Combinatorial Generalizations and Refinements of Euler's Partition Theorem.
MSc

Miehleketo Brighton Ndlovu

A Dissertation submitted to the Faculty of Science, University of the Witwatersrand,
Johannesburg, South Africa,
in fulfilment of the requirement for the award of the Degree of
Master of Science (MSc) in Pure Mathematics.

9 December 2014

DECLARATION

I declare that this Dissertation entitled "Combinatorial Generalizations and Refinements of Euler's Partition Theorem" is my own, unaided work carried out under the supervision of Prof A.O Munagi. It is being submitted for the award of the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signature: ....
Miehleketo Brighton Ndlovu

Date: ....day of..........................2014 at Johannesburg.

Abstract

The aim of this research project is to survey and elaborate on various generalizations and refinements of Euler's celebrated distinct-odd partition theorem which asserts the equality of the numbers of partitions of a positive integer into distinct summands and into odd summands. Although the work is not originally my own, I give clarity where there is obscurity by bridging the gaps on the already existing work. I touch on combinatorial proofs, which are either bijective or involutive. In some cases I give both combinatorial and analytic proofs. The main source of this dissertation is [22, 5, 6, 8]. I start by first summarizing some methods and techniques used in partition theory.

Acknowledgement

I would like to thank Prof A. Munagi for making this work possible through his help and advices on the organization and grammar of this dissertation. I would also like to thank my parents, family, and friends for all their support and understanding they showed till the end of this work.

Contents

0.1 Literature Review

A partition of a positive integer n is a multiset of summands, arranged in a nonincreasing order, totaling n. Theory of Partitions is an active area of research that dates back to around 1674 when Leibniz questioned J. Bernoulli about partitions of positive integers. Surprisingly Euler was the first one to dig deep and make significant discoveries. The Pentagonal number Theorem and Euler's odd-distinct Theorem are amongst the most notable partition identities Euler proved back in the year 1748. Several partition identities containing amongst others the names of Sylvester, Cauchy, Jacobi, Ramanujan, Gauss, Lebesgue and MacMahon were later discovered.

Several people embarked on these theories of Partitions and made major contributions, J.J. Sylvester is one of them after Euler. In the late nineteenth century Sylvester discovered the recent combinatorial theory in the spirit of partitions [2]. Nothing of significance really happened in the field between the eighteenth and nineteenth century. Substantial amount of research was experienced in the twentieth century coupled with significant results from Rademacher, Rogers, MacMahon and Ramanujan to name a few. This theory has famous celebrated beautiful results like Euler's Partition Identity with its generalizations and refinements. Historically, most of these partition identities were first proved using analytical techniques and, only much later, using combinatorial techniques. In this research we will turn our focus on partition identities that are either Euler's generalisation or refinement.

We use the function $p(n)$ to denote the number of partitions of a positive integer n. Andrews in [5] gave some results on Hardy-Ramanujan-Radamacher formula for $p(n)$. A combinatorial explanation of partitions theory can be viewed as distributing n similar balls into n similar urns [18]. In defining a partition, the same parts in two different orders would constitute the same partition, so we regard a preferred order –the descending order– as the representation we work with.

Sylvester and MacMahon viewed fixing the order of partitions important. The descending order has succeeded in this regard. Though choosing the order is arbitrary, having parts in descending order has proven to be more convenient. Tucker in [26] chose to define partitions with parts arranged in increasing order. In various instances partitions has been defined with parts arranged in ascending order, non-squashing partitions [25], M-partitions [20] and Lecture-hall partitions [10] are examples of such instances.

0.2 Tools and Definitions for studying partitions

In this section we introduce tools and definitions that are crucial in what we shall be researching.

0.2.1 Definitions

A partition of a positive integer n is a finite sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_l)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$, where $\lambda_i \geq \lambda_{i+1}$.
The partition function $p(n)$ denotes the number of partitions of a positive integer n.

Thus we shall refer to $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ as a partiton where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ such a λ is a partition of n when the sum of the parts is n. The preceding statements implies that:

$$\sum_{i\geq 1} \lambda_i = |\lambda| = n.$$

Thus for $n = 6$; the partitions are: (1,1,1,1,1,1),(2,1,1,1,1),(2,2,1,1),(3,1,1,1),(4,1,1),(3,2,1), (2,2,2),(3,3),(4,2),(5,1) and (6). Thus p(6)=11.

We will denote by $a(\lambda) = \lambda_1$ the largest part of the partition λ and by $e(\lambda) = \lambda_l$ the smallest parts of the partition λ. We define $l(\lambda) = l$ to mean the number of parts . We define $\mathcal{O}_n$ to be a set of partitions of n in which all parts are odd and $\mathcal{D}_n$ to be a set of partitions of n in which all parts are distinct. Let $m_i = m_i(\lambda)$ denote the number of parts of λ equal to i.

We define the **conjugate partition** $\lambda^{'} = (\lambda_1^{'}, \lambda_2^{'}, \cdots)$ of λ where:

$$\lambda_i^{'} = |\{j : \lambda_j \geq i\}| = m_i + m_{i+1} + \cdots .$$

Let $\lambda = (7, 5, 2, 2)$, then there are:
4 parts ≥ 1
4 parts ≥ 2
2 parts ≥ 3
2 parts ≥ 4
2 part ≥ 5
1 part ≥ 6
1 part ≥ 7

Thus $\lambda^{'} = (4, 4, 2, 2, 2, 1, 1)$.

It follows that,

$$l(\lambda^{'}) = a(\lambda).$$

Given the partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\nu = (\nu_1, \nu_2, \cdots)$ we define sum and union operators as follows:

$$\lambda + \nu = (\lambda_1 + \nu_1, \lambda_2 + \nu_2, \cdots),$$

where the shorter of λ or ν is filled out with parts of size 0.

$$\lambda \cup \nu = \{\lambda_i, \nu_j\} \ i, j \geq 1$$

consequently,

$$(\lambda \cup \nu)^{'} = \lambda^{'} + \nu^{'}.$$

0.2.2 Generating Functions.

Generating functions are powerful in studying integer partitions. Euler used them a lot in his work in partitions. The generating function of a sequence of numbers $\{a(n)\}$ is a formal power series:

$$\sum_{n=0}^{\infty} a(n)q^n, \tag{1}$$

whose coefficient encode information about a sequences of numbers $a(n)$ that is indexed by the natural numbers.
We use this presentation as follows. Let's say we want to explore all possible partitions with strictly an odd part and an even part less than 8. The generating function is:

$$\begin{aligned}(q^2+q^4+q^6+q^8)\cdot(q^1+q^3+q^5+q^7) &= q^{2+1}+q^{2+3}+q^{2+5}+q^{2+7}+q^{4+1}+q^{4+3}+q^{4+5}\\ &+q^{4+7}+q^{6+1}+q^{6+3}+q^{6+5}+q^{6+7}\\ &= q^3+2q^5+3q^7+3q^9+2q^{11}+q^{13} \qquad (2)\end{aligned}$$

Since the exponents represent the actual partitions with an odd part and an even part less than 8, then the coefficients gives us the number of such partitions. For example coefficient of q^9 is 3, which implies that the number of partitions of 9 into one even part and one odd part less than 8 equals 3.

This can be extended more general partitions. For example, The following is a generating function of a partition with distinct parts from $S=\{n_1,n_2,n_3\}$:

$$(1+q^{n_1})(1+q^{n_2})(1+q^{n_3}) = 1+q^{n_1}+q^{n_2}+q^{n_3}+q^{n_1+n_2}+q^{n_1+n_3}+q^{n_2+n_3}+q^{n_1+n_2+n_3} \qquad (3)$$

Now taking $S=\{1,2,3\}$, then (3) becomes:

$$1+q+q^2+2q^3+q^4+q^5+q^6 \qquad (4)$$

If we define:
$p(n,k)$ = number of partitions of n in which no part exceeds k.
$p(n/k)$ = number of partitions of n with exactly k parts.
$p(n/k)_d$ = number of partitions of n with k different parts (i.e., into k part sizes),

then we can extend the generating function presentation to get a generating function of the above partitions.

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,k)q^n &= (1+q^1+q^{1+1}+q^{1+1+1}+\dots)(1+q^2+q^{2+2}+q^{2+2+2}+\dots)\\ &\quad (1+q^3+q^{3+3}+q^{3+3+3}+\dots)\cdots(1+q^k+q^{k+k}+\cdots)\\ &= \prod_{j=1}^{k}(1+q^j+q^{2j}+q^{3j}+\dots)\\ &= \prod_{j=1}^{k}\frac{1}{(1-q^j)} \qquad (5)\end{aligned}$$

Sometimes it is not simple to find a direct generating function for a given partition. It is often helpful to find the conjugate partition for the generating function. For example the generating function for $p(n/k)$ is not easy to find, but we know its conjugate partition equals the number of partitions of n in which the largest part is k. Using the latter characterization we obtain:

$$\sum_{n=0}^{\infty} p(n/k)q^n = q^k(1+q^1+q^{1+1}+q^{1+1+1}+\dots)(1+q^2+q^{2+2}+q^{2+2+2}+\dots)$$
$$(1+q^3+q^{3+3}+q^{3+3+3}+\dots)\cdots(1+q^k+q^{k+k}+\cdots)$$
$$= q^k\prod_{j=1}^{k}(1+q^j+q^{2j}+q^{3j}+\dots)$$
$$= \frac{q^k}{\prod_{j=1}^{k}(1-q^j)}. \tag{6}$$

Sometimes we encounter partition problems which are restricted in such a way that we have to keep count of the parts. Depending on the restriction on the parts, we use a variable x to keep count of the parts, for $p(n/k)$ we have:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n/k)x^k q^n = \prod_{n=1}^{\infty}(1+xq^n+x^2q^{2n}+x^3q^{3n}+\dots)$$
$$= \frac{1}{\prod_{n=1}^{\infty}(1-xq^n)}. \tag{7}$$

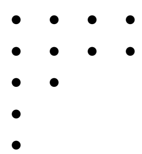
If the restriction on the parts is such that we should keep track of the number of different parts, then we have:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n/k)_d y^k q^n = \prod_{n=1}^{\infty}(1+yq^n+yq^{2n}+yq^{3n}+\dots)$$
$$= \prod_{n=1}^{\infty}\left(1+\frac{yq^n}{1-q^n}\right)$$
$$= \prod_{n=1}^{\infty}\frac{1-(1-y)q^n}{1-q^n}. \tag{8}$$

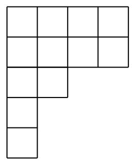
0.2.3 Ferrers Graphs and Ferrers Boards

Partitions can be represented using diagrams which are called Ferrers diagrams. These graphical representations of partitions are very useful in helping us to understand partitions especially for their combinatorial analysis. Many amazing facts about partitions are best explained graphically. There are two common ways of drawing such graphs for a partition, namely Ferrers graphs and Ferrers Boards.

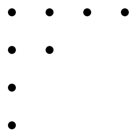
Example $4+4+2+1+1$ has Ferrers graph:



For some uses, the above representation with dots looks best; it's called Farrers graph. On other occasions, the representation where we replace dots with squares comes natural; it's called Farrers boards or Young diagram:



These diagrams are useful in finding transformations and if such a transformation is invertible, then it is a bijection and can be used for proving some partition identities. As a very elementary example of a transformation, take the Ferrers graph above and remove the top row:



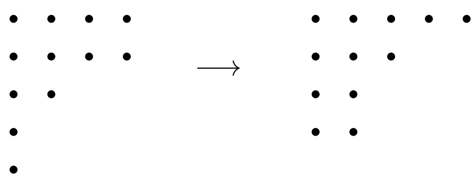
We see that if we remove the top row from a Ferrers graph, we are left with a new Ferrers graph. If r was the length of the removed row, then all rows of the new Ferrers graph have length less than or equal to r. Conversely, for any such Ferrers graph, we can add a row of length r on top and obtain a Ferrers graph. Thus we have a bijection proving the partition identity:

$$p(n| \text{ greatest part } r) = p(n-r| \text{ all parts } \leq r). \quad (9)$$

With this technique (transformation) we can bijectively prove and find lots of partition identities, for example, the variation of this technique that may make sense would be to remove the first column instead of the top row. This would lead to:

$$p(n| \ m \text{ parts}) = p(n-m| \text{ atmost } m \text{ parts}). \quad (10)$$

Conjugate Ferrers Diagram: these are the Ferrers graph obtained by exchanging rows and columns of the old Ferrers graph, i.e: $(4,4,2,1,1) \rightarrow (5,3,2,2)$



This transformation returns a partition with the greatest part equal to the number of parts of the original partition, i.e

$$p(n| \ m \text{ parts}) = p(n| \text{ greatest part is } m). \quad (11)$$

Chapter 1

Euler's Partition Theorem

In combinatorics, integer partitions are of interest, mainly because many questions regarding integer partitions, solved and unsolved, have no simple proofs. In the proof of the following Euler's Theorem we'll see how generating functions are useful in proving partitions identities. One will notice that the use of generating functions is more frequent as we go deep into studying partitions.

Theorem 1 (Euler's Partition Theorem) *[5] The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Example:

Given $n = 9$, then

$$p(9) = 8 = |\mathcal{D}_9|$$

$$\mathcal{D}_9 = (9, 8+1, 7+2, 6+3, 6+2+1, 5+4, 5+3+1, 4+3+2)$$

and

$$p(9) = 8 = |\mathcal{O}_9|$$

$$\mathcal{O}_9 = (9, 7{+}1{+}1, 5{+}3{+}1, 5{+}1{+}1{+}1{+}1, 3{+}3{+}3, 3{+}3{+}1{+}1{+}1, 3{+}1{+}1{+}1{+}1{+}1{+}1, 1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1)$$

Now since:

$$\begin{aligned}\sum_{n=0}^{\infty} p(n \mid \textit{parts distinct})q^n &= (1+q)(1+q^2)(1+q^3)\cdots \\ &= \prod_{n=1}^{\infty}(1+q^n) \qquad (1.1)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} p(n \mid parts\ all\ odd)q^n &= (1+q+q^{1+1}+q^{1+1+1}+\cdots)(1+q^3+q^{3+3}+q^{3+3+3}+\cdots)\\ &\quad (1+q^5+q^{5+5}+q^{5+5+5}+\cdots)\cdots\\ &= (1+q+q^2+q^3+\cdots)(1+q^3+q^6+q^9+\cdots)(1+q^5+q^{10}+q^{15}+\cdots)\cdots\\ &= \prod_{n\ odd} \frac{1}{(1-q^n)} \qquad (1.2)\end{aligned}$$

then Euler's Partition Theorem can be expressed in generating function form as follows:

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n\ odd} \frac{1}{(1-q^n)}. \qquad (1.3)$$

Showing that the two sides are equal will prove that:

$$p(n \mid parts\ distinct) = p(n \mid parts\ all\ odd). \qquad (1.4)$$

This generating function approach is very strong as it eliminates the problem of having to show that the two sides are equal from their exact formulas which are very difficult to find.

1.0.4 Analytical Proof of Euler's Identity

We supply the analytic proof using generating functions [6]. It is sufficient to show that (1.4) holds, i.e

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n\ odd} \frac{1}{(1-q^n)}$$

The proof is as follows:

$$\begin{aligned}
RHS = \prod_{n\ odd} \frac{1}{(1-q^n)} &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})} \\
&= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} \\
&= \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\
&= \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^n)(1+q^n)}{1-q^n} \\
&= \prod_{n=1}^{\infty} (1+q^n) \\
&= LHS
\end{aligned}$$

■

Thus $p(n \mid parts\ distinct) = p(n \mid parts\ all\ odd)$.

This concludes the analytical proof. Below we illustrate a transformation or bijection that proves Euler's identity.

1.0.5 Glaisher's combinatorial description of Euler's Identity

Transformation of a partition with odd parts into a partition with distinct parts, i.e $\underline{\mathcal{O}_n \mapsto \mathcal{D}_n}$. [16]

We start with the partition where the parts are odd. If there are two identical odd parts, then we add them to form one part. We proceed with this procedure until all the parts are distinct.

Example:

For $n = 13$

$$\begin{aligned}
3+3+3+1+1+1+1 &\mapsto (3+3)+3+(1+1)+(1+1) \\
&\mapsto 6+3+2+2 \\
&\mapsto 6+3+(2+2) \\
&\mapsto 6+3+4
\end{aligned}$$

We now consider changing a partition with distinct parts into a partition with odd parts, i.e $\underline{\mathcal{D}_n \mapsto \mathcal{O}_n}$.

We split each even part into two equal halves. We proceed with this procedure until all the parts are odd.

Example:

For $n = 17$

$$\begin{aligned}
8+6+3 &\mapsto (4+4)+(3+3)+3 \\
&\mapsto (2+2)+(2+2)+3+3+3 \\
&\mapsto 3+3+3+(1+1)+(1+1)+(1+1)+(1+1) \\
&\mapsto 3+3+3+1+1+1+1+1+1+1+1.
\end{aligned}$$

∎

We show that this procedure present a mapping that is both surjective and injective $\phi : \mathcal{D}_n \mapsto \mathcal{O}_n$:

Let $\lambda \in \mathcal{D}_n$. Let's express the parts of λ into powers of two, i.e $\lambda_i = 2^{n_i}\tau_i$, where τ_i is an odd integer. Then we have $\lambda = 2^{n_1}\tau_1 2^{n_2}\tau_2 \cdots 2^{n_s}\tau_s$. Since $\lambda_i \neq \lambda_j$, we could have that $\tau_i = \tau_j$ for $n_i \neq n_j$. Then we can always transform λ into a partition of the form $\tau = \tau_{x_1}^{m_1}\tau_{x_2}^{m_2} \cdots \tau_{x_k}^{m_k}$, where $m_i = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}$, i.e m_i is the number of τ_i in λ. Thus $\tau \in \mathcal{O}_n$. ϕ is injective since it could only be the image under ϕ of some λ of the form $2_{n_i}\tau_i$ for each $m_i = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}$ with exactly one sequence of nonnegative integers $i_1 > i_2 > \cdots > i_l$. Since $\lambda \in \mathcal{D}_n$, we have a well-defined mapping $\pi^{-1} : \mathcal{O}_n \mapsto \mathcal{D}_n$, so ϕ is surjective. $\therefore$ ϕ is a bijection. ∎

Chapter 2

Further Proof of Euler's Theorem: Dyson's Iterated Map

Freeman Dyson in a paper published in the journal Eureka [14] (a publication of mathematical students in Cambridge) introduced the notion of rank in a partition. Partitions with specified ranks can also be treated using generating functions, see [13]. The work in this chapter is based on [21].

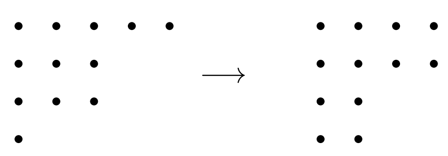
2.1 Dyson's Map

Freeman Dyson defined the rank of a partition λ, $r(\lambda)$, as the largest part minus the number of parts, that is $r(\lambda) = a(\lambda) - l(\lambda)$. Example: for $\lambda = 5 + 3 + 2 + 1$, $r(\lambda) = 5 - 4 = 1$.
Let $\mathcal{P}_{n;r}, \mathcal{G}_{n;r}$ and $\mathcal{H}_{n;r}$ be the sets of partitions of n having a given rank r, at least r and at most r, respectively. We denote their cardinality as follows: $p(n;r) = |\mathcal{P}_{n;r}|$, $h(n;r) = |\mathcal{H}_{n;r}|$ and $g(n;r) = |\mathcal{G}_{n;r}|$.

We define Dyson's Map $\psi_r(\lambda)$ to be the partition whose Young diagram is obtained by first removing the first column of the Ferrers diagram of λ and then adding a new top row of length $l(\lambda) + r(\lambda) - 1$.

Example:

For $\lambda = (5, 3, 3, 1)$, we get $r(\lambda) = 5 - 4 = 1$ and $\psi_r(\lambda) = (4, 4, 2, 2)$:



Nathan Fine[21] gave the following theorem:

Theorem 2 (Fine's theorem) *A bijection exists between* $\mathcal{H}_{n;t+1}$ *and* $\mathcal{G}_{n+t;t-1}$ *for all* $\lambda \in \mathcal{H}_{n;t+1}$*, where* $t \geq r(\lambda) - 1$*.*

2.1.1 First Proof of Fine's Theorem:

We construct a bijection $\psi_t : \mathcal{H}_{n;t+1} \mapsto \mathcal{G}_{n+t;t-1}$. Given the Young diagram of $\lambda \in \mathcal{H}_{n;t+1}$:

(1) Take out $\lambda_1^{'} = l(\lambda)$ from λ.

(2) Add a new row on top of λ with $l(\lambda) + t$ squares. Let $[\mu]$ be the resulting Young diagram.

This gives us $\psi_t : \lambda \mapsto \mu$

μ is a partition because of property (0) below:

(0)

$$\begin{aligned} a(\mu) &= l(\lambda) + t \\ &\geq l(\lambda) + r(\lambda) - 1, \qquad \textit{since } t \geq r(\lambda) - 1 \\ &= l(\lambda) + (a(\lambda) - l(\lambda)) - 1 \\ &= a(\lambda) - 1 \end{aligned}$$

(1)

$$\begin{aligned} |\mu| &= |\lambda| - l(\lambda) + (l(\lambda) + t) \\ &= n + t \end{aligned}$$

(2)

$$\begin{aligned} r(\mu) &= a(\mu) - l(\mu) \\ &= l(\lambda) + t - (\lambda_2^{'} + 1) \\ &\geq t - 1, \qquad \textit{since } l(\lambda) - \lambda_2^{'} \geq 0 \end{aligned}$$

Or

$$\begin{aligned} r(\mu) &= a(\mu) - l(\mu) \\ &= l(\lambda) + t - l(\mu) \\ &\geq l(\lambda) + t - (l(\lambda) + 1), \qquad \textit{since } l(\mu) \leq l(\lambda) + 1 \\ &= t - 1 \end{aligned}$$

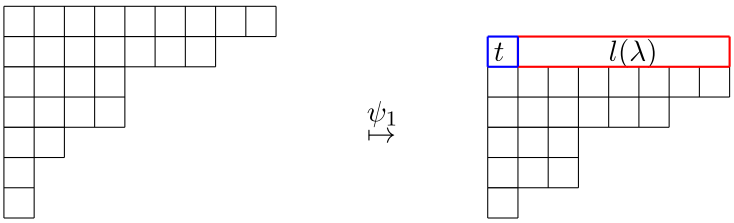
(3)

$$a(\lambda) - l(\lambda) = r(\lambda) \leq t + 1.$$

From property (1) and (2) we therefore conclude that: $\mu = \psi_t(\lambda) \in \mathcal{G}_{n+t;t-1}$.

Example:

$\lambda = (9, 7, 4, 4, 2, 1, 1) = 28 \longrightarrow \mu = (8, 8, 6, 3, 3, 1) = 29$



$\lambda \in \mathcal{H}_{28;2}$, t=1 $\mu \in \mathcal{G}_{28+1;0}$

For $\psi_t^{-1} : \mu \mapsto \lambda$, we should at least know the set $\mathcal{G}_{n+t;t-1}$ where μ comes from, since $r(\mu) \geq t - 1$. We proceed in the following manner:

Step(1):

Compute t by noting that t is one more than the second index of $\mathcal{G}_{n+t;t-1}$.

Step(2):

From μ remove $a(\mu)$ and add 1 to all the remaining parts. Call the resulting partition λ^*.

Step(3):

Note that:

(1) If λ had no part of size one, then

$$|\lambda^*| = |\lambda| = n \text{ and } \lambda^* = \lambda. \tag{2.1}$$

(2) If λ had a part of size one, then

$$|\lambda^*| = |\lambda| - |\mathcal{K}| \text{ and } \lambda = \lambda^* \cup \mathcal{K},$$

where $\mathcal{K}$ is a partition with parts of sizes one only and

$$\begin{aligned} |\mathcal{K}| &= n - |\lambda^*| \\ &= |\mu| - t - |\lambda^*|. \end{aligned} \tag{2.2}$$

Equation (2.1) and (2.2) imply that $|\lambda| = n$. This is clearly a direct reverse of ψ_t with $r(\lambda) \leq t + 1$.

∎

2.1.2 Second Proof of Fine's Theorem:

The first part of this proof $\psi_t : \lambda \mapsto \mu$ is the same as in the first proof. Without wasting time, we'll proceed to the second part $\psi_t^{-1} : \mu \mapsto \lambda$.

Since $a(\mu) = \mu_1$ is the sum of t and the length of the preimage λ which may contain $1's$. We need to obtain the standard form of μ to contain trailing zeros.

We denote the latter by μ^* and let $l(\mu^*) = v(\mu)$, where $v(\mu)$ is the "virtual length" of μ.

Then from property (0) we obtain:

$$a(\mu) = (v(\mu) - 1) + t$$

$$\Rightarrow \qquad v(\mu) = a(\mu) + 1 - t \geq l(\mu)$$

So μ^* contains exactly $v(\mu) - l(\mu)$ zeros, that is

$$\mu^* = \mu \cup (0^{v(\mu)-l(\mu)}).$$

We now obtain the image λ by deleting $a(\mu^*)$ and adding 1 into each remaining part:

$$\lambda = \mu_2 + 1, \mu_3 + 1, \cdots, \mu_{l(\mu)}, 1^{v(\mu)-l(\mu)}. \tag{2.3}$$

Equation (2.3) is the desired image for two reasons:

(i)

$$\begin{aligned} |\lambda| &= (|\mu| - a(\mu)) + v(\mu) - 1 \\ &= n + t - a(\mu) + (a(\mu) + 1 - t) - 1 \\ &= n \end{aligned} \tag{2.4}$$

(ii) $t + 1 \geq r(\lambda)$, since this implies:

$$\begin{aligned} t + 1 &\geq a(\lambda) - l(\lambda) \\ &= \mu_2 + 1 - v(\mu) + 1 \\ &= \mu_2 + 1 - (a(\mu) + 1 - t) + 1 \\ &= \mu_2 + 1 - a(\mu) - t \end{aligned}$$

which gives

$$0 \geq \mu_2 - a(\mu)$$

$$or$$

$$a(\mu) \geq \mu_2,$$

which is trivially true.

(i) and (ii) show that $\lambda \in \mathcal{H}_{n,t+1}$.

∎

Example:

Given $\mu = (8, 8, 6, 3, 3, 1) \in \mathcal{G}_{29,0}$, we have $t - 1 = 0 \Rightarrow t = 1$.

$$v(\mu) = a(\mu) + 1 - t = 8 + 1 - 1 = 8.$$

So

$$\mu^* = (8, 8, 6, 3, 3, 1, 0, 0).$$

Hence

$$\lambda = (9, 7, 4, 4, 2, 1, 1) \in \mathcal{H}_{28,2}.$$

2.2 Iterated Dysons map

Nathan Fine also gave the following theorem on ranks of partitions:

Theorem 3 (Fine) *[21] Let $\mathcal{D}_{n;r}$ be the set of partitions $\mu \in \mathcal{D}_n$ with rank $r(\mu) = r$. Let $\mathcal{U}_{n;2k+1}$ be the set of partitions $\lambda \in \mathcal{O}_n$, such that the largest part $a(\lambda) = 2k + 1$. Then:*

$$|\mathcal{U}_{n;2k+1}| = |\mathcal{D}_{n;2k+1}| + |\mathcal{D}_{n;2k}|.$$

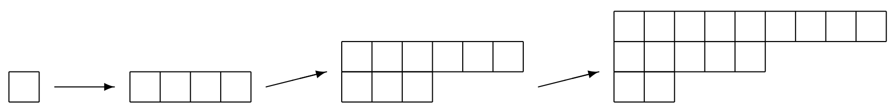
The above theorem is clearly a refinement of Euler's partition theorem. From properties of ψ_t, Andrews in [3] showed how the theorem above directly follow.

Given that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{l(\lambda)}) \in \mathcal{O}_n$. A sequence of partitions $v^1, v^2, \cdots, v^{l(\lambda)}$, such that $v^{l(\lambda)} = (\lambda_{l(\lambda)})$ and v^i is found by simply applying Dyson's map ψ_{λ_i} to v^{i+1}. Let $\mu = v^1$ and we name the new map $\xi : \lambda \mapsto \mu$ the iterated Dyson's map.

For example:

$\lambda = (7, 5, 3, 1) \in \mathcal{U}_{16;7}$, $\mu = (9, 5, 2) \in \mathcal{D}_{16,6}$

$v^4 = (1)$



Theorem 4 *[21] The iterated Dyson's map ξ defined above is a bijection between $\mathcal{O}_n$ and $\mathcal{D}_n$. Moreover, $\xi(\mathcal{U}_{n;2k+1}) = \mathcal{D}_{n;2k+1} \cup \mathcal{D}_{n;2k}$, for all $k \geq 0$.*

2.2.1 Proof of iterated Dysons map

To prove that $\xi : \lambda \mapsto \mu$ is a bijection and $\xi(\mathcal{U}_{n;2k+1}) = \mathcal{D}_{n;2k+1} \cup \mathcal{D}_{n;2k}$, for all $k \geq 0$, we first need to show that $\mu \in \mathcal{D}_{n;2k+1} \cup \mathcal{D}_{n;2k}$ and $r(\mu) \in \{2k+1, 2k\}$, where $a(\lambda) = 2k+1$.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots, \lambda_{l(\lambda)}) \in \mathcal{O}_n$.

By construction we note:

(1)

$$\left|v^i\right| = \lambda_i + \lambda_{i+1}, \cdots + \lambda_{l(\lambda)} \Longrightarrow \left|v^i\right| = \lambda_i + \left|v^{i+1}\right| \ \textit{and} \ \left|v^1\right| = |\lambda| = n, \quad i \geq 0 \tag{2.5}$$

(2)

$$a(v^i) = l(v^{i+1}) + \lambda_i \tag{2.6}$$

(3)

$$l(v^{i+1}) \in \{a(v^{i+1}) - \lambda_{i+1}, \quad a(v^{i+1}) - \lambda_{i+1} + 1\} \tag{2.7}$$

(4)

$$l(v^i) \in \{l(v^{i+1}), \quad l(v^{i+1}) + 1\} \tag{2.8}$$

Let v_1^i and v_2^i denote the length of the first and second row of v^i, respectively.

$$\begin{aligned} v_1^i &= a(v^i) \\ &= l(v^{i+1}) + \lambda_i && \textit{by } (2.6) \\ &\geq a(v^{i+1}) - \lambda_{i+1} + \lambda_i && \textit{by } (2.7) \\ &> a(v^{i+1}) - 1 && \textit{since } \lambda_i - \lambda_{i+1} \geq 0 \\ &= v_2^i \end{aligned} \tag{2.9}$$

Equation (2.9) shows that $v_1^i > v_2^i$. The rest of the rows remain distinct because they lose one box from the removal of the first column. Therefore $v_1^i > v_2^i$ for all i.

$$\begin{aligned} r(v^i) &= a(v^i) - l(v^i) \\ &= (l(v^{i+1}) + \lambda_i) - l(v^i) && \textit{by } (2.6) \\ &\in \{\lambda_i, \lambda_i - 1\} && \textit{by } (2.8) \end{aligned} \tag{2.10}$$

Equation (2.10) shows that:

$$r(v^i) \in \{\lambda_i, \lambda_i - 1\}$$

$\therefore$ we conclude that $\mu \in \mathcal{D}_n$ and moreover $\mu \in \{\mathcal{D}_{n;\lambda_1} \cup \mathcal{D}_{n;\lambda_1 - 1}\}$.

Now we construct ξ^{-1}.

Step(1):

Given $\mu = v^1 \in \{\mathcal{D}_{n;\lambda_1} \cup \mathcal{D}_{n;\lambda_1 - 1}\}$, we know from (2.10) that $r(v^1) \in \{\lambda_1, \lambda_1 - 1\}$. Then we construct λ_1 by computing $r(v^1)$.

Step(2):

Construct $v^{(i+1)*}$ by removing $a(v^i)$ and adding one from the remaining parts in v^i. Note that the second part of $v^{(i+1)*} \geq 0$.

Now construct

$$v^{(i+1)} = v^{(i+1)*} \cup \mathcal{K}^{(i+1)}, \tag{2.11}$$

where $\mathcal{K}^{(i+1)}$ is a partition with part(s) of size one only and

$$|\mathcal{K}^{(i+1)}| = \left|v^i\right| - \left|v^{(i+1)*}\right| - \lambda_i. \tag{2.12}$$

We do this for all i up to the i where $\left|v^i\right| = \lambda_i$ or $\left|v^{i+1}\right| = 0$.

Step(3):

Now construct λ from all the λ_i obtained from step(2). This is clearly the reverse of ξ. $\therefore \xi^{-1} : \mathcal{D}_n \mapsto \mathcal{O}_n$ is well-defined. We now conclude that $\xi : \mathcal{O}_n \mapsto \mathcal{D}_n$ is a bijection.

∎

Note that this theorem implies Euler's odd-distinct parts theorem when $r = \mathbb{Z}$.

Example:

$\lambda = (7, 5, 3, 1, 1, 1, 1)$

$$\begin{aligned} v^7 &= (1) \\ v^6 &= (2) \\ v^5 &= (2, 1) \\ v^4 &= (3, 1) \\ v^3 &= (5, 2) \\ v^2 &= (7, 4, 1) \\ v^1 &= (10, 6, 3) = \mu \end{aligned}$$

Now given $\mu = (10, 6, 3)$ we construct λ :

Step(1)

$$\mu = v^1 = (10, 6, 3) \qquad r(v^1) = 7 \textit{ and } \lambda_1 = 7$$

Step(2)

- $$From\ equations\ (2.11)\ and\ (2.12) \quad , v^{2*} = (7,4)\ and\ \mathcal{K}^2 = (1)$$
  $$v^2 = (7,4) \cup (1) = (7,4,1) \qquad r(v^2) = 4\ and\ \lambda_2 = 5$$

- $$From\ equations\ (2.11)\ and\ (2.12) \quad , v^{3*} = (5,2)\ and\ \mathcal{K}^3 = (0)$$
  $$v^3 = (5,2) \cup (0) = (5,2) \qquad r(v^3) = 3\ and\ \lambda_3 = 3$$

- $$From\ equations\ (2.11)\ and\ (2.12) \quad , v^{4*} = (3)\ and\ \mathcal{K}^4 = (1)$$
  $$v^4 = (3) \cup (1) = (3,1) \qquad r(v^4) = 1\ and\ \lambda_4 = 1$$

- $$From\ equations\ (2.11)\ and\ (2.12) \quad , v^{5*} = (2)\ and\ \mathcal{K}^5 = (1)$$
  $$v^5 = (2) \cup (1) = (2,1) \qquad r(v^5) = 0\ and\ \lambda_5 = 1$$

- $$From\ equations\ (2.11)\ and\ (2.12) \quad , v^{6*} = (2)\ and\ \mathcal{K}^6 = (0)$$
  $$v^6 = (2) \cup (0) = (2) \qquad r(v^6) = 1\ and\ \lambda_6 = 1$$

- $$From\ equations\ (2.11)\ and\ (2.12) \quad , v^{7*} = (1)\ and\ \mathcal{K}^7 = (0)$$
  $$v^7 = (1) \cup (0) = (1) \qquad r(v^7) = 0\ and\ \lambda_7 = 1$$

Step(3)

putting λ_i together we obtain: $\lambda = (7,5,3,1,1,1,1)$.

Chapter 3

Further Proof of Euler's Theorem: Sylvester's Refinement

In this chapter we elaborate on a combinatorial proof of Sylvester's bijection and show how it implies Euler's Partition theorem.

Theorem 5 (Sylvester's refinement) *[22] The number of partitions of n with k sizes of odd part equals the number of partitions of n into k separate sequences of consecutive integers. (A sequence may have only one term).*

This theorem of Sylvester is a refinement of Euler's theorem because for $k \to \infty$ the first part of it is a partition of n into odd parts and the second part of it is the partition of n into distinct parts.

Example: When $n = 15$ and $k = 3$ (i.e using 3 odd parts), the eleven partitions in the first class are:

$11 + 3 + 1, 9 + 5 + 1, 9 + 3 + 1 + 1, 7 + 5 + 3,$
$7 + 5 + 1 + 1 + 1, 7 + 3 + 1 + 1 + 1 + 1 + 1, 7 + 3 + 3 + 1 + 1,$
$5 + 5 + 3 + 1 + 1, 5 + 3 + 3 + 3 + 1, 5 + 3 + 3 + 1 + 1 + 1 + 1,$
$5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

and the eleven partitions in the second class are (i.e 3 separate sequences):

$11 + 3 + 1, 10 + 4 + 1, 7 + 5 + 3, 9 + 5 + 1, 9 + 4 + 2,$
$8 + 6 + 1, 8 + 5 + 2, 8 + 4 + 2 + 1, 7 + 5 + 2 + 1,$
$7 + 4 + 3 + 1, 6 + 5 + 3 + 1.$

3.1 First Proof

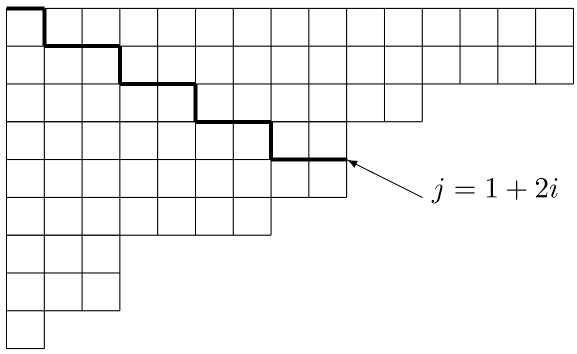
In this proof we attempt to construct a bijection $P : \mathcal{O}_n \to \mathcal{D}_n$. We will do it step by step.

Step one:

We divide the diagram $[\lambda]$ into two parts, along the line $j = 1 + 2i$. (where λ is a partition into odd parts and i increases downwards, while j increases from left to right.)

$$0 \leq i \leq l(\lambda) \textit{ and } 1 \leq j \leq \lambda_i$$

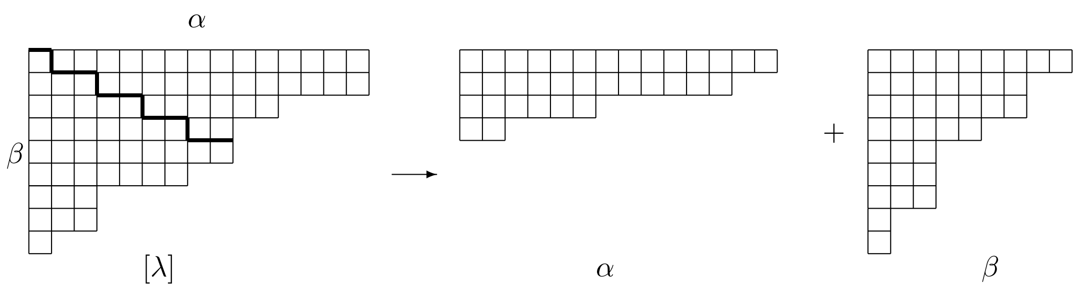
Example:



Step two:

Read the parts above the line $j = 1 + 2i$ as the Young diagram of a partition α. Read the parts below the line $j = 1 + 2i$ as the Young diagram of partition β , obtained after shifting up all parts left-out hanging following the removal of α from $[\lambda]$

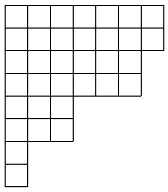
Example:



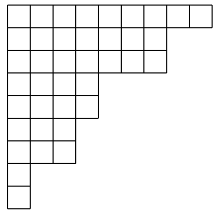
Step three:

Conjugate(inline) each block of two consecutive squares in each row of $[\alpha]$ and denote it by $\bar{\alpha_i} = \left(\frac{\alpha_i}{2}\right) \cup \left(\frac{\alpha_i}{2}\right)$ (i.e half each part of α) and conjugate β normally.

Example:



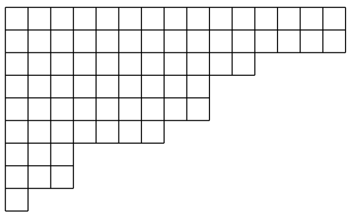
$\bar{\alpha}$



β'

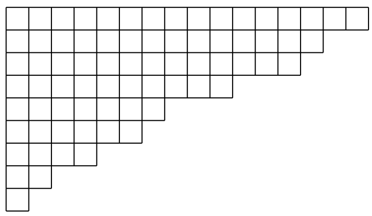
Step four:

$$\lambda \longrightarrow \bar{\alpha} + \beta' = \gamma$$



$[\lambda]$

$\longmapsto$



γ

To complete this proof we need to show that since λ is partitions of n with k sizes of odd part then the resulting partition γ is a partition of n into k separate sequences of consecutive integers. We'll do this by first converting the Young diagram technique we used above into an algebraic expression and then analyzing the difference between consecutive parts.

From the contraction of $[\lambda]$ we see that the parts of $\bar{\alpha}$ and β' are simply constructed from the expressions: $\bar{\alpha}_r = \frac{\lambda_{\lceil \frac{r}{2} \rceil - \delta(r)}}{2}$ and $\beta'_r = \lambda'_r - \lfloor \frac{r}{2} \rfloor$ respectively, where $\delta(r) = \begin{cases} r, & \text{if } r \text{ is odd} \\ r-1, & \text{otherwise} \end{cases}$ and $1 \leq r \leq r_{max}$.

For r_{max} we note that:

$$\left| l\left(\beta'\right) - l(\bar{\alpha}) \right| = 1$$

and

$$r_{max} = \begin{cases} l\left(\beta'\right), & \text{if } l\left(\beta'\right) > l(\bar{\alpha}) \\ l(\bar{\alpha}), & \text{if } l\left(\beta'\right) < l(\bar{\alpha}) \end{cases}$$

From this we obtain the following expression which is an algebraic version of splitting $[\lambda]$ along the line $j = 1 + 2i$ and conjugating α and β to obtain γ:

$$\gamma_r = \frac{\lambda_{\lceil \frac{r}{2} \rceil - \delta(r)}}{2} + \lambda_r^{'} - \left\lfloor \frac{r}{2} \right\rfloor, \qquad 1 \leq r \leq r_{max}$$

Thus we obtain the parts of the partition γ algebraically from it. From this we have that:

$$\gamma_r - \gamma_{r+1} = \frac{\lambda_{\lceil \frac{r}{2} \rceil} - \lambda_{\lceil \frac{r+1}{2} \rceil} + \delta(r+1) - \delta(r)}{2} + \lambda_r^{'} - \lambda_{r+1}^{'} + \left\lfloor \frac{r+1}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor$$

Analyzing the components of this difference for the parity of r we observe the following properties:

if r is odd:

(1)
$$\delta(r+1) - \delta(r) = r - r = 0$$

(2)
$$\left\lfloor \frac{r+1}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor = \frac{r+1}{2} - \frac{r-1}{2} = 1$$

(3)
$$\lambda_{\lceil \frac{r}{2} \rceil} - \lambda_{\lceil \frac{r+1}{2} \rceil} = \lambda_{\frac{r+1}{2}} - \lambda_{\frac{r+1}{2}} = 0$$

(4)
$$\lambda_r^{'} - \lambda_{r+1}^{'} \geq 0$$

if r is even:

(5)
$$\delta(r+1) - \delta(r) = r + 1 - (r-1) = 2$$

(6)
$$\left\lfloor \frac{r+1}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor = \frac{r}{2} - \frac{r}{2} = 0$$

(7)
$$\lambda_{\lceil \frac{r}{2} \rceil} - \lambda_{\lceil \frac{r+1}{2} \rceil} = \lambda_{\frac{r}{2}} - \lambda_{\frac{r}{2}+1} \geq 0$$

(8)
$$\lambda_r^{'} - \lambda_{r+1}^{'} = 0$$

From these properties we get the following:

$$\gamma_r - \gamma_{r+1} = \begin{cases} 1 + \lambda_r^{'} - \lambda_{r+1}^{'} \geq 1, & \text{if } r \text{ is odd} \\ \\ \frac{\lambda_{\frac{r}{2}} - \lambda_{\frac{r}{2}+1}}{2} + 1 \geq 1, & \text{if } r \text{ is even} \end{cases}$$

Now let's analyse the above equation:

For r odd note that each time $\gamma_r - \gamma_{r+1} = 1$, the r^{th} column ends at the same level as the $(r+1)^{th}$ column of λ since $\lambda'_r = \lambda'_{r+1}$. So both columns determine the same odd part. But when $\gamma_r - \gamma_{r+1} > 1$, then since $\lambda'_r > \lambda'_{r+1}$, the r^{th} column ends at a lower level (≥ 1 step) than the $(r+1)^{th}$ column, thus initiating a different odd part.

Similarly, for r even, each time $\gamma_r - \gamma_{r+1} = 1$, the r^{th} row ends at the same position on the right as the $(r+1)^{th}$ row of λ. So both rows determine the same odd part (since $\lambda_{\frac{r}{2}} = \lambda_{\frac{r}{2}+1}$). But when $\gamma_r - \gamma_{r+1} > 1$, then since $\lambda_{\frac{r}{2}} > \lambda_{\frac{r}{2}+1}$, the r^{th} row ends at a further position (≥ 2 steps) on the right than the $(r+1)^{th}$ row, thus initiating a new odd part.

The foregoing discussion implies that the number of different odd parts in λ is equal to the number of separate sequences of consecutive integers in γ.

Now let's construct the inverse mapping.

Let $i' = number\ of\ rows\ of\ the\ partition\ \alpha$

Let $j' = number\ of\ columns\ of\ the\ partition\ \beta$

Basically what the bijection $P(\lambda)$ does with the partition λ is that it divides λ into α and β along the line with equation $j = 1 + 2i$. It then adds the $(j\ and\ j+1)^{th}$ column of β to half the j^{th} row of α, for $j \geq 1$.

Example:

$$1^{st}\ part\ of\ P(\lambda) = \left|1^{st}\ column\ of\ \beta\right| + \left|\frac{1^{st}\ row\ of\ \alpha}{2}\right|$$

$$2^{nd}\ part\ of\ P(\lambda) = \left|2^{nd}\ column\ of\ \beta\right| + \left|\frac{1^{st}\ row\ of\ \alpha}{2}\right|$$

$$3^{rd}\ part\ of\ P(\lambda) = \left|3^{rd}\ column\ of\ \beta\right| + \left|\frac{2^{nd}\ row\ of\ \alpha}{2}\right|$$

$$4^{th}\ part\ of\ P(\lambda) = \left|4^{th}\ column\ of\ \beta\right| + \left|\frac{2^{nd}\ row\ of\ \alpha}{2}\right|$$

$$5^{th}\ part\ of\ P(\lambda) = \left|5^{th}\ column\ of\ \beta\right| + \left|\frac{3^{rd}\ row\ of\ \alpha}{2}\right|$$

$$6^{th}\ part\ of\ P(\lambda) = \left|6^{th}\ column\ of\ \beta\right| + \left|\frac{3^{rd}\ row\ of\ \alpha}{2}\right|$$

$$\vdots$$

Note that each row in α is split into two equal parts and each part is merged with $(j\ and\ j+1)^{th}$ columns for $j \geq 1$.

This will always give us a partition into distinct parts (Euler) because in each j and $j+1$ columns we add the same number, also j and $j+1$ differ by at least one due to the segregation line $j = 1 + 2i$. Therefore we obtain a partition into distinct parts.

We note that:

$$|\# \ of \ colums \ of \ \beta \ - 2(\# \ of \ rows \ of \ \alpha)| = \left| l\left(\beta'\right) - l(\bar{\alpha}) \right| = 1.$$

i.e: $|j^{'} - 2i^{'}| = 1$

Now:

if $l\left(\beta'\right) > l(\bar{\alpha})$ then number of parts of $P(\lambda) = l\left(\beta'\right)$,

if $l\left(\beta'\right) < l(\bar{\alpha})$ then number of parts of $P(\lambda) = l(\bar{\alpha})$.

Since $l(\bar{\alpha}) = 2i^{'}$, then $l(\bar{\alpha})$ is always even, so if the number of parts of $P(\lambda)$ are even, then $l(\bar{\alpha}) > l\left(\beta'\right)$ and if the number of parts of $P(\lambda)$ are odd, then $l\left(\beta'\right) > l(\bar{\alpha})$.

Noting the above analysis, we now construct the inverse map of $P(\lambda)$.

Our key point is the last part of $P(\lambda)$ (the smallest part).

Step one:

Draw the line $j = 1 + 2i$, where

$$\begin{cases} 1 \le j \le l\left(P(\lambda)\right), & if \ l\left(P(\lambda)\right) \ is \ odd \\ 1 \le j \le l\left(P(\lambda)\right) + 1, & if \ l\left(P(\lambda)\right) \ is \ even. \end{cases}$$

Step two:

When the number of parts of $P(\lambda)$ is odd:

Draw the smallest part as a vertical square(s) and place it vertically going down and position it at the end of the line $j = 1 + 2i$, it makes column j of β.

When the number of parts of $P(\lambda)$ is even:

Draw double the smallest part as horizontal square(s) and place it horizontally going to the left and position it at the end of the line $j = 1 + 2i$, it makes row i of α.

Step three:

• When the number of parts of $P(\lambda)$ is odd:

At the $j-1$ position duplicate the j^{th} column. Keep on adding (2 by 1) rectangles at the i^{th} row of α till $|j-1| + \left|\frac{i}{2}\right| = 2^{nd}$ last part. For the next column, calculate the difference between the 3^{rd} last part and the 2^{nd} last part. On the $(j-2)^{th}$ column put the $(j-1)^{th}$ column + the difference calculated . Then $|j-2| + \left|\frac{i}{2}\right| = 3^{rd}$ last part.

For $(j-3)^{th}$ column duplicate $(j-2)^{th}$ column and add (2 by 1) square(s) on the $(i-1)^{th}$ row until the sum $|j-3| + \left|\frac{i-1}{2}\right| = 4^{th}$ last part.

Note that we always duplicate

$$(j, j-1), (j-2, j-3), (j-4, j-5), \cdots$$

since consecutive different parts differ by at least two in a partition with only odd parts.

• When the number of parts of $P(\lambda)$ is even:

For the j^{th} column of β, construct square(s) of the difference between the last and second last part. On the $(j-1)^{th}$ column duplicate j^{th} column. Calculate the difference between the 2^{rd} last part and the 3^{nd} last part then add square(s) of length 2 times the calculated difference to $(i-1)^{th}$ row of α.
We proceed with the process as in the first case of $P(\lambda)$ odd.

Step Four:

Continue with the same analysis and thinking as in step 3 until you get $j-(j-1) = 1^{st}$ column. Then you have $[\lambda]$. ∎

Example

Take our previous example: $P(73) = 16 + 14 + 13 + 10 + 7 + 6 + 4 + 2 + 1$

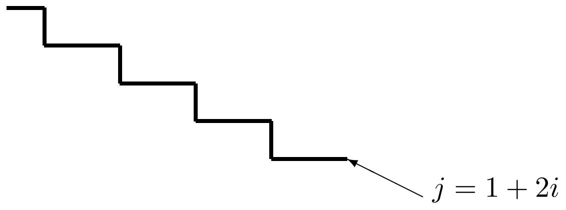
$P(73)$ has 9 parts, hence odd number of parts.

Let's get back to λ by following the inverse mapping of the bijection.

Step one:

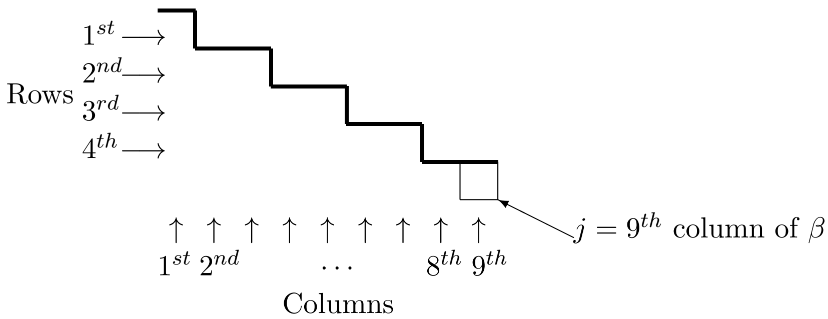
We draw the line

$$j = 1 + 2i$$



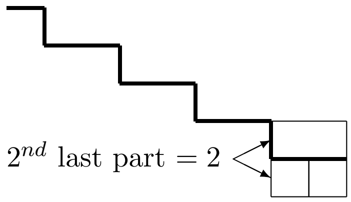
Step two:

Since the number of parts is odd, the smallest part is 1, so we put one square at the end and below the line we drew.



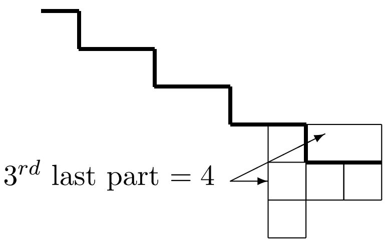
Step three:

On the 8^{th} column we duplicate the 9^{th} column, which is one, and add (2 by 1) rectangles on the 4^{th} row of α until 1+ number of (2 by 1) rectangles $= 2^{nd}$ last part of $P(73)$, i.e $1+1=2$.



The difference between the 3^{rd} last part and the 2^{nd} last part $= 4 - 2 = 2$

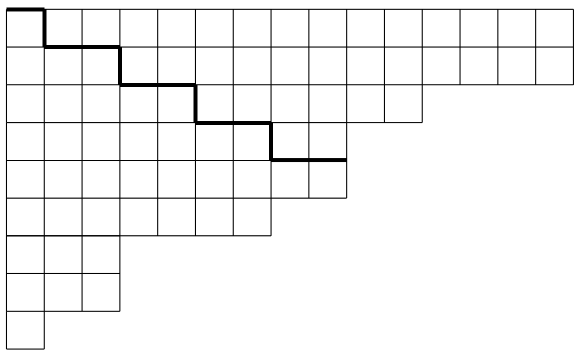
Now put the 7^{th} column of β in such a way that it differs by 2 to 8^{th} column of β, then add (2 by 1) rectangles on the 3^{rd} row of α till the sum of the (2 by 1) square(s) and the 7^{th} column of β is equal to the 3^{rd} last part of $P(73)$.



Step Four:

Continue with the same thinking and analysis as in step three by duplicating the 7^{th} column on the 6^{th} column. Keep on duplicating $(2j)^{th}$ and $(2j+1)^{th}$ columns, $j \geq 1$, and positioning the columns and rows of β and α accordingly.

After all this we are left with the following final diagram:



It is a Young diagram, $[\lambda] = [\mathbf{73}]$, with parts $\mathbf{15+15+11+9+9+7+3+3+1}$, the original partition into odd parts! ∎

3.2 Second Proof [Algebratized reversion]

This proof algebratizes the first proof.

Let $P = (p_1, p_2, p_3, \cdots)$ be a partition into exactly k odd parts and let its image $Q = (q_1, q_2, q_3, \cdots)$ be a partition into k separate sequence of consecutive integers.

Given a partition P, we construct

$$U = (u_1, u_2, \cdots),\ where\ u_{i+1} = p_{i+1} - 2i - 1, u_i \geq 2, i \geq 0,$$

$$V = \left(\frac{U}{2}\right) \cup \left(\frac{U}{2}\right)\ and\ v_i = \frac{u_{\lceil \frac{i}{2} \rceil}}{2} = v_{i+1}, i = 1, 3, 5, 7 \cdots 2l(U) - 1,$$

$$W = (w_1, w_2, w_3, \cdots),\ where\ w_i = p_i^{'} - \left\lfloor \frac{i}{2} \right\rfloor, w_i \geq 1, i \geq 1,$$

and $P^{'}$ denotes the conjugate of P.

Then;

$$Q = V + W.$$

Q is a partition into distinct parts since V is a partition with $v_{2i-1} = v_{2i},\ i \geq 1$ and W is a partition with $w_1 > w_2$ and $w_2 i = w_{2i+1}$.

Given Q, by construction we know the following:

(1) $p_{2i}^{'} = p_{2i+1}^{'}, \quad i \geq 1$

(2) $v_i = v_{i+1}, \quad i = 1, 3, 5, 7 \cdots 2l(U) - 1$

(3) $q_i = p_i^{'} - \lfloor \frac{i}{2} \rfloor + v_i, \quad i \geq 1$

(4) If $l(Q)$ is odd: $q_{l(Q)} = p_{l(Q)}^{'} - \frac{l(Q)-1}{2}$

(5) If $l(Q)$ is even: $q_{l(Q)} = v_{l(Q)} = v_{l(Q)-1}$.

From these properties we construct V and $P^{'}$ as follows:

If $l(Q)$ is odd

(i) From (4) get $p^{'}_{l(Q)}$ and use (1) to get $p_{l(Q)-1}$.

(ii) Use (3) to get v_i and $p^{'}_i$ for all i recursively by using (1) and (2) interchangeably.

If $l(Q)$ is even

(i) From (5) get $v_{l(Q)}$ and use (2) to get $v_{l(Q)-1}$.

(ii) Use (3) to get v_i and $p^{'}_i$ for all i recursively by using (1) and (2) interchangeably.

From V we construct U and from $P^{'}$ we construct W and $W^{'}$ as follows:

$U = (u_1, u_2, u_3, \cdots)$, where $u_{i+1} = 2 \cdot v_{2i+1},\ i \geq 0$
$W = (w_1, w_2, w_3, \cdots)$, where $w_i = p^{'}_i - \lfloor \frac{i}{2} \rfloor$
$W^{'}$ is a conjugate of W

Let W^* be a partition $W^{'}$ but with parts arranged in the following order: The first parts are the first consecutive odd parts up to the part $w^{'}_1$ and the other parts are the remaining parts from $W^{'}$ arranged in decreasing order.

Clearly,

$$P = U + W^*$$

■

Since the parts of P are odd, then the multiplicities of the parts of $P^{'}$ are even except for the largest part which has odd multiplicity, then it follows that the parts of W occur in pairs except for the largest part. This implies that all parts of $W^{'}$ are odd.

P is a partition into odd parts since all parts of U are even and all parts of W^* are odd.

Example:

Let $P = (19, 15, 15, 13, 9, 7, 3, 3, 1)$. and $P^{'} = (9, 8, 8, 6, 6, 6, 6, 5, 5, 4, 4, 4, 4, 3, 3, 1, 1, 1, 1)$

To get U we construct its parts from the fact that: $u_{i+1} = p_{i+1} - 2i - 1, u_i \geq 2, i \geq 0$

$$u_1 = 19 - 2(0) - 1 = 18$$
$$u_2 = 15 - 2(1) - 1 = 12$$
$$u_3 = 15 - 2(2) - 1 = 10$$
$$u_4 = 13 - 2(3) - 1 = 6.$$

V is easily constructed as $v_i = \frac{u_{\lceil \frac{i}{2} \rceil}}{2} = v_{i+1}, i = 1, 3, 5, 7 \cdots 2l(U) - 1$

$$v_1 = \frac{18}{2} = v_2$$
$$v_3 = \frac{12}{2} = v_4$$
$$v_5 = \frac{10}{2} = v_6$$
$$v_7 = \frac{6}{2} = v_8$$

$\therefore V = (9, 9, 6, 6, 5, 5, 3, 3)$.

From the property, $w_i = p_i^{'} - \left\lfloor \frac{i}{2} \right\rfloor, w_i \geq 1, i \geq 1$, we obtain W

$$w_1 = p_1^{'} - \left\lfloor \frac{1}{2} \right\rfloor = 9 - 0 = 9$$
$$w_2 = p_2^{'} - \left\lfloor \frac{2}{2} \right\rfloor = 8 - 1 = 7$$
$$w_3 = p_3^{'} - \left\lfloor \frac{3}{2} \right\rfloor = 8 - 1 = 7$$
$$w_4 = p_4^{'} - \left\lfloor \frac{4}{2} \right\rfloor = 6 - 2 = 4$$
$$w_5 = p_5^{'} - \left\lfloor \frac{5}{2} \right\rfloor = 6 - 2 = 4$$
$$w_6 = p_6^{'} - \left\lfloor \frac{6}{2} \right\rfloor = 6 - 3 = 3$$
$$w_7 = p_7^{'} - \left\lfloor \frac{7}{2} \right\rfloor = 6 - 3 = 3$$
$$w_8 = p_8^{'} - \left\lfloor \frac{8}{2} \right\rfloor = 5 - 4 = 1$$
$$w_9 = p_9^{'} - \left\lfloor \frac{9}{2} \right\rfloor = 5 - 4 = 1$$

$Q = V + W = (9+9, 9+7, 6+7, 6+4, 5+4, 5+3, 3+3, 3+1, 1) = (18, 16, 13, 10, 9, 8, 6, 4, 1)$.

Now that we have Q let's try to make our way back to P:

$l(Q) = 9$**,which is odd**

Step one :

From property (4), page 31, $q_{l(Q)} = p_{l(Q)}^{'} - \frac{l(Q)-1}{2}$

$$1 = q_9 = p_9^{'} - \frac{9-1}{2} \Rightarrow p_9^{'} = 5$$

From property (1), $p_8^{'} = p_{8+1} = 5$.

Step two :

From property (3), with $i = 8$

$$4 = q_8 = p_8^{'} - \left\lfloor \frac{8}{2} \right\rfloor + v_8 \Rightarrow v_8 = 3$$

By property (2) we know that $v_i = v_{i+1}$ for i odd, then:

$$v_8 = v_7 = 3.$$

Substitute v_7 into property (3) to get $p_7^{'}$:

$$q_7 = p_7^{'} - \left\lfloor \frac{7}{2} \right\rfloor + v_7 \Rightarrow p_7^{'} = 6 = p_6^{'}$$

Substitute $p_6^{'}$ into property (3) to get v_6 :

$$q_6 = p_6^{'} - \left\lfloor \frac{6}{2} \right\rfloor + v_6 \Rightarrow v_6 = 5 = v_5$$

Substitute v_5 into property (3) to get $p_5^{'}$:

$$q_5 = p_5^{'} - \left\lfloor \frac{5}{2} \right\rfloor + v_5 \Rightarrow p_5^{'} = 6 = p_4^{'}$$

Substitute $p_4^{'}$ into property (3) to get v_4 :

$$q_4 = p_4^{'} - \left\lfloor \frac{4}{2} \right\rfloor + v_4 \Rightarrow v_4 = 6 = v_3$$

Substitute v_3 into property (3) to get $p_3^{'}$:

$$q_3 = p_3^{'} - \left\lfloor \frac{3}{2} \right\rfloor + v_3 \Rightarrow p_3^{'} = 8 = p_2^{'}$$

Substitute $p_2^{'}$ into property (3) to get v_2 :

$$q_2 = p_2^{'} - \left\lfloor \frac{2}{2} \right\rfloor + v_2 \Rightarrow v_2 = 9 = v_1$$

Substitute v_1 into property (3) to get $p_1^{'}$:

$$q_1 = p_1^{'} - \left\lfloor \frac{1}{2} \right\rfloor + v_1 \Rightarrow p_1^{'} = 9$$

$V = (9, 9, 6, 6, 5, 5, 3, 3)$ and $P^{'} = (9, 8, 8, 6, 6, 6, 6, 5, 5, \cdots)$.

From V we construct $U = (u_1, u_2, u_3, \cdots)$, where $u_{i+1} = 2 \cdot v_{2i+1},\ i \geq 0$,

$$\text{thus } U = (2 \cdot 9, 2 \cdot 6, 2 \cdot 5, 2 \cdot 3) = (18, 12, 10, 6)$$

From $P^{'}$ we construct $W = (w_1, w_2, w_3, \cdots),\ where\ w_i = p_i^{'} - \left\lfloor \frac{i}{2} \right\rfloor, w_i \geq 1, i \geq 1$

$$w_1 = p_1^{'} - \left\lfloor \frac{1}{2} \right\rfloor = 9 - 0 = 9$$

$$w_2 = p_2^{'} - \left\lfloor \frac{2}{2} \right\rfloor = 8 - 1 = 7$$

$$w_3 = p_3^{'} - \left\lfloor \frac{3}{2} \right\rfloor = 8 - 1 = 7$$

$$w_4 = p_4^{'} - \left\lfloor \frac{4}{2} \right\rfloor = 6 - 2 = 4$$

$$w_5 = p_5^{'} - \left\lfloor \frac{5}{2} \right\rfloor = 6 - 2 = 4$$

$$w_6 = p_6^{'} - \left\lfloor \frac{6}{2} \right\rfloor = 6 - 3 = 3$$

$$w_7 = p_7^{'} - \left\lfloor \frac{7}{2} \right\rfloor = 6 - 3 = 3$$

$$w_8 = p_8^{'} - \left\lfloor \frac{8}{2} \right\rfloor = 5 - 4 = 1$$

$$w_9 = p_9^{'} - \left\lfloor \frac{9}{2} \right\rfloor = 5 - 4 = 1$$

$$W = (9,7,7,4,4,3,3,1,1) \; and \; W^{'} = (9,7,7,5,3,3,3,1,1)$$

Finally

$$W^{*} = (1,3,5,7,9,7,3,3,1)$$

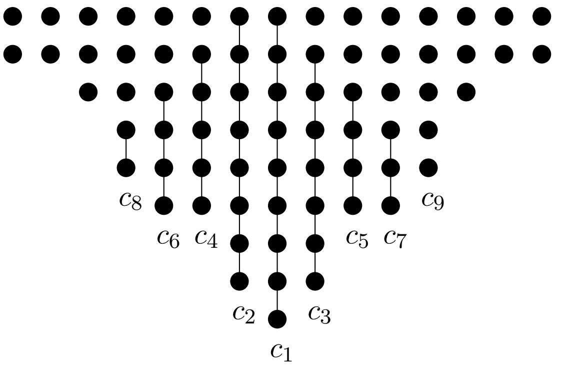
$$\therefore P = U + W^{*} = (18,12,10,6) + (1,3,5,7,9,7,3,3,1) = (19,15,15,13,9,7,3,3,1)$$

3.3 Third Proof

We note that any partition into odd parts can be represented in a centrally justified Ferrers graph. We shall use this special character of partitions into odd parts to give this alternative proof of Sylvester's refinement of Euler's Theorem.

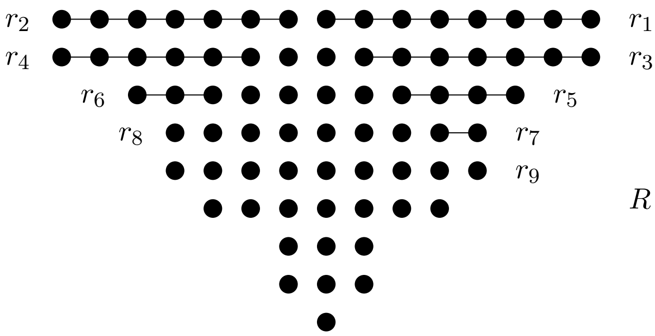
Let $P = (p_1, p_2, p_3, \cdots)$ be a partition into exactly k odd parts and let its image $Q = (q_1, q_2, q_3, \cdots)$ be a partition into k separate sequences of consecutive integers.

Let c_i be the i^{th} column of the central justified Ferrers graph, where c_1 is the middle column. c_i's are counted alternatively on the sides of the middle column, such that c_{2i}'s are on the left. For Example:



$C = \{9, 8, 7, 5, 4, 4, 3, 2, 1\}$

Let r_i be the i^{th} row on the central justified Ferrers graph such that r_{2i}'s are on the left, $p_i = r_{2i-1} + r_{2i} + 2i - 2$ and $r_i \geq 0$. Eg:



$R = \{8, 7, 7, 6, 4, 3, 2, 1, 1\}$

Observe that in the two forgoing diagrams columns and rows start at different places, starting down diagonal from the center of the top row.

Given P, our aim is to find it's image Q. We do this by first finding $C = (c_1, c_2, c_3, \cdots)$ and $R = (r_1, r_2, r_3, \cdots)$. We construct centrally justified Ferrers graph for C and R and then use:

$$q_i = r_i + c_i - 1 \tag{3.1}$$

Given Q, by construction we know the following:

$$c_i = c_{i+1} + 1, \quad i = 2, 4, 6, 8, \cdots \tag{3.2}$$

$$r_i = r_{i+1} + 1, \quad i = 1, 3, 5, 7, \cdots \tag{3.3}$$

We also know that:

(1) If $l(Q)$ is odd, then

$$q_{l(Q)} = c_{l(Q)}.$$

(2) If $l(Q)$ is even, then

$$q_{l(Q)} = r_{l(Q)}.$$

To construct P, we need to find $C = (c_1, c_2, c_3, \cdots)$ and $R = (r_1, r_2, r_3 \cdots)$ from the following steps:

Step(1)

Use (1) or (2), depending on the parity of $l(Q)$, to get $c_{l(Q)}$ or $r_{l(Q)}$.

Step(2)

Use Equations (3.1), (3.2) and (3.3) recursively to get all c_i and r_i.

Step(3)

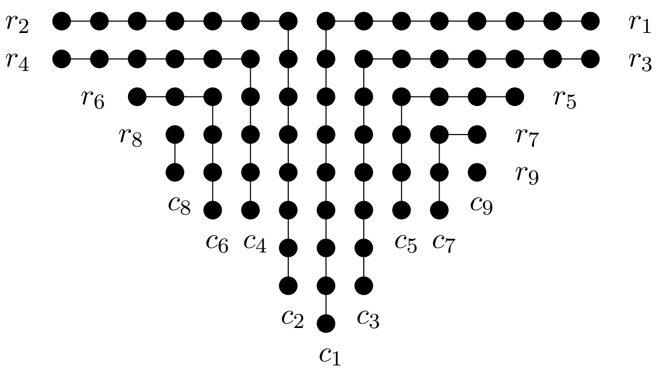
Having C and R, we simple put them together appropriately to construct a centrally justified Ferrers graph of P.

∎

Example:

$$P = (15, 15, 11, 9, 9, 7, 3, 3, 1), \text{ has 9 odd parts.}$$

Given P, it's easy to draw the lines linking c_i and r_i with alternating the columns keeping c_{2i} and r_{2i} to the left:



Thus

$$C = (9, 8, 7, 5, 4, 4, 3, 2, 1)$$

and

$$R = (8, 7, 7, 6, 4, 3, 2, 1, 1)$$

From equation (3.1):

$$q_1 = 8 + 9 - 1 = 16$$

$$q_2 = 7 + 8 - 1 = 14$$

$$q_3 = 7 + 7 - 1 = 13$$

$$q_4 = 6 + 5 - 1 = 10$$

$$q_5 = 4 + 4 - 1 = 7$$

$$q_6 = 3 + 4 - 1 = 6$$

$$q_7 = 2 + 3 - 1 = 4$$
$$q_8 = 1 + 2 - 1 = 2$$
$$q_9 = 1 + 1 - 1 = 1$$

$\therefore\ Q = (16, 14, 13, 10, 7, 6, 4, 2, 1)$.

Now given Q, we should get back to P.

Step(1)

$l(Q) = 9$, which is odd. Then from (1) we have that: $q_9 = c_9 = 1$

Step(2)

- From (3.2): $c_8 = c_9 + 1 = 1 + 1 = 2$ and from (3.1): $1 = r_9 + 1 - 1 \longrightarrow r_9 = 1$
- $$q_8 = r_8 + c_8 - 1 \longrightarrow r_8 = 2 - 2 + 1 = 1$$
  $$and$$
  $$r_7 = r_8 + 1 = 1 + 1 = 2$$
- $$q_7 = r_7 + c_7 - 1 \longrightarrow c_7 = 4 - 2 + 1 = 3$$
  $$and$$
  $$c_6 = c_7 + 1 = 3 + 1 = 4$$
- $$q_6 = r_6 + c_6 - 1 \longrightarrow r_6 = 6 - 4 + 1 = 3$$
  $$and$$
  $$r_5 = r_6 + 1 = 3 + 1 = 4$$
- $$q_5 = r_5 + c_5 - 1 \longrightarrow c_5 = 7 - 4 + 1 = 4$$
  $$and$$
  $$c_4 = c_5 + 1 = 4 + 1 = 5$$
- $$q_4 = r_4 + c_4 - 1 \longrightarrow r_4 = 10 - 5 + 1 = 6$$
  $$and$$
  $$r_3 = r_4 + 1 = 6 + 1 = 7$$

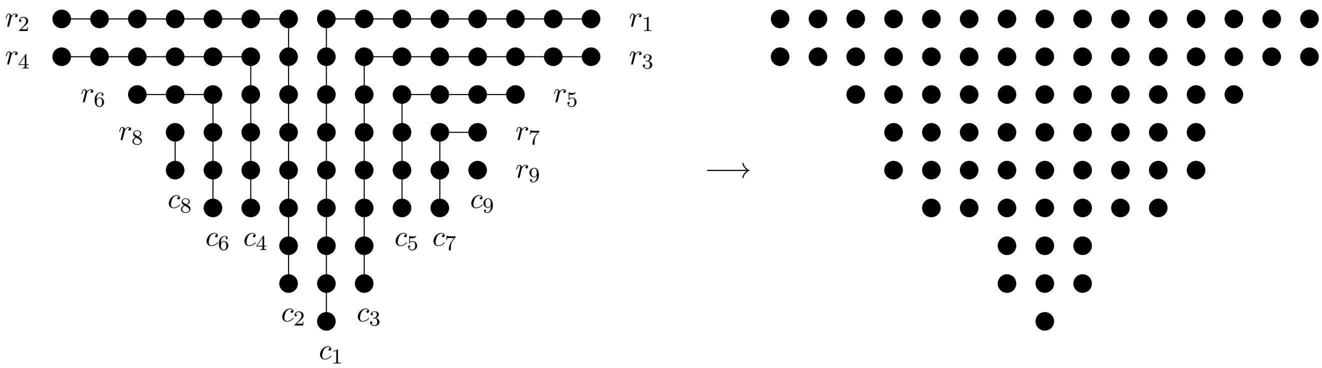
- $$q_3 = r_3 + c_3 - 1 \longrightarrow c_3 = 13 - 7 + 1 = 7$$
$$and$$
$$c_2 = c_3 + 1 = 7 + 1 = 8$$

- $$q_2 = r_2 + c_2 - 1 \longrightarrow r_2 = 14 - 8 + 1 = 7$$
$$and$$
$$r_1 = r_2 + 1 = 7 + 1 = 8$$

- $$q_1 = r_1 + c_1 - 1 \longrightarrow c_1 = 16 - 8 + 1 = 9$$

$\therefore$ $C = (9, 8, 7, 5, 4, 4, 3, 2, 1)$ and $R = (8, 7, 7, 6, 4, 3, 2, 1, 1)$.

Step(3)



$\therefore P = (15, 15, 11, 9, 9, 7, 3, 3, 1)$

Remark:

Note that the bijections in section 3.2 and section 3.3 give the same map since:

• in section 3.2 we construct
$$Q = V + W,$$
with
$$U = (u_1, u_2, \cdots),\ where\ u_{i+1} = p_{i+1} - 2i - 1, u_i \geq 2, i \geq 0$$

$$V = \left(\frac{U}{2}\right) \cup \left(\frac{U}{2}\right) \ \textit{and} \ v_i = \frac{u_{\lceil \frac{i}{2} \rceil}}{2} = v_{i+1}, \quad i = 1, 3, 5, 7 \cdots 2l(U) - 1$$

$$W = (w_1, w_2, w_3, \cdots), \ \textit{where} \ w_i = p_i^{'} - \left\lfloor \frac{i}{2} \right\rfloor, w_i \geq 1, i \geq 1$$

- in section 3.3 we construct

$$q_i = r_i + c_i - 1.$$

By observation

$$r_i - 1 = v_i = \frac{u_{\lceil \frac{i}{2} \rceil}}{2} = v_{i+1} = r_{i+1}, \qquad i = 1, 3, 5, 7 \cdots 2l(U) - 1$$

and

$$c_i = w_i = p_i^{'} - \left\lfloor \frac{i}{2} \right\rfloor, w_i \geq 1, i \geq 1$$

$$\therefore r_i - 1 + c_i = V + W.$$

3.4 Different Refinement of Euler's Theorem

Bousquet-Mélou and Eriksson gave another refinement of Euler's theorem:

Theorem 6 *The number of partitions of n into distinct parts whose alternating sum is r is equal to the number of partitions of n into r odd parts [17].*

On the context of lecture hall partitions, which are briefly explained in Chapter 7, they obtained a general result and showed that the above theorem is a limiting case.

This implies Euler's odd-distinct parts theorem in that when $r \longrightarrow \infty$ we have Euler's partition theorem.

3.4.1 Proof

Let ζ be the mapping,

$$\zeta : \mathcal{D}_n \to \mathcal{O}_n$$

We put a condition on the last part to allow it to be a zero and then establish that any partition, $\lambda \in \mathcal{D}_n$, has even length. We now construct the unique partition

$$\zeta(\lambda) \in \mathcal{O}_n \qquad \text{with } r \text{ parts}$$

where r is the alternating sum of λ, ie

$$r = \sum_{i=1}^{l(\lambda)} (-1)^{i-1}\lambda_i \qquad \text{where } l(\lambda) \text{ is the number of parts of} \lambda.$$

Let $\lambda = \lambda_1\lambda_2\lambda_3\cdots\lambda_{2k}$.

We pile $|\lambda|$ boxes into $l(\lambda)$ rows in the complying fashion:

Step one:

For the first row we put λ_1 boxes horizontally. λ_2 boxes are placed (left justified) on top of the λ_1 boxes to obtain row two. λ_3 boxes are placed(right justified) on top of the λ_2 boxes to obtain row three. We do this process repeatedly, alternating between leftmost and rightmost, until we arrive at a diagram of height $2k$, of which its top row may contain ≥ 0 boxes . It contains 0 boxes when $\lambda_{2k} = 0$.

Step two:

Now separate λ_1 into two pieces, r boxes to the right and $\lambda_1 - r$ boxes to the left by placing a vertical bar through the whole diagram. This vertical bar is called the separator. The columns to the right of the separator now forms a partition into odd parts.
By letting $\alpha = \alpha_1\alpha_2\alpha_3\cdots\alpha_r$ be the partition formed with the columns to the right of the separator. From the construction of the diagram(r boxes to the right of the separator) r is captured by the sum of the horizontal squares between consecutive parts on the right of the separator. Then $\alpha_r = l(\lambda) - 1$ and $\alpha_1 = 1$ because $l(\lambda)$ is even and $\lambda_1 > \lambda_2$, respectively. Any odd integer between α_r and α_1 appears in α at least once as a part because from the right of the separator the steps of the diagram increases upwards by blocks of two.

Step three:

We now construct another partition $\beta = \beta_1\beta_2\beta_3\cdots\beta_{2k}$, where β_i is the number of boxes to the left of the separator in row i. Due to leftmost aligned boxes, we notice that $\beta_{2i-1} = \beta_{2i} > \beta_{2i+1}$. To the rightmost of $2i-1$ in α we evaluate $2i-1+2\beta_{2i-1}$(where $2\beta_{2i-1} = \beta_{2i-1} + \beta_{2i}$), for $i = 1, 2, 3, \cdots, k-1$.

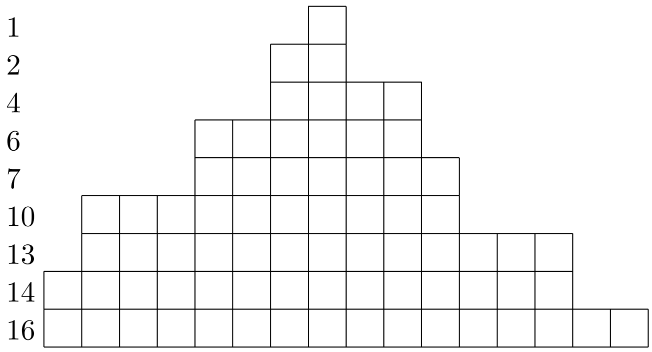
Step four:

Denote by δ the partition we get after rearranging the resulting sequence from step three and the α_i's not used in step three. It's obvious now that each $\delta_i \in \mathcal{O}_n$ and $l(\delta) = r$

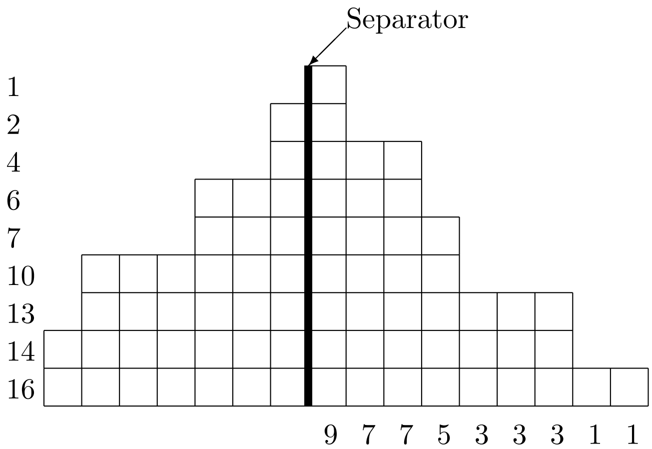
Example

Step one:

Let $\lambda = (16, 14, 13, 10, 7, 6, 4, 2, 1)$, then $r = 9$ and $k = 5$.



Step two:



$$\alpha_9 = 9, \alpha_8 = 7, \alpha_7 = 7, \alpha_6 = 5, \alpha_5 = 3, \alpha_4 = 3, \alpha_3 = 3, \alpha_2 = 1, \alpha_1 = 1$$

Step three:

$$\beta_1 = 7, \beta_2 = 7, \beta_3 = 6, \beta_4 = 6, \beta_5 = 3, \beta_6 = 3, \beta_7 = 1, \beta_8 = 1$$

$$2\beta_1 + \alpha_1 = 15, 2\beta_3 + \alpha_3 = 15, 2\beta_5 + \alpha_6 = 11, 2\beta_7 + \alpha_7 = 9$$

Step four:

$$\delta = 15 + 15 + 11 + 9 + 9 + 7 + 3 + 3 + 1$$

and

$$l(\delta) = 9$$

Remark:

It is not by coincidence that this mapping coincides with the inverse of the Sylvester's bijection studied in section 3.1 above, it is in fact the inverse of Sylvester's refinement of Euler's Theorem.

We now have to prove that the mapping

$$\zeta : \mathcal{D}_n \to \mathcal{O}_n$$

is a bijection where $|\lambda| = |\zeta(\lambda)|$ and $r = |l(\zeta(\lambda))|$.

This is fairly straight forward as it is the reverse of the above. We use the same idea of the "separator".

Since

$$\zeta(\lambda) = \delta$$

let

$$\delta = \delta_1 \delta_2 \cdots \delta_l, \quad \forall i, \ \delta_i \in \mathcal{O}_n.$$

Let α_i be the height of the i^{th} column to the right of the separator, then

$$\alpha = \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_l$$

Let β_i be the number of boxes on the i^{th} row to the left of the separator, then

$$\beta = \beta_1 \beta_2 \beta_3 \cdots \beta_{2k}, \qquad \beta_{2k} \geq 0$$

where k is the largest integer such that

$$\delta_k \geq 2k - 1.$$

We have that

$$\beta_{2i-1} = \beta_{2i}.$$

We know that

$$\delta_i = 2i - 1 + 2\beta_{2i-1}, \qquad 1 \leq i \leq k.$$

From the construction of the above diagram, $\alpha_1 = 1$ and $\alpha_{2k-1} = 2k-1$ and any odd part between 1 and $2k-1$ appears atleast once. The number of parts of α equal the alternating sum of λ,i.e $l(\alpha) = r$. The reason why we add $2\beta_{2i-1}$ to each first appearance of a part of α is to make sure that the new partition formed, δ, is a partition into $r = l(\alpha)$ odd parts and these will form exactly those parts $\delta_k \geq 2k-1$ because $\alpha_i \leq 2i-1$.

We proceed with the next simple steps for $\zeta^{-}(\delta) : \mathcal{O}_n \to \mathcal{D}_n$.

Step one:

Replace $\delta_1 \delta_2 \cdots \delta_k$ with the first k odd numbers such that $\delta_k \geq 2k-1$, and we denote the new partition by δ' once reordered.

i.e

$$\delta' = 1, 3, 5 \cdots 2k-1, \delta_{k+1} \delta_{k+2} \cdots \delta_l,$$

since

$$\alpha_i \leq 2i - 1$$

Step two:

Find all β_i where $1 \leq i \leq 2k$ using the fact that

$$\delta_i = 2i - 1 + 2\beta_{2i-1}, \qquad 1 \leq i \leq k$$

and that

$$\beta_{2i-1} = \beta_{2i}$$

Step three:

Conjugate

$$\delta^{'} = 1, 3, 5 \cdots 2k - 1, \delta_{k+1}\delta_{k+2} \cdots \delta_l$$

to get

$$\alpha = \alpha_1\alpha_2\alpha_3 \cdots \alpha_{2k-1}$$

and note that

$$\alpha_1 = 1 \textit{ and } \alpha_l = l.$$

Step four:

Construct

$$\lambda = \lambda_1\lambda_2\lambda_3 \cdots \lambda_{2k},$$

where $\lambda_i = \alpha_i + \beta_i$ for $1 \leq i \leq 2k$
$\therefore \lambda \in \mathcal{D}_n$ ∎

Example:

Step one:

Let

$$\delta = 15, 15, 11, 9, 9, 7, 3, 3, 1$$

$\delta_k \geq 2k - 1$ at $k = 5$

$$\delta^{'} = 1, 3, 5, 7, 9, 7, 3, 3, 1 = 1, 1, 3, 3, 3, 5, 7, 7, 9$$

Step two:

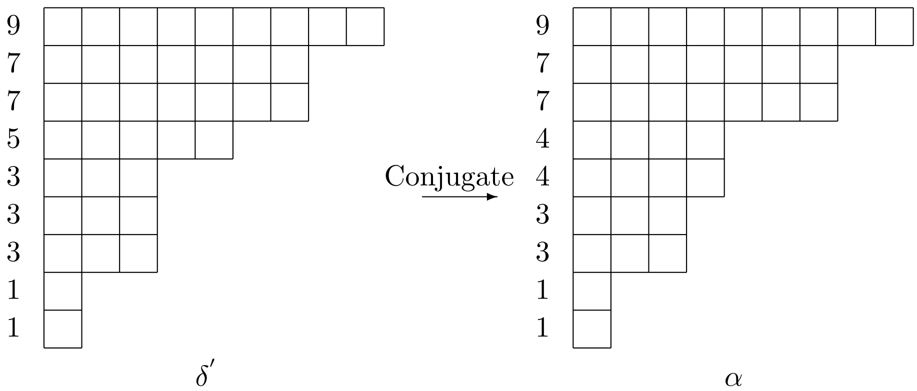
$$\delta_i = 2i - 1 + 2\beta_{2i-1}, \quad 1 \leq i \leq 5$$
$$15 = 1 + 2\beta_1$$

$$15 = 3 + 2\beta_3$$
$$11 = 5 + 2\beta_5$$
$$9 = 7 + 2\beta_7$$
$$9 = 9 + 2\beta_9$$

$\therefore$

$$\beta_1 = 7 = \beta_2$$
$$\beta_3 = 6 = \beta_4$$
$$\beta_5 = 3 = \beta_6$$
$$\beta_7 = 1 = \beta_8$$
$$\beta_9 = 0 = \beta_{10}$$

Step three:



Thus

$$\alpha = \alpha_1\alpha_2\cdots\alpha_9$$
$$= 9, 7, 7, 4, 4, 3, 3, 1, 1.$$

Step four:

We construct:

$$\lambda = \lambda_1\lambda_2\lambda_3\cdots\lambda_{10},$$

where $\lambda_i = \alpha_i + \beta_i$ for $1 \le i \le 10$
From step two and step three:

$$\lambda_1 = \alpha_1 + \beta_1 = 9 + 7 = 16$$

$$\lambda_2 = \alpha_2 + \beta_2 = 7 + 7 = 14$$
$$\lambda_3 = \alpha_3 + \beta_3 = 7 + 6 = 13$$
$$\lambda_4 = \alpha_4 + \beta_4 = 4 + 6 = 10$$
$$\lambda_5 = \alpha_5 + \beta_5 = 4 + 3 = 7$$
$$\lambda_6 = \alpha_6 + \beta_6 = 3 + 3 = 6$$
$$\lambda_7 = \alpha_7 + \beta_7 = 3 + 1 = 4$$
$$\lambda_8 = \alpha_8 + \beta_8 = 1 + 1 = 2$$
$$\lambda_9 = \alpha_9 + \beta_9 = 1 + 0 = 1$$
$$\lambda_{10} = \alpha_{10} + \beta_{10} = 0 + 0 = 0$$

$\therefore$ we obtain

$$\lambda = 16, 14, 13, 10, 7, 6, 4, 2, 1$$

Chapter 4

Further Proof of Euler's Theorem: Glaisher's Generalization

Glaisher's theorem is an identity useful to the study of integer partitions. It is named for James Whitbread Lee Glaisher. The material covered in this chapter comes from [24].

Theorem 7 (Glaisher [16]) *The number of partitions of n, where no part appears more than $d-1$ times is equal to the number of partitions of n into parts which are $\not\equiv 0 \pmod{d}$.*

Glaisher's theorem implies Euler's theorem in that when $d = 2$ we have: The number of partitions of n, where no part appears more than 1 time(distinct parts) is equal to the number of partitions of n into parts which are $\not\equiv 0 \pmod{2}$(odd parts)

In this chapter we are going to demonstrate a proof of Glaisher's theorem based on the uniqueness of binary expansion of an integer. See [24] for more details into this generalization.

4.1 Proof of Glaisher's theorem of Euler

Before outlining this proof, it is very crucial that we introduce the frequency notation of partitions.

Any partition

$$l = l_1 + l_2 \cdots l_t, \tag{4.1}$$

can be written as follows:

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 \cdots \tag{4.2}$$

or

$$\{f_1, f_2, f_3, f_4, \cdots\}, \tag{4.3}$$

where f_i represents the number of times i appears in the partition of n.

Example:

$$5+5+4+3+3+3+2+1 = 2\cdot 5+1\cdot 4+3\cdot 3+1\cdot 2+1\cdot 1$$

$$= \{1,1,3,1,2\}.$$

Now that we have introduced the frequency notation let's do the proof.

Let $l = l_1 + l_2 + \cdots + l_t$ be a partition of n into t odd parts.

We write l in frequency notation:

$$f_1\cdot 1 + f_3\cdot 3 + f_5\cdot 5 + f_7\cdot 7 + \cdots;$$

Now we replace each f_i with its binary expansion which is unique:

$$\cdots + a_{i3}\cdot 2^3 + a_{i2}\cdot 2^2 + a_{i1}\cdot 2^1 + a_{i0}\cdot 2^0. \tag{4.4}$$

So that

$$\begin{aligned} f_1\cdot 1 + f_3\cdot 3 + f_5\cdot 5 + f_7\cdot 7 + \cdots &= (\cdots + a_{13}\cdot 2^3 + a_{12}\cdot 2^2 + a_{11}\cdot 2^1 + a_{10}\cdot 2^0)\cdot 1 \\ &+ (\cdots + a_{33}\cdot 2^3 + a_{32}\cdot 2^2 + a_{31}\cdot 2^1 + a_{30}\cdot 2^0)\cdot 3 \\ &+ (\cdots + a_{53}\cdot 2^3 + a_{52}\cdot 2^2 + a_{51}\cdot 2^1 + a_{50}\cdot 2^0)\cdot 5 \\ &+ (\cdots + a_{73}\cdot 2^3 + a_{72}\cdot 2^2 + a_{71}\cdot 2^1 + a_{70}\cdot 2^0)\cdot 7 \\ &\vdots \\ &= a_{10} + 2a_{11} + 3a_{30} + 4a_{12} + 5a_{50} + 6a_{31} + \cdots \end{aligned}$$

$a_{ij} = 1$ if it's coefficient contribute to $f_i's$ binary expansion , $else\ a_{ij} = 0$.

Since each $a_{ij} \in \{0,1\}$, clearly the above partition is now a partition into distinct parts, where for each distinct part $b_{ij}a_{ij} > 0$, we have $2^j || b_{ij}$.

Since $2^j || b_{ij}$, then to get the original partition into odd parts, we divide each $b_{ij}a_{ij}$ by 2^j and the result is the original odd part and its multiplicity is 2^j. We combine equal odd numbers and finally multiply each odd by it's multiplicity to get the initial partition.i.e:

$$\begin{aligned} a_{10} + 2a_{11} + 3a_{30} + 4a_{12} + 5a_{50} + 6a_{31} + \cdots &= \frac{a_{10}}{2^0} + \frac{2a_{11}}{2^1} + \frac{3a_{30}}{2^0} + \frac{4a_{12}}{2^2} + \frac{5a_{50}}{2^0} + \frac{6a_{31}}{2^1} + \cdots \\ &= 2^0\cdot a_{10} + 2^1\cdot a_{11} + 2^0\cdot 3a_{30} + 2^2\cdot a_{12} + 2^0\cdot 5a_{50} \\ &\quad + 2^1\cdot 3a_{31} + \cdots \\ &= 1\cdot a_{10} + 2\cdot a_{11} + 1\cdot 3a_{30} + 4\cdot a_{12} + 1\cdot 5a_{50} + 2\cdot 3a_{31} + \cdots \\ &= (a_{10} + 2a_{11} + 4a_{12} + \cdots)\cdot 1 + (a_{30} + 2a_{31} + \cdots)\cdot 3 \\ &\quad + (a_{50} + \cdots)\cdot 5 + \cdots \\ &= f_1\cdot 1 + f_3\cdot 3 + f_5\cdot 5 + f_7\cdot 7 + \cdots \end{aligned}$$

∎

Glaisher's theorem can be alternatively stated as: The number of partitions of n into non-multiples of m equals the number of partitions of n where no part appears more than $m-1$ times [24].

4.2 Proof of Generalized Glaisher's Theorem

Let $l = l_1 + l_2 + \cdots + l_t$ be a partition of n into t nonmultiples of m.

We write l in frequency notation (where $f_i = 0 \; if \; m \mid i$) and expand each f_i into base m as follows:

$$\begin{aligned} f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \cdots = & (\cdots + a_{13} \cdot m^3 + a_{12} \cdot m^2 + a_{11} \cdot m^1 + a_{10} \cdot m^0) \cdot 1 \\ & + (\cdots + a_{23} \cdot m^3 + a_{22} \cdot m^2 + a_{21} \cdot m^1 + a_{20} \cdot m^0) \cdot 2 \\ & + (\cdots + a_{33} \cdot m^3 + a_{32} \cdot m^2 + a_{31} \cdot m^1 + a_{30} \cdot m^0) \cdot 3 \\ & + (\cdots + a_{43} \cdot m^3 + a_{42} \cdot m^2 + a_{41} \cdot m^1 + a_{40} \cdot m^0) \cdot 4 \\ & \vdots \end{aligned}$$

Since $a_{ij} \in \{0; m-1\}$, then we conclude that a part can appear at most $m-1$ time. ∎

Chapter 5

Further Proof of Euler's Theorem: Franklin's Extension

5.1 Franklin's extension of the Euler's partition theorem

Theorem 8 (Franklin's extension of Euler's partition theorem) *[22] The number of partitions of n with k even part sizes is equal to the number of partitions of n with k repeated parts.*

This theorem implies Euler's partition theorem when we have $k = 0$

5.1.1 Analytic Proof

Let $\lambda = \eta \cup \tau_k$, where η is a partition into odd parts and τ_k is a partition into k even part sizes. Let $n = |\lambda|$.

$$\begin{aligned}\sum_{n=0}^{\infty} p(n|\lambda = \eta \cup \tau_k) q^n &= \prod_{n_i = odd} (1 + q^{n_i} + q^{n_i + n_i} + q^{n_i + n_i + n_i} + \cdots) \prod_{j=1}^{\infty} (1 + yq^{2j} + yq^{4j} + yq^{6j} + \dots) \\ &= \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} \prod_{j=1}^{\infty} \left(1 + \frac{yq^{2j}}{1 - q^{2j}}\right) \\ &= \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} \prod_{j=1}^{\infty} \frac{1 - (1 - y)q^{2j}}{1 - q^{2j}}. \end{aligned} \tag{5.1}$$

$$\sum_{n=0}^{\infty} p(n| \textit{ into } k \textit{ repeated parts }) q^n = \prod_{n=1}^{\infty}(1+q^n)\prod_{j=1}^{\infty}(1+yq^{j+j}+yq^{2j+2j}+yq^{3j+3j}+\dots)$$
$$= \prod_{n=1}^{\infty}(1+q^n)\prod_{j=1}^{\infty}(1+yq^{2j}+yq^{4j}+yq^{6j}+\dots)$$
$$= \prod_{n=1}^{\infty}(1+q^n)\prod_{j=1}^{\infty}\left(1+\frac{yq^{2j}}{1-q^{2j}}\right)$$
$$= \prod_{n=1}^{\infty}(1+q^n)\prod_{j=1}^{\infty}\frac{1-(1-y)q^{2j}}{1-q^{2j}}. \tag{5.2}$$

∴ To show that

$$\sum_{n=1}^{\infty} p(n|\lambda = \eta \cup \tau_k) q^n = \sum_{n=1}^{\infty} p(n| \textit{ into } k \textit{ repeated parts }) q^n \tag{5.3}$$

we just have to show that

$$\prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}} = \prod_{n=1}^{\infty}(1+q^n). \tag{5.4}$$

Equation (5.4) is true from Euler's Partition theorem.

∴ This concludes the analytic proof of Franklin's theorem. ∎

5.1.2 Combinatorial Proof

We denote $\mathcal{P}_n$ the set of all partitions of n. We then define $\xi : \mathcal{P}_n \leftarrow \mathcal{P}_n$. Let $\lambda = \eta \cup \tau_k$, where η is a partition into odd parts and τ_k is a partition into k even part sizes.

We see that $\lambda \in \mathcal{P}_n$. What ξ does to λ is that it divides each part of τ_k by 2 and convert each part of η into distinct parts by applying Euler's Partition Theorem.

$$\therefore \xi(\lambda) = \xi(\eta \cup \tau_k) = p(\eta|\textit{distinct parts}) \cup \frac{\tau_k}{2} \cup \frac{\tau_k}{2}$$

$\xi^{-1}(\lambda)$ clearly add the repeated part sizes together and convert the distinct parts to odd parts via Euler's Partition Theorem.

$$\xi^{-1}(\lambda) = \xi^{-1}\left(p(\eta|\textit{distinct parts}) \cup \frac{\tau_k}{2} \cup \frac{\tau_k}{2}\right) = p\left(p(\eta|\textit{distinct parts})\right) \cup (\frac{\tau_k}{2} + \frac{\tau_k}{2}) = \eta \cup \tau_k$$

∎

Example

$$\begin{aligned} 48 &= 9+9+8+7+6+3+2+2+1+1 && \textit{3 even part sizes} \\ &= (9+9+7+3+1+1)+(8+6+2+2) \\ &= (18+7+3+2)+(4+4+3+3+1+1+1+1) \\ &= 18+7+4+4+3+3+3+2+1+1+1+1 && \textit{3 repeated part sizes} \\ &= 48 \end{aligned}$$

Chapter 6

The Rogers-Ramanujan Identities

The Rogers-Ramanujan identities have applications in various scientific studies [4, 5]. Rogers, Ramanujan and Schur discovered these identities and later Hardy named them [15]. Most of the material in this chapter are based on [5, 8, 12].

Theorem 9 (The First Rogers-Ramanujan identity) *[5, Chp. 7, p. 109] The partitions of an integer n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into part $\equiv 1$ or $4 \pmod 5$.*

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{6.1}$$

Theorem 10 (The Second Rogers-Ramanujan Identity) *[5, Chp. 7, p. 109] The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts $\equiv 2$ or $3 \pmod 5$.*

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} \tag{6.2}$$

Example:

There are 2 partitions of 9 such that each part differs by at least 2 and where 1 is not a part: (2, 7), (3, 6). There are also exactly 2 partitions of 9 such that each part is congruent to 2 or 3 modulo 5: (2, 2, 2, 3), (2, 7).

The First Rogers-Ramanujan identity is of Euler's type because a set of partitions with difference between any two parts being at least 2 implies a set of partitions with distinct parts. The requirement that a part be congruent to 1 or 4 mod 5 is analogous to the requirement that a part be odd in Euler's theorem, since 1 and 4 are half of the nonzero residues mod 5. A similar analogy applies to the second Rogers-Ramanujan Identity.

We will construct two proofs of these identities, a combinatorial proof and an analytical proof.

6.1 First Proof of the Rogers-Ramanujan identities

We will present and prove a theorem called Gordon's generalization and show that both of these identities are corollaries of this generalization [5]. The proofs in [5] are briefly presented, here we are going to fully elaborate on them and fill in the gaps.

Gordon's Theorem:

Theorem 11 *[5, Chp. 7, p. 109] Let $L_{k,i}(n)$ denote the number of partitions of n of the form $(l_1, l_2, \cdots, l_r)$, where $(l_j - l_{j-k+1}) \geq 2$ and at most $i-1$ of the l_i equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i(\mod 2k+1)$, then $A_{k,i}(n) = L_{k,i}(n)$ for all n.*

To prove this theorem we need to define certain generating functions and do some analytic work on them.

6.1.1 Analytic Tools:

Here we define analytic tools needed for this section.

Standard abbreviations:

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{6.3}$$

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \tag{6.4}$$

$$(a)_0 = 1. \tag{6.5}$$

Cauchy's Theorem:

Theorem 12 *[5, Chp. 2, p. 17] If $|q| < 1, |t| < 1$, then*

$$1 + \sum_{n=1}^{\infty} \frac{(a)_n t^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{(1-atq^n)}{(1-tq^n)}. \tag{6.6}$$

Euler found two special cases of this theorem. We present them in the following corollary.

Corollary 6.1.1 (Euler) *[5, Chp. 2, p. 19]. For $|q| < 1, |t| < 1$,*

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-tq^n)}. \tag{6.7}$$

$$1 + \sum_{n=1}^{\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = \prod_{n=0}^{\infty} (1+tq^n). \tag{6.8}$$

Proof [5, Chp. 2, p. 19]:

Equation (6.7) is trivial as it is a result of setting $a = 0$ in Theorem 12. For equation (6.8) we replace a by $\frac{a}{b}$ and t by bz in Theorem 12:

$$1 + \sum_{n=1}^{\infty} \frac{(1-\frac{a}{b})(1-\frac{a}{b}q)\cdots(1-\frac{a}{b}q^{n-1})(bz)^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{(1-\frac{a}{b}bzq^n)}{(1-bzq^n)}.$$

$$\Downarrow$$

$$1 + \sum_{n=1}^{\infty} \frac{(b-a)(b-aq)\cdots(b-aq^{n-1})z^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-bzq^n)}. \tag{6.9}$$

Finally we get equation (6.8) when we set $b = 0$ and $a = -1$ in (6.9):

$$1 + \sum_{n=1}^{\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = \prod_{n=0}^{\infty} (1 + tq^n).$$

∎

Theorem 13 (Jacobi triple product identity) *[5, Chp. 2, p. 21] For* $z \neq 0, |q| < 1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} (1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}). \tag{6.10}$$

Proof [5, Chp. 2, p. 21]: For $|z| > |q|, |q| < 1$,

$$\begin{aligned}
\prod_{n=0}^{\infty}(1+zq^{2n+1}) &= \sum_{m=0}^{\infty} \frac{z^m q^{m^2}}{(q^2;q^2)_m} \qquad by \;\; (6.8) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} z^m q^{m^2} (q^{2m+2};q^2)_\infty \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} (q^{2m+2};q^2)_\infty \quad (Since\ (q^{2m+2};q^2)_\infty\ dissapear\ for\ m<0) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r}, \text{ replacing } q \text{ and } t \text{ by } q^2 \text{ and } \frac{t}{q} \text{ in (6.8) resp.} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} z^{m+r} q^r q^{m^2+2mr+r^2}}{(q^2;q^2)_r} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} z^{m+r} q^{(m+r)^2} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-q/z)^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \\
&= \frac{1}{(q^2;q^2)_\infty (-q/z;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2}, \text{ on replacing } q \text{ by } q^2 \text{ and } t \text{ by } \frac{-q}{z} \text{ in (6.7).}
\end{aligned}$$

Therefore:

$$\begin{aligned}\sum_{m=-\infty}^{\infty} z^m q^{m^2} &= (q^2;q^2)_\infty(-q/z;q^2)_\infty \prod_{n=0}^{\infty}(1+zq^{2n+1}) \\ &= (1-q^2)(1-q^4)(1-q^6)\cdots(1+q/z)(1+q^3/z)(1-q^5/z)\cdots\prod_{n=0}^{\infty}(1+zq^{2n+1}) \\ &= \prod_{n=0}^{\infty}(1-q^{2n+2})\prod_{n=0}^{\infty}(1+z^{-1}q^{2n+1})\prod_{n=0}^{\infty}(1+zq^{2n+1}) \\ &= \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1})\end{aligned}$$

∎

We now construct the following corollary of Theorem 13:

Corollary 6.1.2 *[5, Chp. 2, p. 22] For* $|q|<1$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i}) \\ &= \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i}).\end{aligned} \tag{6.11}$$

Proof[5, Chp. 2, p. 22]:

We simply replace q by $q^{k+\frac{1}{2}}$ and then let $z=-q^{k+\frac{1}{2}-i}$ in Theorem 13.

$$\sum_{n=-\infty}^{\infty}(-q^{k+\frac{1}{2}-i})^n(q^{k+\frac{1}{2}})^{n^2} = \prod_{n=0}^{\infty}(1-(q^{k+\frac{1}{2}})^{2n+2})(1+(-q^{k+\frac{1}{2}-i})(q^{k+\frac{1}{2}})^{2n+1})(1+(-q^{k+\frac{1}{2}-i})^{-1}(q^{k+\frac{1}{2}})^{2n+1}).$$

$$\Downarrow$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} = \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i})$$

To complete this proof, we show that equation (6.11) is equal to

$$\sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i}).$$

$\therefore$

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i}) &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=1}^{\infty}(-1)^n q^{(2k+1)n(n-1)/2+in} \\ &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} + \sum_{n=-1}^{-\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in}.\end{aligned}$$

∎

6.1.2 Generating Functions

The following generating functions are crucial in our proof and are found in [5, Chp. 7, p. 106].

For $|x| < |q|^{-1}, |q| < 1$,

$$H_{k,i}(a;x;q) = \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n-in}a^n(1-x^iq^{2ni})(axq^{n+1})_\infty(a^{-1})_n}{(q)_n(xq^n)_\infty}, \tag{6.12}$$

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) - xqaH_{k,i-1}(a;xq;q). \tag{6.13}$$

Lemma 1: [5, Chp. 7, p. 106]

$$H_{k,i}(a;x;q) - H_{k,i-1}(a;x;q) = x^{i-1}J_{k,k-i+1}(a;x;q) \tag{6.14}$$

Proof [5, Chp. 7, p. 106]:

We note that applying a few algebraic manipulations:

$$q^{-in}(1-x^iq^{2ni}) - q^{-(i-1)n}(1-x^{i-1}q^{2n(i-1)}) = q^{-in}(1-q^n) + x^{i-1}q^{n(i-1)}(1-xq^n)$$

Now,
$H_{k,i}(a;x;q) - H_{k,i-1}(a;x;q)$:

$$= \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_nq^{-in}(1-q^n)}{(q)_n(xq^n)_\infty} + \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_nx^{i-1}q^{n(i-1)}(1-xq^n)}{(q)_n(xq^n)_\infty}$$
$$= \sum_{n=1}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_nq^{-in}}{(q)_{n-1}(xq^n)_\infty} + \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_nx^{i-1}q^{n(i-1)}}{(q)_n(xq^{n+1})_\infty}$$
$$= \sum_{n=0}^{\infty} \frac{x^{kn+k}q^{kn^2+n+2kn+k+1}a^{n+1}(axq^{n+2})_\infty(a^{-1})_{n+1}q^{-in-i}}{(q)_n(xq^{n+1})_\infty} + \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n}a^n(axq^{n+1})_\infty(a^{-1})_nx^{i-1}q^{n(i-1)}}{(q)_n(xq^{n+1})_\infty}$$
$$= x^{i-1}\sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+in}a^n(axq^{n+2})_\infty(a^{-1})_n}{(q)_n(xq^{n+1})_\infty}\left\{(1-axq^{n+1}) + ax^{k-i+1}q^{2n(k-i)+k-i+1+n}\left(1-\frac{q^n}{a}\right)\right\}$$
$$= x^{i-1}\sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+in}a^n(axq^{n+2})_\infty(a^{-1})_n}{(q)_n(xq^{n+1})_\infty}[1-(xq)^{k-i+1}q^{n[2(k-i+1)-1]}]-$$
$$x^{i-1}\sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+in}a^n(axq^{n+2})_\infty(a^{-1})_n}{(q)_n(xq^{n+1})_\infty}axq^{n+1}[1-(xq)^{k-i}q^{n[2(k-i)-1]}]$$
$$= x^{i-1}[H_{k,k-i+1}(a;xq;q) - axqH_{k,i-1}(a;xq;q)$$
$$= x^{i-1}J_{k,k-i+1}(a;x;q).$$

∎

Lemma 2:[5, Chp. 7, p. 107]

$$J_{k,i}(a;x;q) - J_{k,i-1}(a;x;q) = (xq)^{i-1}(J_{k,k-i+1}(a;xq;q) - aJ_{k,k-i+2}(a;xq;q)). \tag{6.15}$$

Proof [5, Chp. 7, p. 107]:

$$\begin{aligned} J_{k,i}(a;x;q) - J_{k,i-1}(a;x;q) &= (H_{k,i}(a;xq;q) - H_{k,i-1}(a;xq;q)) - axq(H_{k,i-1}(a;xq;q) - H_{k,i-2}(a;xq;q)) \\ &= (xq)^{i-1} J_{k,k-i+1}(a;xq;q) - a(xq)^{i-1} J_{k,k-i+2}(a;xq;q) \\ &= (xq)^{i-1}(J_{k,k-i+1}(a;xq;q) - aJ_{k,k-i+2}(a;xq;q)) \end{aligned}$$

∎

Lemma 3:[5, Chp. 7, p. 108]

For $1 \leq i \leq k, |q| < 1$.

$$J_{k,i}(0;1;q) = \prod_{n=1,\ n\not\equiv 0\ or\ \pm i (mod\ 2k+1)}^{\infty} (1-q^n)^{-1}. \tag{6.16}$$

Proof[5, Chp. 7, p. 108]:

$$\begin{aligned} J_{k,i}(0;1;q) &= H_{k,i}(0;q;q) \quad \textit{by substituting } a = 0 \textit{ and } x = 1 \textit{ into } (6.13) \\ &= \sum_{n=0}^{\infty} \frac{q^{kn^2+n-in+kn} a^n (1-q^{2ni+i})(aq^{n+2})_\infty (a^{-1})_n}{(q)_n (q^{n+1})_\infty} \,|_{a=0} \\ &= \sum_{n=0}^{\infty} \frac{q^{kn^2+n-in+kn}(1-q^{2ni+i})(aq^{n+2})_\infty (a-1)(a-q)(a-q^2)\cdots(a-q^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)(1-q^{n+1})(1-q^{n+2})(1-q^{n+3})\cdots} \,|_{a=0} \\ &= \sum_{n=0}^{\infty} \frac{q^{kn^2+(k-i+1)n}(1-q^{2ni+i})(aq^{n+2})_\infty (-1)^n q q^2 q^3 q^4 \cdots q^{n-1}}{(q)_\infty} \\ &= (q)_\infty^{-1} \sum_{n=0}^{\infty} q^{kn^2+(k-i+1)n}(-1)^n q^{n(n-1)/2}(1-q^{(2n+1)i}) \\ &= (q)_\infty^{-1} \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n(n+1)/2-in}(1-q^{(2n+1)i}) \\ &= \prod_{n=0}^{\infty} \frac{(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i})}{(q)_\infty} \quad \textit{by corollary } 6.1.2 \\ &= \prod_{n=1,\ n\not\equiv 0\ or\ \pm i(mod\ 2k+1)}^{\infty} \frac{1}{(1-q^n)}. \end{aligned}$$

∎

6.1.3 Proving Gordon's Theorem

Now we use the tools we proved above to clarify the proof from [5, Chp. 7, p. 109—111]. Recall this part from Gordon's theorem "Let $l_{k,i}(m,n)$ denote the number of partitions, $(l_1, l_2, \cdots, l_m)$, of n with exactly m parts such that no more than $i-1$ of the l_i equal 1 and where $l_j \geq l_{j+1}$ and $(l_j - l_{j+k-1}) \geq 2$."

Since for $1 \le i \le k$, we know that the only partition that is either of a nonpositive number or has a nonpositive number of parts is the empty partition 0, then;

$$l_{k,i}(m,n) = \begin{cases} 1, & \text{if } n = m = 0 \\ 0, & \text{if } n \le 0 \text{ or } m \le 0 \text{ but } (m,n) \ne (0,0) \end{cases} \tag{6.17}$$

and

$$l_{k,0}(m,n) = 0 \tag{6.18}$$

We also have the following property for $l_{k,i}(m,n)$:

$$l_{k,i}(m,n) - l_{k,i-1}(m,n) = l_{k,k-i+1}(m-i+1, n-m). \tag{6.19}$$

It is straight forward to see that this property holds:

$l_{k,i}(m,n) - l_{k,i-1}(m,n)$ gives those number of partitions from $l_{k,i}(m,n)$ with exactly $i-1$ appearances of 1. This is because subtracting $l_{k,i-1}(m,n)$ from $l_{k,i}(m,n)$ removes all those partitions with at most $i-2$ appearances of 1. Altering $l_{k,i}(m,n) - l_{k,i-1}(m,n)$ by removing the $i-1$ ones and removing one from each of the parts that are remaining, results in partitions of $n-m$ with $m-(i-1)$ parts. The resulting partition is $(l_1', l_2', \cdots, l_{m-i+1}')$, with the property that $l_j{}' \ge l_{j+1}'$ and $(l_j' - l_{j+k-1}') \ge 2$. Due to the difference property of these parts, the frequency of appearances of 1's and 2's in total is at most $k-1$. Since 1 appears at most $i-1$ times, then 2 appears at most $k-1-(i-1)$ times. Therefore in the partition $(l_1', l_2', \cdots, l_{m-i+1}')$, one appears at most $k-i$ times. This establishes a one-to-one correspondence between $l_{k,i}(m,n) - l_{k,i-1}(m,n)$ and $l_{k,k-i+1}(m-i+1, n-m)$.

Now we use the Generating function (6.13) to construct a function $c_{k,i}(m,n)$ and show that it is analogous to $l_{k,i}(m,n)$. We do this in a clever way that will allow us to complete our proof.

Let's define the following generating function:

$$J_{k,i}(0;x;q) = c_{k,i}(0,0) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{k,i}(m,n)x^m q^n \mid_{(m,n)\ne(0,0)} . \tag{6.20}$$

We rewrite the generating function (6.12):

$$H_{k,i}(a;x;q) = \frac{(1-x^i)(axq)_\infty}{(x)_\infty} + \sum_{n=1}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1-x^i q^{2ni})(axq^{n+1})_\infty (a^{-1})_n}{(q)_n (xq^n)_\infty}$$

where,

$$\begin{aligned} J_{k,i}(0;x;q) &= H_{k,i}(a;xq;q) \mid_{a=0} \\ &= \frac{(1-x^i q^i)(axq^2)_\infty}{(xq)_\infty} + \sum_{n=1}^{\infty} \frac{x^{kn} q^{kn^2+n-in+kn} a^n (1-x^i q^{2ni+i})(axq^{n+2})_\infty (a^{-1})_n}{(q)_n (xq^{n+1})_\infty} \mid_{a=0} \\ &= \frac{(1-x^i q^i)}{(xq)_\infty} + \sum_{n=1}^{\infty} \frac{x^{kn} q^{kn^2-in+kn+\frac{n(n+1)}{2}} (1-x^i q^{2ni+i})}{(q)_n (xq^{n+1})_\infty} \end{aligned} \tag{6.21}$$

Then:

$$J_{k,i}(0;x;q) = c_{k,i}(0,0) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{k,i}(m,n)x^m q^n \mid_{(m,n)\neq(0,0)}$$
$$= \frac{(1-x^i q^i)}{(xq)_\infty} + \sum_{n=1}^{\infty} \frac{x^{kn} q^{kn^2-in+kn+\frac{n(n+1)}{2}}(1-x^i q^{2ni+i})}{(q)_n (xq^{n+1})_\infty}$$

For $1 \leq i \leq k$:

$$J_{k,i}(0;0;q) = \frac{(1-0^i q^i)}{(0q)_\infty} = \frac{1}{1} = 1 = J_{k,i}(0;x;0).$$

This implies that:

$$J_{k,i}(0;0;q) = J_{k,i}(0;x;0) = c_{k,i}(0,0) + 0 = 1,$$

and we conclude that:

$$c_{k,i}(m,n) = \begin{cases} 1, & \text{if } n = m = 0 \\ 0, & \text{if } n \leq 0 \textit{ or } m \leq 0 \textit{ but } (m,n) \neq (0,0). \end{cases} \tag{6.22}$$

From (6.21) we see that

$$c_{k,0}(m,n) = \frac{(1-x^0 q^0)}{(xq)_\infty} = 0. \tag{6.23}$$

Substituting $a = 0$ into Lemma 2, yields:

$$J_{k,i}(0;x;q) - J_{k,i-1}(0;x;q) = (xq)^{i-1} J_{k,k-i+1}(0;xq;q),$$

we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{k,i}(m,n)x^m q^n - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{k,i-1}(m,n)x^m q^n = (xq)^{i-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{k,k-i+1}(m,n)(xq)^m q^n,$$
$$\Downarrow$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(c_{k,i}(m,n) - c_{k,i-1}(m,n))x^m q^n = \sum_{m=i-1}^{\infty}\sum_{n=m}^{\infty} c_{k,k-i+1}(m-i+1, n-m)x^m q^n.$$

Comparing coefficients of $x^m q^n$ yields:

$$c_{k,i}(m,n) - c_{k,i-1}(m,n) = c_{k,k-i+1}(m-i+1, n-m). \tag{6.24}$$

$\therefore$ We conclude that $c_{k,i}(m,n) = l_{k,i}(m,n)$, $\forall m,n$ with $0 \leq i \leq k$.

Since $\sum_{m\geq 0} l_{k,i}(m,n) = L_{k,i}(m,n)$, we have that:

$$
\begin{aligned}
\sum_{n\geq 0} L_{k,i}(m,n)q^n &= \sum_{n\geq 0}\sum_{m\geq 0} l_{k,i}(m,n)q^n \\
&= J_{k,i}(0;1;q) \\
&= \prod_{n=1,\ n\not\equiv 0\ or\ \pm i (mod\ 2k+1)}^{\infty} (1-q^n)^{-1} \quad \textit{by Lemma } 3 \\
&= \sum_{n\geq 0} A_{k,i}(n)q^n.
\end{aligned}
$$

∎

Corollary 6.1.3 (First Rogers-Ramanujan identity) *[5, Chp. 7, p. 109] The partitions of an integer n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into part ≡ 1 or 4* $\pmod 5$.

Proof: Substituting $i = 2$ and $k = 2$ into Gordon's Theorem we obtain:

$L_{2,2}(n)$ gives the number of partitions, $(l_1, l_2, \cdots, l_r)$, of n such that no more than 1 of the parts are 1 and where $(l_j - l_{j-1}) \geq 2$. $A_{2,2}(n)$ gives the number of partitions of n into parts not congruent to 0 or $\pm 2(mod\ 5)$, thus parts that are congruent to 1 or 4 modulo 5.

$\therefore A_{2,2}(n) = L_{2,2}(n)$ for all n and this is the first Rogers-Ramanujan identity. ∎

Corollary 6.1.4 (Second Rogers-Ramanujan identity) *[5, Chp. 7, p. 109] The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts ≡ 2 or 3* $\pmod 5$.

Proof: Substituting $i = 1$ and $k = 2$ into Gordon's Theorem we obtain:

$L_{2,1}(n)$ which is the number of partitions, $(l_1, l_2, \cdots, l_r)$, of n with no more than 0 of the parts equal 1 and where $(l_j - l_{j-1}) \geq 2$. $A_{2,1}(n)$ gives the number of partitions of n into parts $\equiv 2$ or $2 \pmod 5$.

$\therefore A_{2,1}(n) = L_{2,1}(n)$ for all n and this is the second Rogers-Ramanujan identity. ∎

6.2 Second Proof of the Rogers-Ramanujan identities

We will split the proof into two independent parts, combinatorial part and algebraic part.

What is important to this proof is our definition of the first Rogers-Ramanujan identity as in equation (6.1):

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

It is very important to prove that this identity in generating function form does indeed hold. To prove this we give and prove a theorem and then show that the identity is just a corollary of this new theorem.

Theorem 14 *(See [5, Chp. 7, p. 111]) For* $1 \leq i \leq k, k \geq 2, |q| < 1$, *then*

$$\sum_{n_1,n_2,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{n=1,\ n\not\equiv 0\ or\ \pm i(mod\ 2k+1)}^{\infty} (1-q^n)^{-1} \quad (6.25)$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$

Proof [5, Chp. 7, p. 109]:

We prove that

$$J_{k,i}(0;x;q) = \sum_{n_1,n_2,\cdots,n_{k-1}\geq 0} \frac{x^{N_1+N_2+\cdots+N_{k-1}} q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}. \quad (6.26)$$

Substituting $x = 1$ into (6.26) and using Lemma (3) clearly proves Theorem 14. Equation (6.26) follows from:

$$J_{k,i}(0;x;q) = \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q)_n} J_{k-1,i}(0;xq^{2n};q), \quad (6.27)$$

which may be seen immediately by induction on k since, setting $i = k+1$ into Lemma (2) we have that:

$$\begin{aligned} J_{k,k+1}(0;x;q) - J_{k,k}(0;x;q) &= (xq)^k (J_{k,0}(0;xq;q) - 0J_{k,1}(0;xq;q)) \\ &= (xq)^k (H_{k,0}(0;xq^2;q)) && from\ (6.13) \\ &= (xq)^k (0) && from\ (6.12) \\ &= 0, \end{aligned}$$

and that

$$J_{1,1}(0;x;q) = 1,$$

since by Lemma (2)

$$J_{1,1}(0;x;q) = J_{1,1}(0;xq;q) = J_{1,1}(0;xq^2;q) = \cdots = J_{1,1}(0;xq^n;q) \to J_{1,1}(0;0;q) = 1$$

Now we only need to prove (6.27).

Let us define

$$R_{k,i}(x;q) = \sum_{n\geq 0} \frac{x^{(k-1)n} q^{(k-1)n^2+(k-i)n}}{(q)_n} J_{k-1,i}(0;xq^{2n};q). \quad (6.28)$$

Setting $1 \leq i \leq k$, then we have:

$$R_{k,i}(0;q) = R_{k,i}(x;0) = 1 \quad (6.29)$$

and

$$R_{k,0}(x;q) = 0 \quad (6.30)$$

We also have that:

$$
\begin{aligned}
R_{k,i}(x;q) - R_{k,i-1}(x;q) &= \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(J_{k-1,i}(0;xq^{2n};q) - q^n J_{k-1,i-1}(0;xq^{2n};q)) \\
&= \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(J_{k-1,i-1}(0;xq^{2n};q)) \\
&+ (xq^{2n+1})^{i-1}J_{k-1,i-1}(0;xq^{2n+1};q) - q^n J_{k-1,i-1}(0;xq^{2n};q) \\
&= \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(1-q^n)J_{k-1,i-1}(0;xq^{2n};q) \\
&+ (xq)^{i-1}\sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k+i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+1};q) \\
&= x^{k-1}q^{2k-i-1}\sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+2};q) \\
&+ (xq)^{i-1}\sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k+i-2)n}}{(q)_n}J_{k-1,k-i+1}(0;xq^{2n+1};q) \\
&- (xq^{2n+2})^{k-i}J_{k-1,i-1}(0;xq^{2n+2};q) \\
&= x^{k-1}q^{2k-i-1}\sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+2};q) \\
&+ (xq)^{i-1}\sum_{n\geq 0} \frac{(xq)^{(k-1)n}q^{(k-1)n^2+(k-(k-i+1)n}}{(q)_n}J_{k-1,k-i+1}(0;xq^{2n+1};q) \\
&- x^{k-1}q^{2k-i-1}\sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(3k-i-2)n}}{(q)_n}J_{k-1,i-1}(0;xq^{2n+2};q) \\
&= (xq)^{i-1}R_{k,k-i+1}(xq;q) \qquad (6.31)
\end{aligned}
$$

Recalling that the coefficients in the expansion of $J_{k,i}(0;x;q)$ were uniquely determined by (6.22), (6.23) and (6.24), we conclude that since $R_{k,i}(x;q)$ satisfies (6.29), (6.30) and (6.31), and thus its coefficients must satisfy (6.17), (6.18) and (6.19). Therefore $R_{k,i}(x;q) = J_{k,i}(0;x;q)$ for $0 \leq i \leq k$. Thus we obtain (6.27) and with it Theorem 14. ∎

Corollary 6.2.1 (First Rogers-Ramanujan Identity) *(See [6, Chp. 5, p. 52—53])*

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \qquad (6.32)$$

Proof:

Substitute $k = i = 2$ in Theorem 14, yields:

$$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$\Downarrow$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

∎

Corollary 6.2.2 (Second Rogers-Ramanujan Identity) *(See [6, Chp. 5, p. 52—53])*

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})} \tag{6.33}$$

Proof:

Substitute $k=2, i=1$ in Theorem 13.

$$1+\frac{q}{1-q^2}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\cdots=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

$$\downarrow$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

∎

We can now use Schur's identity to show its equivalence to the first Rogers-Ramanujan identities. We now proceed to the combinatorial part of the proof.

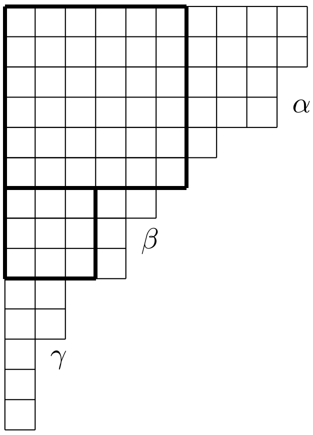
6.2.1 First Part of the Proof [Combinatorial Part]

This section fill in the gaps of the work done in [8].

We shall use the conversion $P=\cup_n P_n$, $p(n)=|P_n|$ where P_n is the set of all partitions λ of n. We have a Rogers-Ramanujan partition if the condition $e(\lambda)\geq l(\lambda)$ holds. A set of Rogers-Ramanujan partitions shall be denoted by $\mathcal{Q}_n$, where $\mathcal{Q}=\cup_n\mathcal{Q}_n$, $q(n)=|\mathcal{Q}_n|$.

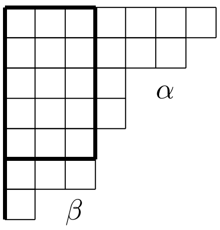
For $m\geq 0$, we shall refer to an m-rectangle to be a rectangle whose height minus its width is m. The largest m-rectangle that fits in the diagram $[\lambda]$ is called the first m-Durfee and the second m-Durfee rectangle is the largest m-rectangle that fits in the diagram $[\lambda]$ just under the first m-Durfee rectangle. Let $s_m(\lambda)$ and $t_m(\lambda)$ represent the height of the first and the second m-Durfee rectangle respectively. Let m-Durfee rectangle's width be ≥ 0 and its height be ≥ 1. The partitions to the right of the m-Durfee rectangles, in the middle of the m-Durfee rectangles and below the m-Durfee rectangles are α, β and γ respectively.

Example(See [8]): Here $m=0$



$$\lambda = (10, 10, 9, 9, 7, 6, 5, 4, 4, 2, 2, 1, 1, 1), m = 0$$
$$\alpha = (4, 4, 3, 3, 1), \beta = (2, 1, 1), \gamma = (2, 2, 1, 1, 1)$$

A case where an m-Durfee rectangle has a width of 0 is obtained when $m > 0$ and we have that $\gamma = (0)$ (see image below)



$$\lambda = (7, 6, 4, 4, 3, 3, 1), m = 2$$
$$\alpha = (4, 3, 1, 1),\ \beta = (3, 1),\ \gamma = (0)$$

Our definition of rank is refined in this chapter to be:

$r_{2,m}(\lambda) = \beta_1 + \alpha_{s_m(\lambda)-t_m(\lambda)-\beta_1+1} - \gamma_1^{'}$, where $\alpha_k = 0$ if $l(\alpha) < k$.

$$r_{2,0}(10, 10, 9, 9, 7, 6, 5, 4, 4, 2, 2, 1, 1, 1) = \beta_1 + \alpha_{6-3-2+1} - \gamma_1^{'} = 2 + 4 - 5 = 1$$
$$r_{2,2}(7, 6, 4, 4, 3, 3, 1) = \beta_1 + \alpha_{5-2-3+1} - \gamma_1^{'} = 3 + 4 - 0 = 7.$$

We note that $r_{2,0}(\lambda)$ is only defined for $\mathcal{P} \setminus \mathcal{Q}$, because otherwise the second m-Durfee rectangle does not exist (and hence β_1) while $r_{2,m}(\lambda)$ exist for all $m > 0$.

The set $\mathcal{H}_{n,m,r}, \mathcal{H}_{n,m,\leq r}$ and $\mathcal{H}_{n,m,\geq r}$ shall represent the set of partitions of n with $r_{2,m}(\lambda) = r,\ r_{2,m}(\lambda) \leq r$ and $r_{2,m}(\lambda) \geq r$ respectively. Equating the values we have:

$$h(n, m, r) = |\mathcal{H}_{n,m,r}|\,,\ h(n, m, \leq r) = |\mathcal{H}_{n,m,\leq r}|\ \textit{and}\ h(n, m, \geq r) = |\mathcal{H}_{n,m,\geq r}|$$

From these definitions we clearly have:

$$h(n,m,\leq r)+h(n,m,\geq r+1)=p(n) \tag{6.34}$$

and

$$h(n,0,\leq r)+h(n,0,\geq r+1)=p(n)-q(n). \tag{6.35}$$

For all $r\in\mathbb{Z}$ and $n\geq 1$, we have:

(First Symmetry)

$$h(n,0,r)=h(n,0,-r).$$

(Second Symmetry)

$$h(n,m,\leq -r)=h(n-r-2m-2,m+2,\geq -r).$$

These equations are crucial for our proof and their proofs can be found in [8].

6.2.2 Second Part of the Proof [Algebraic Part]

From Corollary 6.2.1 we have that:

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

We rewrite the right hand side of this corollary by substituting $q=t^5$ and $z=-t^{-2}$ into the Jacobi triple product identity [1, 5].

Jacobi triple product identity

(See [1, 5, 6])

$$\sum_{-\infty}^{\infty}z^kq^{\frac{k(k+1)}{2}}=\prod_{i=1}^{\infty}(1+zq^i)\prod_{j=0}^{\infty}(1+z^{-1}q^j)\prod_{i=1}^{\infty}(1-q^i) \tag{6.36}$$

Substituting $q=t^5$ and $z=-t^{-2}$ into the left side of the above identity of Jacobi:

$$\begin{aligned}\sum_{k-\infty}^{\infty}(-t^{-2})^k(t^5)^{\frac{k(k+1)}{2}} &= \sum_{-\infty}^{\infty}(-1)^kt^{-2k}t^{\frac{5k(k+1)}{2}}\\ &= \sum_{-\infty}^{\infty}(-1)^kt^{\frac{k(5k+1)}{2}} \end{aligned}\tag{6.37}$$

Substituting $q=t^5$ and $z=-t^{-2}$ into the right hand side of above identity of Jacobi:

$$\prod_{i=1}^{\infty}(1+zq^i)\prod_{j=0}^{\infty}(1+z^{-1}q^j)\prod_{i=1}^{\infty}(1-q^i)=\prod_{i=1}^{\infty}(1+(-t^{-2})(q^5)^i)\prod_{j=0}^{\infty}(1+(-t^{-2})^{-1}(q^5)^j)\prod_{i=1}^{\infty}(1-(q^5)^i)$$
$$=\prod_{i=1}^{\infty}(1-t^{5i-2})\prod_{j=0}^{\infty}(1-t^{5i+2})\prod_{i=1}^{\infty}(1-t^{5i}). \tag{6.38}$$

Equating the above equations we get:

$$\sum_{k=-\infty}^{\infty}(-1)^k t^{\frac{k(5k+1)}{2}}=\prod_{i=1}^{\infty}(1-t^{5i-2})\prod_{j=0}^{\infty}(1-t^{5i+2})\prod_{i=1}^{\infty}(1-t^{5i}),$$

and substituting $k=-m$ and then dividing both sides by $\prod_{i=1}^{\infty}\frac{1}{(1-t^i)}$, we get:

$$\sum_{m=-\infty}^{\infty}(-1)^m t^{\frac{m(5m-1)}{2}}\prod_{i=1}^{\infty}\frac{1}{(1-t^i)}=\frac{\prod_{i=1}^{\infty}(1-t^{5i-2})\prod_{j=0}^{\infty}(1-t^{5i+2})\prod_{i=1}^{\infty}(1-t^{5i})}{\prod_{i=1}^{\infty}(1-t^i)}$$
$$=\prod_{r=0}^{\infty}\frac{1}{(1-t^{5r+1})(1-t^{5r+4})}. \tag{6.39}$$

Schur's identity

[8]

$$1+\sum_{k=1}^{\infty}\frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}=\sum_{m=-\infty}^{\infty}(-1)^m t^{\frac{m(5m-1)}{2}}\prod_{i=1}^{\infty}\frac{1}{(1-t^i)} \tag{6.40}$$

Now it is clear that if Schur's identity is correct, then corollary 6.2.1 is correct as well. We now prove Schur's identity to complete our proof of corollary 6.2.1.

Proof of corollary 6.2.1 [8]

For $j\geq 0$, let

$$a_j=h\left(n-jr-2jm-\frac{j(5j-1)}{2},m+2j,\leq -r-j\right),$$

and

$$b_j=h\left(n-jr-2jm-\frac{j(5j-1)}{2},m+2j,\geq -r-j+1\right).$$

From equations (6.34) and (6.35)we get $a_j+b_j=p\left(n-jr-2jm-\frac{j(5j-1)}{2}\right)$, $\forall j,r>0$. Then:

$$a_j=h\left(n-jr-2jm-\frac{j(5j-1)}{2},m+2j,\leq -r-j\right)=b_{j+1}$$

Thus:

$$\begin{aligned}
h(n, m \leq -r) &= a_0 = b_1 \\
&= b_1 + (a_1 - b_2) - (a_2 - b_3) + (a_3 - b_4) - \cdots \\
&= (b_1 + a_1) - (b_2 + a_2) + (b_3 + a_3) - (b_4 + a_4) + \cdots \\
&= p(n - r - 2m - 2) - p(n - 2r - 4m - 9) + p(n - 3r - 6m - 21) - \cdots \\
&= \sum_{j=1}^{\infty} (-1)^{j-1} p\left(n - jr - 2jm - \frac{j(5j-1)}{2}\right). \qquad (6.41)
\end{aligned}$$

We compute the generating function of the above as:

$$H_{m,\leq -r}(t) := \sum_{n=1}^{\infty} h(n, m, \leq -r) t^n,$$

and for the conditions

- $m, r > 0$
- $m = 0$
- $r \geq 0$,

we have:

$$\begin{aligned}
H_{m,\leq -r}(t) &= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} p\left(n - jr - 2jm - \frac{j(5j-1)}{2}\right) t^n \\
&= \sum_{n=-jr-2jm-\frac{j(5j-1)}{2}}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} p(n) t^{n+jr+2jm+\frac{j(5j-1)}{2}} \\
&= \sum_{n=-jr-2jm-\frac{j(5j-1)}{2}}^{\infty} p(n) t^n \sum_{j=1}^{\infty} (-1)^{j-1} t^{jr+2jm+\frac{j(5j-1)}{2}} \\
&= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1} t^{jr+2jm+\frac{j(5j-1)}{2}} \qquad (6.42)
\end{aligned}$$

In particular, we have:

$$H_{0,\leq 0}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1} t^{\frac{j(5j-1)}{2}}$$

$$H_{0,\leq -1}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{\frac{j(5j+1)}{2}}$$

From equations (6.34) and (6.35) it follows that:

$$H_{0,\leq 0}(t) + H_{0,\leq -1}(t) = H_{0,\leq 0}(t) + H_{0,\geq 1}(t) = P(t) - Q(t),$$

where

$$P(n) = 1 + \sum_{n=1}^{\infty} p(n)t^n = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)},$$

and

$$Q(n) = 1 + \sum_{n=1}^{\infty} q(n)t^n = 1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}$$

$\therefore$ we conclude:

$$\prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \left(\sum_{j=1}^{\infty} (-1)^{j-1} t^{\frac{j(5j-1)}{2}} + \sum_{j=1}^{\infty} (-1)^{\frac{j(5j+1)}{2}} \right) = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)} - \left(1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} \right)$$

Rearranging this equality we have:

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} &= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \left(1 - \sum_{j=1}^{\infty} (-1)^{j-1} t^{\frac{j(5j-1)}{2}} - \sum_{j=1}^{\infty} (-1)^{j-1} \frac{j(5j+1)}{2} \right) \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \left(1 + \sum_{j=1}^{\infty} (-1)^{j} t^{\frac{j(5j-1)}{2}} + \sum_{j=-\infty}^{1} (-1)^{j} \frac{j(5j-1)}{2} \right) \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=-\infty}^{\infty} (-1)^{j} \frac{j(5j-1)}{2} \qquad (6.43) \end{aligned}$$

$\therefore$ this proves Corollary 6.2.1 ∎

Chapter 7

Conclusions and Further work

From our work done in the previous Chapters, we conclude that Euler's Partition Theorem is amongst the beautiful results that are celebrated in partitions theory. We have also seen the power of generating functions in treating partitions. Several generalizations and refinements of Euler's Partition Theorem abound, but the number of different combinatorial proofs remains rather small.

Further work in this project would be to study those generalizations and refinement of Euler's Partition Theorem that require advanced techniques such as abacus diagrams and application of the methods of sampling and simulation to estimate the number of partitions of an integer.

Some of my further work of interest according to the theme of this project are listed below. In each case we indicate the connection with Euler's partition theorem. However, time and scope constraints forbid us from delving into them in detail.

7.1 Lebesgue identity

(See [22, 7])

$$\sum_{r=1}^{\infty} t^{\binom{r+1}{2}} \frac{(1+zt)(1+zt^2)\cdots(1+zt^r)}{(1-t)(1-t^2)\cdots(1-t^r)} = \prod_{i=1}^{\infty}(1+zt^{2i})(1+t^i), \tag{7.1}$$

where it implies Euler's identity:

$$\prod_{i=1}^{\infty}(1+st^i) = 1 + \sum_{r=1}^{\infty} \frac{s^r t^{\frac{r(r+1)}{2}}}{(1-t)(1-t^2)\cdots(1-t^r)},$$

when $z = 0$ *and* $s = 1$ [22].

7.2 The l-Euler theorem

See [23]

For an integer $l \geq 2$, we define the sequence $\{a_n^{(l)}\}_{n\geq 0}$ by

$$a_n^{(l)} = la_{n-1}^{(l)} - a_{n-2}^{(l)},$$

with initial conditions $a_0^{(l)} = 0$ *and* $a_1^{(l)} = 1$. Let c_l be the largest root of the characteristic equation

$$x^2 - lx + 1 = 0.$$

Then the number of partitions of an integer N into parts from the set:

$$\{a_0^{(l)} + a_1^{(l)}, a_1^{(l)} + a_2^{(l)}, a_2^{(l)} + a_3^{(l)}, \cdots\}$$

is the same as the number of partitions of N in which the ratio of consecutive (positive) parts is greater than c_l. This implies Euler's partition theorem when $l = 2$.

7.3 Lecture Hall Partition Theorem

(See [6])

Bousquet-Melou and Eriksson in [11] first presented the theory of Lecture Hall Partitions by looking at the quotient of successive parts.

Theorem 15 (Lecture hall partition theorem) *(See [6, 9] For a fixed length N of the lecture hall, the number of lecture hall partitions equals the number of partitions into odd parts smaller than $2N$. In other words,*

$$p(n|\ lecture\ hall\ of\ length\ N) = p(n|\ odd\ parts\ <\ 2N).$$

Now, in what sense is this result a refinement of Euler's identity? This theorem states that the set of partitions into odd parts less than $2n$ is equinumerous with the set of partitions having at most n distinct parts and satisfying the additional condition of lecture hall-ness" [6]. Euler's theorem states that the number of partitions of N into distinct parts equals the number of partitions of N into odd parts. Then Euler's identity is the limiting case of the lecture hall partition theorem as n tends to infinity: For a fixed N, if we choose n large enough, then the partitions into odd parts less than $2n$ will in fact be all possible partitions into odd parts. On the other hand, the lecture hall partition of N for large n must satisfy conditions of the type

$$\frac{\lambda_{n-k}}{n-k} \leq \frac{\lambda_{n-k+1}}{n-k+1} \tag{7.2}$$

and the numerators are much smaller than the denominators for non-zero parts, so that it is sufficient that $\lambda_{n-k} < \lambda_{n-k+1}$ for the inequality to hold. In other words, the lecture hall partitions of N of length n for large n are all partitions of N into distinct parts. Hence, Euler's identity follows from the lecture hall partition theorem when n tends to infinity.

7.4 The l-lecture hall theorem

See [23]

These lecture halls are the a-Lecture halls but now the a-sequence becomes the l-sequence[23]. For $l \geq 2$ and $n \geq 0$,

$$\sum_{\lambda} q^{\lambda_1+\lambda_2+\cdots+\lambda_n} = \frac{1}{(1-q^{a_0^{(l)}+a_1^{(l)}})(1-q^{a_1^{(l)}+a_2^{(l)}})(1-q^{a_2^{(l)}+a_3^{(l)}})\cdots(1-q^{a_{n-1}^{(l)}+a_n^{(l)}})}, \tag{7.3}$$

where the sum is over all sequences $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ satisfying:

$$\frac{\lambda_1}{a_n^{(l)}} \geq \frac{\lambda_2}{a_{n-1}^{(l)}} \geq \frac{\lambda_3}{a_{n-2}^{(l)}} \geq \cdots \frac{\lambda_{n-1}}{a_2^{(l)}} \geq \frac{\lambda_n}{a_1^{(l)}} \geq 0.$$

We call these λ l-lecture hall partitions.

For more on l-sequence combinatorics see [23].

7.5 Formulas for partition functions

See [6]

This area is concerned with finding the enumerating functions for partitions. I'll demonstrate this by a computational example. Let $p(n,m)$ be the number of partitions of n with parts equal and less than m. Then

$$p(n,m) = p(n|\ parts\ in\ \{1,2,\cdots,m\}). \tag{7.4}$$

Obtaining $p(n,1) = 1$ and $p(n,2) = \lfloor \frac{n}{2} \rfloor + 1$, is straight forward. The problem comes when we have $m \geq 3$.
For $m \geq 3$ the following technique is used:

(1) Write down the generating function of $p(n,m)$, that is;

$$\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{7.5}$$

(2) Alter the right hand side algebraically so that you can obtain tractable power series expansion. You can do this by computing q-Partial Fraction decomposition. See [19] for computation of q-partial fractions.
(3) Extract the coefficients of q^n using known results about series manipulations like **Binomial theorem and series expansion**.

Example for m=3:

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,3)q^n &= \frac{1}{(1-q)(1-q^2)\cdots(1-q^3)} \\ &= \frac{1/6}{(1-q)^3} + \frac{1/4}{(1-q)^2} + \frac{1/4}{1-q^2} + \frac{1/3}{1-q^3} \\ &= \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+2}{2}q^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)q^n + \frac{1}{4}\sum_{n=0}^{\infty}q^{2n} + \frac{1}{3}\sum_{n=3}^{\infty}q^{3n} \\ &= \sum_{n=0}^{\infty}\left(\frac{(n+3)^2}{12} + \epsilon(n)\right)q^n, \qquad (7.6)\end{aligned}$$

where $\epsilon(n)$ *takes only the values* $-1/3, -1/12, 0, 1/4$.

Then when we extract the coefficient of q^n from (7.10) we get that

$$p(n,m) = \frac{(n+3)^2}{12} + \epsilon(n).$$

Therefore

$$p(n,3) = \left\{\frac{(n+3)^2}{12}\right\} \qquad (7.7)$$

since $\epsilon < \frac{1}{2}$.

Bibliography

[1] G. E. Andrews, A simple proof of Jacobi's triple product identity, *Proc. Amer. Math. Soc.* **16** (1965), 333-334.

[2] G.E. Andrews, "J.J. Sylvester, Johns Hopkins and Partitions", *A Century of Mathematics in America, Part I*, Amer. Math. Soc., Providence (1988), 21-40.

[3] G. E. Andrews, On a partition theorem of N. J. Fine, *J. Nat. Acad. Math. India* **1** (1983),105-107.

[4] G. E. Andrews, q-Series, C.B.M.S. *Regional Conference Lecture Series*, Amer. Math. Soc. **66** , Providence, 1986.

[5] G. E. Andrews, The Theory of Partitions, *Encyclopedia Math. Appl.* Addison-Wesley, 1976.

[6] George E. Andrews and Kimmo Eriksson, *Integer Partition*, Cambridge University Press, United Kingdom, 2004.

[7] Christine Bessenrodt, A bijection for Lebesgues partition identity in the spirit of Sylvester, *Discrete Math.* **132** (1994), 1-10.

[8] Cilanne Boulet and Igor Pak, A combinatorial proof of the Rogers-Ramanujan and Schur identities, *J. Comb. Theory Ser. A* 113 (2006), 1019-1030.

[9] Mireille Bousquet-Mélou and Kimmo Eriksson, A refinement of the lecture hall theorem, *J. Comb. Theory Ser. A* **86** 1999), 63-84.

[10] Mireille Bousquet-Mélou and Kimmo Eriksson, Lecture hall partitions, *Ramanujan J.* **1** (1997),101-111.

[11] M. Bousquet-Mélou and K. Eriksson, Lecture hall partitions II, *Ramanujan J.* **1** (1997), 165-185.

[12] D. M. Bressoud, An easy proof of the Rogers-Ramanujan identities, *J. Number Theory* **16** (1983), 235-241.

[13] F. J. Dyson, A new symmetry of partitions, *J. Combin. Theory* **7** (1969), 56-61.

[14] Freeman J. Dyson, Some Guesses in The Theory of Partitions, *Eureka(Cambridge)* **8** (1944), 10-15.

[15] G. H. Hardy, Ramanujan, *Twelve lectures on subjects suggested by his life and work*, Cambridge University Press, England, 1940.

[16] Glaisher J. W. L.,"A theorem in partitions", *Messenger of Math.* **12**, 158-170.

[17] Dongsu Kim and AE JA Yee, A note on partitions into distinct parts and odd parts, *Ramanujan J.* **3** (1999), 227-231.

[18] Donald E. Knuth, *The Art of Computer Programming*, Prefascicle 3B, A draft of sections 7.2.1.4-5: Generating all partitions, Addison-Wesley, 2004.

[19] A.O. Munagi, Computation of q-partial fractions, *Integers* **7** (2007).

[20] Edwin O'Shea, M-partitions: optimal partitions of weight for one scale pan. *Discrete Math.* **289** (2004),81-93.

[21] Igor Pak, On Fine's partition theorems, Dyson, Andrews, and missed opportunities, *Math. Intelligencer* **25** (2003),10-16.

[22] Igor Pak, Partition bijections, a survey, *Ramanujan J.* **12** (2006).

[23] Carla D. Savage and Ae Ja Yeeb, Euler's partition theorem and the combinatorics of l-sequences, *J. Comb. Theory Ser. A* 115 (2008), 967–996.

[24] Andrew Sills, A Generalization of the Euler-Glaisher Bijection, *Electron. J. Comb.* 11(1) (2004).

[25] Neil J. A. Sloane and James A. Sellers, On non-squashing partitions, *Discrete Math.* **294** (2005), 259-274.

[26] Alan Tucker, *Applied Combinatorics*, Wiley, second edition,1984.