An exploratory study into grade 12 learners' understanding of Euclidean Geometry with special emphasis on cyclic quadrilateral and tangent theorems

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A research report submitted to the School of Science Education at the University of the Witwatersrand, Johannesburg, in partial fulfilment of the requirements for the degree Master of Science.

October 2006

Declaration

I, **Ishaak Cassim**, declare that this research report is my own, unaided work. It is being submitted for the degree **Master of Science** at the **University of the Witwatersrand, Johannesburg.** The research has not been submitted before for degree or examination at any other University.

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Signature of Candidate

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Abstract

This research report explored the strategies which grade 12 learners employ to solve geometric problems. The purpose of this research was to gain an understanding of how grade 12 learners begin to solve geometric problems involving cyclic quadrilateral and tangent theorems. A case study method was used as the main research method. The study employed the van Hiele level's of geometric thought as a method for categorising learners levels of understanding. Data about the strategies which learners recruit to solve geometric problems were gathered using learner-based tasks, semi-structured interviews and document analysis.

From the data gathered, the following patterns emerged: learners incorrect use of theorems to solve geometrical problems; learners base their responses on the visual appearance of the diagram; learners "force " a solution when one is not available; learners' views of proof. Each of these aspects is discussed.

The report concludes that learners strategies to solving geometric problems are based largely on the manner in which educators approach the solving of geometrical problems.

Acknowledgements

My sincere thanks to all involved in the completion of this research report and in particular the following:

- My Creator for the strength and guidance He has provided me-
- My supervisor, Dr Willy Mwakapenda for his constant and professional advice and support throughout the study;
- My parents (Mogamat and Mirriam Cassim), brother (Abdul Rafick Cassim), sisters (Yasmeen Abdulla and Aneesa Cassim), wife (Fatima Bibi Cassim) and son (Muhammed Yusuf Cassim) for their support and encouragement to ensure this study is completed successfully;
- Mr M H Allimia and Mr M L Nkwane for proof reading and editing my report;
- The Principal, educator and learners of the school at which the study was undertaken;
- Finally, to all those not mentioned, especially my colleagues at work thank you for the encouragement and support you provided to me during this study.

I.Cassim

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List of Abbreviations Used

AMESA	Association of Mathematics Education of South Africa
DoE	Department of Education
FET	Further Education and Training
GDE	Gauteng Department of Education
HOD	House of Delegates
MASA	Mathematical Association of South Africa
MST	Mathematics, Science and Technology
NCS	National Curriculum Statements
RNCS	Revised National Curriculum Statement
TED	Transvaal Education Department

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Chapter 1: Introduction

1.1 General Introduction

This study is an investigation into grade 12 learners' understanding of Euclidean Geometry at one school in Gauteng, South Africa. In this chapter, I will discuss the problem to be investigated, the purpose of the study, the research questions investigated, and the site of the study. An outline of the research methods and data analysis is also included.

This study investigated an understanding of Euclidean Geometry with specific reference to cyclic quadrilaterals and tangent theorems of a group of grade 12 learners' at an independent school. The order of the discussions outlined is to provide a logical argument on the relevance and importance of this study for mathematics teachers teaching Euclidean geometry at grades 10 – 12 levels.

1.2 The problem to be investigated

When compared with other school mathematics content areas, the topics covered in Euclidean Geometry have remained constant. An analysis of the Interim Core Syllabus (Department of Education (DoE), 2003) is testimony to this. Learners' especially, grade 12's, performance in school geometry has also been reported to be inadequate. Examiners Reports (House of Delegates (HoD); Gauteng Department of Education (GDE) 1995, 2001, 2002, 2003) as well as the Mathematics, Science and Technology (MST) Report (2003), all comment on learners poor performance in Euclidean Geometry, indicating the following as some of the aspects that are typical of the way that learners respond to exam questions in Euclidean Geometry:
When proving theorems, learners omit necessary constructions from their diagrams and statements are not written in a logical sequence. Learners often base their responses to a question on the visual appearance of a given diagram, resulting in learners making assumptions not directly related to the given diagram. When asked to calculate the magnitude (size) of angles, learners' often assign specific values to the measures of angles. The concepts of similarity and congruency are often confused

with one another. Learners have difficulty in identifying the exterior angles of triangles or cyclic quadrilaterals, when they do appear in diagrams. A common problem is the identification of the angle between the tangent and chord and the angle in the alternate segment. The figure below illustrates this typical problem.

In the example, on the right, KT is a tangent to the circle at T.The chord NM is produced to meet the tangent KT at K.Y is a point on the chord NM. The proof below highlights a learner's response:

$\hat{N} = \hat{T}_1 + \hat{T}_2$

$\hat{T}_1 = \hat{Y}_1$

$\hat{T}_4 = \hat{Y}_2$

(Transvaal Education Department (TED), 1994:7)

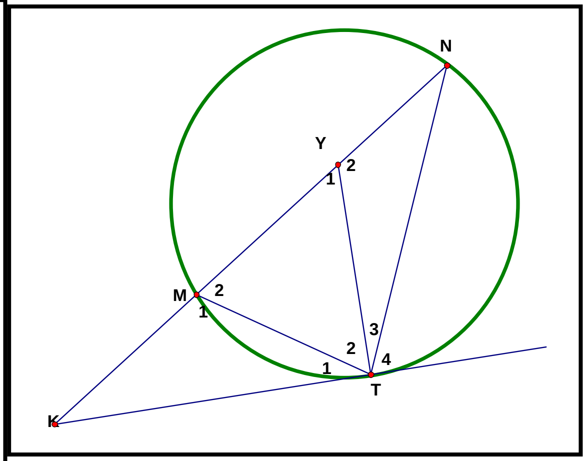

Figure 1: Typical geometry problem involving tangent-chord theorem and a learner's response to such a problem.

In the above proof, the learner incorrectly identifies $\hat{\mathbf{Y}}_1$, as the angle in the alternate segment in relation to $\hat{\mathbf{T}}_{\mathbf{1}}$.

Learners' poor performance in Geometry is not only limited to South Africa. The scope of most writings on Euclidean Geometry focuses on the twin aspects of learners "poor performance of students and an outdated curriculum" (Usiskin, 1987: 17). In an attempt to provide an explanation for learner's poor performance in geometry, Usiskin (1987), cites Allendoefer (1969) who writes:

> "The mathematical curriculum in our elementary and secondary school faces a serious dilemma when it comes to geometry. It is easy to find fault with the traditional course in geometry, but sound advice on how to remedy these difficulties are hard to come by ……. Curricular reform groups at home and abroad have tackled the problem, but with singular lack of success or agreement … …. We are, therefore under pressure to "do something" about geometry; but what shall we do?" (Usiskin; 1987 : 17).

To echo Allendoefer (1969), the question to ask is :**"What shall we do"** to improve learners' performance in Euclidean Geometry?

Why have I decided to focus on Geometry? Some of my reasons are cited below:

- Through school visits as a Subject Advisor for Mathematics, in two different areas of Gauteng province, I have observed that both primary and secondary school educators tend to delay the teaching of Geometry to as late as possible in the school year.

  > "Teaching geometry is very often left to the last term of the year where a few lessons are taught because this is what is being assessed" (Penlington, 2004 : 192).

- The poor performance of grade 12 learners' in geometry as compared to other aspects (sections) of mathematics.

- The present Interim Core Syllabus (GDE, 1995), which is used in grades 10 – 12, does little to advance the improved teaching of Geometry. Educators are provided with a syllabus, which is prescriptive in nature, which enumerates the theorems and their converses that need to be covered in a particular grade. The document lacks instructional (pedagogical) methodology of the teaching of the required concepts. Often learners are instructed by teachers to memorise proof of theorems or properties of geometric figures. A pre-service education student at Wits University, succinctly captures this scenario when he writes in his journal:

  > "(At school) we were given properties to learn by heart and never knew for sure how true is (it) that, for example (the) exterior angle of the triangle is equal to opposite interior angles" (Pournara, 2004 : 208).

  Current textbooks do little in developing the learner's ability to "develop insight into spatial relationships and measurements" (DOE, 1995). They proceed directly to the formal proof of the theorem, without first allowing the learners to get a "feel" of what the theorem is all about.

- The current specific outcomes for Mathematical Literacy Mathematics and Mathematical Sciences (MLMMS) as stated in the policy document for the Intermediate and Senior Phase (DOE, 1997) do not provide clear, lucid, transparent instructions as to what specific content needs to be covered. Specific Outcome 7, dealing with Geometry (space, shape), reads : " Describe and represent experiences with shape, space, time and motion, using all available senses." (DOE, 1997). In the primary grades (i.e. grades 4-9), the emphasis in Geometry is the development of spatial senses, which is important for later studies in Geometry. Van Niekerk (1998) contends that for learners to develop their spatial senses, educators should start with learners ordering and structuring their spatial experiences, which they can encounter in their everyday experiences – which are primarily a three-dimensional experience. This has implications for instructional resources – paper, pencil, scissors, pritt (glue), compasses, etc, which would be required to undertake the activities such as the construction of models. The discrepancies in resourcing of our schools mean that if learners do not have access to these "tools" they may not develop the required skills adequately. Once this basic skill (of visualization) has been adequately developed, it would then be appropriate to lead learners to the more structured and formal geometry encountered at grades 10 – 12 level. "National as well as international research has shown that the majority of learners in schools tend to have a backlog in their intuitive understanding of space in comparison with their intuitive number knowledge" (Van Niekerk, 1998: 70).

- The Revised National Curriculum Statement (RNCS) (DoE, 2002), emphasise a hands – on, practical approach to geometry in the grades R-9 band. "The study of geometry requires thinking and doing" (Penlington, 2004: 192). The learner moves from low level (according to Van Hiele's model) of recognition and description skills to higher order skills of classification and discrimination of two-dimensional objects. In skills such as construction of models, different views of objects are used to be developed in the learner during this phase (in grades 4 – 9).

- In the National Curriculum Statement (NCS), (DoE, 2003) for school for grades 10 -12, learning outcome 3 deals specifically with space, shape and measurement. Learners' are exposed to an enquiring, investigative, developmental approach to Geometry. Learners' are encouraged to investigate, conjecture and discover through guided learning experiences. The

> relationships between, for example, angles and the sides opposite then in any triangle. They are encouraged to test the validity of their conjectures using an array of resources, including computer software like Geometers' Sketchpad and Cabri, for example.

From the preceeding discussion one can trace changes in the approach to how Geometry is to be taught at school level, as well as the skills, knowledge that learners are expected to gain access to. However, the current grade 10 – 12 learners in the system will be assessed using a method that is to be discontinued in 2008. So educators are caught in a "Catch 22" situation: whilst they may be allowed (even encouraged) to use the new methods related to investigations, and discovery, which requires time, and places other demands on both educator and learners, learners are expected to be assessed using methods which do not reflect this type of learning. The positive aspect of this approach, however, is that the learner's understanding will be enriched if the method is appropriately applied.

The purpose of this study is to gain insights into grade 12 learners' understanding of Euclidean Geometry. I have specifically selected grade 12 learners because they would have received at least (8) eight years exposure to Euclidean Geometry.

Whilst other researchers such as, De Villiers, Lubisi and Mudaly, (South African Math educationists), have explored and written extensively on the nature and purpose of Euclidean Geometry, none of these have succinctly explored learners' understanding Euclidean Geometry. Furthermore, the Van Hiele levels of Geometric thinking which is used as a lens through which I undertook this study is restricted to rectilinear shapes like squares and rectangles. It has not been used to investigate learners' understanding of Geometry which involves cyclic quadrilaterals and tangent theorems. The van Hiele model was selected as it has not been designed specifically for Euclidean Geometry only, "but also identifies a way in which the level of geometry argumentation or thinking can be measured" (van der Sandt & Nieuwoudt, 2004:251).

In the South African context, the study undertaken by Human, Nel, De Villiers, Dreyer and Wessels (De Villiers, 1997), which undertook to rethink the manner in which Euclidean Geometry was taught in South African Schools, whilst important did not seem to take flight, what I mean is that, with the exception of the participating schools, their findings regarding the manner in which the geometry ought to be taught at secondary school level was certified ‘dead on arrival’.

In the Human et al (1977) study, (De Villiers, 1997) cited amongst others the following change to the way in which geometry is taught in the South African context is explained:

- Informal geometry as it has been presented in South African schools till the end of grade 9 (std 7).

- Proof restricted to propositions, which really require justification and / or explanation, and assuming without proofs all propositions about which pupils have no doubt.

- Construction of formal, economical definitions for the different types of quadrilaterals, and logically deducing the other properties from the definitions as a first exercise in local axiomatizing

- Local axiomatizing regarding other groups of related propositions, for example, their related to intersecting and parallel lines.

- Global axiomatizing (not included in the experimental course).

(De Villiers, 1997: 37)

Almost thirty years later, and as a result of change in political leadership in South Africa, one does notice a minute, though significant change to the geometry curriculum to be offered in South African schools.

The thrust of this study, affords one a snap view of learners’ understanding of Euclidean Geometry. This understanding is framed to a large extent in the manner it is taught to learners, i.e. as a set of theorems, which need to be memorized for the final grade 12 exams. This study, though limited in its scope, aims to make a meaningful contribution to the pedagogical knowledge of practicing mathematics educators for improving learners' understanding of Euclidean Geometry.

1.3 Focus of the study

The discipline of mathematics, and Euclidean Geometry in particular, offers to the researcher an almost seamless avenue of research prospects. However, time and space limit the focus and scope of this study. The focus of this study was on the learners. However, teacher practice is also considered, but in a very limited way. The research conducted in this study documented learners understanding of Euclidean Geometry with specific reference to cyclic quadrilaterals and tangent-chord theorems.

Whilst learners in secondary schools, i.e. from grade 8 – 12 are exposed to formal geometry this study focuses on grade 12 in particular. The study is further delimited in that it involved only one school from Tshwane South District.

My familiarity with schools and maths educators was significant, as the participating school was selected on the basis of their excellent grade 12 results over the past five years. The selection of grade 12 learners was significant, as it is the culminating point of 12 years of formal schooling as well as at least 5 years exposure to formal geometry. This research would benefit mathematics teachers in the lower grades (i.e. grade 8 – 11) as well as curriculum planners.

The intention of this study is not to generalize findings, but to ensure that the findings are relatable to other grade 12 learners and educators working at this level of the school curriculum.

1.4 Research Questions

An exploratory study of one grade 12 educator and, her group of grade 12 learners in the Tshwane South District in Gauteng, attempted to answer the following research questions:

1. **How do grade 12 learners begin to solve a geometric problem?**

2. **What knowledge and skills do learners employ in order to prove geometric problems?**

The above questions were the focus of this study: to explore and understand how grade 12 learners understand Euclidean Geometry. The aim of the study was to enquire, explore, interpret, understand and report on grade 12 learners understanding of Euclidean Geometry, with specific reference to cyclic quadrilateral and tangent-chord theorems.

1. 5 Research Methods

The research method adopted for this study was an exploratory study in grade 12 learners' understanding of Euclidean Geometry with specific reference to cyclic quadrilaterals and tangent chord theorems. Aspects relating to research method, i.e. the research design, suitability to a qualitative framework, development and reliability of data collection tools are set out in detail in chapter 3. For brevity I used three data collection tools:

1. Interviews.
2. Lesson observation (video recording).
3. Learner -based tasks.

The reliability of the instruments and the triangulation of data emerging from them are key themes that feature in subsequent chapters. The interviews and learner tasks were piloted in order to improve and determine:

1. Their user friendliness to participants;
2. Understanding of key terms;
3. Ensuring relevant generation of data;
4. Analysis and interpretation of data gathered;
5. Researchers' skills in conducting interviews;

6. Validity of instruments and
7. Reliability of findings.

1.6 Analysis of Data

1.6.1 Video Recording of Lesson

The Educator's lesson was recorded to establish her method of teaching geometry. Very often learners mimic the way they are taught a section when working on their own or in groups on an item relevant to the topic. Adhering to the qualitative research paradigm, an open coding system was employed. Fraenkel and Wallen (1993) maintain that open coding takes place when data collected is examined for patterns and / or categories, which are then later further narrowed down.

1.6.2 Interviews

Interview schedules were used to conduct interviews with five learners from the grade 12 class. The interviews generated data on :

- Learners' understanding of the task posed
- The strategies learners' employ when solving geometric problems (tasks).

Furthermore, audiotapes of the learners interviews were employed to prepare the interview transcripts. The questions posed to the learners in the interview were open – ended in order to obtain a rich, thick description of learners' understanding of Euclidean Geometry and the tools they employ to solve geometric problems. Like the video recording of the teacher's lesson, the interviews transcripts were also analysed using an open coding system.

1.6.3 Learner- Based Tasks

The learners were each asked to complete the attached task on their own (see Appendix 2). The tasks involved geometry problems involving tangent – chord theorem and cyclic quadrilateral theorem. The tasks were sourced from Daly (1995).
An open coding system was utilised to develop the categories of learner responses. According to Fraenkel and Wallen (1993), a major benefit regarding the usage of content or document analysis is that its an unobtrusive tool, through which the researcher can peep without being present and that which is analysed is not affected (adversely or otherwise) by the researcher's presence.

1.7 Limitations of the Study

When research is conducted in the classroom such research is inherently constrained. Such limitation is often beyond the control of the researcher. The limitations of this study include the following:

- The problem under investigation focuses on the learner and not on the teacher;
- A single grade 12 class was selected at a particular school in the district.

Such a small sample could begin to question the external validity of the findings.

This study was conducted within the paradigm of qualitative research. Marshall and Rossman (1989), argue that there is a weakness in qualitative research in transferability of results as each qualitative research approach has its own unique features. Although much of the data analysed was based on the researcher's subjective interpretation, it should be noted that this could result in bias of findings. Fraenkel and Wallen (1993), argue that no matter how impartial an observer attempts to be, there is always some element of biasness present.

An audiotape was used to record the interviews with the learners. The presence of the audiotape did at times affect the sincerity of learners' responses. However, as the interviews progressed, learners' became more accustomed to the presence of the audio tape recorder.

In qualitative research the quality of the data collected depends largely on the skills of the researcher. In qualitative research the social climate is always changing and the researcher is not always able to account for changing conditions in the phenomenon being studied. In this study most of the research was undertaken in the third school term.

In a qualitative research paradigm, it is not always easy to overcome the above limitations. It is, however, important to acknowledge such limitations and attempt to minimize their influence on the research process.

1.8 Organisation of the Report

Whilst this chapter has outlined key reasons for the research, further reasons are advanced and clarified in later chapters. The report constitutes the following parts:

Chapter 1: Introduction and purpose of study.

Chapter 2: The literature review which supports this study as well as the theoretical framework which forms the basis of the analysis and arguments put forward in this report.

Chapter 3: A detailed report of the research method, development of materials and processes undertaken to improve on the reliability of the results. This chapter includes an in depth discussion of the different data collection tools used.

Chapter 4: Analysis of the research data and key insights that are flagged for discussion. Issues emanating from the pilot study and key issues from the main study.

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Chapter 5: Discussion platform for key aspects of the research, which are also criticised. .

Chapter 6: Summary of results, recommendations and conclusions based on the one cohort of grade 12 learners' understanding of Euclidean Geometry.

Chapter 2:
Theoretical Framework and Literature Review

2.1 Theoretical Framework

2.1.1. Theoretical Framework: Introduction

Traditional Euclidean geometry teaching focused on the formal write up of proofs to given geometric riders. Whilst most learners performed relatively well in other branches of Mathematics (like Algebra and Trigonometry) they performed dismally when it came to Euclidean Geometry. Comments such as:

- " Had to prove theorems all year long;"
- " Didn't understand what it was all about;"
- "I passed geometry by memorizing proofs;"

are indicative of learners' dissatisfaction for Euclidean Geometry. In a study undertaken by Pournara (2004), a group of prospective teachers at the University of the Witwatersrand were asked, "Why do we study geometry at school?" Some of their responses included:

- To prove theorems;
- *Perhaps to bring marks down a bit.* (emphasis added)
- I don't know but our teacher used to say that we will need the skills we learn in geometry to apply it in our everyday lives
- I don't know why we study geometry at high school because the vast majority of it cannot be applied to everyday life, nor does it have meaning or relevance to the learners' lives
- *Geometry is about using theorems to attain results.* (emphasis added)

The comments above reflect a myopic view of geometry, held by prospective teachers who are expected to teach the subject in the not to distant future. The above comments demonstrate that learners at school cannot see the utilitarian value of geometry outside of the school environment.

In recent times there have been suggestions that Euclidean Geometry should be scrapped from the South African School Curriculum totally. In the July (1996) edition of the Mathematical Digest, one author wrote, " South Africa is the habitat of an endangered species, for Euclidean Geometry has disappeared from the syllabus of most other countries" (p. 26). Despite such pronouncements by certain sceptics, "Geometry is alive and well" (de Villiers, 1996: 37), and experiencing a revival (renaissance) in many countries throughout the world. There is no doubt about the importance of geometry not only to develop logical thinking, but also as a support to developing insights into other branches of Mathematics as well as in fields of study such as engineering, architecture, physics and astronomy.

What we need is a change of strategy to make geometry more easily understood and readily appreciated. To this end, then, we need to develop the linguistic register (vocabulary), concepts, etc, in order to create the necessary insight and understanding of the deductive system, and in this way recapture the fascination of geometry without having to memorise proofs. The question to be answered is ***HOW DO WE DO THIS?***

There is a model called the van Hiele model of geometric thinking, which can be used as a teaching tool as well as an assessment tool.

2.1.2 The Van Hiele Model Of The Development Of Geometric Thinking

The van Hiele model of geometric thinking came out as a result of the doctoral dissertation of the van Hiele couple, Pierre van Hiele and his wife Dina van Hiele – Geldof, at the Dutch University of Utrecht in 1957. It was sad that Dina died shortly after completing her dissertation and it was her husband Pierre who advanced the theory further.

Pierre's dissertation focused primarily on problems experienced by learners in geometry education, while Dina's focused on "teaching experiment" (de Villiers, 1997: 40) i.e. on the sequencing of geometric content and learning activities for learners. The former Soviet Union in the1960's was amongst the first countries in the world to realign her geometry curriculum so that it adhered to the van Hiele model. Since then, however, the model has gained prominence in most countries.

Its distinct levels of understanding characterize the model. The five levels of understanding are labelled from the most basic task to the most cognitively demanding as:

- **Level 0: Visualisation**
- **Level 1: Analysis**
- **Level 2: Informal Deduction**
- **Level 3: Formal Deduction**
- **Level 4: Rigour**

The model contends that through appropriate instruction (teaching) learners' progress from the basic level of visualization, where learners only observe shapes on their physical properties, through to the highest level, which is concerned with "formal abstract aspects of deduction." (Crowley 1987:1).

2.1.3 THE MODEL EXPLAINED

Level 0: Visualisation

At this basic level of the model the learner is basically aware of the space around him / her. Geometric objects are considered in their totality rather than in terms of their properties or constituent parts. A learner at this level has the ability to identify specific shapes, reproduce them and learn the appropriate geometric vocabulary (Crowley, 1987). For example, a learner at this level may be able to identify that figure 1(below left) contains squares and figure 2 (below right) contains rectangles. However, a learner would not be able to state that the opposite sides of a square are parallel or the angles at the vertices are 90^{o}.
Furthermore, given dotty or square paper the learners would be able to reproduce the given sketches to some degree of accuracy.

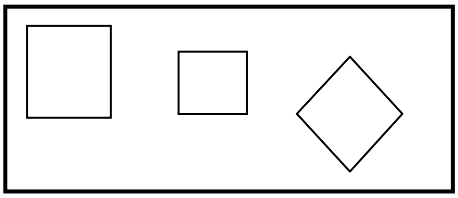


Figure 2: Squares of different sizes.

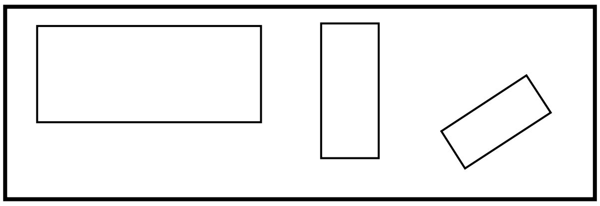


Figure 3: Rectangles of different sizes.

Level 1: Analysis

At the level of analysis properties of geometric shapes are being understood by learners' through experimentation and observation. These new properties are used to conceptualise classes of shapes. For instance a learner is able to recognize that a square is a rhombus, since a square has all the properties of a rhombus. While learners at this level are able to master the relevant technical knowledge to describe figures (shapes), they still lack the capacity to "interrelate figures or properties of figures" (de Villiers, 1997:41), and make sense of definitions.

Level 2: Informal Deduction

At this level of understanding, learners are able to establish the interrelationships that exist between and among figures. For instance learners are able to state that in a quadrilateral, if the opposite sides are parallel, then the opposite angles are equal, as well as that a square is a rectangle as it has all the properties of a rectangle.

At this level then, learners are able to deduce properties of a figure and also recognize classes of figures. Learners are able to understand class inclusion. Definitions begin to make sense for learners and are understood by them. However, at this level, learners are not able to "comprehend the significance of deduction as a whole or the role of axioms" (Crowley, 1987:3). Some Mathematics educationists, and some textbook authors regard axioms as self-evident truths – they do not regard axioms as the initial building blocks of a mathematical system (de Villiers, 1997). As a result of their misrepresentation, learners are also informed incorrectly.

Level 3: Formal Deduction

At this level learners are able to understand "the significance of deduction, the role of axioms, theorems and proof" (de Villiers, 1997: 41). At this level the learners have the ability to construct proofs based on their own understanding. They do not need to memorize readymade proofs and produce them on demand in an exam or test. The learner is able to develop a proof in more than one way. Furthermore, "the interaction of necessary and sufficient conditions is understood; distinctions between a statement and its converse can be made" (Crowley, 1987: 3). However, very few learners reach this stage of "advanced" reasoning.

Level 4: Rigour

The learner at this level does not function at the Euclidean deductive axiomatic system only. The learner has the potential to study non-Euclidean systems such as spherical geometry. By being exposed to other axiomatic systems, the learner is able to compare similarities and differences that exist between the systems. Geometry can be studied / seen in an abstract form.

Of the five levels of the model this last level is the least developed originally. As most high school geometry is taught at level 3 it should not be surprising then that most of the research done focuses on the lower levels of the model.
The key features of the model can be summarized as follows in Table 1.

Table 1: A summary of van Hiele's model of geometric reasoning.

LEVELS	What is studied	How are they studied	Examples
0	Individual objects e.g.: squares or rectangles only.	Visually recognized on the basis of physical appearance.	Rectangles of all sizes having same orientation grouped together on the basis of their orientation or appearance.
1	A class of shapes, e.g. a square is a rhombus since a square has all the properties of rhombuses.	Figures having common characteristics are grouped together, e.g. squares are a subclass of rhombuses.	A square has all adjacent sides equal. Diagonals bisect each other at right angles. Opposite sides are parallel. Opposite angles are equal, etc.
2	Learner begins to define figures / shapes belonging to same grouping (family).	Observing and noticing relationships between properties studied. This is done largely on an informal basis.	Through measurement of diagonals of squares learners will conclude that they intersect at right angles and that they are equal.
3	More formal proofs are studied	Using axiomatic system prove relationships. A more formal approach is adopted.	Prove formally that a square is indeed a rhombus.
4	Geometry is studied on an abstract level. There is a move between systems (e.g. using algebraic system to solve geometry rider)	As an interrelationship of different systems.	Circle in 2-dimension is extended to include a sphere in three-dimensional space.

2.1.4 How does the model work

The different levels of the model do not function independently of each other. The different levels are closely linked to a "network of relations" (van Hiele, 1973 cited in Human et al, 1979:20). Human et al (1979) quote van Hiele (1973) who describes this network of relations between the different levels as: "In a network of relations the words 'rhombus', 'side', 'square', etc, have unique meanings with a distinct collection of properties. Each level is associated with a different network of relations" (Human et al, 1979:20).

It is important for educators to be aware of how the van Hiele model works, since knowledge thereof would impact on the instructional strategies to be used. Below is a brief sketch, according to Crowley (1987), as to how the van Hiele model works.

Sequential: For a learner to function adequately (or competently) at level 1 for instance, the learner should have grasped the basics of the previous level adequately. This type of pre-requisite building blocks is akin to the Piagetian concepts of "assimilation and accommodation" (Helms and Turner, 1981:51). Thus like the van Hiele model which emphasizes an orderly growth path, Piaget's theory also premises all "intellectual behaviour has its beginnings in early infancy, and mature reasoning skills emerge through subsequent phases of conceptual development" (Helms and Turner, 1981:51).

Advancement: The learner's progression from one level to the next is more dependent on the instruction received than the biological maturity of the learner. It should be noted that no method of instruction allows learners to skip levels, i.e. a learner cannot move from level 0 to level 2, without first experiencing level 1. However, methods of instruction (teaching) can "enhance progress, whereas others retard or even prevent movement between levels" (Crowley, 1987:4).

Intrinsic and extrinsic: Initially learners are able to recognize figures and shapes on the basis of their physical appearance. This phase on focusing on some part of the figure/shape is similar to Piaget's concept of centering (Helms and Turner, 1981). The concept of centering is when a learner develops a tendency to "concentrate on a single outstanding characteristic of an object while excluding its other features" (Helms & Turner, 1981:41). For instance a grade 4 learner is shown shapes of squares and told that those are squares. It is not until later in the learner's life that "the figure is analysed and its components and properties are discovered" (Crowley, 1987:4).

Linguistics: Each level of the model has its own set of terminology, which is appropriate for the learner at that particular stage. Crowley (1987) cites van Hiele (1984), in which the latter asserts, "Each level has its own linguistic symbols and its own system of relations connecting them" (van Hiele, 1984, cited in Crowley, 1987:4). For instance learners in lower grades, when taught multiplication are often told that when multiplying two numbers the answer is always more or equal to the two numbers, e.g.: 3 X 4 = 12, 1X 2 = 2, etc. However, as these learners progress in their schooling careers and are exposed to rational numbers then that particular rule is no longer valid, since 2 X ½ = 1, which is less than 2 and more than ½. In a like manner in geometry then, “a figure may have more than one name (class inclusion) - a square is also a rectangle (and a parallelogram)” (Crowley, 1987:4). But for a learner at the van Hiele level 1, this does not make sense. “This type of notion and its accompanying language, however, are fundamental at level 2” (Crowley, 1987:4).

Mismatch: De Villiers (1997) captures the high failure rate in Euclidean Geometry aptly when he states, “the curriculum was presented at a higher level than those of the pupils; in other words they could not understand the teacher nor could the teacher understand why they could not understand” (p.41). If the educator (teacher), teaching materials, subject matter, language, etc. do not cohere with the learners level of development, then the end result will be a learner lacking the ability to “follow the thought processes being used” (Crowley, 1987:4).

2.1.5 Learning Phases

As has been alluded to above, movement through the levels of the van Hiele model is dependent more on the type of instruction received than on the age level or maturation of the learners. “Thus the method and organization of instruction, as well as the content and materials used, are important areas of pedagogical concern” (Crowley, 1987:5). To address the issues around content and instructional tools to be used the van Hiele s had identified five areas (phases) of learning that will assist the educator. The five areas are “inquiry, directed orientation, explication, free orientation, and integration” (Crowley, 1987:5). According to the van Hieles, if a topic or section of geometry is taught according to the above sequence, learners’ will be able to gain mastery of a particular level. The above phases of learning/ teaching are present at each level of the van Hiele model.

Each of the five phases of learning is explained below.

Phase 1: Inquiry / information:
At this primary stage learners and educators are engaged in conversation with each other about the topic at hand. Learners make observations related to the task, ask clarity seeking questions and the educator should introduce vocabulary pertinent to

the specific level at which the task is dealt with. For instance, the educator may ask learners to distinguish between a cyclic quadrilateral and any convex quadrilateral. *Is a parallelogram a cyclic quadrilateral? Is a rectangle a cyclic quadrilateral? When will a parallelogram be a cyclic quadrilateral? Why do you say that?*

Why should a teacher be engaged in such activities? Teachers' need to engage in such an activity as it serves a dual purpose, viz:

(i) "The teacher learns what prior knowledge the students have about a topic; and

(ii) The students learn what direction further study will take" (Crowley, 1987:5).

Phase 2: Directed orientation

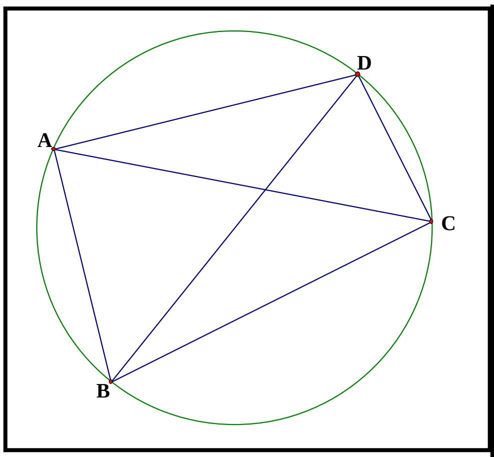
At this phase, the learners' begin to explore a topic using material that has been carefully sequenced by the educator. The activities should reveal to the learners the features peculiar to that particular level "Thus, much of the material will be short tasks designed to elicit specific responses" (Crowley, 1987:5).

Phase 3: Explication

Building on their previous experiences learners begin to express and share their views about the observations made regarding a concept. During this stage the educator plays a minimal role. The educator's role is restricted to assisting learners acquiring and using "accurate and appropriate language" (Crowley, 1987:5). It is during this phase that particular levels of systemic relations begin to become apparent.

Phase 4: Free orientation

During this period of learning, learners are exposed to and engaged in open –ended tasks that can be completed in a variety of ways. The tasks are non-routine, multi-stepped, complex tasks. Consider the following example: *In the given diagram, below* *left, BD is a diameter of the circle. ABCD is a cyclic quadrilateral. BC = 5 cm, AC = 6 cm and BD = 8 cm. Calculate the length of AB.*



Begin by calculating the length of DC in ΔBDC using the Theorem of Pythagoras, i.e. **DC =** $\sqrt{39}$ **.**

Next, calculate the size of $B\hat{D}C$ using the trigonometric ratio of tan, i.e. tan $B\hat{D}C = \frac{5}{\sqrt{39}}$

$B\hat{D}C = 38{,}7^0 = B\hat{A}C$ (angles in same segment).

Using the above results, we proceed to calculate AB, using the cosine rule.

In ΔABC: $(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos B\hat{A}C$
Through substitution and simplification we arrive at the following values for AB: **AB = 8,0** *or* **AB = 1,4**

Figure 4: An example of how learners' knowledge from different section of the Mathematics can be recruited to solve a given problem.

Phase 5: Integration

At this stage of learning, learners need to bring together (synthesize), what they have learnt, with the aim of forming an overview of the new relationships of objects and relations. The educator can assist by "furnishing global surveys" (van Hiele, cited in Crowley, 1987:6). At this stage rules may be formulated and memorized for use later.

By the end of phase five (integration), learners have attained a new level of thinking. This new level of thinking replaces the previous level of thinking and learners are once again ready to repeat the five phases of learning at the next level of the van Hiele model of thinking.

Table 2: The van Hiele model of thinking together with the phases of learning

Van Hiele Level	Phases of learning
Level 4 (Rigour)	Integration Free orientation Explication Directed orientation ↑
Level 3 (Deduction)	Integration Free orientation Explication Directed orientation ↑
Level 2 (Informal deduction)	Integration Free orientation Explication Directed orientation ↑
Level 1 (Analysis)	Integration Free orientation Explication Directed orientation ↑
Level 0 (Visual)	Integration Free orientation Explication Directed orientation ↑

2.1.6 The role of language

"Effective learning occurs as students actively experience the objects of study in appropriate contexts of geometric thinking and as they engage in discussion and reflection using the language of the period" (Teppo, 1991:213). Language is a key component of learning. According to the Department of Education (DoE), one of the aims of mathematics is for learners to "develop the ability to understand, interpret, read, speak and write mathematical language" (1995). The role of language in geometry cannot be understated. Language or language appropriate to the learner's level of thinking, as well as the identification of suitable material, are pivotal aspects in the development of the learners' geometric thinking.

According to van Hiele, the primary reason for the failure of the traditional geometry curriculum can be attributed to the communication gaps between teacher and learner. De Villiers (1997) captures it aptly when he writes, "they could not understand the teacher nor could the teacher understand why they could not understand!" (p.41). In order to enhance conceptual understanding it is important for learners to communicate (articulate) their “linguistic associations for words and symbols and that they use that vocabulary” (Crowley, 1987:13). Verbalizations call for the learners’ to make a conscious effort to express what may be considered vague and incoherent ideas. Verbalisation can also serve as a tool to expose learners “immature and misconceived ideas” (Crowley, 1987:14). At first learners should be encouraged to express their geometric thinking in their own words, e.g.: “Z-angles” for alternate angles; a rectangle that has been kicked” for a parallelogram, etc. As learners advance in their geometry studies at school they should be exposed to the appropriate terminology and encouraged to use it correctly.

A learner’s usage of a word (or term), in mathematics does not imply that the teacher and the listener (the learner) share the same meaning of the word used. For instance when a teacher uses the word parm (short for parallelogram) is the listener thinking of a parallelogram or the palm of his / her hand? As another example, if a learner is given a square in standard position, i.e. □ , the learner is able to identify the figure as a square, but if its rotated 45^0 like ◇ , then it’s no longer a square. In the example the learners’ focused on the orientation of the figure as the determining fact of the “squarness” of the figure. By engaging learners’ in discussion and conversation, educators can expose learners’ misconceptions and incomplete ideas as well as build on correct perceptions.

For the learner to acquire and correctly utilize the appropriate language, the role of the educator becomes paramount. For example, if the learners’ are working at level 1 of the van Hiele model, then the educator should be seen to be using terms such as “ all, some, always, never, sometimes” (Crowley, 1987 : 14). As the learner progress along the van Hiele continuum, appropriate terms need to be used. Terms or phrases, which are typical for some of the levels, are:

Table 3: Terms used at levels 2 and 3 of the van Hiele model

Level	Terms / Phrases
2	If [condition] then [results] it follows that
3	Axioms, postulate, converse, necessary and sufficient; theorem etc.

The type of questions posed by educators' is the key in directing learners' thinking. Questions that require regurgitation of information supplied by the educator will not foster critical thinking, which is needed for geometry. Learners' need to be able to explain and justify their explanation in a critical manner, "Mathematics is based on observing patterns; with rigorous logical thinking, this leads to theories of abstract relations" (DoE, 2003:9). Learners' should be challenged to explain "why" as well as to think about alternative approaches to their initial explanation, by posing appropriate questions, allowing sufficient waiting time and engaging learners' in discussion of their answers and methods which consider learners' level of thinking.

For growth in the learners thinking to happen, the level of instruction of the teacher needs to correspond with the learners' level of development. Thus the educator must be able to ascertain the learner's level of geometric thought, for each of the levels in the van Hiele model are characterized by their own unique vocabulary which is used to identify the concepts, structures and networks at play within a specific level of geometric thinking. "Language is useful, because by the mention of a word parts of a structure can be called up" (van Hiele 1986, cited in Teppo, 1991 : 231) .

2.1.7 Conclusion

Van Hiele's model of geometric thought, as well as their phases of learning is a constructive attempt to assist in identifying a learner's stage of geometric thought as well as the means to progress through the levels. Progression through the levels is dependent more on the type of instruction received than on the learners' physical or biological maturation level. The model has been used extensively in different research studies (e.g. : Burger 1985 ; Burger & Shaunghnessy, 1986) to assess learners' understanding of geometry. The model if applied appropriately throughout the schooling phase [i.e. grade 1 -12], will result in geometric thinking becoming accessible to all.

2.2. Literature Review

2.2.1 Introduction

A snap survey of the writings on school geometry tends to point to two main problems. Firstly, poor learner performance in the subject, e.g. the De Villiers and Njisane's 1987 study of grade 12 learners in KwaZulu Natal whose performance they note that " 45% of black pupils in grade 12 (Std. 10) in Kwazulu had only mastered level 2 or lower, whereas the examination assumed mastering of level 3 and beyond" (De Villiers , 1997 : 42). The poor performance of secondary school learners' in Geometry has also been corroborated by other studies such as Malan (1986); Smith & De Villiers(1990) and Govender(1995).

A second contributing factor to a learner's poor performance in Euclidean geometry is the nature of the curriculum currently in use. I have alluded earlier in the discussion that "South Africa is the habitat of an endangered species, for Euclidean geometry has disappeared from the syllabus of most other countries" (Mathematical Digest, July 1996, as citied in De Villiers , 1997 : 37). While South Africa may be the last surviving bastion of a "not so popular" branch of Mathematics, it is worthy to note that in recent times geometry at all levels has undergone a rebirth or revival in most countries (De Villiers, 1997). It should be noted that the concern about an archaic curriculum is nothing new – as early as 1969, Allendoefer commented on the American curriculum as follows:

"The mathematical curriculum in our elementary and secondary school faces a serious dilemma when it comes to geometry. It is easy to find fault with the traditional course in geometry, but sound advice on how to remedy the difficulties is hard to come by……Curricular reform groups at home and abroad have tackled the problem, but with singular lack of success or agreement……We are therefore, under pressure to " do something" about geometry ; but what shall we do ? (Usiskin, 1987: 17)

Each of the problems identified in the preceeding paragraphs will now be explored in some detail.

2.2.2 The problem about Curriculum

The current grade 10-12 Higher Grade and Standard Grade syllabus for mathematics strives to foster amongst others the following learning and teaching aims:

2.2.2. critical and reflective reasoning ability;

2.2.4. fluency in communicative and linguistic skills e.g. reading, writing, listening and speaking;

2.2.8. to contextualise the teaching and learning in a manner which fits the experience of the pupils" (DoE, 1995)

In addition to the above learning and teaching aims, the same syllabus documents list the following aims peculiar to mathematics, which need to be fostered and developed in learners.

2.3.1. to enable pupils to gain mathematical knowledge and proficiency ;

2.3.3. to develop insight into spatial relationships and measurements ;

2.3.4 to enable pupils to discover mathematical concepts and patterns by experimentation, discovery and conjecture ;

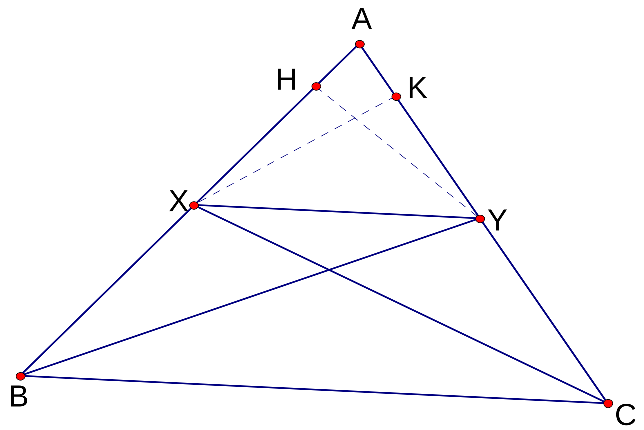
2.3.6. to develop the ability to reason logically, to generalize, socialise, organize, draw analogies and prove;" (DoE , 1995)

Both sets of aims in the syllabus documents are designed to ensure that learners are actively engaged in the "construction" of knowledge. The content should correlate with the learner's experiences *(*see 2.2.8 above*)* and offer learners appropriate opportunities to develop the "geometric eye" (see 2.3.3 above*)* through a process of discovery, experimentation, hypothesizing and so forth. However, when one peruses through a grade 12 textbook, one observes that textbook authors have failed learners by not providing them with sufficient experiences to travel the path of the van Hiele model. Often, textbook authors provide learners with a finished product of the proof of a theorem only.

e.g. 1: "Theorem 1:

A line parallel to one side of a triangle cuts the other two sides, or these sides produced, proportionally."

(Bopape, Hlomuka, Magadla, Shongwe, Taylor, Tshongwe, 1994:186)



Given: Δ ABC, with X on AB and Y on AC. XY // BC

RTP: $\frac{\mathbf{AX}}{\mathbf{XB}} = \frac{\mathbf{AY}}{\mathbf{YC}}$

Construction: Draw XC and YB. Draw altitudes XK $\perp$ AC (or AC produced) and YH $\perp$ AB (or AB produced)

Proof:

Area Δ AXY $= \frac{1}{2}\mathbf{AX.YH}$

Area Δ BXY $= \frac{1}{2}\mathbf{BX.YH}$

$\therefore \frac{\textbf{area } \Delta \textbf{ AXY}}{\textbf{area } \Delta \textbf{BXY}} = \frac{\mathbf{AX}}{\mathbf{XB}}$

Similarly $\frac{\textbf{area } \Delta \textbf{ AXY}}{\textbf{area } \Delta \textbf{CXY}} = \frac{\mathbf{AY}}{\mathbf{YC}}$

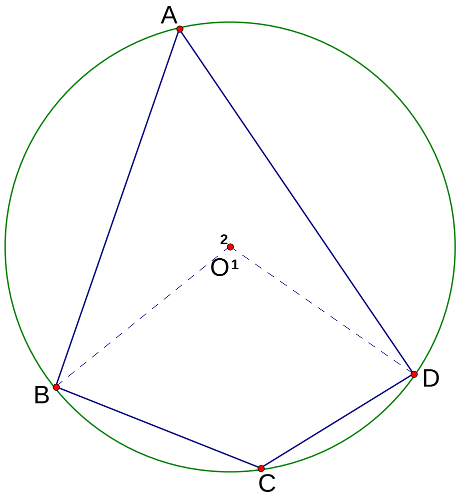
But Area Δ BXY = Area Δ CXY (same base, same height)

$\therefore \frac{\mathbf{AX}}{\mathbf{XB}} = \frac{\mathbf{AY}}{\mathbf{YC}}$

Figure 5: A typical proof which grade 12 learners are expected to reproduce under test / examination conditions.

e.g. 2: "Theorem 5(a) :

The opposite angles of a cyclic quadrilateral are supplementary (Opp. <'s of cyclic quad.)" (Laridon, Brink, Fynn, Jawurek, Kito, Myburg, Pike, Rhodes- Houghton, Van Rooyen, 1987:319)



Given: Circle O containing cyclic quad. ABCD

Required to prove:

$$\hat{A} + \hat{C} = 180^0$$

and $$\hat{B} + \hat{D} = 180^0$$

Proof: Draw BO and DO

$\hat{O}_1 = 2\hat{A}$ (< at centre = 2 X < at circle)

$\hat{O}_2 = 2\hat{C}$ (< at centre = 2 X < at circle)

$$\therefore \hat{O}_1 + \hat{O}_2 = 2\left(\hat{A} + \hat{C}\right)$$

But $\hat{O}_1 + \hat{O}_2 = 360^0$

$$\therefore \hat{A} + \hat{C} = 180^0$$

Similarly, by joining AO and CO, we could prove $\hat{B} + \hat{D} = 180^0$.

Figure 6: A typical proof which is presented to Grade 11 learners' as a finished product.

Both examples, sourced from different textbooks demonstrate how learners are given the theorems as a finished product. When I speak of a product in mathematics, it is meant here "the end-result of some mathematical activity which preceded it" (De Villiers, 1997: 45).In the examples cited above the learners were not (if teachers follow textbooks slavishly) afforded the opportunities to develop the skills articulated under 2.3.4 above. The mathematical processes are not allowed to be developed fully within learners - although official policy documents, such as the syllabus encourages such processes to be nurtured and developed within learners. The process vs. product dichotomy is not new to our schools.

As early as 1978, the predecessor to Association of Mathematics Education of South Africa (AMESA), the Mathematical Association of South Africa (MASA) noted regarding changes to the math's syllabus in South Africa that :

> "The intrinsic value of mathematics is not only contained in the PRODUCTS of mathematical activity (i.e. polished concepts, definitions, structures, and axiomatic systems), but also, and especially, in the PROCESSES OF MATHEMATICAL ACTIVITY leading to such products, e.g. generalization, recognition of pattern, defining, axiomatising. The draft syllabi are intended to reflect and increase emphasis on genuine mathematical activity as opposed to the mere assimilation of the finished products of such activity. This emphasis is particularly reflected in the various sections on geometry" (MASA, 1978, cited in De Villiers, 1997: 45).

The good intentions cited above as well as those cited in the 1995 syllabus seem to have fallen on deaf ears. Teachers and many textbook authors continue providing learners with ready made content, especially in geometry, which learners had to then "assimilate and regurgitate in tests and exams" (De Villiers, 1997 : 45) thereby confirming the assertion that geometry is useless outside the classroom.

By perpetuating the current product-driven approach to geometry, it will only increase learners' negative attitude to the discipline. Sentiments such as the following

> "but I don't see how geometry will be of any value to me ……… We have to get all the statements in just the right order on the left side of the page and always write some reason for the statement on the right-hand side of the page. We memorize definition after definition for things we already know." (Joubert, 1988: 7),

does little to light up the discipline of geometry. Our dogmatic approach to two-column proofs and learners' aimless memorization of definitions of objects they already know add salt to the learners' wounds when dealing with geometry. De Villiers (1998) argues from a theoretical vantage point that instead of teaching learners definitions of geometric concepts such as quadrilaterals, we should rather strive to develop students' ability to define.

Mathematicians such as (Blandford, 1908 and Freudenthal, 1973, as cited in De Villiers, 1998) are strong proponents allowing learners' to be actively engaged in coming up with definitions for geometric concepts. Blandford (1908), in De Villiers (1998) regards the method of giving learners' ready made definitions as a "radically vicious method" (in De Villiers, 1997: 46), and by so doing one is robbing the learners' of the most intellectually enriching activities. "The evolving of the workable definition by the child's own activity stimulated by appropriate questions is both interesting and highly educational." (De Villiers, 1997: 46).

Researchers like Ohtani (1996, in De Villiers, 1998) have argued that the traditional method of providing learners with ready-made definitions by the teacher is an attempt by the teacher to exercise his control over learners, to avoid any dissension, and not having to deal with students' ideas as well as to steer clear of any "hazardous" interaction with learners. The student's ability to regurgitate a definition of a cyclic quadrilateral does not imply that the learner understands the concept (Vinner, 1991, in De Villiers, 1998). For example, a learner may be able to recite the standard definition of a cyclic quadrilateral as "A cyclic quadrilateral is a quadrilateral of which the vertices lie on a circle" (Laridon, et al, 1995: 277), but the learner may not consider that if a quadrilateral with exterior angle equal to the interior opposite angle as being cyclic, since the learners' concept map of cyclic quadrilaterals does not include cases where points are not on a circle. "I would appeal that in order to increase students' understanding of geometric definitions, and of the concepts to which they relate, it is essential to engage them at some stage in the process of defining geometric concepts" (De Villiers, 1998 : 2).

2.2.3 Learners' views of proof for a geometrical problem

There has been a growing interest in mathematics education in recent times regarding the teaching and learning of proof (e.g. Hanna, 2000; Dreyfuss, 1999; De Villiers, 1997; 1990). Whilst there has been this resurgence in proof across mathematics fields at school and tertiary level, there has also been a lack of interest in the mathematical reasoning which learners' engage in when solving geometrical riders.
Traditionally proof has been seen primarily as a means to verify the accuracy (correctness) of mathematical statements (De Villiers, 1990). However, this stereotypical, constrained view of proof has been criticized in recent times by, amongst others, De Villiers (1990), Hanna (2000), and Dreyfuss (1999). De Villiers (1990), for instance argues that proof in mathematics is more than just for verification purposes. He maintains that the view held by most people in mathematics education that verification is the cornerstone of proof is avoiding "the real nature of proof" (Bell, 1976, in De Villiers, 1990:18), as verification in mathematics can be obtained using "quite other means than that of following a logical proof" (De Villiers, 1990:18).

For De Villiers (1990), and others (like Hanna, 2000), proof is made up of the following processes:

- Verification (concerned with the **truth** of a statement);
- Explanation (providing insight into **why** it is true);
- Systematization (the **organization** of various results into a deductive system of axioms, major concepts and theorems);
- Discovery (the discovery or invention of **new** results) ;
- Communication (the **transmission** of mathematical knowledge) (De Villiers, 1990:18).

Whilst the above five aspects will not be dealt with in any detail here, it is sufficient to state that proof is akin to van Hiele's level 3 stage of reasoning. In the van Hiele model of geometric reasoning, level epitomizes the learners' ability to understand "the interrelationship and role of undefined terms, axioms, postulates, definitions, theorems, and proof is seen" (Crowley, 1987:3). Another characteristic of the learner at this level of geometric thinking is the learner's ability to "construct, not just memorize proofs" (Crowley, 1987:3).

The tasks given to the learners in this study were designed not to test their ability to regurgitate theorems, but to check their understanding of the theorems, which they had encountered at school level. Unlike other branches of school mathematics, which are largely algorithmic in nature, geometry is different. The solution to a geometrical rider is in essence a learner's explanation, using theorems, axioms and properties of the figures involved. The thrust of this research project is not on proof per se, but the manner in which learners' present their solutions, which is indicative of the learners' understanding of the nature and purpose of proof.

Like their fellow high school colleagues in other countries, these students' concept of proof corresponds with their international counterparts: "most high school and college students don't know what a proof is or what it is supposed to achieve" (Dreyfuss, 1999:94). At high school level, the distinction between a proof, an explanation, and an argument is not always clear. Whilst this is not the focus of this study, I'd like to end with Hanna's (1995) observation that "while in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation" (p. 47).

2.2.4 The Performance problem

When compared to other branches of mathematics (e.g.: Calculus, Trigonometry, etc), learners' performance in Euclidean Geometry is dismal. Comments such as

- "This [Euclidean Geometry] is probably still the least well-done of all sections" (GDE, 2003 : 145),
- "This [Euclidean Geometry] of the work… is still not well done on the whole. Many candidates write down geometric information but it often does not make sense nor is it relevant to a particular question. They don't appear to understand the words they are using." (GDE, 2002 : 102),

are indicative of learners' poor performance in Euclidean Geometry.

The following tables, which have been extracted from the former House of Delegates examiners' report, shed clearer light on learners' performance on both HG and SG, grade 12 paper 2 sections. Both HG and SG learners at grade 12 level are tested on the following aspects: Trigonometry, Analytic Geometry and Euclidean Geometry.
Table 4 below shows the Higher Grade learners' performance in the different sections of the syllabus relevant to paper 2, written in 1991.

Table 4: Performance of Higher Grade grade 12 learners in Paper 2 (ex HOD, 1993: 1)

	Performance of Candidates				
	TOTAL MARKS	**50% - 100%**	**40% - 49%**	**0% - 39%**	**AV % of PASS**
Trigonometry	**80**	**60%**	**22%**	**18%**	**82%**
Synthetic Geometry[1]	**70**	**22%**	**10%**	**68%**	**32%**
Analytic Geometry	**50**	**36%**	**26%**	**38%**	**62%**

From the above table one can infer that the bulk of the learners' (68%) scripts sampled in 1991, scored between 0% - 39% in Euclidean Geometry. Furthermore, on average, learners' scored 32% for Euclidean Geometry, which translates to a raw mark of 22, 4 out of 70. When compared to Trigonometry, learners' scored on average 65.6/80 (82%) and in Analytic geometry learners scored on average 31/50 (62%) – then the 22, 4/70 is indeed poor – especially for learners on the higher grade.

Table 5, below is an indication of how Standard Grade, grade 12 learners performed in the 1991 examinations.

Table 5: Standard Grade learners average percentage in different sections of Paper 2 (ex HOD, 1993: 1)

SECTION	**TRIGONOMETRY**	**SYNTHETIC GEOMETRY**	**ANALYTIC GEOMETRY**
AVERAGE %	59	32	26,5

<u>NOTE:</u>

1. The data in both Tables 4 and 5 above is based on a 10% sample of learners' who wrote the 1991 grade 12 exams.

[1] Euclidean Geometry is also at times referred to as Synthetic Geometry

2. Prior to 1996, the South African education spectrum was divided into almost 20 different entities. Each entity catered for the specific grouping of the population.
3. The data in Tables 4 and 5 is based on the former House of Delegates report, which catered for the Indian population of the South African community. I have used these results because I had access to them since my first year of teaching in 1992.

From the data in Tables 4 and 5 above we notice that amongst "Indian" learners sitting for the grade 12 exams, Euclidean Geometry was the component in the second paper where both higher grade and standard grade learners faired poorly.

More recently, the Mathematics, Science and Technology (MST) project team in Gauteng was commissioned by the Gauteng Department of Education (GDE) to provide an analysis of learners' performance in Mathematics, Science and Biology at grade 12 level. The MST team used a sample of 2002 Gauteng grade 12 learners for analysis purposes. The team "perused through 30 standard grade and 31 higher grade scripts to get a superficial sense of how the candidates went through the questions" (MST Report, 2003: 3).
Table 6 provides us with an indication of standard grade learner's performance in some of the questions in the second paper.

Table 6: Standard Grade learners performance in some questions of the question paper (Maths Standard Grade Paper 2) (MST, 2003:6)

			NO LEARNERS WHO SCORED BETWEEN			
QUESTION	***MARKS***	***No. of learners not attempting a question***	***0 – 39%***	***40 – 59%***	***60 – 100%***	***TOTAL***
1	18	4	6	3	17	30
3	20	5	4	5	16	30
4	12	6	8	8	8	30
6	13	13	8	5	4	30
8	18	8	12	1	9	30
9	19	7	14	2	7	30
TOTAL	**100**					

Questions 8 and 9, in the above table, are questions involving Euclidean Geometry. We note that question 8 involved a rider dealing with the tangent-chord theorem and cyclic quadrilaterals. From the cohort of 30 learners, 20 of the 30 or 67% of the candidates are unable to deal with riders involving cyclic quadrilaterals and tangent-chord theorems. Question 9 dealt with similar triangles and in this question 70% [21/30] learners are unable to either begin to solve the problem or have attained between 0 and 7 out of a possible 19 marks.

Table 7 below provides us with a snapshot of 31 higher-grade learners' performance in a sample of questions in the 2002 exams. Whilst other questions are reflected as well, our focus is on Euclidean Geometry.

Table 7: HG learners' performance in Mathematics 2002 Grade 12 question paper.

			NO LEARNERS WHO SCORED BETWEEN			
Question	***Marks***	***Not attempted by learners***	***0 – 39%***	***40 – 59%***	***60 – 100%***	***TOTAL***
1	23	10	10	7	10	31
2	25	7	7	1	16	31
4	22	5	10	6	10	31
5	19	4	11	6	10	31
6	24	7	13	2	9	31
7	19	13	13	1	4	31
8	25	10	14	2	5	31
9	12	21	0	4	6	31
TOTAL	**169**					

In the above table (Table 7) we note that 26 / 31 higher-grade learners' are unable to handle geometric riders involving cyclic quadrilaterals (Question 8). Questions 9 and 10 dealt with similar triangles, which are also a cause of concern – but it is not the focus of this study.

We notice that whilst table 4 and 5 may refer to "Indian" learners in the main, Tables 6 and 7 are "race blind" in that the samples of scripts selected would include candidates from across the racial divide. Thus one may have been tempted to regard Euclidean Geometry as problematic only amongst Indian learners, but Tables 6 and 7 suggests that in South Africa, Euclidean Geometry is a problem endemic to all schools and communities within South Africa.

Learners' poor performance in Geometry, especially with regard to proofs of theorems, can be attributed to learners' memorization of proofs of theorems. A typical comment by examiners is that "Pupils are swotting off Theorems and are thus unable to provide a **logical** proof of what is provided" (GDE, 2000: 3). In terms of the van Hiele model of thinking the ability to construct and understand proofs is located within level 3. However, because learners have not gained mastery at the lower levels of the model (i.e. Level 2, 1, 0); they resort to rote learning of the proofs of theorems. Furthermore when learners are asked to prove that a quadrilateral is cyclic and no circle is present learners experience difficulty in accomplishing such tasks (GDE, 2000).
Although, I have been alluding to learners' poor performances at grade 12 levels in Euclidean Geometry, for improvement and a change of attitude towards secondary school geometry, the primary school geometry curriculum needs to be redesigned to be aligned with the levels of geometric thought according to the van Hiele model. "The future of secondary school geometry thus automatically depends on primary school geometry!"(De Villiers, 1997: 43).

A learner's ability to provide meaningful proofs for geometric theorems and riders epitomizes the learner's development in geometry, according to the van Hiele model of thinking. However, in reality this is not the case. Examiners' reports (GDE, 2001, 2002, MST, 2003, TED, 1994, HOD, 1993) all lament learners' poor performance when asked to prove theorems or solve riders. Learners' ability or in-ability to provide successful proofs is best explained in terms of Piaget's theory and van Hiele's model of geometric understanding. Both van Hiele's model and Piaget's theory suggest that learners must progress through the lower levels of geometric thinking before they can gain mastery at higher levels such as the writing and understanding formal proofs. The route travelled from the lower levels to the higher levels of thinking takes a considerable amount of time. "The van Hiele theory suggests that instruction should help students gradually progress through lower levels of geometric thought before they begin a proof – orientated study of geometry" (Battista and Clements, 1995: 50)

Very often, educators bypass the different levels of the van Hiele model and hope that learners will understand what the teacher has taught. This type of naïve, premature, dealing of formal proof results in learners resorting to memorization and "confusion about the purpose of proof" (Battista & Clements, 1995: 50) sets in. Both the van Hiele model and Piaget's theories suggest that learners' can understand and work with an axiomatic deductive system only once they have gained mastery in the highest levels of both theories. "Thus, the explicit study of axiomatic systems is unlikely to be productive for the vast majority of students in high school geometry" (Battista & Clements, 1995:50).

The studies undertaken by other researchers, using the van Hiele model of geometric thought restricted their investigations to linear figures such as squares and rectangles. However, not much research has been undertaken that involves the use of the van Hiele model to ascertain learners' reasoning ability involving shapes other than rectangles and squares and at a level such as grade 12. This study contributes to the existing knowledge base of the Van Hiele levels of understanding. The study deals with the following research questions:

1. How do grade 12 learners begin to solve a geometric problem?
2. What knowledge and skills do learners recruit in order to prove geometric problems?
3. How do learners justify that a proof to a geometric problem is complete?

Chapter 3: Research design and methodology

3.1. The Research Design

There exist three common research designs, which dominate the educational landscape. Reeves and Hedburg (2001) have identified them as the quantitative, qualitative and the eclectic-mixed mode pragmatic research model. Researchers showing preference to work in the quantitative paradigm present their results primarily in the language of numbers. In a quantitative paradigm, the purpose is "not to report data verbally, but to represent those data in commercial values" (Leedy 1993: 243). One just needs to caution the reader here and state that it does not mean that the other paradigms do not use calculations when doing analysis of data. However, calculations are "not the major form in which the data exist" (Leedy, 1993: 243). Data collected in a quantitative paradigm is usually analysed using inferential and descriptive statistical techniques.

This study aims to be exploratory and interpretative in nature, thus data collection and analysis thereof will be primarily determined in relation to the contextual setting and perspectives of the learners. Researchers working within a quantitative paradigm are keen on giving a "more holistic picture of what goes on in a particular situation or setting" (Fraenkel & Wallen, 1993: 10), i.e. they (quantitative researchers) are more interested in gaining a richer understanding of the social phenomena at play than the quantitative data only.

Silverman (2000) cites Hammersly (1992) in which the latter identified five preferences of quantitative researchers. The five preferences are listed in the Table 8, below.

Table 8: The preferences of qualitative researchers (Silverman, 2000:8)

1	A preference for qualitative data understood simply as the analysis of words and images rather than numbers.
2	A preference for naturally occurring data- observation rather than experiment, unstructured rather than structured interviews.
3	A preference for meanings rather than behaviour- attempting "to document the world from the point of view of the people studied" (Hammersly, 1992).
4	A rejection of natural science as a model.
5	A preference for inductive, hypothesis-generating research rather than hypothesis testing (cf. Glasser and Strauss, 1967)

From Table 8, above one should not conclude that practices such as hypothesis testing do not feature as part of the qualitative paradigm. McMillan and Schumacher (2001) argue that in qualitative paradigm researchers' use strategies which are flexible, "using various combinations of techniques to obtain valid data" (p.396). This flexibility of utilization of approaches is encapsulated in the eclectic-mixed mode of research, which characterizes cooperation between the qualitative and quantitative paradigms when collecting data related to educational problems (Reeves and Hedburg, 2001). As alluded to earlier, this study is inquiry- based and interpretive, leaning more towards a qualitative research paradigm than it does with the other paradigms. This does not imply that the study was conducted exclusively without due regard to the quantitative and or eclectic-mixed methods to highlight points of interest that arose in the study, which could be explored in more extensive and further research analysis.

The overarching research approach for this study was a case study design. "Qualitative research uses a case study design meaning that the data analysis focuses on one phenomenon, which the researcher selects to understand in depth regardless of the number of sites or participants for the study" (McMillan and Schumacher, 2001:398). The interpretive nature of case studies allows the researcher to "study and give insight into specific situations or events" (Stake, 1995). In this study the phenomenon explored is grade 12 learners' understanding of Euclidean Geometry. In case studies, generalizations of findings within a wider population or community are not of paramount importance. Fraenkel and Wallen (1993) maintain that a great deal can be "learned from studying just one individual, one classroom, or one school district" (p.392). Elsewhere, Silverman (2000) argues that in qualitative research, the focus should not be on generalizations. Silverman (2000), cites Alasuutari (1995), who notes that

> "Generalization is… [a] word. that should be reserved for surveys only. What can be analysed instead is how the researcher demonstrates that the analysis relates to things beyond the material at hand…extrapolation better captures the typical procedure in qualitative research" (Alasuutari, 1995, in Silverman, 2000:111).

A single case study design was an appropriate research tool in this study as it afforded the researcher the opportunity to explore strategies learners' used in solving geometric riders, discover important questions to ask relative to learners' reasoning strategies employed when solving geometric problems. According to Bell (1987), the great advantage of the case study lies in the fact that it "allows the researcher to concentrate on a specific instance or situation and to identify, or attempt to identify, the various interactive processes at work" (p.6), which may not be easy to identify in a large-scale survey study.

The above discussions on case study design indicate their usefulness as an appropriate and useful method for investigating processes in education, and were thus employed for this study.

3.2 Access to participants

The principal participants in this study were one educator and a cohort of her grade 12 learners from a co-educational school in the Tshwane South District, in Gauteng. Anecdotal evidence, such as discussions with educators around geometry always end or begin with "*My learners hate geometry. I have tried everything but nothing seems to help. I will just teach the way I have been doing –theorem → example → past question papers and that's it!* (emphasis added). Despondency, as manifested above, comes from an educator who has been voted as Gauteng's Teacher (Mathematics / Science) of the year in 2004 and 2003. However, what she did not succeed to do was to improve her learners' performance in Geometry. Her learners scored well in other sections of maths but performed poorly in Geometry as a result their final marks were being affected.

The selection of the teacher was based on the interactions I had with her during grade 12 marking sessions, workshops and on-site school visits. No teacher is an island and schools displaying positive leadership structures and allowed their teachers to make effective and innovative use of resources were also considered. Furthermore feasibility [distance and transport costs, etc] for the duration of the study was also a factor to be considered. In conclusion and perhaps most importantly, the willingness and cooperation of the educator, the learners and the school management team for the duration of the study (six months) was necessary for the case study to be of significant value.

My first interaction with the educator occurred in January 2004, via telephone to set up an informal meeting with the educator and principal. The meeting was scheduled for early February 2004, and took place at the school of the identified educator. At this meeting the educator and principal were briefed about the purpose of the study, the reason(s) for considering the teacher to be an appropriate participant in the study, the data collection tools to be used, the duration of the study and the commitment and willingness of the educator to ensure that the study is meaningful. At this preliminary meeting, the educator (who will henceforth be referred to as Teacher) agreed to participate in the study and was eager to share with me information regarding her learners' performance in Euclidean Geometry. The Teacher raised no objection to the data collection methods to be used, provided prior approval was granted by the school management team to record part or some of the lessons delivered and consent was obtained from the learners' to be interviewed.

Subsequent to our initial meeting, a formal letter (see Appendix 1) was submitted to the Principal, in which permission was sought to conduct research at the school with the identified learners. The letter also explained the nature and expected duration of the study and how at the end, the learners and teachers of mathematics could benefit. Permission from the Gauteng Department of Education (GDE) was also sought to conduct research by informing the relevant units (e.g. Policy and Planning Unit) about the nature and duration of the study. Data collection was tentatively planned to be conducted during August and September 2004.

3.2.1 Data collection instruments

The triangulation of data is used in qualitative research as a means to seek patterns in the data collected (McMillan & Schumacher, 2001). According to McMillan & Schumacher (2001), the triangulation of data in its broadest sense is not restricted to data only. It can include "use of multiple researchers, multiple theories, or perspectives to interpret the data; multiple data sources to corroborate data and multiple disciplines to broaden one's understanding of the method and phenomenon of interest" (McMillan & Schumacher, 2001:408-9). Through the process of triangulation, the researcher is able to find regularities in the data and in so doing improve the reliability of the findings. During the data collection process, an important consideration for me was that the information collected should be triangulated across the three intended methods to be used, viz.: interviews, lesson observation and learner written responses to the given tasks.

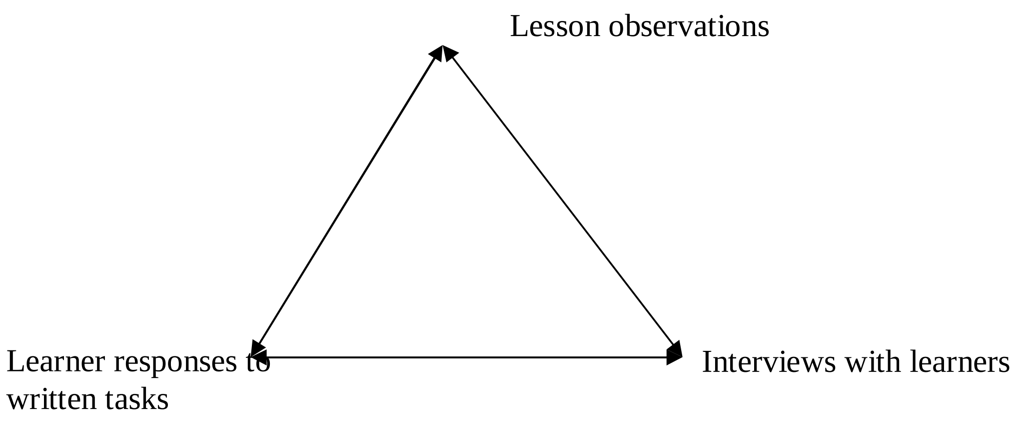


Figure 7: Graphic representation of triangulation process.

The learners' written responses to the tasks were compared to their responses in the interview as well as the alignment to the manner in which the Teacher taught the lesson, and interviews held with learners. The interview, as a data collection tool, corroborated the results obtained by learners through analysis of learners' responses to the task and observations of how the teacher taught. The nature of developing an appropriate interview schedule and fears of researchers' bias resulted in more time and effort than the other two data collection tools.

The three sources of data are closely intertwined [since they are informed by the same research questions] and all three informed emerging patterns, results and conclusions. The triangulation of the three sources is explained in depth in the next chapter. I will now proceed to discuss how each data collection tool was developed, the intended data to be gathered and the reliability of the instrument used.

3.2.2 Interviews

Structure and flexibility

In order to understand how grade 12 learners reason when solving geometric riders, it was necessary for me to gather in-depth information from those learners. The learners were expected to complete the given tasks (see Appendix 2) on their own. The purposes for the learners to complete the tasks are two fold. Firstly, the strategies, which learners employ to solve geometric problems, can be ascertained. Secondly, using the learner's responses to the task I would then be able to identify the learners to be interviewed. Thus the research instrument to be used had to be structured yet flexible. Structure would assist in the gathering of information on pre-determined topics that the researcher regards as valuable to the study. Flexibility on the other hand, would aid in capturing the experiences of learners when solving geometric riders by allowing the researcher to explore and probe unanticipated responses.

Cohen and Manion (1991) are of the opinion that an unstructured interview allows the interviewer greater flexibility and freedom than a structured interview. They go on to argue that although the research purpose shapes the question asked, their content, order and phrasing is left to the researcher's discretion. In a like manner, in semi-structured interviews and the interview guided approach (McMillan & Schumacher, 2001) topics are pre-selected in advance, but flexibility is afforded to the researcher in that the researcher "decides the sequence and wording of the questions during the interview" (McMillan & Schumacher, 2001: 444).

The use of an unstructured interview to collect data, in the hands of a skilled researcher may "produce a wealth of valuable data" (Bell, 1987: 72), but would be difficult for a novice researcher to implement. The adoption of a semi-structured intensive interview guide approach thus seemed feasible to obtain qualitative data from the five (5) grade 12 learners' from a class of approximately 24 learners involved in the study, to explore and interpret their understanding of Euclidean Geometry. The next aspect of selecting the research instrument was its suitability to generate appropriate data within a qualitative framework.

Interviews as a suitable collection tool in a qualitative framework

The intention of the case study was to provide a qualitative analysis of the manner in which grade 12 learners' reason when solving geometric problems. Factors such as how geometry was taught, learners' perception about geometry, and the milieu of the school are all subtle yet important considerations that have an impact on learners understanding of Euclidean Geometry. The choice of interviews was based on how best information about these issues could be generated within a qualitative framework. Concepts about interviews are wide and varied. However, I have identified the following aspects as appropriate for the use of interviews in a qualitative environment.

The interview is a data-collection tool, which is widely used, in qualitative research. The interview is often described as a goal – directed conversation (Macmillan & Schumacher, 2001; Bell, 1987; Marshall and Rossman, 1994). In an interview the researcher's primary goal is to elicit "certain information from the respondent" (Bell, 1987:70); which the researcher regards as important to the research study undertaken. Qualitative researchers pride themselves in discovering and portraying the multi-tired views of the case studied and that the interview is the main artery to multitiered views.

As the research concerns eliciting the "multiple strategies" learners' employed when solving geometric problems, in an in-depth manner, the appropriateness of interviews to a qualitative framework was not difficult to establish. Elsewhere in this chapter, some advantages of interviews over questionnaires are listed to demonstrate its usefulness as a qualitative research tool.

Limitations of the use of interviews as a research tool

Interviews as a data collection tool have certain limitations and weaknesses. It is difficult to generalize research findings to a wider population as the sample sizes are normally small and this adversely impacts on the researchers' validity. In general the interviewing procedure is not standard and the manner in which questions are based would differ from interviewer to interviewer. The lack of standardization during the data collection process, makes interviewing difficult to replicate successfully. Intense interviewing is prone to interviewer bias (Bell, 1987). The flexibility of the researcher in formulating questions and probing issues is a potential source of bias. The quality of the data collected relies heavily on the skills of the interviewer. The lack of standard processes in the analysis of data can lead to opposing interpretations from a single body of data gathered.

In qualitative research it is often difficult to overcome the above listed limitations of interviews. It is significant to acknowledge and at best minimize their influence on the research process. Regarding bias Bell (1987), notes that its best to "acknowledge the fact that bias can creep in than to eliminate it together" (page 73). The limitations listed above need to be viewed in the content of the proposed, general research design. It should be noted that the limitations above are not all applicable to the case study-design or to the study undertaken in this report. Generalization of results is not the focus of case studies in which sample sizes are small deliberately. The purpose of this study is not to replicate the research design, since each case study is unique, but to extend the results and findings to learners in similar environments. Bell (1987) cites Bassy (1981), in which the latter maintains that in case studies, "The relatability of a case study is more important than its generalization".

Cohen and Manion (1991), maintain that validity and researcher bias in interviews are closely related. They (Cohen and Manion, 1991), go on to suggest that the most practical manner to enhance the researcher's validity is to minimize the amount of bias as far as possible. They argue that the sources of bias are the characteristics of the interviewer, the respondents and the substantive content of the questions. Cohen and Manion (1991) suggest that one of the ways to reduce researcher bias is to have multiple (different) interviewers but this would be costly and time-consuming. McMillan and Schumacher (2001), provides us with another view to limit researcher bias, i.e. that if the interview is done correctly, "it does not matter who the interviewer is; any number of different interviewers would obtain the same results" (p. 268). Furthermore, in case studies, researcher bias is best restricted by obtaining actual quotations as well as accurate records from the participants. To further limit potential sources of biases, participants should be afforded the opportunity to check records before the researcher begins to analyse the data gathered.

Bell (1987) states that interviewing in the hands of a capable researcher has potential to generate "a wealth of valuable data, but such interviews require a great deal of expertise to control and a great deal of time to analyse" (p. 72).

An unfortunate reality of using interviews is that it is time consuming to prepare, conduct and analyse and the researcher needs to take cognisance of this before it is selected as a research tool. The final consideration selecting interviews as a research tool was its advantages over other data collection tools like questionnaires.

In a qualitative study like this one, the researcher is often interested in aspects, which are deeply buried in the minds of participants. Thus, in order to reach beyond the physical reach of the participants, researchers normally use either questionnaires or interviews.

In this study personal interviews were selected as the preferred mode of collecting data. Personal interviews are superior to questionnaires because they afford the following benefits to the researcher: it affords the researcher the opportunity to ask structured and open ended questions; responses obtained can be probed if there is a need for such clarification; interactions by participants for clarity and establishing a personal rapport with the participants involved.

3.2.3 Summary of advice about designing and using interviews

Interviews are frequently used in qualitative research designs as an instrument to collect data. However, it is also one of the "frequently misunderstood" (Leedy, 1993:192), research tool. The following advice is a summary to be shared between researchers if researchers would like to conduct successful interviews with the aim of collecting relevant data. The suggestions are clustered according to the following categories:

- The initial planning phase;
- The formulation of questions;
- The pilot study;
- The conducting of the interviews; and
- The analyses of data.

In the *initial planning phase*, it is important for the researcher to do the following: demarcate the area to be explored during the interview and use this as a guide when formulating the questions; decide what you need to know and why you need to know it (Bell, 1987).

When *formulating questions*, it is suggested that the researcher focuses on the sequence of the order in which the questions are to be asked (McMillan and Schumacher, 2001); devise probing questions; pose the same questions to all participants to ensure comparability of results (Fraenkel and Wallen, 1993); questions should be worded clearly and unambiguously (Leedy, 1993; Bell, 1989);a mix of open-ended and direct questions should be used to allow for greater respondent participation (Thompson, 1978).

In the *piloting phase* of the study, the researcher should conduct preliminary interviews with a select few participants who display similar characteristics as the participants in the main study of the research. By so doing, major shortcomings with the interview protocol can be identified early on in the study and rectified (Bell, 1987).

When conducting *the actual interview*, the researcher should state the focus and purpose of the study upfront (McMillan and Schumacher, 2001); demonstrate flexibility by shifting quickly between ideas (Posner and Gertzog, 1982); enhance the participants self-esteem by making positive remarks to their responses (Bell, 1987; Osborne and Gilbert, 1980); encourage participants to ask clarity seeking questions during the interview process. The researcher should not dominate the interview sessions by talking excessively unnecessarily- instead the researcher should afford the participants ample time to air their views without any hindrance (Bell, 1987; Posner and Gertzog, 1982).

When *analysing the data* the researcher should strive to analyse and interpret the collected data in terms of the objectives of the research study (Cohen and Manion, 1991); respect the anonymity of the participants as information gathered should be treated in a highly confidential manner (Bell, 1989); limit their own biases and personal prejudices towards the study (Bell, 1989; Cohen and Manion, 1991); classify open-ended questions into sub-categories and these sub-categories need to be verified by knowledgeable experts (Cohen and Manion, 1991). The researcher should guard against accepting responses at face value, but should instead validate responses by the processes of triangulation (Cohen and Manion, 1991; Schumacher and McMillan, 2001).

According to McMillan and Schumacher (2001), the process of effective interviewing "depends on efficient probing and sequencing of questions" (p. 448). To realise the goal of effective interviewing, McMillan and Schumacher (2001), have offered the following guidelines:

- Interview probes should be used to elicit detailed information, further explanations and classification of responses;
- The researcher needs to articulate the purpose and focus of his/her research from the outset;
- There should be a semi-structured ordering of questions that would allow for flexibility to obtain adequate data;
- Demographic questions should either be dealt with throughout the interview session or in the concluding section of the interview session;

- Questions of a complex, controversial or difficult nature should be catered for during the middle or tail-end of the interview session.

Figure 8 below, indicates the steps to follow when designing, implementing and analysing an interview research instrument in a qualitative study.

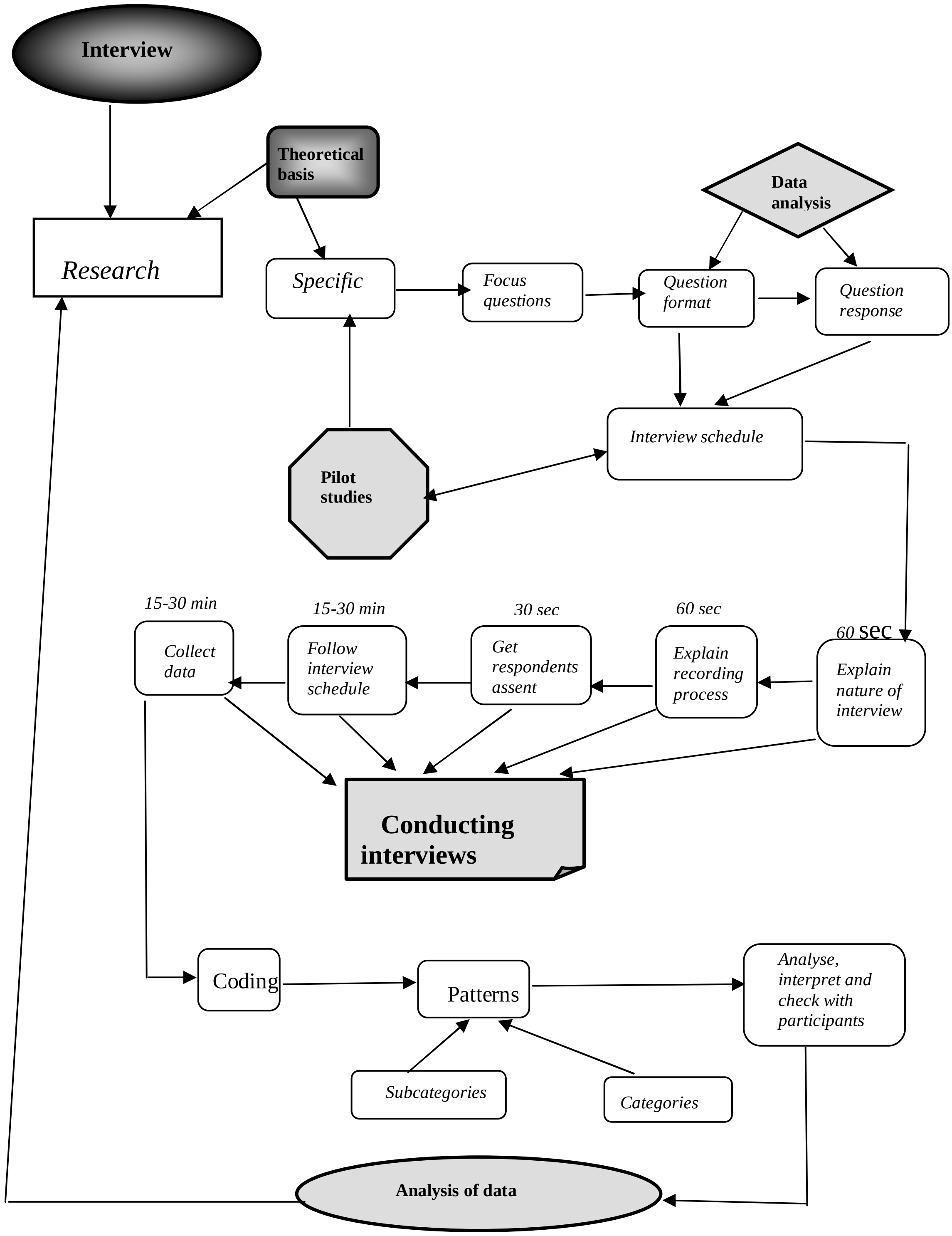


Figure 8: Flowchart for interviews as a research instruments (Chetty, 2003:41)

As stated earlier, elsewhere, the purpose of this study is to gather information through an in-depth qualitative study about learners' reasoning ability when solving geometric riders. Information gathered will be based on learners' reasoning abilities to solve geometric riders involving cyclic quadrilaterals and tangent theorems. The information so gathered, is intended to benefit educators when teaching Euclidean Geometry to high (secondary) school learners'.

Another aim of the study is to translate broad research goals into more detailed and specific research objectives. Hence, the need to prepare the interview schedules. According to Cohen and Manion (1991), this means that research objectives are to be translated into questions that would form the core of the interview. Cohen and Manion (1991), and McMillan and Schumacher (2001), suggest that before the actual questions are prepared, a thought needs to be spared for the format of the questions as well as the possible responses to the questions. In the interview the researcher used both open-ended questions as well as choice selection questions where more direct answers were required. This is in keeping with the balance of structure and flexibility alluded to earlier.

According to Frankel and Wallen (1993); open- ended questions in interviews have a number of advantages. These advantages include the following:

- they are flexible;
- they allow the interviewer to probe respondents for more detail;
- they assist in clearing any misunderstandings; and
- they encourage cooperation and help establish rapport between the researcher and the respondents.

In the interview protocol presented several open-ended questions are included to establish how learners' reason when solving geometric riders. Some choice related questions were included to maintain focus of the interview so that respondents do not talk aimlessly and non-stop.

Cohen and Manion (1991) argues that the type of information sought will frame the response mode as well as the way in which the data collected will be analysed. In the interview protocol the response modes are a mixture of a task based activity to be completed by learners (see end of this Chapter for tasks). The data generated was of a nominal type. The advantage of generating nominal type of data is that it minimises the bias effect while increasing the flexibility (Cohen and Manion, 1991). However, a disadvantage of this type of data is that it becomes more difficult to code.

Following the preparation of the interview schedule was the setting up and conducting of the interviews. Permission, and on site selection of participants was obtained prior to conducting the interviews. Bell (1987), suggests that besides a sense of "common sense and good manners" (p.75), the following courtesies need to be also adhered to by the researcher:

- Introduce yourself as well as the purpose of your study;
- Respondents should be made to feel at ease;
- The manner in which responses are to be recorded need to be explained to the participants;
- If recording devices (tape recorder, video recorder, etc.) are to be used, get participants consent before you actually conduct the interview;
- The interviewer needs to abide by the interview schedule, though not religiously- there should be room for flexibility.

In addition to the above, if the researcher makes any commitments (promises, etc) to the participants, then such promises need to be fulfilled. Researchers' should take Bell's (1987) words to heart and remember them "take care not to promise too much" (p.76).

The final stage in the interview process, once the data has been collected, involves *coding and scoring.* Cohen and Manion (1991), regard the process of coding as the translation of question responses into specific categories for data analysis. In this study, summaries were made after the interviews or through audio recording of participants' responses, which were then catalogued according to pre-determined categories. In the interview protocol listed, responses were post-coded and matched to the pre-determined classes in the content analysis. Learners' responses were then rank-scored to ascertain the frequency of particular responses occurring. Finally, the data was analysed and interpreted according to the objectives of the research study (Cohen and Manion, 1991).

It should be remembered that these stages and processes were not to be adhered to religiously- they are flexible and subject to change- should the need arise. The qualitative researcher, by continuously checking and reflecting on what he/she planned provides valuable insights for his/her research questions. It is however, important for a researcher to plan effectively each step of the interview that he/she will follow in eliciting information from the research question(s). Similar developmental strategies were applied to the instruments for the document analysis and the lesson observation.

3.2.4 Document analysis

Document analysis as its name suggests can be appropriately used to analyse past as well as present records of the participants involved in the study. Fraenkel and Wallen (1993), believe that a “person’s or group’s conscious and unconscious beliefs, attitudes, values, and ideas are often revealed in the documents they produce” (p.389). As stated earlier, information collected from lesson observation session and the interviews was triangulated with information obtained from learners’ responses to the task-based activities. Personal documents, such as learners’ exercise (work) books; educator’s lesson plans, etc are a rich source of valuable information that the researcher can have access if needed.

3.2.5 Participant observation of lessons

Participant observation is commonly used in case studies as it allow the researcher to “actually participate in the situation or setting they are observing” (Fraenkel and Wallen, 1993:390). According to Fraenkel and Wallen (1993), as well as Borg and Gall (1983), the researcher can assume one of two roles when participating in lesson observations. Firstly, the researcher can be fully immersed in the situation in which case his true identity is shielded from the rest of the group; and secondly, the researcher’s participation is partial, meaning that the researcher acts as an observer but is also allowed to participate fully with the group to establish rapport and develop a better understanding of the group’s dynamics (i.e. how the group functions, etc.) and relationships. In this study the researcher assumed the latter role.

Since the aim of this case study was not an intervention, but exploratory and interpretative of conditions as they unfolded, there was little need for me to become fully immersed in the group's functioning. To ascertain how learners reasoned when solving geometric riders, it was necessary to observe learners in practice during the study. An agreed upon time schedule for classroom observations and feedback was established with the educator.

Two modes of recording data were employed for the classroom observation, viz. field notes and video recordings. Field notes were descriptive and informal in nature and selectively used in accordance with the broad categories related to learners' reasoning strategies employed when solving geometric riders. Video recordings were not used in any significant manner.

3.2.6 Learner-based tasks used

The questions selected for the task were typical examination type questions, which the learners may have encountered previously. The questions were well-suited for both higher and standard grade learners doing mathematics at grade 12 levels.

Whilst learners are not usually asked multiple choice questions in an examination, these questions have been designed to gain a peep into the learners thought processes when solving geometric problems. The use of multiple choice questions instead of routine examination type questions was motivated by the fact the in multiple choice questions the answer "lies in front of the pupil" (Daly, 1995). To each of the questions four alternatives are provided, of which only one is a valid response to the question. The challenge for the learner is that the additional three alternatives also appear to be valid responses to the given question.

The questions posed were set or compiled by "experienced teachers who, over the past number of years, have been members of teams of writers, trained to design multiple-choice questions" (Daly, 1995).

Accompanying each question, below is a brief description of what the question entails as well as possible routes learners could embark upon to arrive at a valid response to each question.

3.2.7 The task and the administration thereof

The tasks (see below) that were administered were based on cyclic quadrilaterals and tangent theorems. The tasks consisted of eight (8) multiple questions (Section A) and one open ended type of question (Section B). In the main study learners were asked not to answer task 8 in Section A because of limited time. The tasks were first piloted with a different group of grade 12 learners at the same school. McMillan and Schumacher (2001) argues that conducting a pilot study affords the researcher the opportunity to "Check for clarity, ambiguity in sentences, time for completion, directions, and any problems that may have been experienced"(p.185), before the instruments could be used in the main study.

Participants for the main study were selected on the basis of the school's grade 12 results in Mathematics. It is common knowledge that learners' performance in mathematics in comparison to other subjects is below par. Schools whose average was within a 5% range of the District's (Tshwane South) 2003 average were identified as potential sites of implementation of the research instrument. Consideration of distance, travelling time, and accessibility to learners, educators and school were all factors to be considered before the final selection was made. The latter constraints resulted in a school in close proximity of the researcher's home to be selected.

It was important for the researcher to pilot the main data collection instrument, before actual implementation in the main study. However, due to unsuitable time frames and the reluctance of the School Management at one possible site, the pilot and main study were conducted at the same site, but with different groups of grade 12 learners'.

Table 9: Comparison between the pilot and main study.

Characteristic	**Pilot Study**	**Main Study**
Gender composition of learners	Heterogeneous	Heterogeneous
Ability group	Mainly standard grade learners	Both higher and standard grade learners
Learner performance	Poor to average	Average to good
Learning environment	Enhances learning	Enhances learning
Educator involved	Same educator	Same educator

The above table (Table 9) illustrates the similarities and differences that prevailed between learners in the pilot study and the main study. The same educator taught both groups of learners. The learners' in the pilot study were mainly standard grade learners', whilst the learners' in the main study were from both higher grade and standard grade. Both groups were well represented in terms of gender- both groups had boy-learners and girl-learners in them. As a result of the similarities prevalent in Table 10, as well as the teachers' understanding of the need to improve learners understanding of Euclidean Geometry, it was possible to pilot the main research tool and expect to obtain data which is reliable and meaningful to the main study.

The pilot study of the learner task was used to check on the following:

- User friendliness to the learners;
- Appropriate language used,
- Layout of diagrams and its aesthetic appeal,
- Ease of marking (scoring) of tasks and the analysis thereof; and
- Searching for and identifying patterns to the learner's responses.

The pilot study assisted in the refining of diagrams used as well as the terminology used.

A key aspect of the pilot study was the kind of data generated through the use of the data collection instruments. Meaningful data in this context related to how learners responded to the given tasks and the geometric reasons which they advanced in support of their responses. The data collected indicated that some alternatives had to be refined and in some cases diagrams as well as the given information had to be refined to address the research questions adequately. The interview schedule was expanded to include the following items from the initial three items:

1) Learners' background, attitude to Euclidean Geometry (original);
2) Learners' perception about Geometry (original);
3) Learners' understanding of proof *(added)*,
4) Learners' "tools" used to solve geometric riders *(added)*, and
5) Learners' understanding of theorems (original).

These five items corresponded with the emerging patterns observed through learner's responses to the tasks, interviews and classroom observation.

As data from the pilot study was analysed it became clear to the researcher that the manner in which learners responded to the tasks given could be catalogued under these emerging patterns (numbered 1-5 above). These patterns would be a reliable source of information as they provided rich data about and answers to the following overarching research questions for the study viz:

1) **How do learners (grade 12 learners') begin to solve or write a proof to a given rider?**
2) **What knowledge and skills do grade 12 learners' employ in order to solve geometric riders?**

In this study, these emerging patterns of the pilot study evolved into key aspects against which data in the main study was analysed and key findings identified.

3.3 The main study

3.3.1 Historical data about the educator and learners' concerned.

The educator has a three year Senior Primary Teacher's Diploma after grade 12 (M+3). The educator has taught in a secondary school for the past 20 years. The educator has taught all grades in a secondary school, i.e. grades 8-12. Despite the teacher's qualifications, the teacher has consistently achieved a 100% pass rate at grade 12 level since teaching grade 12's in 1998. The teacher has been twice nominated for the Gauteng Mathematics, Science and Technology Teacher of the year award for consistently producing excellent results at grade 12 levels.

The learners who participated in the main study were 'doing well' in mathematics according to the teacher. Out of the class group of 27 learners' two learners achieved an E symbol as a year mark and two achieved an F symbol as a year mark. The remaining learners in the class obtained symbols A-D as a year mark (see Table 12). Furthermore 16 of the learners entered to write their end of grade 12-year exams on the higher grade and the remaining 11 entered as standard grade candidates. The class had 18 female learners and 9 male learners. Of the 27 learners, only 24 learners completed the tasks- the other three were absent on the day the tasks were given.

MATHEMATICS TASK : 2004

NAME OF STUDENT:

INSTRUCTIONS:

1. Thank you for agreeing to be part of my research project.

2. This is **NOT** a test.

3. The task is divided into two (2) sections.

   - **SECTION A: Multiple choice items.**

   - **SECTION B: Open ended question.**

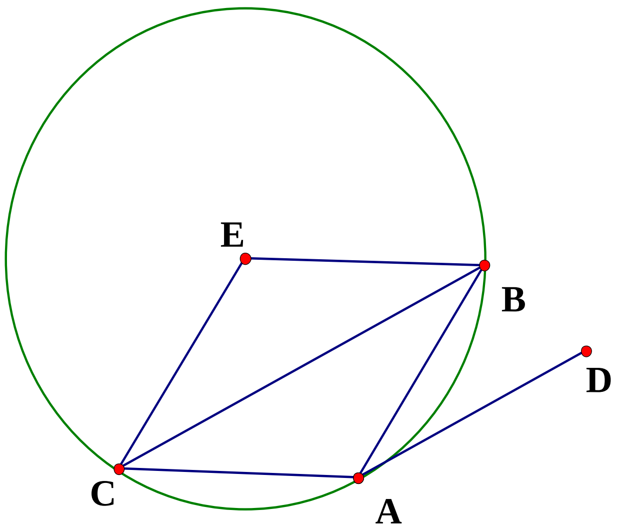
4. Answer **ALL** the questions from both **SECTION A** and **SECTION B**.

5. Show all working details, where necessary, on the blank page opposite each question.

SECTION A:

- Each of the questions has four (4) alternatives to them.
- Circle the letter which you think is the most appropriate response to the question.
- Provide a motivation for your choice to each question on the blank page opposite each question

1. In the given diagram, A, B and C are points on the circumference of the circle. E is the centre of the circle. AD is a tangent to the circle at A. AC and AB are equal chords of the circle. $B\hat{A}D = 30^0$. The size of $C\hat{E}B$ = ...



A) **30^0**

B) **120^0**

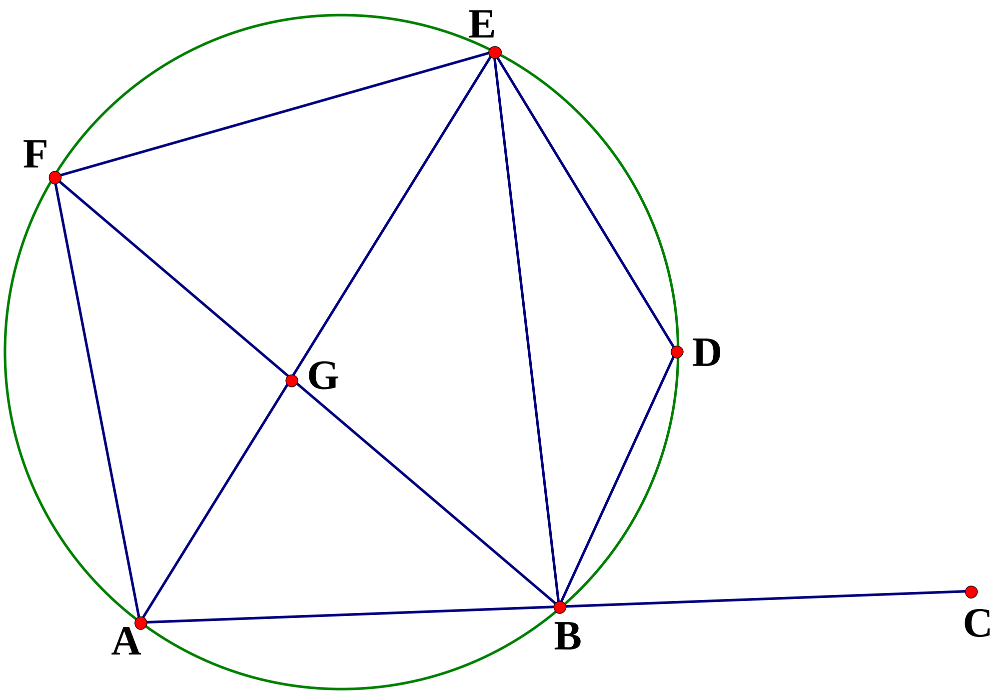
C) **90^0**

D) **60^0**

In order for the learner to be able to solve this rider it is anticipated that the learner will :

- Identify the angle between the tangent and the chord (i.e. angle BAD = 30^0) as been equal to angle BCD;
- Next use the fact that AC = AB to deduce the size of angle BAC =120^0;
- Construct an angle on the major arc of CB which will be equal 60^0;
- Use this fact then to calculate the angle CEB , using the angle at centre theorem ,i.e . angle CEB = 120^0

2. Points A, B, D, E and F lie on the circumference of a circle. $\mathbf{E\hat{B}C = 80^0}$ and $\mathbf{A\hat{E}B = 35^0}$. The magnitude of $\mathbf{E\hat{D}B}$ is ……….



A) **125^0**

B) **100^0**

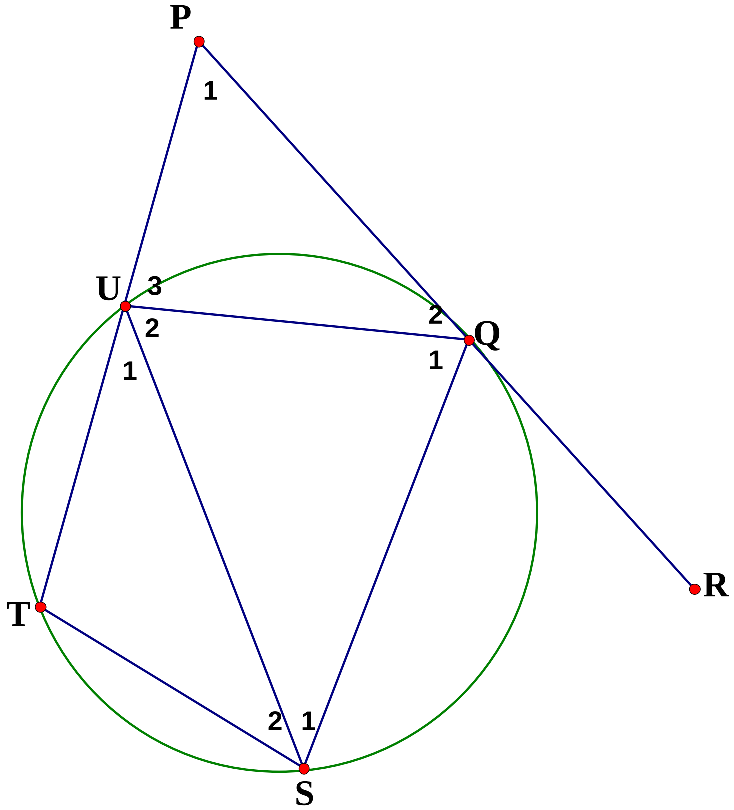
C) **135^0**

D) **45^0**

In order for the learner to be able to solve this rider it is anticipated that the learner will:

- Identify two unique cyclic quadrilaterals, viz. EABD and FABE;
- Next use the fact that angle BEA = angle BFA = 35^0 (angles subtended by chord AB);
- Next they should deduce that angle EBA = AFE = 80^0 (exterior angle of cyclic quad. Equal to interior opposite angle);
- By performing some basic arithmetic i.e. angle AFE – angle AFG = 45^0 and hence angle EDB = $180^0 - 45^0 = 135^0$ (opp.angles of a cyclic quad are supplementary)

3. In the given figure QUTS is a cyclic quadrilateral. PQR is a tangent to the circle at Q. TS = TU ; SU = SQ and TP ∥ SQ. If $\hat{SQR} = x$, which angle is **not** equal to ***x*** .



A) $\hat{P}_1$

B) $\hat{Q}_1$

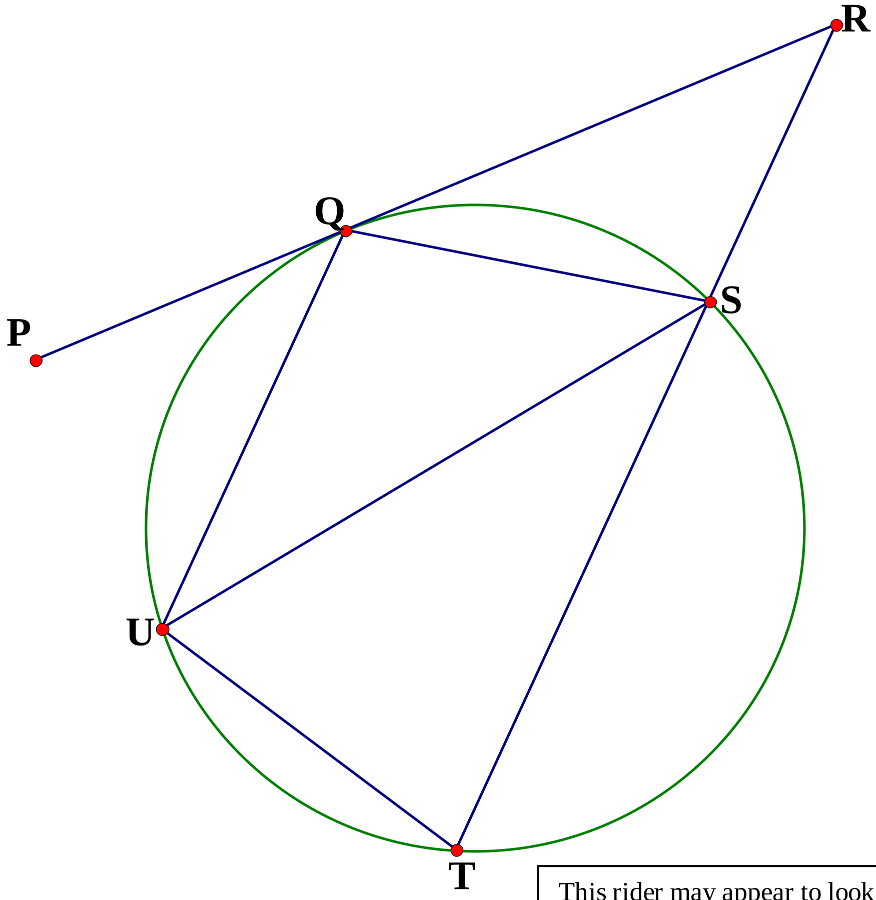
C) $\hat{U}_1$

D) $\hat{S}_1 + \hat{S}_2$

In order for the learner to be able to solve this rider it is anticipated that the learner will:

- Identify the angle between the tangent and the chord, i.e. angle SQR = ***x*** and then the angle in the alternate segment, i.e. angle SUQ;
- Next, using the fact that SU = SQ ,learners can deduce that angle SUQ = angle SQU= ***x***(angles opposite equal sides ***;***
- Next using the fact that TP∥SQ ,learners can identify corresponding angles, viz. angle TPQ and angle SQR;
- Based on the above reasoning then the correct option to choose would be option C.

4. PQR is a tangent to the circle at Q. QU is parallel to RST. UQ = US and UT = TS. If $\mathbf{R\hat{Q}S = \mathit{x}}$, then the value of $\boldsymbol{x}$ is …………..



A) 45^0

B) 36^0

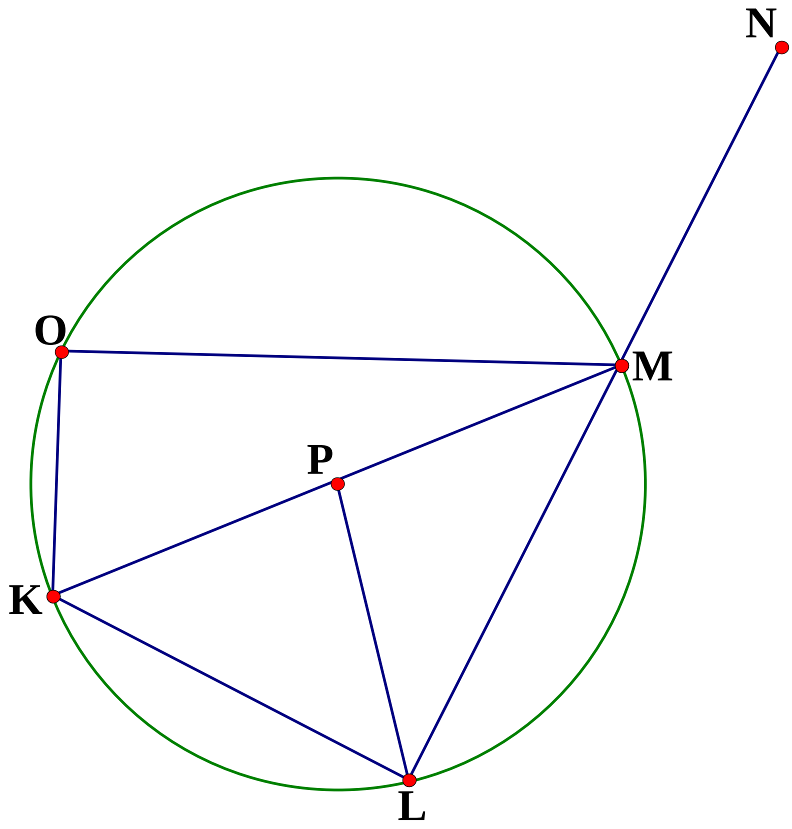
C) Cannot be determined

D) 72^0.

This rider may appear to look like the previous one – but it differs in that the learner is asked to determine the numerical vale of the variable ***x.*** In order for the learner to be able to solve this rider it is anticipated that the learner will:

- Identify the angle between the tangent and the chord, i.e. angle RQS = ***x*** and then the angle in the alternate segment, i.e. angle SUQ;
- Next, using the fact that SU = UQ ,learners can deduce that angle SUQ = angle SQU= ***x***(angles opposite equal sides ***;***
- Next using the fact that TS$\|$UQ ,learners can identify a pair of alternate angles, viz. angle QUS and angle UST;
- Learners can then express angle UTS as $180^0 - 2x$ (remaining angle in ΔUTS);
- Deduce that angle UQS = 2***x*** (opp.angles of a cyclic quad. are supplementary);
- By working with the sum of interior angles of ΔUQS, learners can then deduce that x = 36^0, which is option B.

5. KLMO is a cyclic quadrilateral of a circle with centre P. $N\hat{M}O = 120^0$ and $L\hat{P}M = 80^0$. Which one of the following statements is **FALSE?**



A) OM = ML

B) $M\hat{K}L$ and $K\hat{M}L$ are complementary angles.

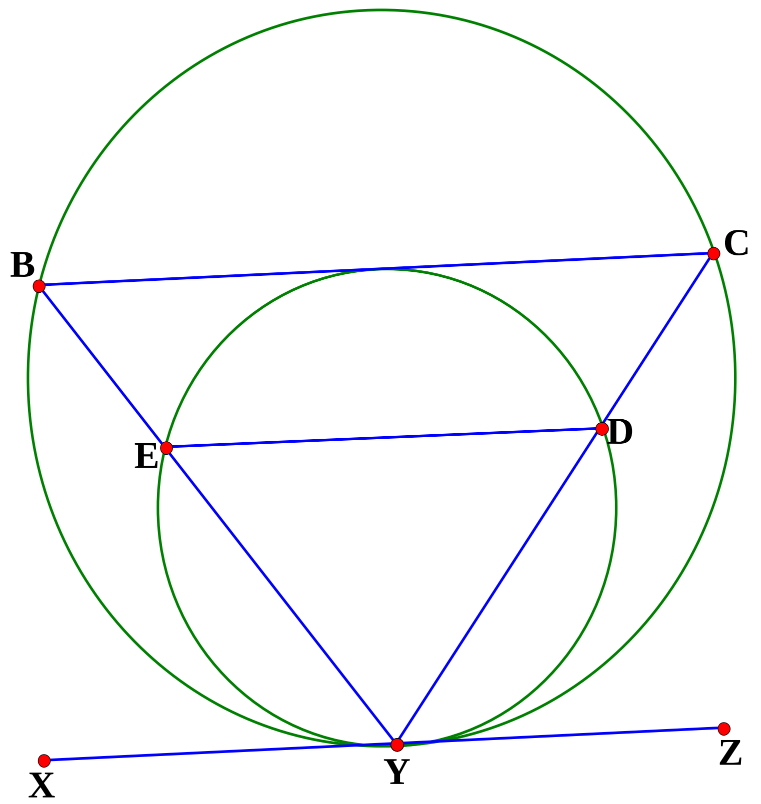
C) $M\hat{P}L = O\hat{K}M$

D) $4\ O\hat{M}K = P\hat{K}L$

This rider is different from the previous one in that the learners are asked to identify the FALSE statement. In this task the term "complementary angles" is used. It is not included to "derail" learners but rather to assess whether they are familiar with terms such as "complementary angles" , which mean that angles add up to 90^0. In order for the learners to be able to solve this rider it is anticipated that the learner will:

- Identify the exterior angle of the cyclic quadrilateral angle NMO = angle OPL (exterior angle of cyclic quad. = interior opp. angle);
- Next, using the fact that P is the centre of the circle then PM = PL = KP = PL (radii);
- Using the above fact, learners can then deduce that angle PLM = angle PML = 50^0. Similarly it can be shown that angle PKL = angle PLK = 40^0
- Base don the above reasoning learners can then deduce that option A is definitely FALSE.

6. XYZ is a common tangent to the two circles. With respect to the given diagram which of the given statements is **TRUE?**



A) Δ **BYC /// Δ DYE**

B) $\frac{BC}{DE} = \frac{BY}{BE}$

C) **ED ‖ XZ**

D) $\frac{BY}{EY} = \frac{CY}{YD}$

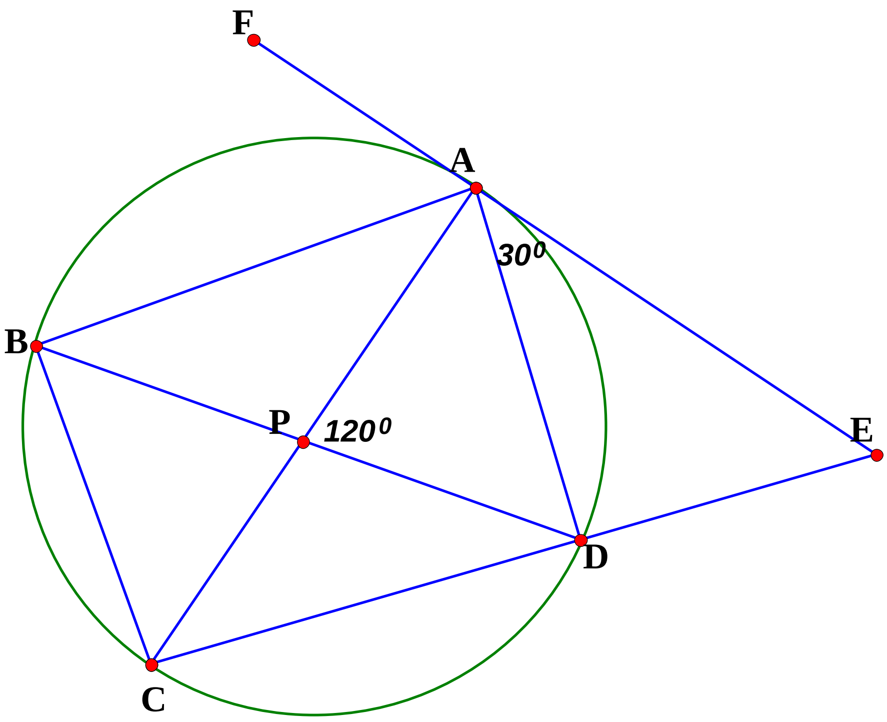
This rider is different from the previous one in that the learners are asked to identify the TRUE statement. Whilst all four alternatives look TRUE at first glance, this may be misleading.. In order for the learners to be able to solve this rider it is anticipated that the learner will:

- Identify the angle between the tangent and the chord relative to the larger circle ,i.e. angle XYB equal to angle YCB (angle in alternate segment),and to angle EDY in the smaller circle. Similarly in relation to the smaller circle the angle between the tangent and chord angle ZYD is equal to angle YED in the smaller circle and to angle YBC in the larger circle;
- Based on the above learners will deduce BC // ED because the re appears to be a pair of corresponding angles equal, which would then make option D the correct one.
- Options A , B and C are incorrect because: Option the order of the triangles are not written properly; option B is not valid deduction and option C looks like something they have dealt with previously.

7. AE is a tangent to the circle at A. CDE is a straight line.

$\mathbf{D\hat{A}E = 30^0}$ and $\mathbf{A\hat{P}D = 120^0}$. The size of $\mathbf{A\hat{D}P}$ is

____



A) 60^0

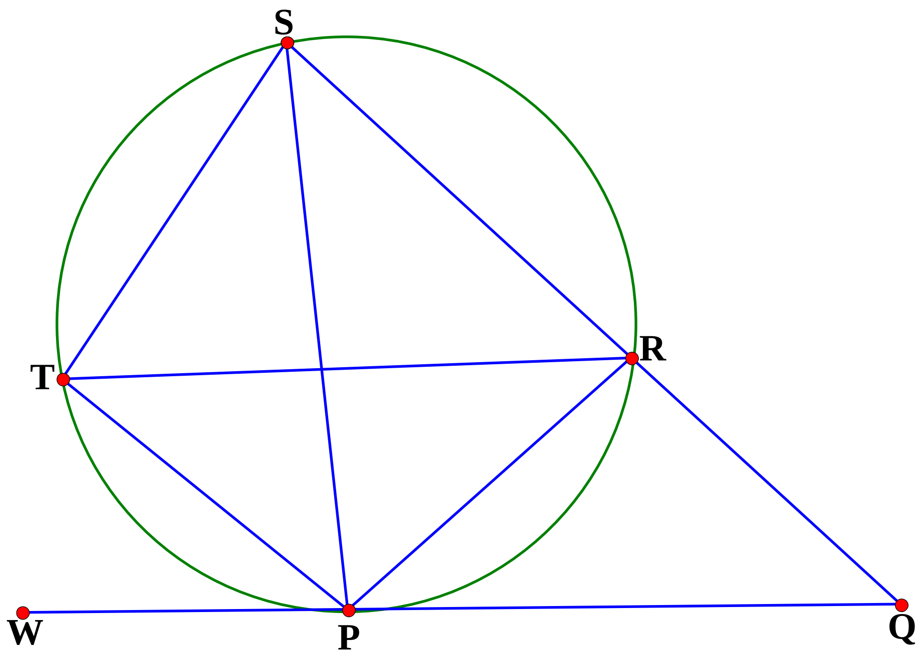
B) 90^0

C) Cannot be determined.

D) 30^0

- This diagram was deliberately distorted by the researcher. The learners were not aware of the fact that the diagram was distorted. The reason for the distortion was to ascertain whether learners could visualise that the measures for the given angles are not applicable to the given diagram. It is anticipated that learners would go for option C because of the nature of the diagram and the values given. This rider wants to explore the “myth” that learners at school believe that there should be a solution to every given rider, especially when numeric values are given.

8. WPQ is a tangent to the circle at P, which meets SR produced in Q. TR $\parallel$ PQ. If ST = 4,5 and SQ = 18 then the length of **SP** **is**_________________



A) **2**

B) $\sqrt{22,5}$

C) $\sqrt{(4,5)^2 + 18^2}$

D) **9**

LEARNERS WERE ASKED NOT TO ANSWER THIS RIDER BECAUSE OF TIME CONSTRAINTS.

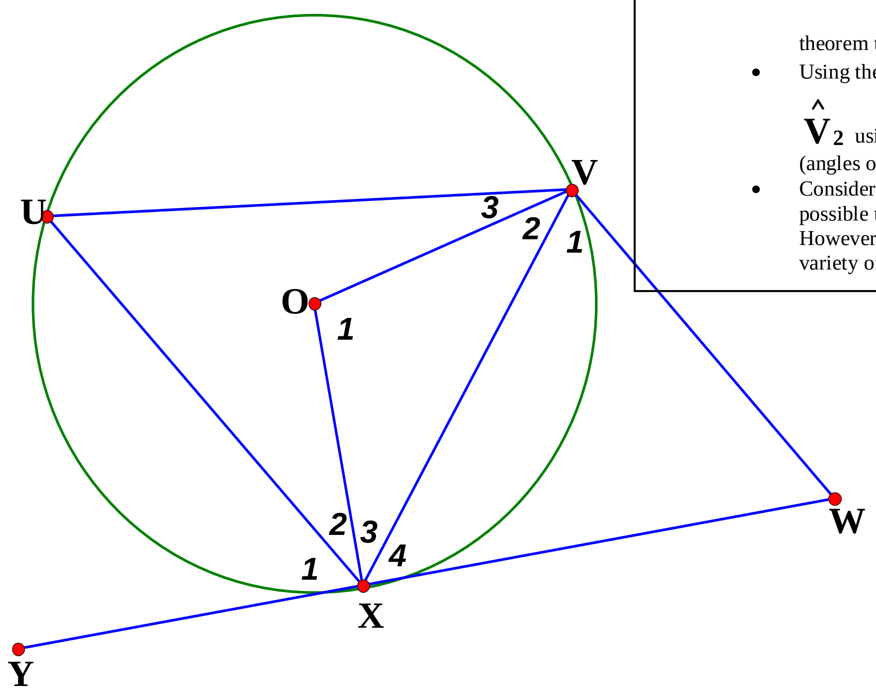
SECTION B

Show all working details on the blank page opposite this one.

1.In the given diagram O is the centre of the circle. $V\hat{O}X = 70^0$. Points U, V and X lie on the circle. Calculate the sizes of the following:

1.1) $\hat{U}$ **1.2)** $\hat{V}_2$ **1.3)** $X\hat{W}V$

- For 1.1. learners had to apply the angle at centre theorem to calculate $\hat{U}$;
- Using the above result learners could calculate $\hat{V}_2$ using the fact the angle V_2 = angle X_3 (angles opp. equal sides) . thus angle $V_2 = 55^0$;
- Considering the limited information given, its not possible to calculate the size of angle XWV. However learners were able to come up with a variety of "solutions" to this question.



THE END

THANK YOU

Chapter 4: Data Analysis and findings

4.1 Analysis of learners responses

Coding process

To safeguard the identity of the participants (learners and teacher) involved, codes were used when referring to either party. The teacher was referred to as ***Teacher***, whilst the learners' responses were coded **L1 –L24**. The **"L"** refers to learner and the numbers **1-24** is the order in which the researcher collected the response sheets from the learners. The responses of the 24 learners were then catagorised and analysed according to the following broad categories.

Table 10: Categories of learner's responses.

Category number	Category
C1	Correct response with **valid reasons** and working shown.
C2	Correct response with **some valid** reasons and working shown.
C3	Correct response with **no valid** reasons and working shown
C4	Incorrect response with **valid reasons** and working shown.
C5	Incorrect response with **some valid** reasons and working shown
C6	Incorrect response with **no valid** reasons and working shown
C7	No response but **some attempt** made to solve question
C8	No response, with **no attempt** to solve question.

Table 11 (below), shows each learner's response to the given task. Table 11 also provides some historical information regarding each learner who completed the task.

The above categories were formulated prior to the learners responding to the tasks. Based on my experience as a maker at grade 12 level as well as my classroom teaching experience, the above categorise were developed. Often learners would provide a correct response for instance without showing any working or reasoning as to how he/she arrived at a solution.

Table 11: Learners responses according to categories outlined in Table 10

Learners	Gender	Level	Year mark	SECTION A								SECTION B		
				1	2	3	4	5	6	7	8	1.1	1.2	1.3
L1	F	H	C	C2	C1	C1	C5	C1	C5	C1	NO RESPONSE	C1	C1	C3
L2	F	S	A	C4	C1	C5	C5	C1	C1	C1		C4	C1	C1
L3	M	S	A	C6	C6	C6	C5	C1	C5	C6		C1	C1	C2
L4	M	H	C	C1	C1	C1	C5	C1	C1	C5		C1	C1	C5
L5	F	H	B	C1	C1	C2	C1	C5	C1	C1		C1	C1	C6
L6	F	H	F	C2	C5	C5	C5	C5	C3	C5		C1	C1	C7
L7	F	H	A	C1	C2	C2	C5	C1	C1	C1		C1	C1	C4
L8	F	S	E	C7	C6	C5	C5	C6	C5	C7		C1	C3	C6
L9	F	H	C	C1	C7	C5	C5	C6	C6	C7		C1	C6	C8
L10	F	H	D	C6	C5	C1	C5	C1	C6	C1		C1	C1	C5
L11	F	H	C	C2	C3	C5	C5	C2	C5	C5		C1	C1	C3
L12	F	S	B	C6	C6	C5	C5	C5	C5	C5		C1	C1	C8
L13	M	H	C	C1	C1	C1	C1	C1	C5	C1		C1	C1	C6
L14	M	S	F	C3	C5	C6	C1	C2	C3	C2		C1	C1	C6
L15	M	H	B	C2	C1	C1	C1	C1	C5	C2		C1	C1	C6
L16	M	S	D	C2	C2	C3	C5	C2	C2	C5		C1	C1	C8
L17	M	S	E	C2	C2	C2	C5	C5	C2	C2		C1	C1	C5
L18	M	S	B	C2	C1	C5	C6	C6	C6	C3		C1	C1	C8
L19	F	H	D	C5	C2	C6	C6	C5	C6	C6		C1	C1	C8
L20	F	S	A	C1	C1	C5	C6	C1	C1	C1		C1	C1	C6
L21	F	H	B	C1	C1	C5	C6	C1	C1	C1		C1	C1	C6
L22	F	H	B	C1	C7	C1	C6	C6	C1	C1		C1	C1	C8
L23	F	H	C	C1	C1	C6	C6	C2	C1	C6		C1	C1	C8
L24	F	S	D	C2	C7	C5	C6	C6	C5	C6		C8	C8	C8

[2]Codes used in above table (See footnote)

The next stage in the classification process was to classify how the learners responded to each of the questions in Section A of the activity. Table 12, below reflects how the learners responded to each of the questions in Section A.

[2] Codes used in Table 11: Level refers to the grade on which learners have entered for the end of year exam (i.e. either higher or standard grade), F = female learner , M = male learner; H = higher grade ; S = Standard grade

Table 12: Learner's response to tasks in Section A

	Number of learners who selected… as a response					
Question No.	**A**	**B**	**C**	**D**	**No Response**	**Total**
1	3 (12, 5%)	16 (66, 67%)	0 (0%)	3 (12, 5%)	2 (8, 33%)	**24 (100%)**
2	0 (0%)	2 (8, 33%)	14 (58, 33%)	3 (12, 5%)	5 (20, 83%)	**24 (100%)**
3	3 (12, 5%)	0 (0%)	11 (45, 83%)	10 (41, 67%)	0 (0%)	**24 (100%)**
4	6 (25%)	4 (16, 67%)	14 (58, 33%)	0 (0%)	0 (0%)	**24 (100%)**
5	14 (58, 33%)	6 (25%)	1 (4, 17%)	3 (12, 5%)	0 (0%)	**24 (100%)**
6	9 (37, 5%)	0 (0%)	3 (12, 5%)	12 (50%)	0 (0%)	**24 (100%)**
7	0 (0%)	2 (8, 33%)	12 (50%)	7 (29, 17%)	3 (12, 5%)	**24 (100%)**
8	**NO RESPONSE BY ALL LEARNERS**					**24**

[3]Cells shaded

The above table (Table 12) indicates at a glance the number of learners who selected each of the given responses (A-D) as their choice as well those who have not provided any response to the given task. Whilst the tables 11-13 are rich quantitatively, they need to be unpacked further so that it can become intelligible to the general readership.

Question by question analysis of learners' responses

What follows is a selection of some learner's responses to the questions in both Section A and Section B of the task. The sample responses are from learners who have provided both correct and incorrect responses. The rationale was to provide a spectrum of responses as well as learners' reasoning ability when solving the tasks.

[3] Shaded in cells in Table 12 refer to the correct solution(s) for a particular question in Section A. For 3 both options C and D are valid options- hence both are shaded .

Question 1 (Section A)

1. In the given diagram (on the right), A, B and C are points on the circumference of the circle. E is the centre of the circle. AD is a tangent to the circle at A. AC and AB are equal chords of the circle. $B\hat{A}D = 30^0$. The size of $C\hat{E}B$ is…………….

   A) **30^0**

   B) **120^0**

   C) **90^0**

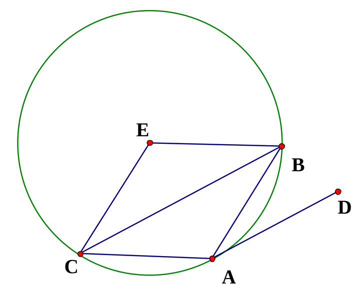


Table 13: A summary of learners' responses to Question 1

	Number of learners who selected… as a response					
	A	**B**	**C**	**D**	**NO Response**	**Total**
No. of learners	3	**16**	0	3	2	**24**
% of learners	12,5%	**66,67%**	0	12,5%	8,3%	**100%**

The above table is a summary of how learners responded to the given question.

One of the learners, L24 supplied the following proof[4] to the given task.

Table 14: L24's response to Question 1

Line	Statement	Reason
1	$\hat{E} = 180^o - 30^o$	Opp angles of cyclic quad
2	$= 150^o$	
3	$\hat{F} = 300^o$	(angle at centre = 2 X angle at circle)
4	$360^o - 300^o = \frac{60^o}{2}$	
5	$\hat{E} = 30^o$	

[4] The learner (and others) has not supplied their proofs in tabular form. The researcher has done so because of convenience only.

When viewing L24's response closer the following aspects are worth noting:

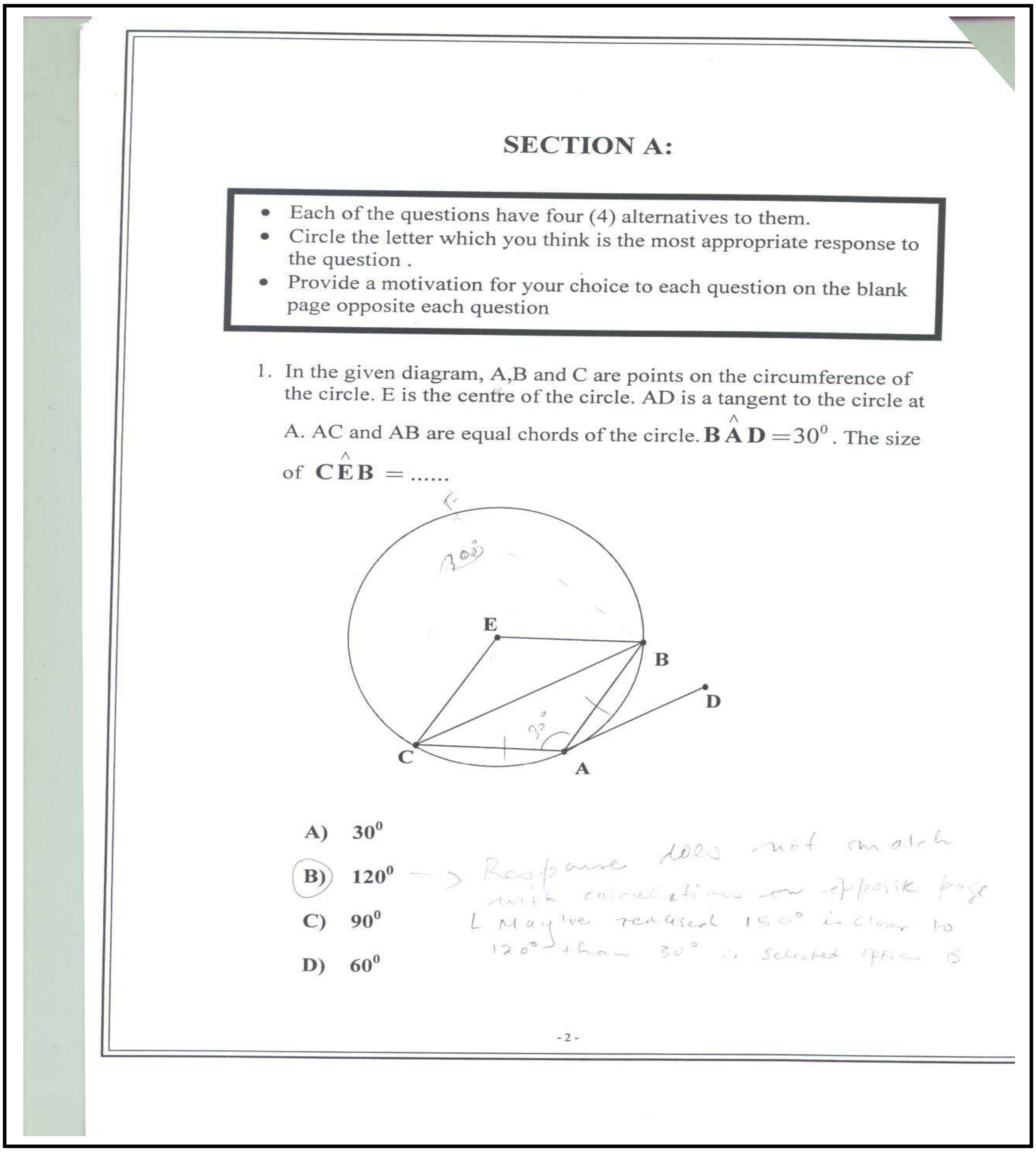
The learner has indicated $\mathbf{C\hat{A}B}$, instead $\mathbf{B\hat{A}D}$ as 30^{o};

a) The learner proceeds to calculate $\mathbf{\hat{E}}$ using that fact that ECAB is acyclic quadrilateral *(according to the learner)*;

b) The learner concludes *(see calculations above)* that $\mathbf{\hat{E}}$ = 30^{o}, but circles option B *(the correct option)* as a response to task 1.

The learner's initial response *(see line 2 above)*, of 150^{o} is closer to the 120^{o} than the other options supplied. Furthermore, both 120^{o} and 150^{o} are three digit numbers and 120^{o} is 30^{o} less than 150^{o}, hence the choice of 120^{o}, despite the working reflecting something else.

Furthermore, L24 has erroneously identified ECAB as a cyclic quadrilateral, because three (C, A and B) of the four vertices lie on the circle. This is a distortion of the definition of a cyclic quadrilateral which states that a "cyclic quadrilateral is a quadrilateral of which the vertices lie on a circle" (Laridon et al, 1995: 277). What needs to be emphasized in this definition is that **ALL four vertices** must lie on a circle. L24's misrepresentation of the definition of a cyclic quadrilateral can be explained in terms of Movshovitz-Hadar, Inbar and Zaslavsky's (1986) notion of "distortions of the consequent" (p.34), in which the original condition is maintained or the original condition is slightly modified to fit in with the learners' view at the time.
Below is L24's diagram used to solve the given task.



SECTION A:

- Each of the questions have four (4) alternatives to them.
- Circle the letter which you think is the most appropriate response to the question .
- Provide a motivation for your choice to each question on the blank page opposite each question

1. In the given diagram, A,B and C are points on the circumference of the circle. E is the centre of the circle. AD is a tangent to the circle at A. AC and AB are equal chords of the circle. $B\hat{A}D = 30^0$. The size of $C\hat{E}B = \ldots\ldots$

A) 30^0

B) 120^0

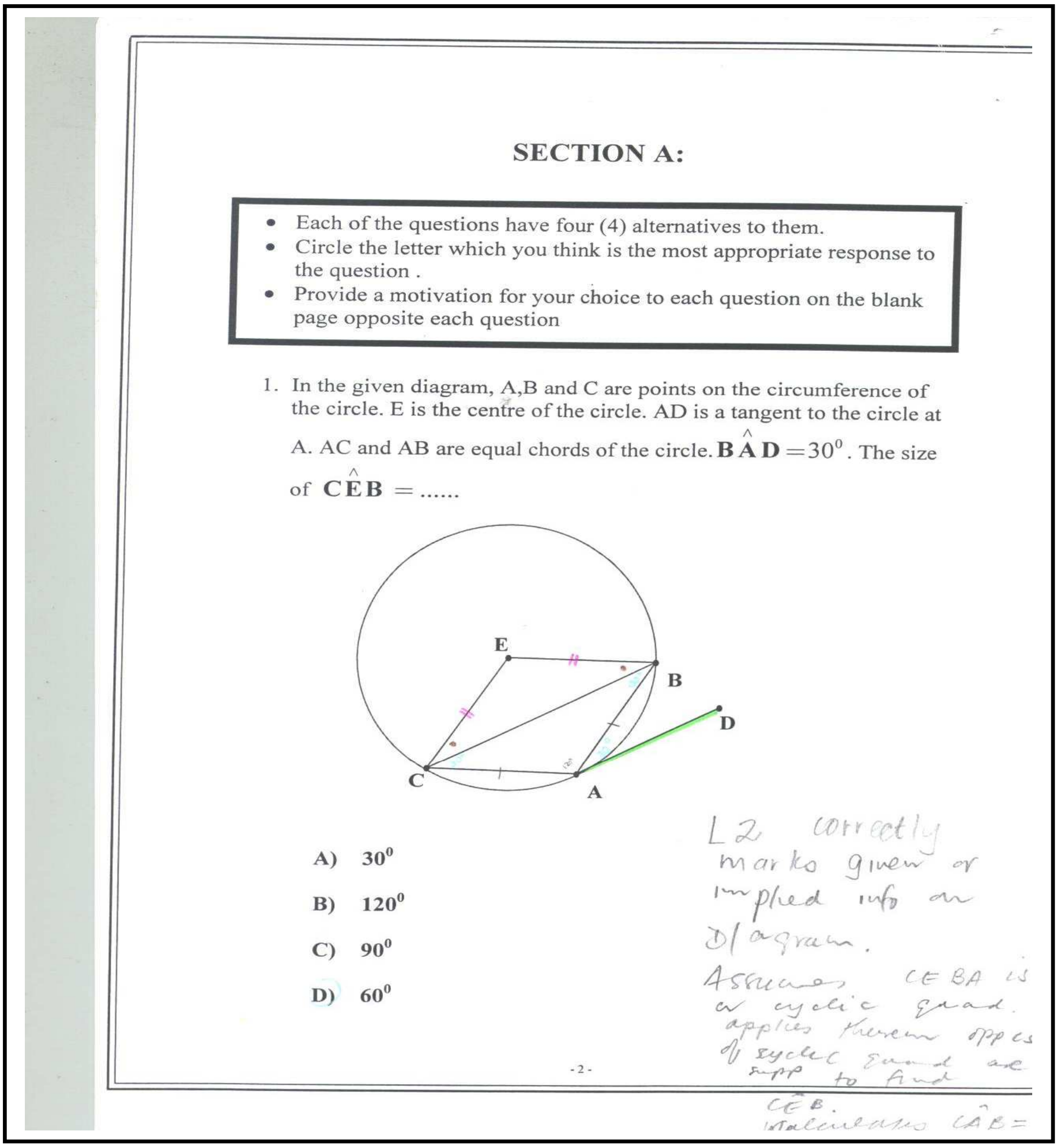
C) 90^0

D) 60^0

-2-

Figure 9: Learner 24's diagram used. Note markings on the diagram which informed her choice of answer.

Another learner, L2 (see Figure 10 below) interprets the given information correctly, i.e. identifies the angle between the tangent and the chord equal to the angle in the alternate segment, radii (CE =EB) as well as angles opposite the equal sides, viz $E\hat{C}B = E\hat{B}C$. Once the learner calculated (determined) the size of $C\hat{A}B$ as 120^o the learner erroneously identified ECAB as a cyclic quadrilateral and hence deduced that $C\hat{E}B = 60^o$. In this case the learner used (applied) the theorem "if a quadrilateral is



SECTION A:

- Each of the questions have four (4) alternatives to them.
- Circle the letter which you think is the most appropriate response to the question .
- Provide a motivation for your choice to each question on the blank page opposite each question

1. In the given diagram, A,B and C are points on the circumference of the circle. E is the centre of the circle. AD is a tangent to the circle at A. AC and AB are equal chords of the circle. $B\hat{A}D = 30^0$. The size of $C\hat{E}B = ...$

A) 30^0

B) 120^0

C) 90^0

D) 60^0

L2 correctly marks given or implied info on Diagram. Assumes CEBA is a cyclic quad. applies theorem opp ∠s of cyclic quad are supp to find $C\hat{E}B$. Calculates $C\hat{A}B$ =

- 2 -

Figure 10: The diagram used by L2 to answer Question 1. Note the markings used on the diagram.

Another learner, L3, proceeded to determine the size of $C\hat{E}B$ using the tangent-chord theorem. However, $C\hat{E}B$ is not the angle in the alternate segment, although AB is a chord of the given circle. Once again Movshovitz-Hadar et al's (1986) notion of "distortions of the consequent" (p.34) is applicable to this learner's thinking.

Question 2 (Section A)

Points A, B, D, E and F lie on the circumference of a circle. $E\hat{B}C = 80^0$ and $A\hat{E}B = 35^0$. The magnitude of $E\hat{D}B$ is ……….

A) **125^0**

B) **100^0**

C) **135^0**

D) **45^0**

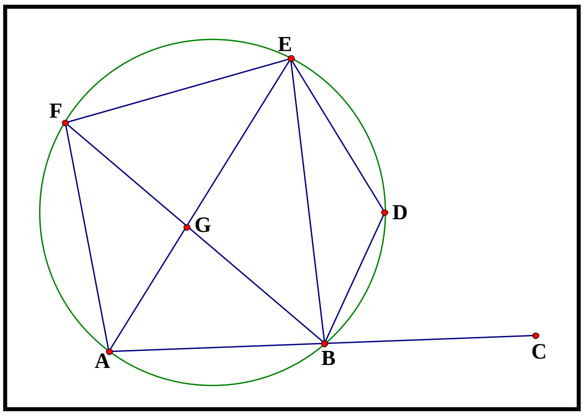


Table 15: A summary of learners' response to Question 2

	Number of learners who selected…… as a response					
	A	**B**	**C**	**D**	**NO Response**	**Total**
No. learners	0	2	**14**	3	5	**24**
% of learners	0%	8,33%	**58,33%**	12,5%	20,83%	**100%**

In this question the learner's understanding of cyclic quadrilateral theorems was assessed. From the above table we can see that 58, 33% (14/24) of the learners were able to provide the correct response to the question posed, while 41, 67% (10/24) of the learners were unable to do so.

Learners who selected option D as a response argued along similar lines to L8 whose proof is given below.

Table 16: L 8's response to Question 2 of Section A

Line no.	Statement	Reason
1	$\hat{F}_1 + \hat{F}_2 = 80^o$	Ext angle of cyclic quadrilateral
2	$\hat{F}_2 = 45^o$	

The proof, which L8 provided above, assisted L8 to determine the size of $E\hat{D}B$. However, L8 then applied properties of other quadrilaterals like squares or rhombus in which the opposite angles are equal to determine the size of $E\hat{D}B$. The learner failed to realize that the properties of squares or rhombus would not as a rule of thumb apply to all quadrilaterals.

L10, whilst attempting to come up with a solution, failed to do so. Judging from Table 11, L10 is an "average" Mathematics higher-grade student, who is expected to cope with the type of riders provided in the task. However, L 10's response "*I don't know how 2 do it!!! Sorry- I gave up!!"*(emphasis added) seems to suggest that he lacks confidence and the necessary skills and know-how to do the task. The learner is able to identify and apply theorems relating to cyclic quadrilaterals (for example angles in the same segment), but lacks the geometric eye to differentiate between an interior and exterior angle of a cyclic quadrilateral.

In this question, learners were given five points, which lie, on the circle. These five points are then joined (*see given sketch above*) and three cyclic quadrilaterals are formed. Learners' were expected to indicate the given information on the sketch, which would have enhanced their ability to solve the given rider successfully. By successfully indicating the given information the learner's "geometric eye" (Godfrey as cited in Fujita and Jones, 2002:384) would have come into play. The learner's ability to "see" a solution is enhanced when the geometric eye is well developed, i.e. when learners are able to visualize how "geometrical properties detach themselves from a figure" (Godfrey in Fujita and Jones, 2002: 385).

Question 3 (Section A)

In the given figure QUTS is a cyclic quadrilateral. PQR is a tangent to the circle at Q. TS = TU ; SU = SQ and TP ∥ SQ. If

$S\hat{Q}R = x$, which angle is **not** equal to x.

A) $\hat{P}_1$

B) $\hat{Q}_1$

C $\hat{U}_1$

D) $\hat{S}_1 + \hat{S}_2$

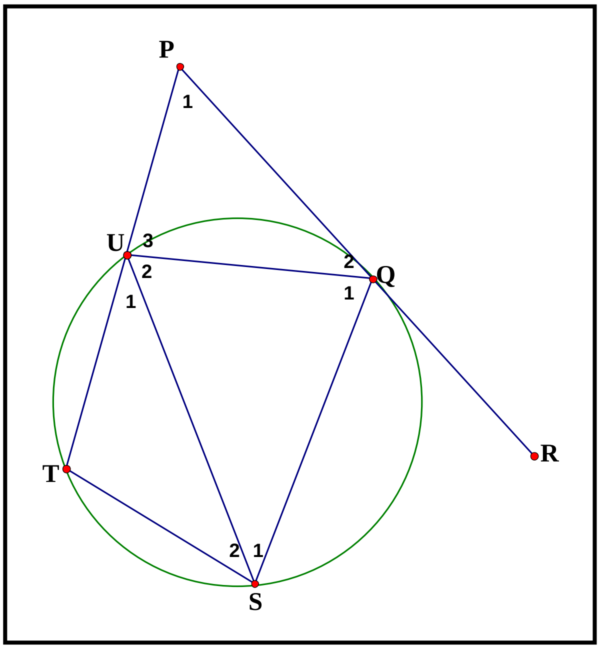


Table 17: Summary of learners' responses to Question 3

	Number of learners who selected…… as a response					
	A	**B**	**C**	**D**	**NO Response**	**Total**
No. learners	3	0	**11**	**10**	0	24
% of learners	12,5%	0%	**45,83%**	**41,67%**	0%	100%

In this question both options **C** and **D** were regarded, as correct responses to the given task.

In this question learners had to apply their knowledge of previously acquired geometric concepts to identify angle(s) not equal to a given angle. Whilst the task may have been straightforward, learners' incorrectly read SU = UQ instead of SU = SQ as was given. L6 for instance committed this type of error. As a result of this incorrect labelling the learner arrives at a correct deduction response to the given question.

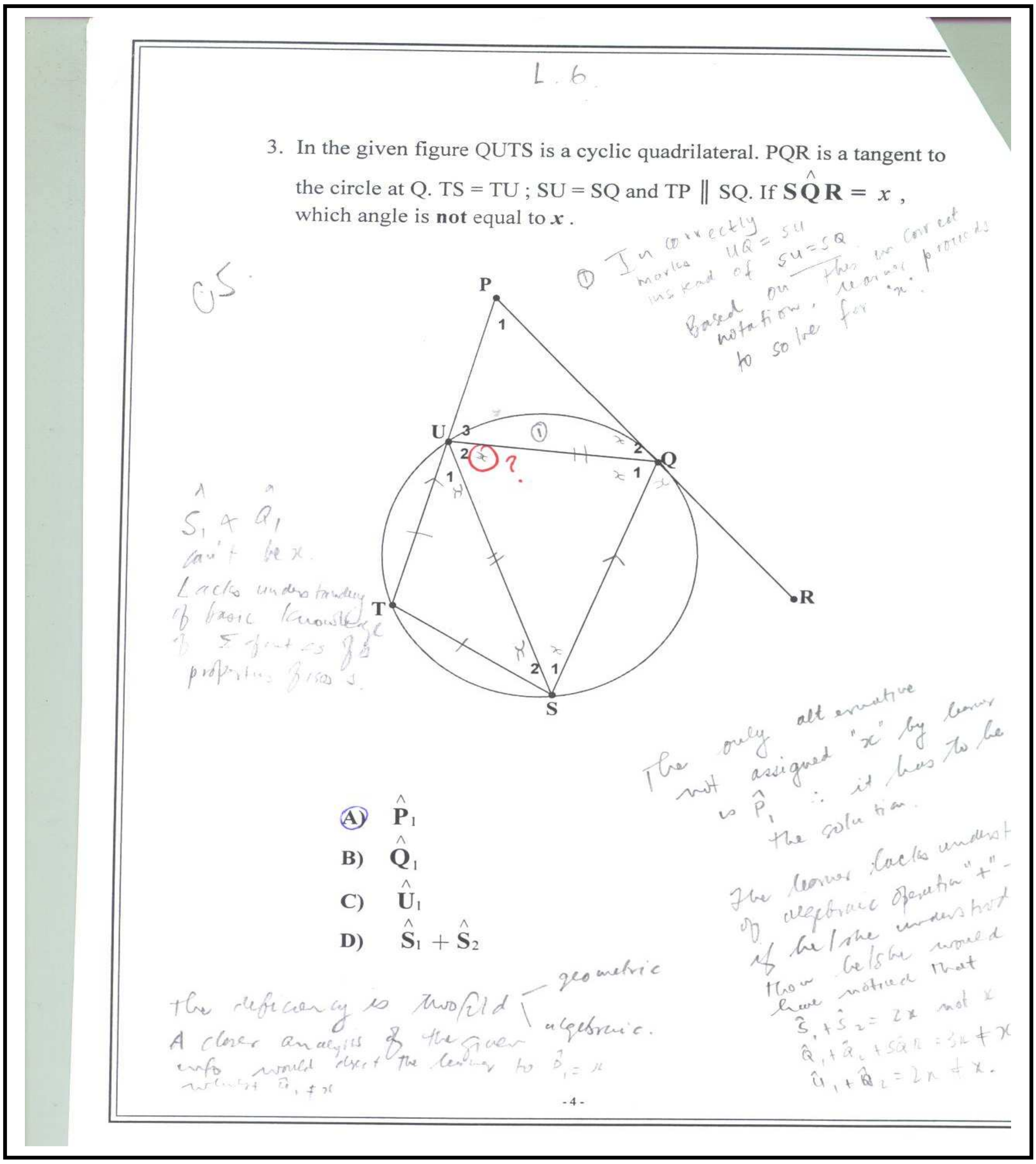


Figure 11: The diagram L 6's used to solve Question 3. Note the markings on the diagram made by the learner.

Question 4 (Section A)

PQR is a tangent to the circle at Q. QU is parallel to RST. UQ = US and UT = TS.

If $R\hat{Q}S = x$, then the value of x is

A) **45^0**

B) **36^0**

C) **Cannot be determined**

D) **72^0.**

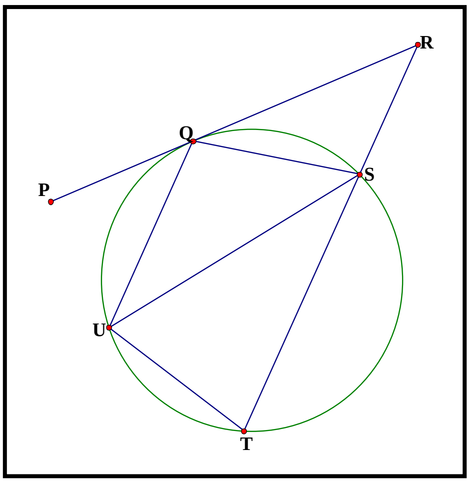


Table 18: Summary of learners' response to Question 4

	Number of learners who selected…… as a response					
	A	**B**	**C**	**D**	**No Response**	**Total**
No. learners	6	4	**14**	0	0	**24**
% of learners	25%	16,67%	**58,33%**	0%	0%	**100%**

In this question the learners knowledge on tangents and cyclic quadrilaterals was integrated with basic geometric facts established in previous grades. In this question learners' were expected to determine the numeric value of the variable $\boldsymbol{x}$. From the above table we can see that 25% (6/24) of learners opted for option A and 58, 3% (14/24) opted for option C as their respective responses to the question.

Learners were expected to formulate a linear equation of the type $5\boldsymbol{p} = 10$ and hence solve for $\boldsymbol{p}$. However the learners may be familiar in doing that when working with algebraic equations but they are not always expected to do likewise when solving geometric problems. A typical response to the given question, which demonstrates the learner's lack of ability to formulate an equation, is supplied below.

Table 19: The proof provided by L12 to Question 4

Line no.	Statement	Reason
1	$\hat{Q}$ = x	
2	x = U	Tan-chord theorem
3	But: U = S = x	Angles opp equal sides
4	U = S = x	Alt. angles QU // RST
5	T = x	Ext angle = int.opp angle
6	Q = 180^o – 2x	Opp angles of cyclic quad supp
7	180^o = 2x	
8	2x = 180^o	
9	x = 90^o ÷2	
10	= 45^o	

The above proof is an attempt by L12 to answer the given question. In line 5, L12 has incorrectly identified R$\hat{Q}$S as an exterior angle of a cyclic quadrilateral QUTS. In terms of the van Hiele model of thinking, L12 is functioning at an inappropriate level for the current grade. Learners at senior secondary level (i.e. at grade 10-12 level) are expected to be functioning at least at level 3 (i.e. informal deduction). However, L12 seems to be functioning at level 1, since he has not yet developed the competency to make sense of the relationships between properties, and "interrelationships between figures are still not seen, and definitions are not yet understood" (Crowley, 1987:2).

Besides L 12's inability to make sense of theorems, L12 attempts to solve the problem using his algebraic knowledge of equations to solve the given rider. L12 recalls solving equations of the type 5***p*** – 10 = 0 and tries to use that knowledge to solve the given task. L12 knows that line 6 ($\hat{Q}$ = 180^o – 2***x***) needs to be rearranged to look like 5***p*** –10 = 0. Line 7 resembles this type of mental activity the learner is engaged in. The mathematical rules used in lines 8 -9 are true, although line 7 is mathematically flawed. The learner's response in line 9, whilst correct, undergoes a further "division" process because the learner's response of 90^o is not on the list of answers provided. Hence, the further division by 2 to match the response of 45^0, which is option A.

Another learner, L10 concluded that the value of ***x*** cannot be determined because "NO values are given", hence he selected option C.

For this learner in order to determine the value of ***x*** some other definite value(s) had to be provided. This would have facilitated the determining of the value of the variable ***x.*** Another learner, L2, who also selected option C, provided the following response.

Table 20: A response to Question 4 by L2

Line No.	Statement	Reason
1	$180^o - (180^o – 2x + x +x)$	
2	$= 180^o$	
3	= 0	

This response by one of the better achieving learners in the class (See Table 11) indicates that learners' only possess procedural knowledge on how to solve geometric riders. They lack the ability to apply knowledge gained to new situations in order to arrive at a valid and plausible solution.

Both L2's and L12's responses can be explained in terms of Bernstein's (1996) recognition and realization rules. Cooper and Dunne (2000), explain Bernstein's twin concepts of recognition and realization as follows: "Recognition rules, 'at the level of the acquirer' are the means by 'which individuals are able to recognize the specialty of the context that they are in'. Realization rules allow the production of 'legitimate text'" (Bernstein, 1996 in Cooper & Dunne, 2000:48). Whilst both learners were able to recognize the given information relating to tangent, and cyclic quadrilaterals, etc., they failed to produce the "legitimate text" (Cooper & Dunne, 2000:48), i.e. appropriate response.

Question 5 (Section A)

KLMO is a cyclic quadrilateral of a circle with centre P. $N\hat{M}O = 120^0$ and $L\hat{P}M = 80^0$. Which one of the following statements is **FALSE?**

A) OM = ML

B) $M\hat{K}L$ **and** $K\hat{M}L$ are complementary angles.

C) $M\hat{P}L = O\hat{K}M$

D) $O\hat{M}K = P\hat{K}L$

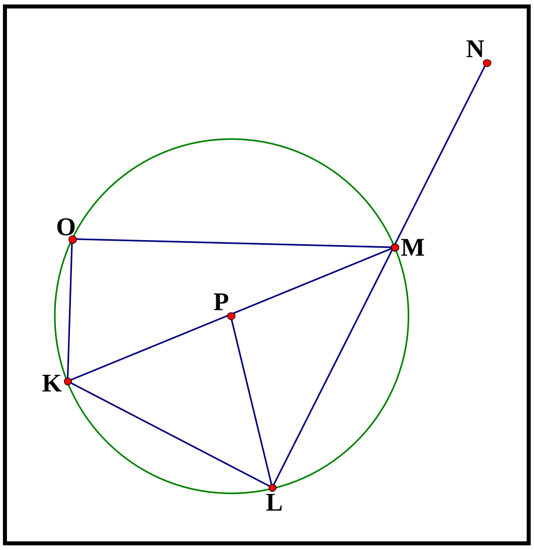


Table 21: Summary of learners' responses to Question 5

	Number of learners who selected…… as a response					
	A	**B**	**C**	**D**	**No Response**	**Total**
No. learners	**14**	6	1	3	0	**24**
% of learners	**58,33%**	25%	4,2%	12,5%	0%	**100%**

In this question learners had to identify using the given information, the statement that is FALSE. From the above table, we can deduce that more than 50% (14/24) of the learners were able to select the correct alternative, i.e. option A (OM = ML). Learners' could have "guessed" the response to be A , which would have been an educated "guess". Learners' could have measured OM and ML and found that they are not equal and deduce accordingly that that statement is FALSE.

Those learners' who had selected option B, could have done so after supplying a proof similar to the one supplied by L8 below:

Table 22: A response to Question 5 by L8

Line no.	Statement	Reason
1	$L\hat{P}M = 80^o$	Given
2	$\hat{K} = 40^o$	Angle at centre = 2X angle at circle
3	$L\hat{P}M = 100^o$	Angles on a straight line
4	$L\hat{M}P = 50^o$	Angle at centre = 2X angle at circle

Learners' such as L8, who have opted for option B, have not mastered the linguistic demands associated with Euclidean Geometry yet. The concept of complementary angles though not frequently encountered refers to angles whose sum adds up to 90^o. It is a concept they have encountered during their study of Trigonometry mainly.

Question 6 (Section A)

XYZ is a common tangent to the two circles. With respect to the given diagram which of the given statements is **TRUE?**

A) Δ **BYC** /// Δ **DYE**

B) $\frac{BC}{DE} = \frac{BY}{BE}$

C) **ED** $\parallel$ **XZ**

D) $\frac{BY}{EY} = \frac{CY}{YD}$

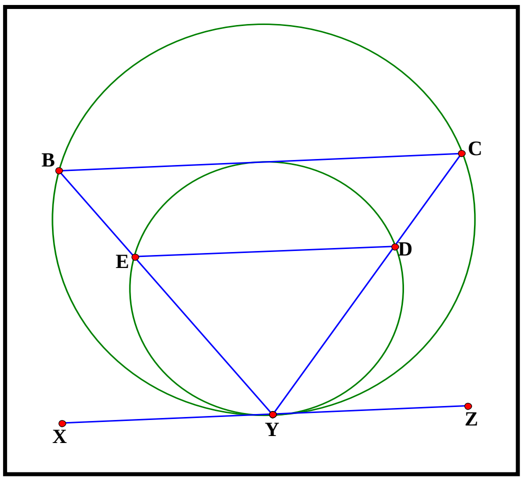


Table 23: Summary of learners' responses to Question 6

	Number of learners who selected…… as a response					
	A	**B**	**C**	**D**	**No Response**	**Total**
No. learners	9	0	3	**12**	0	**24**
% of learners	37,5%	0%	12,5%	**50%**	0%	**100%**

In this question learners had to apply their knowledge of tangent theorems to similar triangles. Although 50% (12/24) of the learners were able to select the correct response, an equal number of learners also selected the incorrect response. Whilst learners in both groups were able to identify the angle(s) between the tangent and chord and the angle in the alternate segment- many of them made the wrong conclusions.

Learners who selected option A failed to adhere to the basic principle when dealing with similar triangles. The order in which the vertices of the triangles are written is of paramount importance. Textbook authors such as Gonin et al (1997) stress this point. When naming similar triangles, "the letters indicating corresponding angles should be *written in the same order for all triangles*"(p.358) (emphasis added). Learners failure to realize that although Δ DYE and Δ EYD refer to the same figure, the order in which the vertices appear do not correspond to the corresponding vertices in Δ BYC. Thus the correct order should be Δ BYC /// Δ EYD and from this the ratio statement $\frac{BY}{EY} = \frac{CY}{YD}$ follows. Attention to order in naming triangles is important, something which a large number of learners have not yet mastered.

Question 7(Section A)

AE is a tangent to the circle at A. CDE is a straight line. $D\hat{A}E = 30^0$ and $A\hat{P}D = 120^0$. The size of $A\hat{D}P$ is ________

A) 60^0

B) 90^0

C) **Cannot be determined.**

D) 30^0

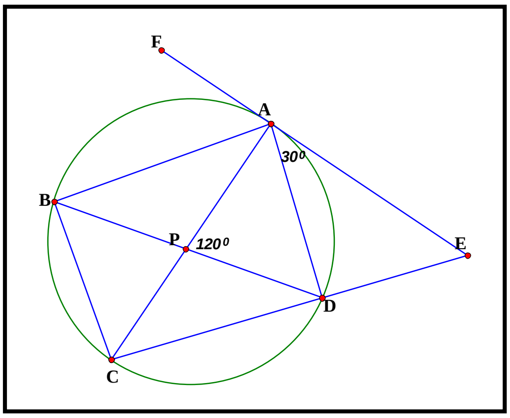


Table 24: Summary of learners' responses to Question 7

	Number of learners who selected…… as a response					
	A	**B**	**C**	D	**No Response**	**Total**
No. learners	0	2	**12**	7	3	**24**
% of learners	0%	8,33%	**50%**	29.2%	12,5%	**100%**

The diagram to this question was deliberately distorted to gain an insight into learners' spatial understanding. The above table reflects that while 50% of learners were able to visualize that the given information does not match the diagram and hence a solution is not possible, an equal number of learners, because of previous experiences, are determined to find a solution at all costs.

Learners' lack of spatial sense more often than not hinders their ability to visualize forms given to them and because of a poor spatial sense make incorrect judgments. Fujita and Jones (2002) cite Atiyah (2000), who writes that:

> "spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics- not only for things that re obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool.." (p.386).

SECTION B

The tasks in Section B were open ended in that no possible values for:

1.1) $\hat{U}$;

1.2) $\hat{V}_2$ and

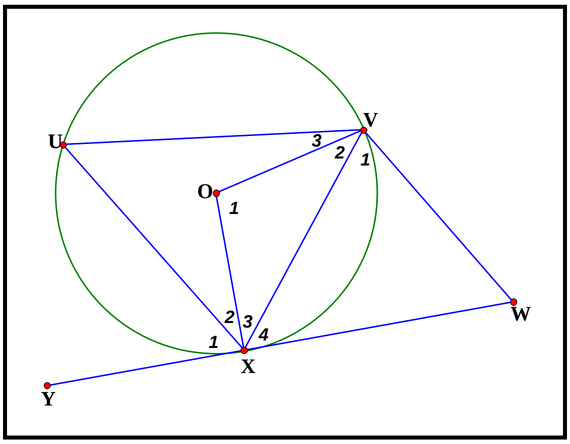
1.3) $X\hat{W}V$ is provided.

In the given diagram O is the centre of the circle.

$V\hat{O}X = 70^0$. Points U, V and X lie on the circle. Calculate the sizes of the following:

1.1) $\hat{U}$ **1.2)** $\hat{V}_2$

1.3) $X\hat{W}V$



The responses to the task 1.3 in Section B, can be categorised into the following:

i) A remark "impossible to find out" (response by L2); "cannot find" (response by L23);

ii) No response or attempt to solve the task; and

iii) Some attempt made by the learners to solve the given task.

For tasks 1.1 and 1.2 all learners' were able to provide legitimate responses to the given tasks. However for task 1.3 some interesting responses were provided. Below is an example of one such "unusual" solution.

L3 provided the following response to task 1.3.

Table 25: L 3's response to 1.3 of Section B

Line	Statement	Reason
1	$\hat{X}_4 = \hat{O}_1 + \hat{V}_2$	Ext angle of triangle
2	$\hat{X}_4 = 70^0 + 55^0$	
3	$\hat{X}_4 = 125^0$	
4	$\hat{V}_1 = \hat{O}_1 + \hat{X}_3$	Ext angle of triangle
5	$= 70^0 + 55^0$	
6	$\hat{V}_1 = 125^0$	
W is impossible to calculate since the sum of $\hat{V}_1$ and $\hat{X}_4$ is greater than 180^0 and the angles of a triangle add up to 180^0.		

In the above proof, L3 regards $\hat{X}_4$ and $\hat{V}_1$ as exterior angles of triangle OXV. One can see that L3 does not have an understanding of the concept of an exterior angle of a triangle. L3 may not have developed the "schema" (Chinnappan, 1998:202) of triangles adequately. Chinnappan (1998), defines the concept of "schema" as "a cluster of knowledge that contains information about core concepts, the relations between these concepts and knowledge about how and when to use these concepts" (p.202). Considering Chinnappan's notion of "schema", L3 lacks insight on how and when to use the knowledge on exterior angles of a triangle. However, despite L3's error in Lines 1 and 4 above, he arrives at a valid conclusion, viz. "the sum of $\hat{V}_1$ + $\hat{X}_4$ is greater than 180^0, hence $\hat{W}$ cannot be determined", demonstrates that L3's schema on the sum of angles of a triangle is better anchored than the schema on exterior angles of a triangle.

In the given task (1.3), neither XO nor OV is produced to warrant the learner's justification for the use of the exterior angle of a triangle theorem. Although the learners' reasoning is flawed, the learner is able to arrive at a valid conclusion, viz. that it is impossible to calculate $\hat{W}$.

Another learner, L10 provided the following solution to task 1.3.

Table 26: L10's response to 1.3 of Section B

Line	Statement	Reason
1	$X\hat{W}V = 180^0 - 70^0$	Properties of a kite
2	$X\hat{W}V = 110$	*This is a clear assumption! I have no idea how 2 do it. I just don't see a solution!!!*

(Learners remark to statement in line 2)

Considering the appearance of the figure OXWV, L10 assumes that the figure is a representation of a kite. The learner then proceeds to use the properties of a kite to calculate the size of $X\hat{W}V$. In this case, L10 draws on the schema of kites to solve the given problem. However, not enough information has been supplied for the learner to conclude positively that OXWV is a kite. Like L10 mentions, an assumption was made to enable him/her to arrive at a solution- because this is what school geometry is all about- there has to be a solution to every rider.

L11, in an attempt to solve the task (1.3), provided a solution based on the appearance of the figure. L11's response was "Look impossible, there are no cyclic quadrilateral and neither a tangent". Furthermore, the learner (L11) expects, because of previous encounters with similar riders, to be provided with a cyclic quadrilateral and/ or a tangent in order to arrive at a solution to the task.

Learners such as L4, L5 and L3 who attempted to solve the task, based their responses on the theorems which they have studied thus far. L4 and L5, for instance based their solution on the angle at centre theorem to determine the size of $\hat{W}$. The point O is given as the centre of the circle passing through the points U, X and V- but the learner expands the circle to include the point W as well. Based on this type of flawed reasoning, L4 provides the following proof.

Table 27: L4's response to 1.3 of Section B

$\hat{O}_2 = 360^0 - 70^0$ (angles around a point)
$= 290^0$
$X\hat{W}V = 145^0$.

From the preceding discussions, it has become apparent that learners employ different strategies in order to solve a given geometric problem. The strategies employed by the learners' are often based on amongst others previous encounters with similar problems, the visual appearance of a given figure, and the incorrect application of (a) theorem(s).

From the responses arriving at a solution ,valid or otherwise meant to the learner that he/she had provided a proof to the given rider. For the learner a proof meant writing a statement with a reason alongside it. A proof had to be in a two column table format for the learner with a reason attached alongside each statement. The learners never considered the mathematical value of their proof to a given rider.

Whilst I had intended to employ the interview method as a data collection tool, I had to disregard that strategy, as the data derived from that exercise was not adding value to my study. The responses to the task, I believe were able to provide me with a much richer source of data then the interviews did.

The learners' strategies to solve a given problem can be clustered according to the following categories:

- Incorrect use of theorems to solve a given problem;
- Responses to a problem are based on the visual appearance of a figure;
- Learners' responses to a problem is based on their previous experience with a similar problem; and
- Learners' view of proof to a given problem.

Each of the above categories will be explored in some detail in the next chapter.

Chapter 5: Discussion of findings

From the preceding analysis of learners' responses to the task in both Sections A and B, the following observations are made regarding learners' responses to the given task:

- Incorrect use of theorems to solve a given task (rider);
- Responses to tasks are based on the visual appearance of a figure;
- Learners "force" a solution to a given task (rider), although no solution is possible at times; and
- Learners demonstrated different views of proof.

5.1 How learners use theorems to solve a given problem.

The tasks given to the learners are typical riders, which they have encountered during their study of Euclidean Geometry. The theorems or tools which learners could draw on, were based on tangents and cyclic quadrilaterals theorems integrated with theorems based on triangles, as well as circle geometry theorems such as angle at the centre of a circle.

The geometric riders consisted of four common geometric concepts regularly encountered by learners: circle; tangent, triangle and quadrilateral. The riders were developed by having these concepts "integrated in a manner which demanded that the solver to recognize a component as serving more than one function" (Chinnappan, 1998:205). For instance, in Question 4 (see Appendix 2), the side QR needs to be identified as (i) a straight line, (ii) a tangent to the circle at Q; and (iii) a side of the quadrilateral RQUS. This recognition of one part of the figure playing multiple roles constitutes an important part in the modelling process before students are able to recruit appropriate theorems in order to generate new information. For instance the recognition of QR as a tangent could aid learners to infer that $R\hat{Q}S = Q\hat{U}S = \boldsymbol{x}$ (tangent-chord theorem). Furthermore, the identification of QU// RST could result in learners using alternate angles, i.e. $Q\hat{U}S = U\hat{S}T$ and the added fact that UT = TS could result in learners inferring that $S\hat{U}T = T\hat{S}U = \boldsymbol{x}$ (base angles of an isosceles triangle) in triangle UTS. Having identified the angles equal to ***x*** and trying to express the unknown angles in terms of ***x***, learners' could have calculated the numerical value of ***x***. However, 80% (20/24) learners' provided option C (i.e. cannot be determined) as a response. This high "negative response" immediately begs the question WHY?

In a research study undertaken in Israel by Movshovitz-Hadar, Inbar and Zaslavsky (1986), about learner's responses to exam type questions, they were astonished at the number of student-invented variations of theorems. A possible contributing factor that hampers learners' in providing an accurate mathematical proof may be learner's lack of understanding of a theorem and then misapplying it. Movshovitz-Hadar et al (1986) refer to this as "distortion of theorems" (p.26). An example of the phenomenon of distortion of theorems is illustrated in the following example.

Table 28: A learner's response to illustrate the phenomenon of "distortion of theorems"

$\hat{X}_4 = \hat{O}_1 + \hat{V}_2$ (ext < of triangle)

$\hat{X}_4 = 70^0 + 55^0$

$\hat{X}_4 = 125^0$

$\hat{V}_1 = \hat{O}_1 + \hat{X}_3$ (ext < of a triangle)

$= 70^0 + 55^0$

$\hat{V}_1 = 125^0$

$\therefore \hat{W}$ is impossible to calculate since the sum of $\hat{V}_1 + \hat{X}_4$ is greater than 180^0 and the angles of a triangle add up to 180^0.

In the above example, L3 identified $\hat{\mathbf{X}}_4$ and $\hat{\mathbf{V}}_1$ as exterior angles of triangle OXV.

What is true is the fact that $\hat{\mathbf{X}}_4$ and $\hat{\mathbf{V}}_1$ are outside (exterior) angles of triangle OXV. However, the theorem related to the exterior angle of a triangle cannot be applied to the given rider in this case. The learner, L3, arrives at a valid conclusion based on incorrect or flawed reasoning. The above example demonstrates what Weber (2002) has identified as a student's "lack of understanding of a theorem or a concept and systematically misapply it" (p. 102). Furthermore, the above example demonstrates this particular learner's lack of appropriate mental schemas that would assist the learner to recruit the appropriate theorems that would result in a valid solution.

Whilst this study is limited in its number of participants, the tasks used were able to demonstrate that some learners were just not able to leave their starting blocks. For instance in:

- Question 1 : 8,33% (2/24) learners did not attempt the task;
- Question 2 : 20,83% (5/24) learners did not attempt the task;
- Question 7: 12, 5% (3/24) learners did not attempt the task.

Learners' lack of attempting the tasks cited above would suggest that the learners' inability to provide an attempted solution is as a result of "they reach an impasse where they simply do not know what to do" (Weber, 2002:102).

5.2 Learners responses are based on the visual appearance of a given figure

According to Monaghan (2000),

> "The conceptual distance that students must cover to move from the stage of recognizing such gross visual features of shapes as straightness or length to more abstract concepts such as parallel ness or perpendicularity is far greater than the mere difference in vocabulary might suggest. Students very early on are able to recognize and distinguish shapes. What is less clear is the basis on which they make such distinctions" (p.184).

From research undertaken by Monaghan (2000), in which secondary school learners had to differentiate between different quadrilaterals, it emerged from the study that the learners used properties of one kind of rectangle for all rectangles. Monaghan (2000) refers to Hasegawa (1997) who provides the following comment in this regard:

> "The prototype is a result of our visual-perceptual limitations which affect the identification ability of individuals, and individuals use the protypical example as a model in their judgments of other instances" (Hasegawa, 1997 in Monaghan, 2000:187).

The above view of Hasegawa (1997) is well illustrated in the manner in which learners responded to the tasks given. To demonstrate this assertion, learners responses to some of the tasks will be dealt with below.

In task 1 of Section A (see Appendix 2), learner's knowledge on:

1. tangent-chord theorem;
2. angle at centre theorem;
3. equal chord properties; and
4. properties of isosceles triangles,

is being probed. The diagram, though, is not a typical prototype diagram found in your textbook, in terms of the above knowledge foci. In terms of the van Hiele model of geometric reasoning, secondary school learners should have surpassed the visual stage (level 0) of the model. The Revised National Curriculum Statements (Grades R-9) (DoE,2002), in their assessment standards for geometry suggest that learners exiting the senior phase(i.e. the end of grade 9) of schooling, should be operating at least at level 2 of the van Hiele model of reasoning. Learners at the senior phase should be able to "describe and represent the characteristics and relationships between 2-D shapes and 3-D objects in a variety of orientations" (p.6). Thus, one would expect that a grade 12 learner would be in a position to identify the key constituent parts of the given diagram and then recruit the necessary theorems to successfully solve the given rider. However, whilst not a generalisation, learners' employed cyclic quadrilateral theorems (e.g.: opposite angles of a cyclic quadrilateral are supplementary, i.e. $C\hat{A}D + C\hat{E}B = 180^0$) to solve the given task- despite there being no cyclic quadrilateral present in the diagram.

Another example where learners made incorrect judgments based on the given diagram is evident in task 6 of Section A (see Appendix 2).
In this task, based on previous encounters with similar examples, 37,5% (9/24) of the learners' deduced that triangle BYC /// triangle DYE and a further 12,5% (3/24) learners' deduced that ED//XZ. Both these responses are typical questions which learners are expected to answer (see Appendix 3).

Jones (undated), maintains that learners' "previous experience and the visual image" of a figure shapes the manner in which a learner would solve or attempt to solve a geometrical rider. Jones, cites Fischbein(1987) in which the latter asserts that "Experience is a fundamental factor in shaping intuitions"(p.82); and visualization is the primary factor "contributing to the production of the effect of immediacy"(p.82). The mental modalities, which learners' recruit in an attempt to solve geometrical riders is thus shaped by the following two assumptions:

1. the visual appearance of the figure; and
2. learners previous engagement (experiences); with similar type of problems.

Thus, the learners intuitive reasoning plays a significant part when formulating a formal argument to a given geometrical rider.

5.3 Learners "force" a solution to a given task although no solution is possible at times

The classroom tasks given to learners are traditionally designed by text book authors to yield a solution. However, some of the tasks in the activity sheet, were designed with the aim of not yielding a solution. Task 7 in Section A and task 1.3 in Section B are two such examples. For task 7, 37, 5% (9/24) of the learners opted for options B (90) and D (30), whilst 12, 5% (3/24) did not provide a solution and the remaining 50% (12/24) selected option C- the correct solution. It is thus note- worthy to explore how the 37, 5% (9/24) learners arrived at option B or D in order to get insights into their reasoning.

The following learners L3, L6; L4 and L23 opted for option D (30^0) – using different theorems related to cyclic quadrilaterals and tangent theorems and generic properties of quadrilaterals. L3 and L23 (see Appendix 3), first used the tan-chord theorem to show that $E\hat{A}D = A\hat{B}BD = 30^0$ and thereafter a potpourri of angles in the same segment and assuming that P is the centre of the circle to deduce that $A\hat{D}P = 30^0$.

L 4 (see Appendix 3*)*, after proving that $A\hat{B}D = 30^0$, indicated that triangle ABP is an equilateral triangle (check markings on sketch*)* - but indicates different measures for the interior angles of triangle ABP. This demonstrates to me that the learner has not yet fully grasped the properties of an equilateral triangle. A similar discrepancy prevails in triangle PCD of the same figure.

Both sets of solutions cited above are indicative of the learners' yearning to provide a numeric solution to a given geometric problem when numeric values are given for angle sizes. This "forced" type of solution indicates that the type of problems learners have been exposed to always resulted in a definite solution.

Similarly in 1.3 of Section B (see Appendix 2), learners were asked to calculate the value of $X\hat{W}V$. Once again, based on the visual appearance of the given diagram and the learners previous engagements with similar problems, learners' (e.g. L4 ; L5) assumed that O is the centre of the circle passing through UXWV. Based on this assumption, learners proceeded to calculate the size of $X\hat{W}V$ using cyclic quadrilateral and angle at centre theorems. Other learners, such as L10, L 18 and L 21 assumed that XW is a tangent to the circle at X and proceeded to use the tangent-chord theorem or tangent perpendicular to the radius to determine the size of $X\hat{W}V$.

Both sets of responses cited above, as a result of the visual appearance of the diagrams and their intuition, learners recruited inappropriate theorems to assist them to solve the given riders. Furthermore, both sets of responses highlighted above suggest that learners (regardless of their ability level) have not yet grasped the theorems they are working with sufficiently. Faulty assumptions made on the basis of the visual appearance of a given diagram suggest that learners have not as yet developed the "geometrical eye" (Godfrey, 1910, cited in Fujita and Jones, 2002:385) for detail. The geometrical eye, as defined by Godfrey (1910), relates to the learners' ability to see "geometrical properties detach themselves from a figure" (Godfrey, as cited in Fujita and Jones, 2002:385).

In addition to developing and nurturing the geometrical eye, learners need to develop and nurture an additional skill to be able to solve mathematical problems in general and geometrical problems in particular, i.e. “geometrical power” (Godfrey, 1910 in Fujita and Jones, 2002:388). The concept of geometrical power relates to the learners “power of seeing geometrical properties detach themselves from a figure” (Godfrey, in Fujita and Jones, 2002:388). As an example of Godfrey’s notion of the “geometrical eye” consider task 6 of Section A (see Appendix 2).

To be able to identify the correct statement, learners had to see (visualize) that triangles BYC and EYD (*note the order of the triangles)* are likely to be similar. Learners were able to deduce that triangle BYC /// triangle DYE (option A), but failed to take note of the order of the second triangle- thus eliminating option A. Similarly, learners failed to see that ED is not parallel to XZ, but ED is parallel to BC. Fujitsa and Jones (2002) cite a study undertaken by Nakashini (1987), with 87 Japanese learners aged between 14-15 years in which they had to prove AZ = BY if triangle XYZ is an isosceles triangle. Although 75% (65/87) of the learners’ were able to provide a correct response to the problem, there were others 25% (22/87), who were unable to “see” the solution.

To be able to successfully solve geometrical riders, one should also have developed a well-trained “geometrical eye” which will assist in arriving at a valid solution. A geometrical eye will not just develop overnight – it’s a process which requires the intervention and support of all educators teaching geometry- from the foundation phase educator right through to the educator who teachers at grade 12 level and beyond.

> “There must be a good foundation of practical work, and recourse to practical and experimental illustration wherever this can be introduced naturally into the later theoretical course. Only in this way can the average boy *[sic]* develop what I will call the ‘geometrical eye’” (Godfrey, 1910 in Fujitsa and Jones, 2002:388).

Chapter 6: Summary, recommendations and conclusion

6.1 Summary and findings

6.1.1 Overview

The aim of this study was to explore, grade 12 learners' understanding of Euclidean Geometry with special reference to cyclic quadrilateral and tangent theorems. The primary focus of this study was to investigate what cognitive tools learners recruit in order to solve geometrical riders.

In **Chapter two**, the van Hiele model of geometric thought was discussed and provided the theoretical framework for collecting, analyzing, interpreting and reporting grade 12 learners' understanding of cyclic quadrilateral and tangent theorems. The findings by examiners of grade 12 exams as well as the MST (2003) Report provided justification for this study. Grade 12 learners' poor performance in exams in geometrical riders involving cyclic quadrilateral and tangent theorems meant that it was important for this study to analyze this group of grade 12 learners' understanding of the mentioned theorems.

In **Chapter three**, the choice of a case study as an appropriate research tool was explained. Using a case study allowed the researcher to explore strategies of how learners' reason, discover important questions to ask during the interview and try to understand learners' thinking processes. The choice of interviews and tasks in the case study design allowed for a rich description and analysis of data.

In **Chapter four**, learners' responses were analyzed, coded and categorised as described in Section 4 .1 (Table 10), see page 77.

Based on the above, the following key aspects were identified in an attempt to understand learners' reasoning skills, namely:

Key Aspect No.	**Aspect**
1	Inappropriate use of theorems to solve a given rider
2	Visual appearance of figure plays a role in solving rider
3	Learners "force" a solution to a rider, even when one is not possible
4	Learners view of proof to a rider

Table 29: Key aspects according to which learner's reasoning skills were categorised.

These four aspects facilitated a detailed analysis of learners reasoning ability. In **Chapter five**, each of the four key aspects identified in Chapter four were discussed in more generic terms.

6.1.2 Primary research questions and sub-questions

The primary research questions in this study are:

1. How do grade 12 learners begin to solve a geometric problem?
2. What knowledge and skills do learners recruit in order to solve geometric problems?

The primary questions were further broken down into the following sub-questions:

1.1 Why did you (the learner) follow a certain route (plan) when solving a given geometrical problem?

1.2 What type of information was provided either directly or by implication in the given tasks?

1.3 How do you (the learner) view proofs?

Why did learners follow a certain route (plan) when solving a given geometrical problem?

In this study learners' engagement with similar types of problems and the visual appearance of the given diagram(s) framed the manner in which learners approached a given problem. The tools (theorems, definitions, axioms, etc.), which learners used, were primarily based on the visual appearance of the figure. Whilst learners were able to identify the theorems to be used, they often came up with their own contrived version of a particular theorem such that it corresponded with the given task.

Learners vision of geometric concepts appears to be blurred at times. Whilst some aspects are obvious from the given information, at other times what learners may see (or read) from the given information may be “blatantly wrong things” (Dreyfuss, 1999:105). However, we need to guard against learners who are able to arrive at correct conclusions using visual reasoning as “correctness of the answer is not the issue, certainly not the main issue” (Dreyfuss, 1999:105).

What type of information was provided either directly or by implication in the given diagram(s) and/ or text?

The given diagrams were preceded by a textual description of the task. Using the description provided, learners were expected to transfer the given information (data) onto the diagram. By so doing learners would then begin to recruit (identify) the appropriate theorems, definitions, etc. that would facilitate the successful solution to the given task. However, this intention was not always realized.

Often learners were not able to “see” the solution to the problem, although the given information was indicated on the diagram(s). As a result of this impediment, learners recruited inappropriate theorems, definitions, etc. and thus arrived at non-valid conclusions. At times the solutions provided were not aligned with the learner’s markings on the diagrams.

Based on their previous engagements with similar type of problems, learners made assumptions; such as XVW is a tangent (see task 1 of Section B*)*, although this was not given directly or indirectly to the learners. As a result of this faulty assumption learners used inappropriate theorems, definitions, etc to solve the given task(s). Similar assumptions were made regarding task 7 of Section A. In that figure *(*task 7 of Section A), learners assumed that P is the centre of the circle- because it looked like the centre of the circle, and thus used theorems such as angle at the centre of the circle to solve the given task.

Thus learners failed to read the question carefully before deciding on a route to the solution. Often learners relied on their "gut-feeling"-based on the appearance of the figure and their previous encounter with similar problems when solving a given task. No consideration was given to the facts given –when learners embarked en-route to solving a task.

Learners view of proof

In this study, for learners, to prove something meant having a neat two-column table-like proof with reasons alongside each statement. The learners were exposed in the main to a deductive axiomatic system of proof. Many learners were afraid to venture out of this protected environment,

> " it is often assumed that students believe that valid proofs may only be constructed by using a chain of deductive reasoning to connect axioms, definitions, and already established theorems within a particular axiomatic system"(Martin, McCrone, Bower and Dindyal, 2005:121),

in this case the deductive –axiomatic system.

Research conducted into mathematical proof concludes that a large body of learners' lack understanding of the nature of proof (see for example, Senk, 1985), and often they (learners') don't reason in a logical, coherent manner. A study conducted by Senk (1985) in the United States of America, for instance, concludes that as much as 70% of high (secondary) school learners do not understand the proofs they study. Learners' often confuse a worked example of a geometrical rider with a proof. The learners' often focused on the appearance of a proof (two column proof), than on the logical, coherent flow of mathematical ideas (content).

Weber (2001), in his research with university students, has identified two generic characteristics of learners' difficulties with proof. The first one has already been alluded to, viz. learners' lack of a clear idea of what constitutes a proof. The second and more relevant and pertinent difficulty which his research has uncovered is the students' lack of "understanding of a theorem or a concept and systematically misapply it" (Weber, 2001:102). Whilst learners were able to identify the theorem to be used, they often distorted it or misapplied it. More often than not learners' despondency leads them to throw their arms up in the air because *"students often fail to construct a proof because they reach an impasse where they simply do not know what to do"* (Webber, 2001:102). (emphasis added)

6.2 Implications and recommendations for classroom practice

It has been argued in this research that Euclidean Geometry poses several challenges to grade 12 learners' reasoning ability. Adopting an alternative teaching strategy for Euclidean Geometry will imply that many educators would be removed from their present comfort zone of presenting geometric theorems as a finished product. In the context of this study, the following recommendations can be put forward as a means to enhance learners reasoning ability in terms of Euclidean Geometry.

How to teach Geometry

To begin with, we need to re-look at the manner in which we teach definitions of geometric concepts to learners. The direct teaching of geometric definitions has come under the spotlight by mathematicians and mathematics educators' alike (De Villiers, 1997). De Villiers (1998), for instance quotes Human (1978), in which the latter calls for a "reconstructive" approach instead of the more regular direct axiomatic-deductive approach to teaching of definitions. Human (1978), as cited in De Villiers (1998), differentiates between the two approaches in the following manner:

> "With this term (reconstructive), we want to indicate that content is not directly introduced to pupils (as finished products of mathematical activity), but that the content is newly reconstructed during teaching in a typical mathematical manner by the teacher and/or pupils" (p.1).

The pedagogical advantages of employing a reconstructive approach are:

- Its implementation accentuates the meaning of the content;
- It allows the learners to become actively engaged in the construction of the content.

Thus employing a reconstructive approach is therefore characterized by not "presenting content as a finished (pre fabricated) product" (De Villiers, 1998:1), but instead to focus on the real mathematical activities through which the content is to be developed.

Researchers, such as Ohtani (1996) (in De Villiers, 1998), for instance have argued that the provision of definitions by educators is to ensure that there is, amongst others a uniformity of the definition as understood by all the students; the educator exerts some kind of control over the learners; and there is no long drawn out debates and discussions concerning learners definitions. These are all "un-constructive" and out of sync with the current curriculum reform practices which encourages learners to

- "communicate appropriately by using descriptions in words, graphs, symbols, tables and diagrams;
- Use mathematical process skills to identify, pose and solve problems creatively and critically" (DoE, 2003:9).

Thus to enhance learners understanding of geometric definitions, it is necessary to encourage learners' to engage in activities that will afford them the opportunities to develop the requisite skills which the curriculum intends to develop within the learners.

It is an accepted fact that all learners are not on the same cognitive level in terms of understanding, ability level, etc. Therefore it is advisable for learners to provide definitions of concepts, which align with their cognitive level. From our discussion of the van Hiele levels of understanding, it was noted that learners develop a clear understanding of definitions from level 3 only. It would thus be futile to attempt to provide learners with formal definitions of concepts when they are not yet cognitively receptive to such demands. At grade 8 level for instance, learners are still at van Hiele levels 1 or 2.It would thus be futile to expect learners at this stage to provide a formal definition for, say a rectangle. At this stage learners would focus on the visual aspect of the figure and provide a definition that corresponds with the visual representation of the figure. A rectangle could be defined in terms of the length of its sides, and a typical definition could be: "A rectangle has all angles 90^0 and two long and two short sides" (De Villiers, 1997:46). Whilst learners at the stage of level 2 of the van Hiele model of understanding, may provide a long, cumbersome definition of a rectangle, it should be noted that such a definition would be in accordance to the learners level of maturity and development. A typical definition at level 2 could be: " A rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90^0, equal diagonals, half turn symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc" (De Villiers, 1997:46).

The process of constructing definitions in mathematics should not be seen as a less important activity than the process of factorization, for instance. To increase learners' understanding of geometric definitions then, it becomes incumbent on every educator to engage learners "in the process of defining geometric concepts" (De Villiers, 1998). Learners should not be provided with ready made definitions of concepts such as cyclic quadrilaterals, tangents, rectangles, etc. By providing learners with such ready-made definitions, a misconception is created in the learners that there exists "only one correct definition for each concept" (De Villiers, 1995). Learners are thus denied the opportunity to search for alternative definitions, in cases where definitions are presented as items cast in stone.

> "Defining concepts accurately in mathematics is certainly not an easy task, and is only developed after lots of experience and practice. However, the educational experience is worth the trouble, and I would like to encourage our authors and teachers out there to seriously rethink their treatment of geometry definitions" (De Villiers, 1995).

It is an established and universally valid fact, that learners' poor performance in Geometry can be attributed to, amongst other factors, the pedagogical (teaching) strategy of the educator, an outdated curriculum and text book authors who perpetuated the cycle of presenting ready-to use, neatly packaged content (theorems, definitions, axioms , etc) , which learners are expected to memorize and accept without question. Martin et al (2005), argues that if learners performance in Geometry is to improve, then the educator needs to be the catalyst for change in this process. To this end, Martin et al (2005), assert that

> "We conclude that pedagogical choices made by the teacher, as manifested in the teacher's actions, are key to the type of classroom environment that is established and, hence, to students' opportunities to hone their proof and reasoning skills" (p.95).

Whilst most traditional geometry courses were product driven, such an approach does not cohere with the current constructivist approach to teaching and learning, which advocates that:

1. " knowledge is not passively received but actively built up by cognizing subject;
2. the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (Jaworski, 1996:16).

Thus to achieve this goal, we need to encourage learners to become actively engaged in the construction of knowledge- especially Euclidean Geometry knowledge.

Teach to prove or teach for proof?

The standard or traditional view of proof in Euclidean Geometry has always been one of verification of the correctness of mathematical statements. However, this limited, naïve view has come in for strong criticism. De Villiers (1990), for instance argues that verification should be seen as one part of the five tiered purpose of proof, which includes:

1. Verification (concerned with the truth of a statement);
2. Explanation (providing insight into why it is true);
3. Systematization (the organization of various results into a deductive system of axioms, major concepts and theorems);
4. Discovery (the discovery or invention of new results);
5. Communication (the transmission of mathematical knowledge) (De Villiers, 1990:18).

Traditionally, learners are given predetermined statements to prove. As a result learners assume then that the statement must be true. For learners to appreciate the value of proof writing and to engage as mathematicians do, they need to know how mathematicians use proof. Learners need to be aware that proof is only one aspect in the process of learning and discovering new mathematics. Mathematicians first make conjectures, which are based on observations, or hunches (one's gut-feeling). The mathematician then tests these observations or hunches before embarking on a formal proof for his conjecture. Before a mathematician can formally publish his/her findings, he/she allows their conjectures to be critiqued by fellow mathematicians before their conjecture is accepted as true. Mathematics in the National Curriculum Statement (NCS) for Further Education and Training (FET), strives to develop in learners the ability to

- “work collaboratively in teams and groups to enhance mathematical understanding;
- collect, analyze and organize quantitative data to evaluate and critique conclusions” (DoE, 2003:10),

this coheres with the tasks and responsibilities of a mathematician. As educators, we need to afford learners the opportunities to evaluate the thinking processes of their colleagues and their own.

Using De Villiers (1990), view of proof educators and textbook authors should then design activities (tasks) that would encourage learners to engage in the above strategies.

Both Piaget and van Hiele have outlined strategies that would enable learners to prove ideas formally. For Piaget, learners thinking in general progresses from a stage of “non-reflective and unsystematic, to empirical and finally logical-deductive” (Battista and Clements, 1995:50). In a similar manner, van Hiele argues that learners geometric thinking processes, progresses from stages of lower levels (visual→ descriptive) of thinking to more complex stages (descriptive → abstract→ formal→ deduction → rigour), a process which is labour intensive and time consuming. The van Hiele model of thinking suggests that the teacher’s instruction (pedagogy)should aid learners to gradually progress from the lower levels of thinking before engaging them (learners) in the rigors of “proof-oriented study of geometry”(Battista and Clements,1995:50). The premature dealing of formal proof will not aid learners’ understanding of proof; instead such an approach will result in “students only to attempts at memorization and to confusion about the purpose of proof” (Battista and Clements, 1995:50).

Gole (2003) argues that textbooks in the USA present solutions to geometric riders in a linear manner implying that learners would be able to make sense of such a solution. That may be a valid approach to simple riders, but such an approach may be less effective when dealing with more complex problems. An alternative approach would be "Sherlock Holmes's backward reasoning" (Gole, 2003:544) type of approach. This process of backward reasoning allows the learner to search for possible solutions to a given rider, using the given information and making valid inferences (deductions) from such information. Consider the following example wherein this process of backward reasoning is used to solve a given rider.

Given: Circle O with POS a diameter and ST a tangent.

Prove: WV // ST (Source: Laridon et al, 1988:338)

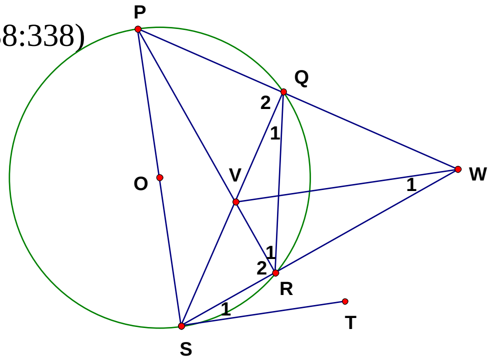


Figure12: A typical geometric rider which grade 12 learners' should be able to solve.

In order to prove WV// ST one would have to either prove:

- a pair of alternate angles equal ;or
- a pair of corresponding angles equal; or
- a pair of co-interior angles supplementary.

A quick glance at the above diagram shows that $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{S}}_1$ are angles in alternate positions of the straight lines WV and ST. Thus if these angles can be proved to be equal to each other, then the task would be solved. But how does one prove something like this by using the information given? The following flow diagram (see next page) represents a possible path to solving the task.

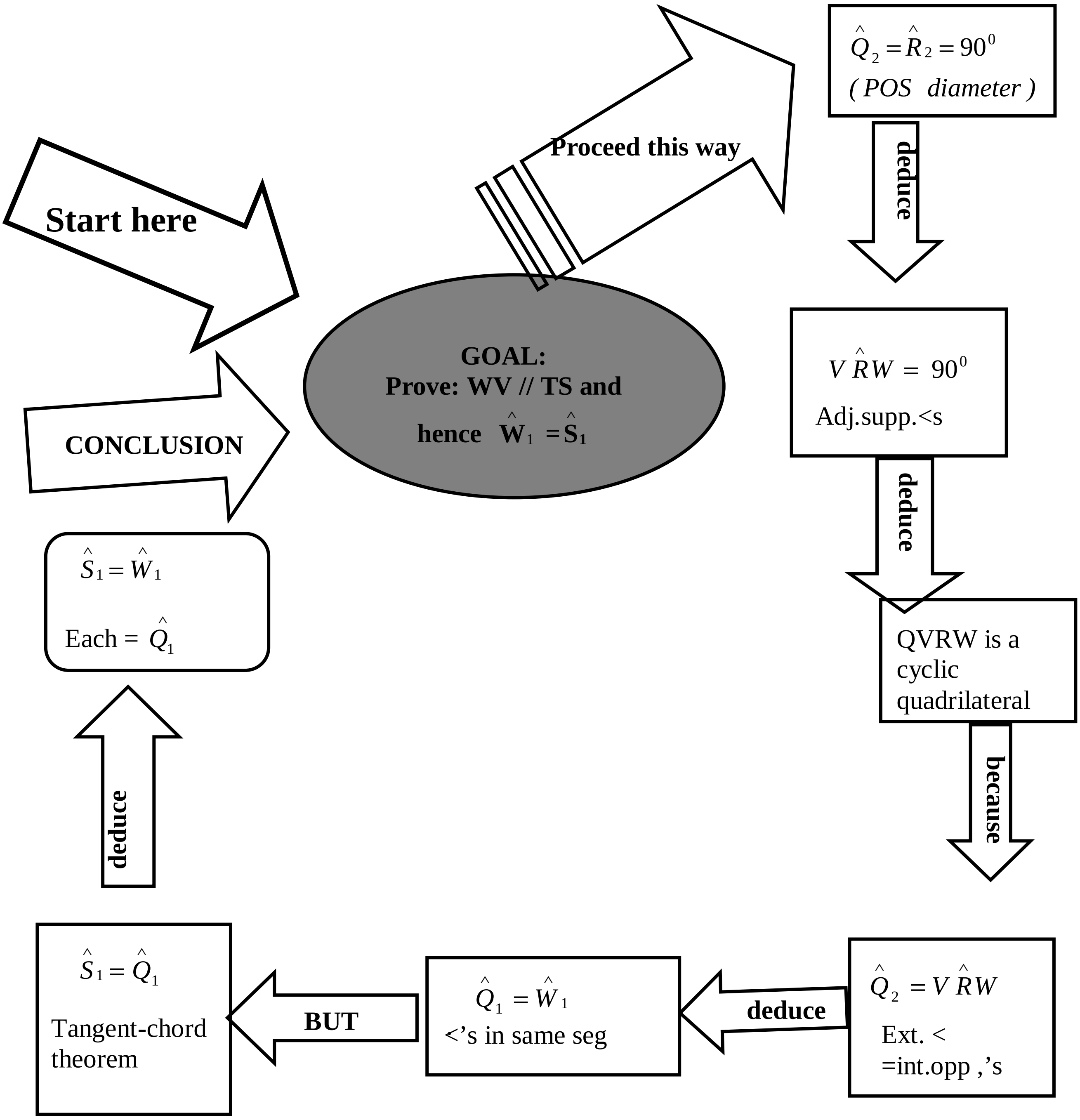


Figure 13: A schematic representation demonstrating the method of "backward reasoning".

The above figure is intended to demonstrate the thought process involved in using the strategy of backward reasoning. One begins at one's destination point and retraces one's footsteps to one's current location before one can begin with the write up of the solution of a given problem.

Often one would hear learners' comment that if only they knew where to start with a given rider then they would not feel frustrated and despondent. A good starting point is to start with what you need to prove and then work backwards. By so doing one limits the element of luck when arriving at a suitable line of argument. This skill of backward reasoning is neither natural nor commonly used in mathematics today. The educator's help may be an essential ingredient in developing and nurturing the skill of backward reasoning by providing learners' with sufficient guided examples over an extended period of time thereby "geometry proofs provide simplified controlled problem-solving contexts and help cultivate through repetition the mental search habit" (Gole, 2003:545).

Learning such skills (backward reasoning), can result in both general and practical rewards for the learner. An important positive spin-off in the development of such skills is that learners levels of confidence "in searching for solutions to unfamiliar and possibly over whelming problems by learning how to limit options" (Gole, 2003:545), increases. If one had to view the skill of backward reasoning as an investment over a period of time, then one can conclude that the returns on such an investment would be far greater than the risks attached to it, since once a learner has mastered the skill, it is highly unlikely for the learner to forget the skill. Furthermore, mastery of such thinking skills in geometry can also develop in learners' important hands-on skills to manage their everyday lives.

Use of interactive technology

In line with seeking workable alternatives to the rigid axiomatic approaches to Euclidean Geometry, the focus on computer programs such as Geometers' Sketchpad is to facilitate and enhance learners' ability to making and testing of conjectures. Researchers such as De Villiers have argued strongly in favour of dynamic software programs such as Geometers' Sketchpad that would be able to revolutionize Geometry at all levels. De Villiers, as cited in Yushan, Mji and Wessels (2005), defends his claim for interactive software as follows "the main advantage of computer exploration of topics…is that it provides powerful visual images and intuitions that can contribute to a person's growing mathematical understanding" (p.17). By visualizing a problem, learners are able to have a global picture of the problem to be solved.

It is well known that the more senses we use in our teaching and learning episodes, the more we understand and retain knowledge. By using interactive software such as Geometers' Sketchpad, educational technology appeals to our senses of sight, sound and touch. From a constructivist point of view, then learners become "more active agents in managing and ensuring the success of their education- invariably sustaining their attention and commitment to mathematics" (Yushan, et al, 2005:18).

Using Sketchpad diagrams for exploration does not only encourage learners to make conjectures, but can also develop "insight for constructing proof" (Battista and Clements, 1995:52). For example, in Sketchpad a learner can construct a circle, locate four points at random on the circle and consider the quadrilateral so formed. If the size (measures) of the angles is measured, the points assume different positions on the circle, and the learners would observe that the sum of the opposite angles of such a quadrilateral approaches 180^0.
The Sketchpad demonstration is convincing, since the size of the circle can be changed and the vertices moved easily. But will our conjecture hold? *If so why? If not, why not?*

6.3 Conclusion

It was the intention of this study to make a meaningful contribution to the body of knowledge related to learners' understanding of Euclidean Geometry. Educators have often lamented learner's poor performance in Geometry- learners having difficulty in "seeing" a solution or a path to a possible solution; or learners are unable to make sense of the theorem(s) to be studied. As a result learners become frustrated, de-motivated and indifferent towards the subject (Geometry), because they felt incompetent in dealing with it.

Researchers like De Villiers (1997) have argued strongly that geometry is alive and well. In fact it is experiencing a renaissance in most countries, including South Africa, at all levels of education. Recent curriculum changes in South Africa, for instance demonstrate a marked shift from the traditional approach to geometry. The study of geometry at grades 10-12 level, for instance, engenders to develop in the learners the ability to:

- "explore relationships, make and test conjectures….;
- investigate geometric properties ….in order to establish, justify and prove conjectures;
- use construction and measurement or dynamic software, for exploration and conjecture" (DoE, 2003:14).

Current curriculum reform initiatives are thus in keeping with changes in the approach to the teaching of geometry in other parts of the world.

For the curriculum reform initiatives to be of any significance, there need to be a radical re-look at the teacher education programmes at both pre-service and in-service level. Most high school educators know a little more geometry than the learners they are expected to teach- through no fault of theirs. Institutions of higher learning offering teacher education programmes need to have compulsory courses in Euclidean and non-Euclidean geometry for both primary and secondary educators if any meaningful change in learner-performance is to be registered.

Whilst this study focused primarily on learners' reasoning when solving geometric problems, I believe that an investigation into:

- how teachers reason when solving geometric problems and when teaching geometry;
- the relevance of the language used by educators and its suitability to the learners' conceptual understanding in geometry;

in terms of the van Hiele model offers one an opportunity then to broaden and deepen one's understanding of the model. I believe that such a study would be able to inform effective teaching of Euclidean Geometry. What I mean is that there is a need to understand geometry teaching practice at the chalk face: how teachers teach geometry, how they use language the language of geometry, and to investigate the extent to which their use of the language of geometry takes into account learners' level of development in terms of the van Hiele model. We need to explore further to the extent to which providing practicing and pre-service educators with opportunities to engage in activities that “require classifying answers by van Hiele levels” (Feza and Webb, 2005:45) might contribute to effective practice in geometry.

Reflections on my research journey

After two years of hard work, I have finally reached the goal I set myself two years earlier. The road to reach this destination was not always smooth. I had to contend with challenges at home, at work, unfavourable deadlines imposed by lecturers, employers and family. Each constituency demanded their due on time. Despite these unfriendly, at times hostile conditions, I was able to come out on the other end academically enriched and fulfilled. The goal of my study was not only to be able to write the letters M.Sc behind my name – there was a greater goal. That greater goal was to assist my colleagues who experience difficulty when teaching Euclidean Geometry, to teach it in a manner which allows for greater learner participation in the discipline. Furthermore, the dilemmas we experience regarding Euclidean Geometry is not restricted to our shores only - it is a universal phenomenon experienced at varying levels by different countries around the world.

To conclude, I would like to impart some advice to the novice researcher. **Firstly,** make sure that if you are married, your spouse has given you his/her consent and support to continue with your studies. Failing which, cancel your registration immediately. **Secondly**, make sure that you have sufficient time assigned for your studies. Trying to fit your studies into your schedule does not work – instead let all activities revolve around your studies. **Thirdly**, select an area or topic, which you would like to explore which your supervisor also shows a keen interest in. Alternatively, try to adjust your topic/area of research which fits into your supervisor’s field of interest. Finally, do not try to do too much – set yourself realistic and achievable goals, **DO NOT TRY TO BE OVER AMBITIOUS**, especially if you are a novice researcher.

Bibliography

Abbott, P. (1970). *Geometry*. London: The English Universities Press Ltd.

Battista, M.T. and Clements, D.H. (1995). Geometry and proof. *The Mathematics Teacher*, 88(1), 44–54.

Bell, J. (1987). *Doing your research project: a guide for first–time researchers in education and social science.* Philadelphia: Open University Press.

Bopape, M., Hlomuka, J., Magadla, L., Shongwe, S., Taylor, N. and Tshongwe, T. (1994). *Understanding Mathematics Std 10.*Cape Town: Maskew Miller Longman

Borg, W.R. and Gall, M.D. (1983). *Educational Research– an introduction.* New York: Longman

Chetty, M. (2003). *Assessment practices in Curriculum 2005: a case study of two senior phase mathematics teachers*. Unpublished MSc dissertation. University of the Witwatersrand, Johannesburg.

Chinnappan, M. (1998). Schemas and mental models in geometry problem solving. *Educational Studies in Mathematics, 36(1),* 201–217.Dordrecht: Kluwer Academic Publishers.

Cohen, L. and Manion, L. (1991). *Research methods in education*. New York: Routledge

Cooper, B. and Dunne, M. (2000). *Assessing children's mathematical knowledge: social class, sex and problem–solving*. Philadelphia: Open University Press.

Crowley, M.L. (1987). The van Hiele model of geometric thought. In Lindquist,M.M., & Shuklte, A.P. (eds.), *Learning and teaching Geometry,K–12. 1987 Year Book*. National Council Of Teachers Of Mathematics:Reston, Virginia.

Daly, R. (1995).*Multiple choice questions: Standard 9 /10.*Randburg: Kagiso Publishers.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17–23.

De Villiers, M. (1995). *The handling of geometry definitions in school textbooks.* http://mzone.mweb.co.za/residents/profmd/geomdef/html. Accessed 8/3/2002

De Villiers, M. (1997). The future of secondary school geometry. *Pythagoras*, 44, 37–54.

De Villiers, M. (1998). *To teach definitions in Geometry or teach to define?* http://mzone.mweb.co.za/residents/profmd/define.html. Accessed 8/3/2002

Dreyfuss, T.(1999). Why Johnny can't prove (with apologies to Morris Kline). *Educational Studies in Mathematics,44*, 5–23. Dordrecht: Kluwer Academic Publishers.

Feza, N.and Webb,P. (2005). Assessment standards, van Hiele levels and grade seven learners' understanding of Geometry. *Pythagoras*, 62, 36–47.

Fraenkel , J.R. and Wallen, N.E. (1993).(Second Edition). *How to design and evaluate research in education.* London: McGraw–Hill Inc.

Fujita, T and Jones, K. (2002). *The bridge between practical and deductive geometry: developing the "geometric eye"*. Proceedings of PME 26

Department of Education. (1995). *Interim Core Syllabus*. Pretoria: Government Printers.

Department of Education. (1997). *Policy document for Specific Outcomes.* Pretoria: Government Printers.

Department of Education. (2002). *Revised National Curriculum Statement. Grades R–9 (Schools) Policy*. Pretoria: Government Printers.

Department of Education. (2003). *National Curriculum Statements Grades 10–12 (General)*. Pretoria: Government Printers.

Gauteng Department of Education. (2000). *Examiners Report: Mathematics – Higher Grade and Standard Grade*. Johannesburg: Assessment & Monitoring Unit.

Gauteng Department of Education. (2001). *Examiners Report: Mathematics – Higher Grade and Standard Grade*. Johannesburg: Assessment & Monitoring Unit.

Gauteng Department of Education. (2002). *Examiners Report: Mathematics – Higher Grade and Standard Grade.* Johannesburg: Assessment & Monitoring Unit

Gauteng Department of Education. (2003). *Examiners Report: Mathematics – Higher Grade and Standard Grade.* Johannesburg: Assessment & Monitoring Unit

Gauteng Department of Education. (2003 *b*). *The MST Strategy:2002 Grade 12 Mathematics, Science and Biology (MSB) paper analysis*. Johannesburg.

Gole, A.M. (2003). Sherlock Holmes, Geometry proofs and backward reasoning. *Mathematics Teacher*, 96(8) , 544–547.

Gonin, A.A., du Plessis, N.M., Kuyler, H.A., de Jager, C.W., Hendricks, W.E, Hawkins, F.C.W, Slabber, G.P.L. & Archer, I.J.M. (1997). *Modern graded mathematics: Standard 10.* Cape Town: Nasou

Hanna, G. (1995). Challenges to the importance of proof. *For the learning of mathematics,* 15 (3), 42 – 49.

Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics,* 44, 5–23. Dordrecht: Kluwer Academic Publishers.

Helms, D.B and Turner, J.S. (1981). *Exploring child behaviour*. London: Holt, Rinehart & Winston.

House of Delegates. (1992 *a*). *Senior certificate examination: Mathematics P 2 (HG): Pupils errors and misconceptions and possible ways to improving pupil performance.* Durban

House of Delegates. (1992 *b*). *Senior certificate examination: Mathematics P 2 (SG): Pupils errors and misconceptions and possible ways to improving pupil performance*. Durban

Human, P.G., Nel, J.H, de Villiers, M.D., Dreyer, T.P. and Wessels, S .F.G. (1977). *Alternative instructional strategies for geometry education: a theoretical and empirical study.* http://mzone.mweb.co.za/residents/profmd/usewo1.html. Accessed 8/3/2002.

Jaworski,B. (1996). *Investigating mathematics teaching: a constructivist enquiry.* London: The Falmer Press.

Jones, K. (undated). Deductive and intuitive approaches to solving geometrical problems. In Mammana, C & Villani, V. (eds.).*Perspectives on the teaching of geometry for the 21st century*.Dordrecht:Kluwer Academic Publishers. Pp 78–83

Joubert,J. (1988). Evaluering van meetkunde.*Pythagoras*, 18, 7–13.

Laridon, P.E.J.M., Brink, M.E., Fynn, C.A., Jawurek, A.M., Kitto,A.L.,Myburgh, M.J., Pike, M.R., Rhodes–Houghton, H.R and van Rooyen, R.P. (1988). *Classroom Mathematics: Standard 9.* Johannesburg: Lexicon Publishers.

Laridon, P.E.J.M., Brink, M.E., Fynn, C.A., Jawurek, A.M., Kitto,A.L.,Myburgh, M.J., Pike, M.R., Rhodes–Houghton, H.R and van Rooyen, R.P. (1996).(Second Edition) *Classroom Mathematics: Grade 12*. Johannesburg: Heinemann.

Leedy, P.D. (1993). (Fifth Edition). *Practical research: planning and design.* New Jersey:Prentice–Hall, Inc.

Marshall, C. and Rossman, G.B. (1994). (Second Edition). *Designing qualitative research.* London: SAGE Publications.

Martin, T.S., McCrone, S.M.S.,Bower,M.L.W., and Dindyal,J.(2005). The interplay of teacher and student in the teaching and learning of geometric proof. *Educational Studies in Mathematics*, 60, 95–124.

Mathematical Digest. (1996). Mathematics Department: University of Cape Town.

McMillan, J.H. and Schumacher, S. (2001). (Fifth Edition). *Research in education: a conceptual introduction.* London: Addison Wesley Longman, Inc.

Merriam, S.B. (1998). *Qualitative research and case study applications in education.* San Francisco, CA:Jossey-Bass.

Monaghan,F. (2000). What difference does it make? Children's views of the differences between some quadrilaterals. *Educational Studies in Mathematics*, 42, 179–196. Dordrecht: Kluwer Academic Publishers.

MovshovitzHadar, N., Inbar,S. and Zaslavsky,O. (1986). Students' distortions of theorems. *Focus on learning problems in mathematics*, 8(1), 25–33.

Osborne,R. and Gilbert, J. (1980). A technique for exploring student's views of the world. *Physics Education,*(15), 376–379.

Penlington, T. (2004). Investigation into the teaching of geometry in the intermediate phase. In Nieuwoudt, S., Froneman, S. and Nkhoma, P. (Eds.). *Proceedings of the Tenth National Congress of the Association for Mathematics Education of South Africa, Vol.1.* pp.199–206. Potchefstroom:Advert Square.

Pournara, C. (2004). Geometry for preservice teachers: some lessons learnt from a pilot module. In Nieuwoudt,S., Froneman,S. and Nkhoma, P. (eds.). *Proceedings of the Tenth National Congress of the Association for Mathematics Education of South Africa, Vol.1.* pp207–215. Potchefstroom: Advert Square

Posner, G. and Gertzog, W. (1982). The clinical interview and measurement of conceptual change. *Science Education*, 66 (2), 195–209.

Reeves, T.C., and Hedburg, J.G. (2001). A pragmatic rationale for evaluation. In Reeves, T. and Hedburg, J. *Interactive learning systems evaluation*. Engelwood Cliffs, NJ:Educational Technology Publications.

Senk, S.L. (1985). 'How well do students write geometry proofs?' *Mathematics Teacher*, 78(6), 448–456.

Silverman, D. (2000). *Doing qualitative research: a practical handbook*. London: SAGE Publications.

Transvaal Education Department. (1994). *Wiskunde 1994– Verslag*. Pretoria.

Teppo,A. (1991). Van Hiele levels of geometric thought revisited. *Mathematics Teacher.* Pp 210–221.

Thompson, P. (1978). *The voice of the past.* Oxford: Oxford University Press.

Van der Sandt, S.and Nieuwoudt,H.(2004). Prospective mathematics teachers' geometry content knowledge: a typical case? In Nieuwoudt, S., Froneman, S. and Nkhoma, P. (Eds.). *Proceedings of the Tenth National Congress of the Association for*

Mathematics Education of South Africa, Vol.1. Pp 250–262. Potchefstroom:Advert Square.

Van Niekerk, R. (1998). What is happening to primary school geometry in South Africa? *Pythagoras*, 46/47, 63–70.

Weber, K. (2001). Student difficulty in constructing proofs: the need for strategic knowledge. *Educational Studies in Mathematics*, 48, Pp 101–119. Dordrecht:Kluwer Academic Publishers.

Yushan,B., Mji,A.,and Wessels, D.C.J. (2005). The role of technology in fostering creativity in the teaching and learning of mathematics. *Pythagoras,* 62. 12–22.

APPENDIX 1 : Letter of permission to conduct research at School

APPENDIX 2: Learner-based tasks

APPENDIX 3: Learners' responses to tasks

APPENDIX 4: Examiners' Reports