Derivatives Pricing with xVAs

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Declaration

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04 October 2021

Dedicated to the loving memory of my mother, Lizah Motsabole Sekgoka.
1968-2006.

Abstract

Derivatives pricing changed drastically in 2007-2008, following the financial crisis. The failure of the Lehman Brothers refuted the myth that ”AAA” rated financial institutions cannot default. Basel III was introduced and implemented to reinforce existing financial market regulations and counterparty credit risk became the primary subject of debate in financial markets' regulations. Prior to the financial crisis, derivatives pricing had its assumptions aligning to those from the Black-Scholes-Merton model. This model assumes that default risk is absent, but this financial crisis highlighted that in reality, this is not true. In this dissertation, we research and offer an analysis of derivatives pricing following the financial crisis whereby we extend the Black-Scholes-Merton model to incorporate default risk, collateral, funding costs, regulatory capital, and initial margin through the implementation of the model of Burgard and Kjaer called the *semi-replication strategy*. The primary emphasis would therefore be on the derivation of xVAs. We also evaluate regulatory capital and initial margin methodologies which are used to compute KVA and MVA, respectively.

Keywords: *Counterparty credit risk, xVA, SIMM, CEM, SA-CCR, BSM model, Regulatory Capital, Initial Margin, LSM*

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List of Abbreviations

BCBS Basel Committee on Banking Supervision

BSM Black-Scholes-Merton

CCP Central counterparty

CCR Counterparty credit risk

CDF Cumulative distribution function

CDS Credit default swap

CE Credit exposure

CEM Current exposure method

CSA Credit support annex

CVA Credit valuation adjustment

COLVA Collateral valuation adjustment

DVA Debit valuation adjustment

EaD Exposure at default

ENE Expected negative exposure

EPE Expected positive exposure

ETD Exchange-traded derivative

ETL Expected tail loss

FCA Funding cost adjustment

FTAP Fundamental theorem of asset pricing

FVA Funding valuation adjustment

FX Foreign exchange

GBM Geometric Brownian motion

IM Initial margin

IMM Internal model method

IRD Interest rate derivative

IRS Vanilla interest rate swap

ITM In the money

KVA Regulatory Capital valuation adjustment

LSM Least squares Monte Carlo simulation

MPoR Margin period of risk

MVA Initial Margin valuation adjustment

OTC Over-the-counter

PDE Partial differential equation

PFE Potential future exposure

RC Replacement cost

RWR Right-way risk

RWA Risk weighted assets

SA-CCR The standardized approach for measuring counterparty credit risk

SDE Stochastic differential equation

SIMM Standard initial margin model

VaR Value at risk

WWR Wrong-way risk

xVA Cross valuation adjustment

ZCB Zero-coupon bond

Main List of Symbols

Symbol	Meaning
S	Underlying asset price
$V; \hat{V}$	Risk-free value; economic value of the derivative
$\Delta\hat{V}_C; \Delta\hat{V}_B$	Change in economic value at counterparty's default; issuer's default
U	The valuation adjustment (xVA)
$J_C; J_B$	Default indicator for counterparty; issuer
$I_B; I_C$	Initial margin posted by B; C
$\mu_S; \sigma_S$	Underlying asset drift; volatility
R_i	Recovery rate on issuer bond i
$R_C; R_B$	Recovery rate on issuer; counterparty derivative positions
$P_1; P_2$	Issuer bond with recovery R_1 and R_2, respectively; where $R_1 \neq R_2$.
P_C	Counterparty bond
P	Own bond value before default $P = \alpha_1 P_1 + \alpha_2 P_2$
P_D	Own bond value after default: $P_D = \alpha_1 R_1 P_1 + \alpha_2 R_2 P_2$
$r; \gamma_K$	Risk-free rate; the cost of capital
r_i	Yield rate on the issuer bond $r_i = r + (1 - R_i)\lambda_B$
r_C	Yield rate on the counterparty bond $r_C = r + \lambda_C$
$\lambda_B; \lambda_C$	Default intensity of issuer B; counterparty C
$K; X$	Regulatory capital; collateral value
ϕ	Fraction of regulatory capital for funding derivatives
ε_h	Hedging error on issuer B's default. This is split into ε_{h_0} and ε_{h_K}
r_{I_B}	Interest rate on initial margin posted by B
$r_F; r_X$	Funding interest rate; interest rate on collateral
$s_X; s_{I_B}$	Spread on collateral $s_X = r_X - r$; issuer B initial margin $s_{I_B} = r_{I_B} - r$
s_F	Funding spread $s_F = r_F - r$
γ_S	The dividend of the underlying asset
q_S	Repo rate of the underlying asset
δ_S	Position of the underlying asset
$q_C; q_S; q_B$	Repo rate for counterparty bond; underlying asset; issuer bond
$\alpha; \alpha_i$	Holding of credit risk-free bonds; issuer bond i
$\alpha_C; \alpha_B$	Holding of counterparty bonds; issuer bonds
$M_C; M_B$	Close-out value of counterparty default; issuer default
$D(t, T)$	Discount factor between time t and T
$H(S)$	Payout of the derivative
$\mathcal{M}$	Financial market model
$B(t, T)$	Zero-coupon bond
g	Close-out value of the derivative
$x_+; x_-$	$x_+ = \max(x, 0)$; $x_- = \min(x, 0)$
$\mathcal{A}; \mathcal{B}$	Differential operators
$\mathbb{Q}; \mathbb{P}$	Risk neutral; real world probability measure
$\mathcal{F}_t$	Filtration
Π	Portfolio value
$\mathcal{P}$	The time interval partition
$\mathbf{A}^T; \mathbf{A}^{-1}; \|A\|$	Transpose; inverse; norm of matrix A

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1 Introduction

> I didn't become interested in derivatives until 1982, 1983.
>
> *John Hull*

This chapter introduces the theory of derivatives, counterparty credit risk (which is also called default risk), and the theory and impact of xVAs on derivatives pricing.

1.1 Theory of Derivatives

From *Theory of Speculation* [5], the paper by Louis Bachelier, the derivatives pricing and valuation grew to greater heights and it became a more complex topic ever since. To this day, this paper is still arguably considered by many practitioners to be the one that gave birth to mathematical finance. In brief, the subject of mathematical finance is concerned with mathematical modelling of financial instruments in the financial markets: a place where practitioners trade (i.e. buy or sell) financial instruments and products. Derivatives are types of financial instruments and are traded on many exchanges all over the world.

Definition 1.1.1 Derivative security
Suppose that t is an arbitrary time where $0 \leq t \leq T < \infty$ and T be the termination or maturity time. The derivative security or simply *derivative* is a financial contract whereby its time t value V_t is derived from the movement of the underlying asset or any set of assets S.

This underlying asset or simply *underlying* can be a credit event, equity, foreign exchange, interest rate, commodity, or another derivative. Most of the time the underlying is tradable (that is to say, it can be sold or bought on the market) but it does not have to be tradable. Derivatives may depend on almost any variable. Some may depend on weather, electricity, or even natural gas. The evolution or movement of the underlying determines the underlying asset price; in turn, this determines the evolution of the value of the derivative. This implies that if S denotes the stochastic price process [1] of the underlying and derivative value is denoted by V then $V = V(S)$. Of course, the value V may depend on other parameters like stochastic interest rates, deterministic volatilities, strike prices, etc.

[1] Price process is a stochastic process used to determine underlying asset price.

A derivative can be viewed as a private contract between two or more parties. In any given derivative contract, two parties exchange payments over a period of time. A derivative is classified by the way it can be traded. Commonly they can either be exchange-traded (ETD) or over-the-counter (OTC) traded. ETDs are only transacted or traded on a regulated platform called an exchange; this exchange act as an intermediary counterparty between the two parties whereas OTC derivatives are only transacted between two counterparties (the issuer and the counterparty) bilaterally with minimal regulation. OTC derivatives are the most traded types of derivatives because they have the flexibility and can be customized to meet particular requirements. For example, two counterparties can agree on the maturity time of the OTC derivative, predetermined prices, etc.

There are four main categories of derivatives. Any existing derivative contract falls at least in one of these categories. These are *forwards*, *futures*, *options*, and *swaps*.

Forwards A forward contract is a non-standardized derivative contract to trade the underlying asset S_T at a future or maturity time T at a certain fixed price E. The price at which the parties agree on is called the delivery price. A forward contract is traded in the OTC market, meaning that it takes place between two parties without a regulated intermediary house. If one party agrees to buy the forward contract we say that they have entered into a long position and if they agree to sell we say they entered into a short position. Suppose that S_T and E are the underlying terminal asset price and delivery price, respectively. The payout of a long position of the forward contract maturing at T is $S_T - E$ and for a short position is given by $E - S_T$.

Futures A futures contract is similar to a forward contract, except that it is traded on a regulated exchange. Essentially, one can say, a futures contract is a regulated forward contract.

Options An option contract is a contract that gives the owner the right, but not the obligation to trade the underlying S before or at maturity at an agreed price E. Options can be traded on an exchange or over-the-counter. There are two primary types of options; call option and put option. A call option contract gives the owner the right, but not the obligation to buy S before or at maturity at a specified price E. A put option contract gives the owner the right, but not the obligation to sell S before or at maturity time T. Options are further divided into several numbers of styles of which the most common are European option and American option. The difference between the two is that a European option is exercised only at maturity and an American option is exercised at any time prior to or at maturity.

Swaps A swap contract is a contract whereby two parties exchange payments or cash flows periodically for an asset or group of assets over a period of time. A common type of swap contract is an interest rate swap. In this contract, two parties exchange interest rate payments by taking advantage of fluctuation of interest rates. For example, a vanilla interest rate swap is an agreement between two parties whereby one party pays fixed interest payments and the other one pays floating or variable interest rate payments periodically.

Figure 1.1 shows the growth of different derivatives with respect to their underlyings. There are four popular underlyings that have been mentioned, however, there are other categories such as weather, natural gas, etc. The last category that will not be discussed in depth is credit derivatives that came onto the market around 1990. Credit derivatives deal with the transfer of credit risk between two parties. We will provide a thorough description of the credit default swap, a type of credit derivative, and how it can be used to mitigate credit risk.

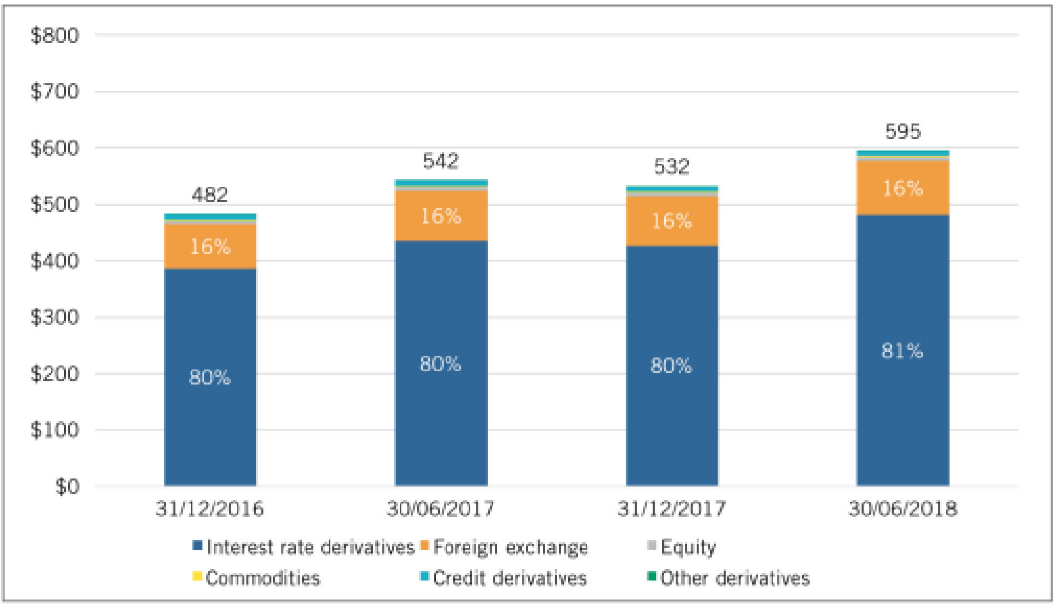


Figure 1: The growth of Global OTC derivatives from 2016 to 2018 in US$ trillions. *Source*: BIS OTC derivatives statistics.

As can be seen from this figure above, the interest rate market is the most competitive and traded in the world; thus IRDs have more than 80% than any other type of derivatives. These types of derivatives depend on the movement of interest rates; an interest rate swap is one of them. Other financial markets that hold a very small portion of the above figure include weather markets where the derivatives depend on the weather.

Derivatives have several uses; the three main uses are speculation, hedging, and arbitraging. Speculation is when you buy a derivative with the expectation that it will become more valuable in the future. Hedging is a risk offset in a derivatives market position. Arbitraging is when you make money without the probability of losing it. Most derivatives pricing models are built in such a way that arbitrage is always absent.

The Black-Scholes-Merton model assumptions

Derivatives pricing involves the use of mathematical models to compute the theoretical value of the derivative. Although there are many papers or models that have been published on derivatives pricing, the BSM model was the first widely used model. It was first published in 1973 by Black and Scholes [10], and later extended by Merton [46] the same year. The following assumptions are central to the BSM model:

- There are no arbitrage opportunities
- There are no transaction costs
- Constant risk-free interest rate (i.e. no default risk)
- The underlying asset pays no dividend
- Returns are lognormally distributed
- The underlying asset has a constant volatility
- The underlying asset price is continuous with no jumps
- The market is complete
- Security trading is continuous

Other assumptions that are not listed in their original papers [10, 46] and which will be the main topic of our discussion are:

- There are no capital requirements
- No margin requirements

From both these lists of assumptions, we can see that the BSM model ignores some of the assumptions that are essential in derivatives pricing, especially following the financial crisis of 2007-2008. For instance, ”no default risk”; default risk is always present in the real world. The default of Lehman Brothers is one example. Not only may a counterparty default but even the issuer (e.g. the bank, government, or any highly rated institution) may also default. Thus it is crucial to incorporate default risk in derivatives pricing. In Chapter 4 we will show how default risk is incorporated in derivatives pricing.

1.2 Counterparty Credit Risk

After the financial crisis, default risk became the main topic of discussion. The primary cause of the financial crisis has been thought of as the OTC derivatives. While ETDs have negligible or no default risk because this default risk is reduced by the exchange house; OTC derivatives are exposed to default risk as they are traded bilaterally between two counterparties.

Definition 1.2.1 Counterparty credit risk
Counterparty credit risk is a type of credit risk that issuer B faces when counterparty C ceases to make agreed payments in a contract.

The counterparty C usually ceases to make payments prior to the expiration of a derivative contract. OTC derivatives can also be exposed to settlement risk whereby one party fails to deliver security or cash.

When issuer B [2] and counterparty C enter into a derivative contract, it is crucial to compute counterparty credit exposure (or simply credit exposure; sometimes exposure). Credit exposure [3] is the maximum amount that issuer B will lose when counterparty C defaults. We compute and consider credit exposure as it helps with making feasible business decisions. For example, we can use credit exposure to decide the amount of money we are allowed to trade with a specific counterparty [17].

Counterparty credit risk mitigation
To ensure that a derivative trade transaction goes smoothly and default risk is reduced, a legal contract, regulated intermediary house, or other counterparty credit risk mitigation tools are used. Counterparty credit risk cannot be mitigated completely, thus we will always have a residual risk. Hence, counterparty credit risk mitigation transfers risk into new types of risks.

Another way to mitigate default risk is by using the Credit Support Annex (CSA). CSA[4] is legalized document that two parties can use when trading an OTC derivative [44]. CSA is introduced to reduce the counterparty credit risk by permitting counterparties to post collateral (credit support). With that in mind, the pledge is to lessen default risk emerging from in-the-money positions from OTC derivatives. One can view CSA as a margin account for

[2]This is normally a highly rated entity, such as a bank.
[3]Credit Exposure = max(derivative value, 0)
[4]CSA is part of the *International Swaps and Derivatives Association* (ISDA) Master Agreement

OTC derivative trade whereby if we post cash as collateral it will earn an interest defined in terms of the CSA agreements. In most cases, the reuse of collateral is prohibited.

There are three types of CSAs in an OTC derivative trade.

- No CSA: In this trade transaction, the CSA is not used at all. In this case, one or both counterparties do not post any collateral. This occurs when one of the counterparties cannot devote themselves to posting collateral.
- Two-way CSA: Both counterparties agree to post collateral.
- One-way CSA: One counterparty posts collateral. This is usually the case when a more creditworthy company is trading with a bank or another entity.

Another way of mitigating counterparty credit risk from an OTC derivative trade is by introducing a clearing house called the central counterparty (CCP). A CCP is like an exchange house because it serves as an intermediary between two parties in the OTC market; CCP steps in between counterparties to ensure good trade performance [30]. CCPs became more relevant following the financial crisis of 2007-2008. An OTC derivative that is cleared through a CCP is called a cleared derivative while a derivative that is not cleared through a CCP is called a non-cleared derivative. One example of this would be a 2 year IRS trade between two counterparties through a CCP.
London Clearing House is one example of the existing CCP and it was founded in 1888.

The diagram below shows the difference between a non-cleared derivative trade and cleared derivatives trade. From this diagram we can see that cash flows are passed through CCP to counterparties, hence CCP serves as a middleman between these two counterparties.

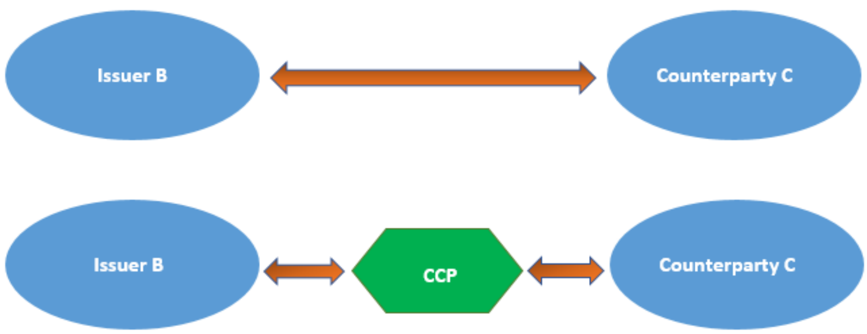


Figure 2: The illustration of the CCP. A CCP will act as a buyer for one counterparty and a seller to the other.

Netting and Aggregation

Suppose that the issuer B and counterparty C traded more than one different trades. For example, let us assume they traded an interest rate swap, currency swap, a European option, Bermudan option, and currency forward with one another. This group of instruments is called a netting set. Normally in this circumstance, some contracts will have negative values and others will have positive values. To compute a single exposure we will need to consider netting all the trades that issuer B traded with counterparty C. Since credit exposure is computed as the maximum of zero and the value of the derivative contract, then when we do netting we will sum all trades that issuer B and counterparty C traded together to calculate the total credit exposure. If V_i is the value of the ith derivative, then, the total netting value of a fully netted derivatives portfolio will become $V = \sum_i V_i$ and credit exposure $CE_{netted} = \max(V, 0)$. If the derivatives portfolio is non-netted then the CE will be computed for each trade and thus is it will become $CE_{non-netted} = \sum_i \max(V_i, 0)$. This will allow both B and C to determine the total amount owed to each other.

Hedging using Credit Default Swap

The Credit Default Swap is the standard and basic type of credit derivative. It is the type of swap contract whose value depends on the credit event of the issuer. Suppose that B is the issuer of the CDS and C is the counterparty or buyer of the CDS. The counterparty C will make a stream of periodic payments until the maturity of the CDS or until a credit event takes place. This stream of period payments is sometimes called premium payments, hence a CDS serves as an insurance policy. From this, we see that a CDS can be used to mitigate default risk between counterparties as it transfers credit risk between counterparties.

1.3 Quantification of Counterparty Credit Risk

Counterparty credit risk is quantified using counterparty credit exposures. Credit exposure metrics are a range of credit exposures used to measure counterparty credit risk. They are crucial as they are used in xVA calculations. For example, we will see the use of CE or expected positive exposure in CVA calculation.

The derivative value V is the main driver of the credit exposure metrics because each of them depends on V. The derivative value V can also be the netting value. We will define and explain different credit exposure metrics by starting to define credit exposure which is also known as current exposure or replacement cost.

Credit Exposure Metrics

Definition 1.3.1 Credit exposure
Credit exposure CE is the maximum loss to an issuer B if counterparty C defaults in a derivative transaction. Mathematically it is given by

$$CE := \max(V, 0) = V_+$$

whereby V denotes the derivative value or total netting value.

Credit exposure can also be computed if we have more than one trade with a certain counterparty. Suppose that the value of the certain derivative i is V_i. Then the formula will become

$$CE_i = \max(V_i, 0)$$

In a collateralized derivative, the credit exposure will be affected and the formula will now become

$$CE = \max(V - X, 0) = (V - X)_+$$

where X is the collateral amount posted by the counterparty. For example, in a fully collateralized trade, where $X = V$, the credit exposure will become zero. If the counterparty also posts the initial margin I then credit exposure will become

$$CE = \max(V - X - I, 0) = (V - X - I)_+$$

The IM aims to reduce the default risk during the MPoR for a derivative trade. We may also calculate the expected credit exposure ECE which is simply given by

$$ECE = \mathbb{E}_{\mathbb{P}}[CE|\mathcal{F}_t]$$

under real probability measure $\mathbb{P}$ and $\mathcal{F}_t$ is a filtration.

Definition 1.3.2 Expected positive exposure
The expected positive exposure EPE is given by

$$EPE := \mathbb{E}_{\mathbb{P}}[\max(V, 0)|\mathcal{F}_t]$$

under $\mathbb{P}$ and $\mathcal{F}_t$ is a filtration.

These types of credit exposure metrics occur in the calculation of CVA. The EPE is the same as ECE.

Definition 1.3.3 Expected negative exposure
The expected negative exposure ENE is defined as

$$ENE := \mathbb{E}_{\mathbb{P}}[\min(V, 0)|\mathcal{F}_t]$$

under $\mathbb{P}$ and $\mathcal{F}_t$ is a filtration.

These types of credit exposure metrics occur in the calculation of DVA. Both EPE and ENE appear in the calculation of FCA.

Definition 1.3.4 Potential future exposure
The potential future exposure PFE is given by

$$PFE_\alpha := \inf\{e : \mathbb{P}(CE \leq e) \geq \alpha\} = \varrho_\alpha(CE)$$

where α is a level of confidence and $\mathbb{P}$ is the real probability measure. PFE is usually calculated using risk metrics ϱ_α such as VaR or ETL. In our case it is a VaR.

To compute these credit exposures, we must start by computing the distribution of the derivative value V_t at times t. This value is however driven by the underlying asset price and thus we need to compute asset prices S_t first.
The credit exposures are important because they are used to calculate regulatory capital used for business decision-making. The new Standardized Approach for measuring Counterparty Credit Risk (SA-CCR), a regulatory capital methodology, uses credit exposure metrics like replacement cost (or current exposure) and PFE to compute the EaD. In fact, in SA-CCR, the usual PFE and PFE add-on are different. Another observation that can be made is that PFE calculation also depends on the distribution of the derivative value. For example, if we assume that the derivative value follows a normal distribution then the risk metric will be that of a normal distribution. It is important not to mix-up distributions when we make this kind of calculation. For instance, if the derivative value is assumed to follow a normal distribution, we cannot apply extreme value distributions to compute credit exposure metrics.

Below is the pseudo-algorithm for calculating credit exposure metrics.

The pseudo-algorithm for credit exposure metrics

(i) Simulate the underlying S_{t_k} in a given fixed set of simulation times n (where $t_k, \forall k = 0, 1, ..., n$ are discrete times)

(ii) Price the derivative at each simulation time t_k and simulation path ω

(iii) Aggregate and net all the prices according to netting and aggregate agreements

(iv) Compute credit exposure metric

The figures 3 and 4 shows the profiles of various credit exposures.

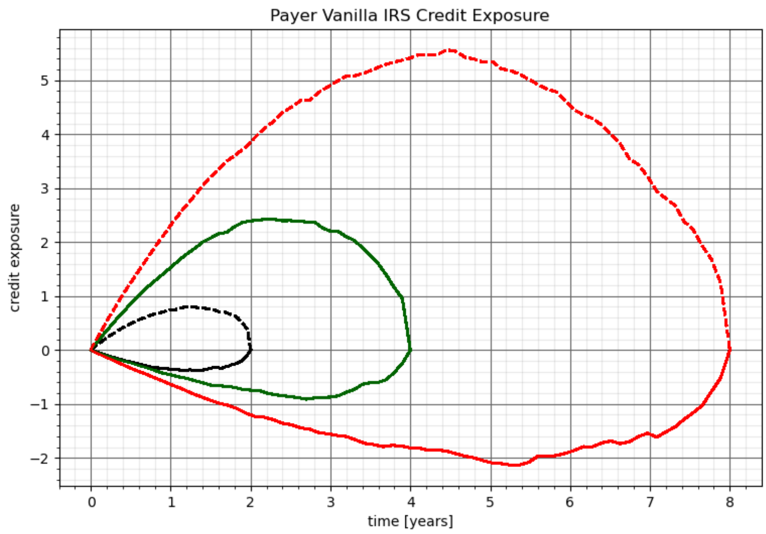


Figure 3: EPE and ENE of uncollateralized payer vanilla IRS of different maturities

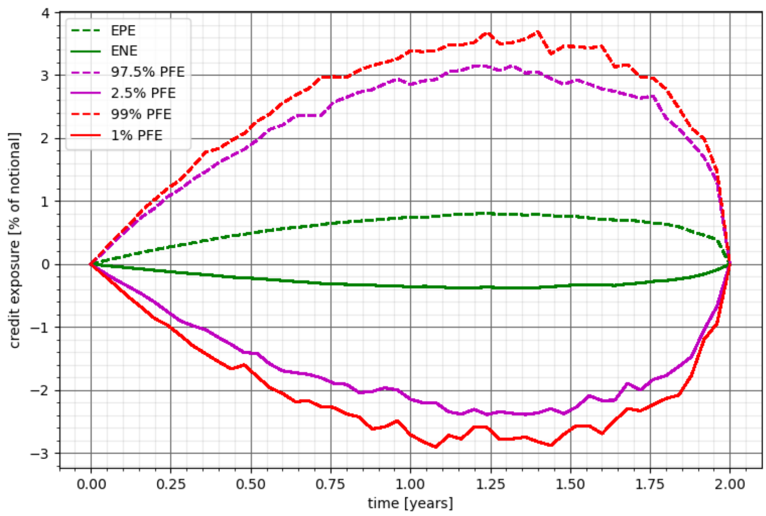


Figure 4: Two-year uncollateralized payer vanilla IRS

Some derivatives do not have closed-form solutions [5] and thus they require the use of Monte Carlo simulation and other numerical methods to compute their values. One of these types of derivatives is the American-style option. If we assume there is no closed-form solution for a particular derivative, then the LSM can be used and modified to price that derivative. This method asserts that the derivative value can simply be expressed as a linear combination of arbitrary basis functions. For example, we can tailor the LSM and use it to price a European option or even a vanilla IRS.

We will price a European option and IRS using their analytical formulae unless stated otherwise. For the American-style option, we will use the LSM.

Wrong-way and Right-way Risk

The WWR(RWR) are crucial elements that should be considered when computing credit exposures. The WWR(RWR) is a type of counterparty credit risk whereby the counterparty's default is positively(negatively) correlated with the counterparty's credit exposure. The figure below illustrates the WWR and RWR.

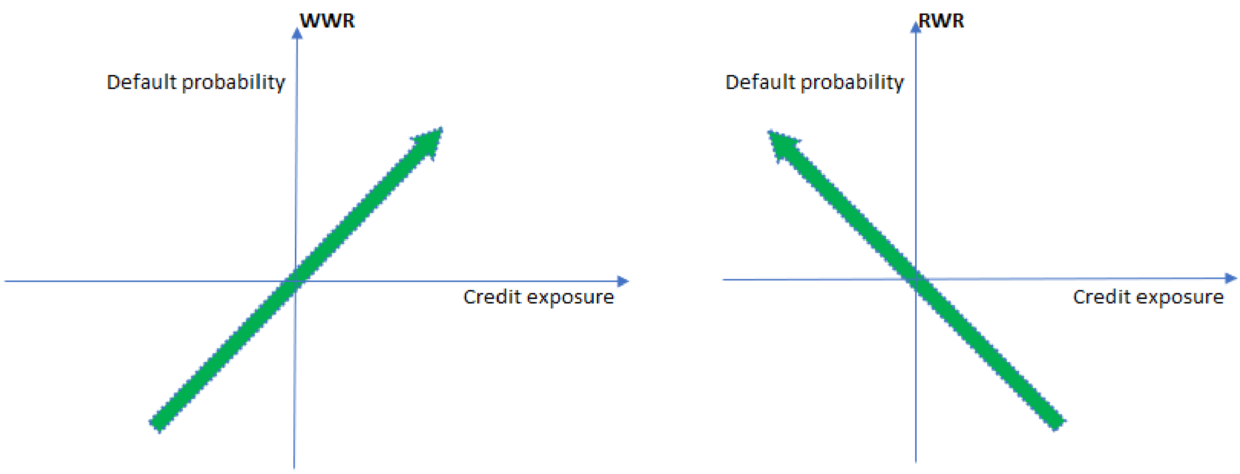


Figure 5: The illustration of WWR and RWR.

We shall assume the absence of the WWR and RWR in this dissertation.

[5] A solution that can be expressed using simple algebraic expressions or equations.

1.4 Theory and impact of xVAs

The xVA is a generic term that was formally adopted after the financial crisis of 2007-2008 to address some of the shortcomings of the BSM model. The xVA stands for valuation adjustments; where x is a generic symbol. There are several types of xVAs, such as Credit Valuation Adjustment (CVA), Debit Valuation Adjustment (DVA), Collateral Valuation Adjustment (COLVA), Funding Valuation Adjustment (FVA), Capital Valuation Adjustment (KVA), and Margin Valuation Adjustment (MVA).

Now we give a brief description of each valuation adjustment and how it impacts the derivative value.

Credit Valuation Adjustment CVA was implemented to incorporate default risk in derivatives pricing. In comparison to other xVAs, CVA has been used and applied for a very long time. CVA is the difference between the derivative value that incorporates default risk and derivative value without default risk [30]. There are two types of CVAs; we have unilateral CVA and bilateral CVA. In unilateral CVA we consider counterparty's default risk while in bilateral CVA we account for both issuer's and counterparty's default risk and thus we include DVA. The CVA is usually calculated at the counterparty level and trade level. However, it can also be adjusted and calculated for a single derivative.

Debit Valuation Adjustment Unlike CVA, DVA reflects the own default risk in derivatives pricing. One can think of DVA as CVA in opposite direction. DVA faced some controversies on whether it can be hedged or monetized in any way, however, it is now a norm to include both CVA and DVA in derivatives pricing [24].

Collateral Valuation Adjustment COLVA reflects the costs and benefits from the collateral agreement. The pricing of derivatives using CSA collateral is discussed by Piterbarg [48].

Funding Valuation Adjustment FVA reflects the funding costs of variation margin for uncollateralized or imperfect collateralized derivatives pricing.

Capital Valuation Adjustment KVA reflects the cost of holding regulatory capital.

Margin Valuation Adjustment MVA reflects the funding costs of posting the initial margin.

Let V, $\hat{V}$ and U be the derivative value, economic value, and valuation adjustment, respectively. The economic value $\hat{V}$ is the total value of the derivative or derivatives portfolio. It includes all or some of the pricing factors [6] such as default risk, funding costs, regulatory capital, initial margin, and/or collateral.

The valuation adjustment U is the sum of all valuation adjustments in the derivative or derivatives portfolio. This assumes that we consider all pricing factors in the derivatives portfolio or derivative pricing. That is

$$U_t = CVA_t + DVA_t + FCA_t + COLVA_t + KVA_t + MVA_t$$

For example, in an uncollateralised derivative trade or portfolio trade, the term $COLVA$ will not be considered. The derivative's economic value $\hat{V}$ is denoted by the sum of BSM model or default-free value V and the valuation adjustment U. That is

$$\hat{V} = V + U \tag{1.4.1}$$

From this equation (1.4.1) we can see that $\hat{V}$ will either increase or decrease.
When $V > 0$: If $U > 0$ then the magnitude of $\hat{V}$ will increase, else it will decrease.
When $V < 0$: If $U < 0$ then the magnitude of $\hat{V}$ will increase, else it will decrease.

Computing V is much easier than U. Although derivative value V is driven by the underlying S, the valuation adjustments U is driven by derivative value V. Mathematically, we say that when $V = V(S)$ and $U = U(V)$, then $U = U(V) = U(V(S)) = (U \circ V)(S)$ [7]. From this observation, it is clear that xVAs may be considered as a family of derivatives whereby derivative value V is an underlying. This simply means that xVAs are also tradable just like any derivative.

To derive valuation adjustment U we will apply the semi-replication strategy. In the next chapter, we provide a theory on how V is computed and we will also derive the BSM PDE for V.

The following is the structure of the remainder of this dissertation:

[6] By pricing factors we refer to factors that give rise to xVAs such as default risk, funding costs, initial margin, collateral, and/or regulatory capital unless stated otherwise.

[7] The small circle between the two functions represents the composition of those functions. For any two real functions f, g we define function composition as $(f \circ g)(x) = f(g(x))$.

Chapter 2: Traditional Derivatives Pricing

To understand the derivation and valuation of xVAs, it is important to know and understand traditional derivatives pricing, hence this chapter is dedicated to default-free continuous-time mathematical tools. The main results are the BSM model and Feynman-Kac formula.

Chapter 3: Literature Review

This chapter gives the literature review for xVAs and also touches on the FVA debate which became a heated debate after the financial crisis.

Chapter 4: Derivatives Pricing with Default Risk

In this chapter, the semi-replication strategy is applied for the first time to derive CVA, DVA, FCA, and COLVA.

Chapter 5: Derivatives Pricing with Regulatory Capital

In this chapter, the semi-replication strategy is extended to incorporate regulatory capital in derivatives pricing. We present the corrected KVA version of Green et al. [29] which is derived by Vauhkonen [54]. We also present regulatory capital methodologies; CEM and SA-CCR approach. These are used to compute CCR and CVA capital. In turn, these are used to compute KVA.

Chapter 6: Derivatives Pricing with Initial Margin

In this chapter, the semi-replication strategy is extended to incorporate initial margin in derivatives pricing by deriving the corrected version of MVA derived in [26] by Vauhkonen [54]. We also present the initial margin methodologies; SIMM and risk measure based models. SIMM model applies to uncleared OTC derivatives while risk measured based models apply to cleared derivatives.

Chapter 7: Numerical Results

This chapter presents the numerical results and analysis of this dissertation. The computer programs are all written in *Python*, which is a free open source programming language that is capable of handling numerical computations in the best possible manner. We used a 4GB RAM Dell computer. The regulatory capital and the initial margin methodologies are assessed and analyzed; xVAs for a European option, IRS, and an American option are presented. We also investigate the difference between the xVAs of OTC uncleared IRS and OTC cleared IRS.

2 Traditional Derivatives Pricing

> Mathematics is the art of giving the same name to different things.
>
> *Henri Poincare*

This chapter presents the default-free continuous-time mathematical tools used in derivatives pricing. The main results of this chapter are the Black-Scholes-Merton model and the Feynman-Kac formula. The pricing formulae for the interest rate swap, European and American options are given and the LSM model is finally presented.

2.1 Principles of Derivatives Pricing

We start by defining the financial market model or simply financial model and conclude by stating and proving the Feynman-Kac formula.

Definition 2.1.1 Financial market model
In continuous-time setting, a financial market model $\mathcal{M} := \big((\mathcal{F}_t)_{t\geq 0}, S_t^1, ..., S_t^n\big)$ is a filtration $\mathcal{F}_t$ and n hedging instruments

$$(S_t^1, ..., S_t^n), \forall\, t \geq 0$$

These hedging instruments may be assets, groups of assets, or other derivatives. As discussed in the first chapter, these instruments drive the value of the derivative. A derivatives pricing model is a type of financial model.

Definition 2.1.2 Risk-free asset
Suppose $r > 0$ is the continuous short interest rate. A risk-free asset M_t is a monotonically increasing price process that follows the differential equation $dM_t := rM_t dt$ and $M_0 = 1$. This risk-free asset is often referred to as the money market or bank account.

If the short rate depends on time t then it will be denoted by r_t and the differential equation will be denoted by $dM_t = r_t M_t dt$. The price process M_t can be obtained using Ito's lemma and its solution is $M_t = \exp(\int_0^t r_v dv)$ where r_t is the stochastic rate process and $M_0 = 1$. If r_t is constant (i.e. $r_t = r$) then $M_t = e^{rt}$. The price process M_t can be used as a normalizing or numeraire asset and this is used mostly in risk neutral pricing as shown in Proposition 2.1.5.

Definition 2.1.3 Discount factor

A stochastic process $D(t,T)$ is called a discount factor if

$$D(t,T) := \frac{M_t}{M_T} = \exp\left(- \int_t^T r_v dv \right)$$

where $0 \leq t \leq T < \infty$ is time period.

The discount factor is crucial in derivatives pricing theory as we shall see this later when we define risk-neutral pricing theorems. The expected value of the discount factor under the risk-neutral measure is the value of the default-free zero-coupon bond (ZCB). The next definition describes this.

Definition 2.1.4 Default-free zero-coupon bond

Let M_t denote the money market account as it is defined in 2.1.2; t and T represent the current time and maturity time of the bond, respectively. Then the value of the ZCB with a face value of 1 at T is

$$B(t,T) := \mathbb{E}_{\mathbb{Q}}\left[\frac{M_t}{M_T} \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(- \int_t^T r(v) dv \right) \middle| \mathcal{F}_t \right]$$

under measure $\mathbb{Q}$.

A ZCB is one of the simplest types of interest rate derivatives. We took the expected value under risk-neutral measure $\mathbb{Q}$ which its definition will be given later in this section.

Definition 2.1.5 Diffusion model

A financial market model $\mathcal{M}^{Diffusion} = \left((\mathcal{F}_t)_{t \geq 0}, S_t^1, ..., S_t^n \right)$ is said to be a diffusion model if for each $j = 1, ..., m$ and $i = 1, ..., n$ the underlying price S_t^i follows the stochastic process

$$\frac{dS_t^i}{S_t^i} := \mu_t^i dt + \sum_{j=1}^{m} \sigma_{j,t}^i dW_t^j$$

where $\mu_t^i, dW_t^j, \sigma_{j,t}^i$ are lognormal drifts, independent Weiner (or Brownian motions) processes and lognormal volatility, respectively.

The special case of the diffusion model is a geometric Brownian motion; which is a one-dimensional diffusion model. Its definition will be provided later in this section.

Definition 2.1.6 Adapted process

A stochastic process ξ_t is said to be an adapted process to the filtration $(\mathcal{F}_t)_{t \geq 0}$ if ξ_t is a random variable in the probability space which is generated by filtration $\mathcal{F}_t$.

The definition of adapted process is crucial when one defines the Ito's integral whereby the integrand needs to be an adapted process.

Definition 2.1.7 The stopping time

Let τ be a $\mathcal{F}_t$-measurable positive random variable. Then τ is said to be a stopping time if and only if the stochastic process A is an adapted process on filtration $(\mathcal{F}_t)_{t\geq 0}$ expressed as

$$A = (A_t)_{t\in[0,T]} := \mathbb{1}_{\{t\geq\tau\}}$$

where $\mathbb{1}$ is an indicator function.

The pricing of American-style options involves the theory of stopping times. The stopping time tells us to decide when to exercise the contract.

Definition 2.1.8 Derivative security

As defined in Chapter 1, in a derivative security, two counterparties will exchange a series of cash flows in future. These cash flows are adapted processes on stopping times in a given derivatives pricing model $\mathcal{M}$. Suppose that ζ_t is an adapted process where $d\zeta_t \in [t, t+dt]$. Then

$$\int_0^t d\zeta_t := \sum_{i=1}^{m} c_i$$

where $\tau_i = \tau_1, ..., \tau_m < t$, (τ_i, c_i) is cash flow streams and c_i is a cash flow.

A derivative like a European call option has cash flows expressed $(S-E)_+$ where S and E are the underlying and strike price, respectively.

Definition 2.1.9 Self-financing replicating strategy

Let H_t be the derivative payout at time t and dH_t be the cash flow stream. Let $(S_t^1, ..., S_t^m)$ be the underlyings or hedging instruments that replicates dH_t. Then a self-financing replicating strategy is a family of adapted processes ξ_t^i whereby the differential of dH_t is

$$dH_t := \sum_{i=1}^{m} \xi_i^t dS_t^i$$

for $i = 1, ..., m$ and $0 \leq t \leq T$.

This strategy will be used to derive the BSM PDE and this strategy is also partially applied in the derivation of xVA models as we shall see in the next chapter.

Definition 2.1.10 Self-financing trading strategy

Let $(S_t^1, ..., S_t^n)$ be the underlyings or hedging instruments and $V_t = \sum_{i=1}^{n} \xi_t^i S_t^i$ for $i = 1, ..., n$ be the value of the asset S_t^i with position ξ_t^i. A self-financing trading strategy holds if the following formula is satisfied

$$\sum_{i=1}^{n} (S_t^i + dS_t^i) d\xi_t^i := 0$$

This self-financing replicating strategy assumes that the cash flows are zero.

Definition 2.1.11 Arbitrage
Let $(\xi_t^1, ..., \xi_t^n)$ be a self-financing trading strategy and $(S_t^1, ..., S_t^n)$ be the underlyings or hedging instruments. Then $(\xi_t^1, ..., \xi_t^n)$ is said to be an arbitrage strategy if one of these two conditions hold:

A1: The process of the derivatives portfolio value is

$$V_t = \sum_{i=1}^{n} \xi_t^i S_t^i$$

such that V_0 and with probability $\mathbb{P}(V_T \geq 0) = 1$.

A2: The process of derivatives portfolio value V_t is assumed to be $V_0 = 0$ and $\mathbb{P}(V_T > 0) > 0$ with $\mathbb{P}(V_t \geq 0) = 1$.

In an arbitrage market, one can make a profit without the risk of losing it. Most derivatives pricing models $\mathcal{M}$ are constructed in such a way that arbitrage opportunities are absent.

Definition 2.1.12 Risk-neutral measure
A measure $\mathbb{Q}$ is referred to as the risk-neutral when these conditions are satisfied:

R1: The discounted underlying asset price $D(t,T)S_t^i$ is a martingale under probability measure $\mathbb{Q}$ for each $i = 1, ..., n$, and

R2: Physical probability measure $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

In any financial market, there must be one risk-neutral measure so that arbitrage opportunities do not occur. The risk-neutral measure is the most useful probability measure that is used in derivatives pricing.

Lemma 2.1.1 Suppose that Π is the derivatives portfolio value. Then the discounted derivatives portfolio value process $D(t,T)\Pi(t)$ at time t is a martingale under $\mathbb{Q}$.

Any derivatives pricing model must be a martingale. This lemma states that the derivatives portfolio's discounted cash flows need to be a martingale under $\mathbb{Q}$.

Theorem 2.1.2 T-forward measure
Let M_t be the money market account as given by 2.1.2 and the discount factor to be $D(t,T)$ as in 2.1.3. Then $\exists$ a measure $\mathbb{Q}_T$ such that

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} := \frac{D(t,T)}{\mathbb{E}_{\mathbb{Q}}[D(t,T)|\mathcal{F}_t]}$$

This probability measure $\mathbb{Q}_T$ is called T-forward measure.

The T-forward measure is obtained using the Radon-Nikodym derivative and this indicates how to move from risk-neutral to T-forward measure. In $\mathbb{Q}_T$ the T-maturity bond acts as a numeraire, thus instead of using a money market account M_t, a T-maturity bond is used.

The definition of this ZCB under this probability measure $\mathbb{Q}_T$ is given in the next definition.

Definition 2.1.13 T-maturity bond
Let M_t be the risk-free asset defined in 2.1.2. We define T-maturity bond $P(t,T)$ as

$$P(t,T) := \mathbb{E}_{\mathbb{Q}_T}\left[\frac{M_t}{M_T}\middle|\mathcal{F}_t\right]$$

under T-forward measure $\mathbb{Q}_T$.

It is easy to show that the value of the T-maturity bond is a martingale under $\mathbb{Q}_T$.

Definition 2.1.14 Complete market
Let $\mathcal{M}$ be the financial market model. Then we say that $\mathcal{M}$ is complete if it is arbitrage-free and every derivative or financial instrument is hedgeable.

It is important to note that the self-financing replicating strategy does not apply to incomplete markets. There must always be one risk-neutral measure for this definition to hold, otherwise, it will fail.

Theorem 2.1.3 The fundamental theorem of asset pricing
The fundamental theorem of asset pricing states that:
P1: $\mathcal{M}$ has a risk-neutral probability measure $\mathbb{Q}$ if arbitrage conditions do not hold,
P2: $\mathcal{M}$ is complete iff $\mathbb{Q}$ is unique,
where $\mathcal{M}$ is the financial market model.

The definition of the complete market in Definition 2.1.14 is the same as ***P2***. The FTAP simply tells us that the derivative value can be written as the expected value of the discounted payout under $\mathbb{Q}$. The proof of the FTAP is presented in [1]. This is also shown by the following corollary and proposition.

Corollary 2.1.4 In any arbitrage-free financial market model $\mathcal{M}$ there exists an asset S_t such that

$$S_t = \mathbb{E}_{\mathbb{Q}}[D(t,T)S_T|\mathcal{F}_t]$$

under risk-neutral probability measure and $0 \leq t \leq T < \infty$.

In a nutshell, this corollary says that the derivative value can be written as the expected discounted cash flows under $\mathbb{Q}$. We can also observe this from the following proposition.

Proposition 2.1.5 Let M_t be the risk-free asset at time t as defined in 2.1.2. Then by above corollary

$$\frac{S_t}{M_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_T}{M_T}\middle|\mathcal{F}_t\right]$$

under $\mathbb{Q}$ and $t \leq T < \infty$.

This proposition shows that the value of any derivative or asset is a martingale. For an asset S_t it means that it must be tradable for this corollary and proposition to hold. Thus we can replace the asset with the derivative value V_t since the asset S_t do not need to be tradable. Hence,

$$\frac{V_t}{M_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{V_T}{M_T}\middle|\mathcal{F}_t\right]$$

We will derive the BSM model in a continuous-time setting. In the BSM framework, the underlying price is assumed to be driven by the following SDE

Definition 2.1.15 Geometric Brownian motion
A stochastic process S_t is said to be a geometric Brownian motion (GBM) if it satisfies the following SDE

$$dS_t := \mu_S S_t dt + \sigma_S S_t dW_t$$

A GBM is the one-dimensional special case of diffusion model $\mathcal{M}^{Diffusion}$. A GBM is used to compute the evolution of the underlying price. The underlying price is primarily driven by the Weiner process W_t, which sometimes is called the Brownian motion.

To derive the solution of S_t, we apply the Ito's formula. Now let $f(t, S_t) = f = \log(S_t)$. The Ito's formula states that

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(dS_t)^2$$

We use product rule to simplify

$$\begin{aligned} dtdW_t &= 0 \\ dt^2 = dtdt &= 0 \\ dW_t^2 = dW_t dW_t &= dt \\ dS_t^2 &= dS_t dS_t \end{aligned}$$

$$
\begin{aligned}
&= (\mu_S S_t dt + \sigma_S S_t dW_t)(\mu_S S_t dt + \sigma_S S_t dW_t) \\
&= \sigma_S^2 S_t^2 dt
\end{aligned}
$$

We now use Ito's formula to compute S_t under GBM

$$
\begin{aligned}
d\log(S_t) &= 0 + \frac{1}{S_t}(\mu_S S_t dt + \sigma_S S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}\sigma_S^2 S_t^2 dt \\
d\log(S_t) &= \left(\mu_S - \frac{\sigma_S^2}{2}\right)dt + \sigma_S dW_t
\end{aligned}
$$

We integrate from 0 to t

$$
\begin{aligned}
\int_0^t d\log(S_u)du &= \int_0^t \left(\mu_S - \frac{\sigma_S^2}{2}\right)du + \int_0^t \sigma_S dW_u \\
\log\left(\frac{S_t}{S_0}\right) &= \left(\mu_S - \frac{\sigma_S^2}{2}\right)t + \sigma_S W_t
\end{aligned}
$$

Finally

$$
S_t = S_0 \exp\left(\left(\mu_S - \frac{\sigma_S^2}{2}\right)t + \sigma_S W_t\right)
$$

The above solution is regarded to be analytical. Note that drift μ_S and volatility σ_S can depend on the underlying asset price and time, in this case, they are presented by $\mu_S(t, S_t)$ and $\sigma_S(t, S_t)$. In the BSM model, they are both assumed to be flat (aka constant).

The best way to obtain the price path of S_t is by using Monte Carlo simulation. The increments of Weiner process W_t follow a normal distribution and thus $W_t = W_t - W_0 \sim \mathcal{N}(0, t)$. If z follows a normal distribution then the GBM equation can be written in discrete form as

$$
S_{t_i} = S_{t_{i-1}} \exp\left(\left(\mu_S - \frac{\sigma_S^2}{2}\right)(t_i - t_{i-1}) + \sigma_S z\sqrt{t_i - t_{i-1}}\right)
$$

for $i = 1, 2, ..., n$ and S_{t_0} is predetermined. We can drop the time index such that $S_i = S_{t_i}$ and rewrite this as

$$
S_i = S_{i-1} \exp\left(\left(\mu_S - \frac{\sigma_S^2}{2}\right)\Delta t + \sigma_S z\sqrt{\Delta t}\right)
$$

where $\Delta t = t_i - t_{i-1}$, $\forall i = 1, 2, ..., n$ is the time step size from t_{i-1} to t_i. This equation is used to model and computed stock prices and we will be apply it when we price options.

Another method that we can use to generate the underlying asset prices for S is the Euler-

Maruyama method (or simply the Euler method). This method is usually used when it is challenging to compute analytical solutions of SDEs. Using this method, GBM is discretisized to

$$S_{t_i} = S_{t_{i-1}} + \mu_S S_{t_{i-1}} \Delta t + \sigma_S S_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \tag{2.1.1}$$

where $\Delta t = t_i - t_{i-1}$ is a time step size from t_{i-1} to t_i, $\forall i = 1, ..., n$.

The Figure 6 below illustrates the simulation of 100 stock prices using equation (2.1.1).

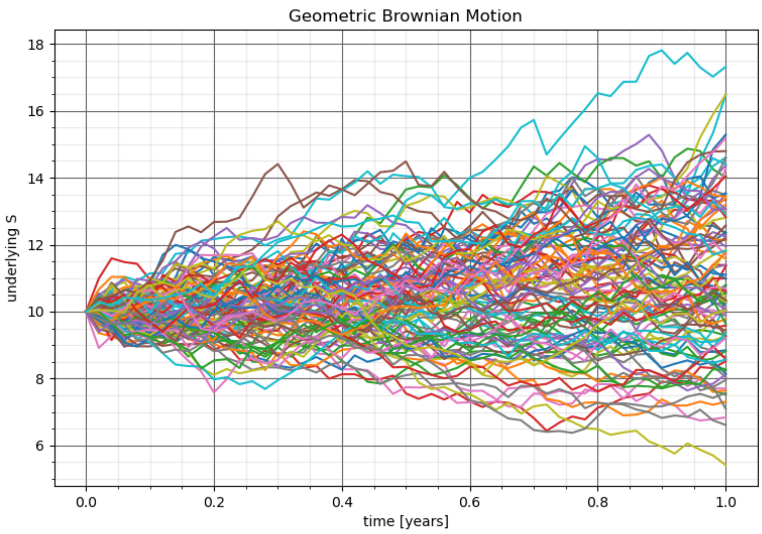


Figure 6: Simulation of 100 stock prices using GBM where $S_{t_0} = S_0 = 10$, $\mu_S = 0.3$, and $\sigma_S = 0.25$.

Now we will state and prove the BSM model in a PDE form.

Theorem 2.1.6 The Black-Scholes-Merton model, $\mathcal{M}^{BSM}$
Let V be the derivative value, S be the underlying at t and $dM = rMdt$ be the evolution of the underlying bank account. If S follows the GBM with constant drift r then the evolution of V is

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$
$$V(T, S_T) = H(S_T)$$

where the derivative payout at maturity time T is $H(S_T)$ and $0 \leq t \leq T < \infty$.

Proof: Consider a self-financing replicating portfolio consisting of δ_S units of the underlying and a derivative contract. Let Π and V be the portfolio value and derivative value, respectively. Then the derivatives portfolio value is

$$\begin{aligned} \Pi &= V + \delta_S S \\ \Rightarrow d\Pi &= dV + \delta_S dS \end{aligned} \tag{2.1.2}$$

We first need to define what δ_S is. In our case δ_S is expressed as $\delta_S = -\frac{\partial V}{\partial S}$. We shall apply the Ito's lemma to compute the evolution of derivative value

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 \\ &= \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma_S S \frac{\partial V}{\partial S}dW \end{aligned} \tag{2.1.3}$$

Now we compute the evolution of portfolio value Π using equation (2.1.2)

$$\begin{aligned} d\Pi &= dV + \delta_S dS \\ &= \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\delta_S\right)dt + \sigma_S S\left(\frac{\partial V}{\partial S} + \delta_S\right)dW \end{aligned} \tag{2.1.4}$$

If Π was invested in riskless asset then $d\Pi = r\Pi dt$. Since $\delta_S = -\frac{\partial V}{\partial S}$ then

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \\ r\Pi dt &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \\ r\left(V - S\frac{\partial V}{\partial S}\right)dt &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \\ rV &= \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} \quad \blacksquare \end{aligned} \tag{2.1.5}$$

Another way of representing this PDE is through the use of differential operator $\mathcal{B}$ which is defined below.

Definition 2.1.16 The BSM model differential operator

In the BSM model $\mathcal{M}^{BSM}$, the differential operator at t is expressed as:

$$\mathcal{B} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r$$

Thus the BSM model PDE may be defined as follows $(\mathcal{B}V)(t, S) = 0$. This equation describes the evolution of derivative value over time. For example, the payout of a forward contract is given by $H(S_T) = S_T - E$ where E is a fixed amount. Then according to the FTAP its value is $V = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}H(S_T)] = S_0 - Ee^{-r(T-t)}$. We can show that this forward contract value V

satisfies above BSM model PDE.

If there exists a derivative contract that its value V follows this BSM PDE and also depends on the underlying asset S, then this derivative is said to be tradable. An example of a tradable contract would be a forward contract. A derivative contract with value $V = S^2$ does not satisfy the BSM PDE, thus it is not tradable.

To compute the value V can be very challenging, however, the Feynman-Kac formula can be used in this case. We state the Feynman-Kac formula.

Theorem 2.1.7 Feynman-Kac formula
Let S_t follow the following stochastic process

$$dS = adt + bdW$$

where $W = W_t$, $a = a(t, S_t)$ and $b = b(t, S_t)$ represents Weiner process on $(\Omega, \mathcal{F}, \mathbb{Q})$, percentage drift and volatility percentage, respectively. Consider a PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} + a\frac{\partial V}{\partial S} - rV + G = 0$$

where $a, r = r(t), b, S = S_t, G = G(t, S)$ are known and $V(T, S_T) = H(S_T)$ is a boundary condition, and $0 \leq t \leq T$. Then the solution of $V = V(t, S)$ is given by

$$V = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(v)dv}H(S_T) + \int_t^T e^{-\int_t^y r(v)dv}G(y, S_y)dy\middle|\mathcal{F}_t\right]$$

Proof: Suppose that $\exists\overline{T}$ such that $0 < t \leq T < \overline{T}$. Wet set $Z_{\overline{T}}$ to
$Z_{\overline{T}} = e^{-\int_t^{\overline{T}} r(v)dv}H(S_{\overline{T}}) - \int_t^{\overline{T}} e^{-\int_t^y r(v)dv}\left(\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} + a\frac{\partial V}{\partial S} - rV + G\right)dy + \int_t^{\overline{T}} e^{-\int_t^y r(v)dv}Gdy$

It is straightforward to show that $Z_{\overline{T}}$ is a martingale and hence $Z_t = \mathbb{E}[Z_T]$.
Finally we have

$$\begin{aligned} V &= V_t = Z_t \\ &= \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r(v)dv}V_T] \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(v)dv}H(S_T) + \int_t^T e^{-\int_t^y r(v)dv}G(y, S_y)dy\middle|\mathcal{F}_t\right] \quad \blacksquare \end{aligned} \tag{2.1.6}$$

The Feynman-Kac formula is related to the FTAP. For example, as we have seen that the forward contract value is expressed as $V = S_0 - Ee^{-r(T-t)}$. We can derive this using Feynman-Kac formula and we will get $V = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}H(S_T)] = S_0 - Ee^{-r(T-t)}$ where payout function is $H(S_T) = S_T - E$ and $G = 0$.

If we set $a = rS$ and $b = \sigma_S S$ so that the underlying process becomes a GBM, then we can see that the BSM PDE satisfies the Feynman-Kac formula and thus this PDE has a solution that can be written as an expectation of the discounted future cash flows of the derivative contract:

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(v)dv}V(T)|\mathcal{F}_t\right]$$

This form is the same as that of Proposition 2.1.5 and it is also similar to the Feynman-Kac formula except that function $G = 0$. The Feynman-Kac formula will play a huge role in later chapters to derive xVAs.

2.2 Pricing of Options and Swaps

We will derive pricing formulae for these three derivatives; a European option, an American option, and a vanilla interest rate swap.

We would like to start by defining the partition[8] of the time interval. Suppose that $t = T$ is derivative maturity and $t = 0$ is today (usually called inception) such that $0 \leq t \leq T < \infty$.

Definition 2.2.1 Partition of the time interval
A partition $\mathcal{P}_n$ of the particular time interval $[0, T]$ is a finite set expressed as

$$\mathcal{P}_n[0, T] := \{0 = t_0, t_1, ..., t_n = T\}$$

such that

$$t_0 < t_1 < ... < t_n$$

for integer $n > 0$.

The times $\{t_i\}$ are cash flow times of the derivative. For example, in a European option contract, the t_i are exercise dates/times while in a swap contract are reset dates/times. Note that any time interval can be further divided into another subinterval.

[8]Partitioning the time interval simply means discretisizing it into new smaller subintervals.

Definition 2.2.2 Time interval step size
Let $[t_{i-1}, t_i]$ be a subinterval of $[0, T]$ induced by $\mathcal{P}_n$. The time interval step size (or simply step size) is

$$\Delta t := t_i - t_{i-1}$$

where $i = 1, ..., n$.

The step size is usually assumed to be constant and sometimes denoted by Δ. Another way of representing the step size is $\Delta = T/n$. Thus this gives us

$$t_i = i\Delta t$$

where $i = 0, 1, ..., n$. We can also write this as

$$t_i = t_{i-1} + \Delta t$$

for $i = 1, ..., n$. By partitioning $[0, T]$ will enable us to work in discrete space (or discrete-time grid). In the previous section, we derived continuous-time derivatives pricing models, however, we must work in a discrete space if we are to write these models in a computer program.

Vanilla European option

A European option is a style of option whereby the holder is given the right but not the obligation to exercise the underlying asset only at the expiration time. There are two types of European options: a call and put option. A European call allows a party to buy an underlying at the expiration time while a European put allows a party to sell an underlying.

Let S_t be the underlying price [9] at time t and E be the exercise price. The payout function of a European call H_T^{call} and put H_T^{put} are given respectively

$$H_T^{call} := (S_T - E)_+$$
$$H_T^{put} := (E - S_T)_+$$

Let Φ be the standard normal cumulative distribution function. To compute the value C_t and P_t we apply the FTAP

[9] This is usually the stock price. Most options are written on a stock.

$$
\begin{aligned}
C_t &= \mathbb{E}_{\mathbb{Q}}[D(t,T)H_T^{call}|\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T - E)_+|\mathcal{F}_t] \\
&= \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} e^{-r(T-t)} \left(S_t \exp\left(\left(r - \frac{\sigma_S^2}{2}\right)(T-t) + \sigma_S W_t \right) - E \right)_+ dy \\
&= S_t \Phi(d_+) - E e^{-r(T-t)} \Phi(d_-)
\end{aligned}
\tag{2.2.1}
$$

and

$$
\begin{aligned}
P_t &= \mathbb{E}_{\mathbb{Q}}[D(t,T)H_T^{put}|\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(E - S_T)_+|\mathcal{F}_t] \\
&= \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} e^{-r(T-t)} \left(E - S_t \exp\left(\left(r - \frac{\sigma_S^2}{2}\right)(T-t) + \sigma_S W_t \right) \right)_+ dy \\
&= E e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+)
\end{aligned}
\tag{2.2.2}
$$

The derivation of these formulae is originally presented in [10] and these formulae are called the Black-Scholes formulae. The underlying follows the GBM $S_T = S_t \exp\big((r - \frac{\sigma_S^2}{2})(T-t) + \sigma_S W_t\big)$ and $d_\pm = \frac{\log(S_t/E) + (r \pm \sigma_S^2/2)(T-t)}{\sigma_S \sqrt{T-t}}$.

American option

The main difference between a European and American option is the exercise style. Unlike a European option, the American option can be exercised at any time prior to or at maturity time. The interesting thing about American options is the flexibility it gives for exercising. Moreover, this feature makes it hard to price and value American-style options. To this day, there are no analytical formulae to price American-style options. However, there are available closed-form approximations such as the one by Bjerksund & Stensland model [9]. While the American option can be exercised in continuous times, the Bermudan option can be exercised at discrete times.

American option price V follows a variation of the BSM model expressed as

$$
\begin{aligned}
\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} &\leq rV \\
V(T, S_T) &= H(S_T)
\end{aligned}
$$

where $V(T, S_T) = H(S_T)$ is the payout.

The payout for American call H_t^{call} and put H_t^{put} are expressed as

$$H_t^{call} := (S_t - E)_+$$
$$H_t^{put} := (E - S_t)_+$$

where S is the underlying price and E is the exercise price. The value $V(t, S_t)$ is

$$\sup_{\tau \in [t,T]} \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^{\tau} r(v)dv\right)H(S_\tau)\right]$$

where τ is the stopping time and $H(S_\tau)$ can either be H_τ^{call} or H_τ^{put}.

American options pricing is usually described as a dynamic programming formulation. We borrow this theory from Glasserman [22].

Let $\tilde{H}_i$ be time t_i payout function. Let $\tilde{V}_i(x)$ represent the option price at time t_i given that $S_i = s$ is the underlying asset. We want to find the value at $t_0 = 0$ (i.e. $\tilde{V}_0(S_0)$). We do recursively as follows

$$\tilde{V}_m(s) = \tilde{H}_m(s)$$
$$\tilde{V}_i(s) = \max\left\{\tilde{H}_i(s), \mathbb{E}[D_{i,i+1}(S_i)\tilde{V}_{i+1}(S_{i+1})|S_i = s]\right\}$$

$\forall i = 0, 1, ..., n$ where $D_{i,i+1}(S_{i+1})$ denote the discount factor between exercise time t_i and t_{i+1} and $D_{0,i}(S_i)D_{i,i+1}(S_{i+1}) = D_{0,i+1}(S_{i+1})$. This discount factor is defined by

$$D_{i,i+1}(S_{i+1}) = D(t_i, t_{i+1})(S_{i+1}) = \exp\left(-\int_{t_i}^{t_{i+1}} r_v dv\right)$$

For any i in $0 \leq i \leq n$ define the following equations,

$$H_i = D_{0,j}(S_j)\tilde{h}_i(s), \ \forall i = 1, ..., n.$$
$$V_i(s) = D_{0,i}(s)\tilde{V}_i(s), \ \forall i = 0, 1, ..., n.$$

Clearly at time zero $V_0 = \tilde{V}_0$ and the V_i will be given by

$$\begin{aligned} V_m(s) &= H_m(s) \\ V_i(s) &= D_{0,i}(s)\tilde{V}_i(s) \\ &= D_{0,i}\max\left\{\tilde{H}_i(s), \mathbb{E}[D_{i,i+1}(S_{i+1})\tilde{V}_{i+1}(S_{i+1})|S_i = s]\right\} \\ &= \max\left\{\tilde{H}_i(s), \mathbb{E}[D_{0,i}(S_i)D_{i,i+1}(S_{i+1})\tilde{V}_{i+1}(S_{i+1})|S_i = s]\right\} \end{aligned}$$

$$= \max\left\{\tilde{H}_i(s), \mathbb{E}[\tilde{V}_{i+1}(S_{i+1})|S_i = s]\right\}$$

for all $i = 0, 1, ..., n$.

Other methods to price the American option include the least squares simulation method [42] which will be presented in the next section. When we price American options by the least squares simulation method we will treat it as a Bermudan option.

Vanilla Interest Rate Swap

A plain vanilla interest rate swap (IRS) is a type of swap contract whereby two parties exchange interest rate payments. The legs are split into two: counterparty leg and issuer leg. One leg is fixed and the other leg is float. We shall derive the formula for linear IRS. Let t, T, N be the current time, maturity time, and notional amount of a vanilla IRS.

Floating leg

Suppose that there are m times of a floating leg for a period of $[0, T]$ and $\mathcal{P}_m$ be the partition of this time period such that

$$0 = t_0 < t_1 < t_2 < ... < t_m = T < \infty$$

Let $F(t; t_{i-1}, t_i)$ be a floating forward rate and m be a number of fixing times of a floating leg. Then the value of a floating leg is

$$V_{float}(t) = N \times \mathbb{E}_{\mathbb{Q}_{t_i}}\left[\sum_{i=1}^{m} F(t; t_{i-1}, t_i)\Delta_i D(t, t_i)\middle|\mathcal{F}_{t_i}\right]$$

where $\Delta_i = T/m = t_i - t_{i-1}$ such that $t_{i-1} < t < t_i$ and $\mathbb{Q}_{t_i}$ is a t_i forward measure. The floating rate $F(t; t_{i-1}, t_i)$ is usually the benchmark rate [10] and will be computed using the Vasicek model.

Fixed leg

Suppose that there are n times of a fixed leg for a period of $[0, T]$ and $\mathcal{P}_n$ be the partition of

[10] An example of a benchmark rate is JIBAR rate which stands for Johannesburg Interbank Agreed Rate

this time period such that

$$0 = t_0 < t_1 < t_2 < ... < t_n = T < \infty$$

Let R_f be the fixed interest rate. Then the value of a fixed leg is

$$V_{fixed}(t) = N \times \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^{n} R_f \Delta_k D(t, t_k)\middle|\mathcal{F}_t\right]$$

where $\Delta_k = T/n = t_k - t_{k-1}$. Although the fixed leg has been taken under the expectation which is under $\mathbb{Q}$, this expectation can be ignored when we price the IRS using a single curve, since $\mathbb{E}[c|\mathcal{F}_t] = c$ for any constant $c \in \mathbb{R}$.

The value of the IRS is

$$V_{IRS}(t) = \kappa(V_{float}(t) - V_{fixed}(t)) \tag{2.2.3}$$

where $\kappa = -1$ for a receiver IRS and $\kappa = 1$ for a payer IRS. It is common practice to set fixed rate R_f in such a way that the price of the IRS is zero when it starts. Hence, for a single curve, the R_f becomes

$$R_f = \frac{\sum_{i=1}^{m} F(t; t_{i-1}, t_i)\Delta_i D(t, t_i)}{\sum_{k=1}^{n} \Delta_k D(t, t_k)}$$

This fixed rate is called a fair swap rate. If $F(t; t_{i-1}, t_i)$ is a simple forward interest rate such that

$$F(t; t_{i-1}, t_i) = \frac{1}{t_i - t_{i-1}}\left(\frac{D(t, t_{i-1})}{D(t, t_i)} - 1\right)$$

Then the value of the floating leg of an IRS will be

$$V_{float}(t) = N[D(t, t_0) - D(t, t_m)] \tag{2.2.4}$$

and fair swap rate becomes

$$R_f = \frac{D(t, t_0) - D(t, t_m)}{\sum_{k=1}^{n} \Delta_k D(t, t_k)}$$

We assumed that both legs have different payment times (i.e. $n \neq m$). However, it is possible to have equal payment times, which is a common practice on the market.

2.3 The Least Squares Monte Carlo Simulation

The LSM is centered around the approximation of conditional expectation using the least squares regression and Monte Carlo simulation. We will show how the LSM is used to price American options.

Let $\mathbf{Y}$ be a response vector of length n and $\mathbf{X}$ be a $n \times k$ matrix of regressors. The regression model is given as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\eta}^T + \mathbf{e} \tag{2.3.1}$$

where $\mathbf{e}$ is a vector of length n called error term. If $\mathbf{X}$ are linear regressors [11], then response vector $\mathbf{Y}$, matrix of regressors $\mathbf{X}$, vector of regression coefficients $\boldsymbol{\eta}$ and vector of error terms $\mathbf{e}$ are given below

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, \mathbf{X} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nk} \end{bmatrix}, \boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \end{bmatrix}, \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}.$$

Theorem 2.3.1 Conditional expectation

Let $\mathbf{Y}$ be a response vector of length n and $\mathbf{X}$ be a $n \times k$ matrix of regressors. Then $\exists f$ such that the expectation of $\mathbf{Y}$ conditional on $\mathbf{X}$ is of the form

$$\mathbb{E}[\mathbf{Y}|\mathbf{X}] = f(\mathbf{X}, \boldsymbol{\eta})$$

where $\boldsymbol{\eta}$ is a k vector of regression coefficients.

Proof: We take the expectation conditional to the matrix of regressors $\mathbf{X}$ and get

$$\begin{aligned} \mathbb{E}[\mathbf{Y}|\mathbf{X}] &= \mathbb{E}[\mathbf{X}\boldsymbol{\eta}^T + \mathbf{e}|\mathbf{X}] \\ &= \mathbb{E}[\mathbf{X}\boldsymbol{\eta}^T|\mathbf{X}] + \mathbb{E}[\mathbf{e}|\mathbf{X}] \\ &= \mathbf{X}\boldsymbol{\eta}^T + \mathbf{0} \\ &= f(\mathbf{X}, \boldsymbol{\eta}) \quad \blacksquare \end{aligned} \tag{2.3.2}$$

because $\mathbb{E}[\mathbf{e}|\mathbf{X}] = \mathbf{0}$. It is important to note that function f is not unique. There are a variety of functions that can be applied to get the best approximation of the conditional expectation.

[11] These regressors can also be non-linear or even trigonometric. However, polynomials functions will give the best approximation of $\mathbf{Y}$.

Theorem 2.3.2 The least squares method

The least squares problem is concerned with finding $\hat{\boldsymbol{\eta}}$ such that

$$\hat{\boldsymbol{\eta}} = \min_{\boldsymbol{\eta}} J(\boldsymbol{\eta})$$

where

$$J(\boldsymbol{\eta}) = \|\mathbf{e}\|^2 = \left\|\mathbf{Y} - \mathbf{X}\boldsymbol{\eta}^T\right\|^2$$

Under the least squares problem $\hat{\boldsymbol{\eta}}$ is given as

$$\hat{\boldsymbol{\eta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

Proof: We set $\frac{\partial J(\hat{\boldsymbol{\eta}})}{\partial \hat{\boldsymbol{\eta}}} = \mathbf{0}$ and compute $\hat{\boldsymbol{\eta}}$

$$\begin{aligned}
\frac{\partial J(\hat{\boldsymbol{\eta}})}{\partial \hat{\boldsymbol{\eta}}} &= \frac{\partial \left\|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\eta}}^T\right\|^2}{\partial \hat{\boldsymbol{\eta}}} \\
&= (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\eta}}^T)^T(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\eta}}^T) \\
&= 2\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\eta}} - 2\mathbf{X}^T\mathbf{Y} \\
&\implies \hat{\boldsymbol{\eta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \quad \blacksquare
\end{aligned} \tag{2.3.3}$$

From this theorem we see that the main aim of the least squares method is to minimize the error of the regression model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\eta}^T + \mathbf{e}$$

Further decomposition of the response $\mathbf{Y}$ is

$$\begin{aligned}
\mathbf{Y} &= \mathbb{E}[\mathbf{Y}|\mathbf{X}] + \mathbf{e} \\
&= f(\mathbf{X}, \boldsymbol{\eta}) + \mathbf{e}
\end{aligned} \tag{2.3.4}$$

The regression function f is expressed as a linear combination

$$f(\mathbf{X}, \boldsymbol{\eta}) = \sum_{i=0}^{n} \eta_i L_i(x) \tag{2.3.5}$$

where L_i is a set of basis regression functions. In our case we choose Laguerre polynomials and its closed form is

$$L_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j!} x^j \tag{2.3.6}$$

One assumptions that we can make so that our approximation becomes accurate is that

$$f(\mathbf{X}, \boldsymbol{\eta}) = \lim_{n\to\infty} \sum_{i=0}^{n} \eta_i L_i(x) \tag{2.3.7}$$

If we suppose that X and Y are stochastic then the approximation of f will be called the LSM function. What makes the LSM powerful is that it can also be used in the pricing of any derivative. This is usually done when the derivative contract does not have analytical value. This is entirely supported by Proposition 2.1.5. The LSM can also be used to price a derivatives portfolio. The derivatives portfolio will depend on whether it contains early-exercise derivatives such as American-style options or not. If it does, then this model can be adjusted to suit this. It is fortunate that Longstaff and Schwartz already presented this in their paper [42]. The below figure is an illustration of the LSM.

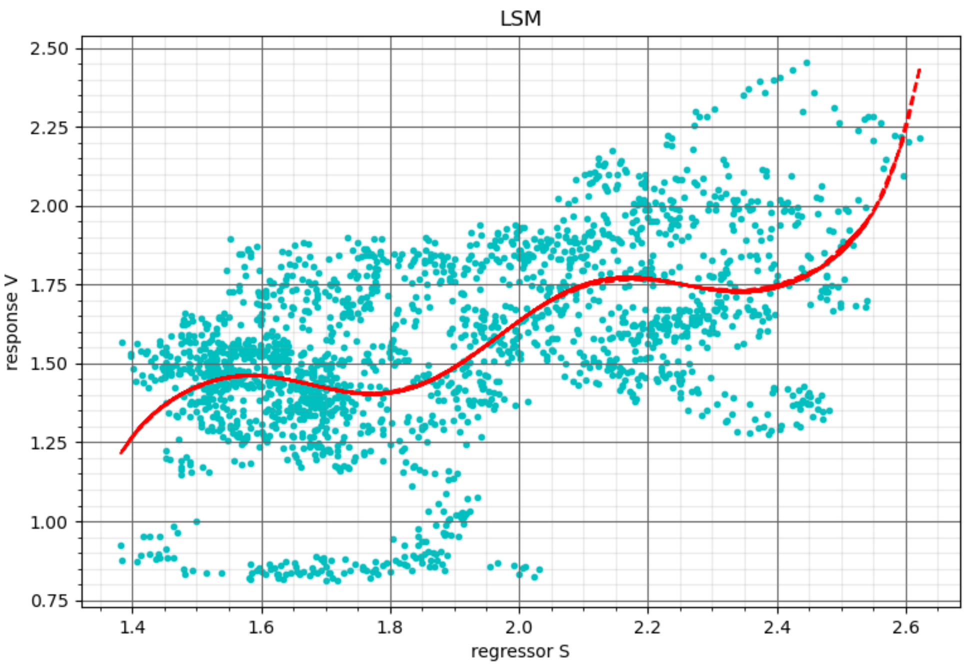


Figure 7: The illustration of the LSM. The line in red is the LSM function.

Now we present the LSM for American-style options from Longstaff and Schwartz. Suppose that $C(s, \omega; t, T)$ represents the cash flows generated by the option with respect to the following stopping strategy $\forall s$, $0 \leqslant t < s < T < \infty$. Moreover, this cash flow $C(s, \omega; t, T)$ depends on the option not being exercised at or before t. Suppose there are M discrete times, $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant \ldots \leqslant t_M = T$.

By the FTAP the value of the option can be valued by taking the expectation of the discounted cash flows $C(\omega, s; t_i, T)$ up until maturity time under $\mathbb{Q}$. Precisely, at any time t_i the continuation value F can be computed using

$$F(t_i, \omega) = \mathbb{E}_{\mathbb{Q}}\left[\sum_{q=i+1}^{M} \exp\left(-\int_{t_i}^{t_q} r(s,\omega)ds\right) C(t_q, \omega; t_i, T)\middle|\mathcal{F}_{t_i}\right], \tag{2.3.8}$$

where $r(s, \omega)$ is the risk-free rate used for discounting and at time t_q the expectation is conditional on $\mathcal{F}_{t_q}$. As we can see F is a conditional expectation similar to that of Theorem 2.3.1. The estimation is done in a discrete space using the LSM

$$\overline{F}(t_j; x) = \lim_{M\to\infty} \sum_{m=0}^{M} \eta_m L_m(S_{t_j} = x) \tag{2.3.9}$$

where η_m and L are Laguerre basis coefficients and polynomials, respectively.

The pseudo-algorithm for American-style LSM

(i) Simulate paths ω_j for $j = 1, 2, ..., N$ for M basis functions

(ii) Compute the option value at the maturity time for each simulated path ω_j

(iii) By taking into consideration of only the in-the-money paths use the cash flow matrix $C(s, \omega_j; t_{k-1}, T)$ to compute the conditional expectation of the continuation $F(t_{k-1}, \omega)$

(iv) Decide whether the immediate exercise is optimal

(v) Compute the cash flow matrix $C(s, \omega_j; t_{k-2}, T)$ and continue this way recursively for each exercise date

(vi) To estimate the value of the option $F(0, \omega)$, find the sum of cash flows and take the average

In this chapter, we provided traditional mathematical tools for derivatives pricing by deriving the BSM model and Feynman-Kac formula. We also presented pricing formulae for a European option, vanilla IRS, and American option. These three derivatives will be applied in computing our xVA numbers. Lastly, we derived the LSM and shown how it is used in pricing the American option.
The following chapter provides the literature review on xVA models.

3 Literature Review

Madam, I have come from a country
where people are hanged if they talk.

Leonhard Euler

This chapter presents the literature review on xVAs and also reviews the FVA debate. We review the semi-replication strategy of Burgard and Kjaer [13] and provide the PDEs of the economic value of xVAs.

Burgard and Kjaer

The first formal model or strategy used to define and derive xVAs was the semi-replication strategy of Burgard and Kjaer [13]. This strategy extends the BSM model to incorporate default risk, collateral and funding costs, regulatory capital, and initial margin. Unlike in the derivation of the BSM model, the semi-replication strategy constructs the hedge portfolio in such a way that it includes two pre-default issuer bonds, one pre-default counterparty bond, the underlying position, cash positions associated with the underlying, counterparty bond, and the relevant pricing factors.

Let B be an issuer and C be the counterparty. The issuer B issues two bonds of different seniorities P_1 and P_2 while counterparty C issues one bond P_C. The hedge portfolio is constructed as follows

$$\Pi = \delta_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta$$

where $\beta = \beta_S + \beta_C + \beta_X$ is the total cash account positions. These cash accounts are prior rebalancing and are expressed as

$$\begin{aligned} d\beta_S &= \delta_S(\gamma_S - q_S)Sdt \\ d\beta_C &= -\alpha_C q_C P_C dt \\ d\beta_X &= -r_X X dt \end{aligned}$$

The semi-replication strategy hedges out the underlying position δ_S, subordinated bond position α_1, senior bond position α_2, counterparty bonds position α_C and cash account β. This is not catered for in the BSM model which does not cater for cash accounts associated with credit, collateral, or other pricing factors. In fact, in the BSM model only the underlying position δ_S and the derivative value V are hedged out. In their paper [13], Burgard and Kjaer derived CVA, DVA, COLVA, FCA and show how these valuation adjustments can be hedged. The PDE of the derivative's economic value $\hat{V}$ is expressed by

$$\begin{aligned}\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} &= s_X X - \lambda_C g_C - \lambda_B g_B + \lambda_B \varepsilon_{h_0} \\ \hat{V}(T, S_T) &= H(S_T) \\ \delta_S + \frac{\partial \hat{V}}{\partial S} &= 0 \\ g_C - \hat{V} &= \alpha_C P_C\end{aligned}$$

where $H(S_T)$ is the derivative payout of the derivative contract at maturity time T, $\mathcal{A}_t \equiv \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2}{\partial S^2} + (q_S - \gamma_S)S\frac{\partial}{\partial S}$, $g_C(V, X) := R_C(V - X)_+ + (V - X)_- + X$, and $g_B(V, X) := (V - X)_+ + R_B(V - X)_- + X$. This economic value is also known as the credit risky value of the derivative and it is given by $\hat{V} = V + U$.

Hull and White

In 2012, the introduction of FVA into the pricing of derivatives became a big controversy. This valuation adjustment arises due to collateral mismatches and the fact that discounting is related to funding [24, 48]. Hull and White were the first to argue that the FVA should not be taken into consideration when pricing derivatives: they said, "we discount at the risk-free rate because this is required by the risk-neutral valuation principle" [32]. Furthermore, they also argued that pricing should be kept apart from funding, using a well-established corporate finance theory concept known as the separation principle. The transfer pricing problem occurs in practice sometimes as this theory is not technically feasible; their argument is more theoretical than practical.

They defined two types of DVA: one arising because a dealer may default on its derivatives portfolio, denoted by DVA_1 and other arising because a dealer may default on its other liabilities, denoted by DVA_2. They went on to define FVA which they classify as anti-Economic Valuation Adjustment as

$$\text{FVA} = \Delta(\text{DVA}_2)$$

where $\Delta(\text{DVA}_2)$ is an increase in DVA_2 resulting from the funding requirements of a derivatives portfolio. In response to their paper, Burgard and Kjaer [14], refuted this definition by clarifying that FVA is actually defined as

$$\text{FVA} = \text{FBA} + \text{FCA}$$

where FBA is funding benefit adjustment and they proven that FBA = DVA. Hull and White refer to FVA as FCA and this contradicts the definition of Burgard and Kjaer. Burgard and Kjaer argued in their paper that if the issuer B can hedge out all risks of their own and counterparty then the FCA term will vanish, however, this is not expected in reality.

Numerous papers defend the inclusion of FVA in derivatives pricing. The first paper to stand out was that of Laughton and Vaisbrot [40]. They argue that the BSM framework does not hold in the real world and that some real markets are incomplete. One of the assumptions of the BSM model is that markets are complete and this means that every derivative must be hedgeable. In reality, however, this is not true as some markets are not complete. Thus the BSM must be adjusted to suit the real world and hence they support that FVA must be considered in derivatives pricing.

Green, Kenyon, and Dennis

The paper by Green et al. [29] was the first to provide a formal definition of KVA in derivatives pricing. In this paper they extended the semi-replication strategy of Burgard and Kjaer to take regulatory capital under the Basel framework into account, thus deriving the valuation adjustment term, KVA, which is defined as the cost of holding regulatory capital. The hedge portfolio and cash account related to regulatory capital are incorporated into the semi-replication strategy. This cash account holds prior rebalancing and is expressed as

$$d\beta_K = -\gamma_K \phi K dt$$

In their original paper, Green et al. excluded the regulatory capital fraction in their definition of cash account associated with regulatory capital. They stated that " To deal with capital itself we introduce a parameter, ϕ, to represent the fraction of the capital, K, used for funding". From this statement, it is clear that derivatives are not funded by the entire capital but just the fraction of it, ϕK.

In Vauhkonen [54], the corrected formula for KVA is derived by including the regulatory capital fraction into the cash account associated with regulatory capital $d\beta_K$. In the base case, where no fraction of regulatory capital is used at all, one would expect the KVA to be zero since the regulatory capital fraction is zero. However, this is not true in Green et al. as they have omitted the regulatory capital fraction in cash account associated with regulatory capital. The PDE of the economic value incorporating the regulatory capital is given by the semi-replication strategy and its PDE form is

$$
\begin{aligned}
0 &= \frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial \hat{V}}{\partial S} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad + g_B\lambda_B + g_C\lambda_C - s_X X - \gamma_K \phi K + r\phi K - \varepsilon_h \lambda_B \\
\hat{V}(T, S_T) &= H(S_T) \\
\delta_S + \frac{\partial \hat{V}}{\partial S} &= 0 \\
g_C - \hat{V} &= \alpha_C P_C
\end{aligned}
$$

Here, the hedging error is $\varepsilon_h = \varepsilon_{h_K} + \varepsilon_{h_0}$; $\varepsilon_{h_K}$is the hedging error that contains regulatory capital and ε_{h_0} does not contain regulatory capital. The function g_C is the same as above.

Green and Kenyon

The formal definition of MVA was first introduced in 2014 by Green and Kenyon [26]. They extended the Burgard and Kjaer semi-replication strategy to take the cost of initial margin into consideration, thus introducing MVA as one of the xVAs. The cash account related to the initial margin prior to rebalancing is

$$
d\beta_{I_B} = r_{I_B} I_B dt
$$

where I_B is the IM posted by issuer B. The calculation of I_B is done using VaR based model in which they apply the LSM also know as *Longstaff-Schwartz Augmented Compression*. This is a similar model as that of Longstaff and Schwartz [42] introduced to price American-style options. The challenging part about computing MVA is the calculation of the initial margin of the derivative or portfolio of different instruments. For simplicity, the distribution of the derivative or portfolio is usually assumed to be a normal distribution. The corrected MVA formula is presented by Vauhkonen [54] which is the same paper that corrected the KVA formula. The economic value partial differential equation that incorporates all these pricing factors is then given by

$$
\begin{aligned}
0 &= \frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial \hat{V}}{\partial S} - (r + \lambda_C + \lambda_B)\hat{V} + s_{I_B} I_B \\
&\quad + g_B\lambda_B + g_C\lambda_C - s_X X - \gamma_K \phi K + r\phi K - \varepsilon_h \lambda_B \\
\hat{V}(T, S_T) &= H(S_T) \\
\delta_S + \frac{\partial \hat{V}}{\partial S} &= 0 \\
g_C - \hat{V} &= \alpha_C P_C
\end{aligned}
$$

where s_{I_B} is the spread on the initial margin. The close-out functions g_B, g_C are now adjusted to $g_B(V, X, I_B) = (V - X + I_B)_+ + R_B(V - X + I_B)_- + X - I_B$ and $g_C(V, X, I_C) = R_C(V - X - I_C)_+ + (V - X - I_C)_- + X + I_C$.

From the above three PDEs, we can see that if we assume that $U = 0$, meaning that no valuation adjustment is considered, then $\hat{V} = V$ and above PDE will be the same as that of the BSM model.

In this chapter, we presented the literature review by reviewing the semi-replication strategy of Burgard and Kjaer and we also provided the PDE for the economic value of the derivative. We discussed the FVA debate which was the heated debate in mathematical finance after the financial crisis of 2007-2008. We also reviewed the work of Vauhkonen [54] and how it corrected the omission of regulatory capital in Green et al. [29].

4 Derivatives Pricing with Default Risk

All models have faults - that doesn't mean you can't use them as tools for making decisions.

Myron Scholes

We have dedicated the previous chapters to the development of default-free pricing models, in particular the BSM model. We will now extend the BSM model to incorporate and account for default risk in derivatives pricing. We derived the BSM model using the self-financing replicating strategy, however, to derive xVAs models we will apply the semi-replication strategy of Burgard and Kjaer [13]. In this chapter, we derive CVA (and DVA) models.

4.1 Deriving CVA by semi-replication strategy

Let B be an issuer and C be the counterparty in a derivatives market such that either B or C can default. Denote an economic value that incorporates default risk by $\hat{V}$. The derivative contract can incorporate collateral and any funding costs the issuer may face before its own default. Thus this strategy is called the semi-replication strategy because it only holds until issuer B default.

Let zero-coupon zero-recovery bond P_C be a bond tradable issued by counterparty C. Denote the underlying price at time t by S and two issuer bonds P_1 and P_2 which have recovery rates denoted by R_1 and R_2. Note that recovery rate P_C is assumed to be zero.

Definition 4.1.1 The evolution of the underlying instruments
The underlying instruments S, P_C, P_1 and P_2 follow these SDEs

$$
\begin{aligned}
dS &= \mu_S S dt + \sigma_S S dW \\
dP_C &= r_C P_C dt - P_C dJ_C \\
dP_i &= r_i P_i dt - (1 - R_i) P_i dJ_B
\end{aligned}
$$

whereby J_B and J_C are default indicators modelled by Poisson process and $i = 1, 2$.

These default indicators are independent processes jumping from 0 to 1. The issuer and counterparty bonds, P_i and P_C are pre-default bond prices and, hence they are default risky bonds (or simply defaultable bonds).

The underlying asset price S still follows GBM where μ_S and σ_S is drift and volatility, respectively. Suppose that P_1 is a junior bond. That is to say $R_1 < R_2$ and $r_1 > r_2$.

Definition 4.1.2 Economic values at default
Suppose that $\hat{V}(t, S, J_B, J_C)$ denotes the economic value of the derivative to the issuer B and the collateral amount be denoted by X. The economic value when issuer B defaults first is given by

$$\hat{V}(t, S, 1, 0) = g_B(M_B, X)$$

and when counterparty C defaults first is given by

$$\hat{V}(t, S, 0, 1) = g_C(M_C, X)$$

where M_B and M_C are general close-out amounts.

If $M_B = M_C = V$, where the closeout value V is the BSM model value, we call these boundary conditions regular. These two functions of g are given by

$$g_B(V, X) := (V - X)_+ + R_B(V - X)_- + X \quad (4.1.1)$$

$$g_C(V, X) := R_C(V - X)_+ + (V - X)_- + X \quad (4.1.2)$$

Definition 4.1.3 The cash account positions prior rebalancing
Let $\beta_S, \beta_C, \beta_X$ be cash account associated with the underlying asset S, counterparty bond and collateral, respectively.
Then the evolution of these cash accounts are given respectively by,

$$d\beta_S = (q_S - \gamma_S)\beta_S dt$$
$$d\beta_C = q_C \beta_C dt$$
$$d\beta_X = r_X \beta_X dt$$

where $\beta_S = -\delta_S S, \beta_C = -\alpha_C P_C, \beta_X = -X$ and $\beta = \beta_S + \beta_C + \beta_X$ is the total cash account positions.

These cash accounts β_S and β_C are used to finance the positions S and P_C [15]. That is to say, $\alpha_C P_C + \beta_C = 0$ and $\delta_S S + \beta_S = 0$. These are assumed to pay net rates of $q_S - \gamma_S$ and q_S

respectively, here γ_S may be the dividend yield.

Definition 4.1.4 The replicating portfolio
Assuming that other parameters are as defined before, then the replicating portfolio is expressed as

$$\Pi = \delta_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta$$

where δ_S is underlying position; α_i for $i \in \{1, 2\}$ and α_C are the fractions of own and counterparty bonds, respectively; and β_S and β_C are cash accounts and X is the collateral.

The collateral amount X is assumed to be fully rehypothecable and is examined from the issuer's point of view. Rehypothecation is when the issuer re-uses the collateral amount to secure their own borrowing. This derivative collateral amount X is assumed to pay a rate of r_X. These pre-default bonds are similar to the one defined in 2.1.4 except that they are conditional on the default of issuer B, hence, we call this the semi-replication strategy.

Definition 4.1.5 The funding constraint
Assume that other parameters are as defined before, then the funding constraint is defined by,

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

where $\hat{V} = \hat{V}(t, S, J_B, J_C)$, $X = X(t)$, $\alpha_i = \alpha_i(t)$ and $P_i = P_i(t)$ for $i \in \{1, 2\}$.

By applying Ito's lemma we arrive at the following evolution of hedge portfolio and economic value, respectively

$$d\Pi = \delta_S dS + \alpha_1 dP_1 + \alpha_2 dP_2 + \alpha_C dP_C + d\beta_S + d\beta_C + d\beta_X \tag{4.1.3}$$

$$d\hat{V} = \frac{\partial \hat{V}}{\partial t} dt + \frac{\partial \hat{V}}{\partial S} dS + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} dt + \Delta \hat{V}_C dJ_C + \Delta \hat{V}_B dJ_B \tag{4.1.4}$$

such that $\Delta \hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) = g_B - \hat{V}$ and $\Delta \hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) = g_C - \hat{V}$.

Since we assumed self-financing portfolio then this equation holds $d(\Pi + \hat{V}) = d\Pi + d\hat{V}$. Here, this sum must always be zero (i.e. $d\Pi + d\hat{V} = 0$).

Definition 4.1.6 Adjusted differential operator
The adjusted differential operator $\mathcal{A}_t$ is given by $\mathcal{A}_t \equiv \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial}{\partial S}$

This is used to simplify the notation and it is similar to the BSM model differential operator; the BSM model PDE will now be modified. The difference is that we are now including the repo rate and the dividend of the underlying.

Theorem 4.1.1 Deriving CVA by semi-replication strategy
Let $\mathcal{M}^{CVA}$, $\mathcal{M}^{DVA}$, $\mathcal{M}^{FCA}$ and $\mathcal{M}^{COLVA}$ represent CVA, DVA, FCA and COLVA models, respectively. If $U_t = CVA_t + DVA_t + FCA_t + COLVA_t$ then

$$
\begin{aligned}
CVA_t &= -\mathbb{E}_t\left[\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[V(y) - g_C(V(y), X(y))]dy\Big|\mathcal{F}_t\right]\\
DVA_t &= -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[V(y) - g_B(V(y), X(y))]dy\Big|\mathcal{F}_t\right]\\
FCA_t &= -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_0}(y)dy\Big|\mathcal{F}_t\right]\\
COLVA_t &= -\mathbb{E}_t\left[\int_t^T s_X(y)e^{-\int_t^y a(v)dv}X(y)dy\Big|\mathcal{F}_t\right]
\end{aligned}
$$

where $a(v) = r(v) + \lambda_B(v) + \lambda_C(v)$ and $\varepsilon_{h_0} = g_B - X + P_D$.

Proof: We start by adding $d\hat{V}$ and $d\Pi$ from equations (4.1.3) and (4.1.4)

$$
\begin{aligned}
d(\hat{V} + \Pi) &= d\hat{V} + d\Pi\\
&= \left(\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} - s_X X + \lambda_C g_C + \lambda_B g_B - \varepsilon_{h_0}\lambda_B\right)dt + \varepsilon_{h_0}dJ_B
\end{aligned}
\tag{4.1.5}
$$

where $\varepsilon_{h_0} = g_B - X + P_D$ is the hedging error that excludes regulatory capital, $P_D = \sum_{i=1}^2 \alpha_i R_i P_i$, λ_C is the default intensity of the counterparty's default hedge position, $s_X \equiv r_X - r$ and $\mathcal{A}_t$ is the adjusted differential operator defined above in Definition 4.1.6, $ds_X = r_X - \gamma_S$, and $s_X = r_X - r$ is the collateral spread.

Since the portfolio is self-financing then $d\hat{V} + d\Pi = 0$, we set $\delta_S + \frac{\partial \hat{V}}{\partial S} = 0$ and $g_C - \hat{V} - \alpha_C P_C = 0$. Finally we get

$$
\begin{aligned}
\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} &= s_X X - \lambda_C g_C - \lambda_B g_B + \lambda_B \varepsilon_{h_0}\\
\hat{V}(T, S_T) &= H(S_T)
\end{aligned}
\tag{4.1.6}
$$

where $H(S_T)$ is the derivative payout maturity time T. The BSM PDE is now modified to

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + (q_S - \gamma_S)S\frac{\partial V}{\partial S} &= rV\\
V(T, S_T) &= 0
\end{aligned}
\tag{4.1.7}
$$

We have stated that valuation adjustment is the correction of the BSM model value V and it is denoted by U. The valuation adjustment is simply expressed as $U = \hat{V} - V$ but its payout at maturity is always zero. The U follows this PDE

$$\begin{aligned}\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_X X - \lambda_C(g_C - V) - \lambda_B(g_B - V) + \lambda_B \varepsilon_{h_0} \\ U(T, S_T) = 0\end{aligned} \tag{4.1.8}$$

We set $a(v) = r(v) + \lambda_B(v) + \lambda_C(v)$ and $U_t = CVA_t + DVA_t + FCA_t + COLVA_t$. Since valuation adjustments U are martingales then we apply Feynman-Kac formula 2.1.7 where $H(S_T) = 0$ such that

$$U_t = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^y a(v)dv} G(y, V_y) dy \Big| \mathcal{F}_t\right]$$

Thus we get each valuation adjustment as

$$CVA_t = -\mathbb{E}_t\left[\int_t^T \lambda_C(y) e^{-\int_t^y a(v)dv}[V(y) - g_C(V(y), X(y))]dy \Big| \mathcal{F}_t\right] \tag{4.1.9}$$

$$DVA_t = -\mathbb{E}_t\left[\int_t^T \lambda_B(y) e^{-\int_t^y a(v)dv}[V(y) - g_B(V(y), X(y))]dy \Big| \mathcal{F}_t\right] \tag{4.1.10}$$

$$FCA_t = -\mathbb{E}_t\left[\int_t^T \lambda_B(y) e^{-\int_t^y a(v)dv} \varepsilon_{h_0}(y) dy \Big| \mathcal{F}_t\right] \tag{4.1.11}$$

$$COLVA_t = -\mathbb{E}_t\left[\int_t^T s_X(y) e^{-\int_t^y a(v)dv} X(y) dy \Big| \mathcal{F}_t\right] \quad \blacksquare \tag{4.1.12}$$

As we can see, function G is unique for each valuation adjustment. For example, for CVA it is given by $\lambda_C(V - g_C(V, X))$.

Throughout this dissertation, we will assume that general close-out values are standard in such a way that $M_B = M_C = V$. It is possible to define other cases such as $M_B = M_C = \hat{V}$, however, for simplicity, we will only focus on $M_B = M_C = V$.

Both CVA and DVA can be simplified further. We will start by simplifying variables within the expectation

$$\begin{aligned}V - g_C(V, X) &= V - [R_C(V - X)_+ + (V - X)_- + X] \\ &= (V - X) - R_C(V - X)_+ - (V - X)_- \\ &= (1 - R_C)(V - X)_+ \\ V - g_B(V, X) &= V - [(V - X)_+ + R_B(V - X)_- + X] \\ &= (1 - R_B)(V - X)_-\end{aligned}$$

Then CVA and DVA terms become

$$CVA_t = -\mathbb{E}_t\left[(1-R_C)\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[(V(y)-X(y))_+]dy\Big|\mathcal{F}_t\right]$$
$$DVA_t = -\mathbb{E}_t\left[(1-R_B)\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[(V(y)-X(y))_-]dy\Big|\mathcal{F}_t\right]$$

As we have postulated before, to calculate CVA, DVA and, FCA, we will need ENE and EPE.

Remark 4.1.1. We have not commented on the nature of default intensities, interest rates and other parameters. In CVA term, for example, we can assume that default intensity λ_C is deterministic. Again, if we assume that only collateral amount $X(t)$ is stochastic then we can further simplify the COLVA term and the main task will be to compute $\mathbb{E}[X(t)|\mathcal{F}_t]$. Some authors such as Burgard and Kjaer [15] denote FVA term as FVA = FCA + DVA while Hull and White [32] denote it as FVA = FCA. In the appendix we will discuss funding strategies according to Burgard and Kjaer [15] and also introduce FVA formula. In Green et al. [29] the hedging error term ε_{h_0} is the term that does not incorporate regulatory capital while ε_{h_K} incorporates regulatory capital. The total hedging error will be $\varepsilon_h = \varepsilon_{h_0} + \varepsilon_{h_K}$.

We presented and derived CVA, DVA, COLVA, and FCA models by using the semi-replication strategy. In the next chapter, we extend the semi-replication strategy to derive KVA. We will see that the hedging error ε_h is broken in two other sub hedging errors as stated in Remark 4.1.1.

5 Derivatives Pricing with Regulatory Capital

> The circulation of capital realizes value, while living labour creates value.
>
> *Karl Marx*

In this chapter, we extend the semi-replication strategy derived in the previous chapter to incorporate regulatory capital K in derivatives pricing. Thus we will introduce KVA.

5.1 Deriving KVA by semi-replication strategy

Definition 5.1.1 Regulatory capital fraction
Let K represent the regulatory capital used for funding and ϕ denoted some number such that $0 \leq \phi \leq 1$. Then the fraction of regulatory capital is ϕK.

The aim is to use this fraction of regulatory capital to fund derivatives or a portfolio of instruments.

Definition 5.1.2 The funding constraint
The funding condition of a collateralized derivative contract with economic value $\hat{V}$ is given by

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 - \phi K = 0$$

As we can see the funding constraint is adjusted to incorporate regulatory capital K.

Definition 5.1.3 The cash account positions prior rebalancing
Let $\beta_S, \beta_C, \beta_X$ and β_K be cash account associated with the underlying asset S, counterparty bond, collateral, and regulatory capital, respectively. Then the evolution of these cash accounts are given by

$$\begin{aligned} d\beta_S &= \delta_S(\gamma_S - q_S)Sdt \\ d\beta_C &= -\alpha_C q_C P_C dt \\ d\beta_X &= -r_X X dt \\ d\beta_K &= -\gamma_K \phi K dt \end{aligned}$$

The economic values for issuer B and counterparty C are defined in 4.1.2.

Definition 5.1.4 The replicating portfolio
Assuming that other parameters are as defined before, then the replicating portfolio Π is defined by

$$\Pi = \delta_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta$$

where $\beta_S = -\delta_S S, \beta_C = -\alpha_C P_C, \beta_X = -X, \beta_K = -\phi K$ and $\beta = \beta_S + \beta_C + \beta_X + \beta_K$ is the total cash account positions.

In Green et al. [29], the cash position β_K does not include the regulatory capital fraction ϕ. In our case we include this regulatory capital fraction, as its exclusion violates Definition 5.1.1.

When we apply the Ito's lemma the evolution of the economic value becomes

$$d\hat{V} = \left(\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2}\right)dt + \frac{\partial \hat{V}}{\partial S} dS + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \tag{5.1.1}$$

Assume that our portfolio, Π, is self-financing, then its evolution is given by

$$\begin{aligned} d\Pi = \delta_S dS + \delta_S(\gamma_S - q_S)S dt + \alpha_1 dP_1 + \alpha_2 dP_2 + \alpha_C dP_C - \alpha_C q_C P_C dt \\ -r_X X dt - \gamma_K \phi K dt \end{aligned} \tag{5.1.2}$$

Theorem 5.1.1 Deriving KVA by semi-replication strategy
Let $\mathcal{M}^{CVA}$, $\mathcal{M}^{DVA}$, $\mathcal{M}^{FCA}$, $\mathcal{M}^{COLVA}$ and $\mathcal{M}^{KVA}$ represent CVA, DVA, FCA, COLVA and KVA models, respectively. If $U_t = CVA_t + DVA_t + FCA_t + COLVA_t + KVA_t$ then

$$\begin{aligned}
CVA_t &= -\mathbb{E}_t\left[\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[V(y) - g_C(V(y), X(y))]dy \middle| \mathcal{F}_t\right] \\
DVA_t &= -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[V(y) - g_B(V(y), X(y))]dy \middle| \mathcal{F}_t\right] \\
FCA_t &= -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_0}(y)dy \middle| \mathcal{F}_t\right] \\
COLVA_t &= -\mathbb{E}_t\left[\int_t^T s_X(y)e^{-\int_t^y a(v)dv}X(y)dy \middle| \mathcal{F}_t\right] \\
KVA_t &= -\mathbb{E}_t\left[\int_t^T e^{-\int_t^y a(v)dv}[(\gamma_K(y) - r(y))\phi K(y) + \lambda_B(y)\varepsilon_{h_K}(y)]dy \middle| \mathcal{F}_t\right]
\end{aligned}$$

where $a(v) = r(v) + \lambda_B(v) + \lambda_C(v)$ and $\varepsilon_h = \Delta\hat{V}_B - (P - P_D) = g_B - X + P_D - \phi K = \varepsilon_{h_0} + \varepsilon_{h_K}$; ε_{h_0} and ε_{h_K} is the non-dependent regulatory capital hedging error that and regulatory error dependent hedging error, respectively.

Proof: Firstly we will compute $d\hat{V} + d\Pi$ by summing equations (5.1.1) and (5.1.2) so that we can apply the self-financing strategy

$$\begin{aligned} d\hat{V} + d\Pi = \Bigg(\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + \delta_S(\gamma_S - q_S)S + \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 \\ + \alpha_C r_C P_C - \alpha_C q_C P_C - r_X X - \gamma_K \phi K \Bigg) dt \\ + \varepsilon_h dJ_B + \left(\delta_S + \frac{\partial \hat{V}}{\partial S} \right) dS \\ + \left[g_C - \hat{V} - \alpha_C P_C \right] dJ_C \end{aligned} \tag{5.1.3}$$

where $\varepsilon_h = \Delta \hat{V}_B - (P - P_D) = g_B - X + P_D - \phi K = \varepsilon_{h_0} + \varepsilon_{h_K}$.

The PDE of the economic value of the derivative is given by

$$\begin{aligned} 0 = \frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial \hat{V}}{\partial S} - (r + \lambda_C + \lambda_B)\hat{V} \\ + g_B \lambda_B + g_C \lambda_C - s_X X - \gamma_K \phi K + r\phi K - \varepsilon_h \lambda_B \\ \hat{V}(T, S_T) = H(S_T) \\ \delta_S + \frac{\partial \hat{V}}{\partial S} = 0 \\ g_C - \hat{V} = \alpha_C P_C \end{aligned} \tag{5.1.4}$$

We can see that $\alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 = rX - (r + \lambda_B)\hat{V} - \lambda_B(\varepsilon_h - g_B) + r\phi K$.

The BSM PDE is modified to

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial V}{\partial S} = rV \\ V(T, S_T) = 0 \end{aligned} \tag{5.1.5}$$

Since we know the economic value $\hat{V}$ and the BSM model value V we can obtain U using this equation $U = \hat{V} - V$. We note that $\frac{\partial U}{\partial S} = \frac{\partial (\hat{V} - V)}{\partial S} = \frac{\partial \hat{V}}{\partial S} - \frac{\partial V}{\partial S}$ [12]. After subtracting equation

[12] This is always true since $\frac{\partial^n (\hat{V} - V)}{\partial S^n} = \frac{\partial^n \hat{V}}{\partial S^n} - \frac{\partial^n V}{\partial S^n}$ and for as long as partial derivatives of $\hat{V}, V$ and S exist. In our case $n = 1, 2$ will always hold.

(5.1.5) from equation (5.1.4) we get the PDE for U expressed as follows

$$\begin{aligned}\frac{\partial U}{\partial S}+\frac{1}{2}\sigma_S^2S^2\frac{\partial^2 U}{\partial S^2}+(q_S-\gamma_S)S\frac{\partial U}{\partial S}-(r+\lambda_B+\lambda_C)U&=V\lambda_C-g_C\lambda_C+V\lambda_B-g_B\lambda_B\\&\quad+\varepsilon_h\lambda_B+s_XX+\gamma_K\phi K-r\phi K\\U(T,S_T)&=0\end{aligned}\tag{5.1.6}$$

We will start by simplifying the right side of equation (5.1.6)

$$\begin{aligned}V\lambda_C-g_C\lambda_C+V\lambda_B-g_B\lambda_B+\varepsilon_h\lambda_B+s_XX+\gamma_K\phi K-r\phi K=\\ \lambda_C(V-g_C)+\lambda_B(V-g_B)\\+\lambda_B(\varepsilon_{h_0}+\varepsilon_{h_K})\\+s_XX+(\gamma_K-r)\phi K\end{aligned}\tag{5.1.7}$$

We set $a(v)=r(v)+\lambda_B(v)+\lambda_C(v)$ and $U_t=CVA_t+DVA_t+FCA_t+COLVA_t+KVA_t$. Since valuation adjustments U are martingales then we apply Feynman-Kac formula 2.1.7 where $H(S_T)=0$ such that

$$U_t=\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^y a(v)dv}G(y,V_y)dy\Big|\mathcal{F}_t\right]$$

Thus we get each valuation adjustment as

$$CVA_t=-\mathbb{E}_t\left[\int_t^T\lambda_C(y)e^{-\int_t^y a(v)dv}[V(y)-g_C(V(y),X(y))]dy\Big|\mathcal{F}_t\right]\tag{5.1.8}$$

$$DVA_t=-\mathbb{E}_t\left[\int_t^T\lambda_B(y)e^{-\int_t^y a(v)dv}[V(y)-g_B(V(y),X(y))]dy\Big|\mathcal{F}_t\right]\tag{5.1.9}$$

$$FCA_t=-\mathbb{E}_t\left[\int_t^T\lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_0}(y)dy\Big|\mathcal{F}_t\right]\tag{5.1.10}$$

$$COLVA_t=-\mathbb{E}_t\left[\int_t^T s_X(y)e^{-\int_t^y a(v)dv}X(y)dy\Big|\mathcal{F}_t\right]\tag{5.1.11}$$

$$KVA_t=-\mathbb{E}_t\left[\int_t^T e^{-\int_t^y a(v)dv}[(\gamma_K(y)-r(y))\phi K(y)+\lambda_B(y)\varepsilon_{h_K}(y)]dy\Big|\mathcal{F}_t\right]\quad\blacksquare\tag{5.1.12}$$

We can also expressed the FCA and KVA equation as

$$\overline{FCA}_t=-\mathbb{E}_t\left[\int_t^T e^{-\int_t^y a(v)dv}[\varepsilon_h(y)\lambda_B(y)-r(y)\phi K(y)]dy\Big|\mathcal{F}_t\right]$$

$$\overline{KVA}_t=-\mathbb{E}_t\left[\int_t^T\gamma_K(y)e^{-\int_t^y a(v)dv}\phi K(y)dy\Big|\mathcal{F}_t\right]$$

These expressions show that FCA may depend on the regulatory capital term K. If we do not incorporate regulatory capital in derivatives pricing then FCA will be the same as that derived in Chapter 4. Also, if we assume perfect hedging such that we do not incorporate regulatory capital, then the hedging error will become zero, and eventually, FCA will be zero. We will see this when we present and derive funding strategies in the appendix.

Remark 5.1.1. As we can see, the KVA term is slightly different from the one derived by Green et al. [29]. From the definition regulatory capital fraction we expect that when $\phi = 0$ then automatically $\text{KVA}_t = 0$. Because it means that no regulatory capital is used at all to fund derivatives. In Green et al. [29] even when $\phi = 0$, $\text{KVA}_t \neq 0$, and this violates the definition of regulatory capital fraction ϕ. As we have stated in the literature, Vauhkonen [54] gives the corrected version of KVA which similar to ours, except that our expression is more general as we do not assume the nature of default intensity, risk-free rate, and other parameters.

The main driver of KVA calculation is the regulatory capital K. In the next section, we will show the difference between CEM and the new regulatory capital methodology called SA-CCR. Lastly, we will present the CVA capital methodology.

5.2 Credit Risk Capital

Under the Basel III regime, regulatory capital is split into three capitals: credit risk capital, market risk capital, and operational risk capital. For KVA calculations the operational risk is not considered because of the complexity of its nature as either it is too hard or too easy to compute [50]. On the other hand, market risk capital is a bank-wide level capital and it will not be considered too in KVA calculations. We will only consider credit risk capital which consists of CCR and CVA capital.

Let K_{CCR} and K_{CVA} be the CCR and CVA capital, respectively. Then regulatory capital K is given by $K = K_{CCR} + K_{CVA}$.

5.2.1 Counterparty Credit Risk capital

We present the CCR capital methodologies, CEM and SA-CCR.

Definition 5.2.1 Counterparty Credit Risk capital
The CCR capital at an arbitrary time is computed using this formula

$$K_{CCR} = 8\% \times RWA$$

and

$$RWA = 12.5 \times w \times EaD$$

where w, EaD and RWA is the risk weight, exposure at default, and risk-weighted assets, respectively.

The EaD is the amount a bank is exposed at the time of default in a transaction. This is the most crucial component to calculate is EaD. There are various ways in which we can compute this credit exposure under CCR. Before we present EaD under CEM and SA-CCR we will give weight values under CCR. We choose S&P rating [53] as our main rating definition.

Weight Calculation

The risk weight calculation in the standardized approach is provided below using S&P credit rating notation.

Table 1: Weights using S&P credit rating notation

Risk-weight	S&P
20%	From AAA to AA-
50%	From A+ to A-
50%	From BBB+ to BBB-
10%	From BB+ to BB-
100%	From B+ to B-
150%	CCC+ and below

EaD Calculation

Under the Basel regime, the EaD is computed in three different ways: via the IMM, the CEM, or the SA-CCR. We will not present or apply IMM in our numerical results. The SA-CCR will be replacing the CEM and is expected to go live by approximately 2022. Each model has its own set of drawbacks. The SA-CCR and CEM are presented below.

a. Current Exposure Method

The CEM was first introduced in Basel I to be used to calculate EaD for derivatives. It is very simple to use compared to other methodologies.

Definition 5.2.2 EaD under CEM
Under CEM, EaD is determined using the formula

$$EaD_{CEM} = RC + N \times \chi \times AddOn$$

where RC, N, χ and $AddOn$ is the replacement cost, notional amount, netting factor, and add-on factor, respectively. The RC can be obtained by

$$RC = (V - C)_+$$

where V is the derivative value in the netting set and C is the collateral amount.

The netting factor depends on the type of netting of derivative trades. There are two types of these; bilateral netting and central clearing netting. The formula is

$$\chi = \begin{cases} 0.4 + 0.6 \times \frac{(\sum_i V_i)_+}{\sum_i (V_i)_+}, & \text{bilateral netting} \\ 0.15 + 0.85 \times \frac{(\sum_i V_i)_+}{\sum_i (V_i)_+}, & \text{central clearing} \end{cases}$$

Finally, the add-on is extracted from the following table for each asset class:

Table 2: Add-on values under Base II

Residual Maturity	Interest Rates	FX Gold	Equity	Precious Metals	Other Commodities
≤ 1 year	0.0%	1.0%	6.0%	7.0%	10.0%
$1 - 5$ years	0.5%	5.0%	8.0%	7.0%	12.0%
> 5 years	1.5%	7.5%	10.0%	8.0%	15.0%

Central clearing derivatives refer to derivatives that are cleared through CCPs.

The add-on values are already calculated and thus it is easy to use CEM. For example, a 2-year IRS will have an add-on of 0.5%, and a 2-year European option written on a stock will have an add-on of 8.0%. For bilateral netting, the netting factor χ is the value between 0.4 and 1 while for central clearing it is between 0.15 and 1.

b. Standardized approach for measuring CCR, SA-CCR
This model can be used for OTC derivatives, security financing transactions, long settlements transactions, and exchange-traded derivatives. We start by defining EaD under SA-CCR.

Definition 5.2.3 EaD under SA-CCR
EaD is calculated separately for each netting set using this formula

$$EaD_{SA-CCR} = \alpha(RC + PFE)$$

where $\alpha = 1.4$ and both RC and PFE depend on whether the derivative trade is margined or unmargined.

The scaling factor $\alpha = 1.4$ is set by BCBS and this explains why the EaD for SA-CCR is always larger than that of CEM. The computation of the replacement cost is easier than potential future exposure as we will see. It is important to note that this PFE is different from the one defined in 1.3.4 and it is called PFE add-on.

Definition 5.2.4 The replacement Cost
The replacement cost for unmargined trade is

$$RC = (V - C)_+$$

where V is the derivative value of transactions in the netting set and C is haircut value of the net collateral held. The replacement cost for margined transaction is modified to

$$RC = \max\{V - C; MTA + TH - NICA; 0\}$$

where MTA is minimum transfer amount under CSA, TH is threshold under CSA and NICA is net independent collateral amount.

The replacement cost is similar to that of CEM. The difference is that in SA-CCR, the trade can either be margined or unmargined. This is one of the reasons why SA-CCR will replace CEM as it does not specify if a trade is margined or not.

Definition 5.2.5 PFE add-on
The PFE add-on will be computed according to

$$PFE = m \times AddOn^{aggregate}$$

where $m = \min\left\{1, Floor + (1 - Floor) \times \exp\left(\frac{V-C}{2(1-Floor)AddOn^{aggregate}}\right)\right\}$ is a multiplier.

The $Floor = 5\%$ guarantees that the PFE add-on can never reach zero but reach the lowest of 5%. To compute m we first need to compute $AddOn^{aggregate}$ since it depends on it. This add-on value depends on a type of asset class.

Definition 5.2.6 Asset class

There are five types of asset classes under SA-CCR

(a) Interest rate, IR

(b) Foreign exchange, FX

(c) Credit, CR

(d) Equity, EQ

(e) Commodity, CO

To allocate a particular derivative to a certain asset class, we need to define the primary risk factor (or driver) [8]. When the primary risk factor is identified the derivative will be categorized according to one of the above asset classes. For example, the primary risk factor for an IRS is interest rate risk. Hence an IRS falls under the IR class. A foreign exchange option's primary risk factor is foreign exchange risk, thus it falls under foreign exchange derivatives.

Definition 5.2.7 Aggregated Add-On

The total aggregated add on is equal to the sum of all add-ons of these five asset classes

$$AddOn^{aggregate} = \sum_{a} AddOn^{(a)}$$

where a is asset class.

The difficult task is to compute an add-on $AddOn^{a}$ for a particular asset class. We will now provide steps to compute an add-on for each of these asset classes. We start by defining four types of dates:

Definition 5.2.8 Time period parameters M_i, S_i, E_i, T_i

The maturity of M_i of the derivative trade shall be the period from now until the last day on which the contract could still be alive.

S_i denotes the period of time starting now until the start of the time period referenced by an interest rate or credit derivative contract.

E_i is the time period from now until the end of time and it applies to an IR and CR derivatives.

T_i is the time period starting now until the latest contractual exercise date as referenced by the contract.

Here S_i and E_i do not refer to the underlying stock price and exercise price. M_i applies to all asset classes, S_i only applies to interest rate and credit derivatives, T_i applies to options in all asset classes, and T_i is used in the calculation of option delta.

Definition 5.2.9 Trade-level adjusted notional

For *IR and CR:*

Let $SD_i = \frac{\exp(-0.05 \times S_i) - \exp(-0.05 \times E_i)}{0.05}$

Then $d_i =$ Notional $\times SD_i$ in domestic currency.

For *FX:*

The adjusted notional is the foreign currency leg converted to the domestic currency. If both of their legs are dominated in non-domestic currencies, then each leg is converted into a domestic currency and thus the adjusted notional is the leg with the larger value of the domestic currency.

For EQ and CO:

The adjusted notional is simply the current price of one unit of the stock or commodity multiply by the number of units referenced by the derivative trade.

This adjusted notional is for the single trade, thus it is called trade-level adjusted notional. In IR and CR, the *Notional* is simply the trade notional.

For example, suppose we have a 2 year IRS trading with a notional of 100, then $S_i = 0$ and $E_i = 2$. The supervisory duration SD_i is 1.9033 and the adjusted notional will then become 190.33 in domestic currency.

Supervisory delta adjustments

Non-options or non-CDO tranches

Table 3: Supervisory deltas for non-options or non-CDO tranches

δ_i	Long in the primary risk factor	Short in the primary risk factor
Instruments that are not options or CDO tranches	$+1$	-1

Delta for options

Let P_i be the underlying price, E_i be the strike price, and T_i be the latest exercise time of the option contract.

Let $z = \frac{\log(P_i/E_i)+\frac{1}{2}\sigma_i^2 T_i}{\sigma_i\sqrt{T_i}}$, where σ_i is the supervisory delta of the option which determined on the basis of supervisory factor and Φ is the cumulative distribution function of the standard normal distribution. Then delta adjustments are given below

Table 4: Delta for option contracts

δ_i	Bought	Sold
Call	$\Phi(z)$	-$\Phi(z)$
Put	-$\Phi(-z)$	$\Phi(-z)$

This delta is similar to the usual option delta for a vanilla European option.

The delta adjustments for CDO tranches

Suppose that we have a CDO tranche with A_i as an attachment point and D_i as the detachment point.

Then delta adjustments are given below

Table 5: Delta for CDO tranches

δ_i	Purchased (long protection)	Sold (short protection)
CDO tranches	$+\frac{15}{(1+15A_i)(1+14D_i)}$	$-\frac{15}{(1+15A_i)(1+14D_i)}$

Definition 5.2.10 Maturity factor

For unmargined transactions, maturity factor is calculated using this formula

$$MF_i^{(unmargined)} = \sqrt{\frac{\min\{M_i; 1\ year\}}{1\ year}}$$

where M_i is the ith remaining maturity expressed in years.

In margined trades, maturity factor is computed using this formula

$$MF_i^{margined} = \frac{3}{2}\sqrt{\frac{MPOR_i}{1\ year}}$$

where $MPoR_i$ is the ith MPoR measured in years.

For example, if we have unmargined trade with a remaining maturity of 0.5 years then its maturity factor will become 0.7071. For margined trades, the maturity factor is often constant because MPoR is usually assumed to be 10 days. If MPoR is 10 days then the maturity factor is equal to 0.2483 where it is assumed that there are 365 days in a year.

Definition 5.2.11 Hedging set
(i) IR asset class consists of separate hedging set for each currency. Additionally, hedging sets are further divided into maturities; and long and short positions are offsetted.

(ii) For each currency pair, the FX asset class consists of a separate hedging set. Same currency positions can be offsetted.

(iii) CR asset class consists of a single hedging set. Full offset is allowed for trades of the same entities and partial offset is allowed for different entities.

(iv) EQ asset class consists of a single hedging set. Same offsetting rules as CR.

(v) CO asset class comprise of 4 hedging sets described for extensive categories of CO derivatives: metals, energy, agriculture, and other commodities.

Now computing of PFE add-on $AddOn^a$ is simply based on hedging set. A hedging set is a series of transactions that fall under a single netting set within which partial or full offsetting is recognized [8]. Hedging sets vary depending on the type of asset class. For example, as seen in the definition above, the commodity asset class has four hedging sets while in FX class are divided according to each currency pair.

Definition 5.2.12 The effective notional
The effective notional D should be computed for each derivative in the netting set. This is calculated using this formula $D = d \times MF \times \delta$,
where d is the adjusted notional, MF is the maturity factor and δ is the supervisory delta.

The effective notional D measures the sensitivity of the trade to the evolution of the underlying risk factor.

Supervisory parameters
The summary of supervisory correlation parameters is given in the paper from BCBS [8]. These parameters are SF, ρ, and δ which are the supervisory factor, supervisory correlation, and supervisory volatility, respectively. These parameters are designed for each asset class.

The final step of computing the PFE add-on is presented below:

Asset class add-on

Finally, we show how an add-on for each asset class is computed. Let HS denote a hedging set, D be effective notional (EN_{HS} is effective notional on the hedging set) and SF_a be a supervisory factor for asset class a.

IR class:

Let D^{B1}, D^{B2}, D^{B3} be effective notional of maturity bucket 1, maturity bucket 2, and maturity bucket 3, respectively. Here, bucket 1, bucket 2, and bucket 3 is less than one year, between one and five years, and more than five years, respectively.
Offset:

$$EN_{HS} = \sqrt{(D^{B1})^2 + (D^{B2})^2 + (D^{B3})^2 + \alpha(D^{B1})^2(D^{B2})^2 + \alpha(D^{B2})^2(D^{B3})^2 + 0.6(D^{B1})^2(D^{B3})^2}$$

No offset: $EN_{HS} = \sum_1^3 |D^{Bi}|$ where $D_i = d_i \times MF_i \times \delta_i$.
Here $AddOn_{HS} = SF_{IR} \times EN_{HS}$ and SF_{IR} is a supervisory factor for IR.
Therefore, $AddOn^{(IR)} = \sum_{HS} AddOn_{HS}$.

FX class:

The add-on for FX is $AddOn^{(FX)} = \sum_{HS} AddOn_{HS}$ where $AddOn_{HS} = SF_{FX}|EN_{HS}|$.
The $EN_{HS_j} = \sum_{j \in HS_j} \delta_i \times d_i^{(FX)} \times MF_i^{(type)}$.

CR class:

$AddOn^{(Credit)} = [(\sum_{entity} \rho_{entity} AddOn_{entity})^2 + \sum_{entity}(1 - \rho_{entity}^2)(AddOn_{entity})^2]^{\frac{1}{2}}$.
The $EN_j^{(Credit)} = \sum_{j \in entity_j} \delta_i \times d_i^{(entity)} \times MF_i$ and $AddOn_{entity_j} = SF_{CR} \times EN_j^{(Credit)}$.

EQ class:

$AddOn^{(Equity)} = [(\sum_{entity} \rho_{entity} AddOn_{entity})^2 + \sum_{entity}(1 - \rho_{entity}^2)(AddOn_{entity})^2]^{\frac{1}{2}}$.
The $EN_j^{Equity} = \sum_{j \in entity_j} \delta_i \times d_i^{(entity)} \times MF_i^{(type)}$ and $AddOn_{entity_j} = SF_{EQ} \times EN_{entity_j}$.

CO class:

$AddOn_{HS} = [(\sum_{comtype} \rho_{comtype} AddOn_{comtype})^2 + \sum_{comtype}(1 - \rho_{comtype}^2)(AddOn_{comtype})^2]^{\frac{1}{2}}$.
Hence , $AddOn^{(Commodity)} = \sum_{HS} AddOn_{HS}$.
The $EN_k^{(Com)} = \sum_{i \in Type_j^k} \delta_i \times d_i^{(Com)} \times MF_i^{(type)}$ and $AddOn_{Type_j^k} = SF_{CO} \times EN_k^{(Com)}$.

5.2.2 CVA Capital

The CVA capital was introduced into Basel III in the CCR calculations due to losses that are caused by CVA. It is computed using either IMM or standardized method. When the issuer or a bank does not have IMM approval for CCR to compute CVA capital then they can resort to standardized method. The CVA capital does not apply to the CCP framework and is calculated at the counterparty level. We shall present CVA capital under Basel III's standardized approach.

Standardised Approach

The derivation of the CVA capital formula under the standard approach can be found in [19]. The CVA capital K_{CVA} is a $VaR_{99\%}$ based formula and is given by

$$K_{CVA} = 2.33\sqrt{h}\sqrt{\left(\sum_i 0.5w_i(M_i EaD_i^{total} - M_i^{hedge}B_i) - \sum_{ind} w_{ind}M_{ind}B_{ind}\right)^2 - \sum_i 0.75w_i^2\left(M_i EaD_i^{total} - M_i^{hedge}B_i\right)^2} \tag{5.2.11}$$

where

- h is the horizon of the risk measure VaR
- i is ith counterparty
- w_i is ith counterparty's weight
- M_i is maturity's exposure
- EaD_i^{total} is exposure at default for ith counterparty
- M_i^{hedge} and M_{ind} are maturities of CDS
- B_i and B_{ind} is a full notional of one or more index CDS

In a case where we do not have a single name [13] or index CDS hedging this formula can be approximated as

$$K_{CVA} \approx \sum_i \frac{2.33}{2}\sqrt{h}w_i M_i EaD_i^{total} \tag{5.2.12}$$

[13] A single name CDS is a type of derivative that its underlying asset is a bond of some issuer.

EaD is obtained using SA-CCR or CEM methodology. The risk weights w_i using S&P rating notation [53] are given in Table 6 below

Table 6: Risk weights for CVA capital

S&P Rating	**Weight w_i**
AAA	0.7%
AA	0.7%
A	0.8%
BBB	1.0%
BB	2.0%
B	3.0%
CCC	10.0%

We have seen that SA-CCR is quite tedious and complex to implement compared to CEM, however, it takes margined trades into account. Unlike in CEM where the add-on values are already provided, in SA-CCR the PFE add-on is computed according to the asset class and this makes SA-CCR more reliable than CEM.

In this chapter, we extended the semi-replication strategy to incorporate regulatory capital, thus deriving KVA as one of the xVAs. We also presented regulatory capital methodologies for CCR capital and CVA capital. In the next chapter, we derive MVA.

6 Derivatives Pricing with Initial Margin

I have discovered a truly remarkable proof of this theorem which this margin is too small to contain.

Pierre de Fermat

We extend the semi-replicating strategy to incorporate initial margin in derivatives pricing. Thus we will introduce MVA.

6.1 Deriving MVA by semi-replication strategy

The evolution of the underlying instruments still are defined in 4.1.1. The funding constraint is adjusted to incorporate the initial margin. Initial margin can also be posted bilaterally by counterparties. Let I_B be the initial margin posted by issuer B and I_C be the initial margin posted by counterparty C.

Definition 6.1.1 Economic values at default
Suppose that $\hat{V}(t, S, J_B, J_C)$ denotes the economic value of the derivative to the issuer B and the collateral amount be denoted by X. The economic value when issuer B defaults first is given by

$$\hat{V}(t, S, 1, 0) = g_B(M_B, X, I_B)$$

and when counterparty C defaults first is given by

$$\hat{V}(t, S, 0, 1) = g_C(M_C, X, I_C)$$

where M_B and M_C are general close-out amounts.

When $M_C = M_B = V$, the values of g are now incorporating initial margins and are expressed as

$$g_B(V, X, I_B) = (V - X + I_B)_+ + R_B(V - X + I_B)_- + X - I_B \tag{6.1.1}$$

$$g_C(V, X, I_C) = R_C(V - X - I_C)_+ + (V - X - I_C)_- + X + I_C \tag{6.1.2}$$

Definition 6.1.2 The funding constraint
The funding condition of a collateralized derivative contract with economic value $\hat{V}$ is given by

$$\hat{V} - X + I_B + \alpha_1 P_1 + \alpha_2 P_2 - \phi K = 0$$

The funding constraint incorporates the initial margin posted by issuer I_B. Note that there is no initial margin posted by the counterparty I_C because rehypothecation is not allowed.

Definition 6.1.3 The cash account positions prior rebalancing
Let $\beta_S, \beta_C, \beta_X, \beta_K, \beta_{I_B}$ be cash account associated with the underlying asset S, counterparty bond, collateral, regulatory capital and initial margin, respectively. Then the evolution of these cash accounts are given respectively by

$$\begin{aligned}
d\beta_S &= \delta_S(\gamma_S - q_S)S dt \\
d\beta_C &= -\alpha_C q_C P_C dt \\
d\beta_X &= -r_X X dt \\
d\beta_K &= -\gamma_K \phi K dt \\
d\beta_{I_B} &= r_{I_B} I_B dt
\end{aligned}$$

The total cash account β is

$$\beta = \beta_S + \beta_C + \beta_X + \beta_K + \beta_{I_B}$$

where $\beta_S = -\delta_S S$, $\beta_C = -\alpha_C P_C$, $\beta_X = -X$, $\beta_K = -\phi K$ and $\beta_{I_B} = I_B$.

These cash accounts are similar to the ones defined in the previous chapter and the cash account associated with the initial margin posted by issuer B has been incorporated. This initial margin cash account earns interest at the rate of r_{I_B}.

Definition 6.1.4 The replicating portfolio
Assuming that other parameters are as defined before, then the replicating portfolio Π is defined by

$$\Pi = \delta_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta$$

where $\beta = \beta_S + \beta_C + \beta_X + \beta_K + \beta_{I_B}$ is the total cash account positions.

We apply Ito's lemma to compute the differential of $\hat{V}$

$$d\hat{V} = \left(\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2}\right)dt + \frac{\partial \hat{V}}{\partial S}dS + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C \tag{6.1.3}$$

As before, we assume that our portfolio, Π, is self-financing, then its evolution is

$$\begin{aligned} d\Pi &= \delta_S dS + \delta_S(\gamma_S - q_S)Sdt + \alpha_1 dP_1 + \alpha_2 dP_2 \\ &\quad + \alpha_C dP_C - \alpha_C q_C P_C dt - r_X X dt \\ &\quad - \gamma_K \phi K dt + r_{I_B} I_B dt \end{aligned} \tag{6.1.4}$$

Theorem 6.1.1 Deriving MVA by semi-replication strategy
Let $\mathcal{M}^{CVA}$, $\mathcal{M}^{DVA}$, $\mathcal{M}^{FCA}$, $\mathcal{M}^{COLVA}$, $\mathcal{M}^{KVA}$ and $\mathcal{M}^{MVA}$ represent CVA, DVA, FCA, COLVA, KVA and MVA models, respectively. If $U_t = CVA_t + DVA_t + FCA_t + COLVA_t + KVA_t + MVA_t$ then

$$\begin{aligned} CVA_t &= -\mathbb{E}\Big[\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[V(y) - g_C(V(y), X(y), I_C(y))]dy\Big|\mathcal{F}_t\Big] \\ DVA_t &= -\mathbb{E}\Big[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[V(y) - g_B(V(y), X(y), I_B(y))]dy\Big|\mathcal{F}_t\Big] \\ FCA_t &= -\mathbb{E}\Big[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_0}(y)dy\Big|\mathcal{F}_t\Big] \\ COLVA_t &= -\mathbb{E}\Big[\int_t^T s_X(y)e^{-\int_t^y a(v)dv}X(y)dy\Big|\mathcal{F}_t\Big] \\ &= +\mathbb{E}\Big[\int_t^T s_{I_B}(y)e^{-\int_t^y a(v)dv}I_B(y)dy\Big|\mathcal{F}_t\Big] \\ KVA_t &= -\mathbb{E}\Big[\int_t^T e^{-\int_t^y a(v)dv}[(\gamma_K(y) - r(y))\phi K(y) + \lambda_B\varepsilon_{h_K}(y)]dy\Big|\mathcal{F}_t\Big] \\ MVA_t &= +\mathbb{E}\Big[\int_t^T s_{I_B}(y)e^{-\int_t^y a(v)dv}I_B(y)dy\Big|\mathcal{F}_t\Big] \end{aligned}$$

where $a(v) = r(v) + \lambda_B(v) + \lambda_C(v)$ and $\varepsilon_h = \Delta\hat{V}_B - (P - P_D) = g_B - X + I_B + R_1\alpha_1 P_1 + R_2\alpha_2 P_2 - \phi K = \varepsilon_{h_0} + \varepsilon_{h_K}$; ε_{h_0} and ε_{h_K} are the hedging errors that does not depend on the regulatory capital and the one depending on the regulatory capital, respectively.

Proof: We want to compute $d\hat{V} + d\Pi$ by using equations (6.1.3) and (6.1.4)

$$\begin{aligned} d\hat{V} + d\Pi = \Big(&\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + \delta_S(\gamma_S - q_S)S + \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 \\ &+ \alpha_C r_C P_C - \alpha_C q_C P_C - r_X X - \gamma_K \phi K + r_{I_B} I_B\Big)dt \\ &+ \varepsilon_h dJ_B + \left(\delta_S + \frac{\partial \hat{V}}{\partial S}\right)dS \\ &+ \big[g_C - \hat{V} - \alpha_C P_C\big]dJ_C \end{aligned}$$

where $\varepsilon_h = \Delta\hat{V}_B - (P - P_D) = g_B - X + I_B + R_1\alpha_1 P_1 + R_2\alpha_2 P_2 - \phi K = \varepsilon_{h_0} + \varepsilon_{h_K}$.

The PDE of the economic value of the derivative contract is given by

$$\begin{aligned} 0 &= \frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial \hat{V}}{\partial S} - (r + \lambda_C + \lambda_B)\hat{V} + s_{I_B} I_B \\ &\quad + g_B \lambda_B + g_C \lambda_C - s_X X - \gamma_K \phi K + r\phi K - \varepsilon_h \lambda_B \\ \hat{V}(T, S_T) &= H(S_T) \\ \delta_S &+ \frac{\partial \hat{V}}{\partial S} = 0 \\ g_C &- \hat{V} = \alpha_C P_C \end{aligned} \tag{6.1.5}$$

The BSM PDE is modified to

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial V}{\partial S} &= rV \\ V(T, S_T) &= 0 \end{aligned} \tag{6.1.6}$$

Since we know the economic value $\hat{V}$ and the BSM model value V we can obtain $U_t = CVA_t + DVA_t + FCA_t + COLVA_t + KVA_t + MVA_t$ using this equation $U = \hat{V} - V$. By noting that $\frac{\partial U}{\partial S} = \frac{\partial (\hat{V} - V)}{\partial S} = \frac{\partial \hat{V}}{\partial S} - \frac{\partial V}{\partial S}$.

We subtract the evolution $\hat{V}$ and V we get the PDE for U

$$\begin{aligned} \frac{\partial U}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 U}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial U}{\partial S} - (r + \lambda_B + \lambda_C) U &= V\lambda_C - g_C\lambda_C + V\lambda_B - g_B\lambda_B \\ &\quad + \varepsilon_h \lambda_B + s_X X - s_{I_B} I_B \\ &\quad + \gamma_K \phi K - r\phi K \\ U(T, S_T) &= 0 \end{aligned} \tag{6.1.7}$$

We will start by simplifying the right hand side of the above equation U is

$$\begin{aligned} V\lambda_C - g_C\lambda_C + V\lambda_B - g_B\lambda_B + \varepsilon_h\lambda_B + s_X X + \phi\gamma_K K - \phi r K = \\ \lambda_C(V - g_C) + \lambda_B(V - g_B) \\ - s_{I_B} I_B + \lambda_B(\varepsilon_{h_0} + \varepsilon_{h_K}) \\ + s_X X + (\gamma_K - r)\phi K \end{aligned} \tag{6.1.8}$$

We now set $a(v) = r(v) + \lambda_B(v) + \lambda_C(v)$.

Since valuation adjustments U are martingales then we apply Feynman-Kac formula 2.1.7

where $H(S_T) = 0$ such that

$$U_t = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^y a(v)dv} G(y, V_y)dy \middle| \mathcal{F}_t\right]$$

Thus we get each valuation adjustment as

$$CVA_t = -\mathbb{E}_t\left[\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[V(y) - g_C(V(y), X(y), I_C(y))]dy \middle| \mathcal{F}_t\right] \tag{6.1.9}$$

$$DVA_t = -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[V(y) - g_B(V(y), X(y), I_B(y))]dy \middle| \mathcal{F}_t\right] \tag{6.1.10}$$

$$FCA_t = -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_0}(y)dy \middle| \mathcal{F}_t\right] \tag{6.1.11}$$

$$COLVA_t = -\mathbb{E}_t\left[\int_t^T s_X(y)e^{-\int_t^y a(v)dv}X(y)dy \middle| \mathcal{F}_t\right] \tag{6.1.12}$$

$$= +\mathbb{E}_t\left[\int_t^T s_{I_B}(y)e^{-\int_t^y a(v)dv}I_B(y)dy \middle| \mathcal{F}_t\right] \tag{6.1.13}$$

$$KVA_t = -\mathbb{E}_t\left[\int_t^T e^{-\int_t^y a(v)dv}[(\gamma_K(y) - r(y))\phi K(y) + \lambda_B(y)\varepsilon_{h_K}(y)]dy \middle| \mathcal{F}_t\right] \tag{6.1.14}$$

$$MVA_t = +\mathbb{E}_t\left[\int_t^T s_{I_B}(y)e^{-\int_t^y a(v)dv}I_B(y)dy \middle| \mathcal{F}_t\right] \quad \blacksquare \tag{6.1.15}$$

As we can see, there is no MVA for the initial margin posted by counterparty C. This is because we assume that rehypothetication is not allowed.

If we assume standard the close-out i.e. $M_B = M_C = V$ then CVA and DVA can be simplified further to

$$CVA_t = -\mathbb{E}_t\left[(1 - R_C)\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[(V(y) - X(y) - I_C(y))_+]dy \middle| \mathcal{F}_t\right]$$

$$DVA_t = -\mathbb{E}_t\left[(1 - R_B)\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[(V(y) - X(y) - I_B(y))_-]dy \middle| \mathcal{F}_t\right]$$

The initial margin is the crucial component for computing MVA. In the next section, we will present two initial margin methodologies: SIMM and risk measure based models.

6.2 Initial Margin

The Initial margin is a type of collateral issued to reduce the credit exposure on derivatives trades. The initial margin covers the change in derivative value over MPoR (set to 10 days) and is a risk-based calculation. There are a variety of models used to compute the initial margin. We shall present two models: the ISDA proposed SIMM model [7] and risk measure based model. The risk measure based model is primarily used by CCPs and it uses historical simulation.

6.2.1 SIMM model

SIMM model is centered around the risk sensitivities of the derivative.

Definition 6.2.1 Risk classes
Under SIMM model, there are six different types of risk classes

(i) Interest Rate (IR),

(ii) Equity,

(iii) Foreign Exchange,

(iv) Commodity,

(v) Credit (Qualifying),

(vi) Credit (non-Qualifying).

These risk classes are categorized according to the basis of the underlying that drives the derivative. An interest rate swap, for example, falls under the Interest Rate risk class and a European option will fall under the Equity risk class since they are mostly traded on stocks. We now define the initial margin for each risk class of the derivative or any financial product.

Definition 6.2.2 Initial margin for each risk class
Let R represent a risk class. The initial margin per risk class R is given by

$$\text{IM}_R = \text{DeltaMargin}_R + \text{VegaMargin}_R + \text{CurvatureMargin}_R + \text{BaseCorrMargin}_R$$

where BaseCorrMargin is present only in the CQ risk class.

There are about seven types of risks in the SIMM model on which the initial margin is computed based on. These are Delta risk, Vega risk, Inter-curve basis risk, Curvature risk, Credit Base Correlation risk, and Concentration risk.

Definition 6.2.3 Product classes
There are four types of product classes

(i) Interest Rate and Foreign Exchange (RatesFX)

(ii) Equities, EQ

(iii) Credit, CR

(iv) Commodities, CO

Once again, for example, IRS will fall under the RateFX because it is a type of IRD. A European option written on a stock will fall under the EQ class.

Definition 6.2.4 Total SIMM
The total margin of a product class is calculated as follows

$$\text{SIMM}_{\text{product}} = \sqrt{\sum_r \text{IM}_r^2 + \sum_r \sum_{s \neq r} \psi_{rs} \text{IM}_r \text{IM}_s}$$

where r and s are risk classes and ψ_{rs} is a correlation matrix between risk classes. This matrix is detailed in [35].

The total SIMM is the sum of all product classes of a portfolio

$$\text{SIMM} = \sum_{\text{product}} \text{SIMM}_{\text{product}}$$

For example, if we have two risk classes, say, Interest Rate and Equity then $\psi_{rs} = 19\%$ and for FX and Commodity $\psi_{rs} = 32\%$.

To capture Delta risk, Vega risk, Curvature risk and Base Correlation risk it is important to follow these summarized steps:

(i) Compute a net sensitivity s

(ii) Weight the net sensitivity WS

(iii) Find aggregated weighted sensitivities $\overline{K}$

(iv) Compute each margin: delta margin, vega margin, curvature margin or base correlation margin where applicable

(v) Sum all the margins of the derivatives portfolio

We will now define what risk sensitivity terms or the underlying market variables that give rise to sensitivities are. Risk sensitivities are often called *Greeks* in mathematical finance.

Definition 6.2.5 Risk sensitivity terms
Define the following terms

(i) V_i is the market value of instrument i,

(ii) r_t is tenor t risk-free interest rate,

(iii) cs_t is tenor t credit spread,

(iv) 1bp is 1 basis point $= 0.01\%$,

(v) EQ_k is the market value of equity k,

(vi) FX_k is the spot exchange rate between currency k and calculation currency,

(vii) CTY_k is the market value of commodity k,

(viii) BC_k is the Base Correlation curve/surface index k.

Note that V_i and r_t are the same as the ones defined in previous chapters. Thus V_i is the risk-free value of the derivative or any financial instrument.

Definition 6.2.6 Delta sensitivity
The Delta risk sensitivities for different risk classes are

$$s_{r_t}(i, r_t) = s(i, r_t) = V(r_t + 1\text{bp}, cs_t) - V(r_t, cs_t)$$

$$s_{cs_t}(i, cs_t) = s(i, r_t) = V(r_t, cs_t + 1\text{bp}) - V(r_t, cs_t)$$

$$s_{ik}(BC_k) = s_{ik} = V_i(BC_k + 1\%) - V_i(BC_k)$$

$$s_{ik}(EQ_k) = s_{ik} = V_i(EQ_k + 1\%.EQ_k) - V_i(EQ_k)$$

$$s_{ik}(CTY_k) = s_{ik} = V_i(CTY_k + 1\%.CTY_k) - V_i(CTY_k)$$

where $s_{r_t}, s_{cs_t}, s_{ik}(BC_k), s_{ik}(EQ_k), s_{ik}(CTY_k)$ is delta risk sensitivity for Interest Rate, Credit non-securitisation, Credit Qualifying Base Correlation, Equity and Commodity, respectively.

The Delta risk sensitivities are types of first-order Greeks. This measures the rate of change of the derivative or portfolio value with respect to the changes in the underlying. There are other ways of computing delta sensitivities; we can using central or backward difference equations. For example, in Equity, Commodity and FX risk we can compute the delta sensitivity using $s = \frac{V(y+1\%.w.y)-V(y)}{w}$, where $0 < |w| \leq 1$. This is similar to computing $\frac{\partial V}{\partial y}$.

Definition 6.2.7 Delta margin for IR risk class

Firstly, we compute risk weight $WS_{k,i}$ using

$$WS_{k,i} = RW_k s_{k,i} CR_b$$

where $s_{k,i}$ is the par rate dv01 for index i of the sub yield curve at tenor k.

The concentration risk factor is expressed as

$$CR_b = \max\left(1, \left(\frac{|\sum_{k,i} s_{k,i}|}{T_b}\right)^{\frac{1}{2}}\right)$$

where T_b is the threshold for each currency b.

The aggregated weighted sensitivities $\overline{K}$ is computed using

$$\overline{K} = \sqrt{\sum_{i,k} WS_{k,i}^2 + \sum_{i,k} \sum_{(j,l)\neq(i,k)} \phi_{i,j} \rho_{k,l} WS_{k,i} WS_{l,j}}$$

Finally, aggregated delta margin is

$$\text{DeltaMargin} = \sqrt{\sum_b \overline{K}_b^2 + \sum_b \sum_{c\neq b} \gamma_{bc} g_{bc} S_b S_c}$$

where

$$S_b = \max\left(\min\left(\sum_{i,k} WS_{k,i}, \overline{K}_b\right), -\overline{K}_b\right)$$

and

$$g_{bc} = \frac{\min(CR_b, CR_c)}{\max(CR_b, CR_c)}$$

for currency b and c.

The parameters $\phi_{i,j}$, T_b and $\rho_{k,l}$ are given in [35]. The concentration factor CR_b is always positive and this makes the function $g_{bc} \leq 1$. The delta margin is easy to compute, however, for exotic instruments like Barrier option it is challenging.

Definition 6.2.8 Delta Margin for non-IR risk class

The DeltaMargin for non-interest rate risk class is similar to the one above, however, there are some changes. For each risk factor k

$$WS_k = RW_k s_k CR_k$$

where CR_k is the concentration risk factor.

This is defined in two ways:
for credit spread risk

$$CR_k = \max\left(1, \left(\frac{|\sum_j s_j|}{T_b}\right)^{\frac{1}{2}}\right)$$

and for equity, commodity and FX risk

$$CR_k = \max\left(1, \left(\frac{|s_j|}{T_b}\right)^{\frac{1}{2}}\right)$$

The aggregated weight sensitivities $\overline{K}$ is now

$$\overline{K} = \sqrt{\sum_k WS_k^2 + \sum_k \sum_{l \neq k} \rho_{kl} f_{kl} WS_k WS_l}$$

and $f_{kl} = \frac{\min(CR_k, CR_l)}{\max(CR_k, CR_l)}$.

Finally, the delta margin is captured using

$$\text{DeltaMargin} = \sqrt{\sum_b \overline{K}_b^2 + \sum_b \sum_{c \neq b} \gamma_{bc} g_{bc} S_b S_c} + \overline{K}_{residual}$$

where $S_b = \max\left(\min\left(\sum_{k=1}^N WS_k, \overline{K}_b\right), -\overline{K}_b\right)$.

As we can see, the difference between delta margin for non-interest rate and interest rate risk class is that the former has an additional aggregated weight sensitivity $\overline{K}_{residual}$.

Definition 6.2.9 Vega sensitivity

The vega risk sensitivity is given by

$$\frac{\partial V_i}{\partial \sigma} = V(\sigma + 1) - V(\sigma)$$

where σ is the implied volatility of the risk factor.

The vega risk sensitivity is also the first-order Greek and it measures the rate of change of the derivative or portfolio value with respect to the changes in the underlying's volatility.

Definition 6.2.10 Vega Margin calculation
Vega margin is captured using

$$\text{VegaMargin} = \sqrt{\sum_b \overline{K}_b^2 + \sum_b \sum_{c \neq b} \gamma_{bc} S_b S_c} + \overline{K}_{residual}$$

where VR_{ik} is vega risk exposure for each derivative i and risk factor k given by

$$VR_{ik} = \sum_j \sigma_{kj} \frac{\partial V_i}{\partial \sigma}$$

The volatility σ_{kj} of the risk factor k at each volatility-tenor j is expressed as

$$\sigma_{kj} = \frac{RW_k \sqrt{365/14}}{\Phi^{-1}(0.99)}$$

The aggregated vega risk exposure $\overline{K}_b$ is computed using

$$\overline{K}_b = VCR_b \sqrt{\sum_k VR_k^2 + \sum_k \sum_{l \neq k} \rho_{kl} VR_k VR_l}$$

where

$$VCR_b = \max\left(1, \left(\frac{|\sum_k VR_k|}{VT_b}\right)^{\frac{1}{2}}\right)$$

,

$$S_b = \max\left(\min\left(VCR_b \sum_{k=1}^{N} VR_k, \overline{K}_b\right), -\overline{K}_b\right)$$

and VR_k is a net vega risk exposure.

For Equities, FX and Commodity vega risk is

$$VR_k = VRW\left(\sum_i VR_{ik}\right) VCR_k$$

and

$$VCR_k = \max\left(1, \left(\frac{|\sum_i VR_{ik}|}{VT_b}\right)^{\frac{1}{2}}\right)$$

For linear instruments such as IRS, we do not consider vega sensitivities. This applies to option contracts; for example, a European option, swaption, or bond option.

Definition 6.2.11 Curvature calculation
The curvature risk sensitivity is

$$CV_{ik} = \sum_j SF(t_{kj})\sigma_{kj}\frac{\partial V_i}{\partial \sigma}$$

for each instrument i and each risk factor k, $SF(t) = \frac{1}{2}\min(1, \frac{14 \text{ days}}{t})$, and volatility σ_{kj} is $\sigma_{kj} = \frac{RW_k\sqrt{365/14}}{\Phi^{-1}(0.99)}$.

The $SF(t)$ is called the scaling function and time t is measured in days. For example, if $t = 1m$ then the scaling function is $SF = 23.0\%$. Here, 12m is assumed to be 365 days.

Definition 6.2.12 Curvature Margin calculation
The curvature margin is split into two components

$$\text{CurvatureMargin} = \text{CurvatureMargin}_{non-res} + \text{CurvatureMargin}_{residual}$$

where $\text{CurvatureMargin}_{non-res}$ and $\text{CurvatureMargin}_{residual}$ are curvature margins without residual and residual margins, respectively.
For the IR risk only, the curvature margin must be multiplied by historical volatility ratio factor HVR_{IR}^{-2}.

The non-residual curvature margin is given by

$$\text{CurvatureMargin}_{non-res} = \max\left(\sum_{b,k} CVR_{b,k} + \lambda\sqrt{\sum_b \overline{K}_b^2 + \sum_b\sum_{c\neq b}\gamma_{bc}^2 S_b S_c}, 0\right)$$

and

$$S_b = \max\left(\min\left(\sum_{k=1}^{N} CVR_{b,k}, \overline{K}_b\right), -\overline{K}_b\right)$$

and the residual curvature margin is given by

$$\text{CurvatureMargin}_{residual} = \max\left(\sum_k CVR_{residual,k} + \lambda_{residual}\overline{K}_{residual}, 0\right)$$

where $\theta_{residual} = \min\left(\frac{\sum_k CVR_{residual,k}}{\sum_k |CVR_{residual,k}|}, 0\right)$ and $\lambda_{residual} = (\Phi^{-1}(0.995)^2 - 1)(1 + \theta_{residual}) - \theta_{residual}$.

Similarly, the curvature margin does not apply to linear instruments such as IRS too. It can only apply to option contracts because it includes vega sensitivities.

Definition 6.2.13 BaseCorrMargin
The Base Correlation Margin is computed as

$$BaseCorrMargin = \sqrt{\sum_k WS_k^2 + \sum_k \sum_{l \neq k} \rho_{kl} WS_k WS_l}$$

where $WS_k = RW_k s_k$ and s_k is the net sensitivity. The correlation parameters ρ_{kl} and the risk weight RW_k are provided in [35].

This applies to instruments that are sensitive to base correlation such as CDO trances. For instruments that are not sensitive to base correlation such as IRS, we will not consider base correlation margin.

6.2.2 Risk measure model

Let $\mathcal{F}_t$ such that $0 \leq t \leq T$ be a filtration of $\mathcal{F}$ with maturity time T and ω be an arbitrary path. Let $V = V_t$ be derivative value at time t which is an $\mathcal{F}_t$ adapted process. Let ν_{IM} such that $0 < \nu_{IM} < T - t$ denote the MPoR. MPoR is the time prior to the default of the counterparty when both counterparties are no longer posting margins. The IM serves as a buffer for counterparties from market fluctuations during MPoR. This is captured by using risk measures such as VaR, ETL, or any other suitable risk measure.

Definition 6.2.14 Initial margin
The IM is a risk measure and therefore is given by

$$IM(t, \omega) := \varrho_\alpha(\Delta V(t, \omega, \nu_{IM})|\mathcal{F}_t)$$

where ϱ_α is the conditional risk measure at the α-level of confidence and $\Delta V(t, \omega, \nu_{IM}) := V(t+\nu_{IM}, \omega) - V(t, \omega)$ is the change in derivative value in the margin period of risk $(t, t+\nu_{IM}]$.

The confidence level is normally set to $\alpha = 0.975$, however, BCBS-IOSCO requires it to be 0.99. Several methodologies are used to compute $IM(t, \omega)$, one of them is nested Monte Carlo simulation, but due to its complexity and computational intensity methods like the LSM are applied. We are going to apply the LSM instead of the nested Monte Carlo simulation. As we can see from Figure 8, the LSM reduces the number of dimensions, thus giving us a proxy.

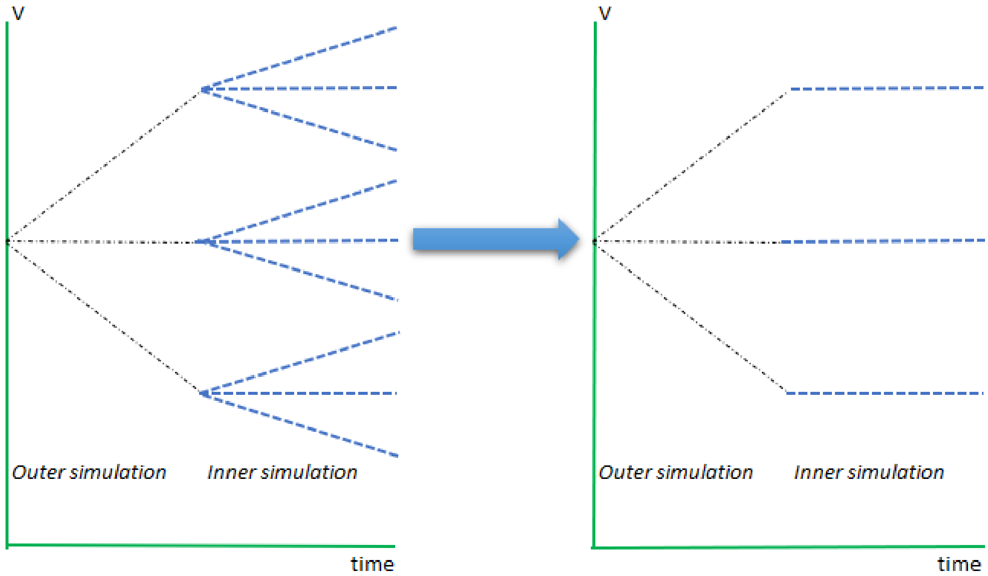


Figure 8: The illustration of Nested Monte Carlo simulation and the LSM.

We denote the conditional risk measure by $\varrho_\alpha(\Delta V_t|V_t)$ i.e., for all $v \in \mathbb{R}$ there exists a real function $f_\alpha : \mathbb{R} \to \mathbb{R}$ such that $f_\alpha(v) := \varrho_\alpha(\Delta V_t|V_t = v)$, thus we have $\mathbb{E}_\mathbb{P}[\varrho_\alpha(\Delta V_t|V_t)|\mathcal{F}_0] = \mathbb{E}_\mathbb{P}[f_\alpha(V_t)|\mathcal{F}_0]$. Our goal is to estimate measure $f_\alpha(v)$.

IM using VaR

If we assume $V(t,\omega)$ to follow a normal distribution and $\Delta V(t,\omega,\nu_{IM})$ to follow a local normal distribution, then the initial margin will be given by the formula

$$(IM(t,\omega)|\mathcal{F}_t) = (\mu(t,\omega) + \sigma(t,\omega)\Phi^{-1}(\alpha)|\mathcal{F}_t) \tag{6.2.1}$$

where $(\sigma(t,\omega)|\mathcal{F}_t) = \sqrt{\mathbb{E}_\mathbb{P}[\Delta^2 V(t,\omega,\nu_{IM})|\mathcal{F}_t]}$ and $\mu(t,\omega) = \mathbb{E}_\mathbb{P}[\Delta V(t,\omega,\nu_{IM})]$. It is convenient to assume that drift $\mu(t,\omega) = 0$ so that the only task left is to compute $\sigma(t,\omega)$. This will be done by applying the LSM as suggested by Anfuso et al. [3]. As we can see, this is a conditional expectation, and the LSM is perfectly suited for this.

IM using ETL

We can also use ETL to compute the IM, in fact, the BCBS proposed that banks should apply ETL instead of VaR. The ETL based IM is computed using

$$(IM(t,\omega)|\mathcal{F}_t) = \left(\mu(t,\omega) + \sigma(t,\omega)\frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}|\mathcal{F}_t\right) \tag{6.2.2}$$

where $\varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$ and $(\sigma(t,\omega)|\mathcal{F}_t)$ is the same as above, and it is also convenient to assume that it is zero.

In this chapter, we derived the MVA model and also presented two initial margin methodologies: SIMM and risk measure based models. We have shown the difference between these two models and where each model applies. In the next chapter, we present our numerical results to compute our xVAs.

7 Numerical Results

We compare the SA-CCR and CEM, regulatory capital methodologies and then use them to compute KVA for various derivatives. We also compare initial margin methodologies, SIMM model, and risk measure based models and use these to compute MVA for derivatives. We consider two underlyings; short interest rates and stocks. The floating interest rates are modelled using the Vasicek model with parameters in Table 7. The options are all written on the stock that follows GBM. The IRS is assumed to be a par IRS where the fixed rate is the fair swap rate. For simulations purpose, we assume 200 simulation time steps per year and 25,000 simulation paths throughout. The parameters of the Vasicek model are given in the table below

Table 7: Input parameters for the Vasicek model

Parameter	**Numerical value**
r_0	0.065
σ	0.015
β	0.02
θ	0.03

We employ the LSM with Laguerre polynomials of order 10 and we used *Numba* package to make our computations fast.

For calculation of CCR capital K_{CCR} we use CEM and SA-CCR and the same methodologies are used to calculate EaD for CVA capital K_{CVA}.

We will consider three derivatives presented in chapter 2. We compute different xVAs and also compare xVAs for cleared and uncleared IRS.

We assume an arbitrary currency for all our calculations. For SIMM methodology we assume that this currency is regular.

7.1 Analyzing regulatory capital methodologies

We analyze and assess the regulatory capital models, CEM and SA-CCR. We then use these to compute CCR and CVA capital. The CEM is quite straightforward to use for computing EaD compared to SA-CCR. The add-on values in the CEM approach are already provided while in SA-CCR are supposed to be calculated. This makes SA-CCR computationally demanding especially when we have a portfolio of different types of derivatives.

We consider an uncollateralized European put option on a non-dividend-paying stock with an exercise price of $E = 11.0$ and a flat rate of $r = 0.070$.

In the CEM approach, since the value of the option V is always greater than zero, then the replacement cost RC can be equated to this value. The $Addon$ value under Basel II is exactly 8% because maturity times fall under maturity bucket 2. The netting factor is 100%, this is due to $V > 0$ for all maturities and the trade is assumed to be bilateral.

The calculation of CVA capital will depend on which approach we use to compute the EaD. In the SA-CCR approach, both the EaD and regulatory capital for this European option are larger than those of the CEM approach and this makes the CVA capital computed using EaD obtained via SA-CCR to be larger than the one computed using the CEM approach. This is shown in the table below. We assume that this put option is unmargined trade in the SA-CCR approach. One of the drawbacks of the CEM approach is that it does not specify if a trade is margined or unmargined. This is one of the reasons why the SA-CCR came into the industry.

Table 8: A European put option value V, EaD, CCR capital, CVA capital, and regulatory capital (in % notional). Maturities are expressed in years. The credit rating is assumed to be BB+.

Maturity	1.4	1.8	2.2	2.6	3.0	3.4	3.8	4.2	4.6	5.0
V	1.161	1.166	1.162	1.151	1.135	1.116	1.095	1.072	1.048	1.023
EaD CEM	1.241	1.246	1.242	1.231	1.215	1.196	1.175	1.152	1.128	1.103
EaD SA-CCR	1.841	1.896	1.935	1.962	1.980	1.992	1.999	2.001	2.001	1.997
K_{CCR} CEM	0.124	0.125	0.124	0.123	0.121	0.120	0.117	0.115	0.113	0.110
K_{CCR} SA-CCR	0.184	0.1896	0.194	0.196	0.198	0.1992	0.1999	0.200	0.2001	0.1997
K_{CVA} CEM	0.040	0.052	0.064	0.075	0.085	0.095	0.104	0.113	0.121	0.128
K_{CVA} SA-CCR	0.060	0.080	0.099	0.119	0.138	0.158	0.177	0.196	0.214	0.233
K CEM	0.165	0.177	0.188	0.198	0.206	0.214	0.221	0.228	0.234	0.239
K SA-CCR	0.244	0.269	0.293	0.315	0.336	0.357	0.377	0.396	0.414	0.432

Figure 9 and 10 show how regulatory capital is affected when using both CEM and SA-CCR.

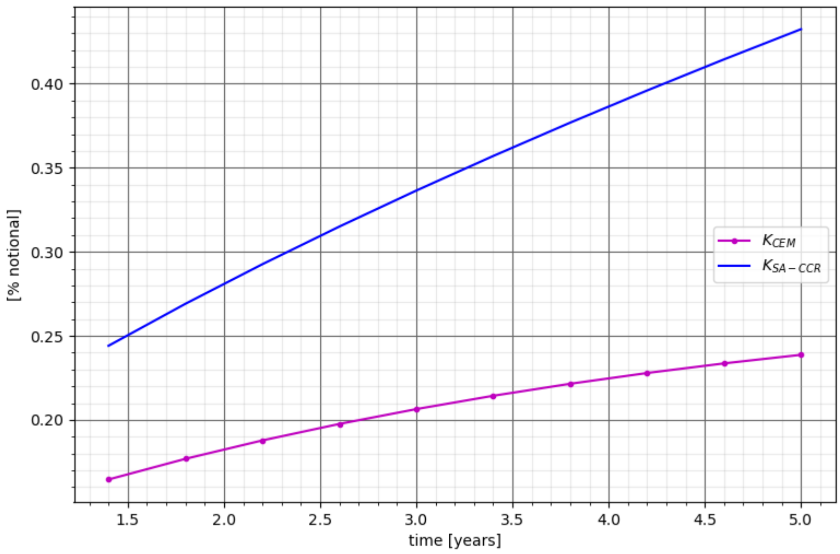


Figure 9: CEM vs SA-CCR capital of a European put option at different maturities.

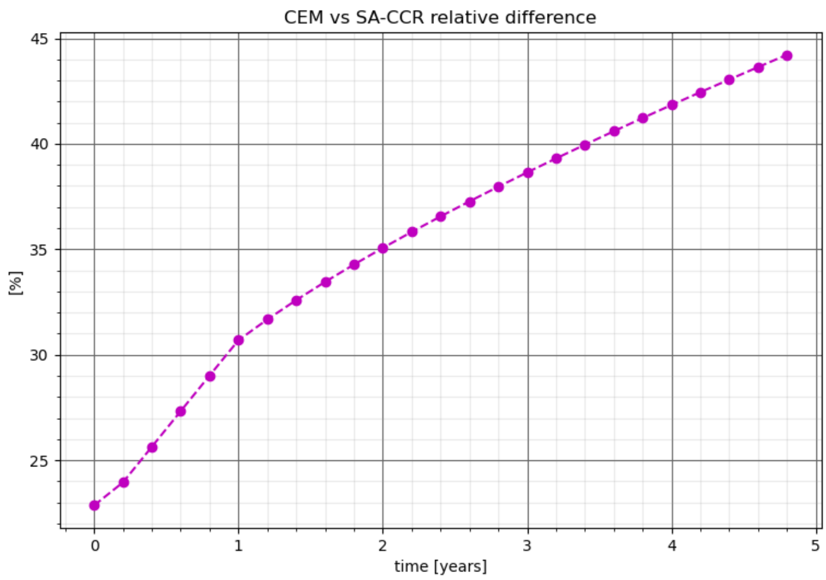


Figure 10: The relative difference between regulatory capital obtained via CEM and SA-CCR for a 5-year uncollateralized European put option.

The EaD for SA-CCR increases much compared to that of CEM and this affect regulatory capital in the same way.

Now we analyze SA-CCR and CEM approach on a 3-year uncollateralized American put option at different strikes ranging from $E = 10$ to $E = 15$. The value of the American put option is obtained via the LSM algorithm using Laguerre polynomials up to order 10, 200 time steps and 25,000 simulations. The flat rate used for discounting is 0.070, the addon for CEM is 8% since we have maturity bucket 2 and trade is assumed to be bilateral and unmargined. We assumed that the credit rating is $A+$. The value of the American option in our case increases when the strike price increases because it is a put option $E > S$. Again since the American put value $V > 0$ at all strikes then this makes both EaD and capitals K also increase with an increase in strike prices.

Table 9: American put option value V, EaD, CCR capital, CVA capital, and regulatory capital (in % notional). The credit rating is assumed to be A+.

Strike	10	10.5	11	11.5	12	12.5	13	13.5	14	14.5	15
V	1.026	1.272	1.582	1.878	2.294	2.636	3.08	3.557	4.023	4.501	5.002
EaD CEM	1.106	1.352	1.662	1.958	2.374	2.716	3.16	3.637	4.103	4.581	5.082
EaD SA-CCR	1.759	2.139	2.605	3.052	3.665	4.172	4.821	5.512	6.188	6.878	7.599
K_{CVA} CEM	0.031	0.038	0.046	0.055	0.066	0.076	0.088	0.102	0.115	0.128	0.142
K_{CCR} CEM	0.553	0.676	0.831	0.979	1.187	1.358	1.580	1.818	2.052	2.290	2.541
K_{CVA} SA-CCR	0.049	0.060	0.073	0.085	0.102	0.117	0.135	0.154	0.173	0.192	0.212
K_{CCR} SA-CCR	0.880	1.070	1.303	1.526	1.832	2.086	2.410	2.756	3.094	3.439	3.800
K CEM	0.584	0.714	0.877	1.034	1.253	1.434	1.668	1.920	2.166	2.418	2.683
K SA-CCR	0.929	1.129	1.376	1.612	1.935	2.203	2.545	2.910	3.267	3.631	4.012

Figure 11 below shows the behaviour of CCR and CVA capital for different strikes. We can see that CVA capital is much smaller than CCR capital even when the option was ATM. It becomes much smaller as the put option gets deep ITM. This is reflected on Figure 12 below.

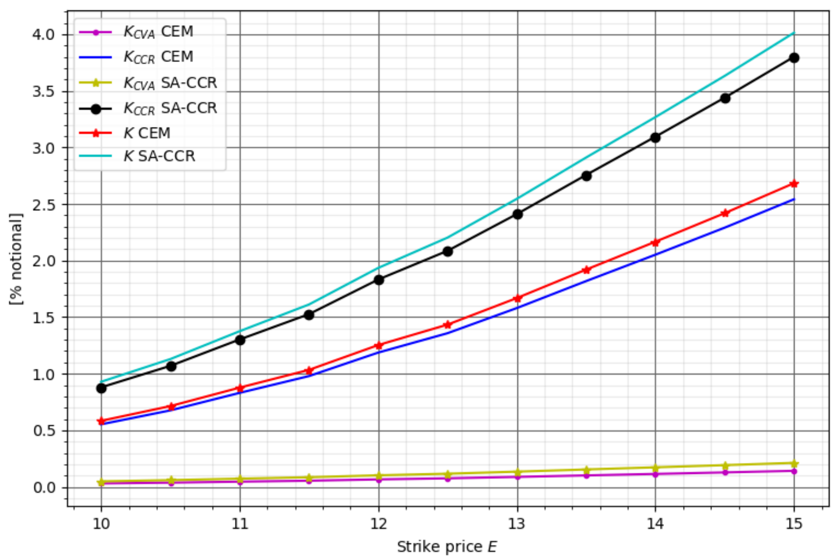


Figure 11: CVA capital, CCR capital, and regulatory capital of a 3-year American put option at different strike prices.

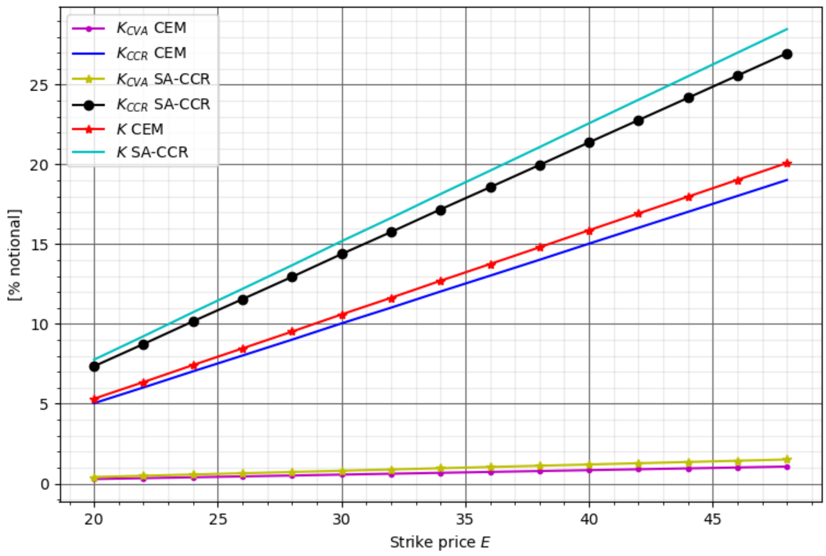


Figure 12: CVA capital, CCR capital, and regulatory capital of a 3-year deep ITM American put option at different strike prices.

Next, we compute the simulated capitals for IRS. Suppose we have a 4-year payer vanilla IRS with the fixed rate being a fair swap rate and floating rates computed using the Vasicek model. We assume that the rating is A+ so that risk weighting w becomes 50%.

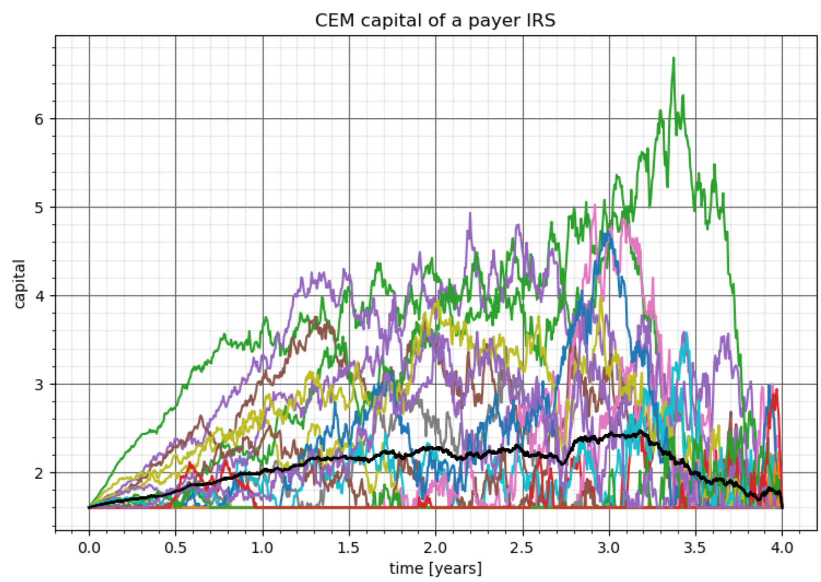


Figure 13: CCR capital obtained via CEM for a 4-year payer IRS. The black line is the expected CCR capital $\mathbb{E}[K_{CCR}]$.

The nature of regulatory capital K depends on the nature of the underlying. If the underlying is stochastic, then both the derivative value and regulatory capital will be stochastic too. This is shown by capital profiles for IRS in Figure 13 and 14.

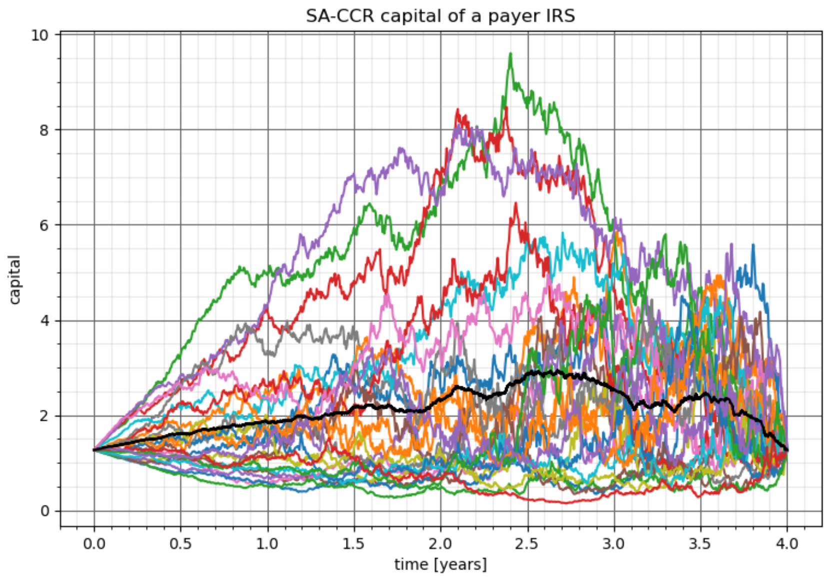
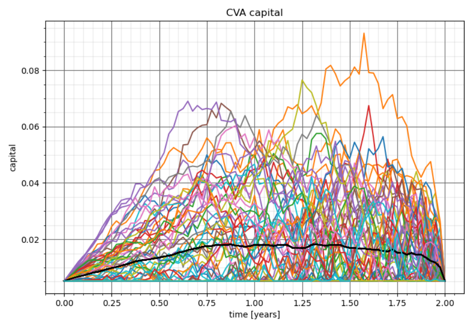
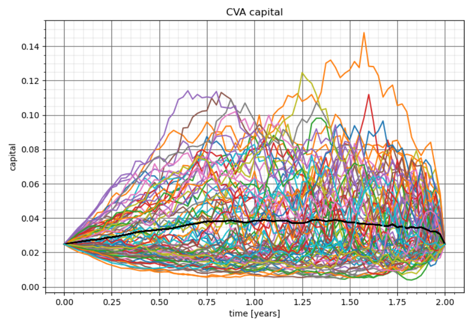


Figure 14: CCR capital obtained via SA-CCR for a 4-year payer IRS. The black line is the expected CCR capital $\mathbb{E}[K_{CCR}]$.



(a) CVA capital obtained via CEM for a 2-year payer IRS. The black line is the expected CVA capital $\mathbb{E}[K_{CVA}]$



(b) CVA capital obtained via SA-CCR for a 2-year payer IRS. The black line is the expected CVA capital $\mathbb{E}[K_{CVA}]$

Figure 15: CVA capital for a 2-year IRS

7.2 Analysing initial margin methodologies

We analyze initial margin methodologies, SIMM, and risk measure models, for an OTC European option and IRS.

Table 10: Input parameters used to calculate the IM for an OTC vanilla European put option

Parameter	**Numerical value**
N	1000
T	4
S_0	10
E	15
σ	0.25
$\mu = r$	0.06

We computed the risk sensitivities using the difference methods. These risk sensitivities are the inputs of the delta margin, vega margin, and curvature margin in the SIMM model. The calculation of curvature and vega margin depends on whether the derivative is volatility sensitive or not. An option contract is volatility sensitive but an IRS is not, so vega and curvature margins are not considered for IRS SIMM calculations.

For this example, we used the LSM to compute vanilla European put option values. From Figure 16 we note that the delta margin is much greater than both vega and curvature margin. Thus the total initial margin will be much greater than the delta, vega, and curvature margin.

Figure 17 shows the comparison of SIMM model, and VaR initial margin which is computed using LSM. The VaR initial margin is greater than SIMM initial margin at all times. The average relative percentage difference at all times is approximately 19.97%. For this example, this means that it will be more expensive to post the initial margin in a cleared OTC European option than in an uncleared one.

In Figure 18 we show the impact of initial margin on EPE.

Table 11: Input parameters used to calculate credit exposure for an OTC fair IRS

Parameter	**Numerical value**
N	100
Type	payer
κ	1
T	2

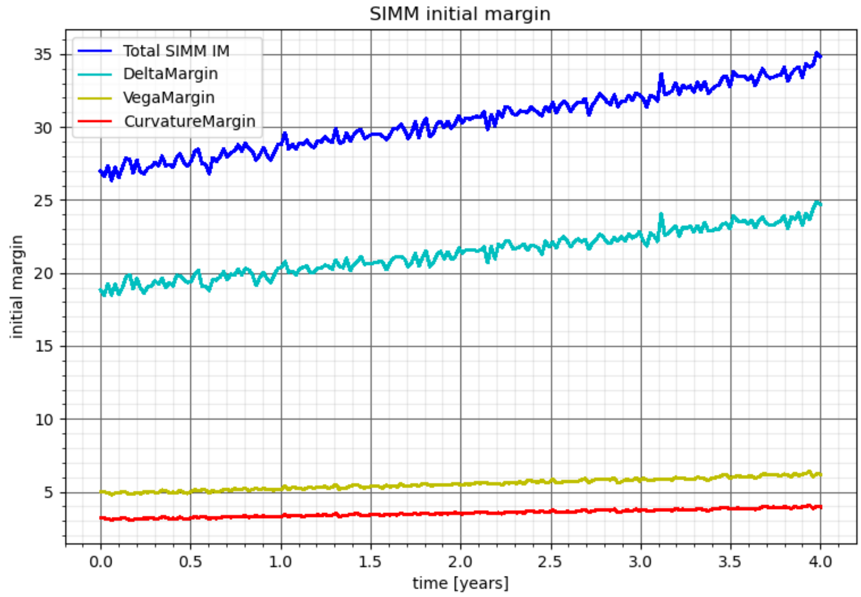


Figure 16: The profile of expected SIMM initial margin $\mathbb{E}[IM]$ for uncleared OTC European put option.

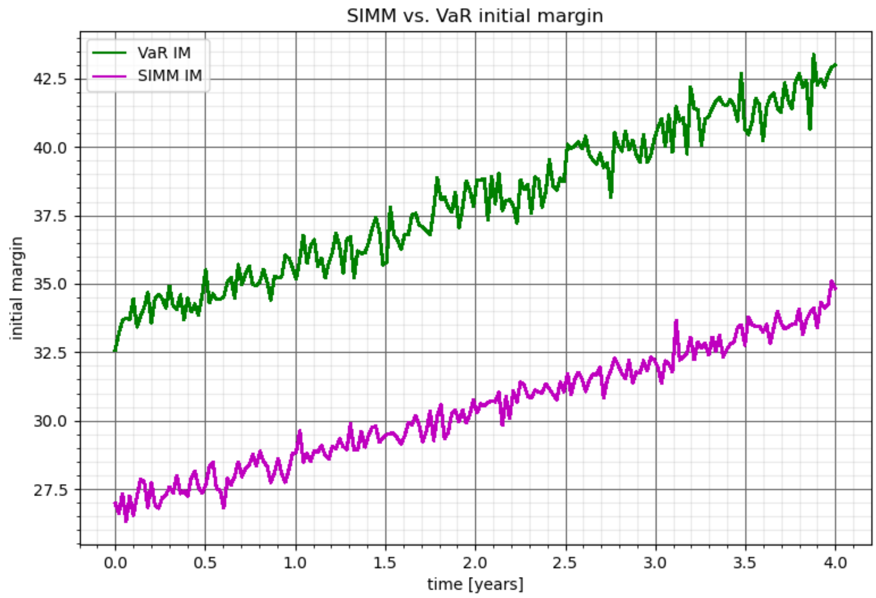


Figure 17: The profile of expected SIMM vs. VaR initial margin $\mathbb{E}[IM]$ for an OTC European put option.

The main aim of the IM is to reduce credit exposure over the MPoR, thus it is used to protect against future credit exposures. Figure 18 shows how the initial margin reduces EPE for 2-year uncollateralized IRS. In this example, the initial margin was computed via the expected tail loss based method with 97.5% confidence interval and MPoR of 5 days. The amount of the IM for risk measure based models also depends on the confidence level. When the confidence level is low, the initial margin will also become small and vice versa.

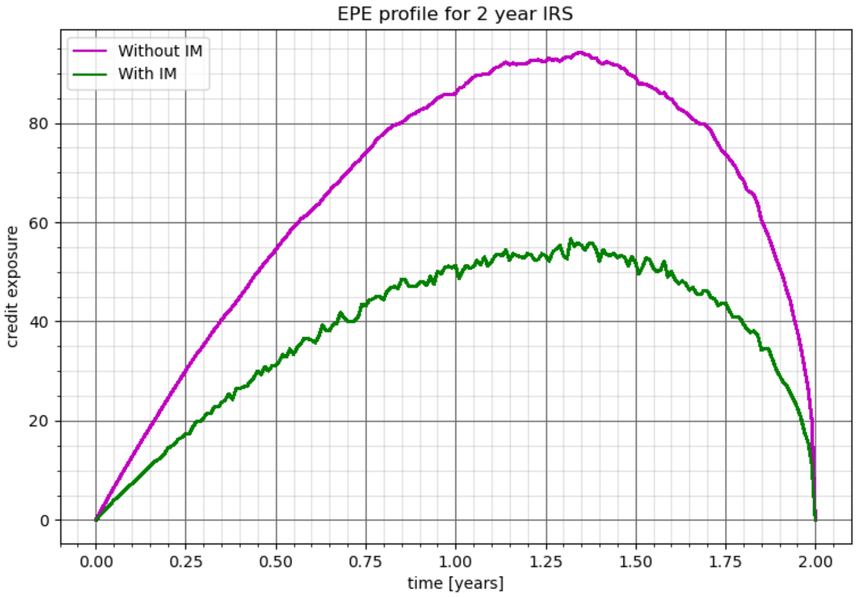


Figure 18: The profile of EPE for 2-year IRS with and without initial margin.

If we consider a collateralized IRS then EPE will be reduced even more reduced because both collateral and initial margin actually EPE.

7.3 xVAs for various derivatives

We compute xVAs on a 3-year European option, 2-year IRS, and 4-year American option. We are only interested in time 0 valuation adjustments (i.e. $U = U_0$).

xVA for a European put option

Table 12: Input parameters used to calculate xVAs for an OTC uncollateralized European put option

Parameter	Numerical value
N	20,000
T	3
S_0	11
E	10
σ	0.25
$\mu = r$	0.07
X	0

We consider the perfect hedge where the hedging error $\varepsilon_h = 0$. We also assume that regulatory capital and initial margin requirements are considered. The initial margin is computed using the VaR-based LSM method and the regulatory capital is calculated using CEM. The derivation of issuer bonds position α_1 and α_2 are shown is Theorem 9.4.1 in the appendix. It is important to note that $FCA \neq 0$ because we consider regulatory capital and this term depends on it, however, if regulatory capital requirements were not considered, this term was going to be zero just like in [15]. The $COLVA$ term is zero because we do not consider collateral value held by issuer B (i.e. $X = 0$).

Table 13: xVA for an uncollateralized European put option using *perfect hedging*. For KVA_{CCR} and KVA_{CVA}, the credit rating is assumed to be BB+, and CEM is used to compute the EaD. Other xVA parameters: $\gamma_K = 0.15$, $\phi = 0.85$, $\lambda_B = 0.035$, $\lambda_C = 0.045$, and $R_B = R_C = 0.4$.

xVA	Value
CVA	-593
DVA	0
$COLVA$	0
FCA	419
KVA_{CCR}	-528
KVA_{CVA}	-369
MVA	105
U	-996

Figure 19 illustrates xVAs for this European put option. The value of this derivative is always greater than zero and also greater than issuer B initial margin I_B at any time t (i.e. $V(t) > I_B(t)$). This makes the $ENE = (V - I_B)_-$ to be zero and thus the DVA term will become zero.

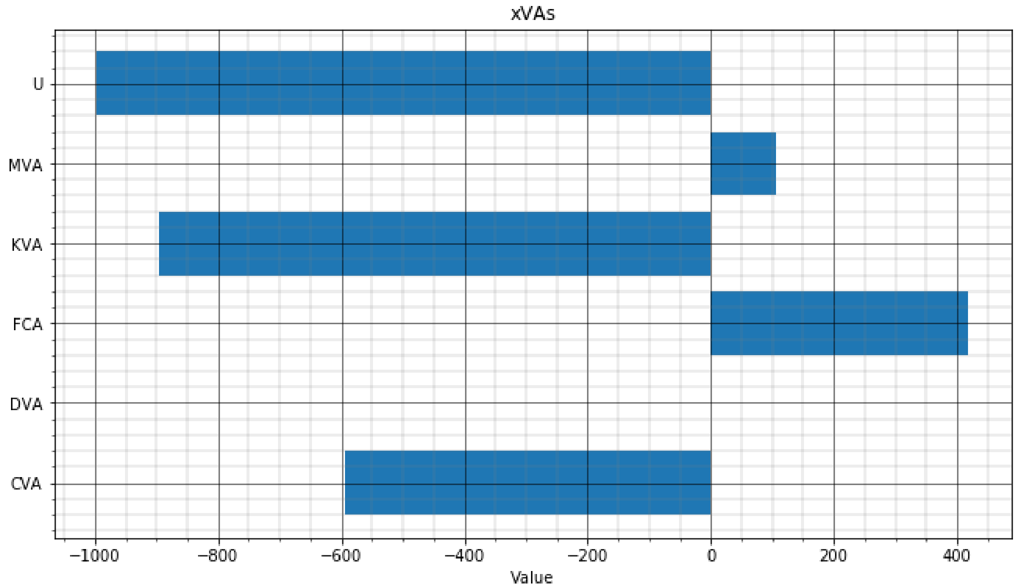


Figure 19: xVAs for an uncollateralized European put option

The value of this contract is $V = 12,166$ and the total economic value $\hat{V} = V + U = 11,200$. This means that this contract's value will relatively decrease by at least 8.19%.

xVA for IRS

We assume that there are no margin and regulatory capital requirements; thus $KVA = 0$ and $MVA = 0$. We also consider a one way CSA where $X = V_-$ and *no shortfall at own default* replicating strategy. In this strategy, $R_1 = 0$, $R_2 = R_B$, and $\varepsilon_h = (1 - R_B)(V - X)_+$.

Table 14: Input parameters used to calculate xVAs for IRS

Parameter	Numerical value
N	500,000
κ	1
T	2
r	0.085
X	V_-

The DVA term is zero and this is because the ENE is also zero at all paths and times t. Since the collateral amount is $X = V_-$ then $ENE = (V - X)_- = (V - \min(V, 0))_- = 0$. In this strategy, we have FVA term which is given by $FVA = DVA + FCA$ but since $DVA = 0$ then $FVA = FCA$. Figure 20 shows the illustration of these xVA terms.

Table 15: xVA for an IRS using *no shortfall at own default* strategy. Other xVA parameters: $R_B = R_C = 0.35$, $\lambda_B = \lambda_C = 0.02$, and $s_X = 0.05$

xVA	Value
CVA	-36
DVA	0
$COLVA$	206
FCA	-2,755
$FVA = FCA + DVA$	-2,755

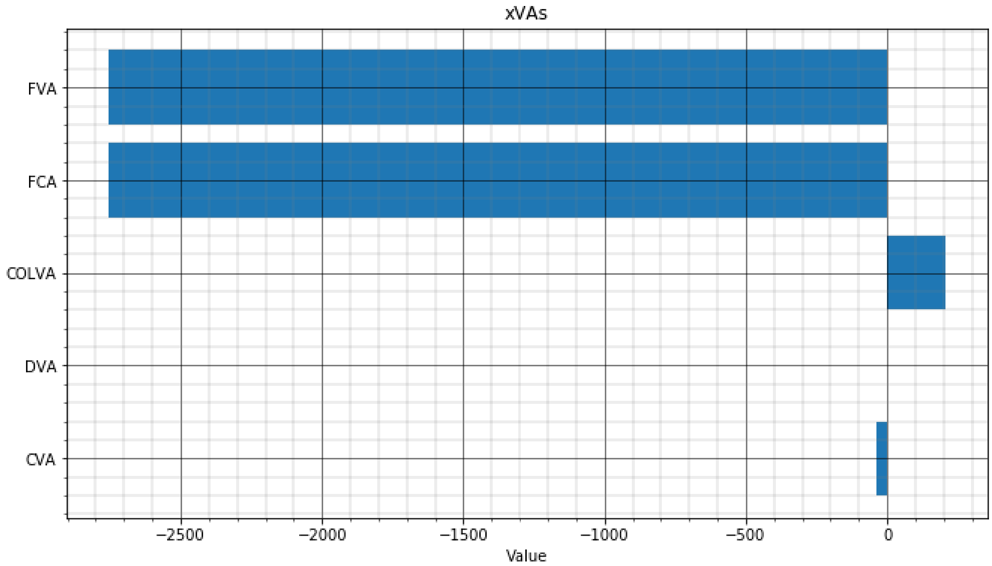


Figure 20: xVAs for a collateralized IRS

We can also consider the *semi-replication with one bond* strategy where $\alpha_1 P_1 = 0$. We now assume that this IRS is uncollateralized so that $X = 0$ and subsequently $COLVA_F = 0$. Table 16 gives the xVA terms for this IRS.

The magnitude of FVA_F is very small compared to other xVA terms.

Table 16: xVA for an IRS using *semi-replication with one bond* strategy. Other parameters are $R_C = 0.35$, $r = 0.085$, $r_F = 0.18$, $\lambda_C = 0.045$, and $X = 0$.

xVA	Value
CVA_F	-121
DVA_F	384
$COLVA_F$	0
FCA_F	-392
$FVA_F = FCA_F + DVA_F$	-8

xVA for American put option

Consider a fully collateralized American put option where $X = V$. This makes the EPE and ENE to be zero. Hence, $CVA = 0$ and $DVA = 0$. The value of this contract V and initial margin I_B are both computed using LSM. We consider the *no shortfall at own default* strategy where R_1 and $R_2 = R_B$.

Table 17: Input parameters used to calculate xVAs for American put option

Parameter	**Numerical value**
N	1,000
T	4
S_0	10
E	15
σ	0.25
$\mu = r$	0.07
X	V

Table 18: xVA for American put option using *semi-replication with one bond* strategy. Other xVA parameters: $R_B = R_C = 0.45$, $r = 0.085$, $\gamma_K = 0.10$, $\phi = 0.5$, $\lambda_B = 0.035$, $\lambda_C = 0.045$, $s_X = 0.008$, $s_{I_B} = 0.008$, and $X = V$.

xVA	Value
CVA	0
DVA	0
$COLVA$	-70.83
FCA	2.92
KVA_{CCR}	-8.02
KVA_{CVA}	-0.59
MVA	2.55
FVA	0

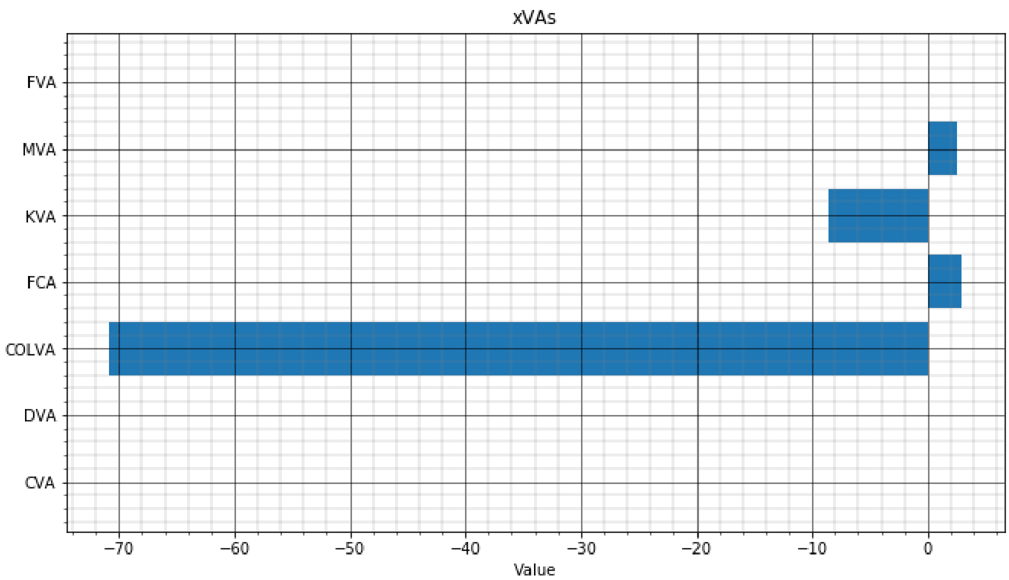


Figure 21: xVAs for a fully collateralized American put option

Unlike in previous results, we can see that $FVA \neq FCA + DVA$. This is because we considered regulatory capital and initial margin on this American put option.

7.4 xVA for cleared OTC IRS vs. uncleared IRS

We investigate the difference between cleared IRS and uncleared IRS. We use and assume *perfect hedging* where $\varepsilon_h = 0$, and since we assume the presence of margin and regulatory capital requirements, this does not mean that FCA will be zero. The initial margin I_B for cleared IRS is calculated by using ETL-based initial margin and for uncleared IRS we used the SIMM model. For regulatory capital K, the CVA capital K_{CVA} is not considered for cleared IRS, hence, $K = K_{CCR}$. The CCR capital is computed using the SA-CCR approach.

Table 19: Input parameters used to calculate xVAs for a receiver IRS. The credit rating is assumed to be A+.

Parameter	**Numerical value**
N	10M
Type	receiver
κ	-1
T	2
r	0.07
X	0

Table 20: xVA for a cleared IRS using *prefect hedging* with $\varepsilon_h = 0$. Other xVA parameters: $\gamma_K = 0.1$, $\phi = 0.2$, $R_B = R_C = 0.5$, $\lambda_B = 0.035$, $\lambda_C = 0.045$, and $s_{I_B} = 0.08$.

xVA	Value
CVA	-954, 086
DVA	704,320
$COLVA$	0
FCA	2,416,938
KVA	-3,452,768
MVA	482,264
U	-803,332

Table 21: xVA for uncleared IRS using *prefect hedging* with $\varepsilon_h = 0$. Other xVA parameters: $\gamma_K = 0.1$, $\phi = 0.2$, $R_B = R_C = 0.5$, $\lambda_B = 0.035$, $\lambda_C = 0.045$, and $s_{I_B} = 0.08$.

xVA	Value
CVA	-953, 086
DVA	704,220
$COLVA$	0
FCA	2,522,088
KVA	-3,602,982
MVA	650,940
U	-648,820

The FCA and KVA for uncleared IRS is larger than those of cleared IRS, this is because we do not consider CVA capital for cleared IRS. Furthermore, the initial margin I_B for uncleared IRS is greater than the one for cleared IRS. This makes the MVA for uncleared IRS to be greater than MVA for cleared. Unlike for uncleared trades where MPoR is assume to be 10 days, in cleared trades this usually assumed to be 5 days or sometimes 10 days.

Although the magnitude of U for cleared receiver IRS is more than the magnitude of uncleared one, this case is not the same as the payer IRS as seen in Table 23 and 24. For a receiver IRS, the U for cleared is 1.2 times the one for uncleared; and for payer IRS, the U for cleared is 0.99 smaller than the one for uncleared. The reason behind this is because the receiver IRS is collateralized while the receiver IRS is uncollateralized.

As we have noted, in both payer and receiver IRS, the COLVA remained constant even when we switched from cleared to uncleared. The reason is that collateral X does not depend on regulatory capital K or even initial margin I_B, thus COLVA does not change.

Table 22: Input parameters used to calculate xVAs for a payer IRS. The credit rating is assumed to be A+.

Parameter	**Numerical value**
N	1M
Type	payer
κ	1
T	1.5
r	0.065
X	V_-

Table 23: xVA for a cleared IRS using *prefect hedging* with $\varepsilon_h = 0$. Other parameters are $\gamma_K = 0.15$, $\phi = 0.3$, $R_B = R_C = 0.4$, $\lambda_B = 0.03$, $\lambda_C = 0.04$, $s_X = 4bps$, and $s_{I_B} = 0.008$.

xVA	Value
CVA	-62.64
DVA	0
$COLVA$	1.63
FCA	213.73
KVA	-489.40
MVA	1.24
U	-335.44

Table 24: xVA for uncleared IRS using *prefect hedging* with $\varepsilon_h = 0$. Other xVA parameters: $\gamma_K = 0.15$, $\phi = 0.3$, $R_B = R_C = 0.4$, $\lambda_B = 0.03$, $\lambda_C = 0.04$, $s_X = 4bps$, and $s_{I_B} = 0.008$.

xVA	Value
CVA	-61.79
DVA	0
$COLVA$	1.63
FCA	214.78
KVA	-493.22
MVA	2.42
U	-336.18

8 Conclusions and Further Research

In this dissertation, we have derived and extended the BSM model to incorporate default risk, regulatory capital, collateral, funding costs, and the initial margin through the implementation of the semi-replication strategy of Burgard and Kjaer. The semi-replication strategy was chosen because of its simple presentation. This strategy makes use of issuer bonds, counterparty bonds, and other pricing factors like collateral, default risk, regulatory capital, funding costs, and the initial margin. The main aim was to deliver a post-financial crisis study of derivatives pricing by introducing and implementing valuation adjustments.

We have analyzed the regulatory capital methodologies, CEM and SA-CCR, by computing regulatory capital for IRS, a European put option, and the American option. The derivative values were computed using 200 time steps and 25,000 simulations. We also analyzed the initial margin methodologies; SIMM and risk measure based models by computing the initial margin for these derivatives. The risk measure based model also known as the historical simulation model was computed using the LSM which used Laguerre polynomials of order 10.

We have shown how xVAs impact the value of the derivatives by computing xVAs for a European put option, American put option, and IRS. We derived and applied three funding strategies similar to Burgard and Kjaer where we took regulatory capital and the initial margin into consideration. We have also shown that in a perfect hedge the FCA term only becomes zero if regulatory capital requirements are absent, this is because FCA depends on regulatory capital when regulatory capital requirements are present.

Finally, we investigated the difference between OTC cleared IRS and OTC uncleared IRS by comparing xVAs using a perfect hedge. We found out that FCA and KVA for cleared OTC IRS are always larger than the ones for uncleared because in an uncleared one the CVA capital is not considered.

Another observation we made is that xVAs can be viewed as derivatives where derivative value is their underlying. This can be expressed mathematically as $U(V)$ where U and V are xVAs and derivative value, respectively. We also assumed the inclusion of FVA in our numerical results, which has been a heated debate before.

Further research on this subject:

- Pricing derivatives and xVAs in extreme market conditions using extreme value distribution instead of a normal distribution,
- Modelling xVA parameters such as default intensity as a stochastic process, thus taking wrong-way and right-way risk into consideration,
- Investigating the difference between KVA and MVA for cleared and uncleared derivatives,
- Compute xVAs for a portfolio of different derivatives.

9 Appendix

9.1 Introductory Probability Theory and Stochastic Calculus

Definition 9.1.1 Sigma algebra
Assume that Ω is a nonempty set and let $\mathcal{F}$ be a family of subsets of Ω. If $\mathcal{F}$ satisfies below properties then we call it σ-algebra:

(i) $\emptyset \in \mathcal{F}$,

(ii) If $F \in \mathcal{F}$ then $F^c \in \mathcal{F}$ where F^c is the complement of F,

(iii) If $F_{i=1}^{\infty} \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

Definition 9.1.2 Filtration
Suppose that Ω is a nonempty set and that there is a σ-algebra $\mathcal{F}_t$ such that each set in $\mathcal{F}_s$ is also in $\mathcal{F}_s$ where $s \leq t$. Then the family of all σ-algebras $\mathcal{F}_t$ is called a filtration.

Definition 9.1.3 Probability space
Assume that Ω is a nonempty set, $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is the probability measure. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be the probability space if it satisfies the following:

(i) $\mathbb{P}(\Omega) = 1$,

(ii) $\mathbb{P}(\bigcap_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mathbb{P}(F_i)$.

Definition 9.1.4 Weiner process
A one dimensional stochastic process $\{W_t\}_{t \geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Weiner process if it satisfies these properties:

(i) $W_0 = 0$ almost surely,

(ii) The map $t \mapsto W_t$ is continuous with probability 1,

(iii) W is stationary and has independent increments,

(iv) W has Gaussian increments.

Theorem 9.1.1 Ito's lemma

Let the stochastic process Y satisfy the following SDE

$$dY = a(t, Y)dt + b(t, Y)dW$$

If $p(t, Y)$ is twice differentiable then p satisfies

$$dp = \left(\frac{\partial p}{\partial t} + a(t, Y)\frac{\partial p}{\partial Y} + \frac{1}{2}b^2(t, Y)\frac{\partial^2 p}{\partial Y^2}\right)dt + b(t, Y)\frac{\partial p}{\partial Y}dW$$

where W is a Weiner process.

Proof: By applying Taylor series

$$\begin{aligned} dp &= \frac{\partial p}{\partial Y}dY + \frac{\partial p}{\partial t}dt + \frac{1}{2}\left(\frac{\partial^2 p}{\partial Y^2}dY^2 + 2\frac{\partial^2 f}{\partial t \partial Y}dtdY + \frac{\partial^2 f}{\partial t^2}dt^2\right) \\ &= \left(\frac{\partial p}{\partial t} + a(t, Y)\frac{\partial p}{\partial Y} + \frac{1}{2}b^2(t, Y)\frac{\partial^2 p}{\partial Y^2}\right)dt + b(t, Y)\frac{\partial p}{\partial Y}dW \end{aligned}$$

since $dY^2 = b^2 dt$.

Theorem 9.1.2 Euler's method

Let the stochastic process Y satisfy the following SDE

$$dY = a(t, Y)dt + b(t, Y)dW$$

Suppose that time interval $0 \leq t \leq T$ and $\mathcal{P}_n$ be a partition of this time interval such that $0 = t_0 < t_n = T$ and $\Delta = T/n = t_i - t_{i-1}$.

The approximation of PDE of Y is obtained via

$$y_i := y_{i-1} + a(t_{i-1}, y_{i-1})\Delta t + b(t_{i-1}, y_{i-1})\Delta w_{i-1}$$

$\forall i = 1, 2, ..., n$ and $x_{t_i} = x_i$.

Theorem 9.1.3 Runge-Kutta method

Let the stochastic process Y satisfy the following SDE

$$dY = a(t, Y)dt + b(t, Y)dW$$

Suppose that time interval $0 \leq t \leq T$ and $\mathcal{P}_n$ be a partition of this time interval such that

$0 = t_0 < t_n = T$ and $\Delta = T/n = t_i - t_{i-1}$. The approximation of this is obtained via

$$y_i := y_{i-1} + a(t_{i-1}, y_{i-1})\Delta t + b(t_{i-1}, y_{i-1})\Delta w_{i-1} + \frac{1}{2}(b(t_{i-1}, \overline{y}_{i-1}) - b(t_{i-1}, y_{i-1}))(\Delta w_{i-1}^2 - \Delta)\Delta^{-1/2}$$

$\forall i = 1, 2, ..., n$ and $x_{t_i} = x_i$. The $\overline{y}_{i-1} = y_{i-1} + a(t_{i-1}, y_{i-1})\Delta t + b(t_{i-1}, y_{i-1})\Delta t^{1/2}$ and $y_{t_0} = y_0$ is the initial value.

Definition 9.1.5 Martingale
An adapted stochastic process X_t is said to be a martingale if

$$X_s = \mathbb{E}[X_t \mid \mathcal{F}_s], \ \forall\, 0 \leq s < t < \infty$$

Definition 9.1.6 Radon-Nikodym derivative
Suppose that A and B are any numeraire assets. The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}_A}{d\mathbb{Q}_B} = \frac{A_T}{B_T}\frac{B_0}{A_0}$$

where probability measures $\mathbb{Q}_A$ and $\mathbb{Q}_B$ are related to numeraire A and B.

9.2 Interest Rate Modeling

Definition 9.2.1 Spot rate
The spot rate $R(t, T)$ is expressed as

$$R(t, T) = -\frac{\log B(t, T)}{T - t}$$

where $B(t, T)$ is a zero coupon bond and $0 \leq t < T < \infty$.

Definition 9.2.2 Simple forward rate
The forward rate $F(t; t_{i-1}, t_i)$ is said to be a simple forward rate if

$$F(t; t_{i-1}, t_i) = \frac{1}{t_i - t_{i-1}}\left(\frac{D(t, t_{i-1})}{D(t, t_i)} - 1\right)$$

where $0 \leq t_{i-1} < t < t_i < T < \infty$.

Definition 9.2.3 The continuously compound forward rate
The forward rate $F(t; T, \overline{T})$ is said to be a continuously compound forward rate if $F(t; T, \overline{T})$ is

$$F(t; T, \overline{T}) = \frac{1}{\overline{T} - T}\log\frac{B(t, T)}{B(t, \overline{T})}$$

where $0 \leq t < T < \overline{T} < \infty$.

Definition 9.2.4 Instantaneous forward rate

The instantaneous forward rate $f(t,T)$ is

$$\begin{aligned} f(t,T) &= \lim_{\overline{T}\to T} F(t;T,\overline{T}) \\ &= -\frac{\partial}{\partial T}\log\frac{B(t,T)}{B(t,\overline{T})} \\ &= -\frac{\partial B(t,T)/\partial T}{B(t,T)} \end{aligned}$$

where $0 \leq t < T < \overline{T} < \infty$.

Definition 9.2.5 Short-rate models

The short-rate model $\mathcal{R}$ is a type of financial market model that describes the evolution of the short rate r_t at future times t given the time interval $[t, t+\Delta t]$. Let $R(t,T)$ such that $0 \leq t < T < \infty$ be the spot rate. Then $r(t)$ is a short rate if it can be expressed as

$$\begin{aligned} r(t) &= \lim_{t\to T} R(t,T) \\ &= f(t,t) \end{aligned}$$

Definition 9.2.6 General form

Any short rate model $\mathcal{R}$ follows Ito's process

$$dr(t) = a(t, r(t))dt + b(t, r(t))dW(t)$$

where $a(t, r_t)$ is drift and $b(t, r_t)$ is volatility.

The short rate model $\mathcal{R}$ is called a one-factor model if its evolution is affected by one market risk factor at a time. This market risk factor is often called stochastic shock and is denoted by dW_t. This shock is usually a Weiner process.

Suppose there is an interest rate derivative with the value v obtained under risk-neutral measure $\mathbb{Q}$ such that it follows the short rate process above. By Ito's lemma this value v follows

$$\begin{aligned} dv &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial r}dr + \frac{1}{2}\frac{\partial^2 v}{\partial r^2}dr^2 \\ &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial r}\big(adt + bdW_t\big) + \frac{1}{2}b^2\frac{\partial^2 v}{\partial r^2}dt \\ &= \left(\frac{\partial v}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 v}{\partial r^2} + a\frac{\partial v}{\partial r}\right)dt + b\frac{\partial v}{\partial r}dW_t \end{aligned}$$

Using similar assumptions as in BSM PDE, this will finally become

$$rv = \frac{\partial v}{\partial t} + a\frac{\partial v}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 v}{\partial r^2}$$

Definition 9.2.7 The Vasicek model
The Vasicek model is a one-factor short rate model that satisfies

$$dr_t = \theta(\beta - r_t)dt + \sigma dW_t,\ r(0) = r_0$$

under the risk neutral measure $\mathbb{Q}$, $0 \leq \theta \leq 1$ and $\sigma,\ \beta > 0$.

Here,

- β is long mean reversion level,
- θ is the speed of the reversion,
- σ is volatility of short rate

In order to obtain an analytical solution for the Vasicek model, we let $g(t, r_t) = e^{\theta t} r_t$. Then by Ito's lemma

$$\begin{aligned}
dg &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial r}dr + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}(dr)^2 \\
&= \theta e^{\theta t} r_t dt + e^{\theta t} dr_t + 0 \\
&= \theta e^{\theta t} r_t dt + e^{\theta t}\theta(\beta - r_t)dt + \sigma e^{\theta t} dW_t \\
&= e^{\theta t}\theta\beta dt + \sigma e^{\theta t} dW_t
\end{aligned}$$

We integrate from 0 to t

$$\begin{aligned}
\int_0^t e^{\theta u} dr_u &= \int_0^t e^{\theta u}\theta\beta du + \int_0^t \sigma e^{\theta u} dW_u \\
e^{\theta t} r_t - r_0 &= \frac{1}{\theta}\beta\theta(e^{\theta t} - 1) + \int_0^t \sigma e^{\theta u} dW_u
\end{aligned}$$

Finally

$$r_t = r_0 e^{-\theta t} + \beta(1 - e^{-\theta t}) + e^{-\theta t}\int_0^t \sigma e^{\theta u} dW_u$$

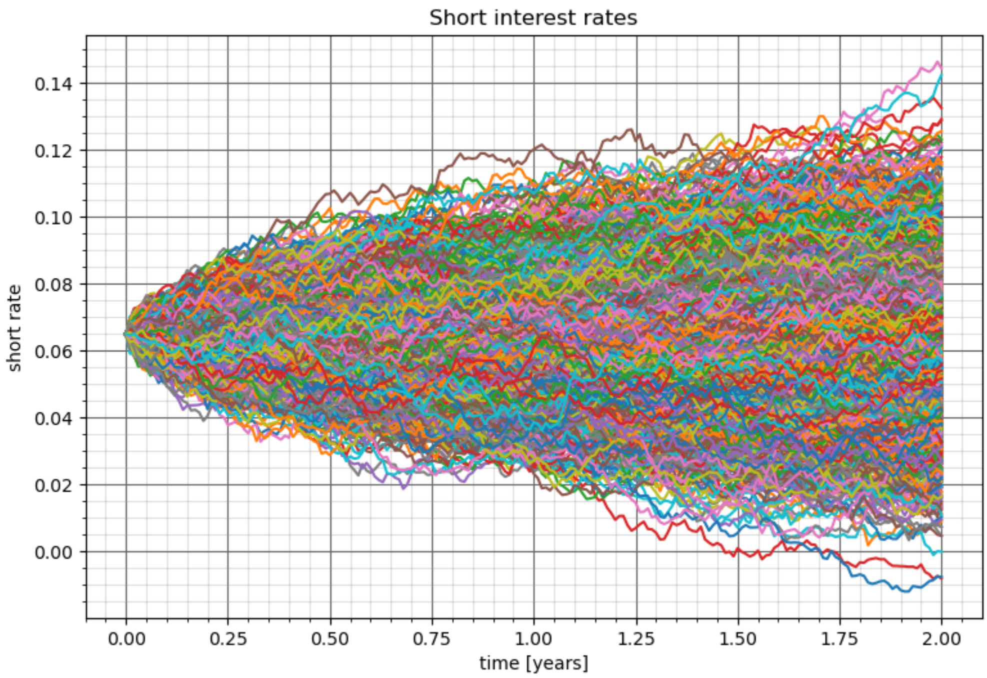


Figure 22: Multiple paths of the Vasicek model short interest rates with parameters $r_0 = 0.065$, $\sigma = 0.015$, $\beta = 0.02$, and $\theta = 0.03$

Another way of obtaining the solution is through the use of Euler's method or Runge-Kutta method which gives us

$$r_{t+\Delta t} - r_t = \theta(\beta - r_t)\Delta t + \sigma\sqrt{\Delta t}z$$

where Δt is the time interval step size and $z \sim \mathcal{N}(0, 1)$.

The mean and variance of r_t are

$$\begin{aligned}
\mathbb{E}[r_t \mid \mathcal{F}_0] &= \mathbb{E}[r_0 e^{-\theta t} + \beta(1 - e^{-\theta t}) + e^{-\theta t}\int_0^t \sigma e^{\theta u} dW_u] \\
&= r_0 e^{-\theta t} + \beta(1 - e^{-\theta t}) + e^{-\theta t}\int_0^t \sigma e^{\theta u}\mathbb{E}[dW_u] \\
&= r_0 e^{-\theta t} + \beta(1 - e^{-\theta t}) \\
\text{Var}[r_t \mid \mathcal{F}_0] &= \mathbb{E}\big[(r_t - \mathbb{E}[r_t])^2\big] \\
&= \mathbb{E}\Big[(\sigma e^{-\theta t}\int_0^t e^{\theta u} dW_u)^2\Big] \\
&= \sigma^2 e^{-2\theta t}\mathbb{E}\Big[\int_0^t e^{2\theta u} du\Big] \\
&= \frac{\sigma^2(1 - e^{-\theta t})}{2\theta}
\end{aligned}$$

As $t \to \infty$ we have

$$\lim_{t\to\infty} \mathbb{E}[r_t \mid \mathcal{F}_0] = \beta$$
$$\lim_{t\to\infty} \text{Var}[r_t \mid \mathcal{F}_0] = \frac{\sigma^2}{2\theta}$$

From this, we can conclude that the Vasicek model is mean-reverting since in a long term the average of all paths will approach β. The Vasicek model is realistic because it allows negative interest rates. Before the financial crisis of 2007-2008, the use of negative rates was scarce. After the crisis, it became a norm. One case of the use of negative rates was when European Central Bank slit its deposit rate to -0.10% in 2014 [6].

The last property of the Vasicek model that we are going to discuss is affinity.

Definition 9.2.8 Affine model

A stochastic short-rate model r_t is said to be affine if the value of the ZCB can be written as

$$B(t, T) = A(t, T)e^{-Z(t,T)r_t}$$

where $A(t, T)$ and $Z(t, T)$ are some arbitrary functions that depend on the type of short-rate models.

The Vasicek model is affine model where $A(t, T)$ and $Z(t, T)$ are

$$Z(t, T) = \frac{1 - e^{\theta(T-t)}}{\theta}$$
$$A(t, T) = \exp\left(\frac{(Z(t, T) + t - T)(\theta^2\beta - \sigma^2/2)}{\theta^2} - \frac{\sigma^2 Z^2(t, T)}{4\theta}\right)$$

9.3 Risk Measures

Definition 9.3.1 Value at Risk

Suppose that L is the portfolio's profit and loss (PnL) distribution and $\alpha \in (0, 1)$ be a level of confidence. Then VaR at α is defined mathematically as

$$VaR_\alpha(L) = \inf\{l \in \mathbb{R} : F_L(l) \leq \alpha\}$$

where F_L is a well defined CDF.

VaR estimates the amount of risk of loss in a portfolio.

If L follows a normal distribution then VaR_α is denoted by

$$VaR_\alpha(L) = \mu_L + \sigma_L \Phi^{-1}(\alpha)$$

where $\mu_L = \mathbb{E}[L]$ and $\sigma_L^2 = \mathbb{E}[(L-\mu)^2]$.

Definition 9.3.2 Expected Tail Loss
The ETL is a risk measure that is regarded as an average VaR. It is defined mathematically as

$$\begin{aligned} ETL_\alpha(L) &= \frac{1}{1-\alpha}\int_0^\alpha VaR_y(L)dy \\ &= \mathbb{E}[L|L \geq VaR_\alpha(L)] \end{aligned}$$

ETL is also called expected shortfall.

If L follows a normal distribution then ETL_α is denoted by

$$ETL_\alpha(L) = \mu_L + \sigma_L \frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}$$

where $\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$. Both VaR and ETL are used to compute the initial margin in CCP frameworks. In our calculations, we assumed that drift μ_L is approximately zero and we computed volatility σ_L using the LSM. In some instances, one can assume that L follows some distribution other than a normal distribution. One of these can be generalized extreme value distribution which can be used in extreme market conditions.

9.4 Funding Strategies and Valuation Adjustments

Three funding strategies used in numerical results are presented according to Burgard and Kjaer [15]. The only difference is that we are considering regulatory capital and initial margin while [15] is not. We will present valuation adjustments according to each strategy. Each strategy has its own unique hedging error ε_h. As we have stated before, we assume the standard close-out values $M_B = M_C = V$.

Theorem 9.4.1 Perfect replication

The strategy is said to be *perfect hedging strategy* if the hedging error is zero i.e. $\varepsilon_h = 0$. The bond positions α_1, α_2 are expressed as

$$\alpha_1 = \frac{\phi K - g_B - I_B + X - R_2(\phi K + X - \hat{V})}{P_1(R_2 - R_1)}$$
$$\alpha_2 = \frac{\phi K - g_B - I_B + X - R_1(\phi K + X - \hat{V})}{P_2(R_2 - R_1)}$$

Proof: We apply funding constraint and the fact that $\varepsilon_h = 0$. We have these two equations

$$\alpha_1 P_1 + \alpha_2 P_2 = \phi K + X - \hat{V}$$
$$\alpha_1 R_1 P_1 + \alpha_2 R_2 P_2 = \phi K + X - g_B - I_B$$

Firstly, we will write these equations as a Wronskian matrix. The determinants of the matrices are

$$\begin{vmatrix} P_1 & P_2 \\ P_1 R_1 & P_2 R_2 \end{vmatrix} = P_1 P_2 R_2 - P_1 P_2 R_1 = P_1 P_2 (R_2 - R_1)$$

$$\begin{vmatrix} P_2 & \phi K + X - \hat{V} \\ P_2 R_2 & \phi K - g_B - I_B + X \end{vmatrix} = P_2(\phi K - g_B - I_B + X - R_2(\phi K + X - \hat{V}))$$

$$\begin{vmatrix} P_1 & \phi K + X - \hat{V} \\ P_1 R_1 & \phi K - g_B - I_B + X \end{vmatrix} = P_1(\phi K - g_B - I_B + X - R_1(\phi K + X - \hat{V}))$$

Finally, the bonds' positions α_1 and α_2 are expressed as

$$\alpha_1 = \frac{\phi K - g_B - I_B + X - R_2(\phi K + X - \hat{V})}{P_1(R_2 - R_1)}$$
$$\alpha_2 = \frac{\phi K - g_B - I_B + X - R_1(\phi K + X - \hat{V})}{P_2(R_2 - R_1)}$$

This setup is more general as it incorporates regulatory capital and initial margin and it does not make any assumptions about underlying bonds P_1 and P_2. If we do not incorporate regulatory capital and initial margin then these equations will be similar to that of Burgard and Kjaer [15]. In our case $\varepsilon_h = 0$ means that $\varepsilon_{h_0} = -\varepsilon_{h_K}$. Hence assuming the presence of regulatory capital makes $FCA \neq 0$. This can be seen below using the FCA equation

$$FCA_t = -\mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_0}(y)dy\middle|\mathcal{F}_t\right]$$
$$= \mathbb{E}_t\left[\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}\varepsilon_{h_K}(y)dy\middle|\mathcal{F}_t\right]$$

Next, we look at funding strategies that involve the underlying bonds. These strategies will also help us to derive FVA for derivatives pricing. We will only present two strategies that involve the underlying bonds.

Theorem 9.4.2 No shortfall at own default

The replicating strategy is described as a *no shortfall at own default* if the recovery rate for bond P_1 is zero (i.e. $R_1 = 0$) and recovery rate for bond P_2 is equal to R_B (i.e. $R_2 = R_B$). Furthermore,

$$\alpha_1 P_1 = -(\hat{V} - V) = -U$$
$$\alpha_2 P_2 = -(V - X + I_B - \phi K)$$

The hedging error will be expressed as $\varepsilon_h = (1 - R_B)(V - X + I_B)_+ - \phi K$.

Proof: By definition of g_B and ε_h

$$g_B = (V - X + I_B)_+ + R_B(V - X + I_B)_- + X - I_B$$
$$\varepsilon_h = g_B - X + I_B + R_1\alpha_1 P_1 + R_2\alpha_2 P_2 - \phi K$$

We now simplify the hedging error

$$\begin{aligned}
\varepsilon_h &= (V - X + I_B)_+ + R_B(V - X + I_B)_- + R_B\alpha_2 P_2 - \phi K \\
&= (V - X + I_B)_+ + R_B(V - X + I_B)_- - R_B(V - X + I_B) - \phi K \\
&= (V - X + I_B)_+ - R_B(V - X + I_B)_+ - \phi K \\
&= (1 - R_B)(V - X + I_B)_+ - \phi K
\end{aligned}$$

If we assume that there is no regulatory capital and initial margin then

$$\varepsilon_h = \varepsilon_{h_0} = (1 - R_B)(V - X)_+$$

This is similar to what is obtained by Burgard and Kjaer [15]. We obtain the following xVA terms

$$CVA_t = -\mathbb{E}_t\left[(1-R_C)\int_t^T \lambda_C(y)e^{-\int_t^y a(v)dv}[(V(y)-X(y))_+]dy\middle|\mathcal{F}_t\right]$$
$$DVA_t = -\mathbb{E}_t\left[(1-R_B)\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[(V(y)-X(y))_-]dy\middle|\mathcal{F}_t\right]$$
$$COLVA_t = -\mathbb{E}_t\left[\int_t^T s_X(y)e^{-\int_t^y a(v)dv}X(y)dy\middle|\mathcal{F}_t\right]$$
$$FCA_t = -\mathbb{E}_t\left[(1-R_B)\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[(V(y)-X(y))_+]dy\middle|\mathcal{F}_t\right]$$

The FVA term is now expressed as

$$FVA_t = DVA_t + FCA_t = -\mathbb{E}_t\left[(1-R_B)\int_t^T \lambda_B(y)e^{-\int_t^y a(v)dv}[(V(y)-X(y))]dy\middle|\mathcal{F}_t\right]$$

where $a(v) = r(v) + \lambda_B(v) + \lambda_C(v)$.

Theorem 9.4.3 Semi-replication with one bond

The funding strategy is said to be *semi-replication with one bond* if the following equations hold

$$\begin{aligned}\alpha_1 P_1 &= 0\\ \alpha_2 P_2 &= -(\hat{V} - X + I_B - \phi K)\\ &= -(V + U - X + I_B - \phi K)\end{aligned}$$

The issuer B issues one bond and the hedging error will become

$$\varepsilon_h = g_B + (1-R_2)[I_B - X - \phi K] - R_2\hat{V}$$

Proof: We apply the definition of ε_h

$$\begin{aligned}\varepsilon_h &= g_B - X + I_B + R_1\alpha_1 P_1 + R_2\alpha_2 P_2 - \phi K\\ &= g_B - X + I_B - R_2(V + U - X + I_B - \phi K) - \phi K\\ &= g_B + (1-R_2)I_B - (1-R_2)X - (1-R_2)\phi K - R_2\hat{V}\\ &= g_B + (1-R_2)[I_B - X - \phi K] - R_2\hat{V}\end{aligned}$$

We can denote this bond by P_F with $s_F = r_F - r$ and thus the hedging error become

$$\varepsilon_h = g_B + (1-\breve{R}_F)[I_B - X - \phi K] - \breve{R}_F\hat{V}$$

where $\breve{R}_F$ is the recovery rate. If we assume that there are no margin and regulatory capital requirements then we have these valuation adjustments

$$\begin{aligned} CVA_F &= -\mathbb{E}_t\left[(1-R_C)\int_t^T \lambda_C(y)e^{-\int_t^y b(v)dv}[(V(y)-X(y))_+]dy\middle|\mathcal{F}_t\right] \\ DVA_F &= -\mathbb{E}_t\left[\int_t^T s_F(y)e^{-\int_t^y b(v)dv}[(V(y)-X(y))_-]dy\middle|\mathcal{F}_t\right] \\ COLVA_F &= -\mathbb{E}_t\left[\int_t^T s_X(y)e^{-\int_t^y b(v)dv}X(y)dy\middle|\mathcal{F}_t\right] \\ FCA_F &= -\mathbb{E}_t\left[\int_t^T s_F(y)e^{-\int_t^y b(v)dv}[(V(y)-X(y))_+]dy\middle|\mathcal{F}_t\right] \end{aligned}$$

And FVA_F term is now expressed as

$$FVA_F = DVA_F + FCA_F = -\mathbb{E}_t\left[\int_t^T s_F(y)e^{-\int_t^y b(v)dv}[(V(y)-X(y))]dy\middle|\mathcal{F}_t\right]$$

where $b(v) = r_F(v) + \lambda_C(v)$. We can further express these equations using normal CVA, DVA, COLVA and FCA terms

$$FCA - FCA_F = -(CVA - CVA_F) - (DVA - DVA_F) - (COLVA - COLVA_F)$$

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