The Analysis of PDEs arising from the Korteweg-de Vries Hierarchies

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WITWATERSRAND,
JOHANNESBURG

2021

DECLARATION

Except when appropriate references have been included, I declare that the contents of this dissertation are original. It is being submitted for a Master of Science degree at the University of Witwatersrand in Johannesburg. It had never been submitted to any other university for a degree or examination.

R Maphanga

Signed at Johannesburg on the 1^{st} day of October 2021.

Publications

Details of contributions to publications that form part of the research presented in this dissertation are:

R. Maphanga and S. Jamal, A conservation law approach to solving KdV hierarchies, submitted, 2021.

Abstract

In this dissertation, we study the hierarchy commonly defined as an infinite sequence of partial differential equations which begins with the Korteweg-de Vries equation and its modified version. We look at how Lie point symmetries and conservation laws of each of these hierarchies aid the solving of higher order partial differential equations through associations and transformations. An important feature of these hierarchies is their highly nonlinear property. In this regard, obtaining solutions for the members of these hierarchies poses a great problem, where in the past, it was impossible to calculate solutions. In this study, we establish a method to allow for the construction of new solutions to the full hierarchy. We begin by determining point symmetries and conserved vectors of each hierarchy, then proceed to testing association between the obtained symmetries and conserved vectors, and finally using transformations to construct solutions.

Keywords: KdV and mKdV Equations, Conserved Vectors, Lie Symmetries, Nonlinear Equations.

ACKNOWLEDGEMENTS

I would like to thank my supervisor Prof. Sameerah Jamal for providing guidance and feedback on time throughout my Masters project. I would also like to thank Usaamah Obaidullah (PhD candidate) for his help with some of the licensed software I didn't have access to. Lastly, I would like to thank the Gauteng City Region Academy (GCRA) for the funding. Honestly, this wouldn't have been possible if it wasn't for the help received from the aforementioned folks - much appreciated.

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Chapter 1

Introduction

The Korteweg-de Vries (KdV) equation, accredited to the Dutch Mathematicians Diederik Korteweg and Gustav de Vries (1895), is a mathematical model of waves on shallow water surfaces [1]. It was originally derived to govern or model the propagation of weakly dispersive nonlinear water waves and serve as a model equation for any physical system in consideration, for which the dispersion relation for frequency against wave number is approximated by

$$\frac{\omega}{k} = c_0 \left(1 - \beta \left(\frac{2\pi}{T} \right)^2 \right) \tag{1}$$

and the nonlinearity is weak and quadratic [1, 2].

A more generalised definition of the KdV equation would be that $u(x,t)$ is said to satisfy

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{2}$$

where α and β are arbitrary constants, and $u(x,t)$ is the function that denotes the elongation of the wave at space x and time t [2].

The origin of the KdV equation has a long history. It began with the experiments of Scott-Russell in 1834 when he was studying the "Wave of Translation", followed by the investigations of Joseph Valentine Boussinesq and Lord Rayleigh in 1870 when they were studying the "Theory of the solitary wave", and finally ending with an article by Diederik Korteweg and Gustav de Vries in 1895 [1, 3]. To this day, the KdV equation is still considered as one of the most important nonlinear partial differential equations, as it has a wide variety of applications [1, 2, 3].

If the nonlinearity is of degree $n+1$, the nonlinear term uu_x in equation (2) must be replaced by $u^n u_x$, and the most important case other than when $n = 1$, is when $n = 2$. This yields the so-called modified Korteweg-de Vries (mKdV) equation, which is given by [4, 5]:

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0. \tag{3}$$

The KdV equation naturally extends to an infinite sequence of integrable nonlinear partial differential equations of solitonic characters [6]. This is known as the KdV hierarchy, where $\alpha = 6$ and $\beta = -1$ as the first member of the hierarchy.

The KdV hierarchies are of great significance and play crucial roles in Mathematical Physics, to study physical systems governing nonlinear water waves, and to study some of the necessary material properties needed to manipulate waves in a desired manner [7]. One of the important features of the KdV Hierarchy is that it can also be used in Fluid Dynamics and Optics, to allow the reduction of the reflexive wave from an interface between two media of differing refractive indices [7, 8].

The modified KdV hierarchy is related to the KdV hierarchy through the Miura transformation, which maps the solutions of the KdV equations to the solutions of mKdV equations [9]. That is,

$$\mathcal{M} : u \to v, \tag{4}$$

which reads

$$u = v_x + v^2, \tag{5}$$

where u is the solution of the KdV equation and v is the solution of the mKdV equation. Both are soliton equations and are nonlinear [9].

1.1 Outline of Chapters

Chapter 2 provides the notation and theory that we utilize to determine point symmetries and conservation laws.

Chapter 3 highlights the general definitions of both the KdV and mKdV hierarchies, along with their point symmetries, Lie brackets of the points symmetries and conservation laws.

Chapter 4 outlines the solutions of the hierarchies obtained in Chapter 3, by application of the symmetries and conservation laws. It also highlights some of the important theorems and lemmas developed to aid the solving of these hierarchies, along with their 2-D and 3-D plots.

Chapter 2

Notation and Theory

In this chapter, we examine the Lie point symmetries and the conservation laws of partial differential equations, which are, in this regard, the primary building blocks of this project.

2.1 Lie Point Symmetries of Differential Equations

The approach for determining point symmetries for an arbitrary system of equations is highlighted as follows [10].

Definition 2.1.1 *Consider q unknown functions u^α which depend on p independent variables x^i, i.e. $u = (u^1, \ldots, u^q)$, $x = (x^1, \ldots, x^p)$, with indices $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$. Let*

$$G_\alpha \left(x, u^{(k)} \right) = 0, \tag{6}$$

be a system of nonlinear differential equations, where $u^{(k)}$ *represents the* k^{th} *derivative of* u *with respect to* x.

Definition 2.1.2 *A one-parameter Lie group of transformations (*ϵ *as the group parameter) that is invariant under (6) is given by*

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \tag{7}$$

Definition 2.1.3 *Invariance of (6) under the transformation (7) implies that any solution* $u = \Theta(x)$ *of (6) maps into another solution* $v = \Psi(x; \epsilon)$ *of (6). Expanding (7) around the identity* $\epsilon = 0$*, generates the following infinitesimal transformations:*

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon \xi^i(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u}^\alpha &= u^\alpha + \epsilon \eta^\alpha(x, u) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{8}$$

Since the Lie group's action can be obtained from that of its infinitesimal generators acting on the space of the dependent and independent variables, we analyze the following vector field:

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{9}$$

Definition 2.1.4 *The infinitesimal criterion for invariance is given by*

$$X\left[G_\alpha\left(x, u^{(k)}\right)\right] = 0, \quad \textit{when} \quad G_\alpha\left(x, u^{(k)}\right) = 0, \tag{10}$$

where X *is extended to all derivatives appearing in the equation through an appropriate prolongation.*

Definition 2.1.5 *The operator in Eq. (9) can be used to define the Lagrange system*

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whose solution provides the zero-order invariants

$$W^{[0]}(x^i, u^\alpha). \tag{11}$$

These invariants can be used in order to reduce the order of the PDE.

2.2 Conservation Laws

Definition 2.2.1 *The total derivative operator D_i with respect to x^i is given by*

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n. \tag{12}$$

Definition 2.2.2 *The Euler operator or variational derivative is given by*

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}. \tag{13}$$

Definition 2.2.3 *A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies*

$$D_i T^i = 0, \tag{14}$$

along the solutions of (6). Eq. (14) is called a local conservation law.

Definition 2.2.4 *The multipliers Λ^α of (6) satisfy the relation [11]*

$$D_i T^i = \Lambda^\alpha G^\alpha, \tag{15}$$

for all functions u^α and the overdetermined equations for Λ^α are

$$\frac{\delta}{\delta u^\alpha}\Big[\Lambda^\alpha G^\alpha\Big] = 0. \tag{16}$$

To solve the KdV equation under study, we first check if its symmetries and conservation laws are associated. Association between symmetries and conservation laws is defined as follows:

Definition 2.2.5 *Suppose that X is a symmetry of the system (6) and T is a conserved vector of (6). Then if X and T satisfy*

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \tag{17}$$

the symmetry X is said to be associated with T [12].

Remark 2.2.1 *If the above condition is not met (meaning no association), then we cannot continue solving using this X and T combination.*

Chapter 3

KdV Hierarchies - Symmetries and Conserved Vectors

Several studies have been conducted on differential equations at large. It has been found and documented that conservation laws have many remarkable uses, specifically with regard to constants of motion, integrability and linearization, analysis of solutions, and numerical solution methods [13]. However, the challenge is then how to calculate those necessarily conservation laws for the PDE under study. This concept shall form the basis of our approach as we consider the general KdV hierarchy equation (18). The hierarchy is defined from all $n \geq 1$, with $n = 1$ being the KdV equation.

Below are the cases of the KdV hierarchy equation when $n = 1$, $n = 2$, along with their symmetries and conservation laws.

3.1 The KdV Hierarchy

The general KdV hierarchy is given by

$$u_t = (D_x^2 - 4u - 2u_x D_x^{-1})^n u_x, \tag{18}$$

where D_x denotes the total derivative and D_x^{-1} denotes the integral with respect to x. For $n = 1$, we have the famous KdV equation

$$u_t = u_{xxx} - 6uu_x. \tag{19}$$

The second member of the hierarchy is given when $n = 2$ in (18), viz.

$$u_t = u_{xxxxx} - 20u_x u_{xx} - 10uu_{xxx} + 30u^2 u_x. \tag{20}$$

Note that for higher-order n, the equations become increasingly nonlinear and higher-order in derivatives.

3.1.1 Point Symmetries

Suppose that the Lie point symmetry is of the general form

$$X = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \eta(x,t,u)\partial_u,$$

if we impose definition (10) on equation (19) and solve, we get the following system of determining equations

$$\xi^1_u = 0,$$
$$\xi^2_u = 0,$$
$$\eta_{u,u} = 0,$$
$$\xi^2_x = 0,$$
$$-3\eta_{x,x,u} + \xi^1_{x,x,x} - \xi^1_t + 6u\xi^2_t - 6u\xi^1_x + 6\eta = 0,$$
$$-\eta_{x,x,x} + \eta_t + 6u\eta_x = 0,$$
$$\eta_{x,u} - \xi^1_{x,x} = 0,$$
$$3\xi^1_x - \xi^2_t = 0,$$

for the symmetry coefficients $\xi^1(x,t,u), \xi^2(x,t,u)$ and $\eta(x,t,u)$. Solving this system shows that equation (19) has the following four symmetries:

$$X_1 = \partial_t, \tag{21}$$

$$X_2 = 6t\partial_x + \partial_u, \tag{22}$$

$$X_3 = \partial_x, \tag{23}$$

$$X_4 = 3t\partial_t - 2u\partial_u + x\partial_x. \tag{24}$$

The commutator table (Lie brackets for all the four symmetries) $[A, B] = AB - BA$, is given in Table 3.1.

Equation (20) has the following three symmetries:

$$X_1 = \partial_t, \tag{25}$$

$$X_2 = \partial_x, \tag{26}$$

$$X_3 = 5t\partial_t + x\partial_x - 2u\partial_u, \tag{27}$$

whose Lie brackets appear in Table 3.2.

Table 3.1: Lie brackets of (19)

[,]	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	0	$-3X_1$
X_2	X_3	0	0	$2X_2$
X_3	0	0	0	$-X_3$
X_4	$3X_1$	$-2X_2$	X_3	0

Table 3.2: Lie brackets of (20)

[,]	X_1	X_2	X_3
X_1	0	0	$-\frac{X_1}{5}$
X_2	0	0	$-X_2$
X_3	$\frac{X_1}{5}$	X_2	0

If we perform the Lie symmetry method again, but considering higher members, we can easily see that the hierarchy (18) have the following Lie point symmetries

$$Y_1 = \partial_t,\ Y_2 = \partial_x,\ X_1^n = (2n+1)\, t\partial_t - 2u\partial_u + x\partial_x, \tag{28}$$

for $n \geq 2$ with Lie bracket relations in Table 3.3.

3.1.2 Conservation Laws

Consider the definitions from the previous chapter, for us to calculate the conserved vector, we use the multiplier approach. Therefore, if (T^t, T^x) is a conserved vector

Table 3.3: Lie brackets of symmetries (28)

[,]	Y_1	Y_2	X_1^n
Y_1	0	0	$(2n+1)Y_1$
Y_2	0	0	X_2
X_1^n	$-(2n+1)Y_1$	$-Y_2$	0

of a conservation law, then

$$D_t T^t + D_x T^x = 0,$$

along the solutions of the differential equation (Eq. (19) in this case). Moreover, if there exists a nontrivial differential function $\Lambda(x, t, u)$, called a multiplier, such that

$$\Lambda(u_t - u_{xxx} + 6uu_x) = D_t T^t + D_x T^x, \tag{29}$$

and

$$\frac{\delta}{\delta u}(\text{LHS of (29)}) = 0,$$

where $\frac{\delta}{\delta u}$ is the standard Euler operator which annihilates divergence expressions, then we may obtain the conservation law.

In each case, T^t is the conserved density and T^x is the conserved flux. The same approach is used in subsequent cases of this dissertation. Below are the cases of the KdV hierarchy equation when $n = 1$, $n = 2$, where the conservation laws are $T_i = (T_i^t, T_i^x)$ where $i = 1, 2, 3$.

By the definitions stated above, Eq. (19) has the determining system for conservation laws as:

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\Lambda\left(x,t,u\right) &= 0,\\
\frac{\partial^2}{\partial u \partial t}\Lambda\left(x,t,u\right) &= \frac{\frac{\partial}{\partial t}\Lambda\left(x,t,u\right)}{u},\\
\frac{\partial^2}{\partial u^2}\Lambda\left(x,t,u\right) &= 0,\\
\frac{\partial}{\partial x}\Lambda\left(x,t,u\right) &= -1/6\,\frac{\frac{\partial}{\partial t}\Lambda\left(x,t,u\right)}{u},
\end{aligned} \tag{30}
$$

which solves to give (C_i are constants),

$$\Lambda\left(x,t,u\right) = \frac{1}{6}\,\left(6\,tu - x\right)C_1 + C_2\,u + C_3,$$

so that we have the following three conservation laws: The multiplier $\Lambda(x,t,u) = 1$, yields the conserved vector

$$T_1^t = u, \quad T_1^x = 3u^2 - u_{xx}, \tag{31}$$

and $\Lambda(x,t,u) = u$ provides the conserved vector

$$T_2^t = \frac{u^2}{2}, \quad T_2^x = 2u^3 + \frac{1}{2}u_x^2 - uu_{xx}, \tag{32}$$

and finally, $\Lambda(x,t,u) = tu - \frac{1}{6}x$ gives us the conservation law

$$T_3^t = \frac{1}{2}tu^2 - \frac{x}{6}u, \quad T_3^x = -\frac{u_x}{6} - tuu_{xx} + 2tu^3 - \frac{x}{2}u^2 + \frac{t}{2}u_x^2 + \frac{x}{6}u_{xx}. \tag{33}$$

Equation (20) has the following two conservation laws:

$$T_1^t = u, \quad T_1^x = -10u^3 + 10uu_{xx} + 5u_x^2 - u_{xxxx}, \tag{34}$$

found from multiplier $\Lambda(x,t,u) = 1$, and from $\Lambda(x,t,u) = u$, we have

$$T_2^t = \frac{u^2}{2}, \quad T_2^x = -\frac{15}{2}u^4 + 10u^2u_{xx} - uu_{xxxx} + u_xu_{xxx} - \frac{1}{2}u_{xx}^2. \tag{35}$$

3.2 The Modified KdV Hierarchy

The general mKdV hierarchy is given by:

$$v_t = (D_x^2 - 4v^2 - 4v_x D_x^{-1} v)^n v_x, \tag{36}$$

where once again, D_x denotes the total derivative and D_x^{-1} denotes the integral with respect to x. For $n = 1$, we have the mKdV equation,

$$v_t = v_{xxx} - 6v^2 v_x, \tag{37}$$

and the case of $n = 2$, gives the second member of the hierarchy,

$$v_t = 30v_x v^4 - 10v^2 v_{xxx} - 40v v_x v_{xx} - 10v_x^3 + v_{xxxxx}. \tag{38}$$

3.2.1 Point Symmetries

Suppose that the Lie point symmetry is of the general form

$$X = \xi^1(x,t,v)\partial_x + \xi^2(x,t,v)\partial_t + \eta(x,t,v)\partial_v,$$

then solving (10) gives a system of determining equations for the symmetry coefficients $\xi^1(x,t,v), \xi^2(x,t,v)$ and $\eta(x,t,v)$. Solving this system shows that equation (37) has the following three symmetries:

$$X_1 = \partial_x, \tag{39}$$

$$X_2 = \partial_t, \tag{40}$$

$$X_3 = \frac{x}{3}\partial_x + t\partial_t - \frac{v}{3}\partial_v, \tag{41}$$

and Lie brackets are presented in Table 3.4.

Table 3.4: Lie brackets of (37)

[,]	X_1	X_2	X_3
X_1	0	0	$-\frac{X_1}{3}$
X_2	0	0	$-X_2$
X_3	$\frac{X_1}{3}$	X_2	0

Eq. (38) has the following three symmetries:

$$X_1 = \partial_x, \tag{42}$$

$$X_2 = \partial_t, \tag{43}$$

$$X_3 = \frac{x}{5}\partial_x + t\partial_t - \frac{v}{5}\partial_v. \tag{44}$$

The corresponding Lie brackets are presented in Table 3.5.

Table 3.5: Lie brackets of (38)

[,]	X_1	X_2	X_3
X_1	0	0	$-\frac{X_1}{5}$
X_2	0	0	$-X_2$
X_3	$\frac{X_1}{5}$	X_2	0

Similarly, if we perform the Lie symmetry method once again, but considering higher members, we can easily see that the hierarchy (36) have the following Lie point symmetries

$$Y_1 = \partial_t,\ Y_2 = \partial_x,\ X_2^n = t\partial_t - \frac{v}{2n+1}\partial_v + \frac{x}{2n+1}\partial_x, \tag{45}$$

for $n \geq 2$ with Lie bracket relations in Table 3.6.

Table 3.6: Lie brackets of symmetries (45)

[,]	Y_1	Y_2	X_2^n
Y_1	0	0	$\frac{Y_1}{2n+1}$
Y_2	0	0	Y_2
X_2^n	$-\frac{Y_1}{2n+1}$	$-Y_2$	0

3.2.2 Conservation Laws

Equation (37) has the following two conservation laws: The multiplier $\Lambda(x,t,v) = v$ gives us the conserved vector

$$T_1^t = \frac{1}{2}v^2, \quad T_1^x = -vv_{xx} + \frac{1}{2}v_x^2 + \frac{3}{2}v^4, \tag{46}$$

and $\Lambda(x,t,v) = 1$ yields the conserved vector components

$$T_2^t = v, \quad T_2^x = 2v^3 - v_{xx}. \tag{47}$$

Similarly, equation (38) has the following two conservation laws, from the multiplier $\Lambda(x,t,v) = v$, we get

$$T_1^t = \frac{1}{2}v^2, \quad T_1^x = -5v^6 + 10v_{xx}v^3 + 5v^2v_x^2 - vv_{xxxx} + v_xv_{xxx} - \frac{1}{2}v_{xx}^2, \tag{48}$$

and from the multiplier $\Lambda(x,t,v) = 1$,

$$T_2^t = v, \quad T_2^x = -6v^5 + 10v^2v_{xx} + 10vv_x^2 - v_{xxxx}. \tag{49}$$

In the next chapter, we consider solutions of the hierarchy by application of the symmetries and conservation laws.

Chapter 4

Solving the KdV Hierarchies

This chapter highlights the step by step procedure to be employed when solving the KdV and mKdV hierarchies.

4.1 The nth Conservation Law

We start by considering the following theorem, as it helps us inaugurate the nth conserved vector of the hierarchy (18).

Theorem 4.1.1 *The KdV hierarchy possesses the conserved vector* $T = (T^t, T^x) = (u, -D_x^{-1}(D_x^2 - 4u - 2u_x D_x^{-1})^n u_x)$ *along the solutions of Eq. (18).*

Proof 4.1.1 *Suppose the conservation law is*

$$D_t T^t + D_x T^x = 0, \tag{49}$$

where T^t *is the conserved density and* T^x *is the conserved flux. Then, from Eq. (18)*

we have

$$\begin{aligned} u_t + (D_x^2 - 4u - 2u_x D_x^{-1})^n u_x &= D_t\,(u) + D_x\Big(- D_x^{-1}(D_x^2 - 4u - 2u_x D_x^{-1})^n u_x\Big) \\ &= 0, \end{aligned} \tag{50}$$

along the solutions of Eq. (18), and the result follows.

Hence, we have that the conservation density is $T^t = u$ for every member of the hierarchy and the conserved flux is $T^x = -D_x^{-1}(D_x^2 - 4u - 2u_x D_x^{-1})^n u_x$. For example, the fluxes for the first few members of the hierarchy, are (31) for $n = 1$ and (34) for $n = 2$, etc.

Similarly, we can prove a result for the mKdV hierarchy, viz.

Theorem 4.1.2 *The mKdV hierarchy possesses the conserved vector* $T = (T^t, T^x) = (v, -D_x^{-1}(D_x^2 - 4v^2 - 4v_x D_x^{-1} v)^n v_x)$ *along the solutions of Eq. (36).*

One can easily check that the conserved vectors (47) for $n = 1$ and (49) for $n = 2$ arise from this theorem.

4.2 Associations and Transformations

To connect the concept of point symmetries and conservation laws, we need the association condition between them. Hence, a symmetry generator X is associated with a conserved vector T of a given equation if X and T satisfy the condition (17). Based on the previous chapter, we noticed that the symmetries $Y_1 = \partial_t$ and $Y_2 = \partial_x$ are possessed by the respective hierarchies for all n. We notice that the point symmetry ∂_t, when it is applied to the condition above is

$$\partial_t \begin{pmatrix} u \\ -D_x^{-1}(D_x^2 - 4u - 2u_x D_x^{-1})^n u_x \end{pmatrix} = \underline{0}, \tag{51}$$

and for the other symmetry,

$$\partial_x \begin{pmatrix} u \\ -D_x^{-1}(D_x^2 - 4u - 2u_x D_x^{-1})^n u_x \end{pmatrix} = \underline{0}. \tag{52}$$

Since they both satisfy the association condition, we therefore, conclude that both symmetries are associated with every conservation law of Theorem 4.1.1, i.e. with any KdV hierarchy member. Similarly, for the mKdV hierarchy, we have

$$\partial_t \begin{pmatrix} v \\ -D_x^{-1}(D_x^2 - 4v^2 - 4v_x D_x^{-1} v)^n v_x \end{pmatrix} = \underline{0}, \tag{53}$$

and for the other symmetry,

$$\partial_x \begin{pmatrix} v \\ -D_x^{-1}(D_x^2 - 4v^2 - 4v_x D_x^{-1} v)^n v_x \end{pmatrix} = \underline{0}. \tag{54}$$

Consequently, both symmetries satisfy the association criterion, and we conclude that they are correlated with each of Theorem 4.2.1's conservation laws.

We now recall the fundamental theorem on double reduction [14, 15], which says that there exist functions T^r such that

$$D_r T^r = 0. \tag{55}$$

The transformed conserved quantity may be expressed as

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}, \tag{56}$$

where r and s are similarity variables connected to an associated symmetry X.

Since ∂_t and ∂_x are associated with the conserved vector T, we consider the linear combination $X = \partial_x + c\partial_t$, ($c$ is a constant) to obtain the similarity transformation

$$r = cx - t, \quad s = x, \quad u(x,t) = u(r), \tag{57}$$

for the KdV hierarchy and similarly,

$$r = cx - t, \quad s = x, \quad v(x,t) = v(r), \tag{58}$$

for the mKdV hierarchy.

Therefore, we may establish the following theorems for T^r.

Lemma 4.2.1 *The conserved quantity of Eq. (18) can be reduced to*

$$u + c\Bigg(D_r^{-1} \left(c^2 D_r^2 - 4u - 2u_r D_r^{-1}\right)^n u_r \Bigg), \tag{59}$$

where $u(r)$ is $u(x,t)$.

Proof 4.2.1 *Since X_1 and X_2 are associated with the conserved vector T, we consider the linear combination $X = X_2 + cX_1$, (c is a constant) to obtain the similarity transformation*

$$r = cx - t, \quad s = x, \quad u(x,t) = u(r). \tag{60}$$

Hence, by (56) we have

$$T^r \;\; = \;\; \frac{u(-1) - D_x^{-1}(D_x^2 - 4u - 2u_x D_x^{-1})^n u_x c}{-1}. \tag{61}$$

In the new variables, Eq. (61) transforms to

$$T^r = u + c\Bigg(D_r^{-1} \left(c^2 D_r^2 - 4u - 2u_r D_r^{-1}\right)^n u_r \Bigg). \tag{62}$$

In particular, T^r corresponding to T_1 of (31) is given by:

$$T^r = u_{rr} \cdot c^3 - 3u^2 c + u, \tag{63}$$

for $n = 1$ and T^r for (34) in the case of $n = 2$ is

$$T^r = u_{rrrr} \cdot c^5 - 10uu_{rr} \cdot c^3 - 5u_r^2 \cdot c^3 + 10u^3 \cdot c + u. \tag{64}$$

Lemma 4.2.2 *The conserved quantity of Eq. (36) can be reduced to*

$$v + c\bigg(D_r^{-1}(c^2 D_r^2 - 4v^2 - 4v_r D_r^{-1} v)^n v_r\bigg), \tag{65}$$

where $v(r)$ is $v(x,t)$.

The proof of Lemma 4.2.2 is similar to Lemma 4.2.1.

Also to this end, examples of T^r for the mKdV hierarchy include, T_2 of (47) given by:

$$T^r = v - 2v^3 \cdot c + v_{rr} \cdot c^3, \tag{66}$$

for $n = 1$ and and T^r for T_2 of (49) is given by:

$$T^r = v + 6v^5 \cdot c - 10v^2 v_{rr} \cdot c^3 - 10v^2 v_r^2 \cdot c^3 + v_{rrrr} \cdot c^5, \tag{67}$$

for $n = 2$.

That is, the above results can be used to find T^r for any value of n, for both the KdV and mKdV hierarchy. Based on Eq. (55), we have that $T^r = \kappa$, κ constant. Therefore, we have reduced the entire partial differential KdV and mKdV hierarchies, to ordinary differential hierarchies. These ordinary differential hierarchies may then be solved for any n. Below, we set $\kappa = 0$ for simplicity.

4.3 Examples

4.3.1 The KdV Hierarchy

Eq. (63) has implicit solution

$$\pm \int^{u(r)} c^2 \frac{1}{\sqrt{c\,(C_1\,c^3 + 2\,a^3 c - a^2)}} da - r - C_2 = 0. \tag{68}$$

Suppose $C_1 = 1, C_2 = 0, c = \frac{1}{2}$. Then the integral is evaluated to be

$$\pm \frac{\sqrt{2}}{2\sqrt{\sqrt{5}-1}} \sqrt{-2\,u+1}\sqrt{-4\,u+\sqrt{5}+1}\sqrt{4\,u+\sqrt{5}-1}\, \mathit{EllipticF}\left(\frac{\sqrt{2}}{\sqrt{\sqrt{5}+1}}\sqrt{-2\,u+1}, \frac{2\,i}{\sqrt{5}-1}\right) \frac{1}{\sqrt{8\,u^3-8\,u^2+1}},$$

where *EllipticF* is the Incomplete Elliptic integral of the first kind, and the solution becomes

$$u(r) = \frac{-2\sqrt{5}-2}{8}\left(\mathit{JacobiSN}\left(\frac{\sqrt{2\sqrt{5}-2}r}{2}, \frac{i}{2} + \frac{i}{2}\sqrt{5}\right)\right)^2 - \frac{\sqrt{5}}{4} + \frac{\sqrt{-10+10\sqrt{5}}}{8} + \frac{1}{2}, \tag{69}$$

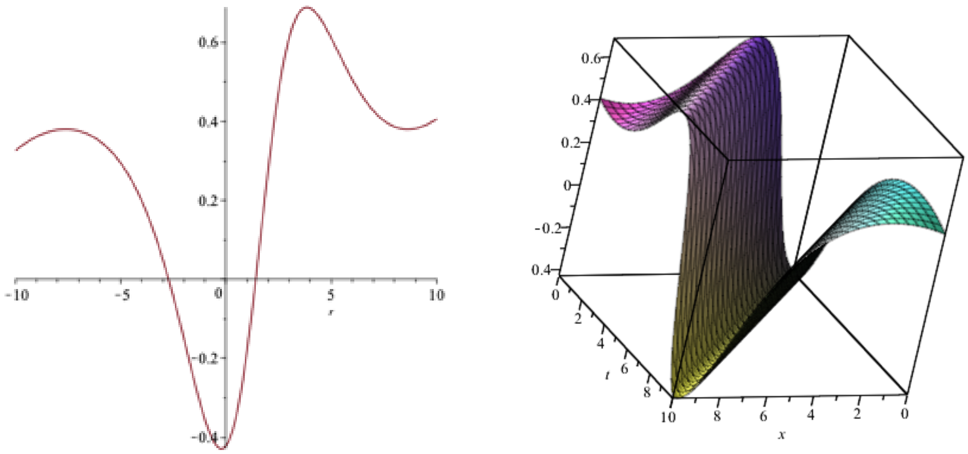
or in the original independent variables

$$u(x,t) = \frac{-2\sqrt{5}-2}{8}\left(\mathit{JacobiSN}\left(\frac{\sqrt{2\sqrt{5}-2}(cx-t)}{2}, \frac{i}{2} + \frac{i}{2}\sqrt{5}\right)\right)^2 - \frac{\sqrt{5}}{4} + \frac{\sqrt{-10+10\sqrt{5}}}{8} + \frac{1}{2}, \tag{70}$$

where *JacobiSN* is an inverse of elliptic integrals and doubly periodic elliptic functions. These solutions appear graphically in Figure 4.1.

4.3.2 The mKdV Hierarchy

The solution of (66) is



(a) 2-D Plot (b) 3-D Plot

Figure 4.1: 2-D and 3-D plots corresponding to (69) and (70), respectively.

$$v(r) = \pm \frac{2i\sqrt{2}c^3C_1\sqrt{-\frac{c}{1-\sqrt{1-4c^4C_1}}}JacobiSN\left(\sqrt{2}\sqrt{\frac{cC_1r^2}{1-\sqrt{1-4c^4C_1}}+\frac{2cC_1C_2r}{1-\sqrt{1-4c^4C_1}}+\frac{cC_1{C_2}^2}{1-\sqrt{1-4c^4C_1}}}|\frac{1-\sqrt{1-4c^4C_1}}{\sqrt{1-4c^4C_1}+1}\right)}{1+\sqrt{1-4c^4C_1}} \tag{71}$$

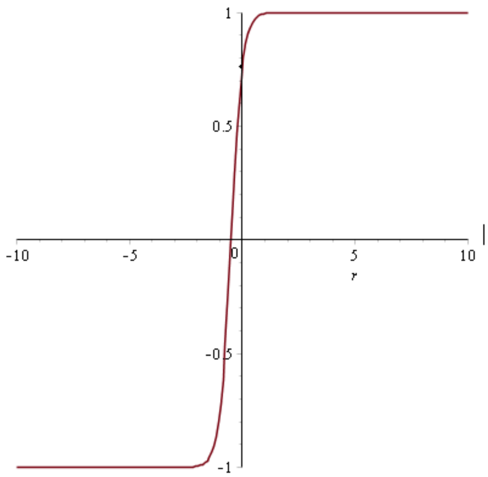
or, alternatively

$$v(r) = \frac{C_2}{\sqrt{c{C_2}^2-c+1}}JacobiSN\left(\frac{\frac{r}{c^2}\sqrt{-c(c-1)}+C_1}{\sqrt{c{C_2}^2-c+1}}, \frac{C_2}{c-1}\sqrt{-c\left(c-1\right)}\right), \tag{72}$$

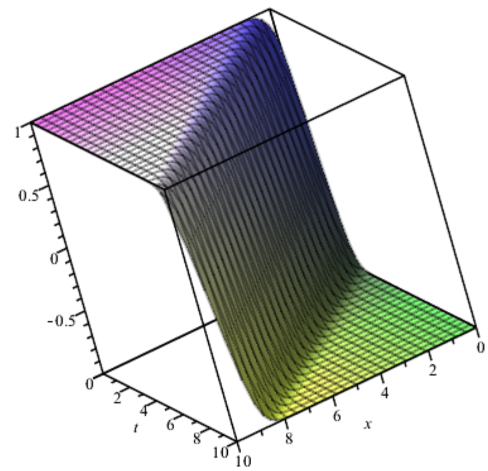
and in original variables

$$v(x,t) = \frac{C_2}{\sqrt{c{C_2}^2-c+1}}JacobiSN\left(\frac{\frac{(cx-t)}{c^2}\sqrt{-c(c-1)}+C_1}{\sqrt{c{C_2}^2-c+1}}, \frac{C_2}{c-1}\sqrt{-c\left(c-1\right)}\right). \tag{73}$$

This solution has the following 2-D and 3-D plots in Figure 4.2.



(a) 2-D Plot



(b) 3-D Plot

Figure 4.2: 2-D and 3-D plots corresponding to (72) and (73), respectively.

4.4 Other Solutions

The previous sections shows that both hierarchies admit conservation laws, independent of Theorems 4.1.1 and 4.2.1. Now, we were unable to transcribe these conservation laws to theorems with a recursion operator as was done to some of the conservation laws. Nonetheless, a T^r function may still be obtained in such cases, using the same formula and transformations.

For example, in the KdV hierarchy the T^r for T_2 of (32) is given by:

$$T^r = uu_{rr} \cdot c^3 - \frac{1}{2}u_r^2 \cdot c^3 - 2u^3c + \frac{u^2}{2}, \tag{74}$$

for $n = 1$ and T^r for T_2 of (35) is given by:

$$T^r = \frac{15}{2}u^4 \cdot c - 10uu_{rr} \cdot c^3 + uu_{rrr} \cdot c^4 - u_r u_{rrr} \cdot c^5 + \frac{1}{2}u_{rr}^2 \cdot c^5 + \frac{u^2}{2}, \tag{75}$$

for $n = 2$.

If we consider the mKdV hierarchy, for $n = 1$ we have T^r for T_1 of (46) is given by:

$$T^r = \frac{1}{2}v^2 + vv_{rr} \cdot c^3 - \frac{1}{2}v_r^2 \cdot c^3 - \frac{3}{2}v^4 \cdot c, \tag{76}$$

and T^r for T_1 of (48) is given by:

$$T^r = \frac{1}{2}v^2 + 5v^6 \cdot c - 10v^3 v_{rr} \cdot c^3 - 5v^2 v_{rr}^2 \cdot c^5 + vv_{rrrr} \cdot c^5 - v_r v_{rrr} \cdot c^5 + \frac{1}{2}v_{rr}^2 \cdot c^5, \tag{77}$$

for $n = 2$. Next, we explore some solutions that arise out of these T^r functions.

4.4.1 The KdV Hierarchy

Eq. (74) has implicit solution

$$\pm \int^{u(r)} c^2 \frac{1}{\sqrt{ca\ (C_1\, c^3 + 2\, a^2 c - a)}} da - r - C_2 = 0. \tag{78}$$

Here, the integral is

$$\frac{1}{2}\,\frac{c\left(-1+\sqrt{-8\,C_1\,c^4+1}\right)\sqrt{2}}{\sqrt{cu\,(r)\left(C_1\,c^3+2\,\left(u\,(r)\right)^2c-u\,(r)\right)}}\times$$
$$\sqrt{\frac{4\,cu\,(r)+\sqrt{-8\,C_1\,c^4+1}-1}{-1+\sqrt{-8\,C_1\,c^4+1}}}\sqrt{\frac{-4\,cu\,(r)+\sqrt{-8\,C_1\,c^4+1}+1}{\sqrt{-8\,C_1\,c^4+1}}}\times$$
$$\times\sqrt{-\frac{cu\,(r)}{-1+\sqrt{-8\,C_1\,c^4+1}}}$$
$$\times \mathit{EllipticF}\left(\sqrt{\frac{4\,cu\,(r)+\sqrt{-8\,C_1\,c^4+1}-1}{-1+\sqrt{-8\,C_1\,c^4+1}}},1/2\,\sqrt{2}\sqrt{\frac{-1+\sqrt{-8\,C_1\,c^4+1}}{\sqrt{-8\,C_1\,c^4+1}}}\right). \tag{79}$$

Suppose $C_1 = 1, C_2 = 0, c = \frac{1}{2}$. Then the solution

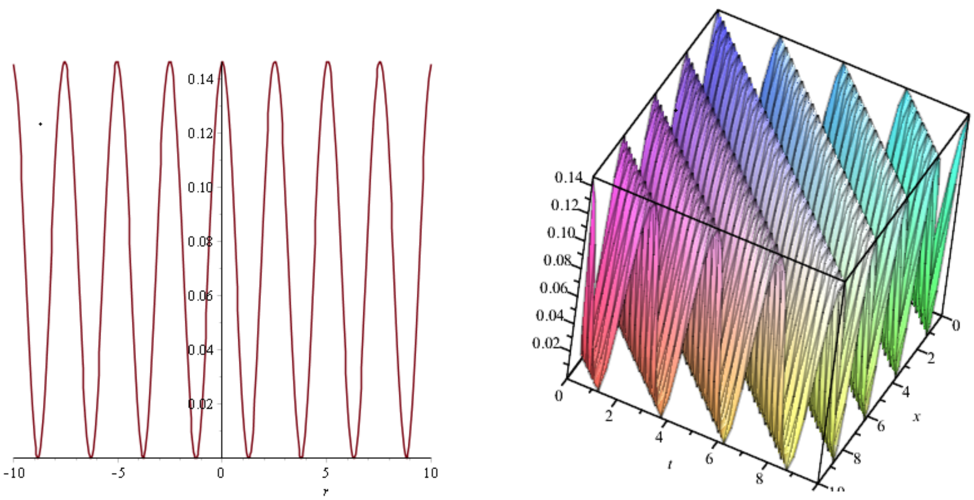
(78) is

$$u(r) = \frac{(-2+\sqrt{2})\left(\left(JacobiSN\left(r\sqrt[4]{2}, \frac{i}{2}\sqrt{2-\sqrt{2}}\sqrt[4]{2}\right)\right)^2 - 1\right)}{4}. \tag{80}$$

The solution in original variables is

$$u(x,t) = \frac{(-2+\sqrt{2})\left(\left(JacobiSN\left((cx-t)\sqrt[4]{2}, \frac{i}{2}\sqrt{2-\sqrt{2}}\sqrt[4]{2}\right)\right)^2 - 1\right)}{4}. \tag{81}$$

The progression of these solutions appear in Figure 4.3.



(a) 2-D Plot

(b) 3-D Plot

Figure 4.3: 2-D and 3-D plots corresponding to (80) and (81), respectively.

4.4.2 The mKdV Hierarchy

Eq. (76) has implicit solution

$$\pm \int^{v(r)} c^2 \frac{1}{\sqrt{ca\,(C_1\,c^3 + a^3 c - a)}} da - r - C_2 = 0. \tag{82}$$

The integral is evaluated to be

$$\frac{1}{2}\,\frac{c\left(-1+\sqrt{-8\,C_1\,c^4+1}\right)\sqrt{2}}{\sqrt{cv\,(r)\left(C_1\,c^3+2\,(v\,(r))^2\,c-v\,(r)\right)}}\times$$
$$\sqrt{\frac{4\,cv\,(r)+\sqrt{-8\,C_1\,c^4+1}-1}{-1+\sqrt{-8\,C_1\,c^4+1}}}\sqrt{\frac{-4\,cv\,(r)+\sqrt{-8\,C_1\,c^4+1}+1}{\sqrt{-8\,C_1\,c^4+1}}}\times$$
$$\sqrt{-\frac{cv\,(r)}{-1+\sqrt{-8\,C_1\,c^4+1}}}\times$$
$$\mathit{EllipticF}\left(\sqrt{\frac{4\,cv\,(r)+\sqrt{-8\,C_1\,c^4+1}-1}{-1+\sqrt{-8\,C_1\,c^4+1}}},1/2\,\sqrt{2}\sqrt{\frac{-1+\sqrt{-8\,C_1\,c^4+1}}{\sqrt{-8\,C_1\,c^4+1}}}\right). \tag{83}$$

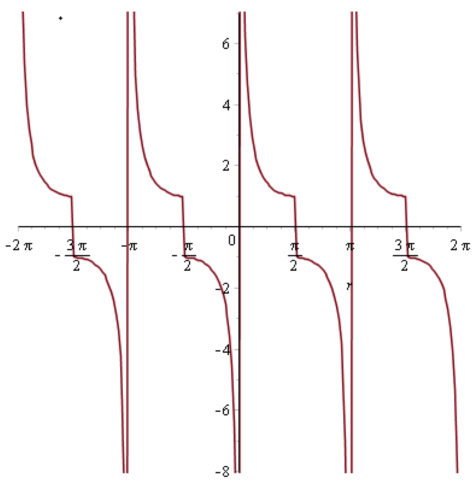
Suppose we let $C_1 = 1, C_2 = 0, c = \frac{1}{2}$. Then the solution (82) is

$$v(r) = \pm\frac{1}{\tan(r)}\sqrt{(\tan(r))^2+1}, \tag{84}$$

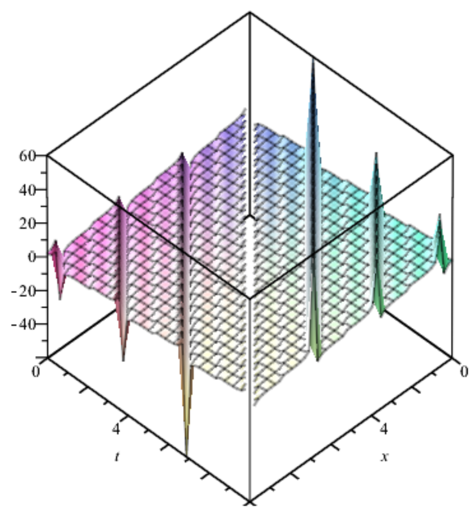
and

$$v(x,t) = \pm\frac{1}{\tan((cx-t))}\sqrt{(\tan((cx-t)))^2+1}, \tag{85}$$

whose graphical representation appears in Figure 4.4.



(a) 2-D Plot



(b) 3-D Plot

Figure 4.4: 2-D and 3-D plots corresponding to (84) and (85), respectively.

Chapter 5

Conclusion

In the study of differential equations, equations that are highly nonlinear and that possess higher-order derivatives are almost impossible to solve. We have proposed a scheme to overcome this problem and solve, in particular, the KdV and mKdV infinite hierarchy. This solving however, involves a series of steps, from which we considered both Lie point symmetries and conservation laws of the KdV and mKdV hierarchies, along with their association.

Checking association is the most fundamental step in the solving procedure, as it determines the next step of the process, as to, whether we may continue using the symmetry-conservation law combination to solve or not. Continuing with the symmetry-conservation law combination under consideration is subject to the association condition being satisfied.

We then used the associated symmetries and conservation laws, along with the fundamental theorem on double reduction to find the reduced conserved quantity, which then resulted in nonlinear ordinary differential equations. The final step was to solve these nonlinear and higher-order differential equations. This was the most challenging part of the project.

Several software were employed to help solve these equations, and implicit solutions were obtained, in which some of them were simplified to get the final solutions. In the analysis of our solutions, we divided our solutions to be of two types. The first type was of the most interesting solution, as it is derived from a recursion operator within the conservation law of the KdV and mKdV infinite hierarchy. The second solution type was independent of a recursion operator.

In both solution types, the knowledge of association between symmetry and conserved components was exploited, and formed the basis of our approach, to reduce the order of the partial differential hierarchy to an ordinary differential hierarchy. Consequently, our method has many significant uses, and can be extended to solve other infinite hierarchies. In future studies, we hope to pursue this idea further.

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