The ARS Algorithm and Invariance Analysis of Ordinary Difference Equations

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Declaration

I declare that the contents of this dissertation are original, except where due references have been made. It is submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

JT Kubayi

Signed at __Johannesburg__ on the __1st__ day of __April__ 2021.

Publications

Details of contributions to publications that form part of the research presented in this dissertation are:

A study on a solvable ordinary difference equations, Journal of Computational Analysis and Applications, submitted, 2021.

An investigation of the symmetry and singularity properties of some third-order boundary flow problems, has been accepted to appear in the Journal of Mathematical Analysis and Applications, 2021.

Abstract

This dissertation will be divided into two parts. The first part will involve the use of symmetries to find exact solutions of higher order difference equations. The second part will deal with important ordinary differential equations, in the search for singularities and integrability testing. We will analyse a Painlevé equation, Ivey's equation, the higher-order Lane-Emden type equation and a class of third-order boundary flow equations.

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Chapter 1

Introduction

During the nineteenth century, the prominent Norwegian mathematician Sophus Lie (1842 - 1899) established remarkable work that became an important core of the theory of groups of transformations (continuous) that leave differential equations invariant [1]. Lie wanted to create a theory of integrating ordinary differential equations that is equivalent to the Abelian theory of computing algebraic equations. He was inspired by Abel and Galois' theory, that stated that the procedure in all exceptional cases of a universal integration on differential equations, is centred on the invariance of the differential equation under continuous symmetries. Lie's group analysis classifies ordinary differential equations in terms of the symmetry group associated with them. He described the set integrable by group-theoretical methods. The concept of invariance of differential equations under infinitesimal transformations involves various techniques which makes it possible to best construct solutions of differential equations, that is, the development permits one to obtain solutions to differential equations systematically by performing symmetry analysis. In 1918, the German Mathematician Emmy Noether proved her theorem that uncovered fundamental justification for conservation laws, that is, conservation laws follow from the symmetry properties of nature.

Back in 1987, Shigeru Maeda showed that Lie's method can be extended to also solve ordinary difference equations(OΔE). He showed that the set of functional equations developed from the linearized symmetry condition of the OΔEs [6]. The philosophy of difference equations and their applications have cemented a central core in applicable analysis.
Maeda [6] showed how to use symmetry methods to obtain the solution of the system of first-order difference equations. In [7], Güven cinar, Ali Gelis and Ozan Özkan investigated the solution and the properties of the difference equation

$$x_n = \frac{x_{n-3k}x_{n-4k}x_{n-5k}}{x_{n-k}x_{n-2k}\left(\pm 1 \pm x_{n-3k}x_{n-4k}x_{n-5k}\right)}. \tag{1}$$

We are going to be using a symmetry based method to solve a generalisation of (1) and compare the solutions of the corresponding cases to that of [7].

In the second part of this dissertation, we investigate the integrability of ordinary differential equation by an integrability algorithm.

The precise meaning of the solution of a system of differential equations can be presented in several ways. One way, is the idea of integrability of a differential equation in terms of a singularity analysis, and this is the focus of our approach. Singularity analysis can be traced back to Painlevé [8] after its success in application by Sophie Kowalevskaya [9]. We consider the Painlevé test for integrability. M.J. Ablowitz, A. Ramani, H. Segur [10, 11, 12] developed an algorithm, called the ARS (Ablowitz-Ramani-Segur) algorithm that tests whether the solution of an ordinary differential equation can be expressed in terms of a Laurent expansion. If so, then the ordinary differential equation is said to pass the Painlevé test and is concluded without proof to be integrable. A detailed description of the algorithm can be found in the work by Conte [13]. In essence, one looks at the existence of a Laurent series for each dependent variable of the equation.

The Laurent series represents an analytic function in a punctured disc with radius centred on the singular point that is a pole or a specific type of a movable point, that is the singular point of a function for which it is possible to assign a complex number in such a way that a function becomes analytic. Once the movable singularity is found, the arbitrary constants of integration x_0 must be determined and the consistency of the proposed Laurent expansion must be checked.

1.1 Outline of Chapters

In Chapter 2, we will firstly look at definitions and some notations that we will be using in this dissertation.

In Chapter 3, we will perform a full Lie analysis of a tenth order ordinary difference equation. We will study the work in [7], where Güven çinar, Ali Geliş and Ozan Özkan investigated the solution and the properties of the difference equation

$$x_n = \frac{x_{n-3k}x_{n-4k}x_{n-5k}}{x_{n-k}x_{n-2k}\left(\pm 1 \pm x_{n-3k}x_{n-4k}x_{n-5k}\right)}. \tag{2}$$

In this chapter, we will derive the solutions of the following difference equation, via the invariant of their group of transformations:

$$x_n = \frac{x_{n-6}x_{n-8}x_{n-10}}{x_{n-2}x_{n-4}\left(a_n + b_n x_{n-6}x_{n-8}x_{n-10}\right)}, \tag{3}$$

where $(a_n)_{\mathbb{N}_0}$ and $(b_n)_{\mathbb{N}_0}$ are non-zero real sequences, using Lie group analysis technique. Our work is inspired by the results in [7] and for the sake of definitions and notations used in this paper, we shall consider:

$$u_{n+10} = \frac{u_n u_{n+2} u_{n+4}}{u_{n+6}u_{n+8}\left(A_n + B_n u_n u_{n+2} u_{n+4}\right)}, \tag{4}$$

where A_n and B_n are real arbitrary sequences. We are going to be using a symmetry based method to solve the general case and compare the solutions of the corresponding cases to that of [7].

In chapter 4, we investigate the singularity analysis of several important ordinary differential equations, such as the Painlevé-Ince equation, Ivey's equation, the higher-order Lane-Emden type equation and a family third-order boundary flow equations. To this end, we follow Kowalevskaya, and seek to determine whether or not the above ordinary differential equations possess movable singularities.

Chapter 2

Mathematical preliminaries

We introduce the relevant theory surrounding Lie point symmetries of differential equations, ordinary difference equations and the ARS algorithm for ordinary differential equations.

2.1 Lie Point Symmetries

The procedure for determining point symmetries for an arbitrary system of equations is as follows [14]. Consider q unknown variables u^a which depend on p independent variables x^i, i.e. $u = (u^1, \ldots, u^q)$, $x = (x^1, \ldots, x^p)$, with indices $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$. Let

$$G_\alpha \left(x, u^{(k)}\right) = 0, \tag{5}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x.

Definition 1 *A one-parameter Lie group of transformations (ϵ as the group parameter) that is invariant under (5) is given by*

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \tag{6}$$

Invariance of (5) under the transformation (6) implies that any solution $u = \Theta(x)$ of (5) maps into another solution $v = \Psi(x; \epsilon)$ of (5). Expanding (6) around the identity $\epsilon = 0$, generates the following infinitesimal transformations:

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon\xi^i(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u}^\alpha &= u^\alpha + \epsilon\eta^\alpha(x, u) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{7}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{8}$$

Definition 2 *The infinitesimal criterion for invariance is given by*

$$X\left[G_\alpha\left(x,u^{(k)}\right)\right]=0, \quad \textit{when} \quad G_\alpha\left(x,u^{(k)}\right)=0, \tag{9}$$

where X is extended to all derivatives appearing in the equation through an appropriate prolongation.

2.2 Definitions and Notations for Ordinary Difference Equations

The definitions and notations in this paper are similar to those adopted by Hydon in [2]. We consider the general form of the ordinary difference equation of order k given by

$$u_{n+k}=\omega\left(n,u_n,u_{n+1},u_{n+2},....,u_{n+k-1}\right), \tag{10}$$

for some function ω with $k\in\mathbb{N}$.

Definition 3 *We define S to be the shift operator acting on n as follows*

$$S:n\rightarrow n+1. \tag{11}$$

That is, if $u_n=F(n,c_1,...,c_N)$ then,

$$S(u_n)=S(F(n,c_1,...,c_N)) \tag{12}$$

$$=F(S(n),c_1,...,c_N) \tag{13}$$

$$=F(n+1,c_1,...,c_N) \tag{14}$$

$$=u_{n+1}, \tag{15}$$

where the c_i's are independent of n, and also $S(u_{n+k})=u_{n+k+1}$, for $k=0,1,....$ Therefore S is an operator on n and hence on u_{n+k}.

Given an equation (or system) whose continuous variables are $x=(x^1,....,x^N)$, a point transformation is a locally defined diffeomorphism

$$\Psi:x\rightarrow\hat{x}(x). \tag{16}$$

The term ‘point’, is used because $\hat{x}$ depends only on the point x.

Theorem 1 *A parametrized set of point transformations,*

$$\Psi_\epsilon : x \to \hat{x}(x;\epsilon), \qquad \epsilon \in (\epsilon_0, \epsilon_1),$$

where $\epsilon_0 \leq 0$ *and* $\epsilon_1 \geq 0$*, is a one-parameter local Lie group if the following conditions are satisfied:*

- Ψ_0 *is the identity map, so that* $\hat{x} = x$ *when* $\epsilon = 0$*.*
- $\Psi_\delta \Psi_\epsilon = \Psi_{\delta+\epsilon}$ *for every* δ*,*ϵ *sufficiently close to zero.*
- *Each* $\hat{x}^\alpha$ *can be represented as a Taylor series in* ϵ *(in the neighbourhood of* $\epsilon = 0$ *that is determined by* x*), and therefore*

$$\hat{x}^\alpha(x,\epsilon) = x^\alpha + \epsilon\xi^\alpha(x) + O(\epsilon^2), \qquad \alpha = 1, ..., N.$$

Now consider the point transformations

$$\Psi_\epsilon : (n, u_n) \to (n, u_n + \epsilon\xi\,(n, u_n))\,, \tag{17}$$

for some continuous characteristic function ξ.

Definition 4 *A symmetry generator is denoted by* X *and is given by*

$$X = \xi\frac{\partial}{\partial u_n} + S\xi\frac{\partial}{\partial u_{n+1}} + \cdots + S^{k-1}\xi\frac{\partial}{\partial u_{n+k-1}}. \tag{18}$$

To solve for the characteristic, we will need the linearized symmetry condition from [2]

$$S^k\xi - X\omega = 0, \tag{19}$$

provided (10) holds.
Given a symmetry generator for the k^{th} order OΔE (10), we have a $(k-1)^{th}$ order invariant

$$v_n = v\,(n, u_n, \ldots, u_{n+k-1})\,,$$

which satisfies

$$Xv_n = 0.$$

Assume that the characteristic ξ is known, then we can solve for the invariant v_n using the characteristics equation

$$\frac{du_n}{\xi} = \frac{du_{n+1}}{S\xi} = \cdots = \frac{du_{n+k-1}}{S^{n+k-1}\xi}\left(= \frac{dv_n}{0}\right). \tag{20}$$

The procedure of finding symmetries is best explained at length in [2], especially for second order difference equations.

Definition 5 *For a given characteristic $\xi(n, u_n) \neq 0$, we define a canonical coordinate, s_n(locally) by*

$$s_n = \int \frac{du}{\xi_i(n, u_n)}.$$

Definition 6 *The product operator multiplies the terms of a partial sequence of a sequence. It is defined as follows*

$$\prod_{k=1}^{n} a_k = (a_1)(a_2)(a_3)...(a_{n-1})(a_n). \tag{21}$$

2.3 The ARS Algorithm

Consider the equation

$$y^{(n)} = E\bigg(x; y'; y'', ..., y^{n-1}\bigg). \tag{22}$$

If there exists a movable singularity, which is a point where the solution of an ordinary differential equation behave in an unfavorable manner and is movable, that is to say that its location depends on the initial conditions of the differential equation, then the solution of (22) will be described by the power- law function $y(x) \simeq (x - x_0)^p$, where p is a negative number and x_0 indicates the singularity's position. It is the initial conditions, that provide us with different positions for the singular point. The algorithm can be described by the following three steps.

Firstly, we substitute $y(x) = a(x - x_0)^p$ into (22), to determine its leading-order behaviour where x_0 is the location of the generally considered movable singularity and look for two or more dominant terms. One way to determine the dominant terms is to look for balance after the above substitution by considering the powers in the equation. Another way is to check that Eq. (22) is invariant under the Lie symmetry $qx\partial_x + y\partial_y$, and the exponent in the leading order term is given by $\frac{-1}{q}$ in [10] and the constant a is determined by the equation. If (22) is invariant separately under $x\partial_x$ and $y\partial_y$ (it has two homogeneity symmetries) the value of p can be determined by the equation. The original ARS algorithm requires that p be a negative integer, or else, the algorithm terminates. After obtaining a, the leading-order term of the solution, we look at the behaviour of the "next-to-leading-order" term. In the ARS algorithm we require that the Laurent series be an increasing series called the right Painlevé series which is of the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+p}, \tag{23}$$

where the singularity at x_0 is a pole of order p. A decreasing series called the left Painlevé series is of the form (not dealt with in our dissertation)

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{-k+p}. \tag{24}$$

Next, one must locate the powers at which the arbitrary constants needed to make the solution a general solution, can be introduced. An expression is introduced,

$$y(x) = a(x - x_0)^p + m(x - x_0)^{p+s},$$

where s is called the resonance, and where the coefficient of terms linear in a variable m is zero. The solution of $s = -1$ always occurs, since $s = -1$ is generic in nature and it is also related to the arbitrariness of the location of the movable singularity. If the rest of the values of the resonance s are not integral, then the ARS algorithm terminates. Lastly we substitute a truncated Laurent series into the original equation, to check that there are no inconsistencies, which involves checking for the correct number of arbitrary constants which depends on the degree of th equation in discussion. For a second order equation with two symmetries $\gamma_1 = x\partial_x$ and $\gamma_2 = -qx\partial_x + y\partial_y$ there can be no inconsistency. We take note of some limitations of the ARS algorithm as explained in more detail in [15]:

- the exponents of the leading-order term needs to be a negative integer or a non-integral rational number,
- the resonances have to be rational and real numbers, because of the movable singularity,
- excluding the resonance $s = -1$, for a right Painlevé series the resonances must be nonnegative, while for a left Painlevé series, the resonances must be nonpositive.
- for a full Laurent expansion the resonances have to be mixed, that is, we can have mixed signs in the nongeneric resonances [16].

Chapter 3

Invariance Analysis of Ordinary Difference Equations

Consider tenth-order difference equations with the form of (4), i.e.,

$$u_{n+10} = \frac{u_n u_{n+2} n_{n+4}}{u_{n+6} u_{n+8} \left(A_n + B_n u_n u_{n+2} u_{n+4}\right)} = \omega(u_n, u_{n+2}, ..., u_{n+8}). \tag{25}$$

where A_n and B_n are real sequences (arbitrary).

3.1 Symmetry Analysis of Tenth Order O$\triangle$Es

By imposing the linearized symmetry condition (19) to (4) with $k = 2$ and replacing n by $n + 10$ from (3) to obtain (4), we get

$$\begin{aligned} &\xi(n+10,\omega) - \frac{\xi(n,u_n)A_n u_{n+2}u_{n+4}}{u_{n+6}u_{n+8}(A_n + B_n u_n u_{n+2}u_{n+4})^2} \\ &- \frac{\xi(n+2,u_{n+2})A_n u_n u_{n+4}}{u_{n+6}u_{n+8}(A_n + B_n u_n u_{n+2}u_{n+4})^2} - \frac{\xi(n+4,u_{n+4})A_n u_{n+2}}{u_{n+6}u_{n+8}(A_n + B_n u_n u_{n+2}u_{n+4})^2} \\ &+ \frac{\xi(n+6,u_{n+6})u_n u_{n+2}u_{n+4}}{u_{n+6}^2 u_{n+8}(A_n + B_n u_n u_{n+2}u_{n+4})} + \frac{\xi(n+8,u_{n+8})u_n u_{n+2}u_{n+4}}{u_{n+6}u_{n+8}^2(A_n + B_n u_n u_{n+2}u_{n+4})} = 0. \end{aligned} \tag{26}$$

We differentiate the functional equation (26) implicitly with respect to u_n in order to solve for ξ, while keeping u_{n+6} fixed, it is important to note that we can choose freely between $u_{n+2}, u_{n+4}, u_{n+6}$ and u_{n+8}. That is, we apply the Linear differential operator, noting first

$$\frac{\partial u_{n+6}}{\partial u_n} = \frac{\frac{\partial \omega}{\partial u_n}}{\frac{\partial \omega}{\partial u_{n+6}}} = \frac{A_n u_{n+6}}{u_n(A_n + B_n u_n u_{n+2}u_{n+4})},$$

then

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+6}}{\partial u_n}\frac{\partial}{\partial u_{n+6}}$$

$$= \frac{\partial}{\partial u_n} + \frac{A_n u_{n+6}}{u_n(A_n + B_n u_n u_{n+2} u_{n+4})}\frac{\partial}{\partial u_{n+6}}$$

on (26). After clearing fractions and then differentiating thrice with respect to u_n, keeping u_{n+6} fixed, we obtain the following

$$\begin{gathered} -2(A_n + 2B_n u_n u_{n+2} u_{n+4}) u_{n+6} \xi^{(3)}(n, u_n) \\ -(A_n + B_n u_n u_{n+2} u_{n+4}) u_n u_{n+6} \xi^{(4)}(n, u_n) = 0. \end{gathered} \tag{27}$$

Now we separate the above, since ξ depends only on u_n, which is a continuous variable, then we will have

$$u_{n+2} u_{n+4} u_{n+6} : u_n \xi^{(4)}(n, u_n) + 4\xi^{(3)}(n, u_n) = 0 \tag{28}$$

$$u_{n+6} : u_n \xi^{(4)}(n, u_n) + 2\xi^{(3)}(n, u_n) = 0 \tag{29}$$

implying that

$$\xi(n, u_n) = \beta^n u_n^2 + \gamma^n u_n + \lambda^n \tag{30}$$

for some functions β^n, γ^n and λ^n of n.

Now we substitute (30) into (26) and then separate the resulting equation by the coefficients of products of shifts of u_n and then setting them to zero. We then obtain the following results

$$\gamma^n + \gamma^{n+2} + \gamma^{n+4} = 0 \tag{31}$$

$$\beta^n = 0 \tag{32}$$

$$\lambda^n = 0. \tag{33}$$

Then the four independent solutions of the fourth order difference equation (31) are given by

$$\gamma_1, \bar{\gamma}_1, \gamma_2, \bar{\gamma}_2,$$

where

$$\gamma_1 = e^{\frac{in\pi}{3}}$$

and $\bar{\gamma_1}$ represents its complex conjugate, also

$$\gamma_2 = e^{\frac{2in\pi}{3}}$$

and $\bar{\gamma_2}$ represents its complex conjugate. Then from (30) we have the characteristics given by

$$\xi_1 = \gamma_1^n u_n,$$

$$\xi_2 = \bar{\gamma_1}^n u_n,$$

$$\xi_3 = \gamma_2^n u_n$$

and

$$\xi_4 = \bar{\gamma_2}^n u_n.$$

Hence we obtain the following prolongation of the spanning vectors of the Lie algebra of (25) given below

$$\begin{aligned} X_1 =& \gamma_1^n u_n \frac{\partial}{\partial u_n} + \gamma_1^{n+1} u_{n+1} \frac{\partial}{\partial u_{n+1}} + \gamma_1^{n+2} u_{n+2} \frac{\partial}{\partial u_{n+2}} + \gamma_1^{n+3} u_{n+3} \frac{\partial}{\partial u_{n+3}} \\ &+ \gamma_1^{n+4} u_{n+4} \frac{\partial}{\partial u_{n+4}} + \gamma_1^{n+5} u_{n+5} \frac{\partial}{\partial u_{n+5}} + \gamma_1^{n+6} u_{n+6} \frac{\partial}{\partial u_{n+6}} + \gamma_1^{n+7} u_{n+7} \frac{\partial}{\partial u_{n+7}} \\ &+ \gamma_1^{n+8} u_{n+8} \frac{\partial}{\partial u_{n+8}} + \gamma_1^{n+9} u_{n+9} \frac{\partial}{\partial u_{n+9}}. \end{aligned}$$

$$\begin{aligned} X_2 =& \bar{\gamma_1}^n u_n \frac{\partial}{\partial u_n} + \bar{\gamma_1}^{n+1} u_{n+1} \frac{\partial}{\partial u_{n+1}} + \bar{\gamma_1}^{n+2} u_{n+2} \frac{\partial}{\partial u_{n+2}} + \bar{\gamma_1}^{n+3} u_{n+3} \frac{\partial}{\partial u_{n+3}} \\ &+ \bar{\gamma_1}^{n+4} u_{n+4} \frac{\partial}{\partial u_{n+4}} + \bar{\gamma_1}^{n+5} u_{n+5} \frac{\partial}{\partial u_{n+5}} + \bar{\gamma_1}^{n+6} u_{n+6} \frac{\partial}{\partial u_{n+6}} + \bar{\gamma_1}^{n+7} u_{n+7} \frac{\partial}{\partial u_{n+7}} \\ &+ \bar{\gamma_1}^{n+8} u_{n+8} \frac{\partial}{\partial u_{n+8}} + \bar{\gamma_1}^{n+9} u_{n+9} \frac{\partial}{\partial u_{n+9}}. \end{aligned}$$

$$\begin{aligned} X_3 =& \gamma_2^n u_n \frac{\partial}{\partial u_n} + \gamma_2^{n+1} u_{n+1} \frac{\partial}{\partial u_{n+1}} + \gamma_2^{n+2} u_{n+2} \frac{\partial}{\partial u_{n+2}} + \gamma_2^{n+3} u_{n+3} \frac{\partial}{\partial u_{n+3}} \\ &+ \gamma_2^{n+4} u_{n+4} \frac{\partial}{\partial u_{n+4}} + \gamma_2^{n+5} u_{n+5} \frac{\partial}{\partial u_{n+5}} + \gamma_2^{n+6} u_{n+6} \frac{\partial}{\partial u_{n+6}} + \gamma_2^{n+7} u_{n+7} \frac{\partial}{\partial u_{n+7}} \\ &+ \gamma_2^{n+8} u_{n+8} \frac{\partial}{\partial u_{n+8}} + \gamma_2^{n+9} u_{n+9} \frac{\partial}{\partial u_{n+9}}. \end{aligned}$$

$$X_4 = \bar{\gamma_2}^n u_n \frac{\partial}{\partial u_n} + \bar{\gamma_2}^{n+1} u_{n+1} \frac{\partial}{\partial u_{n+1}} + \bar{\gamma_2}^{n+2} u_{n+2} \frac{\partial}{\partial u_{n+2}} + \bar{\gamma_2}^{n+3} u_{n+3} \frac{\partial}{\partial u_{n+3}}$$
$$+ \bar{\gamma_2}^{n+4} u_{n+4} \frac{\partial}{\partial u_{n+4}} + \bar{\gamma_2}^{n+5} u_{n+5} \frac{\partial}{\partial u_{n+5}} + \bar{\gamma_2}^{n+6} u_{n+6} \frac{\partial}{\partial u_{n+6}} + \bar{\gamma_2}^{n+7} u_{n+7} \frac{\partial}{\partial u_{n+7}}$$
$$+ \bar{\gamma_2}^{n+8} u_{n+8} \frac{\partial}{\partial u_{n+8}} + \bar{\gamma_2}^{n+9} u_{n+9} \frac{\partial}{\partial u_{n+9}}.$$

3.2 Reduction and Exact solutions

Given that $\xi(n, u_n) = \beta^n u_n^2 + \gamma^n u_n + \lambda^n$ is a characteristic, with $\beta^n = 0$ and $\lambda^n = 0$, then the following condition must be satisfied:

$$\frac{du_n}{\gamma^n u_n} = \frac{du_{n+1}}{\gamma^{n+1} u_{n+1}} = ... = \frac{du_{n+9}}{\gamma^{n+9} u_{n+9}} \left(= \frac{dv_n}{0} \right). \tag{34}$$

That is, given ξ_1, the canonical coordinate 5 that linearizes (25), meaning we can find a linear approximation at this coordinate, is given by

$$S_n = \int \frac{du_n}{\xi_1(n, u_n)} = \frac{1}{\gamma^n} ln|u_n| \tag{35}$$

and from the final constraint (31), we then define the invariant function

$$\begin{aligned} \tilde{V}_n &= \gamma^n S_n + \gamma^{n+2} S_{n+2} + \gamma^{n+4} S_{n+4} \\ &= ln|u_n| + ln|u_{n+2}| + ln|u_{n+4}| \\ &= ln|u_n u_{n+2} u_{n+4}|. \end{aligned} \tag{36}$$

Thus from the fact that

$$X_1 \tilde{V}_n = X_2 \tilde{V}_n = X_3 \tilde{V}_n = X_4 \tilde{V}_n = 0.$$

We must make use of the function V_n defined as

$$|V_n| = \exp(-\tilde{V}_n) \tag{37}$$

that is

$$V_n = \pm \frac{1}{u_n u_{n+2} u_{n+4}}, \tag{38}$$

but we will only consider

$$V_n = \frac{1}{u_n u_{n+2} u_{n+4}}, \tag{39}$$

for the relevancy of our work. Then we must show that V_n is an invariant, it suffices to show that $X_1\tilde{V}_n = 0$, it follows from

$$\begin{aligned} X_1\tilde{V}_n &= \gamma^n u_n \frac{\partial}{\partial u_n}\tilde{V}_n + \gamma^{n+1}u_{n+1}\frac{\partial}{\partial u_{n+1}}\tilde{V}_n \\ &+\gamma^{n+2}u_{n+2}\frac{\partial}{\partial u_{n+2}}\tilde{V}_n + \gamma^{n+3}u_{n+3}\frac{\partial}{\partial u_{n+3}}\tilde{V}_n \\ &+\gamma^{n+4}u_{n+4}\frac{\partial}{\partial u_{n+4}}\tilde{V}_n + \gamma^{n+5}u_{n+5}\frac{\partial}{\partial u_{n+5}}\tilde{V}_n \\ &+\gamma^{n+6}u_{n+6}\frac{\partial}{\partial u_{n+6}}\tilde{V}_n + \gamma^{n+7}u_{n+7}\frac{\partial}{\partial u_{n+7}}\tilde{V}_n \\ &+\gamma^{n+8}u_{n+8}\frac{\partial}{\partial u_{n+8}}\tilde{V}_n + \gamma^{n+9}u_{n+9}\frac{\partial}{\partial u_{n+9}}\tilde{V}_n. \end{aligned} \tag{40}$$

Therefore we have

$$X_1\tilde{V}_n = \gamma^n u_n(\frac{1}{u_n}) + \gamma^{n+2}u_{n+2}(\frac{1}{u_{n+2}}) + \gamma^{n+4}u_{n+4}(\frac{1}{u_{n+4}}), \tag{41}$$

which gives

$$X_1\tilde{V}_n = \gamma^n + \gamma^{n+2} + \gamma^{n+4}. \tag{42}$$

Then invoking the characteristic, that is we take note that (31) gives $\gamma^n + \gamma^{n+2} + \gamma^{n+4} = 0$, then we will have that

$$X_1\tilde{V}_n = 0.$$

Now taking (39)

$$V_n = \frac{1}{u_n u_{n+2} u_{n+4}}, \tag{43}$$

and shift it once, yields

$$V_{n+1} = \frac{1}{u_{n+1} u_{n+3} u_{n+5}}, \tag{44}$$

and shifting V_n six times gives

$$V_{n+6} = \frac{1}{u_{n+6} u_{n+8} u_{n+10}}. \tag{45}$$

We know u_{n+10} from (25), which we then substitute into (45) and obtain

$$V_{n+6} = \frac{1}{u_{n+6}u_{n+8}\frac{u_n u_{n+2} u_{n+4}}{u_{n+6}u_{n+8}(A_n + B_n u_n u_{n+2} u_{n+4})}}, \tag{46}$$

which gives

$$V_{n+6} = \frac{A_n}{u_n u_{n+2} u_{n+4}} + B_n = A_n V_n + B_n. \tag{47}$$

From (39)

$$V_n = \frac{1}{u_n u_{n+2} u_{n+4}}, \tag{48}$$

on which we can perform a shift by two to get

$$V_{n+2} = \frac{1}{u_{n+2} u_{n+4} u_{n+6}}, \tag{49}$$

then making u_{n+6} the subject gives

$$u_{n+6} = \frac{1}{u_{n+2} u_{n+4} V_{n+2}}. \tag{50}$$

Making $u_{n+2}u_{n+4}$ the subject in (25), gives

$$u_{n+2}u_{n+4} = \frac{u_{n+10} u_{n+6} u_{n+8} (A_n + B_n u_n u_{n+2} u_{n+4})}{u_n}. \tag{51}$$

Now substituting (51) into (50) and then simplifying the results, we obtain

$$u_{n+6} = \frac{V_n}{V_{n+2}} u_n. \tag{52}$$

The solution of (47),

$$V_{n+6} = A_n V_n + B_n, \tag{53}$$

in closed form is given by

$$V_{6n+j} = V_j \left(\prod_{k_1=0}^{n-1} A_{6k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{6l+j} \prod_{k_2=l+1}^{n-1} A_{6k_2+j} \right). \tag{54}$$

Starting with

$$u_{n+6} = \frac{V_n}{V_{n+2}} u_n,$$

by iterations that for

$$n = 0 : u_6 = \frac{V_0}{V_2} u_0.$$

$$n = 1 : u_7 = \frac{V_1}{V_3} u_1$$

$$n = 2 : u_8 = \frac{V_2}{V_4} u_2$$

$$n = 3 : u_9 = \frac{V_3}{V_5} u_3$$

$$n = 4 : u_{10} = \frac{V_4}{V_6} u_4$$

$$n = 5 : u_{11} = \frac{V_5}{V_7} u_5$$

$$n = 6 : u_{12} = \frac{V_6}{V_8} u_6 = \frac{V_6 V_0}{V_8 V_2} u_0$$

$$n = 7 : u_{13} = \frac{V_7}{V_9} u_7 = \frac{V_7 V_1}{V_9 V_3} u_1$$

$$\vdots$$

From these iterations, we observe that

$$u_{6n+j} = u_j \left(\prod_{k_1=0}^{n-1} \frac{V_{6k_1+j}}{V_{6k_1+j+2}} \right). \tag{55}$$

To solve (53), we note that it depends on the parity of n, meaning it depends on whether n is odd or even and we have that

$$V_{6n+j} = A_{6n+j} V_{6n+j} + B_{6n+j}, \tag{56}$$

for $j = 0, 1, 2, 3, 4, 5$.
Having that

$$V_n = \frac{1}{u_n u_{n+2} u_{n+4}}$$

and depending on the parity of n to solve

$$V_{n+6} = A_n V_n + B_n,$$

then for $j = 0, 1, 2, 3, 4, 5$, we can have

$$V_{6n+j} = A_{6n+j} V_{6n+j} + B_{6n+j}. \tag{57}$$

Thus the sequence $(V_{6n+j})_{j=0,1,2,3,4,5}$ satisfies the tenth-order linear difference equation

$$\omega_{n+1} = A_{6n+j}\omega_n + B_{6n+j} \tag{58}$$

for $j = 0, 1, 2, 3, 4, 5$. After a few iterations and applying mathematical induction, we have the following solution in closed form as

$$\omega_n = V_{6n+j} = V_j \left(\prod_{k_1=0}^{n-1} A_{6k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{6l+j} \prod_{k_2=l+1}^{n-1} A_{6k_2+j} \right) \tag{59}$$

for $j = 0, 1, 2, 3, 4, 5$. From (59) it can be shown that for $j = 0$,

$$u_{6n} = u_0 \prod_{k=0}^{n-1} \frac{V_{6k}}{V_{6k+2}}$$

$$= u_0 \prod_{k=0}^{n-1} \frac{V_0 \left(\prod_{k_1=0}^{k-1} A_{6k_1} \right) + \sum_{l=0}^{k-1} \left(B_{6l} \prod_{k_2=l+1}^{k-1} A_{6k_2} \right)}{V_2 \left(\prod_{k_1=0}^{k-1} A_{6k_1+2} \right) + \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+2} \right)}$$

$$= u_0 \prod_{k=0}^{n-1} \frac{V_0 \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1} \right) + \frac{1}{V_0} \sum_{l=0}^{k-1} \left(B_{6l} \prod_{k_2=l+1}^{k-1} A_{6k_2} \right) \right]}{V_2 \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1+2} \right) + \frac{1}{V_2} \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+2} \right) \right]}$$

$$= u_0 \prod_{k=0}^{n-1} \frac{\frac{1}{u_0 u_2 u_4} \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1} \right) + u_0 u_2 u_4 \sum_{l=0}^{k-1} \left(B_{6l} \prod_{k_2=l+1}^{k-1} A_{6k_2} \right) \right]}{\frac{1}{u_2 u_4 u_6} \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1+2} \right) + u_2 u_4 u_6 \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+2} \right) \right]}$$

$$= u_0^{1-n} u_6^n \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} A_{6k_1} + u_0 u_2 u_4 \sum_{l=0}^{k-1} \left(B_{6l} \prod_{k_2=l+1}^{k-1} A_{6k_2} \right)}{\prod_{k_1=0}^{k-1} A_{6k_1+2} + u_2 u_4 u_6 \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+2} \right)}.$$

For $j = 1$,

$$
\begin{aligned}
u_{6n+1} &= u_1 \prod_{k=0}^{n-1} \frac{V_{6k+1}}{V_{6k+3}} \\
&= u_1 \prod_{k=0}^{n-1} \frac{V_1 \left(\prod_{k_1=0}^{k-1} A_{6k_1+1} \right) + \sum_{l=0}^{k-1} \left(B_{6l+1} \prod_{k_2=l+1}^{k-1} A_{6k_2+1} \right)}{V_3 \left(\prod_{k_1=0}^{k-1} A_{6k_1+3} \right) + \sum_{l=0}^{k-1} \left(B_{6l+3} \prod_{k_2=l+1}^{k-1} A_{6k_2+3} \right)} \\
&= u_1 \prod_{k=0}^{n-1} \frac{\frac{1}{u_1u_3u_5} \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1+1} \right) + u_1u_3u_5 \sum_{l=0}^{k-1} \left(B_{6l+1} \prod_{k_2=l+1}^{k-1} A_{6k_2+1} \right) \right]}{\frac{1}{u_3u_5u_7} \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1+3} \right) + u_3u_5u_7 \sum_{l=0}^{k-1} \left(B_{6l+3} \prod_{k_2=l+1}^{k-1} A_{6k_2+3} \right) \right]} \qquad (60) \\
&= u_1^{1-n} u_7^n \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} A_{6k_1+1} + u_1u_3u_5 \sum_{l=0}^{k-1} \left(B_{6l+1} \prod_{k_2=l+1}^{k-1} A_{6k_2+1} \right)}{\prod_{k_1=0}^{k-1} A_{6k_1+3} + u_3u_5u_7 \sum_{l=0}^{k-1} \left(B_{6l+3} \prod_{k_2=l+1}^{k-1} A_{6k_2+3} \right)}.
\end{aligned}
$$

For $j = 2$,

$$u_{6n+2} = u_2 \prod_{k=0}^{n-1} \frac{V_{6k+2}}{V_{6k+4}}$$

$$= u_2 \prod_{k=0}^{n-1} \frac{V_2 \left(\prod_{k_1=0}^{k-1} A_{6k_1+2} \right) + \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+12} \right)}{V_4 \left(\prod_{k_1=0}^{k-1} A_{6k_1+4} \right) + \sum_{l=0}^{k-1} \left(B_{6l+4} \prod_{k_2=l+1}^{k-1} A_{6k_2+4} \right)}$$

$$= u_2 \prod_{k=0}^{n-1} \frac{\frac{1}{u_2 u_4 u_6} \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1+2} \right) + u_2 u_4 u_6 \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+2} \right) \right]}{\frac{1}{u_4 u_6 u_8} \left[\left(\prod_{k_1=0}^{k-1} A_{6k_1+4} \right) + u_4 u_6 u_8 \sum_{l=0}^{k-1} \left(B_{6l+4} \prod_{k_2=l+1}^{k-1} A_{6k_2+4} \right) \right]} \tag{61}$$

$$= u_2^{1-n} u_8^n \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} A_{6k_1+2} + u_2 u_4 u_6 \sum_{l=0}^{k-1} \left(B_{6l+2} \prod_{k_2=l+1}^{k-1} A_{6k_2+2} \right)}{\prod_{k_1=0}^{k-1} A_{6k_1+4} + u_4 u_6 u_8 \sum_{l=0}^{k-1} \left(B_{6l+4} \prod_{k_2=l+1}^{k-1} A_{6k_2+4} \right)}.$$

For $j = 3$,

$$u_{6n+3} = u_3 \prod_{k=0}^{n-1} \frac{V_{6k+3}}{V_{6k+5}}$$

$$= u_3 \prod_{k=0}^{n-1} \frac{V_3 \left(\prod_{k_1=0}^{k-1} A_{6k_1+3} \right) + \sum_{l=0}^{k-1} \left(B_{6l+3} \prod_{k_2=l+1}^{k-1} A_{6k_2+3} \right)}{V_5 \left(\prod_{k_1=0}^{k-1} A_{6k_1+5} \right) + \sum_{l=0}^{k-1} \left(B_{6l+5} \prod_{k_2=l+1}^{k-1} A_{6k_2+5} \right)} \tag{62}$$

$$= u_3^{1-n} u_9^n \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} A_{6k_1+3} + u_3 u_5 u_7 \sum_{l=0}^{k-1} \left(B_{6l+3} \prod_{k_2=l+1}^{k-1} A_{6k_2+3} \right)}{\prod_{k_1=0}^{k-1} A_{6k_1+5} + u_5 u_7 u_9 \sum_{l=0}^{k-1} \left(B_{6l+5} \prod_{k_2=l+1}^{k-1} A_{6k_2+5} \right)}.$$

For $j=4$,

$$u_{6n+4}=u_4\prod_{k=0}^{n-1}\frac{V_{6k+4}}{V_{6k+6}}$$

$$=u_4\prod_{k=0}^{n-1}\frac{V_4\left(\prod_{k_1=0}^{k-1}A_{6k_1+4}\right)+\sum_{l=0}^{k-1}\left(B_{6l+4}\prod_{k_2=l+1}^{k-1}A_{6k_2+4}\right)}{V_0\left(\prod_{k_1=0}^{k}A_{6k_1}\right)+\sum_{l=0}^{k}\left(B_{6l}\prod_{k_2=l+1}^{k}A_{6k_2}\right)} \tag{63}$$

$$=\frac{u_4u_2^nu_0^n}{u_6^nu_8^n}\prod_{k=0}^{n-1}\frac{\prod_{k_1=0}^{k-1}A_{6k_1+4}+u_4u_6u_8\sum_{l=0}^{k-1}\left(B_{6l+4}\prod_{k_2=l+1}^{k-1}A_{6k_2+4}\right)}{\prod_{k_1=0}^{k}A_{6k_1}+u_0u_2u_4\sum_{l=0}^{k}\left(B_{6l}\prod_{k_2=l+1}^{k}A_{6k_2}\right)}.$$

For $j=5$,

$$u_{6n+5}=u_5\prod_{k=0}^{n-1}\frac{V_{6k+5}}{V_{6k+7}}$$

$$=u_5\prod_{k=0}^{n-1}\frac{V_5\left(\prod_{k_1=0}^{k-1}A_{6k_1+5}\right)+\sum_{l=0}^{k-1}\left(B_{6l+5}\prod_{k_2=l+1}^{k-1}A_{6k_2+5}\right)}{V_1\left(\prod_{k_1=0}^{k}A_{6k_1+1}\right)+\sum_{l=0}^{k}\left(B_{6l+1}\prod_{k_2=l+1}^{k}A_{6k_2+1}\right)} \tag{64}$$

$$=\frac{u_5u_3^nu_1^n}{u_7^nu_9^n}\prod_{k=0}^{n-1}\frac{\prod_{k_1=0}^{k-1}A_{6k_1+5}+u_5u_7u_9\sum_{l=0}^{k-1}\left(B_{6l+5}\prod_{k_2=l+1}^{k-1}A_{6k_2+5}\right)}{\prod_{k_1=0}^{k}A_{6k_1+1}+u_1u_3u_5\sum_{l=0}^{k}\left(B_{6l+1}\prod_{k_2=l+1}^{k}A6k_2+1\right)}.$$

Then our solution in terms of $\{x_n\}_{n=1}^{\infty}$ is given by following

$$x_{6n-9}=x_{-9}^{1-n}x_{-3}^{n}\prod_{k=0}^{n-1}\frac{\prod_{k_1=0}^{k-1}a_{6k_1}+x_{-9}x_{-7}x_{-5}\sum_{l=0}^{k-1}\left(b_{6l}\prod_{k_2=l+1}^{k-1}a_{6k_2}\right)}{\prod_{k_1=0}^{k-1}a_{6k_1+2}+x_{-7}x_{-5}x_{-3}\sum_{l=0}^{k-1}\left(b_{6l+2}\prod_{k_2=l+1}^{k-1}a_{6k_2+2}\right)} \tag{65a}$$

$$x_{6n-8} = x_{-8}^{1-n} x_{-2}^{n} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+1} + x_{-8}x_{-6}x_{-4} \sum_{l=0}^{k-1} \left(b_{6l+1} \prod_{k_2=l+1}^{k-1} a_{6k_2+1} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+3} + x_{-6}x_{-4}x_{-2} \sum_{l=0}^{k-1} \left(b_{6l+3} \prod_{k_2=l+1}^{k-1} a_{6k_2+3} \right)} \tag{65b}$$

$$x_{6n-7} = x_{-7}^{1-n} x_{-1}^{n} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+2} + x_{-7}x_{-5}x_{-3} \sum_{l=0}^{k-1} \left(b_{6l+2} \prod_{k_2=l+1}^{k-1} a_{6k_2+2} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+4} + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{k-1} \left(b_{6l+4} \prod_{k_2=l+1}^{k-1} a_{6k_2+4} \right)} \tag{65c}$$

$$x_{6n-6} = x_{-6}^{1-n} x_{0}^{n} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+3} + x_{-6}x_{-4}x_{-2} \sum_{l=0}^{k-1} \left(b_{6l+3} \prod_{k_2=l+1}^{k-1} a_{6k_2+3} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+5} + x_{-4}x_{-2}x_{0} \sum_{l=0}^{k-1} \left(b_{6l+5} \prod_{k_2=l+1}^{k-1} a_{6k_2+5} \right)} \tag{65d}$$

$$x_{6n-5} = \frac{x_{-5}x_{-7}^{n}x_{-9}^{n}}{x_{-3}^{n}x_{-1}^{n}} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+4} + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{k-1} \left(b_{6l+4} \prod_{k_2=l+1}^{k-1} a_{6k_2+4} \right)}{\prod_{k_1=0}^{k} a_{6k_1} + x_{-9}x_{-7}x_{-5} \sum_{l=0}^{k} \left(b_{6l} \prod_{k_2=l+1}^{k} a_{6k_2} \right)} \tag{65e}$$

$$x_{6n-4} = \frac{x_{-4}x_{-6}^{n}x_{-8}^{n}}{x_{-2}^{n}x_{0}^{n}} \prod_{k=0}^{n-1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+5} + x_{-4}x_{-2}x_{0} \sum_{l=0}^{k-1} \left(b_{6l+5} \prod_{k_2=l+1}^{k-1} a_{6k_2+5} \right)}{\prod_{k_1=0}^{k} a_{6k_1+1} + x_{-8}x_{-6}x_{-4} \sum_{l=0}^{k} \left(b_{6l+1} \prod_{k_2=l+1}^{k} a_{6k_2+1} \right)}. \tag{65f}$$

Assuming that the denominators are non zero, we can rearrange the equation (64d) above as follows

$$x_{6n} = x_{-6}^{-n} x_{0}^{n+1} \prod_{k=0}^{n} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+3} + x_{-6}x_{-4}x_{-2} \sum_{l=0}^{k-1} \left(b_{6l+3} \prod_{k_2=l+1}^{k-1} a_{6k_2+3} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+5} + x_{-4}x_{-2}x_{0} \sum_{l=0}^{k-1} \left(b_{6l+5} \prod_{k_2=l+1}^{k-1} a_{6k_2+5} \right)}, \tag{66}$$

then the equations in (65) can be rearranged as follows

$$x_{6(n+1)} = x_{-6}^{-(n+1)} x_{0}^{n+2} \prod_{k=0}^{n+1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+3} + x_{-6}x_{-4}x_{-2} \sum_{l=0}^{k-1} \left(b_{6l+3} \prod_{k_2=l+1}^{k-1} a_{6k_2+3} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+5} + x_{-4}x_{-2}x_{0} \sum_{l=0}^{k-1} \left(b_{6l+5} \prod_{k_2=l+1}^{k-1} a_{6k_2+5} \right)} \tag{67a}$$

$$x_{6(n+2)-7} = x_{-7}^{-(n+1)} x_{-1}^{n+2} \prod_{k=0}^{n+1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+2} + x_{-7}x_{-5}x_{-3} \sum_{l=0}^{k-1} \left(b_{6l+2} \prod_{k_2=l+1}^{k-1} a_{6k_2+2} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+4} + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{k-1} \left(b_{6l+4} \prod_{k_2=l+1}^{k-1} a_{6k_2+4} \right)} \tag{67b}$$

$$x_{6(n+2)-8} = x_{-8}^{-(n+1)} x_{-2}^{n+2} \prod_{k=0}^{n+1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+1} + x_{-8}x_{-6}x_{-4} \sum_{l=0}^{k-1} \left(b_{6l+1} \prod_{k_2=l+1}^{k-1} a_{6k_2+1} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+3} + x_{-6}x_{-4}x_{-2} \sum_{l=0}^{k-1} \left(b_{6l+3} \prod_{k_2=l+1}^{k-1} a_{6k_2+3} \right)} \tag{67c}$$

$$x_{6(n+2)-9} = x_{-9}^{-(n+1)} x_{-2}^{n+2} \prod_{k=0}^{n+1} \frac{\prod_{k_1=0}^{k-1} a_{6k_1} + x_{-9}x_{-7}x_{-5} \sum_{l=0}^{k-1} \left(b_{6l} \prod_{k_2=l+1}^{k-1} a_{6k_2} \right)}{\prod_{k_1=0}^{k-1} a_{6k_1+2} + x_{-7}x_{-5}x_{-3} \sum_{l=0}^{k-1} \left(b_{6l+2} \prod_{k_2=l+1}^{k-1} a_{6k_2+2} \right)} \tag{67d}$$

$$x_{6(n+1)-4} = \frac{x_{-4}x_{-6}^{n+1}x_{-8}^{n+1}}{x_{-2}^{n+1}x_0^{n+1}} \prod_{k=0}^{n} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+5} + x_{-4}x_{-2}x_0 \sum_{l=0}^{k-1} \left(b_{6l+5} \prod_{k_2=l+1}^{k-1} a_{6k_2+5} \right)}{\prod_{k_1=0}^{k} a_{6k_1+1} + x_{-8}x_{-6}x_{-4} \sum_{l=0}^{k} \left(b_{6l+1} \prod_{k_2=l+1}^{k} a_{6k_2+1} \right)}. \tag{67e}$$

$$x_{6(n+1)-5} = \frac{x_{-5}x_{-7}^{n+1}x_{-9}^{n+1}}{x_{-3}^{n+1}x_{-1}^{n+1}} \prod_{k=0}^{n} \frac{\prod_{k_1=0}^{k-1} a_{6k_1+4} + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{k-1} \left(b_{6l+4} \prod_{k_2=l+1}^{k-1} a_{6k_2+4} \right)}{\prod_{k_1=0}^{k} a_{6k_1} + x_{-9}x_{-7}x_{-5} \sum_{l=0}^{k} \left(b_{6l} \prod_{k_2=l+1}^{k} a_{6k_2} \right)} \tag{67f}$$

3.3 The case where a_n and b_n are 6-periodic.

Let $x_{-9} = a$, $x_{-8} = b$, $x_{-7} = c$, $x_{-6} = d$, $x_{-5} = e$, $x_{-4} = f$, $x_{-3} = g$, $x_{-2} = h$, $x_{-1} = m$ and $x_0 = p$. We assume that $a_n = a_{n+6}$ and $b_n = b_{n+6}$ for $n \in \mathbb{N}_0$, therefore the solution will be given by

$$x_{6(n+1)} = d^{-(n+1)} p^{n+2} \prod_{k=0}^{n+1} \frac{a_3^k + b_3 dfh \sum_{l=0}^{k-1} a_3^l}{a_5^k + b_5 fhp \sum_{l=0}^{k-1} a_5^l} \tag{68a}$$

$$x_{6(n+2)-7} = c^{-(n+1)} m^{n+2} \prod_{k=0}^{n+1} \frac{a_2^k + b_2 ceg \sum_{l=0}^{k-1} a_2^l}{a_4^k + b_4 egm \sum_{l=0}^{k-1} a_4^l} \tag{68b}$$

$$x_{6(n+2)-8} = b^{-(n+1)}h^{n+2}\prod_{k=0}^{n+1}\frac{a_1^k + b_1bdf\sum_{l=0}^{k-1}a_1^l}{a_3^k + b_3dfh\sum_{l=0}^{k-1}a_3^l} \tag{68c}$$

$$x_{6(n+2)-9} = a^{-(n+1)}h^{n+2}\prod_{k=0}^{n+1}\frac{a_0^k + b_0ace\sum_{l=0}^{k-1}a_0^l}{a_2^k + b_2ceg\sum_{l=0}^{k-1}a_2^l} \tag{68d}$$

$$x_{6(n+1)-4} = \frac{fd^{n+1}b^{n+1}}{h^{n+1}m^{n+1}}\prod_{k=0}^{n}\frac{a_5^k + b_5fhp\sum_{l=0}^{k-1}a_5^l}{a_1^k + b_1bdf\sum_{l=0}^{k}a_1^l} \tag{68e}$$

$$x_{6(n+1)-5} = \frac{ec^{n+1}a^{n+1}}{g^{n+1}m^{n+1}}\prod_{k=0}^{n}\frac{a_4^k + b_4egm\sum_{l=0}^{k-1}a_4^l}{a_0^k + b_0ace\sum_{l=0}^{k}a_0^l}. \tag{68f}$$

3.4 The case a_n and b_n are constant coefficients

Let $a_{6k_1} = a, a_{6k_1+1} = a, a_{6k_1+2} = a, a_{6k_1+3} = a, a_{6k_1+4} = a, a_{6k_1+5} = a$ and $b_{6l} = b, b_{6l+1} = b, b_{6l+2} = b, b_{6l+3} = b, b_{6l+4} = b, b_{6l+5} = b$. Note that this mean $a_n = a$ and $b_n = b$ for all values of n since both a_n and b_n are constant coefficients. Then we will have

$$x_{6(n+1)} = x_{-6}^{-(n+1)}x_0^{n+2}\prod_{k=0}^{n+1}\frac{a^k + x_{-6}x_{-4}x_{-2}\sum_{l=0}^{k-1}\left(ba^l\right)}{a^k + x_{-4}x_{-2}x_0\sum_{l=0}^{k-1}\left(ba^l\right)} \tag{69a}$$

$$x_{6(n+2)-7} = x_{-7}^{-(n+1)}x_{-1}^{n+2}\prod_{k=0}^{n+1}\frac{a^k + x_{-7}x_{-5}x_{-3}\sum_{l=0}^{k-1}\left(ba^l\right)}{a^k + x_{-5}x_{-3}x_{-1}\sum_{l=0}^{k-1}\left(ba^l\right)} \tag{69b}$$

$$x_{6(n+2)-8} = x_{-8}^{-(n+1)} x_{-2}^{n+2} \prod_{k=0}^{n+1} \frac{a^k + x_{-8}x_{-6}x_{-4} \sum_{l=0}^{k-1} \left(ba^l\right)}{a^k + x_{-6}x_{-4}x_{-2} \sum_{l=0}^{k-1} \left(ba^l\right)} \tag{69c}$$

$$x_{6(n+2)-9} = x_{-9}^{-(n+1)} x_{-2}^{n+2} \prod_{k=0}^{n+1} \frac{a^k + x_{-9}x_{-7}x_{-5} \sum_{l=0}^{k-1} \left(ba^l\right)}{a^k + x_{-7}x_{-5}x_{-3} \sum_{l=0}^{k-1} \left(ba^l\right)} \tag{69d}$$

$$x_{6(n+1)-4} = \frac{x_{-4}x_{-6}^{n+1}x_{-8}^{n+1}}{x_{-2}^{n+1}x_{0}^{n+1}} \prod_{k=0}^{n} \frac{a^k + x_{-4}x_{-2}x_{0} \sum_{l=0}^{k-1} \left(ba^l\right)}{a^{k+1} + x_{-8}x_{-6}x_{-4} \sum_{l=0}^{k} \left(ba^l\right)}. \tag{69e}$$

$$x_{6(n+1)-5} = \frac{x_{-5}x_{-7}^{n+1}x_{-9}^{n+1}}{x_{-3}^{n+1}x_{-1}^{n+1}} \prod_{k=0}^{n} \frac{a^k + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{k-1} \left(ba^l\right)}{a^{k+1} + x_{-9}x_{-7}x_{-5} \sum_{l=0}^{k} \left(ba^l\right)} \tag{69f}$$

- When $a_n = 1$ and $b_n = 1$, then (69) we simplifies to give

$$x_{6(n+1)} = d^{-(n+1)} p^{n+2} \prod_{k=0}^{n+1} \frac{1 + (k)dfh}{1 + (k)fhp} \tag{70a}$$

$$x_{6(n+2)-7} = c^{-(n+1)} m^{n+2} \prod_{k=0}^{n+1} \frac{1 + (k)ceg}{1 + (k)egm} \tag{70b}$$

$$x_{(6n+2)-8} = b^{-(n+1)} h^{n+2} \prod_{k=0}^{n+1} \frac{1 + (k)bdf}{1 + (k)dfh} \tag{70c}$$

$$x_{6(n+2)-9} = a^{-(n+1)} h^{n+2} \prod_{k=0}^{n+1} \frac{1 + (k)ace}{1 + (k)ceg} \tag{70d}$$

$$x_{6(n+1)-4} = \frac{fd^{n+1}b^{n+1}}{h^{n+1}m^{n+1}} \prod_{k=0}^{n} \frac{1 + kfhp}{1 + (k+1)bdf} \tag{70e}$$

$$x_{6(n+1)-5} = \frac{ec^{n+1}a^{n+1}}{g^{n+1}m^{n+1}} \prod_{k=0}^{n} \frac{1+kegm}{1+(k+1)ace}. \tag{70f}$$

Therefore we have obtained the same results that Çinar, Gelisken and Özkan obtained for corollary 3.1.1 in [7].

- If $a_n = 1$ and $b_n = -1$, then

$$x_{6(n+1)} = d^{-(n+1)}p^{n+2} \prod_{k=0}^{n+1} \frac{1-(k)dfh}{1-(k)fhp} \tag{71a}$$

$$x_{6(n+2)-7} = c^{-(n+1)}m^{n+2} \prod_{k=0}^{n+1} \frac{1-(k)ceg}{1-(k)egm} \tag{71b}$$

$$x_{6(n+2)-8} = b^{-(n+1)}h^{n+2} \prod_{k=0}^{n+1} \frac{1-(k)bdf}{1-(k)dfh} \tag{71c}$$

$$x_{6(n+2)-9} = a^{-(n+1)}h^{n+2} \prod_{k=0}^{n+1} \frac{1-(k)ace}{1-(k)ceg} \tag{71d}$$

$$x_{6n+2} = \frac{fd^{n+1}b^{n+1}}{h^{n+1}m^{n+1}} \prod_{k=0}^{n} \frac{1-kfhp}{1-(k+1)bdf} \tag{71e}$$

$$x_{6n+1} = \frac{ec^{n+1}a^{n+1}}{g^{n+1}m^{n+1}} \prod_{k=0}^{n} \frac{1-kegm}{1-(k+1)ace}. \tag{71f}$$

Therefore we have obtained the same results that Çinar, Gelisken and Özkan obtained for corollary 3.2.1 in [7].

- If $a_n = -1$, then

$$x_{6(2n)} = d^{-(2n)}p^{2n+1} \left(\frac{-1+bdfh}{-1+bfhp} \right)^{n} \tag{72a}$$

$$x_{6(2n+1)} = d^{-(2n+1)}p^{2n+2} \left(\frac{-1+bdfh}{-1+bfhp} \right)^{n+1} \tag{72b}$$

$$x_{6(2n+2)-7} = c^{-(2n+1)} m^{2n+2} \left(\frac{-1+bceg}{-1+begm} \right)^{n+1} \tag{72c}$$

$$x_{6(2n+3)-7} = c^{-(2n+2)} m^{2n+3} \left(\frac{-1+bceg}{-1+begm} \right)^{n+1} \tag{72d}$$

$$x_{6(2n+2)-8} = b^{-(2n+1)} h^{2n+2} \left(\frac{-1+bbdf}{-1+bdfh} \right)^{n+1} \tag{72e}$$

$$x_{6(2n+3)-8} = b^{-(2n+2)} h^{2n+3} \left(\frac{-1+bbdf}{-1+bdfh} \right)^{n+1} \tag{72f}$$

$$x_{6(2n+2)-9} = a^{-(2n+1)} h^{2n+2} \left(\frac{-1+bace}{-1+bceg} \right)^{n+1} \tag{72g}$$

$$x_{6(2n+3)-9} = a^{-(2n+2)} h^{2n+3} \left(\frac{-1+bace}{-1+bceg} \right)^{n+1} \tag{72h}$$

$$x_{6(2n+1)-4} = \frac{fd^{2n+1}b^{2n+1}}{h^{2n+1}m^{2n+1}} \frac{(-1+bfhp)^n}{(-1+bbdf)^{n+1}} \tag{72i}$$

$$x_{6(2n+2)-4} = \frac{fd^{2n+2}b^{2n+2}}{h^{2n+2}m^{2n+2}} \left(\frac{-1+bfhp}{-1+bbdf} \right)^{n+1} \tag{72j}$$

$$x_{6(2n+1)-5} = \frac{ec^{2n+1}a^{2n+1}}{g^{2n+1}m^{2n+1}} \frac{(-1+begm)^n}{(-1+bace)^{n+1}}. \tag{72k}$$

$$x_{6(2n+2)-5} = \frac{ec^{2n+2}a^{2n+2}}{g^{2n+2}m^{2n+2}} \left(\frac{-1+begm}{-1+bace} \right)^{n+1}. \tag{72l}$$

For $a_n = -1$, $b_n = 1$ and $b_n = -1$ we obtain the same results that Çinar, Gelisken and Özkan obtained for corollary 3.3.1 and corollary 3.4.1 in [7].

Chapter 4

Applications of the ARS algorithm - Singularity Analysis

Many differential equations, do not have solutions in terms of known functions. Over many centuries, mathematicians have delved into different strategies to solve differential equations. We take the ARS algorithm, whereby a differential equation is declared integrable if it possesses the Painlevé property. In other words, if a singularity manifold is determined by $\phi(z_1, ..., z_n) = 0$ and $u = u(z_1, ..., z_n)$ is a solution of the partial differential equation, then we assume that $u = u(z_1, ..., z_n) = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j$, where $\phi = \phi(z_1, ..., z_n)$ and $u_j = u_j(z_1, ..., z_n)$ are analytic functions of $(z_1, ..., z_n)$ in a neighborhood of the manifold, and α and defines the recursion relations for u_j, $j = 0, 1, 2....$ The expansion must be about a movable pole-like singularity in the complex plane of the independent variable, x in our case, with the required number of arbitrary constants. The Laurent expansion implies that the solution is analytic except at the singularities.

In this chapter, prime denotes the differentiation of the dependent variable $y(x)$.

4.1 The Painlevé-Ince Equation

We study the Painlevé-Ince equation of second order, which results from the reduction of the famous Euler-Bernoulli Beam equation and is given by

$$y'' + 3yy' + y^3 = 0. \tag{72}$$

The Painlevé-Ince equation is linearisable by a point transformation and is a member of the Riccati hierarchy [17].

This equation is maximally symmetric and admits the 8 Lie point symmetries

$$
\begin{aligned}
X_1 &= \partial_x, \\
X_2 &= x\,\partial_x - y\,\partial_y, \\
X_3 &= y\,\partial_x - y^3\,\partial_y, \\
X_4 &= \frac{x^2}{2}\,\partial_x + (1 - xy)\,\partial_y, \\
X_5 &= xy\,\partial_x + (y^2 - xy^3)\,\partial_y, \\
X_6 &= \frac{x^2y}{2}\,\partial_x + \left(-\frac{1}{2}x^2y^3 + xy^2 - y\right)\,\partial_y, \\
X_7 &= \frac{1}{6}x^3y\partial_x + \left(-\frac{1}{6}x^3y^3 + \frac{1}{2}x^2y^2 - xy + \frac{2}{3}\right)\,\partial_y, \\
X_8 &= \left(\frac{1}{24}x^4y - \frac{1}{12}x^3\right)\,\partial_x + \left(\frac{1}{6}x - \frac{1}{24}y^3x^4 + \frac{1}{6}x^3y^2 - \frac{1}{4}x^2y\right)\,\partial_y.
\end{aligned}
\tag{73}
$$

Below, we show that Eq. (72) satisfies the ARS algorithm and passes the singularity test [18]. In fact, this equation is among the best examples of the application of the ARS algorithm. We determine the leading-order terms of the above equation by firstly substituting $y(x) = a(x - x_0)^p$ into (72), where x_0 is a constant and considered to be the location of the movable singularity.

We note that

$$y'(x) = ap(x - x_0)^{p-1} \tag{74}$$

$$y''(x) = ap(p - 1)(x - x_0)^{p-2} \tag{75}$$

then we have

$$ap^2(x - x_0)^{p-2} - ap(x - x_0)^{p-2} + 3a^2p(x - x_0)^{2p-1} + [a(x - x_0)^p]^3 = 0. \tag{76}$$

Now to find the values of p that balances the equation, we take the powers of $(x - x_0)$ and equate them and then solve for p, i.e

$$p - 2 = 2p - 1 \implies p = -1.$$

Consequently,

$$
\begin{aligned}
a(-1)^2(x-x_0)^{-1-2} - a(-1)(x-x_0)^{-1-2} + 3a^2(-1)(x-x_0)^{2(-1)-1} + [a(x-x_0)]^3 = 0 \\
a^3 - 3a^2 + 2a = 0 \\
a(a-1)(a-2) = 0
\end{aligned}
$$

then we will have that for the leading-order a, we have $a = 1$ or $a = 2$. This implies that the movable singularity is a simple pole and there are two possibilities for the leading-order behaviour. The arbitrary location of the movable singularity gives one of the constants of integrations. Since (72) is a second-order equation, the second constant of integration has to be determined from a series developed about the singularity. Now for $a = 1$, we take the truncated Laurent series given by

$$y(x) = a(x-x_0)^p + m(x-x_0)^{p+s} \tag{77}$$

and substitute into (72), to find

$$
\begin{aligned}
2\frac{a}{(x-x_0)^3} + \frac{m\,(x-x_0)^{s-1}\,(s-1)^2}{(x-x_0)^2} - \frac{m\,(x-x_0)^{s-1}\,(s-1)}{(x-x_0)^2} \\
+3\left(\frac{a}{x-x_0} + m\,(x-x_0)^{s-1}\right)\left(-\frac{a}{(x-x_0)^2} + \frac{m\,(x-x_0)^{s-1}\,(s-1)}{x-x_0}\right) \\
+\left(\frac{a}{x-x_0} + m\,(x-x_0)^{s-1}\right)^3.
\end{aligned} \tag{78}
$$

Here we take the coefficients of m and set them to be equal to zero to determine the resonances, we get

$$s^2 - 1 = 0$$

and solving for s we obtain $s = 1$ or $s = -1$. The latter value for s should always hold as it is associated with the movable singularity. The value $s = 1$, gives the term in the series at which the second constant of integration occurs.
For $s = 1$ we use the right Painlevé series which is given by

$$y(\chi) = a\chi^p + \sum_{s=1}^{\infty} a_s\chi^{p+s} = \chi^{-1} + \sum_{s=1}^{\infty} a_s\chi^{-1+s}, \tag{79}$$

where $\chi = x - x_0$, which is substituted into (72). Now we test for consistency by solving for some a's in the sum. This involves checking the correct number of arbitrary constants, of which we will have two arbitrary constants since the degree of the equation is two. Hence in the series (79), taking the first few terms of $y(x)$, i.e

$$y(\chi) = \chi^{-1} + a_1 + a_2\chi^1 + a_3\chi^2$$

and substituting them into (72), and then separating according to powers of χ, we have

$\chi^0 : 3a_1^2 + 3a_2 = 0 \implies a_2 = -a_1^2.$

$\chi^1 : a_1^3 + 9a_1a_2 + 8a_3 = 0 \implies a_3 = a_1^3,$
then clearly all other constants are a function of a_1, because of the consistency test.

Hence we may conclude that a_1 is an arbitrary constant, and the consistency test is passed as we have the correct number of arbitrary constants. Thus, the Painlevé-Ince equation of second-order possesses the Painlevé property.
For $a = 2$, one can do a similar analysis to the above.

4.2 Ivey's Equation

Ivey's nonlinear equation appears in space-charge theory, and is given by

$$y'' - \frac{y'^2}{y} + \frac{2}{x}y' + ky^2 = 0, \ k \text{ is a constant.} \tag{80}$$

This equation admits the Lie point symmetry

$$Y = x\partial_x - 2y\partial_y$$

for $k \neq 0$ and the set of 8 symmetries

$$\begin{aligned}
Y_1 &= x\ \partial_x, \\
Y_2 &= x^2\ \partial_x, \\
Y_3 &= y\ \partial_y, \\
Y_4 &= x^2 \ln y\ \partial_x, \\
Y_5 &= \frac{y}{x}\ \partial_y, \\
Y_6 &= y \ln y\ \partial_y, \\
Y_7 &= -\partial_x + \frac{y}{x} \ln y\ \partial_y, \\
Y_8 &= -x \ln y\ \partial_x + \ln(y)^2 y\ \partial_y,
\end{aligned} \tag{81}$$

for $k = 0$. This equation has not been previously analysed in terms of singularity analysis in the literature. We determine the leading-order of the above equation by firstly substituting

$$y(x) = a(x - x_0)^p \tag{82}$$

into (80), where x_0 is a constant and we find

$$\begin{aligned} ap^2(x-x_0)^{p-2} - ap(x-x_0)^{p-2} - \frac{a^2p^2(x-x_0)^{2(p-1)}}{a(x-x_0)^p} \\ + \frac{2}{x}ap(x-x_0)^{p-1} + k[a(x-x_0)^p]^2 = 0. \end{aligned} \tag{83}$$

We need to find the values of p from the dominant terms that balances the equation, we take the powers of $(x - x_0)$, equate them and then solve for p.
Thus

$$p - 2 = 2p \implies p = -2,$$

where all terms in (80) are dominant excluding the third term. We substitute $p = -2$ into the leading terms of (83) and then solve for a to find

$$ka^2 + 2a = 0. \tag{84}$$

Hence we have that $a = 0$(not relevant) or $a = -\frac{2}{k}$, which are the leading order values. For the purpose of this work, we must consider $a = -\frac{2}{k}$, and we take the truncated Laurent series given by

$$y(x) = a\chi^p + m\chi^{p+s}, \tag{85}$$

where $\chi = (x - x_0)$, that is

$$y(x) = -\frac{2}{k}\chi^{-2} + m\chi^{-2+s} \tag{86}$$

and substitute into the dominant terms of (80) to obtain

$$\begin{aligned} -12\,\frac{1}{k\,(x-x_0)^4} + \frac{m\,(x-x_0)^{-2+s}\,(-2+s)^2}{(x-x_0)^2} - \frac{m\,(x-x_0)^{-2+s}\,(-2+s)}{(x-x_0)^2} \\ -\left(4\,\frac{1}{k\,(x-x_0)^3} + \frac{m\,(x-x_0)^{-2+s}\,(-2+s)}{x-x_0}\right)^2 \left(-2\,\frac{1}{k\,(x-x_0)^2} + m\,(x-x_0)^{-2+s}\right)^{-1} \\ +k\left(-2\,\frac{1}{k\,(x-x_0)^2} + m\,(x-x_0)^{-2+s}\right)^2. \end{aligned} \tag{87}$$

We take the coefficient of m to be equal to zero, to get

$$s^2 - s - 2 = 0,$$

where we solve the resulting equation for s. Therefore the resonances are $s = -1$ and $s = 2$. For the value of s, which is -1, is generic in nature and as mentioned, meaning its properties hold for singularity and it is also related to the arbitrariness of the location of the movable singularity. For $s = 2$ we use the right Painlevé series to find y, which is given by

$$y(\chi) = a\chi^p + \sum_{s=1}^{\infty} a_s \chi^{p+s} = -\frac{2}{k}\chi^{-2} + \sum_{s=1}^{\infty} a_s \chi^{-2+s}, \tag{88}$$

which we test for consistency by solving for some a's in the sum. Unfortunately this process leads to no arbitrary constants, and we conclude that Ivey's equation does not pass the singularity test via the ARS algorithm.

When the singularity test fails for a differential equation, it does not mean that the equation is not integrable. Instead it means that the solution cannot be expressed by a Laurent expansion. Also, there exists the possibility that the equation does not admit any singularities. For example, the linear equation

$$y'' - y = 0,$$

that has solution

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

Finally, for $k = 0$, Ivey's equation does not possess enough dominant terms (two or more) to perform the singularity analysis.

4.3 The Higher-Order Lane-Emden Type Equation

A Lane-Emden-Fowler type equation may be defined in the form

$$y'' + \frac{k}{y}y' + f(x)g(y) = 0, \tag{89}$$

where $f(x)$ and $g(y)$ are arbitrary functions, and where $x > 0$. This equation possesses many applications in fluids, patterns, spatial ecology and mechanics, etc. The standard Lane-Emden equation, which arises when $f(x) = 1$ and $g(y) = y^n$ in (89) has been shown to pass the singularity test [19]. Our interest lies in the higher-order Lane-Emden equation [20], which is a third-order differential equation that is given by

$$y''' + \frac{8}{x}y'' + \frac{12}{x^2}y' + y^n = 0. \tag{90}$$

This equation has not been analysed from a singularity perspective before. For arbitrary n, this equation possesses the Lie point symmetry

$$Z_1 = x\partial_x - \frac{3y}{n-1}\partial_y,$$

but for $n = 1$, we have the Lie point symmetry $Z_2 = y\partial_y$ and some strange ∂_y symmetries with Hypergeometric functions for coefficients.
Firstly we have to substitute $y(x) = (x - x_0)^p$ into (90), noting that

$$y(x) = a(x - x_0)^p$$

$$y'(x) = ap(x - x_0)^{p-1}$$

$$y''(x) = ap(p-1)(p-2)(x - x_0)^{p-2}$$

$$y'''(x) = ap(p^2 - 3p + 2)(x - x_0)^{p-3}$$

then we will have

$$\begin{aligned} ap(p^2 - 3p + 2)x^5(x - x_0)^{p-3} + 8ap(p-1)x^4(x - x_0)^{p-2} \\ +12apx^3(x - x_0)^{p-1} + a^n x^5(x - x_0)^{np} = 0, \end{aligned} \tag{91}$$

from which we will compute the value(s) of p from the dominant terms as follows

$$ap(p^2 - 3p + 2)x^5(x - x_0)^{p-3} + a^n x^5(x - x_0)np = 0, \tag{92}$$

to get

$$p = -\frac{3}{n-1},$$

for all $n > 1$. We substitute p into (90) and consider the leading terms to get

$$ap(p^2 - 3p + 2)x^{p-3} + (ax^p)^n = 0 \tag{93}$$

which then gives for a,

$$a = \left[\frac{3(2+n)(1+2n)}{(n-1)^3}\right]^{\frac{1}{n-1}}. \tag{94}$$

Given a above, and that $y = a\,(x-x_0)^p + m\,(x-x_0)^{p+s}$ (77) we have

$$
\begin{aligned}
&-27\,\frac{1}{(n-1)^3\,(x-x_0)^3}\left(3\,\frac{(n+2)\,(2\,n+1)}{(n-1)^3}\right)^{(n-1)^{-1}}(x-x_0)^{-3\,(n-1)^{-1}}\\
&-27\,\frac{1}{(n-1)^2\,(x-x_0)^3}\left(3\,\frac{(n+2)\,(2\,n+1)}{(n-1)^3}\right)^{(n-1)^{-1}}(x-x_0)^{-3\,(n-1)^{-1}}\\
&-6\,\frac{1}{(n-1)\,(x-x_0)^3}\left(3\,\frac{(n+2)\,(2\,n+1)}{(n-1)^3}\right)^{(n-1)^{-1}}(x-x_0)^{-3\,(n-1)^{-1}}\\
&+\frac{m}{(x-x_0)^3}\,(x-x_0)^{-3\,(n-1)^{-1}+s}\left(-3\,(n-1)^{-1}+s\right)^3\\
&-3\,\frac{m}{(x-x_0)^3}\,(x-x_0)^{-3\,(n-1)^{-1}+s}\left(-3\,(n-1)^{-1}+s\right)^2\\
&+2\,\frac{m}{(x-k)^3}\,(x-x_0)^{-3\,(n-1)^{-1}+s}\left(-3\,(n-1)^{-1}+s\right)\\
&+\left(\left(3\,\frac{(n+2)\,(2\,n+1)}{(n-1)^3}\right)^{(n-1)^{-1}}(x-x_0)^{-3\,(n-1)^{-1}}+m\,(x-x_0)^{-3\,(n-1)^{-1}+s}\right)^n.
\end{aligned}
\tag{95}
$$

Taking the coefficients of m and equating them to zero we obtain

$$
\frac{((s-2)\,n-s-1)\,(sn-s-3)\,((s-1)\,n-s-2)}{(n-1)^3}=0. \tag{96}
$$

Solving, will give the following results

$$
s=\frac{3}{n-1},\quad s=\frac{n+2}{n-1},\quad s=\frac{2n+1}{n-1}.
$$

Since we know that one of the s variables should be equal to -1, then we can find all the possible values of n. Hence, we get $n = 0,\ -1/2,\ -2$. Clearly all these values of n give p as a positive integer, where the requisite value of p should be negative. Since we need p to be negative, we will recalculate its value, also a(94) and s from (77) by using $n = 2$ and $n = 4$

To ensure that p is a negative integer, we may select $n = 2$, then we will have $p = -3$, and $n = 4$ yields $p = -1$.

For the first case, $p = -3$, and $n = 4$, we find $a = 60$ and we must solve

$$
s^2 - 13s + 60 = 0, \tag{97}
$$

which unfortunately gives complex conjugate resonances. For $n = 4$ yields $p = -1$, we have that $a = 6^{\frac{1}{3}}$

$$
s^2 - 7s + 18 = 0, \tag{98}
$$

which again gives complex conjugate resonances. In such situations the reliability of the Painlevé test is known to be uncertain. Therefore we state that the integrability analysis for the higher-order Lane-Emden equation is inconclusive via the ARS algorithm.

4.4 Third-Order Boundary Flow Equations

Consider the class of equations related to boundary flow models, given by

$$hy''' + byy'' + cy'^2 + ky' = 0, \qquad h, b, c, k \text{ are constants, and } h \neq 0, \tag{99}$$

Some important equations of this general equation are

$$2y''' + yy'' = 0, \tag{100}$$

$$y''' + yy'' - y'^2 = 0, \tag{101}$$

$$y''' + yy'' - y'^2 - M^2 y' = 0, \quad M \text{ is a magnetic parameter}, \tag{102}$$

$$y''' + yy'' - \beta y'^2 = 0, \quad \beta \text{ is a constant}, \tag{103}$$

where, for example, (100) is the Blasius equation, (101)-(102) are Blasius type equations [21] and (103) is the Falkner-Skan equation.

The symmetries of (99) are

$$\begin{gathered}
\partial_x, \ \text{for } c = 0, \ \text{or } c = -\frac{3}{2}b, \ \text{or } h = 0, \ \text{or } b, c, h, k \neq 0, c \neq -\frac{3}{2}b. \\
\partial_x, \ \partial_y, \ \text{for } b = 0, \ \text{or } b, h = 0. \\
\partial_x, \ F(y)\partial_y, \ \text{for } b, c, h = 0. \\
\partial_x, \ x\partial_x - y\partial_y, \ \text{for } k = 0, \ \text{or } k, c = 0, \ \text{or } h, k = 0. \\
F(x)\partial_x, G(y, x)\partial_y, \ \text{for } b, c, h, k = 0. \\
\partial_x, \ \partial_y, \ x\partial_x - y\partial_y, \ \text{for } b, k = 0, \ \text{or } b, h, k = 0. \\
\partial_x, \ x^2 c\partial_x - (2cxy + 18h)\partial_y, \ x\partial_x - y\partial_y, \ \text{for } c = -\frac{3}{2}b, k = 0. \\
\partial_x, \ x^2\partial_x - 2xy\partial_y, \ x\partial_x - y\partial_y, \ \text{for } c = -\frac{3}{2}b, k, h = 0,
\end{gathered} \tag{104}$$

and lastly

$$\partial_x,\ \partial_y,\ y\partial_y,\ \cos\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y,\ \sin\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y,\ y\sqrt{k}\cos\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y$$
$$+\sqrt{h}\sin\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_x,\ y\sqrt{k}\sin\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_y-\sqrt{h}\cos\left(\frac{\sqrt{k}x}{\sqrt{h}}\right)\partial_x,\ \text{for } b,c=0. \tag{105}$$

In the specific cases Eq. (100), (101) and Eq. (102) ($M=0$), the Lie point symmetries are $\partial_x,\ x\partial_x-y\partial_y$, while for $M\neq 0$ in (102) we only have ∂_x. Eq. (103) admits the same two symmetries for $\beta\neq\frac{3}{2}$ but an extra third symmetry $(xy-6)\partial_y-\frac{1}{2}x^2\partial_x$ for $\beta=\frac{3}{2}$.

A singularity analysis of (99), reveals that $p=-1$ where all terms are dominant excluding the last term in the equation. This p value yields that

$$a=6\,\frac{h}{2\,b+c}.$$

Substitution of the truncated series

$$y\left(x\right)=6\,\frac{h}{\left(2\,b+c\right)\left(x-x_0\right)}+m\left(x-x_0\right)^{-1+s},$$

into the dominant terms of (99), gives a polynomial in m. Taking the coefficients of m as vanishing, we find the equation

$$2\left(s+1\right)\left(\left(b+c/2\right)s^2+\left(-4\,b-7/2\,c\right)s+6\,b+3\,c\right)h=0,$$

which one solves to find the three resonances

$$s=-1,\ s=\frac{1}{4\,b+2\,c}\left(8\,b+7\,c+\pm\sqrt{-32\,b^2+16\,bc+25\,c^2}\right).$$

Recall that the acceptable values for the resonance s, must be real and rational. Note that there are many possibilities for the second and third resonance to be positive, negative or complex. To establish the particular values of the resonance and further progress to the last step the method, we may consider particular values of the free parameters, which facilitate the consistency test for the constants of integration.

Testing equation (100) - (102) leads to complex conjugate resonances, so we conclude that the Painlevé test is unreliable in these cases. For equation (103), we find that the second and third resonances are

$$s=\frac{1}{4-2\,\beta}\left(8-7\,\beta+\pm\sqrt{25\,\beta^2-16\,\beta-32}\right).$$

Taking $\beta=-1$, since the resonance s must be real and rational, we have that $s=2$ and $s=3$. Regarding the consistency test, we substitute

$$y\left(x\right)=\frac{2}{x-x_0}+a_1+a_2(x-x_0)+a_3(x-x_0)^2,$$

into (103), and find that $a_1 = 0$ while a_2 and a_3 are arbitrary constants. Hence, we have the correct number of constants and we conclude that that Falkner-Skan equation passes the singularity test via the ARS algorithm.

Chapter 5

Conclusion

In our first investigation, we performed a Lie analysis of a tenth-order difference equation. We then performed group reductions of the equation using the symmetries of the problem under investigation and the solutions were all given in a unified manner. We separate the single solutions that we obtained into categories with the aim to realize some results that were obtained in the existing literature.

In the second part of this dissertation, we listed the Lie point symmetries and studied integrability analysis of ordinary differential equations via the ARS algorithm. We investigated the Painlevé-Ince equation, Ivey's equation, the higher-order Lane-Emden type equations and some equations related to boundary flow. This algorithm has been used extensively on a wide range of ordinary differential equations to test for singularities and also to find general solutions. A catalogue of equations exist, where equations are deemed Painlevé integrable or not. In the spirit of this catalogue, we have studied the algorithm and applied the method to several equations whose status was unknown in terms of singularity analysis. In showcasing the method, we first considered the famous Painlevé-Ince equation. We then discovered that the singularity test was unsuccessful for Ivey's equation due to a lack of a second arbitrary constant, whilst for the higher-order Lane-Emden type equation, was inconclusive. The latter, because of the presence of complex conjugate resonances. Thereafter, the method was successfully applied to a member of third-order boundary flow equations. We strongly believe that this work can be of high importance in the study of various differential problems and their future advancement.

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