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Genus of a pullback

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Declaration

I, Thandile Tonisi hereby declare that the project work entitled "Genus of a pullback" submitted to the University of the Witwatersrand, Johannesburg is a record of an original work done by me under the guidance of Dr Rugare Kwashira and Dr Jules Mba. This project is submitted for the degree of Master of Science in Mathematics. The work in this report has not been submitted to any other University or Institute for any degree or examination, except where stated otherwise by reference or acknowledgement.

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Date

Abstract

Let Q be a finitely generated nilpotent group. The Mislin genus $\mathscr{G}(Q)$ is defined to be the set of isomorphism classes of finitely generated nilpotent groups R such that for every prime p, the p-localizations R_p and Q_p are isomorphic. If the group Q has a finite commutator subgroup, the genus admits an abelian group structure. In this work, we study the Mislin genus of nilpotent groups through pullbacks. For a relatively prime pair of natural numbers (n, u), let $Q = \langle x, y \mid x^n = 1, yxy^{-1} = x^u \rangle$, where Q is a pullback. We compute the genus $\mathscr{G}(Q)$ and we investigate the relationship between the genera of these nilpotent groups.

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Contents

Chapter 1

Introduction

In [14] Mislin introduced the notion of genus of finitely generated nilpotent groups. He showed that two finitely generated nilpotent groups Q and R belong to the same genus if and only if the two groups are non-isomorphic, but their p-localizations Q_p and R_p are isomorphic at every prime p. For a finitely generated nilpotent group Q with a finite commutator subgroup, Hilton and Mislin [8] defined a finite abelian group structure on the genus $\mathscr{G}(Q)$.
Let G be any group. Define the non-cancellation set, denoted by $\chi(G)$ to be the set of all isomorphism classes of groups K such that $K \times \mathbb{Z} \cong G \times \mathbb{Z}$. According to Warfield's result [17], if Q is a finitely generated nilpotent group with a finite commutator subgroup, then we have that $\chi(Q) = \mathscr{G}(Q)$. Further studies on the non-cancellation set can be found in literature [20], [18] and [19].

Let $\mathscr{X}_0$ denote the class of all finitely generated groups that have a finite commutator subgroup. Let $\mathscr{N}_0$ denote a subclass of nilpotent groups in $\mathscr{X}_0$ which are infinite. Let $\mathscr{K}$ be the class of groups which are of the form $T \rtimes \mathbb{Z}^k$, where T is a finite abelian group and $k \in \mathbb{Z}^+$. Several computations of the genus of $\mathscr{X}_0$-groups which belong to the class $\mathscr{K}$ can be found in [8] and [10]. We do note that there is no general method of computing the genus $\mathscr{G}(Q)$ when $Q \in \mathscr{N}_0$. Hilton and

Casacuberta [2] later introduced a class of nilpotent groups $\mathscr{N}_1 \subset \mathscr{N}_0$, where they calculated the genus $\mathscr{G}(Q)$ for $Q \in \mathscr{N}_1$. Let Q be a nilpotent group. The following sequence was introduced in [2]

$$TQ \rightarrowtail Q \twoheadrightarrow FQ, \tag{1.1}$$

where TQ is the torsion subgroup of Q and FQ is the torsion free quotient. The group Q is in the class $\mathscr{N}_0$ if and only if TQ is finite and FQ is free abelian of finite rank. If, additionally,

a) TQ and FQ are commutative;

b) The sequence in (1.1) splits for the action $\omega : FQ \to Aut(TQ)$;

c) $\omega(FQ) \subseteq Z(Aut(TQ))$, where $Z(Aut(TQ))$ is the center of $Aut(TQ)$,

then we say $Q \in \mathscr{N}_1 \subset \mathscr{N}_0$.

In the presence of (a), property (c) is equivalent to requiring that for each $\zeta \in FQ$ there exists $u \in \mathbb{Z}$ such that $\zeta \cdot x = \omega(\eta)(x) = ux$ for all $x \in TQ$ as observed in [2].
Now define the height of a subgroup $Ker\,\omega$ of a free abelian group F to be the largest positive integer h such that $Ker\,\omega \subseteq hFQ$. Let d be the height of $Ker\,\omega$ in FQ; that is, $d = max\{h \in \mathbb{N} \mid Ker\,\omega \subseteq hFN\}$ and let $(\mathbb{Z}/d)^*$ be the multiplicative group of units of $\mathbb{Z}/d$. The authors in their paper [2] have shown that

$$\mathscr{G}(Q) \cong (\mathbb{Z}/d)^*/\{\pm 1\}. \tag{1.2}$$

Note that if $d = 1$ or 2, then the genus of Q is trivial. We observe that the class $\mathscr{N}_1$ is restricted as shown in [13]. If a group $Q \in \mathscr{N}_1$ has a non-trivial genus, then FQ is cyclic.

In [10] and [12], the calculation of the genus was extended from the class $\mathscr{N}_1$ to a subclass in $\mathscr{N}_1$ of direct products of k copies of Q, where $k \geq 2$. Let $Q \in \mathscr{N}_1$, and Q^k be the k^{th} direct power of Q. There is a surjective homomorphism

$$\rho : \mathscr{G}(Q) \rightarrow \mathscr{G}(Q^k)$$

given by $\rho(R) = R \times Q^{k-1}$, where $R \in \mathscr{G}(Q)$ as shown in [2]. The authors' result in [13] asserts that $\mathscr{G}(Q) = \{0\}$ if FQ is not cyclic, and so, under the same hypotheses, $\mathscr{G}(Q^k) = \{0\}$.

Now assume that FQ is cyclic with generator ζ and let d be of the form $d = p_1^{l_1} p_2^{l_2} \cdots p_\lambda^{l_\lambda}$, where $0 \leq l_j < r_j$, $j = 1, 2, \cdots \lambda$. The authors in [12] calculate $\mathscr{G}(Q^k)$ for $Q \in \mathscr{N}_1$ with FQ cyclic and $k \geq 2$. Their result states that $\mathscr{G}(Q^k) = \mathscr{G}(Q)/H$, where H is an elementary abelian 2-group. Note that in general, H is not a subgroup of the group Q. Therefore we obtain $\mathscr{G}(Q^k)$ from $\mathscr{G}(Q)$ by factoring out those residues $s \, modulo \, d$ such that

$$s \equiv \epsilon_j \; mod \, p_j^{l_j}, \epsilon_j = \pm 1, j = 1, 2, \cdots, \lambda.$$

If $Q \in \mathscr{N}_1$, then for $k \geq 2$ we have that Q^k is in $\mathscr{N}_1$ if and only if Q is itself abelian since Q^k does not inherit property (c) from Q. Also, since Q^k inherits property (a) from Q, we have that $\mathscr{G}(Q^k)$ is finite abelian.

For a pair of relatively prime numbers (n, u), a pullback is a $\mathscr{K}$-group admitting a presentation $Q = \langle x, y \mid x^n = 1, yxy^{-1} = x^u \rangle$. Note that $Q(n, u)$ will denote the group $\mathbb{Z}_n \rtimes_\omega \mathbb{Z}$, where $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n)$ is a non-trivial homomorphism given by $\omega(1) : x \to ux$. Therefore $Q = Q(n, u)$. In particular, a pullback is a subgroup of a direct product of groups. Given that the genus of direct products of nilpotent groups has been computed, our main purpose in this paper is to compute the genus of a pullback and check if the genus of the pullback and that of those groups in the direct product which the pullback is a subgroup of are homomorphic to each other.

In chapter 2, we present some preliminary results, where we give basic definitions and algebraic properties of some mathematical concepts which will be of use for our main results.

Chapter 3 is devoted to the class $\mathscr{N}_1$ of nilpotent groups and to the study of the genus of a group $H \in \mathscr{N}_1$ with an infinite cyclic torsion-free quotient FH. We consider finitely generated nilpotent groups $H_0, H_1, \cdots, H_{s-1}$ with a property that all are in the same Mislin genus but are pairwise non-isomorphic. We show that the genus $\mathscr{G}(H)$ is finite using an exact sequence introduced by the authors in [8]. We further introduce a pullback square and its properties. In particular, we construct a pullback $H_t \in \mathscr{N}_1$, where $t \equiv i + j \, modulo \, s$ and $H_i, H_j \in \mathscr{G}(H_t)$. We show that the genus of H_t coincides with that of H.

In chapter 4, we consider a group $L \in \mathscr{N}_1$ with a cyclic torsion-free quotient. We show that this group is a pullback of groups $K, G \in \mathscr{N}_0$, and we further compute its genus and the genus of K and that of G. Then we show that one of them is a homomorphic image of another.

In Chapter 5, we give a conclusion of our work.

Chapter 2

Preliminaries

In this chapter, we give preliminaries that will enable us to understand some concepts and results that will be required for later sections.

In section 2.1, a group is defined and its well-known basic properties are given. This is followed by a discussion in section 2.2 of a structure-preserving map from one mathematical object to another. In section 2.3, we prove Lagrange's Theorem. In section 2.4, we discuss normal subgroups which are very important in the construction of quotient groups. In Section 2.5, we discuss group actions. To prove the facts on quotient groups and their subgroups, we make use of the three isomorphism theorems, which we discuss in section 2.6.

2.1 Groups

Definition 2.1.1. A **group** T is a non-empty set with a binary operation $*$, where under this binary operation the following properties are satisfied for all $t_1, t_2, t_3 \in T$:

a) $t_1 * t_2 \in T$,

b) $t_1 * (t_2 * t_3) = (t_1 * t_2) * t_3$.

c) There exist a unique element $e_T \in T$, called the identity such that $t * e_T = e_T * t$, for all $t \in T$.

d) For each $t \in T$, there exist a unique element t^{-1} in T called the inverse of t such that $t^{-1} * t = t * t^{-1} = e_T$.

For notation convenience, we will write $t_1 * t_2 = t_1 t_2$ for $t_1, t_2 \in T$.

Definition 2.1.2. A group T is said to be **abelian** if the operation is commutative, that is, $ab = ba$ for all $a, b \in T$.

Definition 2.1.3. The number of elements in a group T is called the **order** or cardinality of the group and is denoted by $|T|$. The group may be of finite or infinite order.

Definition 2.1.4. Let T be a group and $t \in T$. The **order** of t denoted by $|t|$ is defined to be the least positive integer n such that $t^n = e_T$. If there is no such an integer n, then t is said to have infinite order, and we write $|t| = \infty$.

Definition 2.1.5. Let S be a non-empty subset of a group T. The subset S is a **subgroup** of T, denoted $S \leq T$ if the subset S forms a group under the same binary operation of T.

Example 1. Let $\langle T, * \rangle$ be a group, then $\{e\}$ and T are subgroups of $\langle T, * \rangle$. The subgroup $\{e\}$ is called the trivial subgroup.

Proposition 2.1.6. *Let S be a non-empty subset of a group T. The subset S is a subgroup of T if and only if $ab^{-1} \in S$ for all $a, b \in S$.*

Proof. Suppose that S is a subgroup of T, if there is $y \in S$, there is $y^{-1} \in S$. Then by the property of closure, we have that $xy^{-1} \in S$. Conversely, assume that S is non-empty, so there exists $x \in S$. Then by assumption $xx^{-1} = e \in S$. For elements e and x in S, we have that $ex^{-1} = x^{-1}e \in S$. Now for $x, y \in S$, we have $x^{-1} \in S$ and so $x(y^{-1})^{-1} = xy \in S$. Finally, since the binary operation is associative on T and $S \subseteq T$, then S satisfies the associative law. Thus we have proven that $S \leq T$. □

Lemma 2.1.7. *Let T be a group and let S and K be subgroups of T. Then SK is a subgroup of T if and only if $SK = KS$.*

Proof. Suppose that SK is a subgroup of T. Then SK must contain all inverses of the elements of SK. Hence, $SK = (SK)^{-1} = K^{-1}S^{-1} = KS$.

Conversely, let us assume that $SK = KS$. Then since $(SK)^{-1} = K^{-1}S^{-1} = KS = SK$, we have that SK is closed under inverses. And since $(SK)(SK) = SKSK = SSKK = SK$, then SK is closed under products. Associativity is inherited from T, therefore we can conclude that SK is a subgroup of T. □

Definition 2.1.8. A **generating set** of a group T is a subset $X \subset T$ such that for every element $a \in T$ we can write $a = x_1^{m_1} x_2^{m_2} \cdots x_l^{m_l}$ for some $x_i \in X$ and $m_i \in \mathbb{Z}$. We write $T = \langle X \rangle$. If a generating set X is finite, we say the group T is finitely generated.

Definition 2.1.9. A group T is **cyclic** if $T = \{a^r \mid r \in \mathbb{Z}\}$ for some $a \in T$. We may write $T = \langle a \rangle$ and if $a \in T$ has order n, then $|T| = n$ and a is called the group generator. If a has infinite order, then T is of infinite order.

Note. When $X = \{x\}$, we will denote by $\langle x \rangle$ the set of integer powers of an element x in a group T. That is, $\langle x \rangle = \{x^r \mid r \in \mathbb{Z}\}$.

Example 2.

1. In general, $\mathbb{Z}_n$ is a cyclic group under addition with generators 1 or $n-1$.
2. For a prime p, we have that $\mathbb{Z}_p \backslash \{0\}$ is a cyclic group of order $p-1$ under multiplication.

Definition 2.1.10. We define the **multiplicative group of units** in $\mathbb{Z}$, denoted by $\mathbb{Z}_n^*$ to be the set of positive integers that are less than n and co-prime to n, under multiplication modulo n.

Definition 2.1.11. Let T be an abelian group. The **rank** of T denoted by $rank(T)$ is given by the set $min\{|X| : X \subseteq T, \langle X \rangle = T\}$.

Theorem 2.1.12. *Let T be a group and let x be any element of T. Then $\langle x \rangle$ is a subgroup of T.*

Proof. We have that $e \in \langle x \rangle$, therefore $\langle x \rangle$ is non-empty. To complete the proof, we must show that $\langle x \rangle$ is closed under the operation of T and that $\langle x \rangle$ contains the inverse of each of its elements. For any $i, j \in \mathbb{Z}$, let x^i and x^j be elements of $\langle x \rangle$. Then $x^i x^j = x^{i+j} \in \langle x \rangle$. This implies that $\langle x \rangle$ is closed under the operation of T. Now for any $x^i \in \langle x \rangle$, $x^0 = e \in \langle x \rangle$ and $(x^i)^{-1} = x^{-i} \in \langle x \rangle$. Since T is a group and $\langle x \rangle \subseteq T$, then $\langle x \rangle$ obeys the associative law. From this, we can conclude that $\langle x \rangle$ is a group and subsequently a subgroup of T. □

Definition 2.1.13. A group T is **torsion** if every element of T has finite order. If T has no elements of finite order other than the identity of T then T is called **torsion-free**.

Example 3. The group of integers $\mathbb{Z}$ under addition is a torsion-free group.

Definition 2.1.14. Let p be a prime number. The **Prüfer** p-group denoted by $\mathbb{Z}/p^{\infty}$ is defined to be the multiplicative group of all p^nth roots of unity in $\mathbb{C}^*$ for all non-negative n.

Remark 1. If p is a prime, then for a prime $\bar{p}$ not in the set of primes as p, we have that $\mathbb{Z}/p^{\infty}$ is the group $(\mathbb{Z})_{\bar{p}}/\mathbb{Z}$. Moreover, the group is $\bar{p}$-torsion for all primes $\bar{p}$.

2.2 Group homomorphisms

To have a better understanding of groups beyond what we have already discussed above, we employ more structural concepts on groups. This includes the construction and group operations. The construction with relation to groups must be compatible with the group operations. The structure between mathematical objects can be preserved, in other words, the elements of those mathematical objects with relation to the operations do not change the result. Hence we have the following definition.

Definition 2.2.1. Let $\langle T_1, *\rangle$ and $\langle T_2, \circ\rangle$ be any two groups. A map $\delta : T_1 \to T_2$ is called a **group homomorphism** if for any elements $x, y \in T_1$ we have $\theta(x * y) = \theta(x) \circ \theta(y)$.

In addition to the definition above, we say that θ

(i) is an **epimorphism** if θ is surjective.

(ii) is a **monomorphism** if θ is injective.

(iii) is an **isomorphism** if θ is bijective. We write $T_1 \cong T_2$ and say that T_1 is isomorphic to T_2.

(iv) is an **automorphism** if $T_1 = T_2$ and θ is an isomorphism.

Remark 2. In the case of condition (ii), we say θ is an embedding.

Definition 2.2.2. Let T be a group. We define the **automorphism group** $Aut(T)$ to be the set of all group isomorphisms from T to T.

Proposition 2.2.3. *Let T be a group and let $Aut(T)$ be the set of all automorphisms in T. Then $Aut(T)$ is a group under composition of mapppings.*

Proof. Let $\phi_1, \phi_2 \in Aut(T)$. Then ϕ_1 and ϕ_2 are bijections, thus $\phi_1(x * y) = \phi_1(x) * \phi_1(y)$ and similarly, $\phi_2(x * y) = \phi_2(x) * \phi_2(y)$ for any $x, y \in T$. We then have that $\phi_1(\phi_2(x * y)) = \phi_1(\phi_2(x) * \phi_2(y)) = \phi_1(\phi_2(x)) * \phi_1(\phi_2(y))$. Hence $\phi_1\phi_2 \in Aut(T)$, and therefore $Aut(T)$ is closed under composition of mappings. Now consider the identity map $\phi_e : T \to T$ defined by $\phi_e(x) = x$. Then for any $\phi_1 \in Aut(T)$, $\phi_1\phi_e(x) = \phi_1 = \phi_e\phi_1(x)$. Thus $\phi_e \in Aut(T)$. Let $\phi \in Aut(T)$. Then ϕ is a bijection, and $\phi(x*y) = \phi(x)*\phi(y)$, for any $x, y \in T$. We also have that $\phi^{-1}(\phi(x) * \phi(y)) = \phi^{-1}(\phi(x * y)) = x * y = \phi^{-1}(\phi(x)) * \phi^{-1}(\phi(y))$. Therefore $\phi^{-1} \in Aut(T)$, and $Aut(T)$ is closed under inverses. Since the composition of functions is always associative, we have that $Aut(T)$ is associative. Thus $Aut(T)$ is a group. □

Definition 2.2.4. Let T be a group and let S be a subgroup of T. We call S a **characteristic subgroup** if $\phi(S) = S$ for all $\phi \in Aut(T)$.

Definition 2.2.5. Let $x \in T$. A **conjugate** of x in T is an element of the form axa^{-1}, where $a \in T$. Often, we write x^a.

Definition 2.2.6. Let T be a group. An automorphism $\phi : T \to T$ of a group is called an **inner automorphism** if it can be expressed as conjugation by an element of the group. That is, if there is an $a \in T$ such that for all $b \in T$, $\phi(b) = aba^{-1}$.

Definition 2.2.7. Let $\varphi : T_1 \to T_2$ be a group homomorphism. The **kernel** of φ denoted by $Ker\,\varphi$ is the set $\{x \in T_1 \mid \varphi(x) = e_2\}$.

Definition 2.2.8. Let $\varphi : T_1 \to T_2$ be a group homomorphism. The **image** of the homomorphism φ is the set $\{\varphi(x) \mid x \in T_1\}$.

Proposition 2.2.9. *Let $\langle T_1, *\rangle$ and $\langle T_2, \circ\rangle$ be any two groups. Let $\varphi : T_1 \to T_2$ be a group homomorphism. Then $Ker\,\varphi$ and $Im\,\varphi$ are subgroups of T_1 and T_2 respectively.*

Proof. We have that $\varphi(e_1) = \varphi(e_1e_1) = \varphi(e_1)\varphi(e_1)$, so by cancellation, $\varphi(e_1) = e_2$. Thus $Ker\,\varphi \neq \emptyset$. Now let $x, y \in Ker\,\varphi$. Then $\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = e_2e_2 = e_2$. So $xy^{-1} \in Ker\,\varphi$. This implies that $Ker\,\varphi \leq T_1$. Similarly, suppose that $\varphi(x), \varphi(y) \in Im\,\varphi$. Then $\varphi(x)\varphi(y)^{-1} = \varphi(xy^{-1}) \in Im\,\varphi$. Thus $Im\,\varphi \leq T_2$. This completes the proof. □

Example 4. Let $a \in \mathbb{Z}$ and let $n \in \mathbb{Z}^+$. The function $\theta : \mathbb{Z} \to \mathbb{Z}_n$ defined by $\theta(a) = a\,mod\,n$ is a group homomorphism. This homomorphism has a kernel, $Ker\,\theta = n\mathbb{Z}$.

Theorem 2.2.10. *Let $\phi : T_1 \to T_2$ be a group homomorphism. Then ϕ is injective if and only if $Ker\,\phi = \{e_1\}$.*

Proof. Suppose that ϕ is injective and since ϕ is a group homomorphism, at most one element e_1 of T_1 can be sent to the identity element e_2 of T_2. So we have $\phi(e_1) = e_2$. If $t \in Ker\,\phi$, then $\phi(t) = e_2$ and thus $\phi(t) = \phi(e_1)$. Since ϕ is injective, we have $t = e_1$. Hence $Ker\,\phi = \{e_1\}$.

Conversely suppose that $Ker\,\phi = \{e_1\}$ and suppose that there is $x, y \in T_1$ such that

$$\phi(x) = \phi(y). \tag{2.1}$$

Then,

$$\begin{aligned}
\phi(xy^{-1}) =& \phi(x)\phi(y^{-1}) && (\phi \textit{ is} \text{ a homomorphism})\\
=& \phi(x)\phi(y)^{-1}\\
=& \phi(x)\phi(x)^{-1} && \textit{(by 2.1)}\\
=& e_2.
\end{aligned}$$

Thus the element xy^{-1} is in the kernel of ϕ and so $xy^{-1} = e_2$. This implies that $x = y$ and ϕ is injective. □

Example 5. For any $n \in \mathbb{Z}^+$, $Aut(\mathbb{Z}_n) \cong U(n)$.

Proof. Suppose $x \in U(n)$ and define $\phi_x : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\phi_x(a) = xa\,mod\,n$ where $a \in \mathbb{Z}_n$. Now we show that $\phi_x \in \mathbb{Z}_n$. Suppose that $\phi_x(s) = \phi_x(t)$. Then $xs = xt\,mod\,n$. However, there exists $x^{-1}\ mod\,n$. Therefore $x^{-1}xs = x^{-1}xt\,mod\,n \implies es = et\,mod\,n \implies s = t\,mod\,n$. So ϕ_x is injective.

Let $b \in \mathbb{Z}_n$, $b = xx^{-1}b = x(x^{-1}b) = \phi_x(x^{-1}b)$. Then ϕ_x is surjective. Now let $s, t \in \mathbb{Z}_n$. We have that $\phi_x(s+t) = x(s+t)\,mod\,n = xs\,mod\,n + xt\,mod\,n = \phi_x(s) + \phi_x(t)$. Therefore ϕ_x is a homomorphism. Thus $\phi_x \in Aut(\mathbb{Z}_n)$ for all $x \in U(n)$.

For any automorphism ϕ of $\mathbb{Z}_n$, we show that $\phi = \phi_x$ for $x = \phi(1)$. Suppose that $\phi(a) = xa$ for all $a \in \mathbb{Z}_n$. By induction on a, we have that $\phi(1) = x \cdot 1 = x$. Now $\phi(a+1) = \phi(a) + \phi(1) = xa + x = x(a+1)$. Therefore $\phi = \phi_x$ where $x = \phi(1)$. Finally, define $H : Aut(\mathbb{Z}_n) \to U(n)$ by $H(\phi) = \phi(1)$. Since $H(\phi_x) = \phi(1) = x$, we have that H is onto.

Now suppose that $\phi, \psi \in Aut(\mathbb{Z}_n)$. We have that $H(\phi) = H(\psi) \implies \phi(1) = \psi(1)$. For all $a \in \mathbb{Z}_n$, we have $\phi(a) = a\phi(1) = a\psi(1) = \psi(a) \implies \phi = \psi$. Thus H is injective. □

2.3 Cosets and Lagrange's Theorem

Definition 2.3.1. Let S be a subgroup of T and let a, b be elements in T. Define a relation $\equiv$ on T as follows: $v \equiv w$ if and only if $vw^{-1} \in S$. This relation is an equivalence relation. The equivalence class of w is called the **right coset** of S generated by w and is written as $Sw = \{sw \mid s \in S\}$. Similarly, define a relation $\equiv$ on T as follows: $v \equiv w$ if and only if $w^{-1}v \in S$. The **left coset** of S generated by w is written as $wS = \{ws \mid s \in S\}$. Thus T is the union of distinct, disjoint left or right cosets of S.

Note. For an abelian group, there is no distinction between left and right cosets.

Lemma 2.3.2. *Let T be a group and let S be a subgroup of T. For all $x \in T$, we have that $|S| = |Sx|$.*

Proof. Let $\phi : S \to Sx$ be defined by $\phi(s) = sx$ for an element $x \in T$. We show that ϕ is a bijection as follows: $s_1 = s_2 \iff s_1x = s_2x \iff \phi(s_1) = \phi(s_2)$. Then ϕ is well-defined and injective. Next let $sx \in Sx$ for some $s \in S$. Then $\phi(s) = sx$. Therefore ϕ is surjective. So, ϕ is a bijection and $|S| = |Sx|$ for all $x \in T$. □

Definition 2.3.3. Let S be a subgroup of T. The **index** of S in T denoted by $[T : S]$ is the number of distinct right (or left) cosets of S in T.

The following theorem is about the finiteness of groups and their subgroups.

Theorem 2.3.4 (Lagrange's Theorem)**.** *Let T be a finite group and let S be a subgroup of T. Then $|S|$ divides $|T|$. Furthermore, $[T : S] = \frac{|T|}{|S|}$.*

Proof. Any pair of right(or left) cosets of S are either equal or disjoint. Therefore since T is finite, there exists $x_1, x_2, \cdots, x_r \in T$ such that

- $T = \bigcup_{i=1}^{r} Sx_i$
- $Sx_i \bigcap Sx_j = \emptyset$, for all $1 \leq i < j \leq r$.

By Lemma 2.3.2 we know that $|S| = |Sx|$, for all $x \in S$. Then

$$T = \bigcup_{x \in T} Sx.$$

Then,

$$\begin{aligned} |T| &= |\bigcup_{x \in T} Sx| \\ &= \bigcup_{x \in T} |Sx| \\ &= \sum_{x \in T} |S| \\ &= [T : S]|S|. \end{aligned}$$

□

Remark 3. Suppose that T is an infinite group, then the index of a subgroup of T may be finite. For an example, let $T = \mathbb{Z}$ and $2\mathbb{Z}$ be the subgroup of T of even integers. Then there are two cosets, which are the set of even integers and the set of odd integers. Therefore $[\mathbb{Z} : 2\mathbb{Z}] = 2$. In general, for any $n \in \mathbb{Z}^+$, $[\mathbb{Z} : n\mathbb{Z}] = n$.

Corollary 2.3.5. *Let T be a finite group and let $x \in T$ be different from the identity. If the order of T is prime, then T is a cyclic group.*

Proof. Suppose that T is of prime order. Then since $|T| \geq 2$, we may take any $x \in T$ such that $x \neq e_T$. Now consider the subgroup $\langle x \rangle$ of T. Since we have that $e_T, x \in \langle x \rangle$, then $\left|\langle x \rangle\right| \geq 2$. By Lagrange's Theorem $\left|\langle x \rangle\right| = |T|$. Therefore $\langle x \rangle = T$. Thus T is a cyclic group. □

2.4 Normal subgroups and quotient groups

In this section, we study the properties of subgroups whose left and right cosets coincide. For such subgroups, the cosets form a group called the quotient group. Furthermore, we discuss the notions of commutators. We start by defining a normal subgroup.

Definition 2.4.1. Let N be a subgroup of T. Then N is a **normal subgroup** of T if and only if for all $x \in T$, $xN = Nx$. We write $N \lhd T$.

Example 6.

1. Let T be a group. Then we have that $\{e\} \lhd T$ and $T \lhd T$.

2. Let T be an abelian group. Then every subgroup of T is normal in T.

   Proof. Suppose that S is a subgroup of the abelian group T. If $x \in T$ and $s \in S$, then $xsx^{-1} = sxx^{-1} = s \in S$. Then the result follows. □

Example 7. *Any subgroup S of index 2 in a group T is a normal subgroup of T.*

Proof. Suppose S has index 2 in a group T. We want to show that for any $x \in T$, $xS = Sx$. If $x \in S$, then $xS = S = Sx$. So, we assume that $x \notin S$. By Definition 2.3.3, there are exactly two left cosets of S in T, which are S and xS. Since the left cosets are disjoint, we have that $xS = T - S$. Similarly, the right cosets are also disjoint, so $Sx = T - S$. Therefore, $Sx = T - S = Sx$. Thus $H \triangleleft T$. □

Proposition 2.4.2. *Let T be a group and let N be a subgroup of T. Then N is a normal subgroup of T if and only if $xNx^{-1} = N$, for all $x \in T$.*

Proof. Let $x \in T$. Then we have

$$xN = Nx \iff (xN)x^{-1} = (Nx)x^{-1} \iff xNx^{-1} = N.$$

□

Proposition 2.4.3. *Let T be a group. We have that $Inn(T)$ is a subgroup of $Aut(T)$. Moreover, $Inn(T)$ is normal in $Aut(T)$.*

Proof. Suppose that $a, b \in Inn(T)$. Let $a = \psi_t$ and $b = \psi_s$ for some $t, s \in T$. For any $t, s \in T$, we have that $\psi_{ts}(x) = (ts)x(ts)^{-1} = t(sxs^{-1})t^{-1} = \psi_t(sxs^{-1}) = \psi_t[\psi_s(x)] = \psi_t\psi_s(x)$ for all $x \in T$. Thus $\psi_{ts} = \psi_t\psi_s = ab \in Inn(T)$. Now let $a = \psi_t \in Inn(T)$. So $\psi_t(\psi_{t^{-1}}(x)) = t(\psi t^{-1}(x))t^{-1} = t(t^{-1}xt)t^{-1} = x = \psi_e(x)$. Therefore $\psi_{t^{-1}}$ is the inverse in $Inn(T)$. Consider the identity automorphism $\psi : T \to T$ defined by $\psi(x) = x$. Then $\psi(x) = x = exe^{-1} = \psi_e(x)$. So ψ_e is the identity automorphism in $Inn(T)$. We then conclude that $Inn(T)$ is a subgroup of $Aut(T)$.

Now let $\sigma \in Aut(T)$ and $x \in T$, the $\sigma\psi_t\sigma^{-1}(x) = \sigma[t(\sigma^{-1}(x))t^{-1}] =$

$\sigma(t)x(\sigma(t))^{-1} = \psi_{\sigma(t)}(x)$. This implies that $\sigma\psi_t\sigma^{-1} = \psi_{\sigma(t)} \in Inn(T)$. Thus $Inn(T)$ is a normal subgroup of $Aut(T)$. □

Lemma 2.4.4. *Let S and N be subgroups of a group T with either S or N normal in T. Then SN is a subgroup of T.*

Proof. Let us assume that N is normal in T. Then by definition $xN = Nx$, for all $x \in T$. In particular, since $S \leq T$, we have that $SN = NS$. Therefore by Lemma 2.1.7, SN is a subgroup of T. Note that if both S and N are normal in T, then so is SN. □

Proposition 2.4.5. *Let T_1 and T_2 be groups. Let $\phi : T_1 \to T_2$ be a group homomorphism. Then $Ker\,\phi$ is a normal subgroup of T_1.*

Proof. We want to show that if $a \in Ker\,\phi$ and $b \in T_1$, then $bab^{-1} \in Ker\,\phi$. Thus, $\phi(bab^{-1}) = \phi(b)\phi(a)\phi(b)^{-1} = \phi(b)e\phi(b)^{-1} = e$, so $bab^{-1} \in Ker\,\phi$. Therefore $Ker\,\phi$ is a normal subgroup of T. □

Now we discuss the relationship between homomorphisms and quotient groups. Suppose that T_1 and T_2 are groups. Let $\psi : T_1 \to T_2$ be a group homomorphism. We can analyse ψ in terms of quotient groups. We first define a quotient group as follows:

Definition 2.4.6. Let T be a group and let N be a normal subgroup of T. The group of cosets of N in T is called a **quotient group** and is denoted by $T\,/\,N$.

Proposition 2.4.7. *Let T be a group and let N be a normal subgroup of T. The set $T\,/\,N$ is a group under the operation*

$$xN * yN = xyN,\ for\,all\,x, y \in T.$$

Proof. 1. Since $xN * yN = x(Ny)N = (xy)N \in T\,/\,N$, then the set $T\,/\,N$ is closed under the operation $*$.

2. The set $T\,/\,N$ is associative since T is associative.

3. We have that $eN = N$ is the identity in $T\,/\,N$. For any $x \in T$, $xN * N = xN * eN = x(Ne)N = (xe)N = xN$.

4. Now since $(xN)(x^{-1}N) = (xx^{-1})N = N$, then $x^{-1}N$ is the inverse of xN in $T\,/\,N$.

Finally, we show that the operation is well-defined. Take $x' \in xN$ and $y' \in yN$. Then $x' = xn$, for some $n \in N$, and similarly, $y' = xn'$, for some $n' \in N$. Then we have $x'N = xnN = xN$, also, $y'N = yn'N = yN$. Therefore $(x'N)(y'N) = (xN)(yN) = (xy)N$. Hence the proof is complete. □

Commutators are related to nilpotent groups, which we will study in the next chapter.

Definition 2.4.8. Let a and b be any two elements of the group T, we define the **commutator** of a and b denoted by $[a, b]$ as the element $aba^{-1}b^{-1}$.

Definition 2.4.9. Let T be any group. The **commutator subgroup or derived subgroup** of T denoted by $[T, T]$ or T', is the subgroup of T generated by all commutators in T

$$[T, T] = \{[a, b] \mid a, b \in T\}.$$

Definition 2.4.10. Let T be a non-abelian group and let $[T, T]$ be a subgroup of T. The group $T_{ab} = T\,/\,[T, T]$ is the **abelianization** of T.

Remark 4. A group T is abelian if and only if $[T, T] = \{e\}$.

Example 8. Let T be a group. Then the quotient T/T' is an abelian group.

Proof. Let $xT', yT' \in T/T'$. Then $xT'yT' = (xy)T' = yx(x^{-1}y^{-1}xy)T' = (yx)T'$ since $xyx^{-1}y^{-1} \in T'$. But $(yx)T' = (yT')(xT')$ and we have $(xT')(yT') = (yT')(xT')$. Thus T/T' is abelian. □

Example 9. T' is a characteristic subgroup of T.

Proof. Let $\phi : T \to T$ be an automorphism. Let a be a generator of T', such that $a = x^{-1}y^{-1}xy$ for some $x, y \in T$. Then $\phi(a) = \phi(x^{-1}y^{-1}xy) = \phi(x^{-1})\phi(y^{-1})\phi(x)\phi(y) \in T'$. Therefore for every generator, we have that $\phi(a) \in T'$. So $\phi(T') \leq T'$. Similarly, since ϕ is a bijection, there are some $t, s \in T$ with $\phi(t) = x$ and $\phi(s) = y$, so $a = x^{-1}y^{-1}xy = \phi(t)^{-1}\phi(h)^{-1}\phi(t)\phi(s) = \phi(t^{-1})\phi(s^{-1})\phi(t)\phi(s) = \phi(t^{-1}s^{-1}ts) \in \phi(T')$. Therefore $a \in \phi(G')$ for every generator, and so $\phi(T') = T'$. Therefore T' is characteristic and is a normal subgroup of T. □

Definition 2.4.11. Let T be any group. The **derived series** of T is a descending series of subgroups $(T^{(k)} : k = 0, 1, 2, \cdots)$ defined as follows:
$T^{(0)} = T; T^{(1)} = T' \cdots$ and by induction $T^{(k)} = (T^{(k-1)})'$ is the derived subgroup of $T^{(k-1)}$, for all $k = 2, 3, \cdots$. It is clear that by definition, $T^{(k-1)} \leq T^{(k)}$.

Lemma 2.4.12. *Let T be a group and let S be a subgroup of T. Then $S^{(k)} \leq T^{(k)}$ for all $k \in \mathbb{Z}$.*

Proof. We have that $S = S^{(0)}$ is a subgroup of $T = T^{(0)}$. Suppose that for $k > 0$, we have that $S^{(k-1)} \leq T^{(k-1)}$. Let l be any element of $S^{(k)} = (S^{(k-1)})' = [S^{(k-1)}, S^{(k-1)}]$. Then we can write $l = \prod_{j \in I} l_j$, where $l_j = [x_j, y_j]$, with x_j and y_j in $S^{(k-1)} \leq T^{(k-1)}$. This implies that $l \in T^{(k)}$, and the result follows. □

2.5 Group Actions

Definition 2.5.1. Let X be a set with n elements. A **permutation** of X is a bijection $\rho : X \to X$. The set of all permutations on X form a symmetric group denoted by S_n.

Definition 2.5.2. Let T be a group and let X be a set. We define an **action** of a group T on a set X as the function $T \times X \mapsto X$ given by $(t, x) \mapsto t * x$ such that for any $t, s \in T$ and any $x \in X$ the following equations are satisfied :

1. $e_T * x = x$.
2. $(ts) * x = t * (s * x)$.

We say that T acts on X and we call X a T-set.

Remark 5. This action or any other in general on a set is similar to a group homomorphism. Given a group action such as the one in Definition 2.5.2, we can define a group homomorphism $\sigma : T \to Aut(X)$ by $\sigma(t) = \tau : X \to X$, where $\sigma(x) = t * x$. Conversely, a group homomorphism induces a group action.

Definition 2.5.3. Let T be a group that acts on X and let $x \in X$. The **stabilizer** of x in T, denoted by $Stab_G(x)$ is defined as the set of all elements in T that fix x, that is $Stab_T(x) = \{t \in T \mid t * x = x\}$.

Proposition 2.5.4. *Let T be a group that acts on a set X and let $x \in X$. The stabilizer, $Stab_T(x)$ of $x \in X$ is a subgroup of T.*

Proof. We have that $e_T \in Stab_T(x)$. Thus $Stab_T(x)$ is non-empty. Now let $x_1, x_2 \in Stab_T(x)$, then $x_1 * t = x_2 * t = t$, so $(x_1 x_2) * t = x_1 * (x_2 * t) = x_1 * t = t$, so $x_1 x_2 \in Stab_T(x)$. If $x \in Stab_T(x)$, then $x^{-1} * t = x^{-1}(x * t) = (x^{-1}x)t = t$. Therefore $x^{-1} \in Stab_T(x)$. So $Stab_T(x) \leq T$. □

Definition 2.5.5. Let T be a group that acts on X and let $x \in X$. The **orbit** of x, denoted by $Orbit(x)$ is defined as the subset of X given by $Orbit(x) = \{t * x \mid t \in T\}$.

Note. If there is only one orbit, then the action is said to be **transitive**.

Theorem 2.5.6. *Let T be a finite group and let X be a set. Suppose that T acts on X. Then for any $x \in X$ we have*

$$\big|Orb(x)\big| = [T : Stab_T(x)].$$

Proof. We have that $Stab_T(x)$ is a subgroup of T. Then by Lagrange's Theorem, we have that $[T : Stab_T(x)]$ is the number of left cosets of $Stab_T(x)$ in T. Define $\phi : Orb(x) \to T \,/\, Stab_T(x)$ by $\phi(t * x) = t\, Stab_T(x)$. We must show that ϕ is a bijection. Suppose that $t_1, t_2 \in T$. Then

$$t_1 * x = t_2 * x \iff t_1^{-1}(t_2 x) = x \iff (t_1^{-1} * t_2)x = x$$
$$\iff t_1^{-1} * t_2 \in Stab_T(x) \iff t_1\, Stab_T(x) = t_2\, Stab_T(x).$$

Thus ϕ is well-defined and one-to-one. By definition, $Orbit(x) = \{t*x \mid t \in T\}$. Then for any $t\, Stab_T(x) \in T \,/\, Stab_T(x)$ there is $t*x \in Orbit(x)$ such that $\phi(t * x) = t\, Stab_T(x)$. Hence ϕ is onto. □

2.6 Isomorphism theorems and direct products

Theorem 2.6.1. *(First Isomorphism Theorem) Let T_1 and T_2 be groups. Let $\theta : T_1 \to T_2$ be a group homomorphism. Then $T_1 \,/\, Ker\,\theta \cong Im\,\theta$.*

Proof. Put $K = Ker\,\theta$ and define a function $\bar{\theta} : T_1/K \to Im\,\theta$ by $\bar{\theta}(xK) = \theta(x)$, for all $x \in T_1$. We must first check if $\bar{\theta}$ is well-defined. Assume that $xK = yK$; that is $xy^{-1} \in K$. Then $1 = \theta(xy^{-1}) = \theta(x)\theta(y^{-1}) = \theta(x)(\theta(y))^{-1}$. This implies that $\theta(x) = \theta(y)$, and it follows that $\bar{\theta}(xK) = \bar{\theta}(yK)$. Therefore, $\bar{\theta}$ is well-defined. Now we show that $\bar{\theta}$ is a homomorphism as follows:

$$\bar{\theta}(xKyK) = \bar{\theta}(xyK) = \theta(xy) = \theta(x)\theta(y) = \bar{\theta}(xK)\bar{\theta}(yK).$$

If $a \in T_2$, then $a = \theta(x)$ for some $x \in T_1$ and $\bar{\theta}(xK) = \theta(x) = a$. Hence $\bar{\theta}$ is surjective. Finally, we show that $\bar{\theta}$ is injective. If $\bar{\theta}(xK) = \bar{\theta}(yK)$, then $\theta(x) = \theta(y)$; hence $\theta(xy^{-1}) = 1$. This implies that $xy^{-1} \in K$, and $xK = yK$. Therefore, $\bar{\theta}$ is an isomorphism. $\square$

Example 10. Let T be a cyclic group.

1. If T is of infinite order, then $T \cong \langle \mathbb{Z}, + \rangle$.

2. If T is of finite order n, then $T \cong \langle \mathbb{Z}_n, + \rangle$.

Proof. Define $\psi : \mathbb{Z} \to T = \langle a \rangle$ by $\psi(r) = a^r$ with $Ker\,\psi = \{r \in \mathbb{Z} \mid \psi(r) = e_T\} = \{r \in \mathbb{Z} \mid a^r = e\}$ and $Im\,\psi = \{\psi(r) \mid r \in \mathbb{Z}\}$.
By Theorem 2.6.1, we have that $\mathbb{Z} \,/\, Ker\,\psi \cong Im\,\psi$. Now,

$$r_1 = r_2 \iff a^{r_1} = a^{r_2} \iff \psi(r_1) = \psi(r_2). \tag{2.2}$$

Therefore ψ is well defined. To show that ψ is a homomorphism, let $r_1, r_2 \in \mathbb{Z}$. Then we have that

$$\psi(r_1 + r_2) = a^{r_1+r_2} = a^{r_1}a^{r_2} = \psi(r_1)\psi(r_2).$$

By equation 2.2, we have that ψ is one-to-one. Let $a^r \in T$. Then $\psi(r) = a^r$, for all $r \in \mathbb{Z}$. Thus ψ is onto. This implies that ψ is an isomorphism.

1. If T is infinite, then $Ker\,\psi = \{0\}$. This implies that $\mathbb{Z} \,/\, \{0\} = \mathbb{Z}$ and $\mathbb{Z} \cong T$.

2. If T is finite with $|T| = n$, then $Ker\,\psi = n\mathbb{Z}$. This implies that $\mathbb{Z} \,/\, n\mathbb{Z} \cong T$ and $\mathbb{Z} \,/\, n\mathbb{Z} = \mathbb{Z}_n$.

□

The following two theorems are well known in group theory and can be found in [1] and [15].

Theorem 2.6.2. *(Second Isomorphism Theorem) Let S and N be subgroups of T, with N normal in T. Then $S \cap N$ is normal in S and $S \,/\; S \cap N \cong SN \,/\, N$.*

Theorem 2.6.3. *(Third Isomorphism Theorem) Let $K \subset S \subset T$ where S and K are both normal in T. Then $S \,/\, K$ is a normal subgroup of $T \,/\, K$ and $(T \,/\, K) \,/\, (S \,/\, K) \cong T \,/\, S$.*

Definition 2.6.4. Let $\langle T, *\rangle$ and $\langle S, \circ\rangle$ be groups. The **direct product** of T and S denoted by $T \times S$ is the set of ordered pairs $\{(t, s) \mid t \in T, s \in S\}$. The multiplication is given by $(t_1, s_1)(t_2, s_2) = (t_1 * t_2, s_1 \circ s_2)$.

Definition 2.6.5. Let T be a group. We say that T is an **internal direct product** of subgroups S and K if the following hold:

1. $S, K \triangleleft T$;
2. $T = SK$;
3. $S \cap K = \{e_T\}$.

Definition 2.6.6. Let S and K be groups. The **external direct product** of S and K denoted by $S \times K$ is the set of ordered pairs $\{(s, k) \mid s \in S, k \in K\}$ with the binary operation given by $(s, k) \cdot (s', k') = (ss', kk')$. The direct product $S \times K$ is a group under this operation.

Proposition 2.6.7. *Let T, S and K be groups and let $\psi : T \to K$ and $\phi : S \to K$ be group homomorphisms. Define a subset X of $T \times S$ by*

$$X = \{(t, s) \in T \times S \mid \psi(t) = \phi(s)\}.$$

The subset X is a subgroup of $T \times S$.

Proof. Let $(t, s), (m, n) \in X$. By definition we have that $\psi(t) = \phi(s)$ and $\psi(m) = \phi(n)$. We also have that $(t, s) * (m, n) = (tm, sn)$. To show that this is in the subset S, we note the following:

$$\begin{aligned} \psi(tm) &= \psi(t)\psi(m) && (since\, \psi\, is\, a\, homomorphism) \\ &= \phi(s)\phi(n) \\ &= \phi(sn). \end{aligned}$$

Thus $\psi(tm) = \phi(sn)$ and therefore we have that (tm, sn) is contained in S. Therefore S is closed under the group operation. Also,

$$\begin{aligned} \psi(t^{-1}) &= (\psi(t))^{-1} && (since\, \psi\, is\, a\, homomorphism) \\ &= (\phi(s))^{-1} \\ &= \phi(s^{-1}). \end{aligned}$$

Therefore $(t^{-1}, s^{-1}) \in X$ and X is a subgroup of $T \times S$. □

Definition 2.6.8. Let T be a group and let S and N be subgroups of T with N normal in T. The group T is called an **internal semi-direct product** of N by S if

1. $T = NS$ and
2. $N \cap S = \{e_T\}$.

Definition 2.6.9. Let S and N be any two groups. Let $\theta : S \to Aut(N)$ be a group homomorphism. The **external semi-direct product** of N by S, denoted by $N \rtimes_\theta S$ is the set $\{(n, s) \mid n \in N, s \in S\}$ with multiplication given by $(n_1, s_1) \cdot (n_2, s_2) = (n_1\theta(s_1)n_2, s_1s_2)$.

Note. If $\theta : S \to Aut(N)$ is the trivial homomorphism, then $N \rtimes_\theta S$ is the direct product $N \times S$.

Notation. The symbol $\rtimes$ is a combination of the symbol $\times$ and the symbol $\triangleleft$. The notation of the symbol actually tells us which is the normal subgroup. In addition, the structure does not only depend on the two groups, it also depends on the group homomorphism
$\theta : N \to Aut(S)$.

Definition 2.6.10. Let T be a group and let S and N be subgroups of T. The group T is called an **extension** of N by S if

1. $N \triangleleft T$,
2. $T \,/\, N \cong S$.

Definition 2.6.11. Let T, N and S be groups. A sequence of groups and group homomorphisms

$$1 \to N \xrightarrow{\alpha} T \xrightarrow{\beta} S \to 1 \tag{2.3}$$

is **exact** if α is injective, β is surjective and $Ker\,\alpha = Im\,\beta$.

Remark 6. The First Isomorphism Theorem shows that for the group T to be an extension, there must exist an exact sequence (2.3).

Definition 2.6.12. Let T be an extension. We say T is a **split extension** if there exist a homomorphism $\pi : S \to T$ such that $\beta \circ \pi = e_S$. In such a case the sequence in (2.3) is said to be a **split exact sequence**.

Proposition 2.6.13. *Let T be a group. We have that T is a semi-direct product of N and S if and only if T is a split extension of N by S.*

Proof. Suppose that T is a semi-direct product of N and S with N normal in T. For every $x \in T$, we can write $x = ns$, where $n \in N$ and $s \in S$. Define $\beta : T \to S$ by $\beta(x) = s$. We must show that β is a group homomorphism. But first, we show that β is well-defined. Let $x_1, x_2 \in T$. Suppose that $x_1 = x_2 \implies n_1s_1 = n_2s_2$. Then $n_2^{-1}n_1 = s_2s_1^{-1} \in N \cap S = \{e\}$. So $n_2^{-1}n_1 = e$ and $s_2s_1^{-1} = e$. Therefore, $n_1 = n_2$ and $s_2 = s_1$. Hence β is well defined. Note that since N is a normal subgroup of T, we have that $s_1n_2s_1^{-1} = n_2' \in N$ for all $n_2 \in N$ and $s_1 \in T$. Thus,

$$\beta(n_1s_1n_2s_2) = \beta(n_1s_1n_2s_1^{-1}s_1s_2) = \beta(n_1n_2's_1s_2) = s_1s_2 = \beta(n_1s_1)\beta(n_2s_2).$$

Hence β is a homomorphism. Then $Im\,\beta = S$ and $Ker\,\beta = N$. Now let $\beta' : S \hookrightarrow T$ be defined by $\beta'(s) = s$ for all $s \in S$. Then $\beta \circ \beta'(s) = \beta(s) = s$. So T is a split extension of N and S.

Conversely, suppose that T is a split extension. Let $\beta : T \to S$ and $\beta' : S \to T$ be group homomorphisms given by $\beta \circ \beta' = e_S$. Then $N = Ker\,\beta$ is a normal subgroup of T and $\bar{S} = Im\,\beta'$ is a subgroup of T. Now let $x \in N \cap \bar{S}$. Then $\beta(x) = e$ and $\beta'(s) = x$ for some $s \in S$. So we have that $s = \beta(\beta'(s)) = \beta(x) = e$. Thus, $\beta'\beta(x) = x = \beta'(e) = e$. From this, we can deduce that $N \cap \bar{S} = \{e\}$. Lastly, suppose that $x \in T$ and write

$$x = x \cdot \beta'(\beta(x))^{-1} \cdot \beta'(\beta(x)) = ns.$$

It is clear that $s \in \bar{S}$ and

$$\beta(n) = \beta(x \cdot \beta'(\beta(x))^{-1}) = \beta(x) \cdot \beta(\beta'(\beta(x))^{-1}) = \beta(x) \cdot \beta(x)^{-1} = e.$$

So that $n \in Ker\,\beta = N$. Thus T is a semi-direct product of N by $\bar{S}$, where $\bar{S} \cong S$. $\square$

Definition 2.6.14. Let A, B, C and D be groups and let f, g, α and β be group homomorphism. We say the diagram in Figure 2.1

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{\beta} & D \end{array}$$

Figure 2.1

commutes if $gf = \beta\alpha$.

Proposition 2.6.15. *Let A, B and C be groups. Let α, β and τ be* *morphisms such that $\beta \circ \tau = \alpha$ and suppose that*

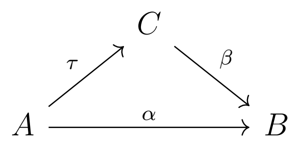

Figure 2.2

is a commutative diagram. If any two of the morphisms are bijections, then so is the third one.

Proof. Let $x \in A, y \in B$, and $z \in C$. Define $\tau : A \to C$ by $\tau(x) = z$, $\beta : C \to B$ by $\beta(z) = y$, and $\alpha : A \to B$ by $\alpha(x) = y$. We will consider 3 cases.

Case 1: Suppose that β and τ are bijective, then $\beta \circ \tau = \alpha$ is bijective. For any $x_1, x_2 \in A$, we have

$$\begin{aligned} & \beta \circ \tau(x_1) = \beta \circ \tau(x_2) & \\ \Rightarrow \quad & \tau(x_1) = \tau(x_2) & (\beta \text{ is injective}) \\ \Rightarrow \quad & x_1 = x_2. & (\tau \text{ is injective}) \end{aligned}$$

So $\beta \circ \tau = \alpha$ is injective. Now $\beta \circ \tau(x) = \beta(z) = y$. This implies that $\beta \circ \tau = \alpha$ is onto. Thus $\beta \circ \tau = \alpha$ is bijective.

Case 2: Suppose that τ and α are bijective, then β is bijective. For any $x_1, x_2 \in A$ and any $z_1, z_2 \in C$, we have

$$\begin{aligned}
& \beta(z_1) = \beta(z_2) & \\
\Rightarrow\quad & \beta \circ \tau(x_1) = \beta \circ \tau(x_2) & \\
\Rightarrow\quad & x_1 = x_2 & (\beta \circ \tau \text{ is injective}) \\
\Rightarrow\quad & \tau(x_1 = \tau(x_2) & (\tau \text{ is well-defined}) \\
\Rightarrow\quad & z_1 = z_2. & (\tau \text{ is onto})
\end{aligned}$$

So β is injective. And $y = \alpha(x) = \beta \circ \tau(x) = \beta(z)$. Therefore, β is surjrctive, and consequently bijective.

Case 3:. Suppose that β and α are bijective, then τ is bijective. For any $x_1, x_2 \in A$, we have

$$\begin{aligned}
& \tau(x_1) = \tau(x_2) & \\
\Rightarrow\quad & \beta \circ \tau(x_1) = \beta \circ \tau(x_2) & (\beta \text{ is well-defined}) \\
\Rightarrow\quad & x_1 = x_2. &
\end{aligned}$$

This implies that τ is injective. Now we have that

$$\begin{aligned}
& \beta \circ \tau(x) = y & \\
\Rightarrow\quad & \beta \circ \tau(x) = \beta(z) & (\beta \text{ is onto}) \\
\Rightarrow\quad & \tau(x) = z. & (\beta \text{ is injective})
\end{aligned}$$

Therefore τ is surjective. It follows that τ is bijective. This completes the proof. □

Chapter 3

Genus of a group

In this chapter, we discuss the results obtained by the author in [5]. The author showed that the genus of a finitely generated nilpotent group Q with a finite commutator subgroup is a cyclic group if the torsion subgroup of Q is a p-group. We start with a brief account of nilpotent groups. In section 3.2, we discuss the results of the localization of nilpotent groups. In section 3.3, we discuss the group structure of the non-cancellation set of $\mathscr{X}_0$-groups. We prove our main result in section 3.4 for a group $H \in \mathscr{N}_1$. Lastly, we show that the genus of a pullback $H_t \in \mathscr{N}_1$ is the same as the genus of H in section 3.5.

3.1 Nilpotent groups

We define and give a brief account of nilpotent groups.

Definition 3.1.1. Let T be a group. The lower central series of subgroups of T is given by

$$\cdots \leq \gamma^{i+1}(T) \leq \gamma^{i}(T) \leq \cdots \leq \gamma^{1}(T) \leq \gamma^{0}(T),$$

where $\gamma^0(T) = T$ and $\gamma^{i+1}(T) = [T, \gamma^i(T)]$ with $i \geq 1$.

Definition 3.1.2. A group T is **nilpotent** if it has a descending central series that terminates at $\{e_T\}$.

Example 11. Every abelian group is nilpotent.

Proposition 3.1.3. *Every subgroup of a nilpotent group is nilpotent.*

Proof. Let $S \leq T$, we then prove by induction on $k \in \mathbb{Z}$ that $\gamma^k(S) \leq \gamma^k(T)$. Trivially our conclusion is valid if $k = 0$ and holds for $k = 1$ since $S \leq T$. Now, assume $\gamma^{k-1}(S) \leq \gamma^{k-1}(T)$ for $k > 0$ and note that

$$\gamma^k(S) = [S, \gamma^{k-1}(S)] \leq [T, \gamma^{k-1}(T)] = \gamma^k(T).$$

Then if $\gamma^n(T) = \{e_T\}$ for $n \in \mathbb{Z}$, also $\gamma^n(S) = \{e_T\}$. Hence the result follows. □

Proposition 3.1.4. *If $N_1, \cdots, N_s$ are nilpotent groups then the direct product $N_1 \times \cdots \times N_s$ is also nilpotent.*

Proof. By induction on k we prove that $\gamma^k(T \times S) = \gamma^k(T) \times \gamma^k(S)$. Trivially the case for $k = 0$ is true. Assume that $\gamma^{k-1}(T \times S) = \gamma^{k-1}(T) \times \gamma^{k-1}(S)$, for $k > 0$. Let $t \in T$, $s \in S$ and (a_{k-1}, b_{k-1}) in $\gamma^{k-1}(T \times S) = \gamma^{k-1}(T) \times \gamma^{k-1}(S)$. We have that $[(t, s), (a_{k-1}, b_{k-1})]$ is a generator of $[G, \gamma^{k-1}(T \times S)] = \gamma^k(T \times S)$. But

$$[(t, s)(a_{k-1}, b_{k-1})] = ([t, a_{k-1}], [s, b_{k-1}]) \tag{3.1}$$

is in $\gamma^k(T) \times \gamma^k(S)$ implying that $\gamma^k(T \times S) \leq \gamma^k(T) \times \gamma^k(S)$. We note that equation 3.1 shows that a generator $([t, a_{k-1}], [s, b_{k-1}])$ of $\gamma^k(T) \times \gamma^k(S)$ is in $\gamma^k(T \times S)$. Thus there is equality. □

3.2 Localization of nilpotent groups

Our main tool in this work is the theory of localization of nilpotent groups. Localization of nilpotent groups has been developed by several authors including G. Mislin, P. Hilton, and J. Roitberg [9]. If Q is a nilpotent group, then it can be localized, this is denoted by Q_p for every prime p.

For notation convenience, we will denote by P, the set of primes p and P' will denote the set of primes not in the set P. By $n \in P'$ we mean that the natural number n is a product of primes from P'.

Definition 3.2.1. A nilpotent group T is called *P-local* if and only if for any $x \in T$, $x \mapsto x^n$ is a bijection for all $n \in P'$.

Definition 3.2.2. Let T be a group. Let $\epsilon : T \to T_p$ be a group homomorphism. We say that ϵ is a P-localizing map if T_p is P-local and for any group S, and a group homomorphism $\psi : T \to S$, there is a homomorphism $\phi : T_p \to S$ such that $\phi \circ \epsilon = \psi$. We denote by T_p the P-localization group of T.

Definition 3.2.3. Let T be any group. An element $x \in T$ is said to be P'-torsion if there exists an $n \in P'$ such that $x^n = 1$.

Definition 3.2.4. Let T_1 and T_2 be finitely generated abelian groups. Suppose that $\theta : T_1 \to T_2$ is a group homomorphism. We have the following.

1. If $Ker\,\theta$ and $Coker\,\theta$ are finite groups with order prime to p, we say that θ is a p-monomorphism and p-epimorphism respectively.

2. If θ is both a p-monomorphism and a p-epimorphism, we say that θ is a p-isomorphism.

Proposition 3.2.5 (Proposition 1.9, [9])**.** *Let T and U be groups. Let $\psi : T \to U$ be a group homomorphism. Then ψ is a p-localizing map if and only if*

1. *U is P-local and*

2. *ψ is a P-isomorphism.*

3.3 Group structure on the non-cancellation set of $\mathscr{X}_0$-groups

We introduce some of the basic properties of $\mathscr{X}_0$-groups. We begin by defining the $\mathscr{X}_0$ class.

Definition 3.3.1. The class $\mathscr{X}_0$, is the class of all finitely generated groups that have finite commutator subgroups.

Proposition 3.3.2. *Let Q be a $\mathscr{X}_0$-group. The torsion subgroup denoted by TQ is a finite normal subgroup of Q.*

Proposition 3.3.3. *Let Q be a $\mathscr{X}_0$-group, then $Q \times \mathbb{Z}$ is a $\mathscr{X}_0$-group. In addition, if a group K satisfies $K \times \mathbb{Z} \cong Q \times \mathbb{Z}$, then $K \in \mathscr{X}_0$.*

Definition 3.3.4. Let Q be any group. The non-cancellation set denoted by $\chi(Q)$ is the set of all isomorphism classes of groups R such that $R \times \mathbb{Z} \cong Q \times \mathbb{Z}$.

Note. If Q is a finite group, then the non-cancellation set $\chi(Q)$ is trivial.

Let Q be a $\mathscr{X}_0$-group and let TQ be the torsion subgroup of Q. In [20] the authors defined a natural number associated with the group Q and is linked to the non-cancellation set of Q. The number is defined as follows: For Q a $\mathscr{X}_0$-group, let n_1 be the exponent of TQ, n_2 be the exponent of $Aut(TQ)$ and n_3 be the exponent of TZQ. The natural number n is defined as $n(Q) = n_1 n_2 n_3$ and the subgroup $FQ = \langle q^n : q \in Q \rangle$, where $n = n(Q)$ is a normal subgroup of Q.

The following are some of the significant results on non-cancellation.

Theorem 3.3.5 (Theorem 3.6, [17]). *Let Q and R be nilpotent groups in the $\mathscr{X}_0$-class. Then $Q \times \mathbb{Z} \cong R \times \mathbb{Z}$ if and only if for every prime p, the p-localization of Q is isomorphisc to the p-localization of R for all primes p.*

Let us consider an infinite group Q in $\mathscr{X}_0$ and $n = n(Q)$. The author in [20] defined a surjective homomorphism from $\mathbb{Z}_n^*$ to the set $\chi(Q)$ by $x \mapsto [S]$, where S is a subgroup of Q of index x. This surjective homomorphism is shown to induce a transitive group action of $\mathbb{Z}_n^*/\{\pm 1\}$ on the set $\chi(Q)$. This transitive group action then induces a group structure on the non-cancellation set $\chi(Q)$.

Theorem 3.3.6 (Theorem 5.2, [20]). *Let Q be any group in $\mathscr{X}_0$ and let $n = n(Q)$. Then there is a group structure on $\chi(Q)$ induced by the function $\phi : \mathbb{Z}_n^*/\{\pm 1\} \to \chi(Q)$.*

The next results show the relationship between certain non-cancellation groups. Note that for an infinite group $Q \in \mathscr{X}_0$ and R a finite group, we have that $Q \times R$ is an infinite group and is in the class $\mathscr{X}_0$.

Theorem 3.3.7 (Theorem 6.2, [20]). *Let Q and R be any $\mathscr{X}_0$-groups with Q infinite. Then $\phi : \chi(Q) \to \chi(Q \times R)$ is a surjective group homomorphism.*

Theorem 3.3.8 (Theorem 6.2, [20]). *Let Q be an infinite $\mathscr{X}_0$-group. Let R be a finite characteristic subgroup of Q. Then $\phi : \chi(S) \to \chi(S/R)$ is a surjective group homomorphism.*

Theorem 3.3.9 (Theorem 2.1, [3]). *Let Q and R be $\mathscr{X}_0$-groups. If Q is infinite and R is finite such that $gcd(|TR|,|TQ|) = 1$, then $\phi : \chi(Q) \to \chi(Q \times R)$ is an isomorphism.*

Proof. By Theorem 3.3.7, we have that ϕ is a surjective homomorphism. Note that since Q is infinite, we can write $TQ = \{e_Q\}$, and since R is finite, then $TR = R$. From this we have that $gcd(|R|,|TQ|) = 1$. Now put $S = Q \times R$. Then we have that $TS = TQ \times R$. Since $gcd(|R|,|TQ|) = 1$ and every torsion subgroup is characteristic, we have that R is a characteristic subgroup of S. Recall that $Q \times R = S$ is infinite in $\mathscr{X}_0$. Therefore by Theorem 3.3.8, $\phi : \chi(S) \to \chi(S/R)$ is a surjective homomorphism. This implies that $\phi : \chi(Q \times R) \to \chi(Q)$ is a surjective homomorphism. We can conclude that $\chi(Q \times R)$ and $\chi(Q)$ are finite so that ϕ is an isomorphism. $\square$

Corollary 3.3.10 (Corollary 6.3, [20]). *Let Q and R be any $\mathscr{X}_0$-groups. If Q is infinite with $\chi(Q)$ trivial, then $\chi(Q \times R)$ is trivial.*

Proof. Suppose that Q is infinite and $\chi(Q)$ is trivial. We have that $\chi(Q) \cong \chi(Q \times R)$ by Theorem 3.3.9. Therefore, we can deduce that $\chi(Q \times R)$ is trivial. $\square$

3.4 A finite genus

Let $\mathscr{X}_0$ denote the class of all finitely generated groups that have a finite commutator subgroup. Let $\mathscr{N}_0$ be the class of all infinite nilpotent groups in $\mathscr{X}_0$ and let $Q \in \mathscr{N}_0$. Following [8], denote by $P-Aut(Q)$ the semigroup of P-automorphisms of Q, where P is the set of prime divisors of the exponent of the torsion of Q. We denote by ZQ the center of Q and by TZQ the torsion subgroup of ZQ. The free center of Q is given by $FZQ = \{x \in ZQ \mid x = y^n, y \in ZQ \text{ with } n = |TZQ|\}$. In particular, we define the finite group QQ by setting $QQ = Q \,/\, FZQ$. Then we denote by QQ_{ab} the abelianization group of QQ. Let e be the exponent of the finite abelian group QQ_{ab}.

The authors in [8] have shown that there is an abelian group structure on $\mathscr{G}(Q)$. This group structure arises from the exact sequence

$$P - Aut(Q) \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathscr{G}(Q). \tag{3.2}$$

The sequence (3.2) is useful in the computation of the genus set $\mathscr{G}(Q)$. We start by defining the Mislin genus $\mathscr{G}(Q)$ for any finitely generated nilpotent group Q.

Definition 3.4.1. Let Q be a finitely generated nilpotent group. The Mislin genus, denoted by $\mathscr{G}(Q)$, is the set of all isomorphism classes of finitely generated nilpotent groups R such that for all primes p, the p-localizations R_p and Q_p are isomorphic.

The genus was calculated in [2] for a group Q in $\mathscr{N}_1 \subset \mathscr{N}_0$. Thus we define the class $\mathscr{N}_1$.

Definition 3.4.2. We say that $Q \in \mathscr{N}_0$ if the torsion subgroup of TQ is finite and the torsion-free quotient FQ is free of finite rank. We say that $Q \in \mathscr{N}_1$ if, additionally

a) TQ and FQ are commutative;

b) the sequence $TQ \rightarrowtail Q \twoheadrightarrow FQ$ splits for the action $\omega : FQ \to Aut(TQ)$;

c) $\omega(FQ) \subseteq Z(Aut(TQ))$, where $Z(Aut(TQ))$ is the center of $Aut(TQ)$ and ω is the action of FQ on TQ.

Note. To avoid triviality of the genus $\mathscr{G}(Q)$, where $Q \in \mathscr{N}_1$, we will consider only those groups Q with a cyclic torsion-free quotient FQ.

Let $H \in \mathscr{N}_1$ and let $FH = \mathbb{Z}$ be a cyclic group generated by ζ. Let $p \in P$ and let $TQ = \mathbb{Z}/p^{n+k} = \langle a \rangle$. Suppose that the action ω of $\mathbb{Z}$ on $\mathbb{Z}/p^{n+k}$ is given by $\zeta \cdot a = ua$ for some $u \in \mathbb{N}$, where u is co-prime with p^{n+k}. Now let $n, k \in \mathbb{N}$ and suppose that $u = 1 + cp^k$, where $p \nmid c$ and $k \geq 1$. We consider the group

$$H = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^u \rangle. \tag{3.3}$$

Let p^n denote the order of $u \, modulo \, p^{n+k}$. We will show that H is nilpotent with a finite commutator subgroup. We will start by showing that $\mu : \mathbb{Z}_{p^n} \times \mathbb{Z}/p^{n+k} \to \mathbb{Z}/p^{n+k}$ is an action of $\mathbb{Z}_{p^n}$ on $\mathbb{Z}/p^{n+k}$.

Notation. For an integer x, we denote by $[x]_{p^{n+k}}$ the class which contains all elements $x \, modulo \, p^{n+k}$.

Proposition 3.4.3. *Let the cyclic group $\mathbb{Z}_{p^n}$ act on an abelian group $\mathbb{Z}/p^{n+k}$. Define $\mu : \mathbb{Z}_{p^n} \to Aut(\mathbb{Z}/p^{n+k})$ by $q \mapsto \tau_q = q[x]_{p^{n+k}}q^{-1}$, $q, r \in \mathbb{Z}_{p^n}$. Then μ is a group homomorphism.*

Proof. Let $[x]_{p^{n+k}} = [y]_{p^{n+k}}$, then we have that

$$\tau_q([x]_{p^{n+k}}) = q[x]_{p^{n+k}}q^{-1} = q[y]_{p^{n+k}}q^{-1} = \tau_q([y]_{p^{n+k}})$$

Therefore τ_q is well defined. Now we have that

$$
\begin{aligned}
\tau_{qr}([x]_{p^{n+k}}) =& qr[x]_{p^{n+k}}(qr)^{-1} \\
=& q(r[x]_{p^{n+k}}r^{-1})q^{-1} \\
=& q(r[x]_{p^{n+k}}r^{-1})q^{-1} \\
=& \tau_q(r[x]_{p^{n+k}}r^{-1}) \\
=& \tau_q(\tau_r([x]_{p^{n+k}})).
\end{aligned}
$$

Therefore we have proven that $\tau_{qr} = \tau_q\tau_r$. Thus, $\mu(qr) = \tau_{qr} = \tau_q\tau_r = \mu(q)\mu(r)$. And we have proven that μ is a homomorphism. Thus μ is an action. □

We give the definition of a nilpotent action.

Definition 3.4.4. Let Q be a cyclic group and let R be a finite abelian group. Let $\Gamma^i_Q(R)$ be the terms of the lower central series of an action of Q on R. Then we have

$$R = \Gamma^1_Q(R) \geq \Gamma^2_Q(R) \geq \cdots \geq \Gamma^{s-1}_Q(R) \geq \Gamma^s_Q(R) \geq \cdots$$

where $\Gamma^{s+1}_Q(R)$ is a subgroup of R generated by elements of the form $\{qr - r \mid q \in Q, r \in \Gamma^s_Q(R)\}$, and $\Gamma^s_Q(R) = \Gamma^2_Q(\Gamma^{s-1}_Q(R))$. The action of Q on R is said to be nilpotent if this series terminates at the identity. In addition, if $c = max\{s \mid \Gamma^s_Q(R) \neq \{0\}\}$ is finite, then we say the action has a nilpotency class of c.

Note. We will exclude the case where $p = 2, k = 1, n > 1$, since the order of $u\, modulo\, p^{n+k}$ is not given by p^n. For example, if $u = 1 + 3 \cdot 2^1$ and $n = 2$, then the order of $u\, modulo\, 2^3$ will be 2, and not 2^2.

Proposition 3.4.5. *Let the group $\mathbb{Z}_{p^n}$ act on $\mathbb{Z}/p^{n+k}$. Then the action*

$\mathbb{Z}_{p^n} \times \mathbb{Z}/p^{n+k} \to \mathbb{Z}/p^{n+k}$ *is nilpotent.*

Proof. Let $\mathbb{Z}_{p^n} = \langle \zeta \rangle$ and let p^n be the order of $u \, modulo \, p^{n+k}$. Define the action of $\mathbb{Z}_{p^n}$ on $\mathbb{Z}/p^{n+k}$ by $\zeta \cdot a = ua$, where $a \in \mathbb{Z}/p^{n+k}$. Let $\Gamma^i_{\mathbb{Z}p^n} \mathbb{Z}/p^{n+k}$, $i = 1, 2, \cdots, s$ be the terms of the lower central series of the action of $\mathbb{Z}_{p^n}$ on $\mathbb{Z}/p^{n+k}$. We must show that this action terminates at the identity. We proceed by induction on i. Suppose that $\Gamma^1_{\mathbb{Z}_{p^n}}(\mathbb{Z}/p^{n+k}) = \mathbb{Z}/p^{n+k}$. Then inductively we show that

$$\begin{aligned}
\Gamma^s_{\mathbb{Z}_{p^n}}(\mathbb{Z}/p^{n+k}) &= \Gamma^2_{\mathbb{Z}_{p^n}}(\Gamma^{s-1}_{\mathbb{Z}_{p^n}} \mathbb{Z}/p^{n+k}) \\
&= \Gamma^2_{\mathbb{Z}_{p^n}}((p^n)^{s-2}(\mathbb{Z}/p^{n+k})) \\
&= p^n (p^n)^{s-2} \mathbb{Z}/p^{n+k} \\
&= p^{ns-n} \mathbb{Z}/p^{n+k} \\
&= (p^n)^{s-1} \mathbb{Z}/p^{n+k} \\
&= i_{\mathbb{Z}/p^{n+k}}. \qquad \text{(since } p^{n+k} \mid u^{p^n} - 1)
\end{aligned}$$

□

Proposition 3.4.6. *Let $\mathbb{Z}$ and $\mathbb{Z}_{p^n}$ be cyclic groups. The function $g : \mathbb{Z} \to \mathbb{Z}_{p^n}$ is a surjection.*

Proof. Define $g : \mathbb{Z} \to \mathbb{Z}_{p^n}$ by $g(x) = x \, modulo \, p^n$. We first show that g is well defined. It is easy to see that if $x = x'$, then

$$g(x) = x \, modulo \, p^n = x' \, modulo \, p^n = g(x').$$

Therefore g is well defined. Now for any class $m \, modulo \, p^n$, there exists $x \in \mathbb{Z}$ such that the class of $x \, modulo \, p^n$ is m. Thus for any $m \in \mathbb{Z}_{p^n}$ there exist $x \in \mathbb{Z}$ such that $g(x) = x \, mod \, p^n$. Therefore g is surjective. □

Lemma 3.4.7. *Let $\mathbb{Z}$ be a group acting on $\mathbb{Z}/p^{n+k}$. Then the action of $\mathbb{Z}$ on $\mathbb{Z}/p^{n+k}$ is nilpotent.*

Proof. We have proven that the action of $\mathbb{Z}_{p^n}$ on $\mathbb{Z}/p^{n+k}$ is nilpotent; that is, an element of $\mathbb{Z}_{p^n}$ acts on $\mathbb{Z}/p^{n+k}$ nilpotently. We have also shown that the mapping $g : \mathbb{Z} \to \mathbb{Z}_{p^n}$ is a surjection. Then since $\mathbb{Z}_{p^n}$ represents all possible residues of integers modulo p^n, we can find $\bar{x} \in \mathbb{Z}_{p^n}$ such that $x \in \mathbb{Z}$ with $g(x) = \bar{x}$. Thus the elements of modulo classes of $\mathbb{Z}_{p^n}$ have pre-images in $\mathbb{Z}$. Then an element of $\mathbb{Z}$ acts on $\mathbb{Z}/p^{n+k}$ nilpotently. We can deduce that the action of $\mathbb{Z}$ on $\mathbb{Z}/p^{n+k}$ is nilpotent. Then since $\mathbb{Z}$ is a nilpotent group that acts nilpotently on $\mathbb{Z}/p^{n+k}$, the semi-direct product $H = \mathbb{Z}/p^{n+k} \rtimes_\omega \mathbb{Z}$ is itself nilpotent. □

Proposition 3.4.8. *Let $H = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^u \rangle$. Then the group H has a finite commutator subgroup.*

Proof. The elements of the group H are of the form $x^m y^l$. Then we have the following commutators,

$$\begin{aligned}
[x^m y^l, x] =& x^m y^l x (x^m y^l)^{-1} x^{-1} \\
=& x^m y^l x x^{-m} y^{-l} x^{-1} \\
=& x^m y^l x^m x y^{-l} x^{-1} \\
=& x^m x^{-m} y^l x y^{-l} x^{-1} \\
=& y^l x y^{-l} x^{-1} \\
=& x^{u^l} x^{-1} \\
=& x^{u^l - 1}.
\end{aligned}$$

We also have that

$$\begin{aligned}[x^m y^l, y] =& x^m y^l y (x^m y^l)^{-1} y^{-1} \\ =& x^m y^l y x^{-m} y^{-l} y^{-1} \\ =& x^m y^l y y^l x^{-m} y^{-1} \\ =& x^m y^l y^{-l} y x^{-m} y^{-1} \\ =& x^m y x^{-m} y^{-1} \\ =& x^m x^{-um} \\ =& x^{m-um}.\end{aligned}$$

Since the order of $u\, modulo\, p^{n+k}$ is p^n, we have that if $p^n \mid l$, then $p^{n+k} \mid u^l - 1$. This implies that $u^l - 1 = hp^{n+k}$, $h \in \mathbb{Z}^+$. Hence we can write $x^{u^l-1} = x^{hp^{n+k}} = (x^{p^k})^{hp^n}$, and $x^{m-um} = (x^{p^k})^{-mc}$. This implies that the commutators are powers of x^{p^k}, so the commutator subgroup is going to be generated by x^{p^k}. Therefore, we deduce that $[H, H] = \langle x^{p^k} \rangle$. We note that since the torsion-free quotient group $\mathbb{Z}$ of H is commutative, we can deduce that the commutator subgroup is finite. □

We have shown that $H \in \mathscr{N}_0$. Now since $\mathbb{Z}/p^{n+k}$ is cyclic, we have that $Aut(\mathbb{Z}/p^{n+k})$ is abelian. Therefore $Z(Aut(\mathbb{Z}/p^{n+k})) = Aut(\mathbb{Z}/p^{n+k})$. Thus we can deduce that $\omega(\mathbb{Z}) \subseteq Z(Aut(\mathbb{Z}/p^{n+k})$. Hence $H \in \mathscr{N}_1$. Now we show that the genus of $H \in \mathscr{N}_1$ is finite and consists of a finite number of classes $[H_i]$, where $i = 0, 1, 2, \cdots, s-1$ for some $s \in \mathbb{Z}^+$.

Since $H \in \mathscr{N}_0$, we will consider the exact sequence [8]

$$P - Aut(H) \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathscr{G}(H). \tag{3.4}$$

Note. In the next result, for $p \in P$, the p-part of the exponent e of QH_{ab}, is given by $e = p^{n+k}$.

Theorem 3.4.9. *Let $H \in \mathscr{N}_1$. The genus of H is a finitely generated cyclic group.*

Proof. We draw our attention to the image of the homomorphism $\theta : P - Aut(H) \to (\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$. Note that since both groups $P - Aut(H)$ and $(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$ are multiplicative, then θ is multiplicative, so $Im\,\theta$ is a subgroup of $(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$. Now we use Theorem 1.4 of [8] to show that the sequence in (3.4) is exact. Let $\phi \in P - Aut(H)$, if $x = y\theta(\phi)$ then $\theta(\phi) = y^{-1}x \implies y^{-1}x \in Im\,\theta$. Therefore $y^{-1}x \in (\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$. The author in [14] defined a surjective homomorphism $\delta : (\mathbb{Z}/p^{n+k})^*/\{\pm 1\} \to \mathscr{G}(H)$ as follows. Let $\bar{x} \in (\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$ be represented by $x \in (\mathbb{Z}/p^{n+k})^*$. Then $\delta(\bar{x}) = H_1$, where $H_1 \in \mathscr{G}(H)$. Now we use the fact that $\delta(y) = \delta(x)$ and that δ is a surjective group homomorphism. Then we show that

$$\delta(y^{-1}x) = \delta(y^{-1})\delta(x) = \delta(y^{-1})\delta(y) = e_{\mathscr{G}(H)}.$$

Therefore $y^{-1}x \in Ker\,\delta$.

Conversely suppose that $y^{-1}x \in Ker\,\delta$. Since $Ker\,\delta$ is a subgroup of $(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$, we have that $y^{-1}x \in (\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$, which implies that $y^{-1}x \in Im\,\theta$. Hence we can deduce that $Ker\,\delta = Im\,\theta$. Implying that sequence in (3.4) is exact. Furthermore, since $Im\,\theta$ is a subgroup of $(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$, then $Im\,\theta$ should consist of the units of $(\mathbb{Z}/p^{n+k}), mod\{\pm 1\}$ which are congruent to 1 modulo p^n. Thus we have that the number of such units are p^k. Therefore $|Im\,\theta| = p^k$. Lastly, since $\delta : (\mathbb{Z}/p^{n+k})^*/\{\pm 1\} \to \mathscr{G}(H)$ is a surjective group homomorphism, by the First Isomorphism Theorem,

$$(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}\Big/_{Ker\,\delta} \cong \mathscr{G}(H).$$

But $Ker\,\delta = Im\,\theta$, so

$$\begin{aligned}\left|\mathscr{G}(H)\right| &= \frac{\left|(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}\right|}{|Im\,\theta|} \\ &= \frac{p^{n+k-1}(p-1)/2}{p^k} \\ &= p^{n-1}(p-1)/2.\end{aligned}$$

This implies that the group $\mathscr{G}(H)$ is finite and cyclic with order s, where $s = p^{n-1}(p-1)\,/\,2$. Therefore $\mathscr{G}(H) \cong (\mathbb{Z}/p^n)^*/\{\pm 1\}$. □

Remark 7. Since the genus $\mathscr{G}(H)$ is a finite cyclic group, it must have a generator. Let $h \in \mathbb{Z}^+$ be the generator and s be the least exponent of h such that $h^s \equiv \pm 1\, mod\, p^n$.

Definition 3.4.10. If two nilpotent groups T_1 and T_2 are p-isomorphic at every prime p, then they are in the same localization genus.

Let $H_i = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^{u^{m^i}} \rangle$, where $0 \leq i \leq s-1, s \in \mathbb{Z}^+$. Then we prove the following lemma.

Lemma 3.4.11. *Let $H, H_1 \in \mathscr{N}_1$. Then $\phi : H \to H_1$ is a homomorphism.*

Proof. Recall that P' is a set of primes not in P. Let $m \in \mathbb{Z}^+$ be coprime to p^n such that $hm \equiv 1\, mod\, p^n$. Now Suppose that H_1 is given by

$$H_1 = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^{u^m} \rangle \tag{3.5}$$

for each $[m] \in (\mathbb{Z}/p^n)^*/\{\pm 1\}$. As in [5], let $\phi : H \to H_1$ be defined by $\phi(x) = x$, $\phi(y) = y^h$. In H_1, we have that $yxy^{-1} = x^{u^m}$, so $y^h x y^{-h} = x^{u^{mh}}$. But since $hm \equiv 1 \, mod \, p^n$, $x^{u^{mh}} = x^u$, therefore $y^h x y^{-h} = x^u$ in H_1. Thus ϕ is a homomorphism. □

Proposition 3.4.12. *Let $H, H_1 \in \mathscr{N}_1$. Let $\phi : H \to H_1$ be a group homomorphism. Then ϕ is a P-equivalence.*

Proof. Let p be a prime and let $\gamma : \mathbb{Z} \to \mathbb{Z}$ be defined by $\gamma(\zeta) = \zeta^h$. Then the homomorphism $\phi : H \to H_1$ yields a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}/p^{n+k} & \rightarrowtail & H & \twoheadrightarrow & \mathbb{Z} \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} \\ \mathbb{Z}/p^{n+k} & \rightarrowtail & H_1 & \twoheadrightarrow & \mathbb{Z} \end{array}$$

Figure 3.1

Since $h \in P'$ then $\mathbb{Z}$ is P-local by Definition 3.2.1. Thus $\gamma_p : \mathbb{Z}_p \cong \mathbb{Z}_p$. Suppose we localize at p, then by Theorem 3.2.5, ϕ is a P-isomorphism. Then we can deduce that $\phi : H \to H_1$ is a P-equivalence. □

We have proven that for $H, H_1 \in \mathscr{N}_1$, $\phi : H \to H_1$ is a P-equivalence where TZH is finite. Therefore ϕ induces an isomorphism $\phi_F : FZH \to FZH_1$. Thus we now show that H_1 generates the genus group $\mathscr{G}(H)$, where H is the neutral element in the group $\mathscr{G}(H)$.

Theorem 3.4.13. *Let $H_1 \in \mathscr{N}_1$ and $H_1 \in \mathscr{G}(H)$. Then H_1 generates the group of the genus $\mathscr{G}(H)$.*

Proof. We first show that $FZH \cong FZH_1$ and $QH \cong QH_1$. To show that $\phi_F : FZH \to FZH_1$ is an isomorphism it is required to show for all primes p that $(\phi_F)_p : (FZH)_p \to (FZH_1)_p$ is an isomorphism.

Now suppose that $p \in P$, then by Lemma 4 of [14] we have that $(\phi_F)_p : (FZH)_p \to (FZH_1)_p$ is an isomorphism. Now we have the following commutative diagram :

$$\begin{array}{ccccc} FZH & \rightarrowtail & H & \twoheadrightarrow & QH \\ \downarrow{\scriptstyle \phi_F} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_Q} \\ FZH_1 & \rightarrowtail & H_1 & \twoheadrightarrow & QH_1 \end{array}$$

Figure 3.2

Since ϕ is a P-equivalence and ϕ_F is an isomorphism with QH and QH_1 finite isomorphic groups, ϕ_Q must be an isomorphism. Thus we can say that ϕ_F is a P-equivalence. This implies that the $\det \phi_F$ is an integer co-prime to P. Hence we can conclude that since $h \in P'$, h is that integer, so $\det \phi_F = h$. Now consider the surjection $\delta : (\mathbb{Z}/p^{n+k})^*/\{\pm 1\} \to \mathscr{G}(H)$ and let $[h] \in (\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$ such that $\delta([h]) = H_1$. Then δ sends $[h]$ to H_1. Thus since h is co-prime to the order of $(\mathbb{Z}/p^{n+k})^*/\{\pm 1\}$, then it generates the group. Hence $H_1 \in \mathscr{G}(H)$ must generate the genus group of H. □

The authors in [5] define H_i, $0 \le i \le s-1, s \in \mathbb{Z}^+$ by

$$H_i = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^{u^{m^i}} \rangle, \tag{3.6}$$

with $\phi_i : H_i \to H_{i+1}$ given by $\phi_i(x) = x$, $\phi(y) = y^h$. Then we prove the following proposition.

Proposition 3.4.14. *The group $\mathscr{G}(H)$ consists of a finite number of classes $[H_i], 0 \le i \le s-1, s \in \mathbb{Z}^+$.*

Proof. We must show that ϕ_i embeds each H_i as a normal subgroup of H_{i+1} with quotient C_h. But first, we show that ϕ_i is injective. Already we have that ϕ_i is a P-equivalence. So ϕ_i must be P-injective. But H_i has no torsion co-prime to P. Thus ϕ_i is injective. Now we are left to show that $\phi_i(H_i)$ is normal in H_{i+1}. By Corollary 1.4 of [7] we must show that for all primes p, $(\phi_i)_p((H_i)_p)$ is normal in $(H_{i+1})_p$. If $p \in P$, then $(\phi_i)_p((H_i)_p) = (H_{i+1})_p$. But $(H_{i+1})_p$ is commutative if $p \in P'$. Therefore as in Example 5.2, we have that $(\phi_i)_p((H_i)_p)$ is normal in $(\phi_i)_p((H_i)_p)$. Thus we can deduce that each $[H_i]$ belongs to $\mathscr{G}(H)$. $\square$

Remark 8. For $H, H_1 \in \mathscr{N}_1$, we have the non-cancellation phenomenon

$$H \times \mathbb{Z} \cong H_1 \times \mathbb{Z}, \quad H \ncong H_1.$$

3.5 The Pullback

Let $H = \langle x, y \mid x^{p^{n+k}} = 1, yxy^{-1} = x^u \rangle$. Our aim in this section is to construct a pullback H_t, where $t \equiv i + j \, mod \, s$ from the l-equivalences $H_i \mapsto H$ and $H_j \mapsto H$, which we will define later. We further show that the genus of H_t is the same as the genus of H from Section 3.4.

Definition 3.5.1. A **category** $\mathcal{C}$ is a collection of objects, morphisms $Hom(A, B)$ between two objects A and B, and a composition of morphisms $Hom(A, B) \times Hom(B, C) \to Hom(A, C)$ written as gf, for every object A, B and C.

The morphisms are subject to the following 3 axioms:

1. each morphism α in $Hom(A, B)$ has a unique domain A and a unique co-domain B;

2. for each object A, there exists an identity morphism 1_A in $Hom(A, A)$ such that for every morphism $\alpha : A \to B$, we have $\alpha 1_A = \alpha$ and $1_A \alpha = \alpha$;

3. given three morphisms $\alpha : A \to B$, $\beta : B \to C$ and $\mu : C \to D$. The associative law

$$\mu(\beta\alpha) = (\mu\beta)\alpha$$

holds.

The methods of computations of our work mainly rely on notions of category theory. Hence we introduce and present properties of a pullback diagram.

Definition 3.5.2. Given two morphisms $f : A \to C$ and $g : B \to C$ in a category $\mathcal{C}$, a pullback is a triple (P, α, β) that satisfies $f\alpha = g\beta$ as indicated in the diagram below

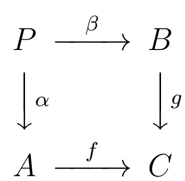

Figure 3.3

Moreover, the pullback satisfies the following universal property: for any other such triple (Q, q_1, q_2) with $fq_1 = gq_2$, there must exist a unique morphism $\delta : Q \to P$ such that the following diagram commute.

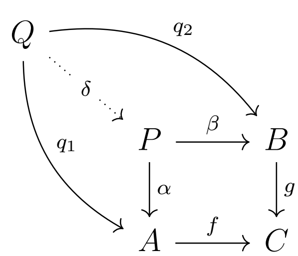

Figure 3.4

The pullback is often written $P = A \times_C B$, and is a subgroup of $A \times B$.

Theorem 3.5.3. *A commutative diagram of groups and morphisms*

$$\begin{array}{ccc} M & \xrightarrow{\psi} & B \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \rho} \\ A & \xrightarrow{\sigma} & C \end{array}$$

Figure 3.5

is a pullback if and only if the following conditions hold:

1) $Ker\,\psi \cap Ker\,\phi = \{0\}$,

2) $\rho^{-1}(\sigma(A)) = \psi(M)$,

3) $\phi(Ker\,\psi) = Ker\,\sigma$.

Proof. The two morphisms ψ and ϕ define a unique morphism $\gamma : M \to A \oplus_C B$. If the diagram given is a pullback, then γ is an isomorphism. We can then say that conditions 1) and 2) are clearly satisfied. Now to prove the third condition, we take $a \in Ker\,\sigma$ and we consider $(a, 0) \in A \oplus_C B$. Assume $\gamma(m) = (a, 0)$, for $m \in M$ which implies that $\phi(m) = a$ and $\psi(m) = 0$. Thus $ker\sigma \subset \phi(Ker\,\psi)$ and automatically $\phi(Ker\,\psi) \subset ker\sigma$ then the result follows.

Conversely, if the three conditions hold, then γ is injective by 1). Now to show surjectivity let $(a, b) \in A \oplus_C B$. Then $b \in \rho^{-1}(\sigma(A))$ since $\sigma(a) = \rho(b)$, so $b = \psi(m)$ for $m \in M$ by condition 2). Now,

$$\sigma(a - \phi(m)) = \sigma(a) - \sigma(\phi(m)) = \rho(b) - \rho(\psi(m)) = 0,$$

so, $a - \phi(m) = \phi(n)$ for some $n \in Ker\,\psi$ by 3). Consequently

$$\gamma(m + n) = (\phi(m + n), \psi(m)) = (a, b).$$

Hence γ is surjective. □

For the following two results we will refer to the diagram in Figure 3.5 of Theorem 3.5.3.

Theorem 3.5.4. *Let the diagram in Figure 3.5 be a pullback. Then we have that ϕ is surjective and $\phi^{-1}(Ker\,\sigma) = Ker\,\psi \oplus Ker\,\phi$. Moreover, ϕ is injective if and only if ρ is injective.*

Proof. Since the diagram in Figure 3.5 is a pullback, conditions 1), 2) and 3) hold. We first show by contradiction that ϕ is surjective. Suppose there exists $a \in A$ such that $\phi(m) \neq a$ for each $m \in M$. By condition 3), we then have that $a \notin Ker\,\sigma$. By condition 2), there exists $m' \in M$ such that $\rho(\psi(m')) = \sigma(a)$. Since the figure commutes, $\sigma(a) = \sigma(\phi(m'))$ and $a - \phi(m') \in Ker\,\sigma$. By condition 3), there exists $m'' \in Ker\,\psi$ such that $a - \phi(m') = \phi(m'')$. Thus, $\phi(m' + m'') = a$, contradicting our initial assumption. Hence, ϕ is surjective.

Now let $m + n \in Ker\,\psi \oplus Ker\,\phi$. Then by condition 3), we have $\phi(m + n) = \phi(m) \in Ker\,\sigma$. Then $Ker\,\psi \oplus Ker\,\phi \subseteq \phi^{-1}(Ker\,\sigma)$. Conversely, let $m \in \phi^{-1}(Ker\,\sigma)$ be arbitrary. Then there exists $a \in Ker\,\sigma$ such that $\phi(m) = a$. We also have $M = (Ker\,\phi)^{\perp} \oplus Ker\,\phi$, since $Ker\,\phi$ is a closed ideal of the algebra M. For some $n \in (Ker\,\phi)^{\perp}$ and $n' \in Ker\,\phi$ we can write m as $n + n'$. Hence, $\phi(m) = \phi(n') + \phi(n) = \phi(n)$ and by conditon 3), there exists $n'' \in Ker\,\psi \subseteq (Ker\,\phi)^{\perp}$ such that $\phi(n'') = a$. By the above it is enough to show that $n \in Ker\,\psi$, but since $\phi(n'' - n) = a - a = 0$, $n'' - n \in Ker\,\phi$. But again, $n'' - n \in (Ker\,\phi)^{\perp}$. Thus, $n = n'' \in Ker\,\psi$.

Suppose that ϕ is injective. By condition 3), $Ker\,\psi = \phi^{-1}(Ker\,\sigma)$.

By condition 2), if $n \in Ker\,\rho = \rho^{-1}(\{0\}) \subseteq \rho^{-1}(\sigma(A)) = \psi(M)$, there exists $m \in M$ such that $\psi(m) = n$. Thus, $\sigma(\phi(m)) = \rho(\psi(m)) = \rho(n) = 0$. This forces that $\phi(m) \in Ker\,\sigma$ or $m \in \phi^{-1}(Ker\,\sigma) = Ker\,\psi$. Therefore, $n = \psi(m) = 0$ and ρ is injective.

Now suppose ρ is injective. To prove that ϕ is injective, we show that $Ker\,\phi = \{0\}$ or $\phi^{-1}(Ker\,\sigma) = Ker\,\psi \oplus \{0\}$. By condition 3), $Ker\,\psi \subseteq \phi^{-1}(Ker\,\sigma)$. To show that $\phi^{-1}(Ker\,\sigma) \subseteq Ker\,\psi$ is enough. So, let $m \in \phi^{-1}(Ker\,\sigma)$. Then there exists $a \in Ker\,\sigma$ such that $\phi(m) = a$, but $0 = \sigma(a) = \sigma(\phi(m)) = \rho(\psi(m))$. Since ρ is injective, $\psi(m) = 0$ and hence, $m \in Ker\,\psi$. Thus the result follows. □

Theorem 3.5.5. *Suppose the diagram in Figure 3.5 is a commutative diagram of groups and morphisms. Then the diagram is a pullback and ϕ is an injection if and only if ϕ is surjective, ρ is injective and $Ker\,\psi \cap Ker\,\phi = \{0\}$.*

Proof. Suppose that the diagram in Figure 3.5 is pullback and that ϕ is an injection. By Theorem 3.5.4 we then have that ϕ is surjective, ρ is an injection, and by Theorem 3.5.3 $Ker\,\psi \cap Ker\,\phi = \{0\}$.

Conversely, let ϕ be surjective, ρ be an injection, and $Ker\,\psi \cap Ker\,\phi = \{0\}$. We want to show that $\rho^{-1}(\sigma(A)) = \psi(M)$ and $\phi(Ker\,\psi) = Ker\,\sigma$. Now let $n \in \psi(M)$, then $n = \psi(m)$ for some $m \in M$. Put $a = \phi(m)$, hence $\rho^{-1}(\sigma(a)) = \rho^{-1}(\sigma(\phi(m))) = \rho^{-1}(\rho(\psi(m))) = \psi(m) = n$. Therefore, $n \in \rho^{-1}(\sigma(A))$.

Conversely, suppose $n \in \rho^{-1}(\sigma(A))$. Then there exists $a \in A$ such that $n = \rho^{-1}(\sigma(a))$. Since ϕ is surjective, there exists $m \in M$ such that $\phi(m) = a$. Thus, $n = \rho^{-1}(\sigma(\phi(m))) = \rho^{-1}(\rho(\psi(m))) = \psi(m)$, since ρ is an injection. So $n \in \psi(M)$. Therefore, $\rho^{-1}(\sigma(A)) = \psi(M)$.

We now prove that $\phi(Ker\,\psi) = Ker\,\sigma$. Let $a \in \phi(Ker\,\psi)$. Then there exists $m \in Ker\,\psi$ such that $a = \phi(m)$. Since the diagram in Figure 3.5 is commutative and ρ is an injection, we have $\sigma(a) = \sigma(\phi(m)) = \rho(\psi(m)) = 0$. Therefore, $a \in Ker\,\sigma$ and hence $\phi(Ker\,\psi) \subseteq Ker\,\sigma$.
Conversely, let $a \in Ker\,\sigma$. Then $\phi(m) = a$ for an element $m \in M$ since ϕ is surjective. However, $\sigma(a) = \sigma(\phi(m)) = \rho(\psi(m)) = 0$. Therefore since ρ is an injection, $\psi(m) = 0$. Thus implying that $m \in Ker\,\psi$ and also, $a = \phi(m) \in \phi(Ker\,\psi)$. Hence, $Ker\,\sigma \subseteq \phi(Ker\,\psi)$ and so $\phi(Ker\,\psi) = Ker\,\sigma$.
Lastly, since ρ is an injection, by Theorem 3.5.4, ϕ is too an injection. And this completes the proof. □

Proposition 3.5.6. *Let G, H and K be nilpotent groups. If the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\phi} & G \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \tau} \\ H & \xrightarrow{\sigma} & K \end{array}$$

Figure 3.6

is a pullback, then L is nilpotent.

Proof. We have that L is a subgroup of $G \times H$. Since G and H are nilpotent, then by Proposition 3.1.4 the group $G \times H$ is also nilpotent. Therefore L is nilpotent by Proposition 3.1.3. In addition, since $Ker\,\tau \cong Ker\,\psi$, it is clear that ψ is a P-injection if and only if τ is a P-injection, which we can verify from Theorem 3.5.4. Then by Theorem 3.5.5, ψ is a P-surjection since ψ is a P-injection. □

Let H_i, H_j and H be nilpotent groups and let $\sigma : H_i \to H$ and $\rho : H_j \to H$. Suppose that

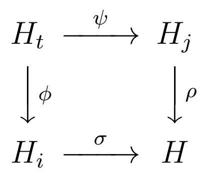

Figure 3.7

is a commutative square. Then we prove the following proposition.

Proposition 3.5.7. *In Figure 3.7, let us assume that ϕ, ψ, ρ and σ are bijective. Then the diagram in figure 3.7 is a pullback.*

Proof. We form the pullback of ρ and σ in $\mathscr{N}$ and obtain the following diagram

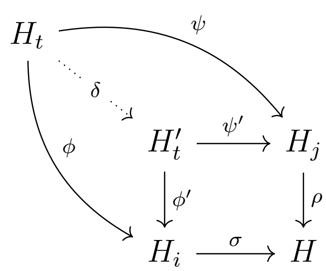

Figure 3.8

We must show that the unique homomorphism $\delta : H_t \to H'_t$ is an isomorphism. Since ρ is bijective, then we have that ϕ' is bijective; and since σ is bijective, then ψ' is also bijective. Thus by Proposition 2.6.15 we conclude that δ is also bijective, therefore δ is an isomorphism. Moreover, $H_t = H_i + H_j$. □

We proceed to show that the genus of H_t is the same as the genus of H.

Corollary 3.5.8. *If $H \in \mathscr{N}_0$, then $\lambda : \mathscr{G}(H) \to \mathscr{G}(H_t)$ is an isomorphism.*

Proof. By Theorem 3.4.9, we know that $\mathscr{G}(H)$ is a finite cyclic group of order s. Now we have that $t \equiv i + j \, mod \, s$, implying that the genus $\mathscr{G}(H_t)$ of H_t is of order s. Since $\mathscr{G}(H)$ and $\mathscr{G}(H_t)$ are of the same order, then $\lambda : \mathscr{G}(H) \to \mathscr{G}(H_t)$ is a bijection. Now since these groups are cyclic, let $\mathscr{G}(H) = \langle H_1 \rangle$ and $\mathscr{G}(H_t) = \langle X \rangle$. Let us define the bijection $\lambda : \mathscr{G}(H) \to \mathscr{G}(H_t)$ by $\lambda(H_1^n) = X^n$, for some $n \in \mathbb{Z}^+$. Let $H_2, H_3 \in \mathscr{G}(H)$. Since H_1 generates $\mathscr{G}(H)$, there exist $r, q \in \mathbb{Z}^+$ such that $H_2 = H_1^r, H_3 = H_1^q$. Then

$$\lambda(H_2H_3) = \lambda(H_1^{r+q}) = X^{r+q} = X^rX^q = \lambda(H_1^r)\lambda(H_1^q) = \lambda(H_2)\lambda(H_3).$$

Thus λ is a homomorphism and therefore an isomorphism. □

Remark 9. We proved that the commutator subgroup of the finitely generated nilpotent group H is finite, which by [2] implies that the genus $\mathscr{G}(H)$ is abelian. Since we have that $FH = FH_t$, then $H_t \in \mathscr{N}_0$ so that $\mathscr{G}(H_t)$ is finite abelian.

Chapter 4

Genus of a pullback

4.1 Pullback

Proposition 4.1.1. *Let n and u be positive integers which are relatively prime. Let $\mu : \mathbb{Z} \to Aut(\mathbb{Z}_n)$ be a group action given by $\mu(1) : \mathbb{Z}_n \to \mathbb{Z}_n$, $t \mapsto ut$. Let G and K be groups such that $G = K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$ and $M = \mathbb{Z}$. Consider the following commutative diagram.*

$$\begin{array}{ccc} L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}_n \rtimes_\mu \mathbb{Z} = G \\ \phi \big\downarrow & & \big\downarrow \alpha \\ K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z} & \xrightarrow[\beta]{} & \mathbb{Z} = M \end{array}$$

Figure 4.1

Then α, β, φ and ϕ are well defined group homomorphisms.

Proof. 1. Let $\alpha, \beta : \mathbb{Z}_n \rtimes_\mu \mathbb{Z} \to \mathbb{Z}$ be defined by $\alpha(t, z) = z$ and $\beta(s, z) = z$ for all $(t, z), (s, z) \in \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$. We note that

$$(t_1, z_1) = (t_2, z_2) \implies t_1 = t_2 \text{ and}$$
$$z_1 = z_2 \implies \alpha(t_1, z_1) = \alpha(t_2, z_2).$$

Hence α is well defined. And we also have that

$$\begin{aligned}\alpha(t_1, z_1) + \alpha(t_2, z_2) &= z_1 + z_2 \\ &= \alpha(t_1 + \mu_{z_1}(t_2), z_1 + z_2) \\ &= \alpha((t_1, z_1) + (t_2, z_2)).\end{aligned}$$

This implies that α is a homomorphism.

Similarly, let $\beta : \mathbb{Z}_n \rtimes_\mu \mathbb{Z} \to \mathbb{Z}$ be defined by $\beta(n, z) = z$ for all $(n, z) \in \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$. Then,

$$\begin{aligned}(s_1, z_1) = (s_2, z_2) &\implies \mathrm{s}_1 = t_2 \textit{ and } z_1 = z_2 \\ &\implies \beta(s_1, z_1) = \beta(s_2, z_2).\end{aligned}$$

Therefore β is well defined. And

$$\begin{aligned}\beta(s_1, z_1) + \beta(s_2, z_2) =& z_1 + z_2 \\ =& \beta(s_1 + \mu_{z_1}(s_2), z_1 + z_2) \\ =& \beta((s_1, z_1) + (s_2, z_2)).\end{aligned}$$

Hence β is a homomorphism.

2. Suppose $\varphi, \phi : (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z} \to \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$ is defined by $\varphi((t, s), z) = (t, z)$ and $\phi((t, s), z) = (s, z)$ for all $((t, s), z) \in (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$. We show that φ and ϕ are well-defined homomorphisms. Note that

$$((t_1, s), z_1) = ((t_2, s), z_2) \implies (t_1, s) = (t_2, s) \text{ and } z_1 = z_2 \implies t_1 = t_2 \implies (t_1, z_1) = (t_2, z_2) \implies \varphi((t_1, s), z_1) = \varphi((t_2, s), z_2)$$

Therefore φ is well defined. And

$$\begin{aligned}
\varphi((t_1,s_1),z_1)+\varphi((t_2,s_2),z_2) &= (t_1,z_1)+(t_2,z_2)\\
&= (t_1+\mu_{z_1}(t_2),z_1+z_2)\\
&= \varphi((t_1,n_1)+\omega_{z_1}(t_2,n_2),z_1+z_2)\\
&= \varphi(((t_1,s_1),z_1)+((t_2,s_2),z_2)).
\end{aligned}$$

Therefore by definition, φ is a homomorphism.

Similarly, let $\phi : (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z} \to \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$ be defined by $\phi((t,s),z) = (s,z)$ for all $((t,s),z) \in (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$. Then,

$$\begin{aligned}
((t,s_1),z_1) = ((t,s_2),z_2) &\implies (\mathrm{t},\, \mathrm{s}_1) = (t,s_2) \text{ and } z_1 = z_2\\
&\implies \mathrm{s}_1 = s_2 \implies (\mathrm{s}_1,z_1) = (s_2,z_2)\\
&\implies \phi((t,s_1),z_1) = \phi((t,s_2),z_2).
\end{aligned}$$

Therefore ϕ is well defined. And

$$\begin{aligned}
\varphi((t_1,s_1),z_1)+\varphi((t_2,s_2),z_2) &= (s_1,z_1)+(s_2,z_2)\\
&= (s_1+\mu_{z_1}(s_2),z_1+z_2)\\
&= \phi((t_1,s_1)+\omega_{z_1}(t_2,s_2),z_1+z_2)\\
&= \phi(((t_1,s_1),z_1)+((t_2,s_2),z_2))
\end{aligned}$$

This implies that ϕ is a homomorphism. This completes our proof.

□

Proposition 4.1.2. *Let K, G and M be groups such that $G = K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$ and $M = \mathbb{Z}$. Let $\alpha : G \to M$ and $\beta : K \to M$ be group* *homomorphisms.*

We define a subset $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$ *of* $K \times G$ *by*

$$L = \{((n,u),z) \in (\mathbb{Z}_n \rtimes_\mu \mathbb{Z}) \times (\mathbb{Z}_n \rtimes_\mu \mathbb{Z}) \mid \beta(n,z) = \alpha(u,z)\}.$$

We prove that L *is a subgroup of* $K \times G$.

Proof. We first show that L is closed under the group operation.
Let $((n_1,u_1),z_1),((n_2,u_2),z_2) \in L$. By definition,

$$\beta(n_1,z_1) = \alpha(u_1,z_1)\,\text{and}\,\beta(n_2,z_2) = \alpha(u_2,z_2).$$

Then

$$((n_1,u_1),z_1) + ((n_2,u_2),z_2) = ((n_1,u_1) + \omega_{z_1}(n_2,u_2)z_1 + z_2), \quad (4.1)$$

is the semi direct product L. We need to show that 4.1) is contained in L.
We note that

$$\begin{aligned}\beta((n_1,z_1)+(n_2,z_2)) =& \beta(n_1,z_1) + \beta(n_2,z_2)\\ =& \alpha(u_1,z_1) + \alpha(u_2,z_2)\\ =& \alpha((u_1,z_1)+(u_2,z_2)).\end{aligned}$$

Therefore $\beta((n_1,z_1)+(n_2,z_2)) = \beta(n_1+\mu_{z_1}(n_2), z_1+z_2) = \alpha(u_1 + \mu_{z_1}(u_2), z_1+z_2) = \alpha((u_1,z_1)+(u_2,z_2))$. Thus, by definition, $((n_1,u_1)+\omega_{z_1}(n_2,u_2), z_1+z_2) \in L$.

Now to show that L is closed under inverses, consider

$$\begin{aligned}\beta((n_1, z_1)^{-1}) =& (\beta(n_1, z_1))^{-1} \\ =& (\alpha(u_1, z_1))^{-1} \\ =& \alpha((u_1, z_1)^{-1}).\end{aligned}$$

Then by definition, $((n_1, u_1)^{-1}, z_1^{-1}) \in L$. And since L is closed under the operation of $K \times G$, we can conclude that L is a subgroup of $K \times G$. $\square$

Proposition 4.1.3. *Recall the commutative diagram in figure 4.2 given by*

$$\begin{array}{ccc} L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}_n \rtimes_\mu \mathbb{Z} = G \\ \phi \downarrow & & \downarrow \alpha \\ K = \mathbb{Z}_n \rtimes_\mu \mathbb{Z} & \xrightarrow[\beta]{} & \mathbb{Z} = M \end{array}$$

Figure 4.2

We show that L is a pullback.

Proof. We show that L is a pullback; that is, we show that the three characterizations given in Theorem 3.5.3 hold.
Let $L = \{((n, u), (n', u')) \in (\mathbb{Z}_n \rtimes_\mu \mathbb{Z}) \times (\mathbb{Z}_n \rtimes_\mu \mathbb{Z}) \mid \beta((n, u)) = \alpha((n', u'))\}$. Then by Proposition 2.6.7, we can see that L is a subgroup of $K \times G$. Now suppose that $\phi : L \to K$ and $\varphi : L \to G$ are group homomorphisms. We are required to prove that $Ker\,\varphi \cap Ker\,\phi = \{((e_n, e_u), e_z)\}$. In order to prove this, we must show that ϕ and φ are injective if and only if their kernels are trivial. $Ker\,\phi = \{(e_K, e_G)\}$ and $Ker\,\varphi = \{(e_K, e_G)\}$ respectively. We have that ϕ is injective and since ϕ is a group homomorphism, at most one element $((e_n, e_u), e_z)$

of L can be sent to the identity element (e_n, e_z) of K. So we have $\phi((e_n, e_u), e_z) = (e_n, e_z)$. If $((n, u), z) \in Ker\,\phi$, then $\phi((n, u), z) = (e_n, e_z)$ and thus $\phi((n, u), z) = \phi((e_n, e_u), e_z)$. Since ϕ is injective, we have $((n, u), z) = ((e_n, e_u), e_z)$. Hence $Ker\,\phi = \{((e_n, e_u), e_z)\}$. Conversely suppose that $Ker\,\phi = \{((e_n, e_u), e_z)\}$ and suppose that there are $((n_1, u_1), z_1), ((n_2, u_2), z_2) \in L$ such that

$$\phi((n_1, u_1), z_1) = \phi((n_2, u_2), z_2). \tag{4.2}$$

Then,

$$\begin{aligned}
\phi(((n_1, u_1), z_1)((n_2, u_2), z_2)^{-1}) &= \phi((n_1, u_1), z_1)\phi(((n_2, u_2), z_2)^{-1}) && (\phi \text{ is a homomorphism})\\
&= \phi((n_1, u_1), z_1)\phi((n_2, u_2), z_2)^{-1} && (\phi \text{ is a homomorphism})\\
&= \phi((n_1, u_1), z_1)\phi((n_1, u_1), z_1)^{-1} && (\text{by equation 4.2})\\
&= ((e_n, e_u), e_z).
\end{aligned}$$

Thus the element $((n_1, u_1), z_1)((n_2, u_2), z_2)^{-1}$ is in the kernel of ϕ and so
$((n_1, u_1), z_1)((n_2, u_2), z_2)^{-1} = e_L$. This implies that $((n_1, u_1), z_1) = ((n_2, u_2), z_2)$, and therefore ϕ is injective.
We do a similar computation for φ, and we obtain $Ker\,\varphi = \{((e_n, e_u), e_z)\}$. Since $Ker\,\varphi$ and $Ker\,\varphi$ are subgroups of L, $Ker\,\varphi \cap Ker\,\phi$ is also a subgroup of L and $Ker\,\varphi \cap Ker\,\phi = \{((e_n, e_u), e_z)\}$. Let ϕ be surjective, α be an injection, and $ker\varphi \cap ker\phi = \{0\}$. We want to show that $\alpha^{-1}(\beta(A)) = \varphi(M)$ and $\phi(ker\varphi) = ker\beta$.

To prove that $\alpha^{-1}(\beta(K)) = \varphi(L)$ and $\phi(Ker\,\varphi) = Ker\,\beta$, let ϕ and α be surjective and injective respectively. Suppose $(u, z) \in \varphi(L)$, then

$(u,z) = \varphi((n,u),z)$ for some $((n,u),z) \in L$. Put $(n,z) = \phi((n,u),z)$, then we have $\alpha^{-1}(\beta(n,z)) = \alpha^{-1}(\beta(\phi((n,u),z))) = \alpha^{-1}(\alpha(\varphi((n,u),z))) = \varphi((n,u),z) = (u,z)$. Therefore, $(u,z) \in \alpha^{-1}(\beta(K))$.
Conversely, suppose $(u,z) \in \alpha^{-1}(\beta(K))$. Then there exists $(n,z) \in K$ such that $(u,z) = \alpha^{-1}(\beta(n,z))$. Since ϕ is surjective, there exists $((n,u),z) \in L$ such that $\phi((n,u),z) = (n,z)$. Also, since α is injective we have

$$\begin{aligned}(u,z) =& \alpha^{-1}(\beta(\phi((n,u),z)))\\ =& \alpha^{-1}(\alpha(\varphi((n,u),z)))\\ =& \varphi((n,u),z).\end{aligned}$$

So $(u,z) \in \varphi(L)$. Therefore, $\alpha^{-1}(\beta(K)) = \varphi(L)$.

We now prove that $\phi(Ker\,\varphi) = Ker\,\beta$. Let $(n,z) \in \phi(Ker\,\varphi)$. Then there exists $((n,u),z) \in Ker\,\varphi$ such that $(n,z) = \phi((n,u),z)$. Since the diagram in Figure 4.2 is commutative and α is injective, we have $\beta((n,z)) = \beta(\phi((n,u),z)) = \alpha(\varphi((n,u),z)) = 0$. Therefore, $(n,z) \in Ker\,\beta$ and hence $\phi(Ker\,\varphi) \subseteq Ker\,\beta$.
Conversely, let $(n,z) \in Ker\,\beta$. Since ϕ is surjective, we have that $\phi((n,u),z) = (n,z)$ for some element $((n,u),z) \in L$. However, we have $\beta((n,z)) = \beta(\phi((n,u),z)) = \alpha(\varphi((n,u),z)) = 0$. Since α is injective, we have $\varphi((n,u),z) = 0$. This implies that $((n,u),z) \in Ker\,\varphi$ and also, $(n,z) = \phi((n,u),z) \in \phi(Ker\,\varphi)$. Hence, $Ker\,\beta \subseteq \phi(Ker\,\varphi)$ and so $\phi(Ker\,\varphi) = Ker\,\beta$. This completes the proof. □

4.2 The genus of a pullback

For n and u relatively prime integers, we consider the group $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$, where $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ is a non-trivial homomorphism given by $\omega(1) : (x, y) \mapsto (ux, uy)$ of $\mathbb{Z}_n \times \mathbb{Z}_n$. Recall the class $\mathscr{N}_0$ of finitely generated infinite nilpotent groups with a finite commutator subgroup. Our objective in this section is to compute the genus of the pullback $L \in \mathscr{N}_1 \subset \mathscr{N}_0$.

Let $\mathbb{Z}_n \times \mathbb{Z}_n = \langle x, y\rangle$ and $\mathbb{Z} = \langle z\rangle$. Let $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ be a non-trivial homomorphism. Then $L = \langle x, y, z\rangle$. The subgroup $\langle x, y\rangle$ is normal in L, so zxz^{-1} and zyz^{-1} are in $\langle x\rangle$ and $\langle y\rangle$ respectively. Then since L is non-abelian, we have that $zxz^{-1} = x^u$ and $zyz^{-1} = y^u$. Thus the generators of L are subject to the relations

$$x^n = 1, y^n = 1, xy = yx, zxz^{-1} = x^u, zyz^{-1} = y^u$$

where $u \equiv 1 \, mod \, n^2$. We note that if $u = 1$, then ω is trivial and $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \times \mathbb{Z}$. Otherwise, L is non-abelian.

Proposition 4.2.1. *Define $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ by $\omega(z) = \phi_z$ with $\phi_z((a, b)) = z((a, b))z^{-1}$, $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_n$. Then ω is a group action.*

Proof. We show that ω is well-defined. Let $z_1, z_2 \in \mathbb{Z}$ and suppose that $z_1 = z_2$. Then,

$$\phi_{z_1}((a, b)) = z_1((a, b))z_1^{-1} = z_2((a, b))z_2^{-1} = \phi_{z_2}((a, b)).$$

Therefore $\omega(z_1) = \omega(z_2)$. This implies that ω is well defined. Now,

$$\begin{aligned}\phi_{z_1 z_2} =& z_1 z_2((a,b))(z_1 z_2)^{-1} \\ =& z_1[z_2((a,b))z_2^{-1}]z_1^{-1} \\ =& \phi_{z_1}(z_2((a,b))z_2^{-1}) \\ =& \phi_{z_1}\phi_{z_2}.\end{aligned}$$

This implies that $\omega(z_1 z_2) = \omega(z_1)\omega(z_2)$. Thus ω is a homomorphism, and therefore a group action. □

Proposition 4.2.2. *Let* $\mathbb{Z}_n \times \mathbb{Z}_n$ *be a non-cyclic group. Then* $Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ *is non-abelian.*

Proof. Suppose that $\mathbb{Z}_n \times \mathbb{Z}_n$ has two generators x and y. We choose x and y such that the order of x divides the order of y. Indeed, since both x and y are of the same order, this is true. We now define the following mappings of $\mathbb{Z}_n \times \mathbb{Z}_n$ to itself :

$\theta : x^r y^s \to x^{r+s} y^s$, $\phi : x^r y^s \to x^r y^{-s}$ and $\psi : x^r y^s \to x^s y^r$.

Since $n \mid n$, we have that θ is always a well defined, non-trivial automorphism of $\mathbb{Z}_n \times \mathbb{Z}_n$. In the case where $n \neq 2$, we have that ϕ is also a non-trivial automorphism of the group $\mathbb{Z}_n \times \mathbb{Z}_n$. If $n = 2$, then we have ϕ to be the identity map, but ψ is a well defined automorphism that does not commute with θ. Therefore we can conclude that $Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ is non-abelian. □

Theorem 4.2.3. *Let* $\mathbb{Z}$ *and* $\mathbb{Z}_n \times \mathbb{Z}_n$ *be abelian groups. The group* $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$ *is nilpotent .*

Proof. Let $u = 1 + cp_i^{\lambda_i}$, with $p_i \nmid c$ and n^2 be the order of $\mathbb{Z}_n \times \mathbb{Z}_n$ where $n = p_i^{\kappa_i}$ for all i. Let us write $\mathbb{Z} = \langle \eta \rangle$ and $\mathbb{Z}_n \times \mathbb{Z}_n = \langle a, b \rangle$.

Let $\Gamma^i_{\mathbb{Z}}\,\mathbb{Z}_n \times \mathbb{Z}_n$, $i = 1, 2, \cdots, r$ be the terms of the lower central series of the action of the group $\mathbb{Z}$ on the abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$. Then $\Gamma^1_{\mathbb{Z}}\,\mathbb{Z}_n \times \mathbb{Z}_n = \langle (a,b)^{(u-1)} \mid (a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle = \mathbb{Z}_n \times \mathbb{Z}_n, \Gamma^2_{\mathbb{Z}}\,\mathbb{Z}_n \times \mathbb{Z}_n = \langle (a,b)^{(u-1)(u-1)} \mid (a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle$. And inductively $\Gamma^r_{\mathbb{Z}}\,\mathbb{Z}_n \times \mathbb{Z}_n = \langle (a,b)^{(u-1)^r} \mid (a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n \rangle$. Therefore for r sufficiently large, we have that $\Gamma^r_{\mathbb{Z}}\,\mathbb{Z}_n \times \mathbb{Z}_n = \{e\}$ if and only if $n^2 \mid (u-1)^r$. Therefore the group L is nilpotent. □

Now that we have proven that the group L given by $(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$, where ω is an action given by $\omega(1) : (x, y) \to (ux, uy)$ is nilpotent, we proceed to show that the commutator subgroup of L is finite. But first, let us fix some notation. Put $n = p_i^{\kappa_i}$, and write $u \equiv 1 \, mod \, n$, where $u = 1 + cp_i^{\lambda_i}$, with $p_i \nmid c$ for all i. Let $d = p_i^{\nu_i}$ be the order of $u \, mod \, p_i^{\kappa_i}$. Then by slightly generalizing the result of a cyclic genus in [5], we set

$$\nu_i = \begin{cases} 0, & \kappa_i \le \lambda_i, \\ \kappa_i - \lambda_i, & \kappa_i > \lambda_i. \end{cases} \tag{4.3}$$

Note that since the order of $\mathbb{Z}_n \times \mathbb{Z}_n$ is n^2, we will write $u \equiv 1 \, mod \, p_i^{2\kappa_i}$, so that $\nu_i = 2\kappa_i - \lambda_i$ if $2\kappa_i > \lambda_i$.

Lemma 4.2.4. *The group L has a finite commutator subgroup.*

Proof. We express elements of L by $(xy)^q z^r$, where $xy = yx$ since the subgroup $\mathbb{Z}_n \times \mathbb{Z}_n$ is abelian. Then we compute the commutators:

$$
\begin{aligned}
[(xy)^q z^r, z] =& (xy)^q z^r z((xy)^q z^r)^{-1} z^{-1} \\
=& (xy)^q z^r z z^{-r} (xy)^{-q} z^{-1} \\
=& (xy)^q z (xy)^{-q} z^{-1} \\
=& (xy)^q z (xy)^{-q} z^{-1} \\
=& (xy)^q (xy)^{-qu} = (xy)^{q(1-u)}.
\end{aligned}
$$

We also have that

$$
\begin{aligned}
[(xy)^q z^r, xy] =& (xy)^q z^r (xy)((xy)^q z^r)^{-1} (xy)^{-1} \\
=& (xy)^q z^r (xy) z^{-r} (xy)^{-q} (xy)^{-1} \\
=& z^r (xy) z^{-r} (xy)^{-1} \\
=& z^r (xy) z^{-r} (xy)^{-1} \\
=& (xy)^{u^r} (xy)^{-1} = (xy)^{u^r - 1}.
\end{aligned}
$$

If $p_i^{\nu_i} \mid r$, we have that $p_i^{2\kappa_i} \mid u^r - 1$. This implies that $u^r - 1 = hp_i^{2\kappa_i}$, for all $h \in \mathbb{Z}^+$. Therefore, we can write $(xy)^{u^r-1} = (xy)^{hp_i^{2\kappa_i}}$. By equation 4.3, $\nu_i = 2\kappa_i - \lambda_i$ for $2\kappa_i > \lambda_i$, so we can write $(xy)^{hp_i^{2\kappa_i}} = (xy)^{hp_i^{\nu_i+\lambda_i}} = ((xy)^{p_i^{\lambda_i}})^{hp_i^{\nu_i}}$. Also, $(xy)^{q(1-u)} = (xy)^{-cqp_i^{\lambda_i}} = ((xy)^{p_i^{\lambda_i}})^{-cq}$. Then the commutators are powers of $(xy)^{p_i^{\lambda_i}}$. Therefore, we have $[L, L] = \langle x^{p_i^{\lambda_i}}, y^{p_i^{\lambda_i}} \rangle$, and is obviously finite since $[L, L]$ lies in the torsion subgroup of L. □

Remark 10. Recall the homomorphism $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ given by $\omega(z) = \phi_z$, for all $z \in \mathbb{Z}$. Here, the image $\omega(\mathbb{Z}) \leq Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ is the inner automorphism group. Since $\mathbb{Z}_n \times \mathbb{Z}_n$ is abelian, we have that $Z(\mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_n \times \mathbb{Z}_n$. Thus $Inn(\mathbb{Z}_n \times \mathbb{Z}_n) \cong \mathbb{Z}_n \times \mathbb{Z}_n / Z(\mathbb{Z}_n \times \mathbb{Z}_n) =$

$\{e\}$, and so $\omega(\mathbb{Z}) = \{e\}$. Now since the identity element commutes with every element of a group, $\omega(\mathbb{Z})$ must commute with every element of $Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$. Therefore we can deduce that $\omega(\mathbb{Z}) \subseteq Z(Aut(\mathbb{Z}_n \times \mathbb{Z}_n))$. Henceforth $L \in \mathscr{N}_1 \subset \mathscr{N}_0$ by Definition 3.4.2.

Let P be the set of primes p. Recall that $P - Aut(L)$ denotes the semigroup of P-automorphisms of L. Then since $L \in \mathscr{N}_0$, we will use the exact sequence

$$P - Aut(L) \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathscr{G}(L) \tag{4.4}$$

in computing the genus of L. Note that we exclude the case where $p = 2$, $\nu_i = 1$, since the genus of L will be trivial. We begin by determining the exponent of QL_{ab} in the following result.

Proposition 4.2.5. *For a prime $p \in P$, the exponent e of the group QL_{ab} is given by $p^{\nu_i+\lambda_i}$.*

Proof. We recall some notation from [8], and with L nilpotent, we find the following,

$ZL = \langle x^{n'}, y^{n'}, z^{p^{\nu_i}} \rangle$, where $n' = \frac{p^{2\kappa_i}}{p^{\lambda_i}}$,

$FZL = \langle z^{p^{\nu_i+\lambda_i}} \rangle$,

$QL = L/FZL = \langle x, y, z \mid x^n = 1, y^n = 1, xy = yx, zxz^{-1} = x^u, zyz^{-1} = y^u, z^{p^{\nu_i+\lambda_i}} = 1 \rangle$ and

$QL_{ab} = \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^{p^{\lambda_i}} = \bar{y}^{p^{\lambda_i}} = 1, \bar{z}^{p^{\nu_i+\lambda_i}} = 1 \rangle$.

Thus it follows that $e = p^{\nu_i+\lambda_i}$ for all i. □

Now we proceed to compute the genus of the pullback L, $\mathscr{G}(L)$. We must examine the image of

$$\theta : P - Aut(L) \to (\mathbb{Z}/p^{\nu_i+\lambda_i})^*/\{\pm 1\}.$$

Theorem 4.2.6. *Let $L \in \mathscr{N}_1$, and suppose that the sequence in (4.4) is exact. Then the genus of L, $\mathscr{G}(L)$ is a finitely generated cyclic group.*

Proof. Since the sequence in (4.4) is exact, by definition, θ must be injective and thus the image of θ is a subgroup of $(\mathbb{Z}/p^{\nu_i+\lambda_i})^*/\{\pm 1\}$. This implies that the image of θ consists of the units of $\mathbb{Z}/p^{\nu_i+\lambda_i}$, $mod\,\{\pm 1\}$, which by [12], are $\equiv 1\, mod\, p^{\nu_i}$. There are p^{λ_i} such units. Therefore $|Im\,\theta| = p^{\lambda_i}$. Since $\delta : (\mathbb{Z}/p^{\nu_i+\lambda_i})^*/\{\pm 1\} \to \mathscr{G}(L)$ is a surjective group homomorphism, by the First Isomorphism Theorem

$$(\mathbb{Z}/p^{\nu_i+\lambda_i})^*/\{\pm 1\} \Big/ {Ker\,\delta} \cong \mathscr{G}(L).$$

But since the sequence is exact, $Ker\,\delta = Im\,\theta$, so

$$\begin{aligned}
\big|\mathscr{G}(L)\big| &= \frac{\big|(\mathbb{Z}/p^{\nu_i+\lambda_i})^*/\{\pm 1\}\big|}{|Im\,\theta|} \\
&= \frac{p^{\nu_i+\lambda_i-1}(p-1)/2}{p^{\lambda_i}} \\
&= p^{\nu_i-1}(p-1)/2.
\end{aligned}$$

Therefore $\mathscr{G}(L) \cong (\mathbb{Z}/p^{\nu_i})^*/\{\pm 1\}$. □

Remark 11. We proved in Lemma 4.2.4 that the commutator subgroup of L is finite. According to the authors' result in [2], this implies that the Mislin genus $\mathscr{G}(L)$ has an abelian group structure.

For the group L in Theorem 4.2.6, we have given an exact description of the genus set $\mathscr{G}(L)$, as a finite cyclic group, and we will deduce from this description the cardinality of $\mathscr{G}(L)$. But first, we must prove results in the next section, which will be of use.

4.3 The genus of a pullback : Application

For a pair of relatively prime positive integers n and u, let $G(n,u)$ denote the group $\mathbb{Z}_n \rtimes_\mu \mathbb{Z}$, where $\mu : \mathbb{Z} \to Aut(\mathbb{Z}_n)$ is a non-trivial homomorphism given by $\mu(1) : x \to ux$ and let $G = \langle x, y \mid x^n = 1, yxy^{-1} = x^u \rangle$. We have that $G = G(n,u)$. Let d be the order of $u\, modulo\, n$, where $n = p_i^{\kappa_i}$ and $u = 1 + cp_i^{\lambda_i}$, $p_i \nmid c$, $\lambda_i \geq 1$ and $d = p_i^{\nu_i}$ for all i. From [2], we know that if $G \in \mathscr{N}_1$, then $\mathscr{G}(G) \cong (\mathbb{Z}/d)^*/\{\pm 1\}$. Let $G \in \mathscr{N}_0$ and let G^k be the kth direct power of G, where $k \geq 2$. From [2], there is a surjective homomorphism $\tau : \mathscr{G}(G) \to \mathscr{G}(G^k)$, given by $\tau(F) = F \times G^{k-1}$, where $F \in \mathscr{G}(G)$. The authors in [2] also prove the following theorem.

Theorem 4.3.1. *Let p be a prime, and let the torsion subgroup of G, TG be a cyclic p-group and the torsion free quotient, FG be cyclic. Then τ is an isomorphism.*

Now recall the pullback $L \in \mathscr{N}_1$ and consider the following diagram

$$\begin{array}{ccc} L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}_{11} \rtimes_\mu \mathbb{Z} = G \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \alpha} \\ K = (\mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z})^{20} & \xrightarrow[\beta]{} & \mathbb{Z}^{20} = M \end{array}$$

Figure 4.3

For notational convenience, we will write $\bar{K} = \mathbb{Z}_{11^2} \rtimes_\mu \mathbb{Z}$. We note that since the torsion subgroup $\mathbb{Z}_{11^2}$ is finite and the torsion free quotient $\mathbb{Z}$ is free abelian of finite rank, we have that $\bar{K}$ is in $\mathscr{N}_0$. Then we prove the following propositions.

Proposition 4.3.2. *Let $\bar{K} \in \mathscr{N}_0$ with an abelian torsion subgroup $\mathbb{Z}_{11^2}$. Then $\bar{K} \in \mathscr{N}_1$.*

Proof. Since $\bar{K} \in \mathscr{N}_0$ and $\mathbb{Z}_{11^2}$ is abelian, we only need to show that $\mu(\mathbb{Z})$ is contained in the center of $Aut(\mathbb{Z}_{11^2})$. We have that $\left|Aut(\mathbb{Z}_{11^2})\right| = 11(11-1) = 110$. Thus $Aut(\mathbb{Z}_{11^2}) \cong \mathbb{Z}_{110}$. Since $\mathbb{Z}_{110}$ is abelian, we have that $Z(Aut(\mathbb{Z}_{11^2})) = \mathbb{Z}_{110}$. This implies that $\mu(\mathbb{Z}) \subseteq Z(Aut(\mathbb{Z}_{11^2}))$. Therefore we can conclude that $\bar{K} \in \mathscr{N}_1$. □

Proposition 4.3.3. *Let $\bar{K} \in \mathscr{N}_1$, then $\mathscr{G}(\bar{K}) \cong \mathscr{G}(K)$.*

Proof. Let $n = 11^2$ and let $u = 1 + 11^{\lambda}c$, $11 \nmid c$, $\lambda \geq 1$. We compute the genus of $\bar{K} \in \mathscr{N}_1$. We will consider two cases:

Case 1: If $2 > \lambda$, then by equation 4.3, $t = 11^{2-\lambda}$ and since $\lambda \geq 1$, then $\lambda = 1$. Finally we have,

$$\begin{aligned}\left|(\mathbb{Z}/11)^*/\{\pm 1\}\right| =& \frac{11^{1-1}(11-1)}{2}\\ =&5\\ =&\left|\mathscr{G}((G(11^2, u))\right|.\end{aligned}$$

Hence $\mathscr{G}(\bar{K})$ is a cyclic group of order 5. According to the authors in [10] we pass from $\mathscr{G}(\bar{K})$ to $\mathscr{G}(K)$ by factoring out the residue class $m\, modulo\, 11$, such that $m \equiv \pm 1\, mod\, 11$. Now since $t = 11$ is a single prime, we will factor out $m \equiv -1\, mod\, 11$. That is, we factor out the group generated by $\{-1, m\}$, where $m = 10$. Thus

$$\mathscr{G}(K) \cong \{0\}.$$

Therefore the genus of K is trivial.

Case 2: If $2 \leq \lambda$, then $t = 11^0 = 1$ by equation 4.3. This implies that $\mathscr{G}(\bar{K})$ is trivial. Therefore, it follows that $\mathscr{G}(K)$ is also trivial.

Finally, we note that in case 1, $\mathscr{G}(\bar{K}) \ncong \mathscr{G}(K)$, and that in case 2,

$\mathscr{G}(\bar{K}) \cong \mathscr{G}(K)$. Therefore, since the torsion subgroup $\mathbb{Z}_{11^2}$ has prime power order and $F\bar{K}$ is cyclic, Theorem 4.3.1 infer that $\mathscr{G}(\bar{K}) \cong \mathscr{G}(K)$ and so $\mathscr{G}(\bar{K})$ must be trivial. □

Remark 12. Although $\mathscr{N}_0$ is closed under direct products, since $\bar{K}$ is not commutative, K is not in $\mathscr{N}_1$. But since the torsion subgroup $\mathbb{Z}_{121}^{20}$ is finite and the torsion free quotient $\mathbb{Z}^{20}$ is free abelian of finite rank, then $K \in \mathscr{N}_0$.

Proposition 4.3.4. *Let $G \in \mathscr{N}_1$. Then $\mathscr{G}(G)$ is trivial.*

Proof. Let $n = 11$ and let $u = 1 + 11^{\lambda}c$, $11 \nmid c$, $\lambda \geq 1$. Then by equation 4.3, $\nu = 1 - \lambda$ if $1 > \lambda$, but since $\lambda \geq 1$, $\nu = 0$. Therefore the order of $u\, modulo\, 11$ is 1, and hence $\mathscr{G}(G) = \{0\}$. This completes the proof. □

Proposition 4.3.5. *Let $G, L \in \mathscr{N}_1$. Then $\phi_* : \mathscr{G}(L) \to \mathscr{G}(G)$ is a surjective group homomorphism.*

Proof. Let $e_2 \in \mathscr{G}(G)$ and let $\phi_* : \mathscr{G}(L) \to \mathscr{G}(G)$ be defined by $\phi_*([x]_s) = e_2$, for all $[x]_s \in \mathscr{G}(L)$, where s is the order of $\mathscr{G}(L)$. Let $[x]_s, [y]_s \in \mathscr{G}(L)$, then

$$\phi_*([x]_s + [y]_s) = e_2 = e_2 + e_2 = \phi_*[x]_s + \phi_*[y]_s.$$

Hence ϕ_* is a homomorphism and is obviously surjective since every element in $\mathscr{G}(L)$ is mapped to the identity. □

We will now use Proposition 4.3.5 to prove the following theorem.

Theorem 4.3.6. *The genus of L, $\mathscr{G}(L)$ is a cyclic group of order 2.*

Proof. We can deduce, using Proposition 4.3.5, that the kernel, $Ker\,\phi_*$ must be the group itself since $\mathscr{G}(L)$ is mapped to the identity element of $\mathscr{G}(G)$, thus $|Ker\,\phi_*| = \left|\mathscr{G}(L)\right|$. Moreover, since the group $\mathscr{G}(L)$ is cyclic by Theorem 4.2.6, then, there can only be one subgroup of order $\left|\mathscr{G}(L)\right|$ in $\mathscr{G}(L)$, which is the group itself. Hence $\mathscr{G}(L)$ must have only two subgroups, which are the trivial subgroup and the group itself. We then deduce that the genus of L, $\mathscr{G}(L)$ is cyclic of order 2.

□

Chapter 5

Conclusions

In this work, we proved that the genus of a pullback $L \in \mathscr{N}_1$ is a cyclic group of finite order. This group appears to be similar to the genus of a group $H \in \mathscr{N}_1$ which we computed in Chapter 3 but with a cyclic torsion subgroup.

In Chapter 3, we studied the results in [5] of a finite genus of a group in $\mathscr{N}_1$. We then discussed these results in more detail for a group $H \in \mathscr{N}_1$, where we showed that the group is nilpotent with a finite commutator subgroup. Moreover, using the exact sequence $P - Aut(H) \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathscr{G}(H)$ from [8], we examined the image of $\theta : P - Aut(H) \to (\mathbb{Z}/e)^*/\{\pm 1\}$. From this we were able to compute the order of the genus $\mathscr{G}(H)$ of the group H. We then deduced that the group $\mathscr{G}(H)$ is cyclic of finite order $s = p^{n-1}(p-1)/2$.

Next, we generalized the results from Chapter 3 to study the group L. We proved in Proposition 4.1.3 that the group $L = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes_\omega \mathbb{Z}$, where $\omega : \mathbb{Z} \to Aut(\mathbb{Z}_n \times \mathbb{Z}_n)$ is a non-trivial homomorphism given by $\omega(1) : (x, y) \mapsto (ux, uy)$ of $\mathbb{Z}_n \times \mathbb{Z}_n$ is a pullback by showing that it satisfies all the conditions given in Theorem 3.5.3. We then proceeded to show that the pullback L belongs to the class $\mathscr{N}_1$. In particular, we proved in Theorem 4.2.3 that the group L is nilpotent and has a finite commutator subgroup. Note that the condition that $\omega(FL) \subseteq Z(Aut(TL))$, where $Z(Aut(TL))$ is the center of $Aut(TL)$

and ω is the action of FL on TL was not automatically satisfied since the torsion subgroup TL is not cyclic.

Now since we proved that L is in $\mathscr{N}_1 \subset \mathscr{N}_0$, we were able to use the exact sequence $P - Aut(L) \xrightarrow{\theta} (\mathbb{Z}/e)^*/\{\pm 1\} \overset{\delta}{\twoheadrightarrow} \mathscr{G}(L)$ to compute the genus of the group L. We proved that $\mathscr{G}(L) \cong (\mathbb{Z}/d)^*/\{\pm 1\}$, where d is the height of $Ker\,\omega$ in FL; that is, $d = max\{h \in \mathbb{N} \mid Ker\,\omega \subseteq hFL\}$. We also showed in Lemma 4.2.4 that the group L has a finite commutator subgroup. From this, we can deduce that the group structure of $\mathscr{G}(L)$ is the same as the group structure of the Mislin genus described by the authors in [2].

We then considered the groups K and G in the class $\mathscr{N}_0$, where K is given by $\bar{K}^{20}$. We showed that $\bar{K}$ is in $\mathscr{N}_1$. We note that since $\bar{K}$ is not abelian, then K is not in $\mathscr{N}_1$. In this case, the condition that $\mu(FK) \subseteq Z(Aut(TK))$, where $Z(Aut(TK))$ is the center of $Aut(TK)$ and μ is the action of FK on TK fails to hold; in general, direct products do not inherit this condition. Therefore, we see the importance of this condition in obtaining the genus of a group in $\mathscr{N}_1$ [Theorem 1.2, [2]]. We computed the genus of $G \in \mathscr{N}_1$ and that of $K \in \mathscr{N}_0$, and we obtained some results; the groups $\mathscr{G}(K)$ and $\mathscr{G}(G)$ are both trivial and therefore are isomorphic to each other.

Finally, in Proposition 4.3.5, we showed that there is a surjective homomorphism between the groups $\mathscr{G}(G)$ and $\mathscr{G}(L)$. This surjective homomorphism enabled us to prove that the genus of L, $\mathscr{G}(L)$ is non-trivial. Our results showed that $\mathscr{G}(L)$ is a cyclic group of order 2.

A problem which we would consider in the future is that of the extended genus. In this case we would require that the groups Q and R not be finitely generated, then define extended actions on abelian groups.

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