Martingales on Riesz Spaces and Banach Lattices [1]

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Contents

Chapter 1

Introduction

The aim of this work is to do a literature study on spaces of martingales on Riesz spaces and Banach lattices, using [16, 19, 20, 17, 18, 2, 30] as a point of departure.

Convergence of martingales in the classical theory of stochastic processes has many applications in mathematics and related areas.

Operator theoretic approaches to the classical theory of stochastic processes and martingale theory in particular, can be found in, for example, [4, 5, 6, 7, 13, 15, 26, 27]. The classical theory of stochastic processes for scalar-valued measurable functions on a probability space (Ω, Σ, μ) utilizes the measure space (Ω, Σ, μ), the norm structure of the associated $L^p(\mu)$-spaces as well as the order structure of these spaces.

Motivated by the existing operator theoretic approaches to classical stochastic processes, a theory of discrete-time stochastic processes has been developed in [16, 19, 20, 17, 18] on Dedekind complete Riesz spaces with weak order units. This approach is measure-free and utilizes only the order structure of the given Riesz space. Martingale convergence in the Riesz space setting is considered in [18]. It was shown there that the spaces of order bounded martingales and order convergent martingales, on a Dedekind complete Riesz space with a weak order unit, coincide.

A measure-free approach to martingale theory on Banach lattices with quasi-interior points has been given in [2]. Here, the groundwork was done to generalize the notion of a filtration on a vector-valued L^p-space to the M-tensor product of a Banach space and a Banach lattice (see [1]).

In [30], a measure-free approaches to martingale theory on Banach lattices is given. The main results in [30] show that the space of regular norm bounded martingales and the space of norm bounded martingales on a Banach lattice E are Banach lattices in a natural way provided that, for the former, E is an order continuous Banach lattice, and for the latter, E is a KB-space.

The definition of a "martingale" defined on a particular space depends on the type of space under consideration and on the "filtration," which is a sequence of operators defined on the space. Throughout this dissertation, we shall consider Riesz spaces, Riesz spaces with order units, Banach spaces, Banach lattices and Banach lattices with quasi-interior points. Our definition of a "filtration" will, therefore, be determined by the type of space under consideration and will be adapted to suit the case at hand.

In Chapter 2, we consider convergent martingale theory on Riesz spaces. This chapter is based on the theory of martingales and their properties on Dedekind complete Riesz spaces with weak order units, as can be found in [19, 20, 17, 18]. The notion of a "filtration" in this setting is generalized to Riesz spaces. The space of martingales with respect to a given filtration on a Riesz space is introduced and an ordering defined on this space. The spaces of regular, order bounded, order convergent and generated martingales are introduced and properties of these spaces are considered. In particular, we show that the space of regular martingales defined on a Dedekind complete Riesz space is again a Riesz space. This result, in this context, we believe is new.

The contents of Chapter 3 is convergent martingale theory on Banach lattices. We consider the spaces of norm bounded, norm convergent and regular norm bounded martingales on Banach lattices. In [30], filtrations (T_n) on the Banach lattice E which satisfy the condition

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} = E,$$

where $\mathcal{R}(T_n)$ denotes the range of the filtration, are considered. We do not make this assumption in our definition of a filtration (T_n) on a Banach lattice. Our definition yields equality (in fact, a Riesz and isometric isomorphism) between the space of norm convergent martingales and $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$. The aforementioned main results in [30] are also considered in this chapter. All the results pertaining to martingales on Banach spaces in subsections 3.1.1, 3.1.2 and 3.1.3 we believe are new.

Chapter 4 is based on the theory of martingales on vector-valued L^p-spaces (cf. [4]), on its extension to the M-tensor product of a Banach space and a Banach lattice as introduced by Chaney in [1] (see also [29]) and on [2]. We consider filtrations on tensor products of Banach lattices and Banach spaces as can be found in [2]. We show that if (S_n) is a filtration on a Banach lattice F and (T_n) is a filtration on a Banach space X, then

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n \otimes S_n)} = \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} \, \widetilde{\otimes}_M \, \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(S_n)}.$$

This yields a distributive property for the space of convergent martingales on the

M-tensor product of X and F. We consider the continuous dual of the space of martingales and apply our results to characterize dual Banach spaces with the Radon-Nikodým property.

We use standard notation and terminology as can be found in standard works on Riesz spaces, Banach spaces and vector-valued L^p-spaces (see [4, 23, 29, 31]). However, for the convenience of the reader, notation and terminology used are included in the Appendix at the end of this work. We hope that this will enhance the pace of readability for those familiar with these standard notions.

Chapter 2

Martingales on Riesz Spaces

Traditionaly martingale theory has been developed on $L^1(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is some probability space. We use the fact that $L^1(\Omega, \mathcal{F}, P)$ is a Riesz space to develop martingale theory for Riesz spaces in general. Before we do this, it is necessary to remind ourselves of some basic classical martingale notions.

2.1 Classical Martingale Notions

For the following definitions, $(\Omega, \mathcal{F}, P)$ is a probability space.

Definition 2.1 An increasing sequence of σ-algebras is called a *filtration*.

Definition 2.2 A Σ-measurable function $\mathbb{E}[f|\Sigma]$, defined for a random variable $f \in L^1(\Omega, \mathcal{F}, P)$ and a sub-σ-algebra Σ is called a *conditional expectation* if

$$\int_A \mathbb{E}[f|\Sigma]\ dP = \int_A f\ dP, \text{ for each } A \in \Sigma.$$

Definition 2.3 A conditional expectation with respect to the trivial σ-algebra $\{\phi, \Omega\}$ is called an *expectation* and is given by $\mathbb{E}(f) = \left(\int_\Omega f\ dP\right)$, where $f \in L^1(\Omega, \mathcal{F}, P)$.

Remark: Since $\mathbb{E}$ is a positive order continuous linear functional on $L^1(\Omega, \mathcal{F}, P)$, it follows that $\mathbb{T}$, defined by $\mathbb{T}(f) = \mathbb{E}(f)\mathbf{1} = (\mathbb{E} \otimes \mathbf{1})f$ for each $f \in L^1(\Omega, \mathcal{F}, P)$ (where $\mathbf{1}$ denotes the constant 1 function), defines an order continuous operator on $L^1(\Omega, \mathcal{F}, P)$. This prompts the definition for expectation operators on a Riesz spaces with weak order unit which will appear in the next section.

Definition 2.4 A pair $(X_n, \mathcal{B}_n)$, where $(X_n) \subseteq L^1(\Omega, \mathcal{F}, P)$ and $(\mathcal{B}_n)$ is a filtration with X_n $\mathcal{B}_n$-measureable for each $n \in \mathbb{N}$ is called a *(sub,super)-martingale* if

$$\mathbb{E}(X_n|\mathcal{B}_n)(\geq, \leq) = X_n, \text{ for each } n \in \mathbb{N}.$$

2.2 Martingale Theory on Riesz Spaces with weak order units

We now look to generalize the above definitions and develop further martingale theory in the abstract Riesz space setting. From [6], we see that conditional expectations in the classical setting are the only positive contractive projections on $L^1(\Omega, \mathcal{F}, P)$ and thus it is natural in our new setting to replace conditional expectations with positive contractive projections.

In [16, Chapters 1-2] these concepts were generalized to Riesz spaces with weak order units and we take a look at how it was done in that context. Thus, we now move on to consider martingales on Riesz spaces with weak order unit as can be found in [16, 19]. However, for reasons which will later become clear, we change the terminology used in [16, 19].

The notion of "filtration" in this setting is motivated by the properties of a conditional expectation in the classical setting of measure spaces.

Definition 2.5 Let T be a positive order continuous projection on a Riesz space E with weak order unit such that the range, $\mathcal{R}(T)$, is a Dedekind complete Riesz subspace of E. If T maps weak order units to weak order units in E, then we call T an *RS-wou-conditional expectation.*

It is known that the condition that T maps weak order units to weak order units is equivalent to the existence of a weak order unit e in E for which $Te = e$ (see [18]).

Definition 2.6 A family of RS-wou-expectations $(T_n)_{n\in\mathbb{N}}$ defined on a Riesz space E, for which

$$T_nT_m = T_mT_n = T_n, \text{ for each } m \geq n,$$

is called an *RS-wou-filtration* on E with respect to (T_n).

Thus an RS-wou-filtration is a commuting family of RS-wou-conditional expectations with increasing ranges, i.e. $\mathcal{R}(T_n) \uparrow_n$.

Definition 2.7 The pair $(f_n, T_n)_{n\in\mathbb{N}}$ on a Riesz space E, where $f_n \in \mathcal{R}(T_n)$ for each $n \in \mathbb{N}$ and (T_n) is an RS-wou-filtration is called a *(sub, super)-martingale* if

$$f_n(\leq, \geq) = T_nf_m, \text{ for each } m \geq n.$$

We now look at the concept of an expectation operator on a Riesz space with weak order unit, which is a special case of a conditional expectation, the definition of which is motivated by our remarks at the end of the previous section (see also [28, page 345]).

Definition 2.8 Let E be a Riesz space with weak order unit. An RS-wou-conditional expectation operator on E with one-dimensional range is called an *RS-wou-expectation.*

Remark: Every RS-wou-expectation operator, $\mathbb{T}$, on a Dedekind complete Riesz space E is of the form $\mathbb{T} = \nu^* \otimes e$ where $\nu^*(e) = 1$, ν^* is a positive order continuous linear functional on E and e is a weak order unit of E invariant under $\mathbb{T}$.

Definition 2.9 Let E be a Dedekind complete Riesz space with weak order unit, $(T_n)_{n=1}^{\infty}$ a RS-wou-filtration on E and $\mathbb{T}$ an expectation operator on E. (T_n) and $\mathbb{T}$ are said to be *compatible* if $T_n\mathbb{T} = \mathbb{T} = \mathbb{T}T_n$, for each $n \in \mathbb{N}$. In this case, if we set $T_0 = \mathbb{T}$, we have that $(T_n)_{n=0}^{\infty}$ is again a RS-wou-filtration on E.

The existence of an expectation operator on E compatible with a given filtration imposes an additional constraint on the space E as shown in the following proposition.

Proposition 2.10 *Let E be a Dedekind complete Riesz space with weak order unit and (T_n) an RS-wou-filtration on E. There exists an expectation operator on E compatible with (T_n) if and only if the set of positive order continuous linear functionals on E has a non-zero element.*

PROOF: If there exists an expectation operator, $\mathbb{T}$, on E then $\mathbb{T} = \nu^* \otimes e$ where ν^* is a non-zero positive order continuous linear functional on E and e is a weak order unit (since $\nu^*(e) = 1$).

Conversely, let (T_n) be a filtration and e a weak order unit invariant under T_n for each $n \in \mathbb{N}$. If there is a positive order continuous linear functional, ν^*, on E then $\nu^*(e) = \alpha > 0$. Setting $\mu^*(f) = \alpha^{-1}\nu^*(T_1 f)$, we obtain that $\mathbb{T}f := \mu^*(f)e$ is an expectation operator that commutes with T_n for each $n \in \mathbb{N}$. In particular

$$\mathbb{T}T_n f = \mu^*(T_n f) = \alpha^{-1}\nu^*(T_1 T_n f)e = \alpha^{-1}\nu^*(T_1 f)e = \mu^*(f)e = \mathbb{T}f$$

and $T_n\mathbb{T}f = \mu^*(f)T_n e = \mu^*(f)e = \mathbb{T}f.$ ■

Let E be a Dedekind complete Riesz space with weak order unit, (T_n) a filtration on E and $\mathbb{T}$ an expectation operator compatible with (T_n). Then $\mathbb{T}$ has representation $\mathbb{T} = \nu^* \otimes e$ where ν^* is a positive order continuous linear functional on E and e a weak order unit of E invariant under $\mathbb{T}$ and T_n for each $n \in \mathbb{N}$.

2.3 Martingale Convergence on Riesz Spaces with weak order units

We now consider some convergence results for martingales on Riesz spaces with weak order units, as can be found in [16, 18], which require the following preliminaries.

Lemma 2.11 *Let E be an Archimedean Riesz space with weak order unit e. Let $0 < f \in E$ and $0 < K < 1, K \in \mathbb{R}$. Then there exists $0 < q \in \mathbb{R}$ with*

$$(qe - f)^+ \wedge (f - Kqe)^+ > 0.$$

Proof: Let P_1 denote the band projection onto the band generated by f, $B_f \neq \{0\}$. Then $P_1 e$ is a weak order unit for B_f and as such $f \wedge nP_1 e \uparrow_n f$. Hence there exists $n \in \mathbb{N}$ with $(nP_1 e - f)^+ > 0$ (in fact, this inequality will hold for all sufficiently large n).

Let P_2 be the band projection onto the band $B_2 = B_{(nP_1 e - f)^+} \neq 0$. Then, as $P_2 P_1 = P_2$ and f is a weak order unit for $B_f \supseteq B_2$, it follows that

$$0 < P_2 f < nP_2 e. \tag{2.1}$$

Now suppose the lemma is false. Then

$$P_{(f-Kqe)^+} - P_{(f-qe)^+} = 0, \text{ for each } q > 0.$$

In particular

$$P_{(P_2 f - K^{j+1} nP_2 e)^+} - P_{(P_2 f - K^j nP_2 e)^+} = 0, \text{ for } j = 0, 1, 2, ... \tag{2.2}$$

Summing the equations in (2.2) for $j = 0, ..., J-1$ and observing from (2.1) that $(P_2 f - nP_2 e)^+ = 0$, we get that

$$P_{(P_2 f - K^J nP_2 e)^+} = P_{(P_2 f - nP_2 e)^+} = 0, \text{ for each } J \in \mathbb{N}.$$

Thus $0 \leq P_2 f \leq K^J nP_2 e$ for each $J \in \mathbb{N}$. But $0 < K < 1$ so $K^J \to 0$ as $J \to \infty$, which, from the Archimedean property, implies that $P_2 f = 0$, contradicting (2.1). ■

Lemma 2.12 *Let $m, M \in E$ with $M > m$ where E is a Dedekind complete Riesz space with weak order unit e. Then there exists $s < t$ such that*

$$(M - te)^+ \wedge (se - m)^+ > 0.$$

PROOF: As $M - m > 0$, by Lemma (2.11), there exists $q > 0$ such that,

$$P_1 = P_{(M-m-5qe)^+} - P_{(M-m-7qe)^+} \neq 0,$$

i.e. $P_1 e \neq 0$. We now restrict attention to the band $\mathcal{R}(P_1)$ where we have $P_1(M - m - 5qe) \geq 0$, and $P_1(M - m - 7qe) \leq 0$. This gives

$$P_1 m + 5qP_1 e \leq P_1 M \leq P_1 m + 7qP_1 e. \tag{2.3}$$

Applying Lemma (2.11) again, we obtain that there exists $r \in \mathbb{R}$ such that

$$P_2 = P_{(P_1 M-(r-q)P_1 e)^+} - P_{(P_1 M-(r+q)P_1 e)^+} \neq 0$$

and thus

$$P_2[(P_1 M - (r - q)P_1 e)^+] \geq 0 \geq P_2[(P_1 M - (r + q)P_1 e)^+].$$

As $P_1 P_2 = P_2 P_1 = P_2$, we have

$$(r + q)P_2 e \geq P_2 M \geq (r - q)P_2 e \tag{2.4}$$

and from (2.3) we see that $P_2 m + 5qP_2 e \leq P_2 M$. Thus

$$P_2 m \leq P_2 M - 5qP_2 e \leq (r - 4q)P_2 e. \tag{2.5}$$

Let $t = r - 2q$ and $s = r - 3q$, then from (2.4) we have $P_2 M - tP_2 e \geq qP_2 e$ and from (2.5) we get $sP_2 e - P_2 m \geq qP_2 e$. Combining the above formulae, we obtain

$$(M - te)^+ \wedge (se - m)^+ \geq qP_2 e > 0,$$

from which the lemma follows. ∎

Lemma 2.13 *Let E be a Dedekind complete Riesz space with weak order unit and T a strictly positive RS-wou-conditional expectation on E. If $g \in \mathcal{R}(T)^+$ such that $T|f| \leq g$ then $f \in B_g$ (the band generated by g).*

PROOF: Let P denote the band projection onto B_g, then, by [17, Lemma 3.1], T and P commute, as $g \in \mathcal{R}(T)$. Applying $I - P$ to $0 \leq T|f| \leq g$ and using the commutation of P and T gives

$$0 \leq (I - P)T|f| = T(I - P)|f| \leq (I - P)g.$$

The strict positivity of T now enables us to conclude that $(I - P)|f| = 0$. ∎

The notion of a "stopping time" in the classical setting has been extended in [19] to the Riesz space setting:

Definition 2.14 *Let (T_i) be a RS-wou-filtration on a Riesz space E with weak order unit. A stopping time adapted to (T_i) is an increasing sequence $P = (P_i)$ such that each P_i is a band projections on E satisfying*

$$T_j P_i = P_i T_j, \text{ for each } i \leq j, \tag{2.6}$$

where we define $P_0 = 0$. The stopping time (P_i) is bounded if there exists N so that $P_n = I$ for all $n \geq N$.

Let P and S be stopping times adapted to the RS-wou-filtration (T_i). We say $S \leq P$ if $P_i \leq S_i$ for each $i \in \mathbb{N}$.

Definition 2.15 *Let E be a Riesz space with weak order unit and let $P = (P_i)$ be a bounded stopping time adapted to the filtration (T_i). For each $(f_i) \subset E$ with $f_i \in \mathcal{R}(T_i)$ for all $i \in \mathbb{N}$, we define the stopped process (f_P, T_P) by*

$$f_P = \sum_i (P_i - P_{i-1}) f_i, \tag{2.7}$$

$$T_P f = \sum_i (P_i - P_{i-1}) T_i f, \quad f \in E. \tag{2.8}$$

We recall the following result from [19], which will be used in the proof of the Upcrossing theorem below:

Theorem 2.16 Hunt's Optional Stopping Theorem *([19, Theorem 5.5])*
Let E be a Riesz space with weak order unit, $(f_n, T_n)_{i\in\mathbb{N}}$ a RS-wou-(sub, super) martingale on E, then (f_P, T_P) is a (sub, super) martingale over the family of all stopping times adapted to the filtration (T_P) and indexed by the partially ordered set of all bounded stopping times adapted to (T_n).

The following result is taken from [16]:

Theorem 2.17 Upcrossing Theorem
Let (f_n, T_n) be a RS-wou-sub (super) martingale in a Riesz space E with weak order unit and let $g, f \in \mathcal{R}(T_1)$ with $g \leq f$. Define the upcrossing yield of (f_n, T_n) across the order interval $[g, f]$ to be

$$Y_N(g, f) = \sum_{n=1}^{N} Q_N^n (f - g)$$

where $S_0^j = 0 = Q_0^j$, $S_i^1 = P_{\bigvee\limits_{n=1}^{i}(g-f_n)^+}$ *and*

$$\begin{aligned} Q_i^j &= \sum_{m=2}^{i} P_{\bigvee\limits_{n=m}^{i}(f_n-f)^+}(S_{m-1}^j - S_{m-2}^j) \\ S_i^{j+1} &= \sum_{m=2}^{i} P_{\bigvee\limits_{n=m}^{i}(g-f_n)^+}(Q_{m-1}^j - Q_{m-2}^j). \end{aligned}$$

Then $T_1 Y_N(g,f) \le T_1(f_N - g)^+$ *(resp.* $T_1(f_N - f)^-$*).*

Proof: Let $N \in \mathbb{N}$ be fixed through out the proof. Let $h_i = (f_i - g)^+$, then (h_i, T_i) is a sub martingale and as $S^{n+1} \ge Q^n$ we have that $S^{n+1} \wedge N \ge Q^n \wedge N$. Thus from Theorem (2.16), $h_{Q^n \wedge N} \le T_{Q^n \wedge N}(h_{S^{n+1}\wedge N})$. Consequently $0 \le T_{Q^n\wedge N}(h_{S^{n+1}\wedge N} - h_{Q^n \wedge N})$, and as $T_1 T_{Q^n \wedge N} = T_1$ we have that $0 \le T_1(h_{S^{n+1}\wedge N} - h_{Q^n\wedge N})$. Thus

$$\begin{aligned} T_1\left[\sum_{n=1}^{N}[h_{Q^n\wedge N} - h_{S^n\wedge N}]\right] &= T_1[h_{Q^N\wedge N} - h_{S^1\wedge N}] + \sum_{n=1}^{N-1} T_1[h_{Q^n\wedge N} - h_{S^{n+1}\wedge N}] \\ &\le T_1[h_{Q^N\wedge N} - h_{S^1\wedge N}]. \end{aligned} \tag{2.9}$$

Direct calculation shows that $(Q^N \wedge N)_i = \begin{cases} I, & i \ge N \\ 0, & i \le N-1 \end{cases}$ and hence the stopped process $h_{Q^N\wedge N} = h_N$. Also $h_{S^1\wedge N} \ge 0$, thus (2.9) can be refined to

$$T_1\left[\sum_{n=1}^{N}[h_{Q^n\wedge N} - h_{S^n\wedge N}]\right] \le T_1 h_N. \tag{2.10}$$

From the definition of S_i^n and the facts that for fixed n, Q_i^n is an increasing family of positive projections,

$$\begin{aligned} S_{i+1}^n - S_i^n &= P_{(f_{i+1}-g)^-}(Q_i^n - Q_{i-1}^n) + \sum_{j=2}^{i}\left[P_{\bigvee_{k=j}^{i+1}(f_k-g)^-} - P_{\bigvee_{k=j}^{i}(f_k-g)^-}\right](Q_{j-1}^n - Q_{j-2}^n) \\ &\le P_{(f_{i+1}-g)^-}. \end{aligned}$$

Thus as $h_{i+1} \ge 0$ and $S_{i+1}^n \ge S_i^n$, we have

$$0 \le (S_{i+1}^n - S_i^n)h_{i+1} \le P_{(f_{i+1}-g)^-}h_{i+1}$$

and the definition of h_{i+1} gives $0 = P_{(f_{i+1}-g)^-}(f_{i+1} - g)^+ = P_{(f_{i+1}-g)^-}h_{i+1}$. Hence

$$(S_{i+1}^n - S_i^n)h_{i+1} = 0. \tag{2.11}$$

Proceeding in a similar manner we obtain that $0 \le Q^n_{i+1} - Q^n_i \le P_{(f_{i+1}-f)^+}$ which when applied to $(f_{i+1} - f)^-$ gives

$$0 \le (Q^n_{i+1} - Q^n_i)(f_{i+1} - f)^- \le P_{(f_{i+1}-f)^+}(f_{i+1} - f)^- = 0$$

and thus proving that

$$0 = (Q^n_{i+1} - Q^n_i)(f_{i+1} - f)^-. \tag{2.12}$$

As a consequence of (2.12),

$$0 \le (Q^n_{i+1} - Q^n_i)(f_{i+1} - f)^+ = (Q^n_{i+1} - Q^n_i)(f_{i+1} - f)$$

from which it follows that $(Q^n_{i+1} - Q^n_i)(f - g) \le (Q^n_{i+1} - Q^n_i)(f_{i+1} - g)$. This inequality along with $0 \le (Q^n_{i+1} - Q^n_i)(f_{i+1} - g)^-$ yields

$$(Q^n_{i+1} - Q^n_i)h_{i+1} \ge (Q^n_{i+1} - Q^n_i)(f - g). \tag{2.13}$$

Combining (2.11) and (2.13) yields

$$[(Q^n_{i+1} - Q^n_i) - (S^n_{i+1} - S^n_i)]h_{i+1} \ge (Q^n_{i+1} - Q^n_i)(f - g). \tag{2.14}$$

Summing over $i = 0, ..., N - 1$ in (2.14) gives

$$\sum_{i=1}^{N}[(Q^n_i - Q^n_{i-1}) - (S^n_i - S^n_{i-1})]h_i \ge Q^n_N(f - g)$$

as $Q^n_0 = 0 = S^n_0$. Adding $[(I - Q^n_N) - (I - S^n_N)]h_N = (S^n_N - Q^n_N)h_N$ to both sides of the above equation we have

$$h_{Q^n \wedge N} - h_{S^n \wedge N} \ge (S^n_N - Q^n_N)h_N + Q^n_N(f - g),$$

which when summed over $n = 1, ..., N$ yields

$$\sum_{n=1}^{N}[h_{Q^n \wedge N} - h_{S^n \wedge N}] + \left[\sum_{n=1}^{N}(Q^n_N - S^n_N)\right] h_N \ge Y_N(g, f). \tag{2.15}$$

Now as $Q^n_N = 0$ for $n = 0, ..., N$ and as $S^n_N h_N \ge 0$ we can deduce from (2.15) that

$$\sum_{n=1}^{N}[h_{Q^n \wedge N} - h_{S^n \wedge N}] \ge Y_N(g, f). \tag{2.16}$$

Applying T_1 to (2.16) and using (2.10) we have $T_1 h_N \ge T_1 Y_N(g, f)$. ■

We now take a brief look at some aspects of order convergence on Riesz spaces.

Let E be a Dedekind complete Riesz space and (f_n) an order bounded sequence in E. If $u_m = \sup\{f_m, f_{m+1}, ...\}$, then u_m exists in E by the Dedekind completeness of E and the fact that (f_n) is order bounded. Furthermore, (u_m) is a decreasing sequence which is order bounded below and hence $\inf u_m = \inf_m \sup\{f_m, f_{m+1}, ...\}$ exists and will be denoted by $\limsup f_m$. Similarly, if $l_m = \inf\{f_m, f_{m+1}, ...\}$, we denote $\sup l_m = \sup_m \inf\{f_m, f_{m+1}, ...\}$ by $\liminf f_n$. Hence the existence of $\limsup f_n$ and $\liminf f_n$ is equivalent to requiring that (f_n) be order bounded. From [24, Proposition 1.1.10], (f_n) is order convergent if and only if $\limsup f_n = \liminf f_n$. It is thus meaningless to consider the concept of order convergence for sequences which are not order bounded.

Lemma 2.18 Local Convergence
Let (f_n, T_n) be a RS-wou-sub(super)-martingale in a Dedekind complete Riesz space E with weak order unit and in which the operators T_n are strictly positive for each $n \in \mathbb{N}$. If there exists $g \in E^+$ such that $T_1|f_n| \le g$ for each $n \in \mathbb{N}$, then, for each $K \in \mathbb{N}$, $(Ke \wedge f_n \vee (-Ke))$ is order convergent and the order limit $L_K \in E$ is given by

$$\limsup_n (Ke \wedge f_n \vee (-Ke)) = L_K = \liminf_n (Ke \wedge f_n \vee (-Ke)).$$

Proof: Let $K \in \mathbb{N}$ then

$$Ke \ge \limsup_n (Ke \wedge f_n \vee (-Ke)) \ge \liminf_n (Ke \wedge f_n \vee (-Ke)) \ge -Ke.$$

We show that

$$\limsup_n (Ke \wedge f_n \vee (-Ke)) = \liminf_n (Ke \wedge f_n \vee (-Ke)).$$

Suppose this equality is false, then

$$M := \limsup(Ke \wedge f_n \vee (-Ke)) > \liminf(Ke \wedge f_n \vee (-Ke)) =: m.$$

By Lemma (2.12) there are $s, t \in \mathbb{R}$ with $-K < s < t < K$ such that $h := (M - te)^+ \wedge (se - m)^+ > 0$.

Let P_h denote the band projection onto the band generated by h. Then the definition of $Q_m^N(e)$ in the upcrossing, along with the definitions of lim sup and lim inf gives that $\bigvee_n Q_n^N(e) \ge P_h(e)$, for all $N \in \mathbb{N}$. Thus

$$H_N := \bigvee_n \left[(t-s)P_h(e) \wedge \frac{1}{N} Y_n(se, te)\right] = (t-s)P_h(e) > 0, \quad \text{for each } N \in \mathbb{N}.$$

From the upcrossing theorem, Theorem (2.17),

$$T_1 Y_n(se, te) \le T_1|f_n| \le g.$$

Now as

$$(t-s)P_h(e) \wedge \frac{1}{N} Y_n(se, te)$$

is an increasing sequence (with respect to n) bounded above by $(t-s)P_h(e)$ it follows that

$$\bigvee_n T_1 \left((t-s)P_h(e) \wedge \frac{1}{N} Y_n(se, te) \right) = T_1 H_N.$$

Thus

$$0 < (t-s)T_1 P_h(e) = T_1 H_N = \bigvee_n T_1 \left((t-s)P_h(e) \wedge \frac{1}{N} Y_n(se, te) \right) \leq \frac{g}{N}$$

for each $N \in \mathbb{N}$, which is not possible as E is an Archimedean Riesz space. Hence proving that

$$Ke \geq \limsup_n (Ke \wedge f_n \vee (-Ke)) = \liminf_n (Ke \wedge f_n \vee (-Ke)) \geq -Ke$$

for each $K \in \mathbb{N}$. ■

Using the above lemma, we get a martingale convergence theorem for order bounded martingales, which is the main result of this section and can be found in [17].

Theorem 2.19 Bounded Martingale Convergence
Let (f_n, T_n) be a RS-wou-sub(super)-martingale in a Dedekind complete Riesz space E with weak order unit and in which the operators T_n are strictly positive for each $n \in \mathbb{N}$. If there exists $g \in E^+$ such that $|f_n| \leq g$ for each $n \in \mathbb{N}$, then (f_n) is order convergent and the order limit $f_\infty \in E$ is given by

$$\limsup f_n = f_\infty = \liminf f_n.$$

PROOF: By Lemma (2.18) we have that for each $K \in \mathbb{N}$, $(Ke \wedge f_n \vee (-Ke))$ is order convergent and the order limit $L_K \in E$ is given by

$$\limsup_n (Ke \wedge f_n \vee (-Ke)) = L_K = \liminf_n (Ke \wedge f_n \vee (-Ke)).$$

It should be noted that $|L_K|$ is an increasing sequence in E. Observe that

$$|Ke \wedge f_n \vee (-Ke)| \leq |f_n| \leq g$$

and hence $|L_K| \leq g$ for each $K \in \mathbb{N}$. Thus $|L_K| \uparrow L \in E$ and in particular

$$\lim_n (|f_n| \wedge Ke) \leq L \text{ for each } K \in \mathbb{N}.$$

Thus $L^S := \limsup f_n$ and $L^I := \liminf f_n$ exist. But from Lemma (2.18)

$$Ke \wedge L^S \vee (-Ke) = \limsup_n (Ke \wedge f_n \vee (-Ke)) = \liminf_n (Ke \wedge f_n \vee (-Ke)) = Ke \wedge L^I \vee (-Ke)$$

for each $K \in \mathbb{N}$. By taking order limits and noting that the terms in the above equation are bounded by $\pm g$ it follows that $F^S = F^I$. ■

2.4 Special classes of Martingales on Riesz Spaces

In this section we consider classes of martingales which play a central role in this dissertation.

With the exception of the definition of order on the space of martingales given below, the material in this section, which is motivated by [16, Chapters 1-2] and [30], is new.

Definition 2.20 A family of positive order continuous projections $(T_n)_{n\in\mathbb{N}}$ defined on a Riesz space E, for which

$$T_nT_m = T_mT_n = T_n, \text{ for each } m \geq n,$$

is called an *order filtration*.

Thus an order filtration is a commuting family of positive projections with increasing ranges.

Definition 2.21 The pair $(f_n, T_n)_{n\in\mathbb{N}}$ on a Riesz space E, where $f_n \in \mathcal{R}(T_n)$ for each $n \in \mathbb{N}$ and (T_n) an order filtration, is called a *(sub, super)-martingale* if

$$f_n(\leq, \geq) = T_nf_m, \text{ for each } m \geq n.$$

Let E be a Riesz space and (T_n) an order filtration on E. Let $M((T_n), E)$ denote the set of all martingales (f_n, T_n) on a Riesz space E. Define order on $M((T_n), E)$ by

$$(f_n, T_n) \leq (g_n, T_n) \iff f_n \leq g_n \text{ for each } n \in \mathbb{N}$$

and vector operations by

$$(f_n, T_n) + (g_n, T_n) = (f_n + g_n, T_n) \text{ and } \lambda(f_n, T_n) = (\lambda f_n, T_n) \text{ for each } \lambda \in \mathbb{R}.$$

Let $M_+((T_n), E)$ denote the positive cone of $M((T_n), E)$ with respect to the ordering defined above.

It is easily verified that $M((T_n), E)$ is an ordered vector space with respect to these operations and order.

2.4.1 Regular Martingales on Riesz Spaces

Let E be a Riesz space and let (T_n) be an order filtration on E. Set

$$M_r((T_n), E) = \left\{(f_n, T_n) \in M((T_n), E) \,\middle|\, \exists (g_n, T_n) \in M_+((T_n), E),\ (f_n, T_n) \leq (g_n, T_n)\right\}.$$

Elements of $M_r((T_n), E)$ are referred to as *regular* martingales with respect to (T_n).

If E is Dedekind complete, then it is readily verified that a martingale (f_n, T_n) is regular if and only if it can be written as the difference of two positive martingales.

The following is the main result of this section and is based on [30, Theorem 13]:

Theorem 2.22 *Let E be a Dedekind complete Riesz space and (T_n) an order filtration on E. Then $M_r((T_n), E)$ is a Dedekind complete Riesz space with lattice operations given by*

$$\begin{aligned}(f_n, T_n)^+ &= (\sup_{m\geq n} T_n f_m^+, T_n);\\ (f_n, T_n)^- &= (\sup_{m\geq n} T_n f_m^-, T_n);\\ (f_n, T_n)\vee(g_n, T_n) &= (\sup_{m\geq n} T_n(f_m\vee g_m), T_n);\\ (f_n, T_n)\wedge(g_n, T_n) &= (\sup_{m\geq n} T_n(f_m\wedge g_m), T_n);\\ |(f_n, T_n)| &= (\sup_{m\geq n} T_n|f_m|, T_n).\end{aligned} \tag{2.17}$$

Proof: Let (f_n, T_n) and (g_n, T_n) be two martingales in $M_r((T_n), E)$. Then there exist positive martingales $(\bar{f}_n, T_n)$ and $(\bar{g}_n, T_n)$ in E such that $f_n \leq \bar{f}_n$ and $g_n \leq \bar{g}_n$ for each $n \in \mathbb{N}$. Hence, for each fixed $n \in \mathbb{N}$, the sequence $(T_n(f_m \vee g_m))_{m=n}^{\infty}$ is order bounded, because

$$f_n \vee g_n \leq T_n(f_m \vee g_m) \leq \bar{f}_n + \bar{g}_n \text{ for each } m \geq n$$

and increasing, since

$$T_n(f_m \vee g_m) = T_n(T_m f_{m+1} \vee T_m g_{m+1}) \leq T_n T_m(f_{m+1} \vee g_{m+1}) = T_n(f_{m+1} \vee g_{m+1}),$$

whenever $m \geq n$. Since E is Dedekind complete, the sequence has a supremum. Let

$$h_n := \sup\{T_n(f_m \vee g_m) \mid m \geq n\} \text{ for each } n \in \mathbb{N}.$$

By the order continuity of T_n for each $n \in \mathbb{N}$, we have for any $k \leq n$ that $T_k h_n = \sup\{T_k T_n(f_m \vee g_m) \mid m \geq n\} = h_k$. Thus (h_n, T_n) is a martingale. Furthermore, (h_n, T_n) is a regular martingale, because $(h_n, T_n) \leq (\bar{f}_n + \bar{g}_n, T_n)$ and $(\bar{f}_n + \bar{g}_n, T_n)$ is a positive martingale.

To conclude that $(h_n, T_n) = (f_n, T_n) \vee (g_n, T_n)$, let (z_n, T_n) be a martingale for which $(z_n, T_n) \geq (f_n, T_n)$ and $(z_n, T_n) \geq (g_n, T_n)$. Then $z_n \geq f_n \vee g_n$ for each $n \in \mathbb{N}$. Thus, it follows from $z_n = T_n z_m \geq T_n(f_m \vee g_m)$ for each $m \geq n$, that $z_n \geq h_n$ for each $n \in \mathbb{N}$; i.e., $(z_n, T_n) \geq (h_n, T_n)$.

Thus $M_r((T_n), E)$ is a Riesz space. The other formulas in (2.18) follow using standard formulas (see [31, Theorem 5.2]).

We still need to verify that $M_r((T_n), E)$ is Dedekind complete.

To this end let $(f_n^{(\alpha)}, T_n)$ be an increasing family (in (α)) of positive elements from $M_r((T_n), E)$ bounded above by $(f_n, T_n) \in M((T_n), E)$. Then $f_n^{(\alpha)} \uparrow_{(\alpha)} \leq f_n$ and by the Dedekind completeness of E, there exists $g_n \in E_+$, for which $f_n^{(\alpha)} \uparrow_{(\alpha)} g_n$. Then $g_n = T_n g_m$ for each $m \geq n$, as $f_n^{(\alpha)} = T_n f_m^{(\alpha)} \uparrow_{(\alpha)} T_n g_m$ and $f_n^{(\alpha)} \uparrow_{(\alpha)} g_n$, showing that (g_n, T_n) is a martingale. Since $(g_n, T_n) \leq (f_n, T_n) \in M_+((T_n), E)$, we have $(g_n, T_n) \in M_r((T_n), E)$, which completes the proof that $M((T_n), E)$ is Dedekind complete. ■

2.4.2 Order Bounded Martingales on Riesz Spaces

Let E be a Riesz space and let (T_n) be an order filtration on E. Set

$$M_{ob}((T_n), E) = \left\{ (f_n, T_n) \in M((T_n), E) \,\middle|\, (f_n) \text{ is order bounded in } E \right\}.$$

Elements of $M_{ob}((T_n), E)$ are referred to as *order bounded* martingales with respect to (T_n).

Any order bounded martingale (f_n, T_n) is regular, for if $|f_n| \leq u$ for each $n \in \mathbb{N}$, then (u_n, T_n), where $u_n = T_n u$ for each $n \in \mathbb{N}$, is a positive martingale. Hence, (f_n, T_n) is regular.

2.4.3 Order Convergent and Generated Martingales

Let E be a Riesz space and (T_n) an order filtration on E. Set

$$M_{oc}((T_n), E) = \left\{ (f_n, T_n) \in M((T_n), E) \,\middle|\, (f_n) \text{ is order convergent in } E \right\}$$

and

$$M_g((T_n), E) = \left\{ (f_n, T_n) \in M((T_n), E) \,\middle|\, \exists f \in E, f_n = T_n f \text{ for each } n \in \mathbb{N} \right\}.$$

Elements of $M_{oc}((T_n), E)$ are referred to as *order convergent* martingales and those of $M_g((T_n), E)$ are referred to as *generated* martingales with respect to (T_n).

Theorem 2.23 *Let E be a Riesz space and (T_n) an order filtration. Then*

$$M_{oc}((T_n), E) \subseteq M_{ob}((T_n), E) \cap M_g((T_n), E) \text{ and } M_g((T_n), E) \subseteq M_r((T_n), E).$$

PROOF: Order convergent sequences are order bounded (see [24, Proposition 1.1.10]). Hence, $M_{oc}((T_n), E) \subseteq M_{ob}((T_n), E)$.

To verify that $M_{oc}((T_n), E) \subseteq M_g((T_n), E)$, let $(f_n, T_n) \in M_{oc}((T_n), E)$. If (f_n, T_n) converges in order to f_∞, then, as T_n is order continuous, $f_n = T_n f_m \to_m T_n f_\infty$, for $m \geq n$, giving that $f_n = T_n f_\infty$ for each $n \in \mathbb{N}$.

If $(f_n, T_n) \in M_g((T_n), E)$, then there exists $f \in E$ such that $(f_n, T_n) = (T_n f, T_n)$. But $(T_n f, T_n) \leq (T_n f^+, T_n)$ and the latter is a positive martingale. Thus $M_g((T_n), E) \subseteq M_r((T_n), E)$. ■

In the special case where E is a Dedekind complete Riesz space with a weak order unit, we get the following:

Theorem 2.24 *Let E be a Dedekind complete Riesz space E with weak order unit and (T_n) a RS-wou-filtration for which the operators T_n are strictly positive for each $n \in \mathbb{N}$. Then $M_{oc}((T_n), E) = M_{ob}((T_n), E)$.*

PROOF: This is just a restatement Theorem 2.19. ■

Chapter 3

Martingales on Banach Lattices

3.1 Special Classes of Martingales on Banach Lattices

In this section we consider classes of martingales on Banach spaces and Banach lattices. We now give generalized definitions of "filtrations" on Banach spaces and Banach lattices (see [2, 30]). These definitions are motivated by the the type of space at hand and the definitions of martingales in the classical setting of measure spaces, the vector-valued space setting (see [4]) and the measure-free setting (see [16, Chapters 1-2] and [2, 30]).

3.1.1 Norm Bounded Martingales

We recall from [4, Chapter 5, §1] some properties about martingales and convergence of martingales in vector-valued L^p-spaces. These properties serve as motivation for the discussion given below on filtrations and convergence of martingales defined on Banach spaces.

Let X be a Banach space, $(\Omega, \mathcal{F}, \mu)$ a probability space and $\mathcal{F}_1$ a sub σ-algebra of $\mathcal{F}$. An element $g \in L^1(\mu, X)$ is called the *conditional expectation* of f relative to $\mathcal{F}_1$ if g is $\mathcal{F}_1$-measurable and

$$\int_A g d\mu = \int_A f d\mu \text{ for all } A \in \mathcal{F}_1.$$

In this case, g is denoted by $\mathbb{E}(f \mid \mathcal{F}_1)$.

Let $(\mathcal{F}_n)$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ and let f_n be $\mathcal{F}_n$-measurable for each $n \in \mathbb{N}$. If $\mathbb{E}(f_{n+1}|\mathcal{F}_n) = f_n$ for each $n \in \mathbb{N}$, then $(\mathbb{E}(f_n|\mathcal{F}_n), \mathcal{F}_n)$ is a *martingale*.

If $f \in L^1(\mu, X)$, it is well known (and readily verified) that $(\mathbb{E}(f|\mathcal{F}_n), \mathcal{F}_n)$ is a martingale. Moreover, if $f \in L^p(\mu, X)$, where $1 \le p < \infty$, then the martingale $(\mathbb{E}(f|\mathcal{F}_n), \mathcal{F}_n)$ converges to f in Bochner norm, denoted by Δ_p. This follows by noting that the martingale is Δ_p-convergent to f in the case where f is a $\bigcup_n \mathcal{F}_n$-step function, because in this case, $(\mathbb{E}(f_n|\mathcal{F}_n), \mathcal{F}_n)$ is eventually constant. Since the $\bigcup_n \mathcal{F}_n$-step functions are dense in $L^p(\mu, X)$ and $\|(\mathbb{E}(\dot{|}\mathcal{F}_n)\|_p \le 1$ (as is well known), $(\mathbb{E}(f|\mathcal{F}_n), \mathcal{F}_n)$ converges to f in Bochner norm for $f \in L^p(\mu, X)$.

These remarks motivate the material in this section. We believe these results to be new.

Definition 3.1 A sequence of contractive projections $T_n : X \to X$ with increasing ranges i.e. $\mathcal{R}(T_n) \subseteq \mathcal{R}(T_m)$ for $n \le m$ is called a *Banach space filtration* (*BS-filtration*).

Definition 3.2 A pair (f_n, T_n), where (T_n) is a BS-filtration and $f_n \in \mathcal{R}(T_n)$ for each $n \in \mathbb{N}$, is called a *martingale* if $T_n f_m = f_n$ for all $m \ge n$.

Let X be a Banach space and let (T_n) be a BS-filtration on X. The map Θ from $M((T_n), X)$ to $X^{\mathbb{N}}$, defined by $(f_n, T_n) \mapsto (f_n)$, is a linear injection.

Recall from [3] that

$$\ell^{\infty}_{bdd}(X) := \{(x_n) \in X^{\mathbb{N}} \mid x_n \in X \text{ and } \sup_n ||x_n||_X < \infty\}$$

is a Banach space with respect to the norm

$$||(x_n)||_{\ell^{\infty}_{bdd}(X)} = \sup_n ||x_n||_X.$$

Define $|| \cdot ||_{\mathcal{M}}$ on $M((T_n), X)$ by

$$||(f_n, T_n)||_{\mathcal{M}} = \sup_n ||f_n||_X$$

and let

$$\mathcal{M}_{nb}((T_n), X) = \{(f_n, T_n) \in M((T_n), X) \mid ||(f_n, T_n)||_{\mathcal{M}} < \infty\}.$$

Then the map $(f_n, T_n) \mapsto (f_n)$ is an isometry from $\mathcal{M}_{nb}((T_n), X)$ into $\ell^{\infty}_{bdd}(X)$ and it is easily verified that $\mathcal{M}_{nb}((T_n), X)$ is norm complete.

Thus if X is a Banach space, $\mathcal{M}_{nb}((T_n), X)$ is itself a Banach space.

Let $I_n = id_X$ for each $n \in \mathbb{N}$, where id_X denotes the identity map on X. Then (I_n) is a filtration on X and

$$(f_n, I_n) \in \mathcal{M}_{nb}((I_n), X) \Longleftrightarrow (f_n) \text{ is a constant sequence in } X.$$

If we define $\Psi \colon X \to \mathcal{M}_{nb}((I_n), X)$ by

$$\Psi(f) = (f_n, I_n), \text{ where } f = f_n \text{ for each } n \in \mathbb{N},$$

then X is isometrically isomorphic to $\mathcal{M}_{nb}((I_n), X)$.

3.1.2 Norm Convergent Martingales on Banach Spaces

The results in this section we believe are new.

Let X be a Banach space and (T_n) a BS-filtration on X. Let $\mathcal{M}_{nc}((T_n), X)$ denote the norm convergent martingales on X; i.e.,

$$\mathcal{M}_{nc}((T_n), X) = \{(f_n, T_n) \in \mathcal{M}_{nb}((T_n), X) \mid (f_n, T_n) \text{ is norm convergent}\}.$$

To characterize $\mathcal{M}_{nc}((T_n), X)$, the following result is used:

Lemma 3.3 *Let X be a Banach space and (T_n) a BS-filtration on X. Then $f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ if and only if $T_n f \to f$ in norm.*

Proof: $\Leftarrow$: If $T_n f \to f$ in norm, then $f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$.

$\Rightarrow$: If $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$, select $(y_i) \subseteq \bigcup \mathcal{R}(T_i)$ such that $y_i \to f$ in norm. Then, for each $i \in \mathbb{N}$, there exists j_i such that $y_i \in \mathcal{R}(T_{j_i})$; moreover, since $\mathcal{R}(T_i) \uparrow$, the j_i may be chosen such that $j_i < j_{i+1}$. Define the sequence (z_i) by

$$z_i = 0 \text{ for } i = 1, 2 \ldots, j_1 - 1 \text{ and } z_i = y_k \text{ for } i = j_k, j_k + 1, \cdots, j_{(k+1)} - 1,$$

for each $k \in \mathbb{N}$. Then $z_i \to f$ in norm and $z_i \in \mathcal{R}(T_i)$ for each $i \in \mathbb{N}$. Furthermore, $T_j z_i = z_i$ for all $i \leq j$, since T_j is a projection and $\mathcal{R}(T_i) \uparrow_i$. It follows from

$$\|T_j f - f\| \leq \|T_j f - T_j z_i\| + \|T_j z_i - f\| \leq \|f - z_i\| + \|z_i - f\| \text{ for } j \geq i$$

that $T_j f \to f$ in norm. ■

Lemma 3.4 *Let X be a Banach space and (T_n) a BS-filtration on X. If (f_n, T_n) is a martingale on X, then the following statements are equivalent:*

(i) $(f_n, T_n) \in \mathcal{M}_{nc}((T_n), X)$.

(ii) *There exists* $f \in X$ *such that* $f_n = T_n f$ *for each* $n \in \mathbb{N}$ *and* $T_n f \to f$.

(iii) *There exists* $f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ *such that* $f_n = T_n f$ *for each* $n \in \mathbb{N}$.

PROOF: (i)⇒(ii) Let $(f_n, T_n) \in \mathcal{M}_{nc}((T_n), X)$. Then there exists $f \in X$ such that $f_n \to f$. But then, for each $m \leq n$, $f_m = T_m f_n \to_n T_m f$, from which we get that $f_n = T_n f$ for each $n \in \mathbb{N}$ and $T_n f \to f$.

(ii)⇒(i) This implication is trivial.

(ii)⇔(iii) This follows from Lemma 3.3. ■

We now prove one of our main results of this subsection:

Proposition 3.5 *Let* X *be a Banach space and let* (T_n) *be a BS-filtration on* X. *Then* $L : \mathcal{M}_{nc}((T_n), X) \to \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ *defined by* $L((f_n, T_n)) = \lim_n f_n$, *is a surjective isometry.*

PROOF: It follows easily that L is well defined and linear. To see that L is a surjection, let $f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$. Then $T_n f \to f$ in norm and $(T_n f, T_n)$ is a BS-martingale on X such that $L((T_n f, T_n)) = f$.

Also, L is injective, because if $f = g \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ then $f_n \to f$, $g_n \to g$ and $(T_n f, T_n) = (T_n g, T_n)$.

Furthermore,

$$||L|| = \sup\{||L(f_n, T_n)||_X \mid ||(f_n, T_n)||_{\mathcal{M}} \leq 1\} \leq 1.$$

and

$$||L^{-1}|| = \sup\{||L^{-1} f||_{\mathcal{M}} \mid ||f||_X \leq 1\} = \sup\{\sup_n ||T_n f||_{\mathcal{M}} \mid ||f||_X \leq 1\} \leq 1.$$

■

The following corollary is easily verified, since (I_n) is a filtration on X:

Corollary 3.6 *Let* X *be a Banach Space and* P *a property on* X *which is inherited by closed subspaces. Then the following are equivalent:*

(i) X has property P;

(ii) $\mathcal{M}_{nc}((T_n), X)$ has property P for every BS-filtration (T_n).

Corollary 3.7 *Let (T_n) be a BS-filtration on a Banach space X. Then $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} = X$ if and only if $\Gamma\colon \mathcal{M}_{nc}((T_n), X) \to \mathcal{M}_{nb}((I_n), X)$, defined by $\Gamma\Big((T_n f, T_n)\Big) = (f_n, I_n)$, where $f_n = f$ and $I_n = id_X$ for each $n \in \mathbb{N}$, is a surjective isometric isomorphism.*

Proof: Suppose $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} = X$. Then, with the notation as above, the composition of the isometric isomorphisms $L\colon \mathcal{M}_{nc}((T_n), X) \to \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ and $\Psi\colon X \to \mathcal{M}_{nc}((T_n), X)$ gives the required isometric isomorphism Γ between $\mathcal{M}_{nc}((T_n), X)$ and $\mathcal{M}_{nb}((I_n), X)$.

Conversely, by their respective definitions, it is readily verified that $\Psi^{-1} \circ \Gamma \circ L^{-1}$ is the inclusion map $J\colon \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} \hookrightarrow X$. It suffices to show that $\Psi^{-1} \circ \Gamma \circ L^{-1}$ is surjective. Let $f \in X$, then

$$f = \Psi^{-1}\Big((f, id_X)\Big) = (\Psi^{-1} \circ \Gamma)\Big((T_n f, T_n)\Big).$$

Since $L\Big((T_n f, T_n)\Big) = f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$, it follows that $f = (\Psi^{-1} \circ \Gamma \circ L^{-1})(f)$ with $f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$. Hence, $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} = X$. ■

3.1.3 Norm Convergent Martingales on Banach Lattices

The ideas in this section are based on [30], but we believe that the results presented are new.

We consider Banach lattices and positive filtrations.

Definition 3.8 Let E be a Banach lattice. A BS-filtration (T_n) on E for which $T_n \geq 0$ for each $n \in \mathbb{N}$ is called a *positive filtration* on E.

The term "filtration" is used in [30] by Troitski for what we call a "positive filtration".

The following is the main result of this subsection:

Proposition 3.9 *Let E be a Banach lattice and (T_n) a positive filtration on E for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E. Then $L : \mathcal{M}_{nc}((T_n), E) \to \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ defined by $L((f_n, T_n)) = \lim_n f_n$ is a surjective isometry for which $L \geq 0$ and $L^{-1} \geq 0$. Consequently, $L : \mathcal{M}_{nc}((T_n), E) \to \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a surjective Riesz isometry.*

PROOF: It was shown above that L is a surjective isometry. To see that L is positive is trivial, because if $(f_n, T_n) \geq 0$, then $f_n \geq 0$ for each $n \in \mathbb{N}$ and $\lim_n f_n \geq 0$. Similarly, L^{-1} is positive, because if $0 \leq f \in \overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ then $T_n f \geq 0$ for each $n \in \mathbb{N}$; hence, $L^{-1}(f) = ((T_n f, T_n)) \geq 0$. ∎

Now, recalling that every convergent martingale is generated (by its limit), i.e.

$$\mathcal{M}_{nc}((T_n), E) = \big\{(f_n, T_n) \in \mathcal{M}_{nb}((T_n), E) \mid \exists f \in E \ , f_n = T_n f \to f\big\}$$

we have the following result:

Corollary 3.10 *Let E be a Banach lattice and (T_n) a positive filtration on E for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E. Then $\mathcal{M}_{nc}((T_n), E)$ is a Banach lattice in which the following formulas hold:*

$$\begin{aligned}
(\lim_{m\to\infty} T_n f_m, T_n)^+ &= (\lim_{m\to\infty} T_n f_m^+, T_n); \\
(\lim_{m\to\infty} T_n f_m, T_n)^- &= (\lim_{m\to\infty} T_n f_m^-, T_n); \\
(\lim_{m\to\infty} T_n f_m, T_n) \vee (\lim_{m\to\infty} T_n g_m, T_n) &= (\lim_{m\to\infty} T_n (f_m \vee g_m), T_n); \\
(\lim_{m\to\infty} T_n f_m, T_n) \wedge (\lim_{m\to\infty} T_n g_m, T_n) &= (\lim_{m\to\infty} T_n (f_m \wedge g_m), T_n); \\
|(\lim_{m\to\infty} T_n f_m, T_n)| &= (\lim_{m\to\infty} T_n |f_m|, T_n).
\end{aligned} \tag{3.1}$$

PROOF: Since $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a Riesz space, it follows that $\mathcal{M}_{nc}((T_n), E)$ is also a Riesz space. Indeed, for $f, g \in \mathcal{M}_{nc}((T_n), E)$, it is readily verified that $L^{-1}(L(f) \vee L(g))$ is the least upper bound in $\mathcal{M}_{nc}((T_n), E)$ of $\{f, g\}$.

Thus, by the preceding theorem, L is a surjective Riesz isometry.

Since $|| \cdot ||_E$ is a Riesz norm, $|| \cdot ||_{\mathcal{M}}$ is also a Riesz norm. Furthermore, since $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a Banach lattice, $\mathcal{M}_{nc}((T_n), E)$ is a Banach lattice.

The formulas are easy to prove. Since L is a bijective Riesz homomorphism, it follows from $L(T_n|f|, T_n) = |f| = |L(T_nf, T_n)|$ that

$$(T_n|f|, T_n) = L^{-1}(L(T_n|f|, T_n)) = L^{-1}(|L(T_nf, T_n)|) = |(L^{-1}(L(T_nf, T_n))| = |(T_nf, T_n)|.$$

The other formules follow in a similar manner. ■

We get the following order analogues of Corollaries 3.6 and 3.7:

Corollary 3.11 *Let E be a Banach lattice and P a property on E which is inherited by closed Riesz subspaces. Then the following are equivalent:*

(i) E has property P;

(ii) $\mathcal{M}_{nc}((T_n), E)$ has property P for every positive filtration (T_n) for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E.

Proof: The first implication is trivial. We have already seen that there exists a surjective Riesz isometry between $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ and $\mathcal{M}_{nc}((T_n), E)$. Now taking $f_n = f$ and $T_n = id_E$ for each $n \in \mathbb{N}$, we get the reverse implication. ■

Corollary 3.12 *Let E be a Banach lattice. Then:*

(i) E is order continuous if and only if $\mathcal{M}_{nc}((T_n), E)$ is order continuous for every positive filtration (T_n) for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace on E.

(ii) E is a KB-space if and only if $\mathcal{M}_{nc}((T_n), E)$ is a KB-space for every positive filtration (T_n) for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E.

(iii) E is an AL-space if and only if $\mathcal{M}_{nc}((T_n), E)$ is an AL-space for every positive filtration (T_n) for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E.

(iv) E has the RNP if and only if $\mathcal{M}_{nc}((T_n), E)$ also has the RNP for every positive filtration (T_n) for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E.

Proof: For the first three properties, it is well known that they are inherited by closed Riesz subspaces. For the last property see [24, Corollory 5.4.8]. ■

Corollary 3.13 *Let (T_n) be a positive filtration on a Banach lattice E. Then $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} = E$ if and only if $\Gamma\colon \mathcal{M}_{nc}((T_n), E) \to \mathcal{M}_{nb}((I_n), E)$, defined by $\Gamma\Big((T_n f, T_n)\Big) = (f_n, I_n)$, where $f_n = f$ and $I_n = id_X$ for each $n \in \mathbb{N}$, is a Riesz and surjective isometric isomorphism.*

Troitski considers positive filtrations (T_n) on a Banach lattice E, in [30], for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)} = E$. Such filtrations are called *dense* filtrations in [30]. Corollary 3.13 shows that such filtrations give rise to very simple convergent martingales.

3.1.4 Regular Martingales on Order Continuous Banach Lattices

Let (T_n) be a positive filtration on a Banach lattice E. We now consider the space

$$\mathcal{M}_r((T_n), E) = \{(f_n, T_n) \in \mathcal{M}_{nb}((T_n), E) \mid \exists (g_n, T_n) \in M_+((T_n), E), f_n \leq g_n \ \forall\ n \in \mathbb{N}\},$$

the elements of which are called *regular norm bounded* martingales.

Proposition 3.14 *Let (T_n) be a positive filtration on a Banach lattice E for which $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E. Then*

$$\mathcal{M}_{nc}((T_n), E) \subseteq \mathcal{M}_r((T_n), E).$$

Proof: Since $\mathcal{M}_{nc}((T_n), E)$ is a Riesz space, each of its element can be written as the difference of its positive and negative parts. ■

We now prove the main result of this section, noted by Troitsky in [30, Theorem 13].

Theorem 3.15 *Let E be a Banach lattice with order continuous norm and (T_n) a positive filtration on E. Then $\mathcal{M}_r((T_n), E)$ is a Dedekind complete Banach lattice with lattice operations given by*

$$\begin{aligned}
(f_n, T_n)^+ &= (\lim_{m\to\infty} T_n f_m^+, T_n); \\
(f_n, T_n)^- &= (\lim_{m\to\infty} T_n f_m^-, T_n); \\
(f_n, T_n) \vee (g_n, T_n) &= (\lim_{m\to\infty} T_n (f_m \vee g_m), T_n); \\
(f_n, T_n) \wedge (g_n, T_n) &= (\lim_{m\to\infty} T_n (f_m \wedge g_m), T_n); \\
|(f_n, T_n)| &= (\lim_{m\to\infty} T_n |f_m|, T_n).
\end{aligned} \tag{3.2}$$

PROOF: Let (f_n, T_n) and (g_n, T_n) be two martingales in $\mathcal{M}_r((T_n), E)$. By the order continuity of the norm of E, it follows that T_n is order continuous for each $n \in \mathbb{N}$. Then, by Theorem (2.22),

$$(f_n, T_n) \vee (g_n, T_n) = (\sup_{m \geq n} T_n(f_m \vee g_m), T_n) := (h_n, T_n) \in M_r((T_n), E).$$

Since

$$||T_n(f_m \vee g_m)|| \leq ||f_m \vee g_m|| \leq ||\,|f_m| + |g_m|\,|| \leq ||f_m|| + ||g_m||,$$

we have

$$\sup_m ||T_n(f_m \vee g_m)|| \leq \sup_m \Big[||f_m|| + ||g_m||\Big] \leq \sup_m ||f_m|| + \sup_m ||g_m||.$$

But $T_n(f_m \vee g_m) \uparrow_m$ and E has order continuous norm, so that

$$\|h_n\| = \sup_{m \geq n} \|T_n(f_n \vee g_n)\| = \lim_{m \to \infty} \|T_n(f_n \vee g_n)\|.$$

Hence

$$||(h_n, T_n)|| = \sup_n ||h_n|| = \sup_n \sup_{m \geq n} ||T_n(f_m \vee g_m)|| \leq ||(f_n, T_n)|| + ||(g_n, T_n)||;$$

i.e., (h_n, T_n) is norm bounded.

Thus $\mathcal{M}_r((T_n), E)$ is a Riesz space.

To verify that $\mathcal{M}_r((T_n), E)$ is Dedekind complete, is easy. Let

$$(0, T_n) \leq (f_n^{(\alpha)}, T_n) \uparrow_{(\alpha)} \leq (f_n, T_n) \text{ in } \mathcal{M}_r((T_n), E).$$

By Theorem 2.22, $\sup_\alpha (f_n^{(\alpha)}, T_n) = (\sup_\alpha f_n^{(\alpha)}, T_n) \in M_r((T_n), E)$. Since $(\sup_\alpha f_n^{(\alpha)}, T_n) \leq (f_n, T_n)$ and the latter is norm bounded, so is the former. Thus $\sup_\alpha (f_n^{(\alpha)}, T_n) \in \mathcal{M}_r((T_n), E)$.

To show that $||\,|(f_n, T_n)|\,|| = ||(f_n, T_n)||$ for every $(f_n, T_n) \in \mathcal{M}_r((T_n), E)$, i.e. to show that the norm on $\mathcal{M}_r((T_n), E)$ is indeed a lattice norm, we note that since $T_n|f_m| \uparrow_{m \geq n}$ for $n \in \mathbb{N}$ fixed and since E has order continuous norm, it follows that

$$||\,|(f_n, T_n)|\,|| = ||(\sup_{m \geq n} T_n|f_m|, T_n)|| = \sup_n \sup_{m \geq n} ||T_n|f_m|\,|| \leq \sup_n ||f_n||.$$

On the other hand, for $m \geq n$, $|f_n| \leq |T_n f_m| \leq T_n|f_m| \leq \sup_{m \geq n} T_n|f_n|$, from which we get

$$||\,|(f_n, T_n)|\,|| \geq ||\sup_{m \geq n}(T_n|f_n|, T_n)|| \geq ||(|f_n|, T_n)|| = \sup_n ||f_n||,$$

i.e.,

$$|| \, |(f_n, T_n)| \, || = ||(f_n, T_n)||.$$

It is left only to show that $\mathcal{M}_r((T_n), E)$ is a Banach space. It suffices to prove that $\mathcal{M}_r((T_n), E)$ is a closed subspace of $\mathcal{M}_{nb}((T_n), E)$. Suppose that $(f_n^{(k)}, T_n)$ is a sequence of regular martingales in k which converges in norm to some $(f_n^{(\infty)}, T_n) \in \mathcal{M}_{nb}((T_n), E)$. We need to show that (f_n, T_n) is regular. Without loss of generality we can assume that $||(f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)|| < 2^{-k}$ for each k. Since $(f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)$ is also regular, its modulus exists and

$$|| \, |(f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)| \, || = ||(f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)|| < 2^{-k},$$

so that the series $\sum\limits_{k=1}^{\infty} |(f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)|$ converges in $\mathcal{M}_{nb}((T_n), E)$. If we set $(g_n, T_n) = \sum\limits_{k=1}^{\infty} |(f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)|$ then it is clear that $(g_n, T_n) \geq 0$. Now

$$\begin{aligned}
(f_n^{(m)}, T_n) &= (f_n^{(1)}, T_n) + \sum_{k=1}^{m-1}((f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n)) \\
&\leq (f_n^{(1)}, T_n) + \sum_{k=1}^{m-1} |((f_n^{(k+1)}, T_n) - (f_n^{(k)}, T_n))| \\
&\leq |(f_n^{(1)}, T_n)| + (g_n, T_n).
\end{aligned}$$

It follows that $(f_n^{(\infty)}, T_n) \leq |(f_n^{(1)}, T_n)| + (g_n, T_n)$. ■

Corollary 3.16 *Let E be a Banach lattice with order continuous norm and (T_n) a positive filtration on E such that $\overline{\bigcup\limits_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E. Then $\mathcal{M}_{nc}((T_n), E)$ is an ideal in $\mathcal{M}_r((T_n), E)$.*

PROOF: It is clear from Theorems 3.15 and 3.10 that $\mathcal{M}_{nc}((T_n), E)$ is a Riesz subspace of $\mathcal{M}_r((T_n), E)$. If $|(f_n, T_n)| \leq |(g_n, T_n)|$, where $(f_n, T_n) \in \mathcal{M}_{nb}((T_n), E)$ and $(g_n, T_n) \in \mathcal{M}_r((T_n), E)$, then $|(f_n, T_n)|$ is regular, because $|(g_n, T_n)|$ is a positive martingale. Thus $\mathcal{M}_{nb}((T_n), E)$ is an ideal of $\mathcal{M}_r((T_n), E)$. ■

3.2 Krickeberg's Decomposition for KB-Spaces

We now establish a version of Krickeberg's decomposition theorem for positive filtrations.

Note that we do not assume the space E to have a quasi-interior point.

Lemma 3.17 *Let E be a KB-space, (T_n) a positive filtration on E and (f_n, T_n) a sub-martingale on E with the property that $K = \sup_n ||f_n^+|| < \infty$. Then*

(i) For $n \in \mathbb{N}$ fixed, $\{T_n f_m^+ \mid m \geq n\}$ is increasing and norm bounded by K.

(ii) (g_n, T_n), where $g_n = \lim_{m\to\infty} T_n f_m^+$, is a positive martingale with the property that $g_n \geq f_n^+$, for each $n \in \mathbb{N}$. Furthermore, (g_n, T_n) is the smallest martingale satisfying this property.

Proof: (i) Since (f_n, T_n) is a sub-martingale, we have that $T_n f_m \geq f_n$ whenever $m \geq$ n. Thus (f_n^+, T_n) is a sub-martingale because

$$T_n f_m^+ \geq (T_n f_m)^+ \geq (f_n)^+ = f_n^+.$$

To see that $\{T_n f_m^+ \mid m \geq n\}$ is a monotone increasing sequence in m, notice that

$$T_n f_{m+1}^+ = T_n T_m f_{m+1}^+ \geq T_n f_m^+$$

by the sub-martingale property. Let $K = \sup_n ||f_n^+|| < \infty$. Then for $m \geq n$,

$$||T_n f_m^+|| \leq ||T_n|| ||f_m^+|| \leq ||f_m^+|| \leq K < \infty.$$

So that K becomes a norm bound for $\{T_n f_m \mid m \geq n\}$. Because this sequence is increasing we have

$$\lim_{m\to\infty} T_n f_m^+ = \sup_{m\to\infty} T_n f_m^+.$$

Fix $n \in \mathbb{N}$. Since E is a KB-space, there exists $g_n \in E$ such that $\lim_{m\to\infty} T_n f_m^+ = g_n$ in norm. From the fact that $T_n f_m \uparrow_{m\geq n}$, we get $T_n f_m \uparrow_m g_n$.

(ii) We show that (g_n, T_n) is a positive norm bounded martingale.

By the positivity of the T_n's, $g_n \geq 0$ for each $n \in \mathbb{N}$. Furthermore, the norm of g_n is finite for each $n \in \mathbb{N}$ since

$$||g_n|| = ||\sup_{m\to\infty} T_n f_m^+|| \leq \sup_{m\to\infty} ||T_n f_m^+|| < \infty.$$

We also note that (g_n, T_n) satisfies the martingale property:

$$T_n g_{n+1} = T_n(\sup_{m\geq n+1} T_{n+1} f_m^+) = \sup_{m\geq n+1} T_n T_{n+1} f_m^+ = \sup_{m\geq n+1} T_n f_m^+ = g_n$$

by the order continuity and commutivity of the T_n's respectively, and

$$g_n \in \mathcal{R}(T_n) \text{ for each } n \in \mathbb{N}.$$

Note also that

$$g_n = \sup_{m \geq n} T_n f_m^+ \geq T_n f_m^+ \geq f_n^+$$

where the last step is by the sub-martingale property.

Thus (g_n, T_n) is a positive martingale which satisfies the required property. To complete the proof we need only show that (g_n, T_n) is minimal. Suppose that (g_n', T_n) is some arbitrary positive martingale satisfying $g_n' \geq f_n^+$ for each $n \in \mathbb{N}$. Let $m \geq n$, then we have

$$g_n' = T_n g_m' \geq T_n f_m^+$$

and since this holds for each $m \geq n$ it holds for the supremum as well. Hence

$$g_n' \geq \sup_{n \leq m} T_n f_m^+ = g_n$$

and we are done. ■

This puts us in a position to prove Krickeberg's decomposition for KB-spaces.

Theorem 3.18 Krickeberg's Decomposition for KB-spaces
Let (T_n) be a positive filtration on a KB-space E and (f_n, T_n) be a sub-martingale on E with the property $\sup_n ||f_n^+|| < \infty$. Then there exists a positive martingale (g_n, T_n) and a positive super-martingale (h_n, T_n) such that

$$(f_n, T_n) = (g_n, T_n) - (h_n, T_n), \text{ for each } n \in \mathbb{N}.$$

Furthermore, this decomposition is minimal with respect to (g_n, T_n).

PROOF: Take (g_n, T_n) to be the minimal positive martingale in (3.17(ii)) and define (h_n, T_n) such that $h_n = g_n - f_n$. We claim that (h_n, T_n) is a positive super-martingale. Clearly $h_n \in \mathcal{R}(T_n)$. Since $T_n f_m^+ \uparrow_m g_n$ and $(T_n f_n^+, T_n)$ is a sub-martingale, we get

$$g_n \geq T_n f_n^+ \geq f_n^+ \geq f_n,$$

so that (h_n, T_n) is positive. Lastly, to show that the super-martingale property holds, simply note that

$$T_n h_{n+1} = T_n g_{n+1} - T_n f_{n+1} \leq g_n - f_n = h_n.$$

The decomposition is minimal by our choice of (g_n, T_n). ■

We will now show, as a consequence of Lemma (3.17) that in the case where E is a KB-space, every bounded sub-martingale is dominated by a unique martingale, justifying the term "sub-martingale".

Corollary 3.19 *Let E be a KB-space and (T_n) a positive filtration on E. For every bounded sub-martingale (f_n, T_n) on E, there exists a martingale (g_n, T_n) such that $(f_n, T_n) \leq (g_n, T_n)$ and $||(g_n, T_n)|| \leq ||(f_n, T_n)||$.*

PROOF: Since (f_n, T_n) is a bounded sub-martingale, we apply Lemma (3.17(i)) which yields, for $n \in \mathbb{N}$ fixed, that the sequence $(T_n f_m)_{m=n}^{\infty}$ is increasing and norm bounded by $||(f_n, T_n)||$. Since E is a KB-space, this sequence converges, say to g_n. By applying Lemma (3.17(ii)) to (f_n, T_n) we see that (g_n, T_n) is a martingale and furthermore, it is the least martingale such that $(f_n, T_n) \leq (g_n, T_n)$.

For the last part, notice that

$$||g_k|| = \lim_{m\to\infty} ||T_k f_m|| \leq \lim_{m\to\infty} ||f_m|| = ||(f_n, T_n)|| \text{ for each } k \in \mathbb{N},$$

so that (g_n, T_n) is bounded and $||(g_n, T_n)|| \leq ||(f_n, T_n)||$. ■

Remark: In general, $||(g_n, T_n)|| \neq ||(f_n, T_n)||$. If we take $f_n = -\frac{1}{n}\chi_{[0,1]}$, then (f_n, T_n) is a sub-martingale of constant functions with $||(f_n, T_n)|| > 0$ but $(g_n, T_n) = 0$ where $g_n = \lim_{m\to\infty} T_n f_m$. However, if (T_n) preserves the norms of positive vectors and (f_n, T_n) is a positive sub-martingale, then $||g_n|| = \lim_{m\to\infty} ||T_n f_m|| = \lim_{m\to\infty} ||f_m|| = ||(f_n, T_n)||$ so that $||(g_n, T_n)|| = ||(f_n, T_n)||$.

3.3 Martingales and KB-Spaces

The following is the main result of this section, which was noted by Troitsky in [30]:

Theorem 3.20 *If E is a KB-space and (T_n) is a positive filtration on E, then $\mathcal{M}_{nb}((T_n), E)$ is a Dedekind complete Banach lattice with lattice operations given by*

$$\begin{aligned}
(f_n, T_n)^+ &= (\lim_{m\to\infty} T_n f_m^+, T_n); \\
(f_n, T_n)^- &= (\lim_{m\to\infty} T_n f_m^-, T_n); \\
(f_n, T_n) \vee (g_n, T_n) &= (\lim_{m\to\infty} T_n(f_m \vee g_m), T_n); \\
(f_n, T_n) \wedge (g_n, T_n) &= (\lim_{m\to\infty} T_n(f_m \wedge g_m), T_n); \\
|(f_n, T_n)| &= (\lim_{m\to\infty} T_n|f_m|, T_n).
\end{aligned} \tag{3.3}$$

and martingale norm

$$||(f_n, T_n)|| = \sup_n ||f_n||_E.$$

PROOF: We have already noted in §3.2 that $\mathcal{M}_{nb}((T_n),E)$ is a Banach space with respect to the martingale norm. It remains to show that $\mathcal{M}_{nb}((T_n),E)$ is a Dedekind complete Riesz space in which the formulas hold as stated and that the martingale norm is a Riesz norm.

If $(f_n,T_n),(g_n,T_n)\in\mathcal{M}_{nb}((T_n),E)$, then $\sup_n ||f_n^+||<\infty$ and $\sup_n ||g_n^+||<\infty$. Since the supremum of two sub-martingales is again a sub-martingale, we use (3.17(ii)) to get the third equation in (3.3). The remaining four equations are derived from standard formulas (see [31, Theorem 5.2]).

To show that $\mathcal{M}_{nb}((T_n),E)$ is Dedekind complete, let

$$(0,T_n)\le (f_n^{(\alpha)},T_n)\uparrow_{(\alpha)}\le (f_n,T_n)\in\mathcal{M}_{nb}((T_n),E).$$

By the Dedekind completeness of E, we get $\sup_{(\alpha)} f_n^{(\alpha)}\in E$ for all $n\in\mathbb{N}$. Since $(\sup_{(\alpha)} f_n^{(\alpha)},T_n)\le (f_n,T_n)$, it is readily verified that $\sup(f_n^{(\alpha)},T_n)=(\sup_{(\alpha)} f_n^{(\alpha)},T_n)\in\mathcal{M}_{nb}((T_n),E)$,

To show that the martingale norm is a Riesz norm, let $(f_n,T_n)\in\mathcal{M}_{nb}((T_n,E)$. Then, by the first part of the proof, $|(f_n,T_n)|=(\lim_{m\to\infty} T_n|f_m|,T_n)=:(h_n,T_n)\in\mathcal{M}_{nb}((T_n),E)$ and

$$||h_k||=\lim_{m\to\infty}||\,T_k|f_m|\,||\le\lim_{m\to\infty}||f_m||=||(f_n,T_n)||,\ \text{for each } k\in\mathbb{N}.$$

Thus,

$$\big\||(f_n,T_n)|\big\|=\|(h_n,T_n)\|\le||(f_n,T_n)||.$$

Also, from $\pm(f_n,T_n)\le|(f_n,T_n)|=(h_n,T_n)$, we get that

$$||f_k||=||\,|f_k|\,||\le||h_k||\le\sup_k||h_k||=||\,|(f_n,T_n)|\,||\ \text{for each } k\in\mathbb{N}.$$

Hence $||\,|(f_n,T_n)|\,||=||(h_n,T_n)||=||(f_n,T_n)||$.

Thus $\mathcal{M}_{nb}((T_n),E)$ is a Dedekind complete Banach lattice and the proof is complete. ■

Proposition 3.21 *If E is a KB-space and (T_n) is a positive filtration on E, then*

$$\mathcal{M}_r((T_n),E)=\mathcal{M}_{nb}((T_n),E).$$

PROOF: Now if E is a KB-space then from Theorem (3.20) it follows that for every norm bounded martingale (f_n,T_n) in E, $|(f_n,T_n)|$ is a norm bounded martingale with $(f_n,T_n)\le|(f_n,T_n)|$ so that (f_n,T_n) is a regular norm bounded martingale. Hence $\mathcal{M}_r((T_n),E)=\mathcal{M}_{nb}((T_n),E)$. ■

The following result was also noted by Troitsky in [30], where he assumed that the (positive) filtration (T_n) on E has the property that $\overline{\bigcup_{n=1}^{\infty}\mathcal{R}(T_n)}=E$:

Theorem 3.22 *Let (T_n) be a positive filtration on a KB-space E such that $\overline{\bigcup_{n=1}^{\infty} \mathcal{R}(T_n)}$ is a closed Riesz subspace of E. Then $\mathcal{M}_{nc}((T_n), E)$ is a projection band in $\mathcal{M}_{nb}((T_n), E)$.*

Proof: It is readily verified that $\mathcal{M}_{nc}((T_n), E)$ is an ideal of $\mathcal{M}_{nb}((T_n), E)$. Let (f_n) be any increasing, order bounded sequence in $\mathcal{M}_{nc}((T_n), E)^+$. Then (f_n) is also norm bounded and has supremum in $\mathcal{M}_{nc}((T_n), E)$, since $\mathcal{M}_{nc}((T_n), E)$ is a KB-space (3.12) and in particular, Dedekind complete, it follows that $\mathcal{M}_{nc}((T_n), E)$ is a projection band in $\mathcal{M}_{nb}((T_n), E)$. ■

Remark: We have the following inclusions

$$\mathcal{M}_{nc}((T_n), E) \subseteq \mathcal{M}_{r}((T_n), E) \subseteq \mathcal{M}_{nb}((T_n), E).$$

It is interesting to note that the larger the space of martingales, the stronger the assumption required on E for the former to be a Banach lattice.

We now give requirements for which $\mathcal{M}_{nb}((T_n), E)$ is in fact a KB-space.

For the Banach lattice E, we say that the positive filtration (T_n) is **bounded below** on E_+ if there exists $1 \leq n \in E$ and a constant $c > 0$ such that $||T_n f|| \geq c||f||$ for each $f \in E_+$. Notice that if (T_n) is bounded below then $||f_m|| \leq \frac{1}{c}||T_n f_m|| = \frac{1}{c}||f_n||$ for each $m \geq n$, giving $||(f_n, T_n)|| \leq \frac{1}{c}||f_n|| < \infty$ so that every positive martingale (and therefore every regular martingale) is bounded. Notice in addition that if (T_n) is bounded below, then T_n is strictly positive for some $n \in \mathbb{N}$.

Lemma 3.23 *If E has order continuous norm and (T_n) a positive filtration on E which is bounded below on E_+, then the martingale norm is again order continuous.*

Proof: Suppose that $(f_n^{(\alpha)}, T_n)$ is a decreasing net of martingales such that $\inf_\alpha (f_n^{(\alpha)}, T_n) = 0$. For $n \in \mathbb{N}$ fixed, the net $(f_n^{(\alpha)})$ is decreasing in α so that the order continuity of E yields that $\lim_\alpha f_n^{(\alpha)}$ exists, we will call it f_n. It is easily verified that $f_n \geq 0$ and (f_n, T_n) is a martingale since

$$T_m f_n = T_m(\lim_\alpha f_n^{(\alpha)}) = \lim_\alpha T_m f_n^{(\alpha)} = \lim_\alpha f_m^{(\alpha)} = f_m.$$

Clearly (f_n, T_n) is bounded and it follows from $0 \leq (f_n, T_n) \leq (f_n^{(\alpha)}, T_n)$ for each α and $\inf_\alpha (f_n^{(\alpha)}, T_n) = 0$ that $(f_n, T_n) = 0$. Thus $\lim_\alpha f_n^{(\alpha)} = 0$ for each $n \in \mathbb{N}$.

Since (T_n) is bounded below, there exists $n \geq 1$ and a constant $c > 0$ such that $||T_n f|| \geq c||f||$ for each $f \geq 0$. Now for every $m \geq n$ we have $||f_n^{(\alpha)}|| = ||T_n f_m^{(\alpha)}|| \geq c||f_m^{(\alpha)}||$, so that $||(f_n^{(\alpha)}, T_n)|| \leq \frac{1}{c}||f_n^{(\alpha)}|| \to 0$. ■

Theorem 3.24 *If E is a KB-space and (T_n) is a positive filtration which is bounded below on E_+, then $\mathcal{M}_{nb}((T_n), E)$ is itself a KB-space.*

Proof: It follows from Theorem (3.20) that $\mathcal{M}_{nb}((T_n), E)$ is a Banach lattice. We consider an increasing sequence of martingales $(f_n^{(k)}, T_n)_{k=1}^{\infty}$ with uniformly bounded norms $||(f_n^{(k)}, T_n)|| \leq K \in \mathbb{R}$. For $n \in \mathbb{N}$ fixed, the sequence $(f_n^{(k)})$ is increasing in k and is norm bounded by K. Since E is a KB-space, the sequence converges to some f_n where $||f_n|| \leq K$. Consider (f_n, T_n). For $n \leq m$ we have

$$T_n f_m = T_n(\lim_{k\to\infty} f_m^{(k)}) = \lim_{k\to\infty} T_n f_m^{(k)} = \lim_{k\to\infty} f_n^{(k)} = f_n,$$

so that $(f_n, T_n) \in \mathcal{M}_{nb}((T_n), E)$. Clearly, $(f_n, T_n) = \sup_k (f_n^{(k)}, T_n)$ and hence Lemma (3.23) gives that $||(f_n, T_n) - (f_n^{(k)}, T_n)|| \to 0$. ■

Chapter 4

Martingales on Tensor Products

4.1 Preliminaries

Important applications of classical martingale theory can be found in Banach space-valued L^p-spaces (see [4]). It is well known that for a Banach space X and $1 \leq p < \infty$ we have

$$L^p(\mu, X) = L^p(\mu)\widetilde{\otimes}_{\Delta_p} X,$$

where $L^p(\mu)\widetilde{\otimes}_{\Delta_p} X$ denotes the norm completion of $L^p(\mu) \otimes_{\Delta_p} X$ with respect to the Bochner norm Δ_p. In this setting, interest has mainly been in filtrations of the form $(T_i \otimes id_X)$, where (T_i) is a filtration on $L^p(\mu)$ and id_X is the identity operator on X (see [4]).

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice and a Banach space (see [1] and [29]). If E is Banach lattice and X is a Banach space, then the norm on $X \otimes E$, defined by

$$\|u\|_{|\mu|} = \inf \left\{ \left\| \sum_{i=1}^{n} \|x_i\| \, |y_i| \right\| \;\middle|\; u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

and

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_M = \left\| \sup \left\{ \sum_{i=1}^{n} \langle x', x_i \rangle y_i \;\middle|\; x' \in X' \text{ and } \|x'\| \leq 1 \right\} \right\|$$

where $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in E$, are equal and are reasonable cross norms on $X \otimes E$ (cf. [1, Theorem 1.4]. Both these norms are also equal to Schaefer's m-norm ([29, Ch 4, §7]) on $E \otimes F$ (cf. [14, Ch 4, §5]). Their transposition is equal to Schaefer's l-norm; i.e.,

$$\|u\|_l = \inf \left\{ \left\| \sum_{i=1}^{n} \|y_i\| \, |x_i| \right\| \;\middle|\; u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

and

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_l = \left\| \sup \left\{ \sum_{i=1}^{n} \langle y', y_i \rangle x_i \;\middle|\; y' \in Y' \text{ and } \|y'\| \leq 1 \right\} \right\|$$

where $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in F$.

For $1 < p < \infty$, the Bochner norm Δ_p on $L^p(\mu, X)$ has the property that if $0 \leq T : L^p(\mu) \to L^p(\mu)$ and $S : X \to X$ are bounded maps, then $T \otimes S : L^p(\mu, X) \to L^p(\mu, X)$ has the property that $\|T \otimes S\| \leq \|T\|\|S\|$ (see [4], [21]).

This extends to the l tensor product of a Banach lattice and a Banach space, as the l-norm has the following property (a proof can be found in [21]):

(i) Let E_1 and E_2 be Banach lattices and X_1 and X_2 be Banach spaces. If $T_1 : E_1 \to E_2$ be a positive linear operator and $T_2 : X_1 \to X_2$ be a bounded linear operator. Then $\left\|(T_1 \otimes T_2)u\right\|_l \leq \|T_1\| \, \|T_2\| \, \|u\|_l$ for each $u \in E_1 \widetilde{\otimes}_l X_1$.

The l-norm is injective in the following sense (see [21] for a proof):

(ii) Let E be a Banach lattice and Y a Banach space. If E_0 is a Banach lattice and $i_0 : E_0 \to E$ a Riesz isometry, then $i_0 \otimes id_Y : E_0 \widetilde{\otimes}_l Y \to E \widetilde{\otimes}_l Y$ is an isometry.

(ii′) Let E and F be Banach lattices. If E_0 is a Banach lattice and $i_0 : E_0 \to E$ a Riesz isometry, then $i_0 \otimes id_F : E_0 \widetilde{\otimes}_l F \to E \widetilde{\otimes}_l F$ is a Riesz isometry.

(iii) Let E be a Banach lattice and Y a Banach space. If Y_0 is a Banach space and $j_0 : Y_0 \to Y$ an isometry, then $id_E \otimes j_0 : E \widetilde{\otimes}_l Y_0 \to E \widetilde{\otimes}_l Y$ is an isometry.

(iii′) Let E and F be Banach lattices. If F_0 is a Banach lattice and $j_0 : F_0 \to F$ a Riesz isometry, then $id_E \otimes j_0 : E \widetilde{\otimes}_l F_0 \to E \widetilde{\otimes}_l F$ is a Riesz isometry.

It should be noted that similar properties hold for the M-norm.

We would like to consider "filtrations" of the form $(T_n \otimes S_n)$ on $E\widetilde{\otimes}_l X$, rather than just "filtrations" of the form $(T_n \otimes id_X)$.

4.2 Filtrations on Tensor Products

The results in this section are based on [2], where "quasi-interior point preserving filtrations" were considered.

Lemma 4.1 *Let E be a Banach lattice and X a Banach space with $0 \leq S : E \to E$ and $T : X \to X$ contractive projections on E and X respectively. If $\mathcal{R}(S)$ is a closed Riesz subspace of E, then* *(i)* $(S \otimes_l T) : E \widetilde{\otimes}_l X \to E \widetilde{\otimes}_l X$ *is a contractive projection with range,* $S(E) \widetilde{\otimes}_l T(X)$, *a closed subspace of* $E \widetilde{\otimes}_l X$.

(ii) $(T \otimes_M S) : X \widetilde{\otimes}_M E \to X \widetilde{\otimes}_M E$ *is a contractive projection with range,* $T(X) \widetilde{\otimes}_M S(E)$, *a closed subspace of* $X \widetilde{\otimes}_M E$.

PROOF: Since S is positive and T is continuous it follows that $\|S \otimes T\| \leq \|S\| \|T\| \leq 1$; consequently, the continuous extension $(S \otimes_l T)$ has norm less than or equal to one.

Since $E \otimes X$ is l-dense in $E \widetilde{\otimes}_l X$, $(S \otimes T)(E \otimes X) \subset E \otimes X$ and $S \otimes T : E \otimes_l X \to E \otimes_l X$ is continuous, we get that $S \otimes_l T : E \widetilde{\otimes}_l X \to E \widetilde{\otimes}_l X$.

To see that $(S \otimes_l T)$ is a projection, let $u \in E \widetilde{\otimes}_l X$. Then there exists a sequence $(u_j) \subset E \otimes X$ such that $u_j \to u$ in norm. Representing each u_j as $\sum_{i=1}^{n_j} x_i^{(j)} \otimes y_i^{(j)}$, we see that

$$(S \otimes_l T)^2(u_j) = \sum_{i=1}^{n_j} S^2(x_i^{(j)}) \otimes T^2(y_i^{(j)}) = \sum_{i=1}^{n_j} S(x_i^{(j)}) \otimes T(y_i^{(j)}) = (S \otimes_l T)(u_j).$$

By the continuity of $(S \otimes_l T)$, it follows that $(S \otimes_l T)^2(u) = (S \otimes_l T)(u)$.

As $S(E)$ is a closed Riesz subspace and $T(X)$ a closed subspace of E and X respectively, the injectivity of the l-norm gives

$$(S \otimes_l T)(E \otimes_l X) = S(E) \otimes_l T(X) \subset S(E) \widetilde{\otimes}_l T(X) \hookrightarrow E \widetilde{\otimes}_l X \quad \text{(isometrically)} .$$

Thus

$$S(E) \otimes_l T(X) \subset (S \otimes_l T)(E \widetilde{\otimes}_l X) \subset S(E) \widetilde{\otimes}_l T(X).$$

As $(S \otimes_l T)$ is a projection and thus has closed range, $(S \otimes_l T)(E \widetilde{\otimes}_l X) = S(E) \widetilde{\otimes}_l T(X)$.
■

The injectivity of the l-norm and Lemma 4.1 motivate the following definition:

Definition 4.2 A BS-filtration (T_i) for which T_i is positive and $\mathcal{R}(T_i)$ is a closed Riesz subspace of E for each $i \in \mathbb{N}$ is called a *Banach lattice filtration (BL-filtration).*

It is trivial to remark that every BL-filtration (T_i) on E is a positive filtration such that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is a closed Riesz subspace of E.

Proposition 4.3 *Let* E *be a Banach lattice and* X *a Banach space. If* (S_i) *is a BL-filtration on* E *and* (T_i) *is a BS-filtration on* X, *then* *(i)* $(S_i \otimes_l T_i)$ *is a BS-filtration on* $E \widetilde{\otimes}_l X$.

(ii) $(T_i \otimes_M S_i)$ *is a BS-filtration on* $X \widetilde{\otimes}_M E$.

Proof: By the above Lemma, each $S_i \otimes_l T_i$ is a contractive projection with $\mathcal{R}(S_i \otimes_l T_i) = S_i(E) \widetilde{\otimes}_l T_i(X)$. Since $S_i(E)$ is a closed Riesz subspace of $S_j(E)$ and $T_i(X)$ is a closed subspace of $T_j(X)$ for each $i \leq j$ and furthermore, the l-norm is injective, it follows that $\mathcal{R}(S_i \otimes T_i) \uparrow_i$. ■

Let E and F be Banach lattices, then $E \widetilde{\otimes}_l F$ is a Banach lattice with positive cone the l-closure of the projective cone

$$E_+ \otimes F_+ = \left\{ \sum_{i=1}^{n} x_i \otimes y_i : (x_i, y_i) \in E_+ \times F_+ \right\}$$

(cf. [1] and [29]). Also, $F \widetilde{\otimes}_M E$ is a Banach lattice with positive cone the M-closure of the projective cone $F_+ \otimes E_+$.

Lemma 4.4 *Let* E *and* F *be Banach lattices. If* $S : E \to E$ *and* $T : F \to F$ *are positive contractive projections with ranges that are closed Riesz subspaces of* E *and* F *respectively, then*

(i) $(S \otimes_l T) : E \widetilde{\otimes}_l F \to E \widetilde{\otimes}_l F$ *is a positive contractive projection with range,* $S(E) \widetilde{\otimes}_l T(F)$*, a closed Riesz subspace of* $E \widetilde{\otimes}_l F$.

(ii) $(T \otimes_M S) : F \widetilde{\otimes}_M E \to F \widetilde{\otimes}_M E$ *is a positive contractive projection with range,* $T(F) \widetilde{\otimes}_M S(E)$*, a closed Riesz subspace of* $F \widetilde{\otimes}_M E$.

Proof: From (4.1), it suffices to show that $S \otimes_l T \geq 0$ and $S(E)\widetilde{\otimes}_l T(F)$ is a Riesz subspace of $E \widetilde{\otimes}_l F$.

Since $E_+ \otimes F_+$ is l-dense in $(E \widetilde{\otimes}_l F)_+$, $(S \otimes T)(E_+ \otimes F_+) \subset E_+ \otimes F_+$ and $S \otimes T : E \otimes_l F \to E \otimes_l F$ is continuous, we get that $0 \leq S \otimes_l T : E \widetilde{\otimes}_l F \to E \widetilde{\otimes}_l F$.

Since Fremlin's Riesz tensor product $S(E)\overline{\otimes}T(F)$ of $S(E)$ and $T(F)$ is a Riesz subspace of the Riesz tensor product $E\overline{\otimes}F$ of E and F, $S(E)\overline{\otimes}T(F)$ is dense in $S(E)\widetilde{\otimes}_l T(F)$ and $E\overline{\otimes}F$ is dense in $E\widetilde{\otimes}_l F$ (see [8, 10, 21]), we get that $S(E)\widetilde{\otimes}_l T(F)$ is a closed Riesz subspace of $E\widetilde{\otimes}_l F$. ■

Proposition 4.5 *Let* E *and* F *be Banach lattices and let* (S_i) *and* (T_i) *be BL-filtrations on* E *and* F *respectively. Then*

(i) $(S_i \otimes_l T_i)$ *is a BL-filtration on* $E \widetilde{\otimes}_l F$.

(ii) $(T_i \otimes_M S_i)$ *is a BL-filtration on* $F \widetilde{\otimes}_M E$.

4.3 Filtrations on Banach Lattices - Revisited

To investigate the connection to RS-filtrations which map weak order units to weak order units, we introduce the following:

Definition 4.6 A BL-filtration (T_i) on E for which there exists a quasi-interior point e of E_+ such that $T_i(e) = e$ for each $i \in \mathbb{N}$ is called a *BL-qip-filtration*.

Theorem 4.7 *Let E and F be Banach lattices with quasi-interior points e and f, of E_+ and F_+, respectively. If (S_i) and (T_i) are BL-filtrations which leave e and f invariant, then $e \otimes f$ is a quasi-interior point of $E \widetilde{\otimes}_l F$ which is invariant under the $BL-$qip-filtration $(S_i \otimes_l T_i)$ on $E \widetilde{\otimes}_l F$.*

Proof: Select quasi-interior points e and f in E_+ and F_+ respectively such that $S_i e = e$ and $T_i f = f$ for each $i \in \mathbb{N}$. It is easily verified that $e \otimes f$ is a quasi-interior point of $E \widetilde{\otimes}_l F$ and $(S_i \otimes_l T_i)(e \otimes f) = e \otimes f$ for each $i \in \mathbb{N}$. ■

It is well known that if e is a quasi-interior point of E_+ then e is also a weak order unit of E. If E is an order continuous Banach lattice, then the notions of quasi-interior point and weak order unit coincide. Furtheremore, if $T : E \to E$ is continuous, then T is also order continuous.

The following result shows that the notion of "BL-qip-filtration" on order continuous Banach lattices are special cases of the notion "RS-wou-filtration" on Dedekind complete Riesz spaces with weak order units, as given in Chapter 2.

Proposition 4.8 *Let E be a Banach lattice and $T : E \to E$ be a positive projection. Then the following conditions are equivalent:*

(i) There exists a quasi-interior point e of E_+ such that $Te = e$.

(ii) For every quasi-interior point q of E_+ we have that Tq is a quasi-interior point of E_+.

Proof: (ii)⇒(i) If q is a quasi-interior point in E_+, then $e = T(q)$ and since T is idempotent, $T(e) = e$.
(i)⇒(ii) Assume that e is a quasi-interior point of E and $Te = e$. Note that a positive projection on the Banach lattice E is automatically continuous. Thus for any quasi-interior point q of E^+ we have, by the continuity of T, that $\lim_{n\to\infty} T(nq \wedge e) = T(e) = e$. Notice that

$$0 \le T(nq \wedge e) \le nTq \wedge e \le e$$

and since $T(nq \wedge e)$ is an increasing sequence, it follows that $\lim_{n\to\infty} (nTq) \wedge e = e$.

Let $p \in E_+$. Then

$$\begin{aligned} p &= \lim_{m\to\infty}(me)\wedge p = \lim_{m\to\infty}(m\lim_{n\to\infty}(nTq)\wedge e)\wedge p = \lim_{m\to\infty}\lim_{n\to\infty}(m(nTq)\wedge e)\wedge p \\ &\leq \lim_{m\to\infty}\lim_{n\to\infty}(mnTq)\wedge p \leq p, \end{aligned}$$

so that Tq is a quasi-interior point of E_+. ■

4.4 Strictly Positive Filtrations on Tensor Products

The main result of this section we believe is new.

To access the Upcrossing Theorem for martingales on Dedekind complete Riesz spaces with weak order units, as given in Chapter 2 one needs to consider filtrations (T_i), where each T_i is strictly positive. Hence we address the following question.

If S and T are strictly positive, is $S \otimes T$ strictly positive ?

Here we resort to the fundamental works [10, 8, 11] of D.H. Fremlin. If E is a Banach lattice with order continuous norm, then E has the Fatou property (see [11]). Also, if E is a locally solid Riesz space and has the Fatou property, then E is order dense in its topological completion $\widetilde{E}$ (see [11]).

Popa showed, in [25], that if E and F are order continuous Banach lattices [KB-spaces] then $E \widetilde{\otimes}_l F$ is an order continuous Banach lattice [a KB-space].

Lemma 4.9 *Let E and F be order continuous Banach lattices and let $S : E \to E$ and $T : F \to F$ be strictly positive. Then $S \otimes T : E \widetilde{\otimes}_l F \to E \widetilde{\otimes}_l F$ is strictly positive.*

PROOF: Let $0 < u \in E \widetilde{\otimes}_l F$. The Riesz tensor product of E and F, denoted by $E\overline{\otimes}F$, has the property that the l-closure of $E\overline{\otimes}F$ is $E \widetilde{\otimes}_l F$. Since $E \widetilde{\otimes}_l F$ has order continuous norm, it has the Fatou property. But then $E\overline{\otimes}F$ is order dense in $E \widetilde{\otimes}_l F$. Hence, there exist $v \in E\overline{\otimes}F$ such that $u \geq v > 0$. By the properties of $E\overline{\otimes}F$, there exists $(x, y) \in E_+ \times F_+$ such that $0 < x$, $0 < y$ and $x \otimes y \leq v$ (see [8]). It follows from $0 < S(x)$ and $0 < T(y)$ that $0 < S(x) \otimes T(y) \leq (S \otimes T)(v) \leq (S \otimes T)(u)$. ■

As a consequence of the preceding result, the following result holds:

Corollary 4.10 *Let E and F be Banach lattices. If (S_i) and (T_i) are BL-filtrations on E and F respectively such that S_i and T_i are strictly positive for each $i \in \mathbb{N}$, then*

(i) $(S_i \otimes_l T_i)$ is a strictly positive BL-filtration on $E \widetilde{\otimes}_l F$.

(ii) $(T_i \otimes_M S_i)$ is a strictly positive BL-filtration on $F \widetilde{\otimes}_M E$.

4.5 Norm Convergent martingales

We recall from chapter 3 that $\mathcal{M}_{nc}((T_n), E)$ is the space of all norm convergent martingales on the Banach lattice E.

4.5.1 A Distributive Property

Lemma 4.11 *Let E be a Banach lattice and let X be a Banach space. If (S_i) is a BL-filtration on E and (T_i) is a BS-filtration on X, then*

$$\text{(i)}\quad \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X} = \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes T_i)}^{l}$$

$$\text{(ii)}\quad \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X}\ \widetilde{\otimes}_M\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E} = \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i\otimes S_i)}^{M}$$

Proof: We will prove the first equality, the second is derived similarly.

($\supseteq$) Let $\epsilon > 0$ be given. Let $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes T_i)}^{l}$ and select $y_0 \in \mathcal{R}(S_i \otimes T_i)$ for some $i \in \mathbb{N}$ such that $\|y - y_0\|_l < \epsilon$. Since $\mathcal{R}(S_i \otimes_l T_i) = S_i(E)\,\widetilde{\otimes}_l\,T_i(X)$ and $S_i(E)\,\widetilde{\otimes}_l\,T_i(X) \subseteq \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X}$ by the injectivity of the l-norm, it follows that $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X}$.

($\subseteq$) Let $\epsilon > 0$ be given. Let $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X}$ and select $y_0 \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E} \otimes \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X}$ such that $||y - y_0||_l < \epsilon/2$. Let $y_0 = \sum_{i=1}^{n_0} a_i \otimes y_i$, where $a_i \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}^{\|\cdot\|_E}$ and $y_i \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}^{\|\cdot\|_X}$. Select $v_i \in \bigcup_{i=1}^{\infty}\mathcal{R}(T_i)$ such that $||y_i - v_i||_X < \frac{\epsilon}{4n_0\sum_{i=1}^{n_0}||a_i||}$. Select $b_i \in \bigcup_{i=1}^{\infty}\mathcal{R}(S_i)$ such that $||a_i - b_i||_E < \frac{\epsilon}{4n_0\sum_{i=1}^{n_0}||v_i||}$. Let $z_1 = \sum_{i=1}^{n_0} b_i \otimes v_i$. Then $z_1 \in \bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes T_i)$, $y_0 - z_1 = \sum_{i=1}^{n_0}\Big(a_i \otimes (y_i - v_i) + (a_i - b_i)\otimes v_i\Big)$,

$$||y_0 - z_1||_l \le \left\| \sum_{i=1}^{n_0}\Big(\|y_i - v_i\|\,|a_i| + \|v_i\|\,|a_i - b_i|\Big)\right\|_E < \epsilon/4 + \epsilon/4 = \epsilon/2,$$

and

$$\|y - z_1\|_l \le \|y - y_0\|_l + \|y_0 - z_1\|_l \le \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $y \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes T_i)}^{l}$. ■

Thus, one has the following distributive property:

Proposition 4.12 *Let E and F be Banach lattices and let X be a Banach space. If (S_i) is a BL-filtration on E, (T_i) is a BS-filtration on X and (Q_i) is a BL-filtration on F, then*

(i) $\mathcal{M}_{nc}\Big((S_i \otimes_l T_i), E \widetilde{\otimes}_l X\Big) \simeq \mathcal{M}_{nc}((S_i), E) \widetilde{\otimes}_l \mathcal{M}_{nc}((T_i), X)$;

(i′) $\mathcal{M}_{nc}\Big((T_i \otimes_M S_i), X \widetilde{\otimes}_M E\Big) \simeq \mathcal{M}_{nc}((T_i), X) \widetilde{\otimes}_M \mathcal{M}_{nc}((S_i), E)$;

(ii) $\mathcal{M}_{nc}\Big((S_i \otimes_l Q_i), E \widetilde{\otimes}_l F\Big) \cong \mathcal{M}_{nc}((S_i), E) \widetilde{\otimes}_l \mathcal{M}_{nc}((Q_i), F)$;

(ii′) $\mathcal{M}_{nc}\Big((Q_i \otimes_M S_i), F \widetilde{\otimes}_M E\Big) \cong \mathcal{M}_{nc}((Q_i), F) \widetilde{\otimes}_M \mathcal{M}_{nc}((S_i), E)$;

where $\cong$ denotes Riesz and isometrically isomorphic and $\simeq$ only the latter.

Proof: (i) By Propositions (3.5) and (3.9), $\mathcal{M}_{nc}\Big((S_i \otimes_l T_i), E \widetilde{\otimes}_l X\Big) \simeq \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes T_i)}^{l}$.
By the Lemma above,

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes T_i)}^{l} = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\|\cdot\|_E} \widetilde{\otimes}_l \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_X}.$$

Furtheremore, $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\|\cdot\|_E} \cong \mathcal{M}_{nc}((S_i), E)$ and $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_X} \simeq \mathcal{M}_{nc}((T_i), X)$,
from which it follows that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\|\cdot\|_E} \widetilde{\otimes}_l \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_X} \simeq \mathcal{M}_{nc}((S_i), E) \widetilde{\otimes}_l \mathcal{M}_{nc}((Q_i), F)$.

(ii) The proof is similar. ■

4.5.2 Convergent martingales and L^p-spaces

We extend the following characterization of $L^p(\mu)$-spaces, which can be found in [22], to suitable spaces of convergent martingales:

Theorem 4.13 *Let E be a Banach lattice. The following statements are equivalent for $1 \le p < \infty$:*

(i) *E is Riesz and isometrically isomorphic to an $L^p(\mu)$-space, for some measure space (Ω, Σ, μ).*

(ii) *The norms Δ_p and ${}^t\Delta_p$ are equivalent on $L^p(\mu_1) \otimes E$ for all finite measure spaces $(\Omega_1, \Sigma_1, \mu_1)$.*

(iii) *The norms Δ_p and ${}^t\Delta_p$ are equivalent on $L^p(\mu_1) \otimes E$ for all σ-finite measure spaces $(\Omega_1, \Sigma_1, \mu_1)$.*

(iv) *The norms Δ_p and ${}^t\Delta_p$ are equivalent on $\ell^p \otimes E$.*

In terms of convergent martingales, L^p-spaces can be characterized as follows, which we believe is a new result:

Proposition 4.14 *Let E be a Banach lattice and $1 \le p < \infty$. Then the following statements are equivalent:*

(i) *E is Riesz and isometrically isomorphic to an $L^p(\mu)$-space, for some measure space (Ω, Σ, μ).*

(ii) $\mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, E\Big) \cong \mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu)\,\widetilde{\otimes}_M\, E\Big)$ *for any σ-finite measure space (Ω, Σ, μ), for all BL-filtrations (S_i) on $L^p(\mu)$ and for all BL-filtrations (T_i) on E.*

(ii′) $\mathcal{M}_{nc}\Big((S_i \otimes T_i), \ell^p\,\widetilde{\otimes}_{{}^tM}\, E\Big) \cong \mathcal{M}_{nc}\Big((S_i \otimes T_i), \ell^p\,\widetilde{\otimes}_M\, E\Big)$ *for all BL-filtrations (S_i) on ℓ^p and for all BL-filtrations (T_i) on E.*

(iii) $\mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, E\Big) \cong \mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu)\,\widetilde{\otimes}_M\, E\Big)$ *for any probability space (Ω, Σ, μ), for all BL-filtrations (S_i) on $L^p(\mu)$ for which $S_i(\mathbf{1}) = \mathbf{1}$ for each $i \in \mathbb{N}$ and for all BL-filtrations (T_i) on E.*

where $\cong$ denotes Riesz and isometrically isomorphic to.

Proof: *(i)* $\Rightarrow$ *(ii)* Let (Ω, Σ, μ) be a σ-finite measure space, (S_i) a BL-filtration on $L^p(\mu)$ and (T_i) a BL-filtration on E. If E is isometrically isomorphic to an L^p-space, then $L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, E \cong L^p(\mu)\,\widetilde{\otimes}_M\, E$. But then $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes T_i)}^{{}^tM} \cong \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes T_i)}^{M}$ and consequently

$$\mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, E\Big) \cong \mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu)\,\widetilde{\otimes}_M\, E\Big).$$

(ii) $\Rightarrow$ *(iii)* This implication is trivial.

(iii) $\Rightarrow$ *(i)* Let (Ω, Σ, μ) be a probability space. If (iii) holds, then for for all BL-filtrations (S_i) on $L^p(\mu)$ for which $S_i(\mathbf{1}) = \mathbf{1}$ for each $i \in \mathbb{N}$ and for all BL-filtrations (T_i) on E, we have that

$$\mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu) \,\widetilde{\otimes}_{^tM}\, E\Big) \cong \mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu) \,\widetilde{\otimes}_M\, E\Big).$$

But

$$\begin{aligned} \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\Delta_p} \,\widetilde{\otimes}_{^tM}\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_E} &\cong \mathcal{M}_{nc}((S_i), L^p(\mu)) \,\widetilde{\otimes}_{^tM}\, \mathcal{M}_{nc}((T_i), E) \\ &\cong \mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu) \,\widetilde{\otimes}_{^tM}\, E\Big) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu) \,\widetilde{\otimes}_M\, E\Big) &\cong \mathcal{M}_{nc}((S_i), L^p(\mu)) \,\widetilde{\otimes}_M\, \mathcal{M}_{nc}((T_i), E) \\ &\cong \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\Delta_p} \,\widetilde{\otimes}_M\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_E}. \end{aligned}$$

Hence

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\Delta_p} \,\widetilde{\otimes}_{^tM}\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_E} \cong \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)}^{\Delta_p} \,\widetilde{\otimes}_M\, \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}^{\|\cdot\|_E}$$

for all BL-filtrations (S_i) on $L^p(\mu)$ and for all BL-filtrations (T_i) on E. In particular, if $S_i = id_{L^p(\mu)}$ and $T_i = id_E$ for each $i \in \mathbb{N}$, then $L^p(\mu) \,\widetilde{\otimes}_{^tM}\, E \cong L^p(\mu) \,\widetilde{\otimes}_M\, E$. But then E is Riesz and isometrically isomorphic to an L^p-space.

(ii) $\Rightarrow$ *(ii′)* This implication is trivial.

(ii′) $\Rightarrow$ *(i)* This implication follows in a similar way to the proof of *(iii)* $\Rightarrow$ *(i)*. ■

4.6 Continuous Duals

The preceding result enables us to describe the continuous duals of

$$\mathcal{M}_{nc}\Big((S_i \otimes_l T_i), E \,\widetilde{\otimes}_l\, X\Big) \text{ and } \mathcal{M}_{nc}\Big((T_i \otimes_M S_i), X \,\widetilde{\otimes}_M\, E\Big).$$

For E a Banach lattice and Y a Banach space, recall from [29, Ch.4,§3] that a linear map $T : E \longrightarrow Y$ is called *cone absolutely summing* if for every positive summable sequence (x_n) in E we have that the sequence (Tx_n) is absolutely summable in Y.

Then $L^{\text{cas}}(E,Y) = \{T : E \to Y \mid T \text{ is cone absolutely summing}\}$ is a Banach space with respect to the norm defined by

$$\|T\|_{\text{cas}} = \sup\left\{\sum_{i=1}^{n} \|Tx_i\| \;\middle|\; x_1,\ldots,x_n \in E_+,\ \Big\|\sum_{i=1}^{n} x_i\Big\| = 1,\ n \in \mathbb{N}\right\}$$

for all $T \in L^{\text{cas}}(E,Y)$. It is well known that the norm on $E \otimes Y$ induced by the norm $||\cdot||_{\text{cas}}$ on $L^{\text{cas}}(E',Y)$ is the l-norm, $(E \widetilde{\otimes}_l Y)'$ is isometrically isomorphic to $L^{\text{cas}}(E,Y')$ and $(E \widetilde{\otimes}_l F)'$ is Riesz and isometrically isomorphic to $L^{\text{cas}}(E,F')$ (cf. [1], [29] and [3]).

For X a Banach space and F a Banach lattice, a linear map $T : X \to F$ is called *majorizing* if for every null sequence (x_i) in X, $(|Tx_i|)$ is a majorized sequence in F. Then $L^{\text{maj}}(X,F) = \{T : X \to F \mid T \text{ is majorizing}\}$ is a Banach space with respect to the norm defined by

$$\|T\|_{\text{maj}} = \sup\left\{\|\sup_i |Tx_i|\| \;\middle|\; x_1 \ldots, x_n \in E_+,\ \|x_i\| \le 1, n \in \mathbb{N}\right\}$$

for each $\in L^{\text{maj}}(X,F)$. It is well known that the norm on $X \otimes F$ induced by the norm $||\cdot||_{\text{maj}}$ on $L^{\text{maj}}(X',F)$ is the M-norm of Chaney (= m-norm of Schaefer),$(X \widetilde{\otimes}_M F)'$ is isometrically isomorphic to $L^{\text{maj}}(X,F')$ and $(F \widetilde{\otimes}_M E)'$ is Riesz and isometrically isomorphic to $L^{\text{maj}}(F,E')$(cf. [1], [29] and [3]).

Corollary 4.15 *Let E and F be Banach lattices and let X be a Banach space. If (S_i) is a BL-filtration on E, (T_i) is a BS-filtration on X and (Q_i) is a BL-filtration on F, then*

(i) $$\Big[\mathcal{M}_{nc}\Big((S_i \otimes_l T_i), E \widetilde{\otimes}_l X\Big)\Big]' \simeq L^{\text{cas}}\Big(\mathcal{M}_{nc}((S_i),E), \Big[\mathcal{M}_{nc}((T_i),X)\Big]'\Big).$$

(i′) $$\Big[\mathcal{M}_{nc}\Big((T_i \otimes_M S_i), X \widetilde{\otimes}_M E\Big)\Big]' \simeq L^{\text{maj}}\Big(\mathcal{M}_{nc}((T_i),X), \Big[\mathcal{M}_{nc}((S_i),E)\Big]'\Big).$$

(ii) $$\Big[\mathcal{M}_{nc}\Big((S_i \otimes_l Q_i), E \widetilde{\otimes}_l F\Big)\Big]' \cong L^{\text{cas}}\Big(\mathcal{M}_{nc}((S_i),E), \Big[\mathcal{M}_{nc}((Q_i),F)\Big]'\Big).$$

(ii′) $$\Big[\mathcal{M}_{nc}\Big((Q_i \otimes_M S_i), F \widetilde{\otimes}_M E\Big)\Big]' \cong L^{\text{maj}}\Big(\mathcal{M}_{nc}((Q_i),F), \Big[\mathcal{M}_{nc}((S_i),E)\Big]'\Big).$$

where $\cong$ denotes Riesz and isometrically isomorphic and $\simeq$ only the latter.

The following theorem is due to Chaney.

Theorem 4.16 *E has RNP iff $L^p(\mu) \widetilde{\otimes}_l E = L^{\text{cas}}(L^q(\mu),E)$ for all finite measure spaces (Ω,Σ,μ), $1 < p < \infty$ and $1/p + 1/q = 1$.*

It is well known that a Banach space X' has the Radon-Nikodým property if and only if $\Big[L^p(\mu, X)\Big]' = L^q(\mu, X')$, (see [4]) or equivalently, $L^{\text{cas}}(L^p(\mu), X') = L^q(\mu, X')$, (see [1]), where (Ω, Σ, μ) is any finite measure space, $1 < p < \infty$ and $1/p + 1/q = 1$.

We derive the following analogue for spaces of norm convergent martingales which we believe is a new result:

Proposition 4.17 *Let X be a Banach space, $1 < p < \infty$ and $1/p + 1/q = 1$. Then $\mathcal{M}_{nc}((T_i), X)'$ has the Radon-Nikodým property for any BS-filtration (T_i) on X if and only if*

$$\Big[\Big(\mathcal{M}_{nc}((S_i \otimes_l T_i), L^p(\mu, X))\Big)\Big]' = \Big[\mathcal{M}_{nc}((S_i), L^p(\mu))\Big]' \widetilde{\otimes}_l \Big[\mathcal{M}_{nc}((T_i), X)\Big]'$$

for any finite measure space (Ω, Σ, μ) and any BL-filtration (S_i) on $L^p(\mu)$ that leaves **1** *invariant.*

Proof: First note that since the image under L (notation as in Chapter 3) of $\mathcal{M}_{nc}(S_i, L^p(\mu))$ is a closed Riesz subspace of $L^p(\mu)$ and L maps $(S_i\mathbf{1}, S_i)$ to $\mathbf{1}$, it follows that $\mathcal{M}_{nc}(S_i, L^p(\mu))$ is a closed Riesz subspace of $L^p(\mu)$ and has a quasi-interior point. But then there exists a finite measure space $(\Omega_1, \Sigma_1, \mu_1)$ such that $\mathcal{M}_{nc}(S_i, L^p(\mu))$ is Riesz and isometrically isomorphic to $L^p(\mu_1)$ (cf. [22] and [23]).

Suppose $\Big[\mathcal{M}_{nc}((T_i), X)\Big]'$ has the Radon-Nikodým property for any BS-filtration (T_i) on X. Let (Ω, Σ, μ) be a probability space, (S_i) a BL-filtration on $L^p(\mu)$ that leaves the $\mathbf{1}$-function invariant and (T_i) a BS-filtration on X. Then

$$\Big[\mathcal{M}_{nc}((S_i), L^p(\mu))\Big]' \widetilde{\otimes}_l \Big[\mathcal{M}_{nc}((T_i), X)\Big]' = L^{\text{cas}}\Big(\mathcal{M}_{nc}((S_i), L^p(\mu)), \Big[\mathcal{M}_{nc}((T_i), X)\Big]'\Big)$$

and consequently

$$\Big[\mathcal{M}_{nc}((S_i), L^p(\mu))\Big]' \widetilde{\otimes}_l \Big[\mathcal{M}_{nc}((T_i), X)\Big]' = \Big[\mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu) \widetilde{\otimes}_l X\Big)\Big]'.$$

Conversely, suppose

$$\Big[\Big(\mathcal{M}_{nc}((S_i), L^p(\mu))\Big)\Big]' \widetilde{\otimes}_l \Big[\Big(\mathcal{M}_{nc}((T_i), X)\Big)\Big]' = \Big[\mathcal{M}_{nc}\Big((S_i \otimes T_i), L^p(\mu) \widetilde{\otimes}_l X\Big)\Big]'$$

for any finite measure space (Ω, Σ, μ) and any BL-filtration (S_i) on $L^p(\mu)$ that leaves $\mathbf{1}$ invariant. By our remark at the beginning of the proof, $\Big(\mathcal{M}_{nc}((S_i), L^p(\mu))\Big)$ is an $L^p(\mu_1)$-space, where $(\Omega_1, \Sigma_1, \mu_1)$ is a finite measure space, and

$$\Big[\mathcal{M}_{nc}((S_i), L^p(\mu))\Big]' \widetilde{\otimes}_l \Big[\mathcal{M}_{nc}((T_i), X)\Big]' = L^{\text{cas}}\Big(\mathcal{M}_{nc}((S_i), L^p(\mu)), \Big[\mathcal{M}_{nc}((T_i), X)\Big]'\Big).$$

Thus, by Chaney's result, $\Big[\mathcal{M}_{nc}((T_i), X)\Big]'$ has the Radon-Nikodým property for any BS-filtration (T_i) on X. ■

4.7 Norm Bounded Martingales and Tensor Products

The following is the main result of this section:

Theorem 4.18 *Let E and F be Banach lattices and let (T_i) and (S_i) be BL-filtrations on E and F respectively.*

(i) If E and F are KB-spaces, then $\mathcal{M}_{nb}((T_i \otimes S_i), E\widetilde{\otimes}_l F)$ is a Banach lattice and $\mathcal{M}_{nc}((T_i \otimes S_i), E\widetilde{\otimes}_l F)$ is a projection band in $\mathcal{M}_{nb}((T_i \otimes S_i), E\widetilde{\otimes}_l F)$.

(ii) If E and F are order continuous Banach lattices, then $\mathcal{M}_r((T_i \otimes S_i), E\widetilde{\otimes}_l F)$ is a Banach lattice and $\mathcal{M}_{nc}((T_i \otimes S_i), E\widetilde{\otimes}_l F)$ is a ideal in $\mathcal{M}_r((T_i \otimes S_i), E\widetilde{\otimes}_l F)$.

PROOF: Since E and F are KB-spaces [order continuous Banach lattices], $E\widetilde{\otimes}_l F$ is a KB-space [order continuous Banach lattice], by Popa's result. Because (T_i) and (S_i) are BL-filtrations on E and F respectively, $(T_i \otimes S_i)$ is a BL -filtration on $E\widetilde{\otimes}_l F$ and so $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes T_i)}^{l}$ is a closed Riesz subspace of $E\widetilde{\otimes}_l F$ (see Lemma 4.4).

But then $\mathcal{M}_{nc}((T_i \otimes S_i), E\widetilde{\otimes}_l F)$ is a projection band [ideal] in the Banach lattice $\mathcal{M}_{nb}((T_i \otimes S_i), E\widetilde{\otimes}_l F)$ [$\mathcal{M}_r((T_i \otimes S_i), E\widetilde{\otimes}_l F)$], by Theorem 3.22 [Corollary 3.16]. If E and F are KB-spaces, then $\mathcal{M}_{nb}((T_i \otimes S_i), E\widetilde{\otimes}_l F) = \mathcal{M}_r((T_i \otimes S_i), E\widetilde{\otimes}_l F)$, by Corollary 3.21. ■

Appendix A

Glossary of Definitions

We begin by recalling all the theory which is relevent to this dissertation, the majority of which is discussed in more detail in [31, Chapters 1-5].

A.1 Riesz Space Theory

Ordered Vector Space
A real vector space on which there exists a partial ordering such that the vector space structure and the order structure are compatible. That is for every, $g, h \in V$ the following hold:

(i) $f \leq g \Rightarrow f + h \leq g + h, \quad \forall h \in V.$

(ii) $f \geq 0 \Rightarrow \alpha f \geq 0, \quad \forall \alpha \in \mathbb{R}.$

Riesz Space
An ordered vector space on which a supremum and infimum, with respect to the partial ordering, of any pair of elements exists. We denote $\sup\{f, g\} = f \vee g$ and $\inf\{f, g\} = f \wedge g$.

In the remainder of the section **E is a Riesz space**.

Positive Cone
$E^+ = \{f \in E \mid f \geq 0\}$. $f \vee 0$ is the positive part of f and $f^- = (-f) \vee 0$ is the negetive part of f. Also $E = E^+ - E^+$ uniquely so that for any $f \in E$ we can write $f = f^+ - f^-$.

Archimedean Riesz Space
A Riesz space with the property that $\inf\limits_{n \in \mathbb{N}} \frac{1}{n} u = 0$ for each $u \in E^+$.

Riesz Subspace
A subset of a Riesz space which is itself a Riesz space.

Ideal
A linear subspace $S \subset E$ such that if $f \in S$ and $|g| \leq |f|$ then $g \in S$.

Principal Ideal
An ideal A_f generated by a single element $f \in E$. This is the smallest ideal containing f, and is given explicitly by $A_f = \{g \in E \mid |g| \leq |\alpha f|, \alpha \in \mathbb{R}\}$.

Band
An ideal $B \subset E$ for which $\sup(D) \in B$ for every subset $D \subset B$.

Principal Band
A band B_f generated by a single element $f \in E$. This is the smallest band containing f, which is also the smallest band containing A_f.

Strong Order Unit
An element $f \in E, f > 0$ for which $A_f = E$.

Weak Order Unit
An element $f \in E, f > 0$ for which $B_f = E$.

Quasi-Interior Point
An element $f \in E, f > 0$ for which A_f is dense in E.

Disjoint Complement
The set $B^d = \{f \in E \mid |f| \wedge |g| = 0 \text{ for each } g \in B\}$ with respect to a band B.

Upwards (Downwards) Directed Set
A non-empty subset, say $D \subset E$ such that for every two elements f and g in D there exists an element h in D such that $h \geq f \vee g$ ($h \leq f \wedge g$). We denote this as $D \uparrow$ ($D \downarrow$) and if $f_0 = \sup D$ ($f_0 = \inf D$) exists in E we shall write $D \uparrow f_0$ ($D \downarrow f_0$).

Dedekind Complete
A Riesz space for which every non-empty upwards directed bounded above subset of E^+ has a supremum.

Principal Projection Property
A Riesz Space E such that $E = B \oplus B^d$ for each principal band B in E.

Theorem A.1 *Let E be a Riesz space, then*

1. *E Dedekind complete $\Rightarrow$ E has the projection property $\Rightarrow$ E has the principle projection property $\Rightarrow$ E is Archimedean.*

2. *E Dedekind complete $\Rightarrow$ E Dedekind σ-complete $\Rightarrow$ E has the principle projection property $\Rightarrow$ E is Archimedean.*

Order Convergent Sequence
A sequence $(f_n) \subset E$ for which there exists a sequence $(u_n) \downarrow 0 \subset E$ and some f_∞ such that $|f_n - f_\infty| \leq u_n$. We call f_∞ the order limit of (f_n) and write $f_n \to f_\infty$ in order.

Order Bounded Sequence
A sequence $(f_n) \subset E$ for which there exists an element $u \in E^+$ such that $|f_n| \leq u$ for all n.

A.2 Norms and Banach Space Theory

Norm
A map $||\cdot||$ from a Riesz space E to $\mathbb{R}$ with the following properties:

(i) $f = 0$ if and only if $||f|| \geq 0$ and $||f|| = 0$ for each $f \in E$;

(ii) $||\alpha f|| = |\alpha|||f||$ for each $f \in E$ and $\alpha \in \mathbb{R}$;

(iii) $||f + g|| \leq ||f|| + ||g||$ for each $f, g \in E$.

In the remainder of the section $||\cdot||$ **is a norm.**

Riesz Norm
A norm for which one of the following equivalent properties holds for each $f, g \in E$

(iv) $f \leq g \Rightarrow ||f|| \leq ||g||$;

(v) $0 \leq f \leq g \Rightarrow ||f|| \leq ||g||$ and $||\,|f|\,|| = ||f||$.

Order Continuous Norm
A norm for which $\inf ||A|| = 0$ for each downward directed set A.

L - Norm
A norm for which $||f|| + ||g|| = ||f + g||$ for all positive elements f and g.

Normed vector Space
A linear vector space on which a suitable norm is defined.

In the remainder of the section **E is a normed Riesz Space.**

Cauchy Sequence
A sequence (f_n) in a normed vector space such that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $||f_n - f_m|| < \epsilon$ for all $m, n \geq N$.

Norm Convergence Sequence
A sequence (f_n) in a normed vector space for which there exists some f_∞, called the limit, such that $||f_n - f_\infty|| \to 0$. We write $f_n \to f_\infty$(norm).

Banach Space
A normed vector space in which every Cauchy sequence converges in norm.

Banach Lattice
A Riesz and Banach space endowed with a Riesz norm.

Order Continuous Space
A vector space endowed with an order continous norm.

AL - Space
A vector space endowed with an L-norm.

Norm Bounded Sequence
A sequence (f_n) on a normed vector space for which there exists some constant K such that $||f_n|| \leq K$ for each $n \in \mathbb{N}$.

KB - Space
A Banach lattice in which every increasing, norm bounded sequence is norm convergent.

Radon Nikodým Property (RNP)
A Banach space in which for every finite measure space (Ω, Σ, μ) and for each bounded linear operator $T : L^1(\mu) \to E$, there exists a $g \in L^\infty(\mu)$ such that $Tf = \int fgd\mu$ for each $f \in L^1(\mu)$.

Fatou Property
A locally solid Riesz space (E, τ) has the Fatou property if τ has a basis of zero consisting of solid and order closed sets.

A.3 Operator Theory

Linear Operator
A mapping between two vector spaces, say V and W, such that $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$.

For the remainder of the section **T is a linear operator**.

Positive Operator
A linear operator for which $f > 0$ implies $Tf > 0$.

Strictly Positive Operator
A linear operator for which $\{f \in E \mid T|f| = 0\} = 0$.

Order Continuous Operator
A positive operator such that for each set $f_\alpha \downarrow_{\alpha \in D} 0$ we have $T(f_\alpha) \downarrow_{\alpha \in D} 0$.

Norm Bounded Operator

A linar operator on a normed vector space V for which there exists a constant K such that $||Tf|| \leq K||f||$ for each $f \in V$.

Norm of an Operator
The infimum of all numbers K in the above definition, i.e

$$||T|| = \inf_{f \in V}\{K \mid ||Tf|| \leq K||f||\}.$$

Projection
An operator for which $TT = T^2 = T$.

Isometry
A linear operator for which $||Tf|| = ||f||$ for each f.

Range of an Operator
A vector subspace of X defined by $\mathcal{R}(T) = \{y \in Y \mid \exists\, x \in X \text{ so that } Tx = y\}$, where $T : X \to Y$. We call the dimension of this subspace the rank.

Kernel of an Operator
$\mathcal{N}(T) = \{x \in X \mid Tx = 0\}$. Furthermore, the kernel is a vector subspace of X the dimension of which is called the nullity.

Linear Functional
A linear operator for which the target space is $\mathbb{R}$.

Dual Space
The space of all bounded linear functionals on a normed vector space. For a normed vector space V we denote the dual space by $V' = \{f' \mid f' : V \to \mathbb{R}\}$.

Universally Complete with respect to T
A Dedekind complete Riesz space in which every increasing net $(f_\alpha) \subset E^+$ with Tf_α bounded, is order convergent. We require that T be a strictly positive contractive projection.

A.4 Measure Theory and L^p-spaces

We now present the classical L^p-space as an example of a Riesz space, in fact, it is an example of an order continuous Banach lattice for $1 \leq p < \infty$.

Let Ω denote a non empty point set.

Power Set $\mathcal{P}$
The set of all subsets, denoted for a set Ω, $\mathcal{P}(\Omega)$.

σ-Algebra
A subset of $\mathcal{P}(\Omega)$ call it $\mathcal{M}$ for which the following conditions hold:

(i) $\Omega \in \mathcal{M}$,

(ii) $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$ and

(iii) $\{A\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Measurable set
The elements of a σ-algebra.

Measure
A positive real valued set function $\mu : \mathcal{M} \to [0, \infty]$ defined on a measurable space $(\Omega, \mathcal{M})$ such that

(i) For every mutually disjoint collection $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{M}$ we have that $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

(ii) $\mu(\phi) = 0$ where ϕ is the empty set.

Measure Space
The triple $(\Omega, \mathcal{M}, \mu)$ where Ω is a point set, $\mathcal{M}$ is a σ-algebra and μ a measure.

Measurable Function with respect to the measure space $(\Omega, \mathcal{M}, \mu)$
A function $f : \Omega \to Y$, where Y is a topological space, whose inverse maps open sets to open sets. That is if U is open in Y then $f^{-1}(U)$ is measurable in Ω.

Finite Measure Space
A measure space $(\Omega, \mathcal{M}, \mu)$ for which $\mu(\Omega) < \infty$.

Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space and let $M(\Omega, \mathcal{M}, \mu)$ denote the set of all real-valued, μ-measurable functions on Ω. Then $M(\Omega, \mathcal{M}, \mu)$ becomes a real vector space under pointwise addition and scalar multiplication. Furthermore, we identify functions that differ only on a set of measure zero, i.e. $f \sim g$ if and only if $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$ for all $f, g \in M(\Omega, \mathcal{M}, \mu)$. In this case we shall write $f = g$ a.e.$[\mu]$.

In the remainder of the section $(\Omega, \mathcal{M}, \mu)$ **is a finite measure space**.

$\mathbf{L^0}(\boldsymbol{\Omega}, \mathcal{M}, \mu)$
The the collection of equivalence classes of a.e.$[\mu]$ equivalent functions in $M(\Omega, \mathcal{M}, \mu)$. If there is no confusion, we shall simply use the shorthand notation $L^0(\mu)$.

$L^0(\Omega, \mathcal{M}, \mu)$ is a real vector space under $[f] + [g] = [f + g]$ and $\alpha[f] = [\alpha f]$ and is called the space of μ-measurable functions. We treat the elements of $L^0(\Omega, \mathcal{M}, \mu)$ as functions rather than equivalence classes and denote the space by $L^0(\Omega, \mathcal{M}, \mu)$.

Charactaristic Function $\chi_{\mathbf{A}}$
The function defined by $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.

Simple Function
A finite linear combination of charactaristic functions. Thus $s : \Omega \to \mathbb{R}$ is simple iff s is measurable and $\mathcal{R}(f) = \{\alpha_1, \ldots, \alpha_n\}$ such that

$$s(x) = \sum_{n=1}^{n} \alpha_n \chi_{A_n}(x) \quad \text{where} A_n = s^{-1}(\alpha_n).$$

Note that s is measurable if and only if the A_n's are measurable sets.

Define order on $L^0(\mu)$ pointwise, i.e. $f \leq g \Leftrightarrow f(x) \leq g(x)$ for all $x \in \Omega$. Then $L^0(\mu)$ is a Riesz space under the lattice operations $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$ for all $f, g \in L^0(\mu)$.

Integral with respect to μ
The number

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

defined for a simple measurable function $s : \Omega \to [0, \infty)$.

Lebegue Integral with respect to μ
The number

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu : 0 \leq s \leq f,\ s \text{ a simple function} \right\}.$$

defined for a measurable function $f : \Omega \to [0, \infty]$.

Integrable Function
A function $f : \Omega \to [-\infty, \infty]$ for which $\int_\Omega |f| d\mu < \infty$.

We are now in a position to define the L^p-spaces where $1 \leq p \leq \infty$.

$|| \cdot ||_p$ **Norm where $\mathbf{1 \leq p < \infty}$**
The function $\| \cdot \|_p : L^0(\Omega, \mathcal{M}, \mu) \to \mathbb{R}_+$ defined by

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu \right)^{1/p}.$$

$\mathbf{L^p(\Omega}, \mathcal{M}, \mu)$ **where $\mathbf{1 \leq p < \infty}$**
The Lebesgue space

$$L^p(\Omega, \mathcal{M}, \mu) = \{f \in L^0(\Omega, \mathcal{M}, \mu) : \|f\|_p < \infty\}$$

of integrable functions. When unambiguous use the notation $L^p(\mu)$.

$\|\cdot\|_\infty$ Norm
The function $\|\cdot\|_\infty : L^0(\Omega, \mathcal{M}, \mu) \to \mathbb{R}_+$ defined by

$$\|f\|_\infty = \inf\{M > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > M\}) = 0\}.$$

$\mathbf{L}^\infty(\boldsymbol{\Omega}, \mathcal{M}, \mu)$
The Lebesgue space

$$L^\infty(\Omega, \mathcal{M}, \mu) = \{f \in L^0(\Omega, \mathcal{M}, \mu) : \|f\|_\infty < \infty\}.$$

of essentially bounded functions. When unambiguous we use the notation $L^\infty(\mu)$.

Notice that if $p = 1$ in the above definition, then $L^p(\mu) = L^1(\mu)$ is just the vector space of integrable functions on Ω.

We recall here that each L^p-space $1 \le p \le \infty$ is a Banach lattice with respect to the norms defined above.

Vector-Valued $\mathbf{L}^\mathbf{p}$−spaces
If we change the target space from $\mathbb{R}$ to a more general Banach space, say X, then we can define the space $L^p(\Omega, \mathcal{M}, \mu, X)$ of Vector valued functions in a similar way.

A.5 Tensor Products of Banach Spaces

We now introduce some concepts from a seemingly unrelated field of mathematics, but it turns out that these notions are useful for describing the vector-valued L^p-spaces introduced in the previous section.

Bilinear Map
A map $\varphi : X \times Y \to Z$ which is linear in both components, i.e. the following hold

(i) $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z)$ for each $\alpha, \beta \in \mathbb{R}$, $x, y \in X$ and $z \in Y$;

(ii) $\varphi(x, \gamma y + \eta z) = \gamma\varphi(x, y) + \eta\varphi(x, z)$ for each $\gamma, \eta \in \mathbb{R}$, $x \in X$ and $y, z \in Y$.

We write $B(X \times Y, Z)$ for the vector space of all bilinear mappings from $X \times Y$ into Z. If Z is $\mathbb{R}$, then we just write $B(X \times Y)$.

Tensor Product of Banach Spaces
The vecotor subspace $X \otimes Y$ of $B(X \times Y)^\#$ generated by the set

$$\{\psi \in B(X \times Y)^\# : \exists (x, y) \in X \times Y \text{ such that } \psi(f) = f(x, y) \text{ for each } f \in B(X \times Y)\}.$$

Tensor
An element $u \in X \otimes Y$, denoted $x \otimes y$ and is of the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in X$, $y_i \in Y$ and $i = 1, \dots n$.

It is easy to see that the map $\otimes : X \times Y \to X \otimes Y$, defined by $(x, y) \mapsto x \otimes y$, is bilinear and thus exhibits properties of a multiplication.

Rank of a Tensor
The smallest number $n \in \mathbb{N}$ for which the tensor say u can be written in the form $u = \sum_{i=1}^{n} x_i \otimes y_i$. A tensor of rank one (i.e. $u = x \otimes y$) is called an elementary tensor. Note that $0 \otimes y = x \otimes 0 = 0$ and has rank zero.

A.6 Vector valued L^p-spaces

The classical L^p-spaces of real valued funtions can be generalized to functions which take on values in Banach spaces.

Let $(\Omega, \mathcal{M}, \mu)$ denote a finite measure space (μ is a non-negative real measure), Y denote a Banach space and $f : \Omega \to Y$ denote a function on Ω taking on values in Y. The material in this section is taken from [4, 2].

Simple Vector Function
A function $s : \Omega \to Y$ for which there exist $x_1, x_2, \ldots, x_n \in Y$ and $A_1, A_2, \ldots, A_n \in \mathcal{M}$ disjoint such that $s = \sum_{i=1}^{n} x_i \chi_{A_i}$, where χ_{A_i} denotes the characteristic function of A_i.

Measurable Vector Function with respect to μ
A function $f : \Omega \to Y$ for which there exists a sequence of simple vector functions (s_n) such that $\lim_{n\to\infty} \|s_n - f\| = 0$ a.e.$[\mu]$.

$\mathbf{L^0(\Omega, \mathcal{M}, \mu, Y)}$
The space of a.e.$[\mu]$ equivalent measurable functions. When there is no ambiguity we will simply write $L^0(\mu, Y)$.

Integral
The number $\int_E s \, d\mu = \sum_{i=1}^{n} x_i \mu(A_i \cap E)$, where $E \in \mathcal{M}$, defined for a simple function $s : \Omega \to Y$.

Bochner Integrable Function
A measurable function $f : \Omega \to Y$ for which there exists a sequence (s_n) of Y valued step-functions such that $\lim_{n\to\infty} \int_\Omega \|s_n - f\| \, d\mu = 0$.

Bochner Integral
The number $\int_E f \, d\mu = \lim_{n\to\infty} \int_E s_n \, d\mu$, where $E \in \mathcal{M}$, defined for a Bochner integrable function f. A measurable function $f : \Omega \to Y$ is Bochner integrable if and only if $\int_\Omega \|f\| \, d\mu < \infty$.

Δ_p Norm where $1 \le p < \infty$
The function $\Delta_p : L^0(\Omega, \mathcal{M}, \mu, Y) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\Delta_p(f) = \left(\int_\Omega \|f\|^p d\mu \right)^{1/p}.$$

$L^p(\Omega, \mathcal{M}, \mu, Y)$ where $1 \le p < \infty$
The Bochner space of integrable functions $f \in L^0(\Omega, \mathcal{M}, \mu, Y)$ for which $\Delta_p(f) < \infty$. When unambiguous we use the notation $L^p(\mu, Y)$.

Δ_∞ Norm
The function $\Delta_\infty : L^0(\Omega, \mathcal{M}, \mu, Y) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\Delta_\infty(f) = \inf\{M > 0 : \mu(\{\omega \in \Omega : \|f(\omega)\| > M\}) = 0\}.$$

$L^\infty(\Omega, \mathcal{M}, \mu, Y)$
The Bochner space of essentially bounded functions $f \in L^0(\Omega, \mathcal{M}, \mu, Y)$ for which $\Delta_\infty(f) < \infty$. When unambiguous we use the notation $L^\infty(\mu, Y)$.

Notice that since the Banach space Y has no order structure, we cannot endow $L^p(\mu, Y)$ with a pointwise ordering as in the scalar case; thus, we only have that $L^p(\mu, Y)$ is a Banach space.

Bibliography

[1] J. CHANEY, Banach lattices of compact maps, *Math. Z.* **129** (1972), 1–19.

[2] S. F. CULLENDER, *Properties of Generalized Vector-valued L^p-spaces,* MSc Dissertation, University of the Witwatersrand, Johannesburg, 2004.

[3] J. DIESTEL, H. JARCHOW, A. TONGE, *Absolutely summing operators*, Cambridge studies in advanced mathematics : 43, Cambridge University Press, 1995.

[4] J. J. DIESTEL, J. UHL, *Vector Measures*, American Mathematical Society, 1977.

[5] P. G. DODDS, C. B. HUIJSMANS, B. DE PAGTER, Charaterizations of conditional expectation-type operators, *Pacific J. Math.,* **141** (1990), 55-77.

[6] R.G DOUGLAS, *Contractive Projections on an L_1 Space,* Pacific J. Math, 1965.

[7] G. A. EDGAR, L. SUCHESTON, Martingales in the limit and amarts, *Proc. Amer. Math. Soc.,* **67** (1977), 315- 320.

[8] D.H. FREMLIN, Tensor products of Archimedean vector lattices, *Amer. J. Math.* **94** (1972), 778–798.

[9] D. H. FREMLIN, On the completion of locally solid vector lattices, *Pacific J. Math.* **43** (1972), 341-347.

[10] D.H. FREMLIN, Tensor products of Banach lattices, *Math. Ann.* **212** (1974), 87–106.

[11] D. H. FREMLIN, On the completion of locally solid vector lattices, *Pacific J. Math.* **43** (1972), 341-347.

[12] N. GHOUSSOUB, Riesz-space-valued measures and processes, *Bull. Soc. Math. France,* **110** (1982), 233-257.

[13] J. J. GROBLER, B. DE PAGTER, Operators representable as multiplication-conditional expectation operators, *J. Operator Theory,* **48** (2002), 15-40.

[14] G.A.M. Jeurnink, Integration of functions with values in a Banach lattice, Thesis, University of Nijmegen, The Netherlands, 1982.

[15] I. Karatzas, S Shreve, *Brownian Motion and Stochastic Calculus,* Springer Verlag, 1991.

[16] W.-C. Kuo, *Stochastic Processes on Riesz Spaces,* MSc Dissertation, University of the Witwatersrand, Johannesburg, 2004.

[17] W.-C. Kuo, C. C. A. Labuschagne, B. A. Watson, Conditional Expectations on Riesz Spaces, *J. Math. Anal. Appl.*, in press.

[18] W.-C. Kuo, C. C. A. Labuschagne, B. A. Watson, Convergence of Riesz Space Martingales, *preprint.*

[19] W.-C. Kuo, C. C. A. Labuschagne, B. A. Watson, Discrete-time stochastic processes on Riesz spaces, *Indag. Math. (N.S.),* **15** (2004), 435–451.

[20] W.-C. Kuo, C. C. A. Labuschagne, B. A. Watson, Riesz space and fuzzy upcrossing theorems, In: *Proceedings of the Conference on Soft Methods in Probability and Statistics* 2004, *Soft Methodology and Random Information Systems*, Series: Advances in Soft Computing, Springer-Verlag, 101–108, 2004.

[21] C.C.A. Labuschagne, Characterizing the one-sided tensor norms Δ_p and ${}^t\Delta_p$, *Quaestiones Math.*, **27** (2004), 339–363.

[22] C.C.A. Labuschagne, Equivalence of the Bochner norm and its transpose characterizes L^p-spaces, *J. Math. Anal. Appl.*, in press.

[23] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces II,* Springer Verlag, 1979

[24] P. Meyer-Nieberg, *Banach Lattices*, Springer Verlag, 1991.

[25] N. Popa, *Die Permanenzeigenschaften der Tensorprodukte von Banachverbänden*, Romanian-Finnish Seminar on Complex Analysis (Proc., Bucharest, 1976), Lecture Notes in Mathematics **743**, 627–647, Springer Verlag, 1979.

[26] M.M. Rao, Conditional Measures and Operators, *J. of Multivariate Anal.,* **5** (1975), 330-413.

[27] M.M. Rao, Two Characterizations of Conditional Probability, *Proc. Amer. Math. Soc.,* **19** (1976), 75-80.

[28] S.I. Resnick, *A probability path*, Birkhäuser, 2001.

[29] H. H. Schaefer, *Banach Lattices and Positive Operators,* Springer Verlag, 1974.

[30] V. Troitsky, Martingales in Banach lattices, *preprint.*

[31] A. C. Zaanen, *Introduction to Operator Theory in Riesz Space,* Springer Verlag, 1997.