Results Math (2024) 79:241
Online First

https://doi.org/10.1007/s00025-024-02266-5

Results in Mathematics

Lower and Upper Bounds for the Generalized Csiszár f-divergence Operator Mapping

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Abstract. Let $\mathbf{A} = \{A_1, ..., A_n\}$ and $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ and f, $h : \mathbb{I} \to \mathbb{R}$ continuous functions with $h > 0$.. We consider the generalized Csiszár f-divergence operator mapping defined by

$$\mathbf{I}_{f\triangle h}(\mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} P_{f\triangle h}(A_i, B_i),$$

where

$$P_{f\triangle h}(A, B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

is introduced for every strictly positive operator A and every self-adjoint operator B, where the spectrum of the operators

$$A, A^{-1/2} B A^{-1/2} \text{ and } h(A)^{-1/2} B h(A)^{-1/2}$$

are contained in the closed interval $\mathbb{I}$. In this paper we obtain some lower and upper bounds for $\mathbf{I}_{f\triangle h}(\mathbf{A}, \mathbf{B})$ with applications to the geometric operator mean and the relative operator entropy. We verify the information monotonicity for the Csisz ár f-divergence operator mapping and the generalized Csiszár f-divergence operator mapping.

Mathematics Subject Classification. Primary 81P45, 15A39; Secondary 47A63, 15A42, 81R15.

Keywords. Perspective function, generalized perspective function, relative operator entropy, operator geometric mean.

Published online: 05 September 2024 Birkhäuser

1. Introduction

The classical perspective function associated to a continuous function $f : [0, \infty) \to \mathbb{R}$ is a function of two variables defined by $P_f(s,t) := sf(\frac{t}{s})$, cf. [19]. For two discrete probability distributions $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ the f-divergence functional

$$I_f(p,q) = \sum_{i=1}^{n} P_f(p_i, q_i)$$

was introduced by Csiszár [6] as a distance function on the set of discrete probability distributions.

Let f and h be two real valued continuous functions defined on the closed interval $\mathbb{I}$ and $h > 0$. The value $f(A)$ is defined via the functional calculus as usual for a self–adjoint operator A whose spectrum is contained in $\mathbb{I}$. A fully noncommutative perspective of two variables (associated to f), by choosing an appropriate ordering, was introduced in [12] by setting

$$P_f(A,B) := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

and the operator version of a fully noncommutative generalized perspective of two variables (associated to f and h) was also introduced by setting

$$P_{f\Delta h}(A,B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

for every strictly positive operator A and every self-adjoint operator B on a Hilbert space $\mathcal{H}$, where the spectrum of the operators

$$A, A^{-1/2} B A^{-1/2} \text{ and } h(A)^{-1/2} B h(A)^{-1/2}$$

are contained in the closed interval $\mathbb{I}$. Note that in this situation $P_{f\Delta h}(A,B) = P_f(h(A), B)$. Then, several striking matrix analogues of a classical result for operator convex functions were proved. More precisely, the necessary and sufficient conditions for the joint convexity of a fully noncommutative perspective and generalized perspective function were proved where restricting to the positive commuting matrices ensures Effros' approach announced in [13].

Throughout this section we assume that f and h are continuous real valued functions defined on $[0, \infty)$ and $h > 0$ unless we note otherwise.

Let $\mathbf{A} = \{A_1, ..., A_n\}$ and $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators and $f : \mathbb{I} \to \mathbb{R}$ a continuous function. We consider the Csiszár f-divergence operator mapping by setting

$$\mathbf{I}_f(\mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} P_f(A_i, B_i)$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} P_{f\Delta h}(A_i, B_i).$$

The joint convexity of the perspective and generalized perspective was proved in [12,25,29].

Theorem 1. *The following statements hold:*

(i) *If f is operator convex, then P_f is jointly convex.*
(ii) *If f is operator convex with $f(0) \leq 0$ and h is operator concave, then $P_{f\Delta h}$ is jointly convex.*
(iii) *If f and h are operator concave with $f(0) \geq 0$, then $P_{f\Delta h}$ is jointly concave.*

These results can be generalized to the Csiszár f-divergence operator mappings [22]. The following corollary is a simple application of the joint convexity of the perspective.

Corollary 1. *The following statements hold:*

(i) *If f is operator convex, then $\mathbf{I}_f$ is jointly convex.*
(ii) *If f is operator convex with $f(0) \leq 0$ and h is operator concave, then $\mathbf{I}_{f\Delta h}$ is jointly convex.*
(iii) *If f and h are operator concave with $f(0) \geq 0$, then $\mathbf{I}_{f\Delta h}$ is jointly concave.*

For a continuous function $g : (0, \infty) \to \mathbb{R}$ the transpose function $\tilde{g}$ of g is defined by

$$\tilde{g}(x) = xg(x^{-1}),\ x > 0.$$

Corollary 2. *Let $g : (0, \infty) \to \mathbb{R}$ be a continuous function. Then,*

$$\mathbf{I}_{\tilde{g}}(\mathbf{A}, \mathbf{B}) = \mathbf{I}_g(\mathbf{B}, \mathbf{A})$$

for two finite sequences of strictly positive operators $\mathbf{A}$ *and* $\mathbf{B}$.

2. Upper and Lower Bounds

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x - a)\varphi(a) \text{ for any } x, a \in I. \tag{2.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

For a finite sequence of strictly positive operators $\mathbf{A} = \{A_1, ..., A_n\}$ on a Hilbert space $\mathcal{H}$ and a continuous function f, we set

$$S_{\mathbf{A}} := \sum_{i=1}^{n} A_i \text{ and } S_{f(\mathbf{A})} := \sum_{i=1}^{n} f(A_i),$$

where $f(\mathbf{A}) := \{f(A_1), ..., f(A_n)\}$ is a finite sequence of operators on $\mathcal{H}$.

We have:

Theorem 2. *Let* $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ *be two finite sequences of strictly positive operators. If* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *is a convex function and* $h : [a, b] \to (0, \infty)$ *is a continuous function such that*

$$mh(A_i) \leq B_i \leq Mh(A_i), \quad i = 1, ..., n \tag{2.2}$$

for some m, $M \in [a, b]$ *with* $0 < m < M$, *then*

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \geq f\,(t)\, S_{h(\mathbf{A})} + \varphi\,(t) \left(S_{\mathbf{B}} - tS_{h(\mathbf{A})}\right) \tag{2.3}$$

for all $t \in (a, b)$ *and* $\varphi \in \partial f$.

In particular,

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \geq f\left(\frac{m + M}{2}\right) S_{h(\mathbf{A})} + \varphi\left(\frac{m + M}{2}\right)\left(S_{\mathbf{B}} - \frac{m + M}{2} S_{h(\mathbf{A})}\right).$$

Proof. From (2.1) we have

$$f\,(x) \geq f\,(t) + (x - t)\,\varphi\,(t) \tag{2.4}$$

for any $x \in [m, M]$ and $t \in (a, b)$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp\,(X) \subseteq [m, M] \subset (a, b)$ we have from (2.4) in the operator order that

$$f\,(X) \geq f\,(t)\, 1_H + \varphi\,(t)\,(X - t1_H) \tag{2.5}$$

for any $t \in (a, b)$.

If the condition (2.2) is valid, then by multiplying both sides by $h(A_i)^{-1/2}$ we get

$$m1_H \leq h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} \leq M1_H.$$

Now, if we take $X = h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}$, $i = 1, ..., n$, in (2.5), then we get

$$f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \geq f\,(t)\, 1_H + \varphi\,(t)\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t1_H\right) \tag{2.6}$$

for any $t \in (a, b)$ and $i = 1, ..., n$.

By multiplying both sides of (2.6) with $h(A_i)^{1/2}$ we get

$$\begin{aligned} &h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2} \\ &\quad \geq f(t)\, h(A_i) + \varphi(t)\, h(A_i)^{1/2} \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} \\ &\quad = f(t)\, h(A_i) + \varphi(t)\,(B_i - t h(A_i)) \end{aligned}$$

for any for any $t \in (a, b)$ and $i = 1, ..., n$, namely

$$P_{f\Delta h}(A_i, B_i) \geq f(t)\, h(A_i) + \varphi(t)\,(B_i - t h(A_i))$$

for all $t \in (a, b)$ and $i = 1, ..., n$.

If we sum over i from 1 to n, we derive the desired result (2.3). □

Corollary 3. *With the assumptions of Theorem 2, we have for any $x \in H \setminus \{0\}$ that*

$$\begin{aligned} \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \geq\; & f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle}\right) S_{h(\mathbf{A})} \\ & + \varphi\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle}\right) \left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle} S_{h(\mathbf{A})}\right). \end{aligned} \tag{2.7}$$

In particular,

$$\frac{\langle \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle} \geq f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle}\right). \tag{2.8}$$

Proof. For $x \in H \setminus \{0\}$ we have

$$\begin{aligned} t_{A,B} = \frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{h(\mathbf{A})} x, x\rangle} &= \frac{\left\langle S_{h(\mathbf{A})}^{1/2} \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) S_{h(\mathbf{A})}^{1/2} x, x\right\rangle}{\left\langle S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle} \\ &= \frac{\left\langle \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle}{\left\langle S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle} \\ &= \frac{\left\langle \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) S_{h(\mathbf{A})}^{1/2} x, S_{h(\mathbf{A})}^{1/2} x\right\rangle}{\left\| S_{h(\mathbf{A})}^{1/2} x\right\|^2}. \end{aligned}$$

If we put

$$u = \frac{S_{h(\mathbf{A})}^{1/2} x}{\left\| S_{h(\mathbf{A})}^{1/2} x\right\|} \neq 0,$$

then $\|u\| = 1$ and

$$t_{\mathbf{A},\mathbf{B}} = \left\langle \left(S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2}\right) u, u\right\rangle \in [m, M] \subset (a, b),$$

since, by summing in (2.2), we get

$$mS_{h(\mathbf{A})} \leq S_{\mathbf{B}} \leq MS_{h(\mathbf{A})},$$

which gives

$$mI \leq S_{h(\mathbf{A})}^{-1/2} S_{\mathbf{B}} S_{h(\mathbf{A})}^{-1/2} \leq MI.$$

By taking $t = t_{\mathbf{A},\mathbf{B}}$ in (2.3) we obtain (2.7).

The inequality (2.7) is equivalent to

$$\begin{aligned} &\langle \mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})y, y\rangle \\ &\quad \geq f\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{h(\mathbf{A})}x, x\rangle}\right)\langle S_{h(\mathbf{A})}y, y\rangle \\ &\qquad + \varphi\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{h(\mathbf{A})}x, x\rangle}\right)\left(\langle S_{\mathbf{B}}y, y\rangle - \frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{h(\mathbf{A})}x, x\rangle}\langle S_{h(\mathbf{A})}y, y\rangle\right) \end{aligned} \tag{2.9}$$

for any $y \in H$. This is an inequality of interest in itself.

In particular, if we take in (2.9) $y = x$, then we get the desired result (2.8). □

We also have:

Corollary 4. *With the assumptions of Theorem 2, we have*

$$\begin{aligned} \mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) &\geq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})} \\ &\quad - \frac{1}{M-m}\left[f(M)\left(MS_{h(\mathbf{A})} - S_{\mathbf{B}}\right) + f(m)\left(S_{\mathbf{B}} - mS_{h(\mathbf{A})}\right)\right]. \end{aligned} \tag{2.10}$$

Proof. If we take the integral mean in the inequality (2.3), then we get

$$\begin{aligned} &\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) \\ &\quad \geq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})} \\ &\qquad + \left(\frac{1}{M-m}\int_m^M \varphi(t)\,dt\right) S_{\mathbf{B}} - \left(\frac{1}{M-m}\int_m^M t\varphi(t)\,dt\right) S_{h(\mathbf{A})}. \end{aligned} \tag{2.11}$$

Observe that, since $\varphi \in \partial f$, hence

$$\frac{1}{M-m}\int_m^M \varphi(t)\,dt = \frac{f(M)-f(m)}{M-m}$$

and

$$\frac{1}{M-m}\int_m^M t\varphi(t)\,dt = \frac{1}{M-m}\left[tf(t)|_m^M - \int_m^M f(t)\,dt\right]$$
$$= \frac{Mf(M)-mf(m)}{M-m} - \frac{1}{M-m}\int_m^M f(t)\,dt$$

and by (2.11) we get

$$\begin{aligned}\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) &\geq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})} + \frac{f(M)-f(m)}{M-m} S_{\mathbf{B}} \\ &\quad - \left(\frac{Mf(M)-mf(m)}{M-m} - \frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})} \\ &= 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{h(\mathbf{A})} \\ &\quad - \frac{1}{M-m}\left[f(M)(MA - S_{\mathbf{B}}) + f(m)\left(S_{\mathbf{B}} - mS_{h(\mathbf{A})}\right)\right]\end{aligned}$$

that proves the desired result (2.10). □

Remark 1. If we take $h(t) = t$, and assume also that f is differentiable on (a,b) while

$$mA_i \leq B_i \leq MA_i, i = 1, ..., n \tag{2.12}$$

then

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq f(t)\, S_{\mathbf{A}} + f'(t)\,(S_{\mathbf{B}} - tS_{\mathbf{A}}) \tag{2.13}$$

for all $t \in (a,b)$.

In particular,

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq f\left(\frac{m+M}{2}\right) S_{\mathbf{A}} + f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}} - \frac{m+M}{2} S_{\mathbf{A}}\right). \tag{2.14}$$

Moreover, for any $x \in H \backslash \{0\}$ we have that

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right) S_{\mathbf{A}} + f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle} S_{\mathbf{A}}\right). \tag{2.15}$$

In particular,

$$\frac{\langle \mathbf{I}_f(\mathbf{A},\mathbf{B})x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle} \geq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right). \tag{2.16}$$

Also, we have

$$\mathbf{I}_f(\mathbf{A},\mathbf{B}) \geq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) S_{\mathbf{A}} - \frac{1}{M-m}\left[f(M)(MS_{\mathbf{A}} - S_{\mathbf{B}}) + f(m)(S_{\mathbf{B}} - mS_{\mathbf{A}})\right]. \tag{2.17}$$

We observe that for $n=1$, we recapture the results obtained in [10].

The following reverse of inequality (2.3) is as follows:

Theorem 3. *Let $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators. If $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function and $h : [a,b] \to (0,\infty)$ is a continuous function such that the condition (2.2) is valid, then for any $t \in (a,b)$ we have*

$$\begin{aligned}
&\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) \\
&\leq f(t)\,S_{h(\mathbf{A})} + \mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B}) - t\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B}) \\
&\leq f(t)\,S_{h(\mathbf{A})} + f'(t)\left(S_{\mathbf{B}} - tS_{h(\mathbf{A})}\right) + \left[f'_-(M) - f'_+(m)\right]\mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B}) \\
&\leq f(t)\,S_{h(\mathbf{A})} + f'(t)\left(S_{\mathbf{B}} - tS_{h(\mathbf{A})}\right) + (M-m)\left[f'_-(M) - f'_+(m)\right]S_{h(\mathbf{A})}
\end{aligned} \tag{2.18}$$

where ℓ is the identity function, i.e. $\ell(t) = t$ and

$$\mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B}) := \sum_{i=1}^n h(A_i)^{1/2}\left|h(A_i)^{-1/2}\left(B_i - th(A_i)\right)h(A_i)^{-1/2}\right|h(A_i)^{1/2}.$$

In particular, we have

$$\begin{aligned}
\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) &\leq f\left(\frac{m+M}{2}\right)S_{h(\mathbf{A})} + \mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B}) - \frac{m+M}{2}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B}) \\
&\leq f\left(\frac{m+M}{2}\right)S_{h(\mathbf{A})} + f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}} - \frac{m+M}{2}S_{h(\mathbf{A})}\right) \\
&\quad + \left[f'_-(M) - f'_+(m)\right]\mathbf{I}_{\left(|\cdot|,\frac{m+M}{2}\right)\Delta h}(\mathbf{A},\mathbf{B}) \\
&\leq f\left(\frac{m+M}{2}\right)S_{h(\mathbf{A})} + f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}} - \frac{m+M}{2}S_{h(\mathbf{A})}\right) \\
&\quad + \frac{1}{2}(M-m)\left[f'_-(M) - f'_+(m)\right]S_{h(\mathbf{A})}.
\end{aligned} \tag{2.19}$$

Proof. By the gradient inequality we have

$$f'(x)(x-t) + f(t) \geq f(x) \tag{2.20}$$

for any $x \in [m,M]$ and $t \in (a,b)$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp(X) \subseteq [m,M] \subset \mathring{I}$ we have from (2.13) in the operator order that

$$f'(X)(X - t1_H) + f(t)\,1_H \geq f(X) \tag{2.21}$$

for any $t \in (a,b)$.

Now, if we take $X = h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}$, $i = 1, ..., n$, in (2.21), then we get

$$
\begin{aligned}
&f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right)\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) + f(t)\, 1_H \\
&\quad \ge f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right)
\end{aligned}
\tag{2.22}
$$

for any for any $t \in (a,b)$ and $i = 1, ..., n$.

If we multiply both sides of (2.22) by $h(A_i)^{1/2}$, then we obtain

$$
\begin{aligned}
&h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right)\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) \\
&\quad h(A_i)^{1/2} + f(t)\, h(A_i) \\
&\quad \ge h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2}
\end{aligned}
$$

for any $t \in (a,b)$ and $i = 1, ..., n$.

If we sum over i from 1 to n, then we get

$$
\begin{aligned}
&\sum_{i=1}^{n} h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \\
&\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} + f(t) \sum_{i=1}^{n} h(A_i) \\
&\quad \ge \sum_{i=1}^{n} h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2}.
\end{aligned}
\tag{2.23}
$$

Since

$$
\begin{aligned}
&\sum_{i=1}^{n} h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \\
&\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} + f(t) \sum_{i=1}^{n} h(A_i) \\
&\quad = \mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A}, \mathbf{B}) - t \mathbf{I}_{f'\Delta h}(\mathbf{A}, \mathbf{B}) + f(t)\, S_{h(\mathbf{A})}
\end{aligned}
$$

and

$$
\sum_{i=1}^{n} h(A_i)^{1/2} f\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) h(A_i)^{1/2} = \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}),
$$

then by (2.23) we get the first inequality in (2.18).

Now, observe also that

$$\begin{aligned}
&\sum_{i=1}^{n} h(A_i)^{1/2} f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) \\
&\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} + f(t) \sum_{i=1}^{n} h(A_i) \\
&\quad = \sum_{i=1}^{n} h(A_i)^{1/2} \left[f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) - f'(t) 1_H\right] \\
&\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} \\
&\qquad + f'(t) \left(\sum_{i=1}^{n} B_i - t \sum_{i=1}^{n} h(A_i)\right) + f(t) \sum_{i=1}^{n} h(A_i)
\end{aligned}$$

for any $t \in (a, b)$.

Since f' is nondecreasing on (a, b) we have for any $x \in [m, M]$ and $t \in (a, b)$ that

$$\begin{aligned}
0 &\leq (f'(x) - f'(t))(x - t) = |(f'(x) - f'(t))(x - t)| \\
&= |f'(x) - f'(t)| \, |x - t| \leq \left[f'_-(M) - f'_+(m)\right] |x - t| \\
&\leq (M - m) \left[f'_-(M) - f'_+(m)\right],
\end{aligned}$$

which, as above, implies in the operator order that

$$\begin{aligned}
&h(A_i)^{1/2} \left[f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) - f'(t) 1_H\right] \\
&\times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} \\
&\leq \left[f'_-(M) - f'_+(m)\right] h(A_i)^{1/2} \left|h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right| h(A_i)^{1/2} \\
&= \left[f'_-(M) - f'_+(m)\right] h(A_i)^{1/2} \left|h(A_i)^{-1/2} (B_i - t h(A_i)) h(A_i)^{-1/2}\right| h(A_i)^{1/2} \\
&\leq (M - m) \left[f'_-(M) - f'_+(m)\right] h(A_i).
\end{aligned}$$

If we sum over i from 1 to n, then we get

$$\begin{aligned}
&\sum_{i=1}^{n} h(A_i)^{1/2} \left[f'\left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}\right) - f'(t) 1_H\right] \\
&\qquad \times \left(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - t 1_H\right) h(A_i)^{1/2} \\
&\quad \leq \left[f'_-(M) - f'_+(m)\right] \sum_{i=1}^{n} h(A_i)^{1/2} \left|h(A_i)^{-1/2} (B_i - t h(A_i)) \right. \\
&\qquad \left. h(A_i)^{-1/2}\right| h(A_i)^{1/2}
\end{aligned}$$

$$\leq (M-m)\left[f'_-(M)-f'_+(m)\right]\sum_{i=1}^{n} h(A_i)$$

This proves the second inequality in (2.18).

We need to prove only the last part of (2.19).

Since $x \in [m, M]$, then $\left|x - \frac{m+M}{2}\right| \leq \frac{1}{2}(M-m)$ that implies in the operator order

$$\left|h(A_i)^{-1/2} B_i A h(A_i)^{-1/2} - \frac{m+M}{2} 1_H\right| \leq \frac{1}{2}(M-m)\, 1_H,$$

which by multiplication on both sides with $h(A_i)^{1/2}$ and summing over i gives that

$$\mathbf{I}_{\left(|\cdot|, \frac{m+M}{2}\right)\Delta h}(\mathbf{A}, \mathbf{B}) \leq \frac{1}{2}(M-m)\, S_{h(\mathbf{A})}.$$

□

Corollary 5. *With the assumptions of Theorem 3, we have for any $x \in H\setminus\{0\}$ that*

$$\begin{aligned}
&\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \\
&\quad \leq f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right) S_{h(\mathbf{A})} + \mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A}, \mathbf{B}) - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\mathbf{I}_{f'\Delta h}(\mathbf{A}, \mathbf{B}) \\
&\quad \leq f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right) S_{h(\mathbf{A})} \\
&\qquad + f'\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right)\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle} S_{h(\mathbf{A})}\right) \\
&\qquad + \left[f'_-(M) - f'_+(m)\right] \mathbf{I}_{\left(|\cdot|, \frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right)\Delta h}(\mathbf{A}, \mathbf{B}) \\
&\quad \leq f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right) S_{h(\mathbf{A})} \\
&\qquad + f'\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right)\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle} S_{h(\mathbf{A})}\right) \\
&\qquad + (M-m)\left[f'_-(M) - f'_+(m)\right] S_{h(\mathbf{A})}.
\end{aligned} \tag{2.24}$$

In particular,

$$\begin{aligned}
\langle \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) x, x\rangle &\leq f\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\right)\left\langle S_{h(\mathbf{A})} x, x\right\rangle + \left\langle \mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A}, \mathbf{B}) x, x\right\rangle \\
&\quad - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\left\langle S_{h(\mathbf{A})} x, x\right\rangle}\langle \mathbf{I}_{f'\Delta h}(\mathbf{A}, \mathbf{B}) x, x\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\left\langle S_{h(\mathbf{A})}x,x\right\rangle}\right)\left\langle S_{h(\mathbf{A})}x,x\right\rangle \\
&\quad+\left[f'_{-}\left(M\right)-f'_{+}\left(m\right)\right]\left\langle \mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\left\langle S_{h(\mathbf{A})}x,x\right\rangle}\right)\Delta h}(\mathbf{A},\mathbf{B})x,x\right\rangle \\
&\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\left\langle S_{h(\mathbf{A})}x,x\right\rangle}\right)\left\langle S_{h(\mathbf{A})}x,x\right\rangle \\
&\quad+\left(M-m\right)\left[f'_{-}\left(M\right)-f'_{+}\left(m\right)\right]\left\langle S_{h(\mathbf{A})}x,x\right\rangle \qquad (2.25)
\end{aligned}$$

for any $x\in H\backslash\{0\}$.

We also have:

Corollary 6. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}
\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) &\leq \left(\frac{1}{M-m}\int_m^M f\left(t\right)dt\right)S_{h(\mathbf{A})} \\
&\quad+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B}) \\
&\leq 2\left(\frac{1}{M-m}\int_m^M f\left(t\right)dt\right)S_{h(\mathbf{A})} \\
&\quad-\frac{1}{M-m}\left[\Phi\left(M\right)\left(MS_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+\Phi\left(m\right)\left(S_{\mathbf{B}}-mS_{h(\mathbf{A})}\right)\right] \\
&\quad+\frac{f'_{-}\left(M\right)-f'_{+}\left(m\right)}{M-m}\int_m^M \mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B})dt. \qquad (2.26)
\end{aligned}$$

Proof. If we take the integral mean in (2.12), then we get

$$\begin{aligned}
&\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) \\
&\quad\leq \left(\frac{1}{M-m}\int_m^M f\left(t\right)dt\right)S_{h(\mathbf{A})}+\mathbf{I}_{(f'\ell)\Delta h}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'\Delta h}(\mathbf{A},\mathbf{B}) \\
&\quad\leq \left(\frac{1}{M-m}\int_m^M f\left(t\right)dt\right)S_{h(\mathbf{A})}+\frac{1}{M-m}\int_m^M f'\left(t\right)\left(S_{\mathbf{B}}-tS_{h(\mathbf{A})}\right)dt \\
&\qquad+\frac{f'_{-}\left(M\right)-f'_{+}\left(m\right)}{M-m}\int_m^M \mathbf{I}_{(|\cdot|,t)\Delta h}(\mathbf{A},\mathbf{B})dt. \qquad (2.27)
\end{aligned}$$

Since, as in the proof of Corollary 4, we have

$$\begin{aligned}
&\frac{1}{M-m}\int_m^M f'\left(t\right)\left(S_{\mathbf{B}}B-tS_{h(\mathbf{A})}\right)dt \\
&\qquad=\left(\frac{1}{M-m}\int_m^M f\left(t\right)dt\right)S_{h(\mathbf{A})}
\end{aligned}$$

$$-\frac{1}{M-m}\left[f(M)\left(MS_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+f(m)\left(S_{\mathbf{B}}-mS_{h(\mathbf{A})}\right)\right],$$

then by (2.27) we get the last part of (2.26). □

Remark 2. If we take $h(t)=t$, and assume also that f is differentiable on (a,b) while $mA_i \le B_i \le MA_i$, $i=1,...,n$, then

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B}) &\le f(t)\,S_{\mathbf{A}}+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-t\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\ &\le f(t)\,S_{\mathbf{A}}+f'(t)\,(S_{\mathbf{B}}-tS_{\mathbf{A}})+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{(|\cdot|,t)}(\mathbf{A},\mathbf{B})\\ &\le f(t)\,S_{\mathbf{A}}+f'(t)\,(S_{\mathbf{B}}-tS_{\mathbf{A}})+(M-m)\left[f'_-(M)-f'_+(m)\right]S_{\mathbf{A}}\end{aligned}\tag{2.28}$$

for all $t\in(a,b)$, where ℓ is the identity function, i.e. $\ell(t)=t$ and

$$\mathbf{I}_{(|\cdot|,t)}(\mathbf{A},\mathbf{B}):=\sum_{i=1}^{n}{A_i}^{1/2}\left|{A_i}^{-1/2}(B_i-tA_i)\,{A_i}^{-1/2}\right|{A_i}^{1/2}.$$

In particular, we have

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B}) &\le f\left(\frac{m+M}{2}\right)S_{\mathbf{A}}+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\ &\le f\left(\frac{m+M}{2}\right)S_{\mathbf{A}}+f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}}-\frac{m+M}{2}S_{\mathbf{A}}\right)\\ &\quad+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{\left(|\cdot|,\frac{m+M}{2}\right)}(\mathbf{A},\mathbf{B})\\ &\le f\left(\frac{m+M}{2}\right)S_{\mathbf{A}}+f'\left(\frac{m+M}{2}\right)\left(S_{\mathbf{B}}-\frac{m+M}{2}S_{\mathbf{A}}\right)\\ &\quad+\frac{1}{2}(M-m)\left[f'_-(M)-f'_+(m)\right]S_{\mathbf{A}}.\end{aligned}\tag{2.29}$$

We also have

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B}) &\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}}+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\ &\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}}+f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\left(S_{\mathbf{B}}-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}S_{\mathbf{A}}\right)\\ &\quad+\left[f'_-(M)-f'_+(m)\right]\mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)}(\mathbf{A},\mathbf{B})\\ &\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}}+f'\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\left(S_{\mathbf{B}}-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}S_{\mathbf{A}}\right)\\ &\quad+(M-m)\left[f'_-(M)-f'_+(m)\right]S_{\mathbf{A}}\end{aligned}\tag{2.30}$$

and

$$\begin{aligned}&\langle\mathbf{I}_f(\mathbf{A},\mathbf{B})x,x\rangle\\ &\quad\le f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle+\langle\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})x,x\rangle-\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\langle\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})x,x\rangle\end{aligned}$$

$$\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle+\left[f'_{-}(M)-f'_{+}(m)\right]\left\langle \mathbf{I}_{\left(|\cdot|,\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)}(\mathbf{A},\mathbf{B})x,x\right\rangle$$

$$\leq f\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)\langle S_{\mathbf{A}}x,x\rangle+(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]\langle S_{\mathbf{A}}x,x\rangle \quad (2.31)$$

for $x\in H\backslash\{0\}$.

Also, we have

$$\begin{aligned}\mathbf{I}_f(\mathbf{A},\mathbf{B}) &\leq \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{\mathbf{A}}\\ &\quad+\mathbf{I}_{f'\ell}(\mathbf{A},\mathbf{B})-\frac{m+M}{2}\mathbf{I}_{f'}(\mathbf{A},\mathbf{B})\\ &\leq 2\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)S_{\mathbf{A}}\\ &\quad-\frac{1}{M-m}\left[\Phi(M)(MS_{\mathbf{A}}-S_{\mathbf{B}})+\Phi(m)(S_{\mathbf{B}}-mS_{\mathbf{A}})\right]\\ &\quad+\frac{f'_{-}(M)-f'_{+}(m)}{M-m}\int_m^M \mathbf{I}_{(|\cdot|,t)}(\mathbf{A},\mathbf{B})dt. \end{aligned}\quad (2.32)$$

We observe that for $n=1$, we recapture the results obtained in [10].

3. Information Monotonicity

We now prove the information monotonicity of the generalized Csiszár f-divergence operator as follows. In our main results we find the complementary inequalities and some other applications. Note that for a positive linear map Φ and a finite sequence of strictly positive operators $\mathbf{A}=\{A_1,\dots,A_n\}$ by $\Phi(\mathbf{A})$ we mean the finite sequence of strictly positive operators $\{\Phi(A_1),\dots,\Phi(A_n)\}$.

Theorem 4. *If $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is a positive linear map, f an operator convex function on an interval $[a,b]$ and $h:[a,b]\to(0,\infty)$ a continuous function such that the condition (2.2) is valid for some m, $M\in[a,b]$ with $0<m<M$, then*

$$\mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))\leq\Phi(\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}))$$

for any strictly positive operators A_i, any self-adjoint operator B_i with

$$Sp(h(A_i)^{-1/2}B_ih(A_i)^{-1/2}),Sp(h(\Phi(A_i))^{-1/2}\Phi(B_i)h(\Phi(A_i))^{-1/2})\subseteq[m,M]$$

and any Hilbert spaces H, K.

Proof. Following an idea of Ando [2] to a fixed strictly positive operator $A_i\in\mathcal{B}(H)$ we set

$$\Psi(X)=h(\Phi(A_i))^{-1/2}\Phi(h(A_i)^{1/2}Xh(A_i)^{1/2})h(\Phi(A_i))^{-1/2}$$

and notice that $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a unital linear map. By operator convexity of f and Choi-Davis-Jensen inequality, we realize that

$$f(\Psi(X)) \leq \Psi(f(X))$$

for all positive invertible operator X. Therefore,

$$f\left(h(\Phi(A_i))^{-1/2}\Phi(h(A_i)^{1/2}Xh(A_i)^{1/2})h(\Phi(A_i))^{-1/2} \right) \leq h(\Phi(A_i))^{-1/2}\Phi(h(A_i)^{1/2}f(X)h(A_i)^{1/2})h(\Phi(A_i))^{-1/2}$$

For $X = h(A_i)^{-1/2}B_iA_i^{-1/2}$ this entails

$$P_{f\Delta h}(\Phi(A_i), \Phi(B_i)) \leq \Phi(P_{f\Delta h}(A_i, B_i).$$

If we sum over i from 1 to n, then we get

$$\sum_{i=i}^{n} P_{f\Delta h}(\Phi(A_i), \Phi(B_i)) \leq \sum_{i=i}^{n} \Phi(P_{f\Delta h}(A_i, B_i) = \Phi\left(\sum_{i=i}^{n} P_{f\Delta h}(A_i, B_i) \right),$$

which entails the result. □

We now prove a complementary inequality for the information monotonicity of the operator perspective.

Corollary 7. *Let f be a differentiable function on $[a, b]$ and $h : [a, b] \to (0, \infty)$ a continuous function such that the condition (2.2) is valid for some m, $M \in [a, b]$ with $0 < m < M$, any strictly positive operators A_i and any self-adjoint operator B_i.*

(i) *If f' is decreasing, then*

$$f(m)S_{h(\mathbf{A})} + f'(b)(S_{\mathbf{B}} - mS_{h(\mathbf{A})}) \leq \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \leq f(m)S_{h(\mathbf{A})} + f'(a)(S_{\mathbf{B}} - mS_{h(\mathbf{A})}).$$

(ii) *If f' is increasing, then the reverse inequalities in part (i) hold.*

Proof. (i) Regarding the assumption f' is decreasing and so $f'(b) \leq f'(t) \leq f'(a)$ for any $t \in [a, b]$. Define $g(t) := f(t) - f'(b)t$ and $k(t) := f'(a)t - f(t)$ for any $t \in [a, b]$. Since g is increasing, $g(m) \leq g(t)$ for any $t \in [m, M]$ and so

$$f(m) - f'(b)m \leq f(t) - f'(b)t,$$

which entails

$$f(m) + f'(b)(t - m) \leq f(t) \tag{3.1}$$

for any $t \in [m, M]$. Since k is increasing, $k(m) \leq k(t)$ for any $t \in [m, M]$ and hence

$$f'(a)m - f(m) \leq f'(a)t - f(t).$$

This inequality signifies

$$f(t) \leq f(m) + f'(a)(t-m) \tag{3.2}$$

for any $t \in [m, M]$. Combining the inequalities (3.1) and (3.2), one can yield

$$f(m) + f'(b)(t-m) \leq f(t) \leq f(m) + f'(a)(t-m) \tag{3.3}$$

for any $t \in [m, M]$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp\,(X) \subseteq [m, M] \subset (a, b)$ we have from (3.3) in the operator order that

$$f(m)1_H + f'(b)(X - m1_H) \leq f(X) \leq f(m)1_H + f'(a)(X - m1_H) \tag{3.4}$$

for any $t \in [m, M]$. If we take $X = h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}$, $i = 1, ..., n$, in (3.4) and multiplying both sides by $h(A_i)^{-1/2}$ we get

$$\begin{aligned} f(m)h(A_i) + f'(b)(B_i - mh(A_i)) &\leq h(A_i) f(h(A_i)^{-1/2} B_i A h(A_i)^{-1/2}) h(A_i) \\ &\leq f(m)h(A_i) + f'(a)(B_i - mh(A_i)) \end{aligned}$$

If we sum over i from 1 to n, then we get the result.

(ii) Applying a similar way as in the proof of part (i) we reach the result. □

Corollary 8. *If $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a positive linear map, f an operator convex function on $[a, b]$ and $h : [a, b] \to (0, \infty)$ a continuous function such that the condition (2.2) is valid for some m, $M \in [a, b]$ with $0 < m < M$, any strictly positive operators A_i and any self-adjoint operator B_i, then*

$$\begin{aligned} \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})) \leq \mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}), \Phi(\mathbf{B})) &+ f(m)(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) \\ &+ (f'(b) - f'(a))S_{\Phi(\mathbf{B})} \\ &+ m(f'(a)S_{h(\Phi(\mathbf{A}))} - f'(b)S_{\Phi(h(\mathbf{A}))}). \end{aligned}$$

In particular, if $\Phi \circ h = h \circ \Phi$, then

$$\begin{aligned} \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})) &\leq \mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}), \Phi(\mathbf{B})) \\ &\quad + (f'(b) - f'(a))(S_{\Phi(\mathbf{B})} - mS_{\Phi(h(\mathbf{A}))}). \end{aligned}$$

Moreover, if $h(t) = t$, then

$$\begin{aligned} \Phi(\mathbf{I}_f(\mathbf{A}, \mathbf{B})) &\leq \mathbf{I}_f(\Phi(\mathbf{A}), \Phi(\mathbf{B})) \\ &\quad + (f'(b) - f'(a))(S_{\Phi(\mathbf{B})} - mS_{\Phi(\mathbf{A})}). \end{aligned}$$

Proof. Since f is operator convex, so that f is differentiable and f' is increasing. Using Corollary 7 (ii) we find that

$$\begin{aligned} f(m)S_{h(\mathbf{A})} + f'(b)(S_{\mathbf{B}} - mS_{h(\mathbf{A})}) &\geq \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) \\ &\geq f(m)S_{h(\mathbf{A})} + f'(a)(S_{\mathbf{B}} - mS_{h(\mathbf{A})}). \end{aligned} \tag{3.5}$$

Substituting $\Phi(\mathbf{A}), \Phi(\mathbf{B})$ by $\mathbf{A}, \mathbf{B}$ in the second inequality one can deduce

$$\mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}), \Phi(\mathbf{B})) \geq f(m)S_{h(\Phi(\mathbf{A}))} + f'(a)(S_{\Phi(\mathbf{B})} - mS_{h(\Phi(\mathbf{A}))}). \tag{3.6}$$

It follows from (3.6) that

$$\mathbf{I}_{f\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))-f(m)S_{h(\Phi(\mathbf{A}))}-f'(a)(S_{\Phi(\mathbf{B})}-mS_{h(\Phi(\mathbf{A}))})\geq 0. \quad (3.7)$$

By linearity and positiveness of Φ, the second inequality in (3.5) leads to the following inequality

$$f(m)\Phi(S_{h(\mathbf{A})})+f'(b)(\Phi(S_{\mathbf{B}})-m\Phi(S_{h(\mathbf{A})}))\geq \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})). \quad (3.8)$$

Since $\Phi(S_{h(\mathbf{A})})=S_{\Phi(h(\mathbf{A}))}$ and $\Phi(S_{\mathbf{B}})=S_{\Phi(\mathbf{B})}$, from (3.8) we have

$$f(m)S_{\Phi(h(\mathbf{A}))}+f'(b)(S_{\Phi(\mathbf{B})}-mS_{\Phi(h(\mathbf{A}))})\geq \Phi(\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})). \quad (3.9)$$

Summing the inequalities (3.7) and (3.9), we reach the desired result.

Note that if $\Phi\circ h=h\circ\Phi$, then $S_{\Phi(h(\mathbf{A}))}=S_{h(\Phi(\mathbf{A}))}$ and the result follows. □

4. Applications for Operator Geometric Mean

Assume that A, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_\nu B:=(1-\nu)\,A+\nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_\nu B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted operator geometric mean,* where $\nu\in[0,1]$. When $\nu=\frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The definition $A\sharp_\nu B$ can be extended accordingly for any real number ν.

The following inequality is well as the operator *Young inequality* or operator *ν-weighted arithmetic–geometric mean inequality*: [16]

$$A\sharp_\nu B\leq A\nabla_\nu B \text{ for all } \nu\in[0,1]. \quad (4.1)$$

We consider the continuous function $f_\nu:[0,\infty)\to[0,\infty)$, $f_\nu(x)=x^\nu$ for $\nu\in[0,1]$. Consider the convex function $f=-f_\nu$. We consider the Csisz ár f-divergence operator mapping by setting

$$\mathbf{I}_{f_\nu}(\mathbf{A},\mathbf{B})=-\sum_{i=1}^n A_i\sharp_\nu B_i$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{f_\nu\Delta h}(\mathbf{A},\mathbf{B})=-\sum_{i=1}^n h\,(A_i)\,\sharp_\nu B_i.$$

Assume that $\mathbf{A}=\{A_1,...,A_n\}$, $\mathbf{B}=\{B_1,...,B_n\}$ satisfy the condition

$$mA_i\leq B_i\leq MA_i,\quad i=1,...,n$$

then by (2.13) we derive

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \le t^\nu (1-\nu) S_{\mathbf{A}} + \nu t^{\nu-1} S_{\mathbf{B}} \tag{4.2}$$

for all $t \in (a,b)$ which contains the interval $[m,M] \subset (0,\infty)$.

In particular,

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \le (1-\nu) \left(\frac{m+M}{2}\right)^\nu S_{\mathbf{A}} + \nu \left(\frac{m+M}{2}\right)^{\nu-1} S_{\mathbf{B}}. \tag{4.3}$$

Moreover, for any $x \in Hc\{0\}$ we have that

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \le (1-\nu) \left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right)^\nu S_{\mathbf{A}} + \nu \left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right)^{\nu-1} S_{\mathbf{B}}. \tag{4.4}$$

In particular,

$$\left\langle \sum_{i=1}^{n} A_i \sharp_\nu B_i x, x \right\rangle \le \langle S_{\mathbf{B}} x, x\rangle^\nu \langle S_{\mathbf{A}} x, x\rangle^{1-\nu}, \tag{4.5}$$

which holds for any $A_{i,}$ $B_i \ge 0$. Moreover, if we take the supremum over $\|x\| = 1$, we deduce the norm inequality

$$\left\| \sum_{i=1}^{n} A_i \sharp_\nu B_i \right\| \le \|S_{\mathbf{A}}\|^{1-\nu} \|S_{\mathbf{B}}\|^\nu$$

for $\nu \in [0,1]$.

We observe that, if we write the inequality (4.1) for A_i, B_i an sum, then we get

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \le (1-\nu) S_{\mathbf{A}} + \nu S_{\mathbf{B}},$$

which is equivalent to

$$\sum_{i=1}^{n} \langle A_i \sharp_\nu B_i x, x\rangle \le (1-\nu) \langle S_{\mathbf{A}} x, x\rangle + \nu \langle S_{\mathbf{B}} x, x\rangle \tag{4.6}$$

for any $x \in H \backslash \{0\}$.

Since by the scalar *A-G-inequality* we have

$$\langle S_{\mathbf{B}} x, x\rangle^\nu \langle S_{\mathbf{A}} x, x\rangle^{1-\nu} \le (1-\nu) \langle S_{\mathbf{A}} x, x\rangle + \nu \langle S_{\mathbf{B}} x, x\rangle,$$

for any $x \in H \backslash \{0\}$, then we can conclude that the inequality (4.5) is better than (4.6).

For $x \ne y$ and $p \in \mathbb{R} \backslash \{-1, 0\}$, we define the *p- logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

From (2.17) we derive

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \leq 2L_\nu^\nu(m,M)S_{\mathbf{A}} - \frac{1}{M-m}\left[M^\nu\left(MS_{\mathbf{A}} - S_{\mathbf{B}}\right) + m^\nu\left(S_{\mathbf{B}} - mS_{\mathbf{A}}\right)\right]. \tag{4.7}$$

From the inequality (2.28) we derive

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \geq t^\nu S_{\mathbf{A}} + \nu t^{\nu-1}\left(S_{\mathbf{B}} - tS_{\mathbf{A}}\right) - \nu\left(M-m\right)\left(m^{\nu-1} - M^{\nu-1}\right)S_{\mathbf{A}}$$

for all $t \in (a,b)$ which contains the interval $[m,M] \subset (0,\infty)$.

From (2.29) we get

$$\sum_{i=1}^{n} A_i \sharp_\nu B_i \geq \left(\frac{m+M}{2}\right)^\nu S_{\mathbf{A}} + \nu\left(\frac{m+M}{2}\right)^{\nu-1}\left(S_{\mathbf{B}} - \frac{m+M}{2}S_{\mathbf{A}}\right) - \frac{1}{2}\nu\left(M-m\right)\left(m^{\nu-1} - M^{\nu-1}\right)S_{\mathbf{A}}. \tag{4.8}$$

From (2.31) we derive

$$\left\langle \sum_{i=1}^{n} A_i \sharp_\nu B_i x, x\right\rangle \geq \left\langle S_{\mathbf{B}}x, x\right\rangle^\nu \left\langle S_{\mathbf{A}}x, x\right\rangle^{1-\nu} - \nu\left(M-m\right)\left(m^{\nu-1} - M^{\nu-1}\right)\left\langle S_{\mathbf{A}}x, x\right\rangle \tag{4.9}$$

for any $x \in H \setminus \{0\}$.

5. Applications for Relative Operator Entropy

Kamei and Fujii [14], defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}, \tag{5.1}$$

which is a relative version of the operator entropy [3,15,21,32].

Consider the logarithmic function $\ln$. Then the relative operator entropy can be interpreted as the perspective of $-\ln$, namely

$$P_{-\ln}(A,B) = -S(A|B).$$

Consider the convex function $f = -\ln$. We consider the Csiszár f-divergence operator mapping by setting

$$\mathbf{I}_{-\ln}(\mathbf{A},\mathbf{B}) = -\sum_{i=1}^{n} S(A_i|B_i) = -\sum_{i=1}^{n} A_i^{1/2}\left(\ln\left(A_i^{-1/2}B_iA_i^{-1/2}\right)\right)A_i^{\frac{1}{2}}$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{(-\ln)\Delta h}(\mathbf{A},\mathbf{B}) = -\sum_{i=1}^{n} S\left(h\left(A_i\right)|B_i\right) = -\sum_{i=1}^{n} h\left(A_i\right)^{1/2}\left(\ln\left(h\left(A_i\right)^{-1/2}B_i h\left(A_i\right)^{-1/2}\right)\right)h\left(A_i\right)^{1/2}.$$

From Remark 1 we get

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq (\ln t)\, S_{\mathbf{A}} + t^{-1}\left(S_{\mathbf{B}} - tS_{\mathbf{A}}\right) \tag{5.2}$$

for all $t > 0$.

In particular, if $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ satisfy the condition (2.12), then

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq \ln\left(\frac{m+M}{2}\right)S_{\mathbf{A}} + \left(\frac{m+M}{2}\right)^{-1}\left(S_{\mathbf{B}} - \frac{m+M}{2}S_{\mathbf{A}}\right).$$

Moreover, for any $x \in H\setminus\{0\}$ we have that

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq \ln\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)S_{\mathbf{A}} + \left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right)^{-1}\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}S_{\mathbf{A}}\right).$$

In particular,

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right)x, x\right\rangle \leq \langle S_{\mathbf{A}}x,x\rangle \ln\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right). \tag{5.3}$$

The following inequality for the relative operator entropy is well known

$$S\left(A|B\right) \leq B - A$$

for any A, B positive invertible operators. This inequality is equivalent to

$$\langle S\left(A|B\right)x, x\rangle \leq \langle Bx, x\rangle - \langle Ax, x\rangle \tag{5.4}$$

for any $x \in H$.

By summing over i in (5.4) we get

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right)x, x\right\rangle \leq \langle S_{\mathbf{B}}x,x\rangle - \langle S_{\mathbf{A}}x,x\rangle. \tag{5.5}$$

We know the following elementary inequality that holds for the logarithm

$$\ln t \leq t - 1 \text{ for any} t > 0.$$

If we take in this inequality $t = \frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle} > 0$, $x \in H$, $x \neq 0$ and multiply with $\langle S_{\mathbf{A}}x,x\rangle > 0$, then we get

$$\langle S_{\mathbf{A}}x,x\rangle \ln\left(\frac{\langle S_{\mathbf{B}}x,x\rangle}{\langle S_{\mathbf{A}}x,x\rangle}\right) \leq \langle S_{\mathbf{B}}x,x\rangle - \langle S_{\mathbf{A}}x,x\rangle \tag{5.6}$$

for any $x \in H$, $x \neq 0$.

Therefore, by (5.3) and (5.6) we have

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right) x, x \right\rangle \leq \langle S_{\mathbf{A}} x, x\rangle \ln\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right) \leq \langle S_{\mathbf{B}} x, x\rangle - \langle S_{\mathbf{A}} x, x\rangle$$

for any $x \in H$, $x \neq 0$.

This shows that the inequality (5.3) is better than (5.5).

Also, we have

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \leq 2\ln I\left(m, M\right) S_{\mathbf{A}} - \frac{1}{M-m}\left[\ln\left(M\right)\left(MS_{\mathbf{A}} - S_{\mathbf{B}}\right) + \ln\left(m\right)\left(S_{\mathbf{B}} - mS_{\mathbf{A}}\right)\right], \quad (5.7)$$

where $I\left(m, M\right)$ is the *identric mean* defined for x, $y > 0$ by

$$I(x, y) := \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)} \quad (5.8)$$

and satisfies the equality

$$\frac{1}{M-m}\int_m^M \ln t dt = \ln I\left(m, M\right).$$

From Remark 2 we obtain

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \geq \ln\left(t\right) S_{\mathbf{A}} + t^{-1}\left(S_{\mathbf{B}} - tS_{\mathbf{A}}\right) - \left(M - m\right)\left(m^{-1} - M^{-1}\right) S_{\mathbf{A}}$$

for $t \in [m, M] \subset (0, \infty)$.

From (2.29) we get the operator inequality

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \geq \ln\left(\frac{m+M}{2}\right) S_{\mathbf{A}} + \left(\frac{m+M}{2}\right)^{-1}\left(S_{\mathbf{B}} - \frac{m+M}{2} S_{\mathbf{A}}\right) - \frac{1}{2}\left(M-m\right)\left(m^{-1} - M^{-1}\right) S_{\mathbf{A}}.$$

From (2.30) we obtain

$$\sum_{i=1}^{n} S\left(A_i|B_i\right) \geq \ln\left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right) S_{\mathbf{A}} + \left(\frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle}\right)^{-1}\left(S_{\mathbf{B}} - \frac{\langle S_{\mathbf{B}} x, x\rangle}{\langle S_{\mathbf{A}} x, x\rangle} S_{\mathbf{A}}\right) - \left(M-m\right)\left(m^{-1} - M^{-1}\right) S_{\mathbf{A}}$$

while from (2.31)

$$\left\langle \sum_{i=1}^{n} S\left(A_i|B_i\right)x, x\right\rangle \geq \ln\left(\frac{\langle S_{\mathbf{B}}x, x\rangle}{\langle S_{\mathbf{A}}x, x\rangle}\right)\langle S_{\mathbf{A}}x, x\rangle - (M-m)\left(m^{-1}-M^{-1}\right)\langle S_{\mathbf{A}}x, x\rangle$$

for $x \in H$, $x \neq 0$.

6. Some Applications of the Information Monotonicity

Assume that $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ satisfy the condition

$$mA_i \leq B_i \leq MA_i, \quad i = 1, ..., n.$$

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a positive linear map and $h : [a, b] \to (0, \infty)$ a continuous function such that the condition (2.2) is valid for some $m, M \in [a, b]$ with $0 < m < M$. Let $f_\nu : [0, \infty) \to [0, \infty)$, $f_\nu(x) = x^\nu$ for $\nu \in [0, 1]$ be a continuous function. Consider the convex function $f = -f_\nu$. We consider the information monotonicity for the Csisz ár f-divergence operator mapping and the generalized Csiszár f-divergence operator mapping. By Theorem 4 we derive

$$-\sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \leq \Phi\bigg(-\sum_{i=1}^{n} h(A_i)\sharp_\nu B_i\bigg)$$

and hence

$$\sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \geq \Phi\bigg(\sum_{i=1}^{n} h(A_i)\sharp_\nu B_i\bigg). \tag{6.1}$$

Moreover, by Corollary 8, we get a complementary for this inequality as follows:

$$\begin{aligned}\Phi\left(-\sum_{i=1}^{n} h(A_i)\sharp_\nu B_i\right) \leq & -\sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \\ & - m^\nu(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) \\ & + (-\nu b^{\nu-1} + \nu a^{\nu-1})S_{\Phi(\mathbf{B})} \\ & + m(-\nu a^{\nu-1}S_{h(\Phi(\mathbf{A}))} + \nu b^{\nu-1}S_{\Phi(h(\mathbf{A}))})\end{aligned}$$

and so

$$\begin{aligned}\sum_{i=1}^{n} \Phi\bigg(h(A_i)\sharp_\nu B_i\bigg) \geq & \sum_{i=1}^{n} h(\Phi(A_i))\sharp_\nu \Phi(B_i) \\ & + m^\nu(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) \\ & + \nu(b^{\nu-1} - a^{\nu-1})S_{\Phi(\mathbf{B})} \\ & + m\nu(a^{\nu-1}S_{h(\Phi(\mathbf{A}))} - b^{\nu-1}S_{\Phi(h(\mathbf{A}))}).\end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2) one can reach the information monotonicity and its complementary for the generalized Csiszár f-divergence operator mapping. In particular, when $h(t)=t$, we obtain the information monotonicity for the Csiszár f-divergence operator mapping as follows

$$\begin{aligned}\sum_{i=1}^{n}\Phi(A_i)\sharp_\nu\Phi(B_i) &\geq \sum_{i=1}^{n}\Phi\Big(A_i\sharp_\nu B_i\Big)\\ &\geq \sum_{i=1}^{n}\Phi(A_i)\sharp_\nu\Phi(B_i)\\ &\quad+\nu(b^{\nu-1}-a^{\nu-1})(S_{\Phi(\mathbf{B})}-mS_{\Phi(\mathbf{A})}).\end{aligned}$$

W know that the relative operator entropy can be interpreted as the perspective of $-\ln$, namely

$$P_{-\ln}(A,B)=-S(A|B).$$

Consider the convex function $f=-\ln$. We consider the information monotonicity for the Csisz ár f-divergence operator mapping and the generalized Csiszár f-divergence operator mapping.

By Theorem 4 we derive

$$\begin{aligned}-\sum_{i=1}^{n}S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right) &= \mathbf{I}_{(-\ln)\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))\\ &\leq \Phi(\mathbf{I}_{(-\ln)\Delta h}(\mathbf{A},\mathbf{B}))\\ &= \Phi\Bigg(-\sum_{i=1}^{n}S\left(h\left(A_i\right)|B_i\right)\Bigg)\end{aligned}$$

and hence

$$\sum_{i=1}^{n}S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right)\geq\sum_{i=1}^{n}\Phi\Big(S\left(h\left(A_i\right)|B_i\right)\Big). \tag{6.3}$$

On the other hand, by Corollary 8, we deduce

$$\begin{aligned}\Phi\Bigg(-\sum_{i=1}^{n}S\left(h\left(A_i\right)|B_i\right)\Bigg) &= \Phi(\mathbf{I}_{(-\ln)\Delta h}(\mathbf{A},\mathbf{B}))\\ &\leq \mathbf{I}_{(-\ln)\Delta h}(\Phi(\mathbf{A}),\Phi(\mathbf{B}))\\ &\quad(-\ln m)(S_{\Phi(h(\mathbf{A}))}-S_{h(\Phi(\mathbf{A}))})+\left(-\frac{1}{b}+\frac{1}{a}\right)S_{\Phi(\mathbf{B})}\\ &\quad+m\left(-\frac{1}{a}S_{h(\Phi(\mathbf{A}))}+\frac{1}{b}S_{\Phi(h(\mathbf{A}))}\right)\\ &= -\sum_{i=1}^{n}S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right)\end{aligned}$$

$$(-\ln m)(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) + \left(\frac{1}{a} - \frac{1}{b}\right) S_{\Phi(\mathbf{B})}$$
$$+ m\left(\frac{1}{b} S_{\Phi(h(\mathbf{A}))} - \frac{1}{a} S_{h(\Phi(\mathbf{A}))}\right).$$

It follows that

$$\sum_{i=1}^{n} \Phi\Big(S\left(h\left(A_i\right)|B_i\right)\Big) \geq \sum_{i=1}^{n} S\left(h\left(\Phi(A_i)\right)|\Phi(B_i)\right)$$
$$(-\ln m)(S_{\Phi(h(\mathbf{A}))} - S_{h(\Phi(\mathbf{A}))}) + \left(\frac{1}{a} - \frac{1}{b}\right) S_{\Phi(\mathbf{B})}$$
$$+ m\left(\frac{1}{b} S_{\Phi(h(\mathbf{A}))} - \frac{1}{a} S_{h(\Phi(\mathbf{A}))}\right). \tag{6.4}$$

Combining (6.3) and (6.4) we reach the information monotonicity and its complementary for the generalized Csiszár f-divergence operator mapping. In particular, when $h(t) = t$, we have

$$\sum_{i=1}^{n} S\left(\Phi(A_i)|\Phi(B_i)\right) \geq \sum_{i=1}^{n} \Phi\Big(S\left(A_i|B_i\right)\Big)$$
$$\geq \sum_{i=1}^{n} S\left(\Phi(A_i)|\Phi(B_i)\right)$$
$$+ \left(\frac{1}{a} - \frac{1}{b}\right)(S_{\Phi(\mathbf{B})} - m S_{\Phi(\mathbf{A})}).$$

Funding The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Declarations

Conflict of interest The authors remark that potential conflict of interest and data sharing does not apply to this article and no data sets were generated during the current study.

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Received: March 10, 2024.
Accepted: August 15, 2024.