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On the arguments of the roots of the generalized Fibonacci polynomial*

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Abstract. We revisit the classical subject of equidistribution of the roots of Littlewood-type polynomials. More precisely, we show that the roots of the family of polynomials $\Psi_k(z) = z^k - z^{k-1} - \cdots - 1$, $k \geqslant 1$, are uniformly distributed around the unit circle in the strong quantitative form, confirming a conjecture from [C.-A. Gómez and F. Luca, *Commentat. Math. Univ. Carol.*, On the distribution of roots of $z^k - z^{k-1} - \cdots - z - 1$, 62(3):291–296, 2021].

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1 Introduction

Location of zeros of polynomials with restricted coefficients is a central subject in analysis with vast literature; see Ambrose [1], Bloch and Polya [2], Bombieri and Vaaler [3], Erdős and Turan [5], Borwein and Erdeley [4], Littlewood [7], Schur [9], Odlyzko and Poonen [8], and many others. In particular, if we restrict attention to the polynomials with coefficients ± 1, the so-called Littlewood polynomials, we expect that for a "typical polynomial" in this family, the zeros become "evenly distributed around the unit circle." Erdős and Turan [5] made this statement precise by proving their beautiful theorem. See also a recent paper of Soundararajan [10] with an new Fourier-analytic proof of their result. It is however a rather nontrivial task to show the equidistribution of zeros *for any particular* family of polynomials. One example when this task is essentially obvious is the family of polynomials

$$\psi_n(z) = z^n + z^{n-1} + \cdots + 1 = \frac{z^{n+1} - 1}{z - 1}.$$

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In this case the corresponding roots are the roots of unity $\alpha_n^k = \mathrm{e}^{2\pi \mathrm{i}k/(n+1))}$ with $k = \overline{1, n+1}$, which are evenly distributed around the circle.

Gómez and Luca [6] looked at the roots of the Littlewood polynomials $\Psi_k(z) = z^k - z^{k-1} - \cdots - 1$ for $k \geqslant 1$. Sometimes these polynomials are called the k-generalized Fibonacci polynomials since they are the characteristic polynomials of any k-length Fibonacci recurrence $\{F_n^{(k)}\}_{n\geqslant 0}$ with $F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)}$ for $n \geqslant 0$. Since $F_n^{(k)}$ is generally given by a Binet-type formula, which is a linear combination of the nth powers of the roots of $\Psi_k(z)$, it is important to understand the behavior of these roots if we want to answer Diophantine questions with k-generalized Fibonacci numbers. Let these roots be labeled $\alpha_1, \ldots, \alpha_k$. It is known that such a polynomial has exactly one real root larger than 1, which belongs to the interval $(2-1/2^k, 2)$. The other roots are inside the disc $|z| < 1$. In fact, a bit more is is known about their modulus, namely that they satisfy the inequality $1 - (\log 3)/k < |z| < 1 - 1/(2^8 k^3)$. This inequality was proved in [6]. As for the arguments of these roots, based on numerical evidence, Alexis Gómez, one of the authors of [6] conjectured the following:

Conjecture 1. For each $h \in \{0, 1, \ldots, k-1\}$, there is a root $\alpha_j = \rho_j \mathrm{e}^{\mathrm{i}\theta_j}$ of $\Psi_k(z)$ with $\theta_j \in [0, 2\pi)$ and

$$\left|\theta_j - \frac{2h\pi}{k}\right| < \frac{1}{k}.$$

The conjecture has been verified numerically by Gómez for all $k \in [2, 1000]$. Here we prove Conjecture 1 up to a small constant factor.

Theorem 1. *For $k \geqslant 2$ and every $h \in \{0, 1, \ldots, k-1\}$, there is θ_j such that*

$$\left|\theta_j - \frac{2h\pi}{k}\right| < \frac{\pi}{k}.$$

Furthermore, it appears that in the above inequality the right-hand side $O(1/k)$ improves to $O(1/k^2)$ as roots get closer to the imaginary axis.

Our proof avoids complex analysis. It only uses real analysis, more precisely, the fact that a continuous function of a real variable changing sign in an interval must have a zero in that interval.

2 Proof

Consider the related polynomial

$$g_k(z) = (z-1)\Psi_k(z) = z^{k+1} - 2z^k + 1.$$

Let $z = \rho \mathrm{e}^{\mathrm{i}\theta}$, $0 < \rho < 1$, and let $\theta \in [0, \pi)$ be a root of $g_k(z)$ in the upper half-plane. Rewrite the equation $g_k(z) = 0$ as

$$z + z^{-k} = 2$$

and consider the real and imaginary parts of both sides:

$$\rho \cos\theta + \rho^{-k}\cos(k\theta) = 2, \tag{2.1}$$

$$\rho \sin\theta - \rho^{-k}\sin(k\theta) = 0. \tag{2.2}$$

From (2.2) we obtain

$$\rho^{k+1} = \frac{\sin(k\theta)}{\sin\theta}. \tag{2.3}$$

Multiplying Eqs. (2.1) and (2.2) by $\sin(k\theta)$ and $\cos(k\theta)$, respectively, and then adding them, we obtain

$$\rho(\cos\theta\sin(k\theta) + \sin\theta\cos(k\theta)) = 2\sin(k\theta)$$

or

$$\rho\sin\bigl((k+1)\theta\bigr) = 2\sin(k\theta), \tag{2.4}$$

which gives

$$\rho = \frac{2\sin(k\theta)}{\sin((k+1)\theta)}. \tag{2.5}$$

Elimination of ρ using (2.3) and (2.5) yields

$$\frac{2^{k+1}\sin^{k+1}(k\theta)}{\sin^{k+1}((k+1)\theta)} = \frac{\sin(k\theta)}{\sin\theta}. \tag{2.6}$$

By (2.2) and (2.4), $\sin\theta$, $\sin(k\theta)$, and $\sin((k+1)\theta)$ are all nonzero for $\theta \notin \{0,\pi\}$. Then by (2.6) θ must be one of the zeros of the function

$$H(x) = 2^{k+1}\sin x \cdot \sin^k(kx) - \sin^{k+1}\bigl((k+1)x\bigr).$$

In the interval $x \in (0,\pi)$, $\sin(kx)$ vanishes at the points $\beta_\ell = \pi\ell/k$, $1 \leqslant \ell \leqslant k-1$, whereas $\sin((k+1)x)$ vanishes at the points $\alpha_m = \pi m/(k+1)$, $1 \leqslant m \leqslant k$. In view of the inequalities

$$\frac{j}{k+1} < \frac{j}{k} < \frac{j+1}{k+1} < \frac{j+1}{k},$$

these vanishing points are interlaced:

$$0 < \alpha_1 < \beta_1 < \cdots < \alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1} \cdots < \beta_{k-1} < \alpha_k < \pi.$$

Since $\sin((k+1)x) > 0$ for $x \in (\alpha_{2j}, \alpha_{2j+1})$, by interlacing we obtain

$$H(\beta_{2j}) = -\sin^{k+1}\bigl((k+1)\beta_{2j}\bigr) < 0.$$

Likewise, $\sin(kx) > 0$ for $x \in (\beta_{2j}, \beta_{2j+1})$, whereas $\sin x > 0$ in $(0,\pi)$. By interlacing

$$H(\alpha_{2j+1}) = 2^{k+1}\sin(\alpha_{2j+1})\sin^k(k\alpha_{2j+1}) > 0.$$

This means that $H(x)$ changes sign in each interval $(\beta_{2j}, \alpha_{2j+1})$, and therefore it has at least one zero there, say, $\theta_j \in (\beta_{2j}, \alpha_{2j+1})$. By interlacing the functions $\sin(kx)$ and $\sin((k+1)x)$ are both positive precisely in the intervals $(\beta_{2j}, \alpha_{2j+1})$. If $H(\theta_j) = 0$, then Eq. (2.6) holds for $\theta = \theta_j$. Consequently, the positive real number $\rho_j = \rho(\theta_j)$ defined by Eq. (2.4) satisfies Eqs. (2.3) and (2.4). This means that the pair (θ_j, ρ_j) is a solution to the system of equations (2.1)–(2.2). Therefore the complex number $z_j = \rho_j e^{i\theta_j}$ is the root of the polynomial $g_k(z)$. Conversely, by Eqs. (2.3) and (2.5) each nonreal root z_j with positive imaginary part must have an argument $\theta_j \in (\beta_{2j}, \alpha_{2j+1})$ for some j in the range $1 \leqslant j \leqslant \lfloor (k-1)/2 \rfloor$. Since the number of such intervals $\lfloor (k-1)/2 \rfloor$ matches the number of complex roots of $g_k(z)$, namely $(k-2)/2$ when k is even and $(k-1)/2$ when k is odd, there is a one-to-one correspondence between the set of roots and the set of intervals. Now let z be an arbitrary nonreal root of $g_k(z)$. By replacing z with its complex conjugate if necessary, we can find an integer j in $[1, \lfloor (k-1)/2 \rfloor]$ such that $z = \rho_j e^{i\theta_j}$ with $\rho_j < 1$ and $\theta_j \in (\beta_{2j}, \alpha_{2j+1})$, as it was shown above. So the distance between β_{2j} to θ_j is at most

$$0 \leqslant \theta_j - \beta_{2j} \leqslant \alpha_{2j+1} - \beta_{2j} = \frac{\pi(2j+1)}{k+1} - \frac{\pi(2j)}{k} = \frac{\pi(k-2j)}{k(k+1)} \leqslant \frac{\pi(k-2)}{k(k+1)} \leqslant \frac{\pi}{k+1}. \tag{2.7}$$

This concludes the proof.

Remark. Note that as j approaches $\lfloor (k-1)/2 \rfloor$, the approximations get even better until the error decreases to about $2\pi/k^2$ near the imaginary axis. Namely, substituting $j = \lfloor (k-1)/2 \rfloor$ in (2.7) yields

$$0 \leqslant \theta_j - \beta_{2j} \leqslant \frac{\pi(k - 2\lfloor (k-1)/2 \rfloor)}{k(k+1)} \leqslant \frac{2\pi}{k(k+1)} < \frac{2\pi}{k^2}.$$

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