direction, p_g is the gas plus radiation pressure, p_t is the turbulent pressure and $F_{c,z}$ is the convective heat flux in the vertical direction. Using the following approximations,

$$\overline{p_t} \equiv \overline{\rho w^2} \simeq \overline{\rho}\,\overline{w^2} \tag{5.1.23}$$

$$\overline{F_{c,z}} \equiv \overline{\rho h' w} \simeq \overline{\rho c_p}\,\overline{wT'} \tag{5.1.24}$$

$$\overline{F_{r,z}} = -\overline{\rho c_p \kappa}\frac{d\overline{T}}{dz} \tag{5.1.25}$$

where w is the vertical component of the velocity and κ is the thermal diffusivity, we can rewrite the mean equations as

$$\frac{d\overline{p_g}}{dz} = -g\overline{\rho} - \overline{\rho}\frac{\partial \overline{w^2}}{\partial z} \tag{5.1.26}$$

and

$$\overline{wT'} + \kappa\frac{d\overline{T}}{dz} = H \tag{5.1.27}$$

respectively, where H is an integration constant. Equation (5.1.26) is just the law of hydrostatic equilibrium with the inclusion of a turbulent pressure and equation (5.1.27) expresses the fact that the sum of radiative and convective transfers of heat is constant over a given layer of fluid.

Using the Boussinesq equation of state,

$$\frac{\rho'}{\rho} = -\overline{\delta}\frac{T'}{T} \tag{5.1.28}$$

we can write the fluctuation equations as,

$$\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i' u_j' - \overline{u_i' u_j'}\right) = -\frac{1}{\overline{\rho}}\frac{\partial p_g'}{\partial x_i} - g_i\frac{\overline{\delta}}{\overline{T}}T' \tag{5.1.29}$$

and

$$\frac{\partial T'}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i' T' - \overline{u_i' T'}\right) - \beta w = -\frac{1}{\overline{\rho c_p}}\frac{\partial F_{r,i}'}{\partial x_i} \tag{5.1.30}$$

We now need a way of resolving the non-linear advection terms that appear in equations (5.1.29) and (5.1.30). In eddy-viscosity models, the effect of the turbulent mixing of momentum is thought to be analogous to the molecular transport of momentum, which leads to a laminar stress. The role of turbulence is then to increase the effective viscosity by an amount, ν_t, called the "eddy-viscosity". This idea, which dates back to Boussinesq's work in the 1870's, is widely used in engineering models of turbulence.

Of course, ν_t is a property of the turbulence and not the fluid. Prandtl was the first to propose a way of estimating this quantity using a model based on the concept of a "mixing length". Inspired by the kinetic theory of gases, he suggested that

$$\nu_t = w\lambda \tag{5.1.31}$$

where λ is the mixing length or "mean free path" of a turbulent eddy. The Reynolds stresses are then approximated as

$$-\rho\overline{u_i^{'}u_j^{'}} \simeq \rho\nu_t\frac{dw}{dz} \tag{5.1.32}$$

We can now simplify equations (5.1.29) and (5.1.30) as follows (see for eg. [91]):

$$\frac{\partial}{\partial x_j}\left(u_i^{'}u_j^{'} - \overline{u_i^{'}u_j^{'}}\right) \simeq \nu_t\frac{d^2w}{dz^2} \simeq \frac{2w^2}{\lambda} \tag{5.1.33}$$

$$\frac{\partial}{\partial x_i}\left(u_i^{'}T^{'} - \overline{u_i^{'}T^{'}}\right) \simeq \kappa_t\frac{d^2w}{dz^2} \simeq \frac{2wT^{'}}{\lambda} \tag{5.1.34}$$

where κ_t in (5.1.34) is the eddy diffusivity, which accounts for the turbulent exchange of heat. It was first introduced by Öpik in 1950 [68], and is defined as

$$\kappa_t = \frac{\nu_t}{P_r} \tag{5.1.35}$$

where P_r is the Prandtl number.

Note that in deriving (5.1.33) and (5.1.34) we have replaced the spatial derivatives of the fluctuating quantities by λ^{-1} so that $\partial^2/\partial x_i^2 = \lambda^{-2}$ (see for eg. [47]). The additional factors of 2 in the equations appear because we have assumed that a typical parcel at any instant might have travelled half the distance λ.

From here on we omit the overbars on mean quantities in so far as no ambiguity results. This is done both for the sake of simplicity, and to reconcile our notation with that found in the literature. In a static atmosphere in a steady state, we can ignore the time derivatives of the momentum and temperature fluctuations that appear in (5.1.29) and (5.1.30) respectively. If the pressure fluctuations in the momentum equation are ignored then the coupling between the vertical and horizontal motion will be removed. The fluctation equations can then be simplified with the aid of (5.1.33) and (5.1.34) to give the following expressions for the convective velocity and temperature fluctuation:

$$w^2 = \frac{1}{2} g \frac{\delta \lambda}{T} T^{'} \tag{5.1.36}$$

and

$$T^{'} = \frac{1}{2} \left(\beta - \frac{\kappa}{\lambda^2} \frac{T^{'}}{w} \right) \lambda \tag{5.1.37}$$

It can then be shown [30] that, in the limit of highly efficient convection, the convective flux may be approximated as

$$F_c \simeq \frac{1}{4} \rho c_p \left(\frac{g\delta}{T} \right)^{1/2} \lambda^2 \beta^{3/2}. \tag{5.1.38}$$

Note that the numerical factor in this formula may differ from paper to paper since it depends on the assumed geometry of the fluid element and the precise definition of λ.

Chapter 6

The Numerical Implementation of Convection

> *"...It would thus clearly be safer if we stopped our discussion of stellar evolution here and waited for the results from the big computers, which we may expect in the nearest future..."* M. Schwarzschild.

Numerical simulations have become a tool of vital importance in astrophysics. They give us invaluable information about complex systems and physical processes under extreme conditions which cannot be realized in laboratory based experiments. The origins of computational astrophysics can be traced back to 1956 when Haselgrove and Hoyle performed the first numerical simulation of stellar evolution on an electronic computer. In 1964 Louis Henyey proposed a relaxation method for solving the stellar structure equations in a way that was better suited to the two-boundary value nature of the problem. His method was a significant improvement over the direct numerical integration performed by Haselgrove and Hoyle, and has since been adopted by the majority of stellar evolution codes to date.

In this chapter we give a brief review of the implementation of convection in several well known computer codes for stellar evolution. A thorough discussion of these codes may be found in [65].

6.1 The Aarhus Stellar Evolution Code (ASTEC)

The development of ASTEC began at Cambridge in 1974 as part of an investigation of solar stability, following earlier work by Christensen-Dalsgaard et al. [16]. The code drew some inspiration from the Eggleton stellar evolution code [21], but was developed independently of it. ASTEC has found widespread use in the field of helio-seismology and has been carefully tested for the computation of solar models however considerable development is still required in the treatment of convective mixing.

The code solves the structure equations in the following form:

$$\frac{\partial \log_{10} r}{\partial x} = \frac{m}{4\pi\rho r^3} \tag{6.1.1}$$

$$\frac{\partial \log_{10} P}{\partial x} = -\frac{Gm^2}{4\pi r^4 P} \tag{6.1.2}$$

$$\frac{\partial \log_{10} T}{\partial x} = \nabla \frac{\partial \log_{10} P}{\partial x} \tag{6.1.3}$$

$$\frac{\partial \log_{10} L}{\partial x} = \left(\epsilon - \frac{\partial H}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial t}\right)\frac{m}{L} \tag{6.1.4}$$

where the independent variable, x, is defined by

$$x = \log_{10}\left(\frac{m}{M}\right) \tag{6.1.5}$$

and m is the mass that is interior to the point under consideration, r is the distance to the stellar centre, H is the enthalpy per unit mass, and the rest of the symbols have their usual meaning.

6.1.1 The Treatment of Convection

The value of ∇ in (6.1.3) depends on whether or not the region is convectively stable. If it is, then

$$\nabla = \nabla_{rad} = \frac{3}{16\pi ac}\frac{\kappa}{T^4}\frac{Lp}{Gm} \tag{6.1.6}$$

In convective regions, the calculation of ∇ is carried out using the Vitense [95] and Böhm-Vitense [5] version of MLT. In addition, emulations of the Canuto and Mazzitelli formulation [10], established by Monteiro et al. [64] can also be used. As discussed in section (4.1), the mixing length is taken as a constant multiple of the pressure scale height and is either set by the user or, in the case of solar models, is automatically fitted so as to match the observational data.

The treatment of convective cores is still an area of active development in ASTEC [18]. Due to the lack of a satisfactory numerical prescription for diffusion near the core, an explicit calculation of its chemical evolution is performed according to the following formula,

$$\frac{dX_k}{dt} = \overline{\mathcal{R}}_k + \frac{1}{q_c}\frac{dq_c}{dt}\left[X_k(x_c) - X_k\right] \tag{6.1.7}$$

where X_k is the mass fraction of element k, R_k is the rate of change of X_k due to nuclear reactions, q_c is the mass fraction in the convective core (ie. the ratio of the mass of the core to the total mass of the star), $x_c = \log_{10} q_c$, and

$$\overline{\mathcal{R}}_k = \frac{1}{q_c}\int_0^{q_c} \mathcal{R}_k dq \tag{6.1.8}$$

is the reaction rate averaged over the core.

The term $X_k(x_c)$ in (6.1.7) is evaluated just outside the core and only has an effect if there is a composition discontinuity at its edge. This can happen in models of intermediate mass stars (with masses up to $1.7M_\odot$) where the gradual conversion of ^{16}O to ^{14}N causes an increase in the importance of the CNO cycle and a discontinuity in the hydrogen abundance at the edge of the core [18]. This also leads to discontinuities in ρ and κ, and since κ increases with X and ρ, while ρ decreases with increasing X, it is not *a priori* clear how ∇_{rad} behaves at this discontinuity. In practice, ∇_{rad} should increase in going from the value of X in the convective core to the higher value in the radiative region just outside it. The problem here is that, if the edge of the core is defined using the composition of the core, then the region immediately outside it will be convectively unstable. As a consequence, ASTEC defines the edge of the convective core by the chemical abundances in the radiative region, leaving a small convectively stable region (which is assumed to be fully mixed in the standard ASTEC implementation [18]) below that boundary. This may be regarded as an example of semi-convection, the effects of which are still not fully understood.

The code has several options for the treatment of convective overshooting. The overshoot region may be taken to be either adiabatically or radiatively stratified and the implementation of overshoot from a convective envelope follows the work by Monteiro et al. [63].

One of the shortcomings of ASTEC is its failure to treat adequately models with convective cores, especially when both diffusion and the settling of heavy elements are included (cf. [17]). This problem may be related to the issues of semi-convection where convective stability is closely related to the details of the composition profile [62].

6.2 The Yale Rotating Stellar Evolution Code (YREC)

The original rotating version of YREC was developed by Pinsonneault [70] in 1988 as part of his Ph.D thesis, and was based on earlier work by Kippenhahn and Thomas [45] and Endal and Sofia [25]. It has since undergone several revisions and has been used extensively in the fields of asteroseismology and stellar rotation. Notable features of YREC include its treatment of convective core overshoot and its implementation of the effects of turbulence on the structure of the surface layers of stars with convective enevelopes.

YREC uses the well known Henyey method [36] to solve the structure equations in Lagrangian form [1]

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{6.2.1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \tag{6.2.2}$$

$$\frac{\partial L}{\partial m} = \epsilon - c_p \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t} \tag{6.2.3}$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla \tag{6.2.4}$$

The system is treated as a two point boundary value problem, and a relaxation technique based on a finite difference approximation is used.

6.2.1 The Treatment of Convection

Convective stability is tested for using the Schwarzschild criterion, and the abundance of chemical species in convective cores is treated under the assumption of instantaneous mixing. The code implements some novel treatments of convective overshoot and turbulence which we discuss below.

6.2.1.1 Core Overshoot

YREC has several options related to the treatment of overshoot (OS), all of which assume that the mixing of chemical species in the OS region is instantaneous. This assumption is well founded due to the small characteristic timescale of convection relative to the star's thermal and nuclear timescales. The OS options may be divided into 2 categories:

[1] See for example [46]

- A parametric treatment in which the extent of the OS is taken as a multiple of H_p according to

$$r_{os} = r_s + \alpha H_p(r_s) \tag{6.2.5}$$

  where r_s is distance from the centre of the star to the border of the convective instability and $H_p(r_s)$ is the pressure scale height at that point.

- A physically motivated treatment in which the extent of the OS is calculated from a non-local convection theory originally developed by Kuhfuss [48] and later extended by Wuchterl and Feuchtinger [97]. This approach is based on an equation for the turbulent kinetic energy which Kuhfuss derives by taking spherical averages of the first-order perturbed Navier-Stokes equations within the framework of the anelastic approximation. The extent of the convective core can be determined from the solution of this equation, which also incorporates the effects of OS.

  The theory contains a set of five parameters which must first be calibrated before use. In the strictly local limit, Kuhfuss' treatment is equivalent to the MLT equations when based on the Ledoux criterion. A detailed discussion of the implemented equations and associated numerical techniques can be found in Straka et al. [87].

6.2.1.2 The Inclusion of Turbulence

YREC can incorporate the effects of turbulence into the outer layers of one-dimensional stellar models. The method requires a detailed three-dimensional hydrodynamical simulation of the atmosphere and superadiabatic regions of the star in order to extract the required paramaters from the velocity field. It also incorporates work by Li et al. [55] who use a self consistent approach introduced by Lydon and Sofia [58] to include the effects of magnetic fields in the calculation of the convective temperature gradient within the MLT framework.

6.3 The ATON 3.1 Stellar Evolution Code

ATON is a versatile stellar evolution code that uses the Newton-Raphson method to solve the structure equations in Lagrangian form. It was originally developed and used by Mazzitelli in his study of the helium content and mixing length in solar models [59]. The code was later updated to include the Full Spectrum of Turbulence (FST) model of convection [10] which he co-developed with Canuto. By the end of the nineties treatments of OS and non-instantaneous mixing were also incorporated [93]. Other improvements were made by taking advantage of the grid model of atmospheres, originally computed by Heiter et al. [35], which also employed the Canuto and Mazzitelli convection model. Version 2.4 of the code contains routines that allow for the effects of rotation according to the approach followed by Endal and Sofia [24]. This approach improves upon the method of Kippenhahn and Thomas [45] by adding a potential function that includes a term related to the distortion of the geometry of the star. These methods, although not included in version 3.1, are to be updated and added to ATON 3.4 [94]. The current version differs substantially from other codes in the following areas:

- The treatment of convection.
- The handling of diffusive mixing and OS.
- The input physics of the equation of state.
- The technique used to compute the opacity.

6.3.1 The Treatment of Convection

The code allows for the calculation of the temperature gradient within the instability region using either the traditional MLT, or the FST model. The mixing in these regions is taken to be non-instantaneous and is treated as a diffusion process according to the equation

$$\frac{dX_k}{dt} = R_k + \frac{\partial}{\partial m}\left[\left(4\pi r^2 \rho\right)^2 D \frac{\partial X_k}{\partial m}\right] \tag{6.3.1}$$

where X_k and R_k are defined as in (6.1.7). Note that the diffusion coefficient, D, is taken to be $D = \frac{1}{3}vl$, where v and l are the convective velocity and length scale respectively. Non-instantaneous mixing coupled with nuclear reactions is important for the description of all phases of stellar evolution for which the lifetime of nuclear species is comparable to the mixing lifetime (as in the case of lithium production in giants). It does not affect the duration of core hydrogen burning in stars with a convective core, but does have an influence on the He burning phase (which has been shown to be longer in the non-instantaneous treatment [94]). An attractive feature of this implementation is that non-instantaneous mixing spontaneously mimicks semiconvection profiles at the border of the helium burning core [94].

6.3.1.1 Overshooting

ATON's handling of OS is based on the following formula,

$$v = v_b \exp \pm \left(\frac{1}{\zeta f} \ln \frac{P}{P_b}\right) \tag{6.3.2}$$

which describes the exponential decay of the convective velocity as measured from the Schwarzschild boundary. v_b and P_b are the turbulent velocity and pressure at the boundary, P is the local pressure, and f refers to the thickness of the convective region measured in fractions of H_p. ζ is a free parameter which defines the distance over which the velocities in the OS region go to zero. It is related to the e-folding distance of the decay. A detailed discussion of this parameter and the description of convective velocities in OS regions may be found in [93].

6.4 Modules for Experiments in Stellar Astrophysics (MESA)

MESA is a highly modularized, open-source library for stellar astrophysics that began as an effort to improve upon the now defunct EZ stellar evolution code [21], [69]. The MESA modules are "thread safe", meaning that more than one process can execute the module routines at the same time. This allows for the utilization of multicore processors which is particularly useful in simulations of stellar evolution.

The library includes a one-dimensional code called MESA Star which serves the same purpose as EZ but has a much richer set of features. The numerical and computational methods used in MESA star, which include adaptive mesh refinement and sophisticated timestep adjustment, allow it to evolve stellar models consistently through phases of stellar evolution that have posed substantial challenges for evolutionary codes in the past. These include the helium core flash in low mass stars and the advanced nuclear burning phases in massive stars.

6.4.1 The Treatment of Convection

The convection module uses the standard model of MLT as discussed in chapter (4). Given the total luminosity, it can compute the actual temperature gradient or, alternatively, the convective flux given the actual temperature gradient. The methods employed are identical to those presented in section (4.7).

MESA also incorporates the variation of MLT due to Henyey et al. [38], which allows the convective efficiency to vary with the opaqueness of the convecting element. This is an important effect in the outer convection zones of stars.

6.4.1.1 Overshoot

OS is treated as a time-dependent diffusion process with a diffusion coefficient, D, given by

$$D = D_0 \exp\left(-\frac{2z}{f\lambda_0}\right) \tag{6.4.1}$$

where D_0 is the MLT derived diffusion coefficient at a user-defined location near the Schwarzschild boundary, λ_0 is the pressure scale height at that boundary, z is the length scale over which the OS occurs, and f is an adjustable parameter [39].

The parameter, f, may have different values at the upper and lower convective boundaries for non-burning, H-burning, He-burning, and metal-burning convection zones. It essentially allows the user to set a lower limit on D below which overshoot mixing is neglected, and to limit the region of the star over which OS mixing will be considered

Chapter 7

Discussion

> *"Despite the great achievements of the stellar evolution theory, there are many points of disagreement between theory and observations which are ultimately related to our poor knowledge of the extension of convectively unstable regions and associated mixing processes"* C. Chiosi.

The convection problem has vexed stellar astrophysics for several decades. It is the prototype of many astrophysical problems in which the bottle-neck preventing significant progress is the difficulty involved in solving the hydrodynamic equations. Any convective flow, whether turbulent or not, will be non-linear due to the presence of the advection term, $u \cdot \nabla u$, in the momentum equation and it is this non-linearity that makes all but the very simplest problems almost impossible to solve. Our present inability to derive analytic solutions of the Navier-Stokes equations in the case of turbulent flows has forced us to depend on computer simulations. However, numerical solutions of the equations are themselves extremely difficult to obtain due to the large range of spatial and temporal scales that need to be resolved. In direct numerical simulations (DNS) the mesh resolution required to resolve length scales ranging from the Kolmogorov microscales up to the integral scale of the flow results in a computing time that is currently not feasible for practical applications [81]. This limitation in computing power is something that the astrophysics community has had to contend with for some time now and has forced researchers in the field to develop simplified models of convection that are based on rather drastic approximations. The mixing length theory is one such model.

The mixing-length treatment has been proven to be fairly adequate when building stellar models whose sole purpose is the description of the salient features of stellar evolution. But it has well known shortcomings in specific applications. Almost all versions of MLT approximate the equations of motion in the manner set out by Boussinesq [8]. The Boussinesq approximation is known to be valid only when the mixing-length is much less than the pressure and density scale heights, and implies in particular that the motion is subsonic. However, there is a disagreement between stellar models and observations unless the mixing length is of the same order as these scale heights, so the theory is internally inconsistent. Moreover, the energy equation does not admit terms such as viscous heating which can be important in deep layers, even when the motion is subsonic [31]. It also appears that supersonic convective velocities may be achieved in certain classes of variable stars, making the generation of acoustic energy an important effect. If the flux of acoustic energy becomes a significant proportion of the total flux, then pulsational effects can no longer be ignored. This is not something that can be described within the Boussinesq regime since it does not allow for the possibility of acoustic waves.

Another major deficiency of the MLT is that it describes the convective heat flux in terms of the local properties of the fluid. This makes it particularly inadequate for studies of non-local effects such as convective overshoot, which is known to play a significant role in almost all stages of stellar evolution. Moreover, MLT provides little information about the dynamical properties of convection apart from the mean size of the convective eddies and their turnover time.

MLT is therefore inadequate when dealing with:

- The description of photospheric and subphotospheric layers.
- Convective overshoot and penetration.
- The coupling between convection and pulsation.
- The interaction between convection and rotation.
- The generation and transport of magnetic fields in stars.

A number of papers in the literature have started to remedy the situation. Alternate prescriptions for the treatment of convection in the stellar interior have for example been proposed by Canuto and Mazittelli [10], who claim that the MLT can be significantly improved by considering the full spectrum of eddies with the appropriate convective flux distributions. Recently, more sophisticated models which not only include the turbulent pressure but the full Reynolds stress and other higher order moments in the velocity and temperature fluctuations have also received attention [11], [12].

Moreover, rapid advances in computer technology are finally beginning to provide scientists with the tools necessary to perform full 3-dimensional hydrodynamic simulations of turbulent flow [86]. Such simulations unquestionably provide a more realistic picture of convection, but this approach is still too expensive to be implemented in current stellar evolution codes, which is why it has not yet replaced the mixing-length treatment.

In light of the above, it is evident that despite its many shortcomings, the mixing-length theory of convection is, for moment, likely to remain the model of choice in stellar evolution calculations. It remains to be seen whether the use of more sophisticated closure models based on higher order moments will result in predictions that are in better agreement with observation than the MLT. This is now becoming an area of active research and it is hoped that the development of these models will provide a solution to some of the severe limitations present in MLT. It is however, my opinion that significant progress can be made only by use of full 3-dimensional hydrodynamic simulations, which in the near future may finally become a viable alternative to MLT, a theory which has indeed served us well, but which we must agree has reached the end of its tour of duty.

> *"Little attention is paid to assessing the accuracy of the models, partly because there is a general feeling that mixing-length theory is so uncertain that the task would be fruitless, and partly, perhaps, because of an optimism that the theory will soon be superseded by something better."* D. Gough.

Appendix A

The Adiabatic Temperature Gradient

Consider a bubble of gas located a distance r from the stellar centre. If the bubble is slightly hotter than its surroundings, it will begin to rise under the action of the buoyancy force. We will assume that that gas is ideal and that the bubble moves adiabatically through the stellar medium in such way that it always maintains a state of complete pressure equilibrium with its environment. We now to determine how the temperature of the gas *inside* the bubble changes as it rises and expands adiabatically.

A.1 Adiabatic Temperature Gradient in Terms of Distance

The ideal and adiabatic gas laws can be expressed in terms of ρ as

$$P = \rho \Re T \tag{A.1.1}$$

$$P = k\rho^{\gamma} \tag{A.1.2}$$

respectively, where $\gamma \equiv c_p/c_v$ is known as the adiabatic index. From (A.1.1),

$$\begin{aligned} \frac{dP}{dr} &= \frac{P}{\rho}\left(\frac{d\rho}{dr}\right) + \frac{P}{T}\left(\frac{dT}{dr}\right) \\ &= \frac{1}{\gamma}\frac{dP}{dr} + \frac{P}{T}\left(\frac{dT}{dr}\right) \end{aligned} \tag{A.1.3}$$

using (A.1.2). We therefore have that

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \tag{A.1.4}$$

A.2 The Adiabatic Temperature Gradient in Terms of Pressure

One often defines the temperature stratification within a convective element in terms of P since it is a more direct measure of the thermodynamic state of the material and is itself a function of r. In this approach, the temperature gradient of the convective element, as it moves adiabatically through a region of the star, is defined to be

$$\nabla_{ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)_s \tag{A.2.1}$$

We now derive an expression for (A.2.1) noting that for an adiabatic process, $dq = 0$.

Assume that the equation of state is $\rho = \rho(P,T)$ and the heat equation is $u = u(\rho, T)$. The change in internal energy of the system can be expressed as,

$$du = \left(\frac{\partial u}{\partial v}\right)_T dv + \left(\frac{\partial u}{\partial T}\right)_v dT \tag{A.2.2}$$

This allows us to write dq as,

$$dq = \left(\frac{\partial u}{\partial T}\right)_v dT + \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv \tag{A.2.3}$$

Now $(\partial u/\partial v)_T$ can be expressed in terms of T by considering the change $ds = dq/T$ in the specific entropy. Using (A.2.3), ds can be written as:

$$ds = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_v dT + \frac{1}{T}\left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv \tag{A.2.4}$$

Since dv and dT determine ds, they may be regarded as independent differentials. Now, the differential ds is exact, so (A.2.4) must arise from the differentiation of a function of the form

$$s = s(T, v) \tag{A.2.5}$$

Differentiating (A.2.5) gives us,

$$ds = \left(\frac{\partial s}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial v}\right)_T dv \tag{A.2.6}$$

Comparing (A.2.4) with (A.2.6) we get

$$\left(\frac{\partial s}{\partial T}\right)_v = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_v dT$$

and

$$\left(\frac{\partial s}{\partial v}\right)_T = \frac{1}{T}\left[\left(\frac{\partial u}{\partial v}\right)_T + P\right]$$

Since ds is a total differential form,

$$\frac{\partial^2 s}{\partial T \partial v} = \frac{\partial^2 s}{\partial v \partial T}$$

which means that,

$$\left(\frac{\partial}{\partial T}\right)_v \left[\frac{1}{T}\left(\frac{\partial u}{\partial v}\right)_T + \frac{P}{T}\right] = \frac{1}{T}\frac{\partial^2 u}{\partial T \partial v} \tag{A.2.7}$$

But,

$$\left(\frac{\partial}{\partial T}\right)_v \left[\frac{1}{T}\left(\frac{\partial u}{\partial v}\right)_T + \frac{P}{T}\right] = -\frac{1}{T^2}\left(\frac{\partial u}{\partial v}\right)_T + \frac{1}{T}\frac{\partial^2 u}{\partial T \partial v} - \frac{P}{T^2} + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_v$$

Substituting the above into (A.2.7) gives us *the first energy equation* for the system,

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_v - P$$

We can now rewrite (A.2.3) as,

$$dq = \left(\frac{\partial u}{\partial T}\right)_v dT + T\left(\frac{\partial P}{\partial T}\right)_v dv$$

Thus,

$$\left(\frac{\partial u}{\partial T}\right)_v = \left(\frac{dq}{dT}\right)_v \equiv c_v$$

where c_v is the *specific heat* at constant volume. Hence

$$dq = c_v dT + T\left(\frac{\partial P}{\partial T}\right)_v dv \tag{A.2.8}$$

Now define an isothermal compressibility coefficient, α, as:

$$\alpha \equiv -\frac{P}{v}\left(\frac{\partial v}{\partial P}\right)_T = \left(\frac{\partial \ln \rho}{\partial \ln P}\right)_T \tag{A.2.9}$$

and a volume coefficient of expansion, δ, as:

$$\delta \equiv \frac{T}{v}\left(\frac{\partial v}{\partial T}\right)_P = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P \tag{A.2.10}$$

dq can be expressed in terms of α and δ by noting that,

$$\left(\frac{\partial v}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial T}{\partial v}\right)_P = -1$$

This is called the *First Cyclic Relation*. Solve this equation for $(\partial P/\partial T)_v$ using the *Reciprocal Relation*, and introduce α and δ into the resulting expression as follows:

$$\left(\frac{\partial P}{\partial T}\right)_v = -\frac{\left(\frac{\partial v}{\partial T}\right)_P}{\left(\frac{\partial v}{\partial P}\right)_T} = \frac{P\delta}{T\alpha} \tag{A.2.11}$$

The expression for dq given in (A.2.8) then becomes,

$$dq = c_v dT + T\frac{P\delta}{T\alpha}dv \tag{A.2.12}$$

We still can not however, determine (A.2.1) from (A.2.12) in its current form. We need to introduce a term containing dP, so that we can eventually solve for dT/dP. This can be achieved by noting that,

$$\begin{aligned} dv &= \frac{d\rho}{\rho^2} \\ &= \frac{1}{\rho}\left[\left(\frac{\partial \ln \rho}{\partial \ln P}\right)_T d\ln P + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P d\ln T\right] \\ &= \frac{1}{\rho}\left[\alpha\frac{dP}{P} - \delta\frac{dT}{T}\right] \end{aligned} \tag{A.2.13}$$

Using (A.2.13), (A.2.12) becomes,

$$\begin{aligned} dq &= c_v dT - \frac{P\delta}{\rho\alpha}\left(\alpha\frac{dP}{P} - \delta\frac{dT}{T}\right) \\ &= \left(c_v + \frac{P\delta^2}{\rho T\alpha}\right)dT - \frac{\delta}{\rho}dP \\ &= c_p dT - \frac{\delta}{\rho}dP \end{aligned} \tag{A.2.14}$$

Now,

$$0 = dq = c_p dT - \frac{\delta}{\rho}dP$$

which implies that

$$\left(\frac{dT}{dP}\right)_s = \frac{\delta}{\rho c_p}$$

and hence

$$\nabla_{ad} \equiv \frac{P}{T}\left(\frac{dT}{dP}\right)_s = \frac{P\delta}{T\rho c_p} \tag{A.2.15}$$

Note also that (A.2.15) can be substituted into (A.2.14) to yield an expression for dq in terms of the adiabatic temperature gradient

$$\begin{aligned} dq &= c_p dT - \frac{Tc_p\nabla_{ad}}{P}dP \\ &= c_p T\left[\frac{dT}{T} - \nabla_{ad}\frac{dP}{P}\right] \end{aligned} \tag{A.2.16}$$

Bibliography

[1] Aizenman, M.L., Cox, J.P. (1975): Ap. J., 202, 137.

[2] Biermann, L. (1937): Astr. Nachr., 263, 185.

[3] Biermann, L. (1951): Z. Astrophys. 28, 304.

[4] Boesgaard, A.N., Tropicco, M.J. (1986): Ap. J., 303, 724.

[5] Böhm-Vitense, E. (1958): Z. Astrophys., 46, 108.

[6] Böhm-Vitense, E. (1989): Introduction to Stellar Astrophysics, vol.II, Cambridge University Press.

[7] Böhm-Vitense E. (1992): Introduction to Stellar Astrophysics, vol.III, Cambridge University Press.

[8] Boussinesq J. (1903): Theorie Analytique de la Chaleur, vol. II, Gauthier-Villars.

[9] Caloi V., Mazzitelli I. (1990): Astron. Astrophys., 240, 305.

[10] Canuto V.M., Mazzitelli I. (1991): Ap. J., 370, 295.

[11] Canuto V.M. (1992): Ap. J., 392, 218.

[12] Canuto V.M. (1993): Ap. J., 416, 331.

[13] Castellani V., Chieffi A., Pulone L., Tornambe, A. (1985): Ap. J. 296, 204.

[14] Chandrasekhar S. (1965): Ap. J., 142, 1519.

[15] Chen Y.Q., Nissen P.E., Benoni T., Zhao G. (2001): arXiv:astro-ph/0104053v1

[16] Christensen-Dalsgaard J., Dilke F.W.W., Gough D.O. (1974): M.N.R.A.S., 169, 429.

[17] Christensen-Dalsgaard J. (2007): E.A.S. Publ. Ser., 26, 177

[18] Christensen-Dalsgaard J. (2008): Astrophys. Space Sci., 316, 13.

[19] Chiosi C., Maeder A. (1986): ARA. & A., 24, 239.

[20] Cox J.P., Giuli R.T. (2004): Principles of Stellar Structure, Cambridge Scientific.

[21] Eggleton P.P. (1971): M.N.R.A.S., 151, 351.

[22] Eggleton P.P. (1972): M.N.R.A.S., 156, 361

[23] Eggleton P.P., Dearborn D.S.P, Lattanzio J.C. (2006): Science, 314, 5805.

[24] Endal A.S., Sofia S. (1976): Ap. J., 210, 184.

[25] Endal A.S., Sofia S. (1981): Ap. J., 243, 625.

[26] Faulkner J., Griffiths K., Hoyle F. (1965): M.N.R.A.S., 129, 323.

[27] Freytag B., Ludwig H.G., Steffen M. (1996): Astron. Astrophys., 313, 497.

[28] Galloway D.J., Weiss N.O. (1981): Ap. J., 243, 945.

[29] Gough D.O., (1969): J. Atmos. Sci., 26, 448.

[30] Gough D.O., (1976): Proc. IAU Colloq. 38, 15.

[31] Gough D.O., (1977): Ap. J., 214, 196.

[32] Hansen C.J. (1978): Ann. Rev. Astron. Astrophys., 16, 15.

[33] Hayashi C., (1961): Publ. Astron. Soc. Japan, 13, 450.

[34] Hayashi C., Hoshi R., Sugimoto D. (1962): Suppl. Progr. Theor. Phys., 22, 1.

[35] Heiter U. (2002): Astron. Astrophys., 392, 619.

[36] Henyey L.G., Wilets L., Böhm K.H., et al. (1959): Ap. J., 129, 628.

[37] Henyey L.G., Forbes J.E., Gould N.L. (1964): Ap. J., 139, 306.

[38] Henyey L.G., Vardya M.S., Bodenheimer P. (1965): Ap. J., 142, 841.

[39] Herwig F. (2000): Astron. Astrophys., 360, 952.

[40] Hofmeister E., Kippenhahn R., Weigart A. (1964): Z.Ap., 59, 215.

[41] Hofmeister E., Weigart A. (1964): Zs. f. Ap., 59, 119.

[42] Iben I., Ehrman J.R. (1962): Ap. J., 135, 770.

[43] Jones W.P., Launder B.E. (1972): Int. J. Heat Mass Transfer, 16, 1119.

[44] Kippenhahn R., Weigert A., Hofmeister E. (1967): Methods in Computational Physics, vol. 7, Academic Press.

[45] Kippenhahn R., Thomas H. (1970): Proc. IAU Colloq. 4, 20.

[46] Kippenhahn R., Weigert A., (1994): Stellar Structure and Evolution, Springer-Verlag.

[47] Kraichnan R.H. (1962): The Physics of Fluids, 5, 11.

[48] Kuhfuss R. (1986): Astron. Astrophys., 160, 116.

[49] Landau L.D., Lifshitz E.M. (1987): Fluid Mechanics (Pergamon Press).

[50] Launder B.E., Reece G.J., Rodi W. (1975): J. Fluid Mech., 68, 537.

[51] LeBlanc F., Michaud G. (1995): Astron. Astrophys., 303, 166.

[52] Lebovitz N.R. (1965): Ap. J., 142, 229.

[53] Langer N., El Eid M.F., Fricke K.J. (1985): Astron. Astrophys., 145, 179.

[54] Ledoux P. (1947): Ap. J., 105, 305.

[55] Li L.H., Robinson F.J., Demarque P., et al. (2002): Ap. J., 567, 1192.

[56] Limber D. (1958): Ap. J., 127, 363.

[57] Ljuboja M., Rodi W. (1981): A.S.M.E. J. Heat Transfer, 103, 343.

[58] Lydon T.J., Sofia S. (1995): Ap. J. Suppl. Ser., 101, 357.

[59] Mazzitelli I. (1979): Astron. Astrophys., 79, 251.

[60] Mazzitelli I. (1999): A.S.P.C., 173, 77.

[61] Mohanty S., Baraffe I., Chabrier G. (2007): IAU Symposium, 239, 197.

[62] Montalbán J., Théado S., Lebreton Y. (2007): E.A.S. Publ. Ser., 26, 167.

[63] Monteiro M.J.P.F.G., Christensen-Dalsgaard J., Thompsom M.J., (1994): Astron. Astrophys., 283, 247.

[64] Monteiro M.J.P.F.G., Christensen-Dalsgaard J., Thompsom M.J., (1996): Astron. Astrophys., 307, 624.

[65] Monteiro M.J.P.F.G., et. al. (2008): Evolution and Seismic Tools for Stellar Astrophysics, Springer.

[66] Mowlavi N., Forestini M. (1994): Astron. Astrophys. 282, 843.

[67] Orszag S.A. (1977): Fluid Dynamics, Gordon and Breach.

[68] Opik E.J., (1950): M.N.R.A.S., 110, 559.

[69] Paxton B. (2004): P.A.S.P., 116, 699.

[70] Pinsonneault M.H. (1988): Ph.D. Thesis, Yale University.

[71] Plaskett H.H., (1936): M.N.R.A.S., 96, 402.

[72] Prandtl L. (1925): Z. Angew. Math. Mech., 5, 136.

[73] Robertson J.W., Faulkner D.J. (1972): Ap. J., 171, 309.

[74] Rodi W. (1976): Z.A.M.M., 56, 219.

[75] Rotta J. (1951): Z. Phys., 129, 547.

[76] Schwarzschild, K. (1916): Sitzber. Deutsch. Akad. Wiss. 584.

[77] Schwarzschild M., Härm, R. (1958): Ap. J., 128, 348.

[78] Schwarzschild M. (1958): Structure and Evolution of the Stars, Princeton University Press.

[79] Shaviv G., Chitre S.M. (1968): M.N.R.A.S., 140, 61.

[80] Simpson E.E. (1971): Ap. J., 165, 295.

[81] Speziale C.G. (1998): A.I.A.A. 36, 2.

[82] Spiegel E. (1966): Stellar Evolution, Stein R.F., Cameron A.G. (eds.), Plenum Press, New York.

[83] Spiegel E. (1963): Ap. J., 138, 216.

[84] Spruit H.C. (1992): Astron. Astrophys., 253, 131.

[85] Srineevasan S.R., Ziebarth K.E. (1974): Astrophys. Space Sci., 30, 57.

[86] Stein R.F., Nordlund A. (1998): Ap. J., 342, 95.

[87] Straka C.W., Demarque P., Guenther D.B. (2005): Ap. J., 629, 1075.

[88] Taylor G.I., (1915): Phil. Trans. R. Soc. London, A., 215, 1.

[89] Thorne K.S. (1966): Ap. J. 144, 201.

[90] Trimble V. (1992): PASP, 104, 1.

[91] Unno W. (1967): P.A.S.J., 19, 140.

[92] Unsöld A. (1931): Z. Astrophys. 2, 209.

[93] Ventura P., Zeppieri A., Mazzitelli I., et al. (1998): Astron. Astrophys., 334, 953.

[94] Ventura P., D'Antona F., Mazzitelli I. (2008): Astrophys. Space Sci., 316, 93.

[95] Vitense E. (1953): Zs. f. Ap. 32, 135.

[96] Weiss A. (2002): E.S.A. S.P., 485, 57.

[97] Wuchterl G., Feuchtinger M.U. (1998): Astron. Astrophys., 340, 419.

[98] Xiong D. (1986): Astron. Astrophys., 167, 239.

[99] Xu H.Y., Li Y., (2004): Astron. Astrophys., 418, 213.