Flow of a thin ribbon of molten glass on a bath of molten tin

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Declaration

I declare that the content of this dissertation is original except where due references have been made. It has not been submitted before for any degree to any other institution.

Zinhle B. Sangweni 07 November 2016

Abstract

The equations for the flow of a thin film of molten glass on a bath of molten tin are extended to the case in which the viscosity of the molten glass depends on the temperature. The continuity equation for an incompressible fluid, the Navier-Stokes equation and the energy balance equation are written in the lubrication (thin fluid film) approximation. The kinematic boundary condition and the boundary conditions for the normal and tangential stress and the normal heat flux are derived on the upper and lower surfaces of the glass ribbon. It is found for the lubrication approximation that only one equation is obtained for four unknowns which are the two horizontal velocity components, the absolute temperature difference and the thickness of the molten glass ribbon. The remaining three equations are obtained by taking the calculation to the next order in the square of the ratio of the thickness to length of the glass ribbon. The kinematic edge condition and the edge conditions for the normal and tangential stress and the normal heat flux are derived. The four edge conditions and the boundary conditions at the inlet and outlet give the boundary conditions for the four partial differential equations. It is not the aim of the dissertation to solve the boundary value problem which has been derived, either numerically or analytically.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

A float glass furnace is the modern way of producing large quantities of glass. The float glass manufacturing process was developed by Pilkington in 1959, taking flat glass technology to a revolutionary new level [1]. The glass gains its lustrous finish and perfect flatness by floating on a bath of molten tin in a chemically controlled atmosphere. The ribbon of glass is then cooled, while still moving, until the surfaces are hard enough for it to be taken out of the bath without the roller marking the surface. The glass is then automatically cut and stacked, ready to be packed for distribution to local and international customers. The users of the glass are: processors, merchants, appliance manufacturers and furniture manufacturers, as well as producers of automotive glass products. The increasing use of glass has led to the application of more analytical and numerical techniques to improve the quality of the glass produced.

This dissertation is an extension of the work that was done by Howell [2] who solved equations derived from the Navier-Stokes equation for slender threads and sheets of viscous fluid. He considered the two-dimensional slow flow of a Newtonian fluid of constant viscosity between two surfaces. The kinematic boundary condition was imposed on the upper surface while the condition of zero tangential stress was imposed on the lower surface. Because of the high temperature in the float glass process the surface tension is neglected except at the edge of the glass ribbon flow. He derived a system of partial differential equations governing the stretching of a viscous fibre and a two-dimensional thin sheet. The model is known as the Trouten Model [3]. The most crucial component of this dissertation is that viscosity is dependent on temperature, whereas in the work that was done by Howell viscosity is assumed to be constant. In this dissertation the glass ribbon flow is modeled with a temperature dependent viscosity which is given by:

$$\mu(T) = Kexp\left[\frac{E_0}{T - T_0}\right] \tag{1.1}$$

where $T(x, y, z, t)$ denotes absolute temperature, T_0 is the melting temperature and E_0 is a constant. Another investigation that has been considered is the one-dimensional model including inertia effects which was analyzed using finite-element methods by Motoichi and Hiroshi [4]. Foster [5] explored the dynamics of films of floam known as logs, that spread across the surface of the pool of molten glass. He considered three cases : when the problem has a large Reynolds number, the flow of slender geometry films and the two-dimensional nature of the flow allowing the boundary integral method to be employed. Another investigation has been made by Gramberg et al [6] who considered the structure of slow two-dimensional convective flow in a rectangular container, the upper horizontal boundary of which was subject to a prescribed temperature gradient. In this work they studied convective flows that are driven by horizontal non-uniform heating from above and assumed that the viscosity and the thermal conductivity are constant and that the Boussinesq approximation is valid.

Figure 1.1 gives an overview of the float glass process that is used to produce the glass. Once the raw material is put into the furnace, the furnace melts it to form a glass melt. The glass melt ($1100°C$) is poured on a bath of molten tin. The molten glass spreads out to form a perfect flat sheet. The width and the thickness of the glass ($900°C$) is adjusted by top rollers. The glass which is at $600°C$ is then lifted from the tin bath on to a conveyor.

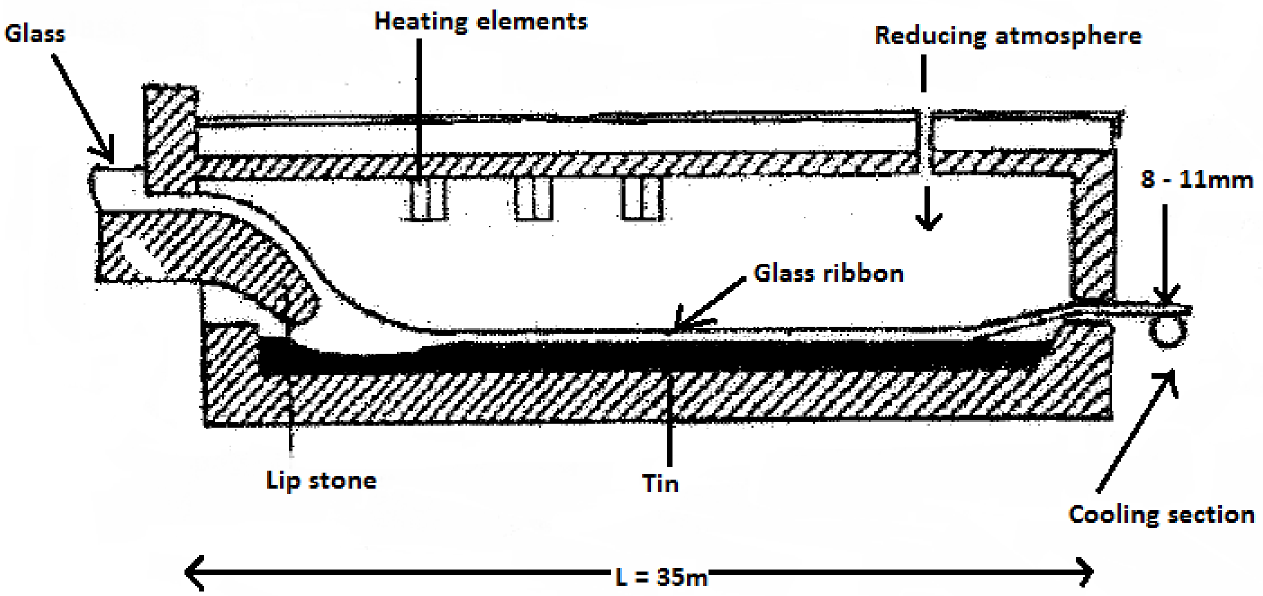

Figure 1.1: Float glass process. Adapted from Pilkington[1].

In Table 1.1 some of the fluid variables and characteristic quantities that are used throughout the dissertation are listed. In Appendix B an estimate of the magnitude of the physical parameters and dimensionless numbers is given.

Variable	**Description**
u	velocity in the x-direction
v	velocity in the y-direction
w	velocity in the z-direction
t	time
p	pressure
μ^*	shear viscosity of glass at some reference temperature
g	acceleration due gravity
L	characteristic length in x and y-directions
H^*	characteristic height in z-direction
ρ_G	density of glass
ρ_T	density of tin
v^*	kinematic viscosity of glass at some reference temperature
$Re = \frac{\rho_G U L}{\mu^*}$	Reynolds number for glass ribbon

Table 1.1: Fluid variables and physical quantities in the float glass process.

The objective of the dissertation is to derive the thin fluid film equations and edge conditions for the flow of a thin ribbon of molten glass on a bath of molten tin including temperature dependence of the viscosity of the molten glass. It is not an objective to solve either numerically or analytically, the boundary value problem which is obtained.

This dissertation is outlined as follows. In Chapter 2, the mathematical model is derived. The thin fluid film (lubrication) approximation is imposed. The kinematic boundary condition and the normal and tangential stress boundary conditions on the upper and lower surfaces of the glass ribbon are derived. In the Chapter 3 the energy equation for the temperature is derived in the thin fluid film approximation. The thermal boundary conditions on the upper and lower surfaces are derived. In Chapter 4 the thin fluid film equations are considered. They are reduced to one equation for four unknown functions. In Chapter 5 the remaining three equations are obtained by considering a perturbation expansion and terms in the next order of approximation. In Chapter 6 four conditions at the edge of the glass ribbon are derived and the boundary conditions at the inlet and outlet and the initial conditions are stated. In Chapter 7 the thin fluid film equations are summarized and conclusions are made. In Appendix A an outline of the derivation of the energy equation is given. In Appendix B the values of the physical quantities and dimensionless numbers are listed.

Through out the text there are several places where the dimensionless numbers are required to be small. These assumptions can be checked by referring to the values of the dimensionless numbers in Table $B.1$ in Appendix B.

Chapter 2

Mathematical model

2.1 Introduction

We consider the two-dimensional slow flow of Newtonian fluid with viscosity depending on temperature which models the flow of glass melt on a bath of molten tin. Due to the high temperature in the glass ribbon, surface tension can be neglected except at the edge of the ribbon, the edge conditions are considered in Chapter 6. The upper surface of the fluid is given by

$$z = H(x, y, t) + \frac{1}{2}h(x, y, t) \tag{2.1}$$

and lower surface by

$$z = H(x, y, t) - \frac{1}{2}h(x, y, t), \tag{2.2}$$

as illustrated in Figure 1.1 where

$$z = H(x, y, t) \tag{2.3}$$

denotes the equation of the centre-line and $h(x, y, t)$ is the thickness of the fluid film. At the edge of the glass ribbon, $h = 0$ and therefore $H = 0$. We let

$$\epsilon = \frac{H^*}{L} \ll 1, \tag{2.4}$$

where,from Table 1.1, H^* is the characteristic length in the z-direction and L is the characteristic length in the x- and y- directions.

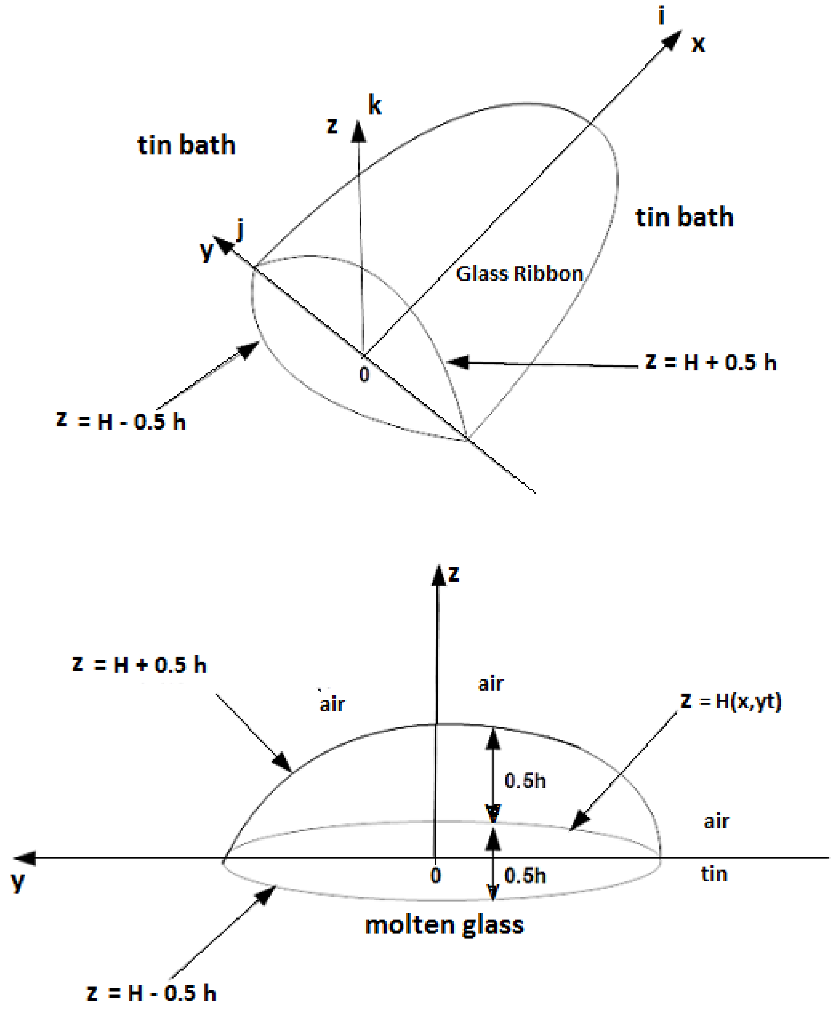

Figure 2.1: Coordinate system and centre-line, $z = H(x, y, t)$, for a glass ribbon on a bath of molten tin. At the edge of the glass ribbon, $H = 0$ and $h = 0$.

2.2 Dimensionless equations

The fluid glass melt is assumed to be incompressible. The flow of the glass melt is governed by the conservation of mass equation and the Navier-Stokes equation[7]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \ , \tag{2.5}$$

$$\rho_G \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right)$$
$$= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \tag{2.6}$$

$$\rho_G \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right)$$
$$= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) \tag{2.7}$$

$$\rho_G \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right)$$
$$= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - \rho_G g, \tag{2.8}$$

where $\mu = \mu(x, y, z, t)$ is the shear viscosity of the glass which depends on x,y and z through the temperature. We introduce the dimensionless variables denoted by an overhead bar and defined as follows:

$$x = L\bar{x}, \quad z = \epsilon L\bar{z}, \quad u = U\bar{u}, \quad v = U\bar{v}$$

$$w = \epsilon U\bar{w}, \quad p = \rho_T g\epsilon L\bar{p}, \quad t = \frac{L}{U}\bar{t}, \quad \mu = \mu^*\bar{\mu}. \tag{2.9}$$

The characteristic pressure in the glass ribbon is $\rho_T g H^* = \rho_T g\epsilon L$ which is an estimate of the pressure exerted on the lower surface of the glass ribbon by the molten tin in the bath. The dimensionless conservation of mass and Navier-Stokes equations are given by:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{2.10}$$

$$\epsilon^2 Re\left[\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right]$$
$$= -\epsilon^2 \frac{\rho_T}{\rho_G} A \frac{\partial \bar{p}}{\partial \bar{x}} + \epsilon^2 \frac{\partial}{\partial \bar{x}}\left(\mu \frac{\partial \bar{u}}{\partial \bar{x}}\right) + \epsilon^2 \frac{\partial}{\partial \bar{y}}\left(\mu \frac{\partial \bar{u}}{\partial \bar{y}}\right) + \frac{\partial}{\partial \bar{z}}\left(\mu \frac{\partial \bar{u}}{\partial \bar{z}}\right), \tag{2.11}$$

$$\epsilon^2 Re\left[\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{z}}\right]$$
$$= -\epsilon^2 \frac{\rho_T}{\rho_G} A \frac{\partial \bar{p}}{\partial \bar{y}} + \epsilon^2 \frac{\partial}{\partial \bar{x}}\left(\mu \frac{\partial \bar{v}}{\partial \bar{x}}\right) + \epsilon^2 \frac{\partial}{\partial \bar{y}}\left(\mu \frac{\partial \bar{v}}{\partial \bar{y}}\right) + \frac{\partial}{\partial \bar{z}}\left(\mu \frac{\partial \bar{v}}{\partial \bar{z}}\right), \tag{2.12}$$

$$\epsilon^2 Re\left[\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right]$$
$$= -\frac{\rho_T}{\rho_G} A \frac{\partial \bar{p}}{\partial \bar{z}} - A + \epsilon^2 \frac{\partial}{\partial \bar{x}}\left(\mu \frac{\partial \bar{w}}{\partial \bar{x}}\right) + \epsilon^2 \frac{\partial}{\partial \bar{y}}\left(\mu \frac{\partial \bar{w}}{\partial \bar{y}}\right) + \frac{\partial}{\partial \bar{z}}\left(\mu \frac{\partial \bar{w}}{\partial \bar{z}}\right), \tag{2.13}$$

where the Reynolds number of the flow of the glass melt,Re, and the dimensionless number A are defined by

$$Re = \frac{\rho_G g U L}{\mu^*}, \tag{2.14}$$

$$A = \frac{\epsilon \rho_G g L^2}{\mu^* U}. \tag{2.15}$$

We obtain the dimensionless number A by considering the relative order of magnitude of the gravity and viscous terms. Considering the z-component of the Navier-Stokes equation, the gravitational force per unit volume is given by $\rho_G g$ and the viscous force per unit volume is given by the dominant viscous term

$$\frac{\partial}{\partial z}\big(\mu \frac{\partial w}{\partial z}\big) \simeq \frac{\mu^* W}{H^{*2}} = \frac{\mu^* U}{\epsilon L^2}. \tag{2.16}$$

Therefore the relative magnitude of gravity and viscous force equals

$$\frac{gravity\ force}{viscous\ force} \sim \frac{\rho_G g H^{*2}}{\mu^* W} \sim \frac{\epsilon \rho_G g L^2}{\mu^* U}. \tag{2.17}$$

We assume the lubrication or thin fluid film approximation [8]

$$\epsilon = \frac{H^*}{L} \ll 1, \quad \epsilon^2 Re \ll 1. \tag{2.18}$$

We also assume that the gravity force and the viscous force are the same order of magnitude so that A is order of magnitude one. The continuity equation and the three components of the Navier-Stokes equation reduce to

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{2.19}$$

$$\frac{\partial}{\partial \bar{z}}\left(\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{z}}\right) = 0, \tag{2.20}$$

$$\frac{\partial}{\partial \bar{z}}\left(\bar{\mu}\frac{\partial \bar{v}}{\partial \bar{z}}\right) = 0, \tag{2.21}$$

$$-\frac{\rho_T}{\rho_G} A \frac{\partial \bar{\rho}}{\partial \bar{z}} - A + \frac{\partial}{\partial \bar{z}}\left(\bar{\rho}\frac{\partial \bar{w}}{\partial \bar{z}}\right) = 0 \tag{2.22}$$

2.3 Kinematic boundary conditions

Depicted in Figure 2.1 is the glass ribbon showing both the upper and the lower surfaces and the unit vectors **i**, **j** and **k**. We consider the upper surface and the lower surface of the glass ribbon. The following kinematic boundary conditions were identified on each surface.

On upper surface

$$z = H(x, y, t) + \frac{1}{2}h(x, y, t), \tag{2.23}$$

the dimensional boundary condition is the kinematic condition which states that a fluid particle on the surface remains on the surface as the fluid evolves in time:

$$\frac{D}{Dt}\left[z-\left(H(x,y,t)+\frac{1}{2}h(x,y,t)\right)\right]\Bigg|_{z=H+\frac{1}{2}h}=0, \tag{2.24}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u(x,y,z,t)\frac{\partial}{\partial x}+v(x,y,z,t)\frac{\partial}{\partial y}+w(x,y,z,t)\frac{\partial}{\partial z} \tag{2.25}$$

is the total time derivative. Expanding (2.24) gives

$$\begin{gathered}w(x,y,H+\frac{1}{2}h,t)=\frac{\partial H}{\partial t}+\frac{1}{2}\frac{\partial h}{\partial t}\\ +u(x,y,H+\frac{1}{2}H,t)\frac{\partial}{\partial x}\left(H+\frac{1}{2}h\right)+v(x,y,H+\frac{1}{2}h,t)\frac{\partial}{\partial y}\left(H+\frac{1}{2}h\right).\end{gathered} \tag{2.26}$$

On the lower surface

$$z=H(x,y,t)-\frac{1}{2}h(x,y,t), \tag{2.27}$$

the dimensional boundary condition is again the kinematic condition that a fluid particle on the surface remains on the surface for all time:

$$\begin{gathered}w(x,y,H-\frac{1}{2}h,t)=\frac{\partial H}{\partial t}-\frac{1}{2}\frac{\partial h}{\partial t}\\ +u(x,y,H-\frac{1}{2}H,t)\frac{\partial}{\partial x}\left(H-\frac{1}{2}h\right)+v(x,y,H-\frac{1}{2}h,t)\frac{\partial}{\partial y}\left(H-\frac{1}{2}h\right).\end{gathered} \tag{2.28}$$

There is no slip between the molten glass and the molten tin. The molten tin is therefore not at rest. It is drawn along by the glass. Thus

$$u(x,y,H-\frac{1}{2}h,t)\neq 0, \quad v(x,y,H-\frac{1}{2}h,t)\neq 0. \tag{2.29}$$

When expressed in terms of the dimensionless variables in (2.9) the kinematic boundary conditions on the upper and lower surfaces become respectively.

$$\bar{w}(\bar{x},\bar{y},\bar{H}+\frac{1}{2}\bar{h},\bar{t})=\frac{\partial\bar{H}}{\partial\bar{t}}+\frac{1}{2}\frac{\partial\bar{h}}{\partial\bar{t}}$$
$$+\bar{u}(\bar{x},\bar{y},\bar{H}+\frac{1}{2}\bar{h},\bar{t})\frac{\partial}{\partial\bar{x}}\left(\bar{H}+\frac{1}{2}\bar{h}\right)+\bar{v}(\bar{x},\bar{y},\bar{H}+\frac{1}{2}\bar{h},\bar{t})\frac{\partial}{\partial\bar{y}}\left(\bar{H}+\frac{1}{2}\bar{h}\right) \tag{2.30}$$

and

$$\bar{w}(\bar{x},\bar{y},\bar{H}-\frac{1}{2}\bar{h},\bar{t})=\frac{\partial\bar{H}}{\partial\bar{t}}-\frac{1}{2}\frac{\partial\bar{h}}{\partial\bar{t}}$$
$$+u(\bar{x},\bar{y},\bar{H}-\frac{1}{2}\bar{h},\bar{t})\frac{\partial}{\partial\bar{x}}\left(\bar{H}-\frac{1}{2}\bar{h}\right)+\bar{v}(\bar{x},\bar{y},\bar{H}-\frac{1}{2}\bar{h},\bar{t})\frac{\partial}{\partial\bar{y}}\left(\bar{H}-\frac{1}{2}\bar{h}\right). \tag{2.31}$$

2.4 Unit normal and tangential vectors to upper and lower surfaces

In order to calculate the stress on the upper and lower surfaces of the glass ribbon the unit normal vector $\mathbf{n}$ and the unit tangent vectors in two perpendicular directions, $\mathbf{t}^1$ and $\mathbf{t}^2$, are required at each point on the two surfaces. These vectors are derived in the section.

Consider first the upper surface. Its equation (2.23) can be written as

$$\phi(x,y,z,t)=z-H(x,y,t)-\frac{1}{2}h(x,y,t)=0. \tag{2.32}$$

Now for a displacement on the upper surface,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \tag{2.33}$$

and therefore the vectors

$$\mathbf{a} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}, \quad \mathbf{b} = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}, \tag{2.34}$$

are orthogonal. Since $\mathbf{b}$ lies in the tangent plane at the point (x, y, z) it follows that $\mathbf{a}$ is normal to the tangent plane at (x, y, z). From (2.32),

$$\frac{\partial \phi}{\partial x} = -\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right), \quad \frac{\partial \phi}{\partial y} = -\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right), \quad \frac{\partial \phi}{\partial z} = 1 \tag{2.35}$$

and therefore the unit normal vector $\mathbf{n}$ to the tangent plane at (x, y, z) is

$$\mathbf{n} = \frac{-\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\mathbf{i} - \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)^2 + 1}}. \tag{2.36}$$

Consider now the unit tangent vector $\mathbf{t}^{(1)}$ which lies in the xz-plane. Then

$$\mathbf{t}^{(1)} = \frac{\alpha\mathbf{i} + \beta\mathbf{k}}{\sqrt{\alpha^2 + \beta^2}} \tag{2.37}$$

where α and β are constants to be determined. Now

$$\mathbf{t}^{(1)} \cdot \mathbf{n} = 0 \tag{2.38}$$

gives

$$\beta = \alpha\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right). \tag{2.39}$$

Hence

$$\mathbf{t}^{(1)} = \frac{\mathbf{i} + \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\mathbf{k}}{\sqrt{1 + \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}}. \tag{2.40}$$

Similarly the unit tangent vector $\mathbf{t}^{(2)}$ which lies in the yz- plane is

$$\mathbf{t}^{(2)} = \frac{\mathbf{j} + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\mathbf{k}}{\sqrt{1 + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)^2}}. \tag{2.41}$$

Expressed in terms of the dimensionless variables (2.9) the unit normal vector $\mathbf{n}$ and the unit tangent vectors $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ become

$$\mathbf{n} = \frac{-\epsilon\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\mathbf{i} - \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)\mathbf{j} + \mathbf{k}}{\sqrt{1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)^2}}, \tag{2.42}$$

$$\mathbf{t}^{(1)} = \frac{\mathbf{i} + \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\mathbf{k}}{\sqrt{1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2}}, \tag{2.43}$$

$$\mathbf{t}^{(2)} = \frac{\mathbf{j} + \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)\mathbf{k}}{\sqrt{1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)^2}}. \tag{2.44}$$

Also

$$\mathbf{t}^{(1)} \cdot \mathbf{t}^{(2)} = \frac{\epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)}{\sqrt{\left(1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2\right)\left(1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)^2\right)}} \tag{2.45}$$

and therefore when terms of order ϵ^2 are neglected, the unit tangent vectors $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ are orthogonal.

For the lower surface,

$$\phi(x, y, z, t) = z - H(x, y, t) + \frac{1}{2}h(x, y, t) = 0. \tag{2.46}$$

The unit outward normal vector is

$$\mathbf{n} = \frac{-\frac{\partial \phi}{\partial x}\mathbf{i} - \frac{\partial \phi}{\partial y}\mathbf{j} - \frac{\partial \phi}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}}. \tag{2.47}$$

In the same way as for the upper surface it can be shown for the the lower surface that the unit outward normal vector $\mathbf{n}$ and the unit tangent vector $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ in the xz-plane and yz-plane expressed in dimensionless variables are

$$\mathbf{n} = \frac{\epsilon\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\mathbf{i} + \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)\mathbf{j} - \mathbf{k}}{\sqrt{1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)^2}}, \tag{2.48}$$

$$\mathbf{t}^{(1)} = \frac{\mathbf{i} + \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\mathbf{k}}{\sqrt{1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)^2}}, \tag{2.49}$$

$$\mathbf{t}^{(2)} = \frac{\mathbf{j} + \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)\mathbf{k}}{\sqrt{1 + \epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)^2}}. \tag{2.50}$$

When terms of the order ϵ^2 are neglected the unit tangent vectors $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ are orthogonal.

2.5 Cauchy stress tensor for fluid glass ribbon

The Navier-Poisson law for the Cauchy stress tensor for an incompressible fluid is

$$\tau_{ik} = -p\delta_{ik} + 2\mu D_{ik}, \tag{2.51}$$

where D_{ik} is the rate-of-strain tensor.

$$D_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \tag{2.52}$$

We write the components of the Navier-Poisson law (2.52) in terms of the dimensionless variables (2.9). The characteristic stress is the same as the characteristic pressure so that

$$\tau_{ik} = \rho_T g \epsilon L \bar{\tau}_{ik}. \tag{2.53}$$

It can be shown that

$$\bar{\tau}_{11} = -\bar{p} + \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{x}}, \tag{2.54}$$

$$\bar{\tau}_{22} = -\bar{p} + \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{v}}{\partial \bar{y}}, \tag{2.55}$$

$$\bar{\tau}_{33} = -\bar{p} + \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{w}}{\partial \bar{z}}, \tag{2.56}$$

$$\bar{\tau}_{12} = \frac{1}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right), \tag{2.57}$$

$$\bar{\tau}_{13} = \frac{1}{\epsilon A}\frac{\rho_G}{\rho_T}\bar{\mu}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \epsilon^2\frac{\partial \bar{w}}{\partial \bar{x}}\right), \tag{2.58}$$

$$\bar{\tau}_{23} = \frac{1}{\epsilon A}\frac{\rho_G}{\rho_T}\bar{\mu}\left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2\frac{\partial \bar{w}}{\partial \bar{y}}\right), \tag{2.59}$$

2.6 Normal and tangential stress applied on upper surface

In this section we consider the upper surface of the fluid glass ribbon and derive expressions for the normal and tangential applied stress on the surface. Cauchy's formula for the stress vector $\mathbf{T}(\mathbf{n})$ on a surface element with unit outward normal vector $\mathbf{n}$ is

$$T_i(\mathbf{n}) = n_k \tau_{ki}. \tag{2.60}$$

Thus the normal stress N applied to the surface element in the direction of the outward normal $\mathbf{n}$ is

$$N = \mathbf{n} \cdot \mathbf{T}(\mathbf{n}) = n_i T_i(\mathbf{n}) = n_i n_k \tau_{ki}. \tag{2.61}$$

Hence expanding (2.61) and using dimensionless variables,

$$\bar{N} = {n_1}^2 \bar{\tau}_{11} + {n_2}^2 \bar{\tau}_{22} + {n_3}^2 \bar{\tau}_{33} + 2n_1 n_2 \bar{\tau}_{12} + 2n_1 n_3 \bar{\tau}_{13} + 2n_2 n_3 \bar{\tau}_{23}. \tag{2.62}$$

We calculate $\bar{N}$ correct to order ϵ. For the upper surface, from (2.42),

$$n_1 = -\epsilon \left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2} \frac{\partial \bar{h}}{\partial \bar{x}} \right) + O(\epsilon^3)),\ n_2 = -\epsilon \left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2} \frac{\partial \bar{h}}{\partial \bar{y}} \right) + O(\epsilon^3),\ n_3 = 1 + O(\epsilon^3). \tag{2.63}$$

Thus using the dimensionless stress components τ_{ik} derived in Section 2.5 we find that

$$ {n_1}^2 \bar{\tau}_{11} = O(\epsilon^2), \tag{2.64}$$

$$ {n_2}^2\bar{\tau}_{22} = O(\epsilon^2), \tag{2.65} $$

$$ {n_3}^2\bar{\tau}_{33} = -\bar{p} + \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{w}}{\partial \bar{z}} + O(\epsilon^2), \tag{2.66} $$

$$ 2n_1n_2\bar{\tau}_{12} = O(\epsilon^2), \tag{2.67} $$

$$ 2n_1n_3\bar{\tau}_{13} = -\frac{2}{A}\frac{\rho_G}{\rho_T}\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon^2), \tag{2.68} $$

$$ 2n_2n_3\bar{\tau}_{23} = -\frac{2}{A}\frac{\rho_G}{\rho_T}\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)\bar{\mu}\frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon^2). \tag{2.69} $$

Thus

$$ \bar{N} = -\bar{p} - \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\left[\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right)\frac{\partial \bar{u}}{\partial \bar{z}} + \left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)\frac{\partial \bar{v}}{\partial \bar{z}} - \frac{\partial \bar{w}}{\partial \bar{z}}\right] + O(\epsilon^2). \tag{2.70} $$

Consider next, $T^{(\alpha)}$, the component of the tangential stress in the direction of the unit vector $\mathbf{t}^{(\alpha)}$ where α takes the values 1 and 2. Now

$$ T^{(\alpha)} = t^{(\alpha)} \cdot \mathbf{T}(\mathbf{n}) = {\mathbf{t}_i}^{(\alpha)}T_i(\mathbf{n}) = {t_i}^{(\alpha)}n_k\tau_{ki}. \tag{2.71} $$

Expanding (2.71) gives, expressed in dimensionless variables,

$$ \bar{T}^{(\alpha)} = n_1{t_1}^{(\alpha)}\bar{\tau}_{11} + n_2{t_2}^{(\alpha)}\bar{\tau}_{22} + n_3{t_3}^{(\alpha)}\bar{\tau}_{33} + (n_1{t_2}^{(\alpha)} + n_2{t_1}^{(\alpha)})\bar{\tau}_{12} $$

$$ +(n_1{t_3}^{(\alpha)} + n_3{t_1}^{(\alpha)})\bar{\tau}_{13} + (n_2{t_3}^{(\alpha)} + n_3{t_2}^{(\alpha)})\bar{\tau}_{23}. \tag{2.72} $$

Consider first $\alpha = 1$. Neglecting terms of order ϵ^2 it follows from (2.43) that

$$t_1{}^{(1)} = 1 + O(\epsilon^2), \quad t_2{}^{(1)} \equiv 0, \quad t_3{}^{(1)} = \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right) + O(\epsilon^3). \tag{2.73}$$

We evaluate (2.72) using the dimensionless components $\bar{\tau}_{ik}$ derived in Section 2.5, equation (2.63) for $\mathbf{n}$ and (2.73) for $\mathbf{t}^{(1)}$. It is found that the largest term is of order $\frac{1}{\epsilon}$. We therefore calculate $T^{(1)}$ correct to zero order in ϵ . Now

$$n_1 t_1{}^{(1)} \bar{\tau}_{11} = O(\epsilon), \tag{2.74}$$

$$n_2 t_2{}^{(1)} \bar{\tau}_{22} = 0, \tag{2.75}$$

$$n_3 t_3{}^{(1)} \bar{\tau}_{33} = O(\epsilon), \tag{2.76}$$

$$(n_1 t_2{}^{(1)} + n_2 t_1{}^{(1)}) \bar{\tau}_{12} = O(\epsilon), \tag{2.77}$$

$$(n_1 t_3{}^{(1)} + n_3 t_1{}^{(1)}) \bar{\tau}_{13} = \frac{1}{\epsilon A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon), \tag{2.78}$$

$$(n_2 t_3{}^{(1)} + n_3 t_2{}^{(1)}) \bar{\tau}_{32} = O(\epsilon). \tag{2.79}$$

Thus

$$\bar{T}^{(1)} = \mathbf{t}^{(1)} \cdot \bar{\mathbf{T}}(\mathbf{n}) = \frac{1}{\epsilon A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon). \tag{2.80}$$

Consider next $\alpha = 2$. Neglecting terms of order ϵ^2, (2.44) gives

$$t_1{}^{(2)} \equiv 0, \quad t_2{}^{(2)} = 1 + O(\epsilon^2), \quad t_3{}^{(2)} = \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right) + O(\epsilon^3). \tag{2.81}$$

Thus

$$n_1 {t_1}^{(2)} \bar{\tau}_{11} = 0, \tag{2.82}$$

$$n_2 {t_2}^{(2)} \bar{\tau}_{22} = O(\epsilon), \tag{2.83}$$

$$n_3 {t_3}^{(2)} \bar{\tau}_{33} = O(\epsilon), \tag{2.84}$$

$$(n_1 {t_2}^{(2)} + n_2 {t_1}^{(2)}) \bar{\tau}_{12} = O(\epsilon), \tag{2.85}$$

$$(n_1 {t_3}^{(2)} + n_3 {t_1}^{(2)}) \bar{\tau}_{13} = O(\epsilon), \tag{2.86}$$

$$(n_2 {t_3}^{(2)} + n_3 {t_2}^{(2)}) \bar{\tau}_{23} = \frac{1}{\epsilon A} \frac{\rho_G}{\rho_T} \bar{\mu} \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon). \tag{2.87}$$

Hence from (2.72),

$$\bar{T}^{(2)} = \mathbf{t}^{(2)} \cdot \bar{\mathbf{T}}(\mathbf{n}) = \frac{1}{\epsilon A} \frac{\rho_G}{\rho_T} \bar{\mu} \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon). \tag{2.88}$$

Atmospheric pressure, p_A, is neglected. The stress boundary condition on the upper surface of the glass ribbon are that the tangential and normal stresses vanish.

$\bar{z} = \bar{H}(\bar{x}, \bar{y}, \bar{t}) + \frac{1}{2}\bar{h}(\bar{x}, \bar{y}, \bar{t})$:

$$\bar{T}^{(1)} = 0: \qquad \frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon^2) = 0, \tag{2.89}$$

$$\bar{T}^{(2)} = 0: \qquad \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon^2) = 0, \tag{2.90}$$

$$\bar{N} = 0: \qquad -\bar{p} + \frac{2}{A} \frac{\rho_G}{\rho_T} \bar{\mu} \frac{\partial \bar{w}}{\partial \bar{z}} + O(\epsilon^2) = 0. \tag{2.91}$$

2.7 Normal and tangential stress applied on lower surface

In this section we consider the lower surface of the fluid glass ribbon and derive the normal and tangential stress on the surface.

The normal stress is given by (2.62). For the lower surface, from (2.48),

$$n_1 = \epsilon \left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}} \right) + O(\epsilon^3), \quad n_2 = \epsilon \left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}} \right) + O(\epsilon^3), \; n_3 = -1 + O(\epsilon^2). \tag{2.92}$$

Using the dimensionless stress components $\bar{\tau}_{ik}$ derived in Section 2.5, we obtain

$${n_1}^2 \bar{\tau}_{11} = O(\epsilon^2), \tag{2.93}$$

$${n_2}^2 \bar{\tau}_{22} = O(\epsilon^2), \tag{2.94}$$

$${n_3}^2 \bar{\tau}_{33} = -\bar{p} + \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{w}}{\partial \bar{z}} + O(\epsilon^2), \tag{2.95}$$

$$2 n_1 n_2 \bar{\tau}_{12} = O(\epsilon^2), \tag{2.96}$$

$$2 n_1 n_3 \bar{\tau}_{13} = -\frac{2}{A}\frac{\rho_G}{\rho_T} \left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}} \right) \bar{\mu} \frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon^2), \tag{2.97}$$

$$2 n_2 n_3 \bar{\tau}_{23} = -\frac{2}{A}\frac{\rho_G}{\rho_T} \left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}} \right) \bar{\mu} \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon^2), \tag{2.98}$$

and (2.62) becomes

$$\bar{N} = -\bar{p} - \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu} \left[\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}} \right) \frac{\partial \bar{u}}{\partial \bar{z}} + \left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}} \right) \frac{\partial \bar{v}}{\partial \bar{z}} - \frac{\partial \bar{w}}{\partial \bar{z}} \right] + O(\epsilon^2). \tag{2.99}$$

Consider next the component of the tangential stress $\bar{T}^{(1)}$ in the direction of the unit vector $\mathbf{t}^{(1)}$ given by (2.72). Neglecting terms of order ϵ^2 we have from (2.49),

$$t_1{}^{(1)} = 1 + O(\epsilon^2), \quad t_2{}^{(1)} \equiv 0, \quad t_3{}^{(1)} = \epsilon\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right) + O(\epsilon^3). \qquad (2.100)$$

Equation (2.72) is evaluated using the dimensionless stress components $\bar{\tau}_{ik}$ given in Section 2.5, equation (2.92) for $\mathbf{n}$ and (2.100) for $\mathbf{t}^{(1)}$. Now

$$n_1 t_1{}^{(1)} \bar{\tau}_{11} = O(\epsilon), \qquad (2.101)$$

$$n_2 t_2{}^{(1)} \bar{\tau}_{22} = 0, \qquad (2.102)$$

$$n_3 t_3{}^{(1)} \bar{\tau}_{33} = O(\epsilon), \qquad (2.103)$$

$$(n_1 t_2{}^{(1)} + n_2 t_1{}^{(1)}) \bar{\tau}_{12} = O(\epsilon), \qquad (2.104)$$

$$(n_1 t_3{}^{(1)} + n_3 t_1{}^{(1)}) \bar{\tau}_{13} = -\frac{1}{\epsilon A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon), \qquad (2.105)$$

$$(n_2 t_3{}^{(1)} + n_3 t_2{}^{(1)}) \bar{\tau}_{23} = O(\epsilon). \qquad (2.106)$$

Hence

$$\bar{T}^{(1)} = \mathbf{t}^{(1)} \cdot \mathbf{T}(\mathbf{n}) = -\frac{1}{\epsilon A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon). \qquad (2.107)$$

Finally, consider the component of the tangential stress $T^{(2)}$ in the direction of the unit vector $\mathbf{t}^{(2)}$. Neglecting terms of order ϵ^2, it follows from (2.50) that

$$t_1{}^{(2)} \equiv 0, \quad t_2{}^{(2)} = 1 + O(\epsilon^2), \quad t_3{}^{(2)} = \epsilon \left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2} \frac{\partial \bar{h}}{\partial \bar{y}} \right) + O(\epsilon^3). \tag{2.108}$$

Equation (2.72) is evaluated using the stress tensor components $\bar{\tau}_{ik}$ in Section 2.5, (2.92) for $\mathbf{n}$ and (2.108) for $\mathbf{t}^{(2)}$. We have

$$n_1 t_1{}^{(2)} \bar{\tau}_{11} = 0, \tag{2.109}$$

$$n_2 t_2{}^{(2)} \bar{\tau}_{22} = O(\epsilon), \tag{2.110}$$

$$n_3 t_3{}^{(2)} \bar{\tau}_{33} = O(\epsilon), \tag{2.111}$$

$$(n_1 t_2{}^{(2)} + n_2 t_1{}^{(2)}) \bar{\tau}_{12} = O(\epsilon), \tag{2.112}$$

$$(n_1 t_3{}^{(2)} + n_3 t_1{}^{(2)}) \bar{\tau}_{13} = O(\epsilon), \tag{2.113}$$

$$(n_2 t_3{}^{(2)} + n_3 t_2{}^{(2)}) \bar{\tau}_{23} = -\frac{1}{\epsilon A} \frac{\rho_G}{\rho_T} \bar{\mu} \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon). \tag{2.114}$$

and therefore

$$\bar{T}^{(2)} = \mathbf{t}^{(2)} \cdot \mathbf{T}(\mathbf{n}) = -\frac{1}{\epsilon A} \frac{\rho_G}{\rho_T} \bar{\mu} \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon). \tag{2.115}$$

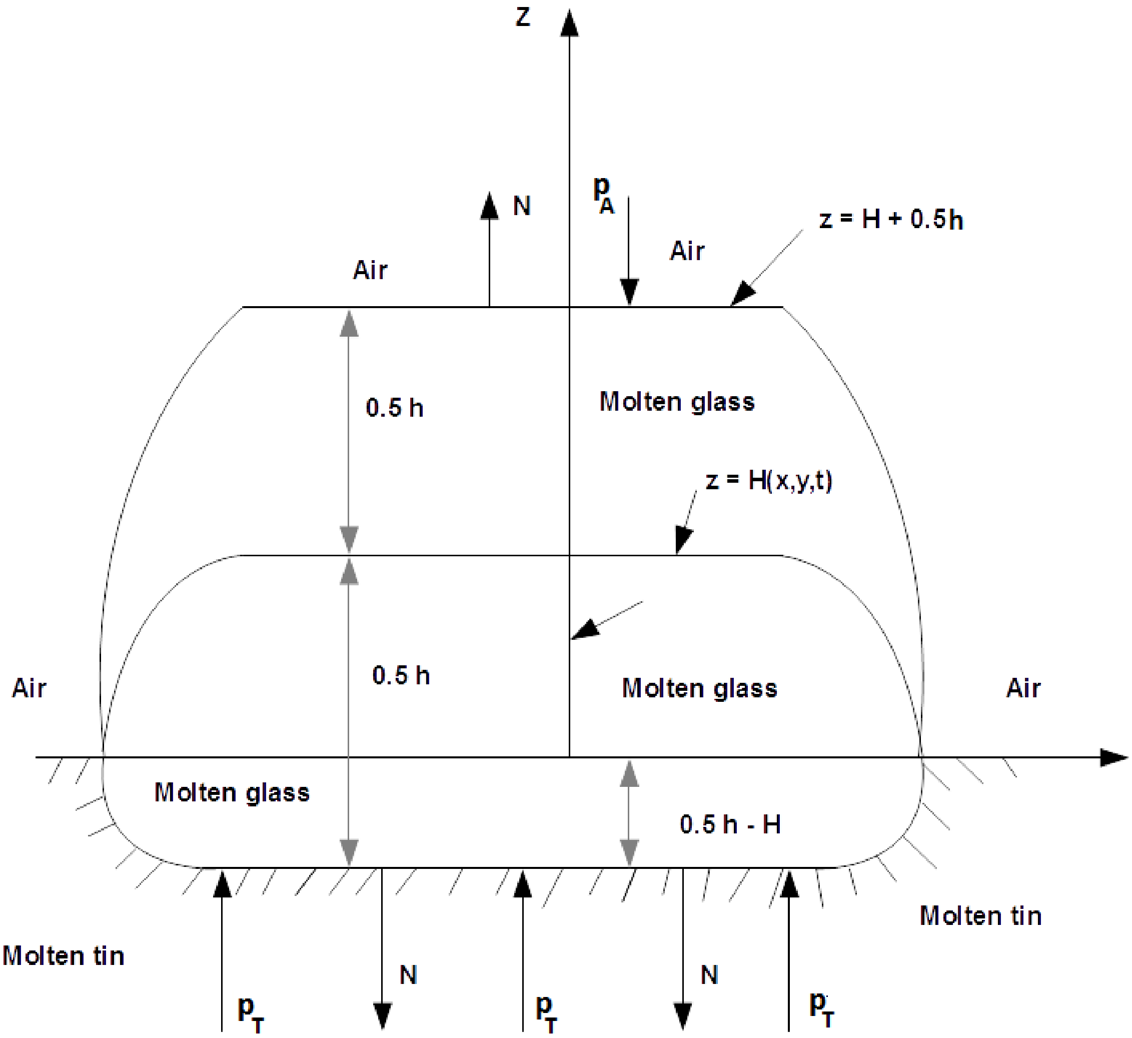

Figure 2.2: Normal stress boundary condition on the upper and lower surfaces of the molten glass ribbon

We can now consider the boundary conditions on the lower surface of the glass ribbon. The normal stress on the glass ribbon is illustrated in Figure 2.2. The pressure, p_T, exerted by the molten tin on the lower surface of the glass ribbon is the pressure at depth $\frac{1}{2}h - H$ in the molten tin. In dimensional form

$$p_T = \rho_T g \left(\frac{1}{2} h - H \right). \tag{2.116}$$

Since the characteristic pressure is $\rho_T g H^*$, where H^* is the characteristic distance in the z- direction, the dimensionless pressure on the lower surface of the glass ribbon is

$$\bar{p}_T = \frac{1}{2}\bar{h} - \bar{H}. \tag{2.117}$$

Thus from Figure 2.2,

$$\bar{N} = -\bar{p}_T = \bar{H} - \frac{1}{2}\bar{h}. \tag{2.118}$$

The viscosity of the molten tin is much less than the viscosity of the molten glass. The approximation is made that the tangential stress exerted by molten tin on the lower surface of the glass ribbon is zero. We therefore have the following boundary conditions on the lower surface.

$z = H(x, y, t) - \frac{1}{2}h(x, y, t)$:

$$\bar{T}^{(1)} = 0 : \qquad \frac{\partial \bar{u}}{\partial \bar{z}} + O(\epsilon^2) = 0, \tag{2.119}$$

$$\bar{T}^{(2)} = 0 : \qquad \frac{\partial \bar{v}}{\partial \bar{z}} + O(\epsilon^2) = 0, \tag{2.120}$$

$$\bar{N} = \bar{H} - \frac{1}{2}\bar{h} : \qquad -\bar{p} + \frac{2}{A}\frac{\rho_G}{\rho_T}\bar{\mu}\frac{\partial \bar{w}}{\partial \bar{z}} = \bar{H} - \frac{1}{2}\bar{h}. \tag{2.121}$$

2.8 Conclusions

We have derived the thin fluid film equations for a thin ribbon of molten glass on a bath of molten tin. The glass ribbon was modelled as an incompressible viscous Newtonian fluid. The kinematic boundary conditions on the upper and lower surfaces of the glass ribbon were derived by imposing the condition that a fluid particle on a surface remains on the surface as the fluid evolves. The boundary conditions on the normal stress and the two tangential stress components on the upper and lower surfaces were also derived. The atmospheric pressure was neglected on the upper surface and the tangential stress exerted by the molten tin on the lower surface was neglected.

Chapter 3

Energy equation

3.1 Introduction

The viscosity of the molten glass depends on the absolute temperature $T(x, y, z, t)$. The temperature dependence of viscosity is modelled by the equation

$$\mu(T) = K \exp\left[\frac{E_0}{T - T_0}\right] \tag{3.1}$$

where T_0 is a reference temperature, K and E_0 are constants. The energy balance for an incompressible viscous Newtonian fluid is considered. It gives the equation for the absolute temperature in the molten glass.

3.2 Mathematical model

Thermal radiation effects are neglected. An outline of the derivation of the energy balance equation for an incompressible viscous Newtonian fluid is given in Appendix A [9, 10]. The energy balance equation can be written in the form of an equation for the absolute temperature T as equation $(A.25)$:

$$c_v \rho_G \frac{DT}{Dt} = k_G \bigtriangledown^2 T + 2\mu D_{ik} D_{ih}, \tag{3.2}$$

where D_{ik} is the rate of strain tensor,

$$D_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \tag{3.3}$$

c_v is the specific heat at constant volume of the molten glass and k_G is the coefficient of thermal conductivity of the molten glass, both assumed constant. The terms in (3.2) are expanded in Appendix A. In expanded form the energy balance equation is given by equation $(A.27)$:

$$\begin{aligned} c_v \rho_G \left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] &= k_G \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] \\ +\mu \Bigg[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial x}\right)^2 &+ 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} \\ +\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 &+ \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\Bigg]. \end{aligned} \tag{3.4}$$

We define the absolute temperature difference

$$S(x, y, z, t) = T(x, y, z, t) - T_0, \tag{3.5}$$

where T_0 is the reference temperature. The energy balance equation is made dimensionless by transforming to the dimensionless variables defined in (2.9). We also let

$$\bar{S} = \frac{T - T_0}{T_0} \tag{3.6}$$

Written in dimensionless form the energy balance equation (3.4) becomes

$$\epsilon^2 Re P_r\left[\frac{\partial \bar{S}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{S}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{S}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{S}}{\partial \bar{z}}\right]=\epsilon^2\frac{\partial^2 \bar{S}}{\partial \bar{x}^2}+\epsilon^2\frac{\partial^2 \bar{S}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{S}}{\partial \bar{z}^2}$$
$$+E_c P_r \bar{\mu}\left[2\epsilon^2\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2+2\epsilon^2\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2+2\epsilon^2\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^2+2\epsilon^2\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{x}}+2\epsilon^2\frac{\partial \bar{v}}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial \bar{y}}+2\epsilon^2\frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{z}}\right.$$
$$\left.+\epsilon^2\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2+\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2+\epsilon^2\left(\frac{\partial \bar{v}}{\partial \bar{x}}\right)^2+\left(\frac{\partial \bar{v}}{\partial \bar{z}}\right)^2+\epsilon^4\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2+\epsilon^4\left(\frac{\partial \bar{w}}{\partial \bar{y}}\right)^2\right] \quad (3.7)$$

where

$$\epsilon=\frac{H^*}{L} \quad (3.8)$$

is the ratio of the characteristic length perpendicular to the glass ribbon, H^*, to the characteristic length along the glass ribbon, L. The dimensionless Reynolds number was defined in (2.14) for the flow along the glass ribbon:

$$Re=\frac{\rho_G U L}{\mu^*}=\frac{inertia\ per\ unit\ volume}{viscous\ force\ per\ unit\ volume}. \quad (3.9)$$

The dimensionless Prandtl number is defined by

$$P_r=\frac{\mu^* c_v}{k_G}=\frac{viscous\ diffusion\ rate}{thermal\ diffusion\ rate} \quad (3.10)$$

and the dimensionless Eckert number is defined by

$$E_c=\frac{U^2}{c_v T_0}=\frac{kinematic\ energy\ per\ unit\ mass}{internal\ energy\ per\ unit\ mass}. \quad (3.11)$$

We again make the lubrication approximation [8]

$$\epsilon \ll 1, \qquad \epsilon^2 Re \ll 1. \quad (3.12)$$

We consider a molten glass ribbon for which the product $E_c P_r$ is order of magnitude unity. We see from Appendix B, Table $B.1$, that this is satisfied for a glass ribbon in a temperature range around $850°C$. The energy balance equation (3.7)reduces to

$$\frac{\partial^2 \bar{S}}{\partial \bar{z}^2} + E_c P_r \bar{\mu} \left[\left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{z}} \right)^2 \right] = 0. \tag{3.13}$$

3.3 Thermal boundary condition on upper surface

The thermal boundary condition on the upper surface is that the normal component of the heat flux vector is continuous across the interface:

$$z = H + \frac{1}{2}h : \qquad \mathbf{n} \cdot \mathbf{q_G} = \mathbf{n} \cdot \mathbf{q_A}. \tag{3.14}$$

where $\mathbf{q_A}$ is the heat flux vector in air. Using Fourier's law of heat conduction,($A.3$), equation (3.14) can be written as

$$z = H + \frac{1}{2}h : \qquad \mathbf{n} \cdot \boldsymbol{\nabla} T \Big|_{glass} = \frac{k_A}{k_G} \mathbf{n} \cdot \boldsymbol{\nabla} T \Big|_{air}. \tag{3.15}$$

But from Appendix B, estimates of the magnitude of the thermal conductivities, k_A and k_G, of air and glass are

$$k_A \simeq 2.4 \times 10^{-2} W m^{-1} K^{-1}, \quad k_G = 1.35 W m^{-1} K^{-1}$$

and therefore

$$\frac{k_A}{k_G} = 1.8 \times 10^{-2}.$$

Heat flow from the molten glass into the air can therefore be neglected. The thermal boundary condition on the upper surface is therefore

$$z = H + \frac{1}{2}h : \qquad \mathbf{n} \cdot \boldsymbol{\nabla} T \bigg|_{glass} = 0, \tag{3.16}$$

which expressed in terms of S is

$$z = H + \frac{1}{2}h : \qquad \mathbf{n} \cdot \nabla S = 0. \tag{3.17}$$

In expanded form (3.17) is

$$z = H + \frac{1}{2}h : \qquad n_1 \frac{\partial S}{\partial x} + n_2 \frac{\partial S}{\partial y} + n_3 \frac{\partial S}{\partial z} = 0. \tag{3.18}$$

Introducing dimensionless variables,

$$x = L\bar{x}, \quad y = L\bar{y}, \quad z = \epsilon L\bar{z},$$

$$H = \epsilon L\bar{H}, \quad h = \epsilon L\bar{h}, \quad S = T_0\bar{S}. \tag{3.19}$$

The boundary condition (3.18) becomes

$$\bar{z} = \bar{H} + \frac{1}{2}\bar{h} : \qquad n_1 \frac{\partial \bar{S}}{\partial \bar{x}} + n_2 \frac{\partial \bar{S}}{\partial \bar{y}} + \frac{1}{\epsilon} n_3 \frac{\partial \bar{S}}{\partial \bar{z}} = 0. \tag{3.20}$$

But from (2.63),

$$n_1 = -\epsilon \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x} \right) + O(\epsilon^3), \tag{3.21}$$

$$n_2 = -\epsilon \left(\frac{\partial H}{\partial y} + \frac{1}{2} \frac{\partial h}{\partial y} \right) + O(\epsilon^3), \tag{3.22}$$

,

$$n_3 = 1 + O(\epsilon^2). \tag{3.23}$$

Multiplying (3.20) by ϵ the boundary condition becomes

$$\bar{z} = \bar{H} + \frac{1}{2}\bar{h} : \quad \left[-\epsilon^2 \left(\frac{\partial \bar{H}}{\partial \bar{x}} + \frac{1}{2} \frac{\partial \bar{h}}{\partial \bar{x}} \right) + O(\epsilon^4) \right] \frac{\partial \bar{S}}{\partial \bar{x}}$$

$$+ \left[-\epsilon^2 \left(\frac{\partial \bar{H}}{\partial \bar{y}} + \frac{1}{2} \frac{\partial \bar{h}}{\partial \bar{y}} \right) + O(\epsilon^4) \right] \frac{\partial \bar{S}}{\partial \bar{y}} + \left[1 + O(\epsilon^2) \right] \frac{\partial \bar{S}}{\partial \bar{z}} = 0. \tag{3.24}$$

In the lubrication approximation terms of order ϵ^2 are neglected. The thermal boundary condition on the upper surface reduces to

$$\bar{z} = \bar{H} + \frac{1}{2}\bar{h} : \qquad \frac{\partial \bar{S}}{\partial \bar{z}} = 0. \tag{3.25}$$

3.4 Thermal boundary condition on lower surface

The thermal boundary condition on the lower surface is that the normal component of the heat flux vector is continuous across the interface:

$$z = H - \frac{1}{2}h : \qquad \mathbf{n} \cdot \mathbf{q_G} = \mathbf{n} \cdot \mathbf{q_T}, \tag{3.26}$$

which by using Fourier's law of heat conduction can be written as

$$z = H - \frac{1}{2}h : \qquad \mathbf{n} \cdot \nabla T \Big|_{glass} = \frac{k_T}{k_G} \mathbf{n} \cdot \nabla T \Big|_{tin}, \tag{3.27}$$

But from Appendix B, estimates of the magnitudes of the thermal conductivities are

$$k_T = 34\ Wm^{-1}K^{-1}, \qquad k_G = 1.35\ Wm^{-1}K^{-1} \tag{3.28}$$

and therefore

$$\frac{k_T}{k_G} = 25. \tag{3.29}$$

The heat flux from the molten glass into the molten tin is therefore not small as it was for the flow from glass to air on the upper surface. However, in order not to make the model too complicated, we will neglect the heat flow from the glass into the tin and take for the thermal boundary condition on the lower surface

$$z = H - \frac{1}{2}h : \qquad \mathbf{n} \cdot \nabla T \Big|_{glass} = 0. \tag{3.30}$$

We express (3.30) in terms of S using (3.5) and proceed as for the boundary condition (3.17) on the upper surface by expanding it and expressing it in terms of the dimensionless variables (3.19). The boundary condition (3.30) becomes

$$\bar{z} = \bar{H} - \frac{1}{2}h : \qquad n_1 \frac{\partial \bar{S}}{\partial \bar{x}} + n_2 \frac{\partial \bar{S}}{\partial \bar{y}} + \frac{1}{\epsilon} n_3 \frac{\partial \bar{S}}{\partial \bar{z}} = 0. \tag{3.31}$$

But from (2.92),

$$n_1 = \epsilon \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x} \right) + O(\epsilon^3), \tag{3.32}$$

$$n_2 = \epsilon \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y} \right) + O(\epsilon^3), \tag{3.33}$$

$$n_3 = -1 + O(\epsilon^2) \tag{3.34}$$

and multiplying (3.31) by ϵ gives

$$\bar{z} = \bar{H} - \frac{1}{2}\bar{h}: \qquad \left[\epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{x}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{x}}\right) + O(\epsilon^4)\right]\frac{\partial \bar{S}}{\partial \bar{x}}$$

$$+\left[\epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right) + O(\epsilon^4)\right]\frac{\partial \bar{S}}{\partial \bar{y}} + \left[-1 + O(\epsilon^2)\right]\frac{\partial \bar{S}}{\partial \bar{z}} = 0. \tag{3.35}$$

Neglecting terms of order ϵ^2 in the lubrication approximation, the thermal boundary condition on the lower surface is

$$\bar{z} = \bar{H} - \frac{1}{2}\bar{h}: \qquad \frac{\partial \bar{S}}{\partial \bar{z}} = 0, \tag{3.36}$$

3.5 Conclusions

In the lubrication approximation the dimensionless absolute temperature difference, $\bar{S}(x, y, z, t)$, satisfies the second order partial differential equation

$$\frac{\partial^2 \bar{S}}{\partial \bar{z}^2} + E_c P_r \bar{\mu}\left[\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{z}}\right)^2\right] = 0, \tag{3.37}$$

subject to the boundary conditions on the upper and lower surfaces of the molten glass ribbon,

$$\bar{z} = \bar{H} + \frac{1}{2}\bar{h}: \qquad \frac{\partial \bar{S}}{\partial \bar{z}} = 0, \tag{3.38}$$

$$\bar{z} = \bar{H} - \frac{1}{2}\bar{h}: \qquad \frac{\partial \bar{S}}{\partial \bar{z}} = 0. \tag{3.39}$$

In Chapter 2 the equations for the fluid flow in the glass ribbon and the boundary condition on the upper and lower surfaces of the ribbon were derived with the lubrication equation. The fluid flow equations are coupled to the energy balance equation through the temperature dependence of the viscosity. In Chapter 4 the solution of the combined system of lubrication equations and boundary conditions will be investigated.

In order to keep the notation simple, unless otherwise indicated, the overhead bars on dimensionless quantities will be suppressed in the remainder of the dissertation.

Chapter 4

Lubrication approximation

4.1 Introduction

In this chapter the equations and boundary conditions derived in Chapter 2 and 3 for the lubrication approximation are combined and their solution is investigated. The equations and boundary conditions are coupled by the viscosity, μ,which depends on the absolute temperature through equation (3.1). Expressed in dimensionless variables, (3.1) becomes

$$\bar{\mu} = \bar{K} exp \left[\frac{\bar{E}_0}{\bar{S}} \right], \tag{4.1}$$

where

$$\bar{\mu} = \frac{\mu}{\mu^*}, \quad \bar{K} = \frac{K}{\mu^*}, \quad \bar{E}_0 = \frac{E_0}{T_0}, \quad \bar{S} = \frac{T - T_0}{T_0}. \tag{4.2}$$

In order to keep the notation simple, unless otherwise stated, the overhead bars on dimensionless quantities will be suppressed in the remainder of the dissertation.

4.2 Lubrication approximation

The lubrication(thin fluid film) approximation is given by (2.18):

$$\epsilon = \frac{H^*}{L} \ll 1, \qquad \epsilon^2 Re \ll 1. \tag{4.3}$$

Terms of order ϵ^2 are neglected. The unknown dependent variables are

$$u = u(x, y, z, t), \quad v = v(x, y, z, t), \quad w = w(x, y, z, t), \quad p = p(x, y, z, t),$$

$$S = S(x, y, z, t), \quad H = H(x, y, t), \quad h = h(x, y, t) \tag{4.4}$$

The partial differential equations are the continuity equation (2.19), the three components of the Navier-Stokes equation,(2.20) to (2.22) and the energy balance equation,(3.37):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.5}$$

$$\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) = 0, \tag{4.6}$$

$$\frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) = 0, \tag{4.7}$$

$$0 = -\frac{\rho_T}{\rho_G} A \frac{\partial p}{\partial z} - A + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right), \tag{4.8}$$

$$\frac{\partial^2 S}{\partial z^2} + E_c P_r \mu \left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right] = 0, \tag{4.9}$$

where

$$\mu(x, y, z, t) = K exp\left[\frac{E_0}{S}\right]. \tag{4.10}$$

The kinematic boundary conditions on the upper and lower surfaces are (2.30) and (2.31):

$$z = H + \frac{1}{2}h: \qquad W = \frac{\partial H}{\partial t} + \frac{1}{2}\frac{\partial h}{\partial t} + u\frac{\partial}{\partial x}\left(H + \frac{1}{2}h\right) + v\frac{\partial}{\partial y}\left(H + \frac{1}{2}h\right), \tag{4.11}$$

$$z = H - \frac{1}{2}h: \qquad W = \frac{\partial H}{\partial t} - \frac{1}{2}\frac{\partial h}{\partial t} + u\frac{\partial}{\partial x}\left(H - \frac{1}{2}h\right) + v\frac{\partial}{\partial y}\left(H - \frac{1}{2}h\right), \tag{4.12}$$

The boundary conditions derived from the normal and tangential stresses applied to the upper and lower surfaces are obtained from (2.89) to (2.91) and (2.119) to (2.121) by neglecting terms of order ϵ^2:

$z = H + \frac{1}{2}h$:

$$T^{(1)} = 0, \qquad \frac{\partial u}{\partial z} = 0, \tag{4.13}$$

$$T^{(2)} = 0, \qquad \frac{\partial v}{\partial z} = 0, \tag{4.14}$$

$$N = 0, \qquad -p + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\frac{\partial w}{\partial z} = 0, \tag{4.15}$$

$z = H - \frac{1}{2}h$:

$$T^{(1)} = 0, \qquad \frac{\partial u}{\partial z} = 0, \tag{4.16}$$

$$T^{(2)} = 0, \qquad \frac{\partial v}{\partial z} = 0, \tag{4.17}$$

$$N = H - \frac{1}{2}h, \qquad -p + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\frac{\partial w}{\partial z} = H - \frac{1}{2}h. \tag{4.18}$$

The thermal boundary conditions on the upper and lower surfaces are (3.38) and (3.39) :

$$z = H + \frac{1}{2}h : \qquad \frac{\partial S}{\partial z} = 0, \tag{4.19}$$

$$z = H - \frac{1}{2}h : \qquad \frac{\partial S}{\partial z} = 0. \tag{4.20}$$

This completes the formulation of the equations and boundary conditions for the lubrication approximation.

4.3 Solution of the lubrication equations

Consider first the components $u(x, y, z, t)$ and $v(x, y, z, t)$ of the velocity of the molten glass. From (4.6),

$$\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) = 0 \tag{4.21}$$

and therefore

$$\mu\frac{\partial u}{\partial z} = f(x, y, t). \tag{4.22}$$

But from boundary condition (4.13)

$$z = H + \frac{1}{2}h : \qquad \frac{\partial u}{\partial z} = 0 \tag{4.23}$$

and therefore $f(x, y, t) = 0$. Thus

$$\frac{\partial}{\partial z}u(x, y, z, t) = 0. \tag{4.24}$$

The boundary condition (4.16) on the lower surface is identically satisfied. By (4.24)

$$u = U(x, y, t). \tag{4.25}$$

Similarly, by considering (4.7) and the boundary condition (4.14) it can be shown that

$$v = V(x, y, t). \tag{4.26}$$

Since u and v are independent of z, the flow is an extensional flow.

Consider next the component $w(x, y, z, t)$ of the velocity of the molten glass. From the continuity equation (4.5)

$$\frac{\partial w}{\partial z}(x, y, z, t) = -\left[\frac{\partial U}{\partial x}(x, y, t) + \frac{\partial V}{\partial y}(x, y, t)\right] \tag{4.27}$$

and therefore

$$w(x, y, z, t) = -\left(\frac{\partial U}{\partial x}(x, y, t) + \frac{\partial V}{\partial y}(x, y, t)\right) z + g(x, y, t). \tag{4.28}$$

In order to obtain $g(x, y, t)$ consider the kinematic boundary condition at the upper and the lower surfaces, (4.11) and (4.12). Substituting equation (4.28) into (4.11) and (4.12) we obtain

$$-\left(\frac{\partial U}{\partial x}(x, y, t) + \frac{\partial V}{\partial y}(x, y, t)\right)\left(H + \frac{1}{2}h\right) + g(x, y, t)$$

$$= \frac{\partial H}{\partial t} + \frac{1}{2}\frac{\partial h}{\partial t} + U(x, y, t)\left(\frac{\partial H}{\partial t} + \frac{1}{2}\frac{\partial h}{\partial x}\right) + V(x, y, t)\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) \tag{4.29}$$

and

$$-\left(\frac{\partial U}{\partial x}(x,y,t)+\frac{\partial V}{\partial y}(x,y,t)\right)\left(H-\frac{1}{2}h\right)+g(x,y,t)$$

$$=\frac{\partial H}{\partial t}-\frac{1}{2}\frac{\partial h}{\partial t}+U(x,y,t)\left(\frac{\partial H}{\partial x}-\frac{1}{2}\frac{\partial h}{\partial x}\right)+V(x,y,t)\left(\frac{\partial H}{\partial y}-\frac{1}{2}\frac{\partial h}{\partial y}\right). \quad (4.30)$$

In order to obtain $g(x,y,t)$ add equations (4.29) and (4.30). This gives

$$g(x,y,t)=\left[\frac{\partial U}{\partial x}(x,y,t)+\frac{\partial V}{\partial y}(x,y,t)\right]H(x,y,t)+\frac{\partial H}{\partial t}$$

$$+U(x,y,t)\frac{\partial H}{\partial x}+V(x,y,t)\frac{\partial H}{\partial y}. \quad (4.31)$$

Substituting (4.31) into (4.28) we obtain

$$w(x,y,z,t)=\left[\frac{\partial U}{\partial x}(x,y,t)+\frac{\partial V}{\partial y}(x,y,t)\right](H(x,y,t)-z)+\frac{\partial H}{\partial t}$$

$$+U(x,y,t)\frac{\partial H}{\partial x}+V(x,y,t)\frac{\partial H}{\partial y}. \quad (4.32)$$

By subtracting (4.30) from (4.29) we obtain

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hU)+\frac{\partial}{\partial y}(hV)=0. \quad (4.33)$$

Equation (4.33) is independent of z. It contains three unknowns which depend only on x,y and t.

Now the viscosity μ depends on $S(x,y,z,t)$ by equation (4.10). Before we solve for the pressure using the $z-$ component of the Navier-Stokes equations (4.8) it is first necessary to investigate the solution for S. Using (4.25) and (4.26) for u and v, the energy balance equation (4.9) reduces to

$$\frac{\partial^2 S}{\partial z^2} = 0 \tag{4.34}$$

an therefore

$$\frac{\partial S}{\partial z} = A(x, y, t). \tag{4.35}$$

But from the thermal boundary condition on the upper surface (4.19),

$$z = H + \frac{1}{2}h : \qquad \frac{\partial S}{\partial z} = 0, \tag{4.36}$$

it follows that $A(x, y, t) = 0$ and from (4.35)

$$\frac{\partial S}{\partial z}(x, y, z, t) = 0 \tag{4.37}$$

The boundary condition (4.20) on the lower surface is identically satisfied. From (4.37),

$$S = S(x, y, t). \tag{4.38}$$

We see from (4.10) that the viscosity μ is independent of z:

$$\mu = \mu(x, y, t) = Kexp\left[\frac{E_0}{S(x, y, t)}\right]. \tag{4.39}$$

Consider now the fluid pressure $p(x, y, z, t)$. From (4.8)

$$\frac{\partial p}{\partial z} = \frac{\rho_G}{\rho_T}\frac{1}{A}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - \frac{\rho_G}{\rho_T}. \tag{4.40}$$

But from the continuity equation (4.27) and (4.39) for μ,

$$\frac{\partial}{\partial z}\left(\mu\frac{\partial W}{\partial z}\right) = 0 \tag{4.41}$$

and therefore (4.40) reduces to

$$\frac{\partial p}{\partial z} = -\frac{\rho_G}{\rho_T}. \tag{4.42}$$

Hence

$$p(x, y, z, t) = -\frac{\rho_G}{\rho_T}z + B(x, y, t). \tag{4.43}$$

In order to obtain $B(x, y, t)$ consider the boundary condition for the applied normal stress on the upper surface, (4.15):

$$z = H + \frac{1}{2}h : \qquad p = \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\frac{\partial W}{\partial z}, \tag{4.44}$$

which becomes using the continuity equation (4.27),

$$p(x, y, H + \frac{1}{2}h, t) = -\frac{2}{A}\frac{\rho_G}{\rho_T}\mu\left[\frac{\partial U}{\partial x}(x, y, t) + \frac{\partial V}{\partial y}(x, y, t)\right]. \tag{4.45}$$

Substituting (4.43) into the boundary condition (4.45) gives

$$B(x, y, t) = \frac{\rho_G}{\rho_T}\left(H + \frac{1}{2}h\right) - \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\left[\frac{\partial U}{\partial x}(x, y, t) + \frac{\partial V}{\partial y}(x, y, t)\right] \tag{4.46}$$

and therefore (4.43) becomes

$$p(x, y, z, t) = \frac{\rho_G}{\rho_T}\left(H + \frac{1}{2}h - z\right) - \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\left[\frac{\partial U}{\partial x}(x, y, t) + \frac{\partial V}{\partial y}(x, y, t)\right]. \tag{4.47}$$

One further condition remains to be imposed, the boundary condition for the applied normal stress on the lower surface, (4.18):

$$z = H - \frac{1}{2}h: \qquad -p + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\frac{\partial W}{\partial z} = H - \frac{1}{2}h. \tag{4.48}$$

On substituting (4.47) for p and the continuity equation (4.27) into (4.48) we obtain

$$H = \left(\frac{1}{2} - \frac{\rho_G}{\rho_T}\right)h. \tag{4.49}$$

This completes the solution for the lubrication approximation.

4.4 Summary of the results for the lubrication approximation

All of the equations and boundary conditions, (4.5) to (4.20), have been used. The results derived from these equations and boundary condition can be summarized as follows:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU) + \frac{\partial}{\partial y}(hV) = 0, \tag{4.50}$$

$$u = U(x, y, t), \tag{4.51}$$

$$v = V(x, y, t), \tag{4.52}$$

$$S = S(x, y, t), \tag{4.53}$$

$$h = h(x, y, t), \tag{4.54}$$

$$H = \left(\frac{1}{2} - \frac{\rho_G}{\rho_T}\right)h, \tag{4.55}$$

$$w(x, y, z, t) = \left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right](H - z) + \frac{\partial H}{\partial t} + U\frac{\partial H}{\partial x} + V\frac{\partial H}{\partial y}, \tag{4.56}$$

$$p = p(x, y, z, t) = \frac{\rho_G}{\rho_T}\left(H + \frac{1}{2}h - z\right) - \frac{2}{A}\frac{\rho_G}{\rho_T}\mu\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right], \tag{4.57}$$

$$\mu = \mu(x, y, t) = Kexp\left[\frac{E_0}{S}\right]. \tag{4.58}$$

The unknown functions are

$$U = U(x, y, t), \quad V = V(x, y, t), \quad S = S(x, y, t), \quad h = h(x, y, t). \tag{4.59}$$

Once U, V, S and h have been found the remaining quantities H, w, p and μ can be obtained from (4.55) to (4.58). There is only one partial differential equation, (4.50), for the four unknown functions U, V, S and h.

4.5 Conclusions

The lubrication (thin fluid film) approximation does not give a complete solution to the problem. There is only one partial differential equation for the four unknown functions. Three further equations are required relating the four unknowns. The derivation of these equations will be considered in Chapter 5.

An important result obtained in this Chapter is that U, V, S and μ do not depend on z.

Chapter 5

Perturbation expansion and system of equations for zero order solution

5.1 Introduction

We saw in Chapter 4 that there is only one equation for the four unknown functions U,V,S and h. In this chapter the remaining three equations are obtained by considering a perturbation expansion in powers of ϵ^2. The zero order terms in the expansion are the lubrication approximation solution derived in Chapter 4. Three consistency conditions have to be satisfied by the first order components of the fluid velocity and absolute temperature. These three conditions are independent of first order quantities. They are expressed only in terms of U,V,S and h and are the three remaining zero order equations.

The aim of the investigation is not to solve the problem to first order in ϵ^2. It is to complete the system of equations for the zero order solution.

5.2 Partial differential equations to first order

In dimensionless form the continuity equation is given by (2.10), the x,y and z components of the Naiver- Stokes equation by (2.11) to (2.13) and the energy balance equation by (3.7). We consider a perturbation expansion in ϵ^2 instead of in ϵ because the components of the Navier - Stokes equation and the energy balance equation depend on ϵ^2:

$$u(x,y,z,t) = U_0(x,y,t) + \epsilon^2 u_1(x,y,z,t) + O(\epsilon^4), \tag{5.1}$$

$$v(x,y,z,t) = V_0(x,y,t) + \epsilon^2 v_1(x,y,z,t) + O(\epsilon^4), \tag{5.2}$$

$$w(x,y,z,t) = W_0(x,y,z,t) + \epsilon^2 w_1(x,y,z,t) + O(\epsilon^4), \tag{5.3}$$

$$p(x,y,z,t) = P_0(x,y,z,t) + \epsilon^2 p_1(x,y,z,t) + O(\epsilon^4), \tag{5.4}$$

$$S(x,y,z,t) = S_0(x,y,t) + \epsilon^2 S_1(x,y,z,t) + O(\epsilon^4), \tag{5.5}$$

$$\mu(x,y,z,t) = \mu_0(x,y,t) + \epsilon^2 \mu_1(x,y,z,t) + O(\epsilon^4), \tag{5.6}$$

$\epsilon \to 0$. The zero order terms are the solutions for the lubrication approximation. The function U_0,V_0 and S_0 are independent of z which greatly simplifies the analysis. From (4.56), (4.57) and $(4,58)$,

$$W_0(x,y,z,t) = \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)(H - z) + \frac{\partial H}{\partial t} + U_0\frac{\partial H}{\partial x} + V_0\frac{\partial H}{\partial y}, \tag{5.7}$$

$$P_0(x,y,z,t) = \frac{\rho_G}{\rho_T}\left(H + \frac{1}{2}h - z\right) - \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right), \tag{5.8}$$

$$\mu_0(x,y,t) = Kexp\left[\frac{E_0}{S_0(x,y,t)}\right]. \tag{5.9}$$

The perturbation expansion is substituted into the dimensionless continuity equation,(2.10), the three components of the Navier - Stokes equations, (2.11) to (2.13) and the energy balance equation (3.7). This gives

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} + \frac{\partial W_0}{\partial z} + \epsilon^2\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right) + O(\epsilon^4) = 0, \tag{5.10}$$

$$\begin{aligned}
\epsilon^2 Re\left[\frac{\partial U_0}{\partial t} + U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial y} + W_0\frac{\partial U_0}{\partial z} + O(\epsilon^2)\right] \\
= -\epsilon^2\frac{\rho_T}{\rho_G}A\left[\frac{\partial P_0}{\partial x} + \epsilon^2\frac{\partial p_1}{\partial x} + O(\epsilon^4)\right] \\
+\epsilon^2\frac{\partial}{\partial x}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial U_0}{\partial x} + \epsilon^2\frac{\partial u_1}{\partial x} + O(\epsilon^4)\right)\right] \\
+\epsilon^2\frac{\partial}{\partial y}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial U_0}{\partial y} + \epsilon^2\frac{\partial u_1}{\partial y} + O(\epsilon^4)\right)\right] \\
+\epsilon^2\frac{\partial}{\partial z}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial U_0}{\partial z} + \epsilon^2\frac{\partial u_1}{\partial z} + O(\epsilon^4)\right)\right],
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\epsilon^2 Re\left[\frac{\partial V_0}{\partial t} + U_0\frac{\partial V_0}{\partial x} + V_0\frac{\partial V_0}{\partial y} + W_0\frac{\partial V_0}{\partial z} + O(\epsilon^2)\right] \\
= -\epsilon^2\frac{\rho_T}{\rho_G}A\left[\frac{\partial P_0}{\partial y} + \epsilon^2\frac{\partial p_1}{\partial y} + O(\epsilon^4)\right] \\
+\epsilon^2\frac{\partial}{\partial x}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial V_0}{\partial x} + \epsilon^2\frac{\partial v_1}{\partial x} + O(\epsilon^4)\right)\right] \\
+\epsilon^2\frac{\partial}{\partial y}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial V_0}{\partial y} + \epsilon^2\frac{\partial v_1}{\partial y} + O(\epsilon^4)\right)\right] \\
+\epsilon^2\frac{\partial}{\partial z}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial V_0}{\partial z} + \epsilon^2\frac{\partial v_1}{\partial z} + O(\epsilon^4)\right)\right],
\end{aligned} \tag{5.12}$$

$$\epsilon^2 Re\left[\frac{\partial W_0}{\partial t} + U_0\frac{\partial W_0}{\partial x} + V_0\frac{\partial W_0}{\partial y} + W_0\frac{\partial W_0}{\partial z} + O(\epsilon^2)\right]$$

$$= -\frac{\rho_T}{\rho_G}A\left[\frac{\partial P_0}{\partial z} + \epsilon^2\frac{\partial p_1}{\partial z} + O(\epsilon^4)\right] - A$$

$$+\epsilon^2\frac{\partial}{\partial x}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial W_0}{\partial x} + \epsilon^2\frac{\partial w_1}{\partial x} + O(\epsilon^4)\right)\right]$$

$$+\epsilon^2\frac{\partial}{\partial y}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial W_0}{\partial y} + \epsilon^2\frac{\partial w_1}{\partial y} + O(\epsilon^4)\right)\right]$$

$$+\epsilon^2\frac{\partial}{\partial z}\left[\left(\mu_0 + \epsilon^2\mu_1 + O(\epsilon^4)\right)\left(\frac{\partial W_0}{\partial z} + \epsilon^2\frac{\partial w_1}{\partial z} + O(\epsilon^4)\right)\right], \tag{5.13}$$

$$\epsilon^2 Re P_r\left[\frac{\partial S_0}{\partial t} + U_0\frac{\partial S_0}{\partial x} + V_0\frac{\partial S_0}{\partial y} + W_0\frac{\partial S_0}{\partial z} + O(\epsilon^2)\right]$$

$$= \epsilon^2\frac{\partial^2 S_0}{\partial x^2} + \epsilon^2\frac{\partial^2 S_0}{\partial y^2} + \frac{\partial^2 S_0}{\partial z^2} + \epsilon^2\frac{\partial^2 S_1}{\partial z^2} + O(\epsilon^4)$$

$$+E_r P_r \mu_0\left[2\epsilon^2\left(\frac{\partial U_0}{\partial x}\right)^2 + 2\epsilon^2\left(\frac{\partial V_0}{\partial y}\right)^2 + 2\epsilon^2\left(\frac{\partial W_0}{\partial z}\right)^2\right.$$

$$+2\epsilon^2\frac{\partial U_0}{\partial y}\frac{\partial V_0}{\partial x} + 2\epsilon^2\frac{\partial V_0}{\partial z}\frac{\partial W_0}{\partial y} + 2\epsilon^2\frac{\partial W_0}{\partial x}\frac{\partial U_0}{\partial z} + \epsilon^2\left(\frac{\partial U_0}{\partial y}\right)^2$$

$$\left.+\left(\frac{\partial U_0}{\partial z} + \epsilon^2\frac{\partial u_1}{\partial z}\right)^2 + \epsilon^2\left(\frac{\partial V_0}{\partial x}\right)^2 + \left(\frac{\partial V_0}{\partial z} + \epsilon^2\frac{\partial v_1}{\partial z}\right)^2 + O(\epsilon^4)\right] \tag{5.14}$$

We assume that $Re \ll 1$ so that the inertia term in (5.11) to (5.13) can be neglected even at order ϵ^2. This is a stronger assumption than the lubrication approximation $\epsilon^2 Re \ll 1$. The zero order equations in (5.10) to (5.14) are the equations of the lubrication approximation, (2.19) to (2.22) and (3.13). Since U_0, V_0 and S_0 are independent of z, the equations for order ϵ^2 are :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{5.15}$$

$$0 = -\frac{\rho_T}{\rho_G} A \frac{\partial P_0}{\partial x} + \frac{\partial}{\partial x}\left[\mu_0 \frac{\partial U_0}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu_0 \frac{\partial U_0}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu_0 \frac{\partial u_1}{\partial z}\right], \tag{5.16}$$

$$0 = -\frac{\rho_T}{\rho_G} A \frac{\partial P_0}{\partial y} + \frac{\partial}{\partial x}\left[\mu_0 \frac{\partial V_0}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu_0 \frac{\partial V_0}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu_0 \frac{\partial v_1}{\partial z}\right], \tag{5.17}$$

$$0 = -\frac{\rho_T}{\rho_G} A \frac{\partial p_1}{\partial z} + \frac{\partial}{\partial x}\left[\mu_0 \frac{\partial W_0}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu_0 \frac{\partial W_0}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu_0 \frac{\partial W_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu_1 \frac{\partial W_0}{\partial z}\right], \tag{5.18}$$

$$0 = \frac{\partial^2 S_0}{\partial x^2} + \frac{\partial^2 S_0}{\partial y^2} + \frac{\partial^2 S_1}{\partial z^2}$$
$$+ E_c P_r \mu_0 \left[2\left(\frac{\partial U_0}{\partial x}\right)^2 + 2\left(\frac{\partial V_0}{\partial y}\right)^2 + 2\left(\frac{\partial W_0}{\partial z}\right)^2 + 2\frac{\partial U_0}{\partial y}\frac{\partial V_0}{\partial x} + \left(\frac{\partial U_0}{\partial y}\right)^2 + \left(\frac{\partial V_0}{\partial x}\right)^2 \right] \tag{5.19}$$

We derived all five perturbation equations to order ϵ^2 in order to give an overview of the perturbation method. Three of the equations, (5.16),(5.17) and (5.19) lead to the remaining three equations for U_0,V_0,S_0 and h. The other two equations, (5.15) and (5.18), will not be used.

Using (5.7) for W_0 and (5.8) for P_0, equations (5.16), (5.17) and (5.19) can be written as

$$\frac{\partial}{\partial z}\left(\mu_0\frac{\partial u_1}{\partial z}\right)$$

$$= A\left(\frac{\partial H}{\partial x}+\frac{1}{2}\frac{\partial h}{\partial x}\right)-3\frac{\partial}{\partial x}\left(\mu_0\frac{\partial U_0}{\partial x}\right)-\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial y}\right)-2\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial y}\right), \quad (5.20)$$

$$\frac{\partial}{\partial z}\left(\mu_0\frac{\partial v_1}{\partial z}\right)$$

$$= A\left(\frac{\partial H}{\partial y}+\frac{1}{2}\frac{\partial h}{\partial y}\right)-\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial x}\right)-3\frac{\partial}{\partial y}\left(\mu_0\frac{\partial V_0}{\partial y}\right)-2\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial x}\right), \quad (5.21)$$

$$\frac{\partial^2 S_1}{\partial z^2} = -\frac{\partial^2 S_0}{\partial x^2}-\frac{\partial^2 S_0}{\partial y^2}$$

$$-E_c P_r \mu_0\left[2\left(\frac{\partial U_0}{\partial x}\right)^2+2\left(\frac{\partial V_0}{\partial y}\right)^2+2\left(\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)^2+\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)^2\right]. \quad (5.22)$$

The right hand sides of (5.20), (5.21) and (5.22) depend only on zero order quantities. The procedure is to integrate equations (5.20) to (5.22) with respect to z and impose the boundary conditions on the upper and lower surfaces. By eliminating the arbitrary functions of integration from each equation the three remaining equations for the four zero order quantities U_0, V_0, S_0 and h are derived.

In the perturbation solution to first order in ϵ^2, which will not be considered in this dissertation, the solution of (5.20), (5.21) and (5.22) gives expressions for u_1, v_1 and S_1. The velocity component w_1 is obtained from (5.15) and p_1 is obtained from (5.18). We can expect that to complete the system of equations for u_1, v_1, S_1 and p_1 it will be necessary to consider the perturbation solution to order ϵ^4.

We will now derive the boundary conditions for $\frac{\partial u_1}{\partial z}$, $\frac{\partial v_1}{\partial z}$ and $\frac{\partial S_1}{\partial z}$ by extending the results of Section 2.6, 2.7, 3.3 and 3.4 to the next order in ϵ^2.

5.3 Boundary conditions on the tangential stress

On the upper and lower surfaces of the molten glass ribbon we calculate the tangential stresses $T^{(1)}$ and $T^{(2)}$ correct to order ϵ and obtain boundary conditions for $\frac{\partial u_1}{\partial z}$ and $\frac{\partial v_1}{\partial z}$.

Consider first the upper surface. We first calculate $T^{(1)}$ correct to order ϵ . The unit normal and tangential vectors are given by (2.63) and (2.73) :

$$n_1 = -\epsilon\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right) + O(\epsilon^3), \quad n_2 = -\epsilon\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) + O(\epsilon^3),$$

$$n_3 = 1 + O(\epsilon^2) \tag{5.23}$$

and

$$t_1{}^{(1)} = 1 + O(\epsilon^2), \quad t_2{}^{(1)} \equiv 0, \quad t_3{}^{(1)} = \epsilon\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial y}\right) + O(\epsilon^3). \tag{5.24}$$

From (2.72) with $\alpha = 1$, since $t_2{}^{(1)} \equiv 0$,

$$T^{(1)} = n_1 t_1{}^{(1)}\tau_{11} + n_3 t_3{}^{(1)}\tau_{33} + n_2 t_1{}^{(1)}\tau_{12} + n_1 t_3{}^{(1)}\tau_{13} + n_3 t_1{}^{(1)}\tau_{13} + n_2 t_3{}^{(1)}\tau_{23}. \tag{5.25}$$

The dimensionless components of the Cauchy stress tensor, τ_{ik}, are given by (2.54) to (2.59). They are expanded in powers of ϵ^2 using (5.1) to (5.4) and (5.6). The zero order quantities W_0 and P_0 are given by (5.7) and (5.8) and depend on z. Those zero order quantities and their partial derivatives are evaluated at $z = H + \frac{1}{2}h$. Now

$z = H + \frac{1}{2}h$:

$$n_1 t_1{}^{(1)} \tau_{11} = -2\epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial H}{\partial x} + \frac{1}{2} \frac{\partial h}{\partial x} \right) \left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} \right) + O(\epsilon^3), \tag{5.26}$$

$$n_3 t_3{}^{(1)} \tau_{33} = O(\epsilon^3), \tag{5.27}$$

$$n_2 t_1{}^{(1)} \tau_{12} = -\epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial H}{\partial y} + \frac{1}{2} \frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right) + O(\epsilon^3), \tag{5.28}$$

$$n_1 t_3{}^{(1)} \tau_{13} = O(\epsilon^3), \tag{5.29}$$

$$n_1 t_1{}^{(1)} \tau_{13} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial u_1}{\partial z} + \frac{\partial W_0}{\partial x}(x, y, H + \frac{1}{2}h) \right) + O(\epsilon^3), \tag{5.30}$$

$$n_2 t_3{}^{(1)} \tau_{23} = O(\epsilon^3). \tag{5.31}$$

Thus on the surface $z = H + \frac{1}{2}h$,

$$T^{(1)} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left[\frac{\partial u_1}{\partial z} + \frac{\partial W_0}{\partial x} \left(x, y, H + \frac{1}{2}h \right) \right.$$

$$\left. -2 \left(\frac{\partial H}{\partial x} + \frac{1}{2} \frac{\partial h}{\partial x} \right) \left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} \right) - \left(\frac{\partial H}{\partial y} + \frac{1}{2} \frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right) \right] + O(\epsilon^3). \tag{5.32}$$

Since the tangential stress vanishes on the upper surface we obtain the boundary condition,

$$z = H + \frac{1}{2}h : \qquad \frac{\partial u_1}{\partial z} = -\frac{\partial W_0}{\partial x}\left(x, y, H + \frac{1}{2}h\right)$$

$$+2\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right). \quad (5.33)$$

Consider next $T^{(2)}$ correct to order ϵ on the upper surface. From (2.81) the tangent vector is

$$t_1{}^{(2)} \equiv 0, \quad t_2{}^{(2)} = 1 + O(\epsilon^2), \quad t_3{}^{(2)} = \epsilon\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) + O(\epsilon^3). \quad (5.34)$$

From (2.72) with $\alpha = 2$, since $t_1{}^{(2)} \equiv 0$,

$$T^{(2)} = n_2 t_2{}^{(2)}\tau_{22} + n_3 t_3{}^{(2)}\tau_{33} + n_1 t_2{}^{(2)}\tau_{12} + n_1 t_3{}^{(2)}\tau_{13} + n_2 t_3{}^{(2)}\tau_{23} + n_3 t_2{}^{(2)}\tau_{23}. \quad (5.35)$$

Now,

$z = H + \frac{1}{2}h$:

$$n_2 t_2{}^{(2)}\tau_{22} = -2\epsilon\frac{\rho_G}{\rho_T}\frac{\mu_0}{A}\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right) + O(\epsilon^3), \quad (5.36)$$

$$n_3 t_3{}^{(2)}\tau_{33} = O(\epsilon^3), \quad (5.37)$$

$$n_1 t_2{}^{(2)}\tau_{12} = -\epsilon\frac{\rho_G}{\rho_T}\frac{\mu_0}{A}\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right) + O(\epsilon^3), \quad (5.38)$$

$$n_1 t_3{}^{(2)}\tau_{13} = O(\epsilon^3), \quad (5.39)$$

$$n_2 {t_3}^{(2)} \tau_{23} = O(\epsilon^3), \tag{5.40}$$

$$n_3 {t_2}^{(2)} \tau_{23} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial v_1}{\partial z} + \frac{\partial W_0}{\partial y}(x, y, H + \frac{1}{2}h) \right) + O(\epsilon^3) \tag{5.41}$$

Thus on the upper surface $z = H + \frac{1}{2}h$,

$$T^{(2)} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left[\frac{\partial v_1}{\partial z} + \frac{\partial W_0}{\partial y}(x, y, H + \frac{1}{2}h) - \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right) \right.$$

$$\left. +2 \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y} \right) \right] + O(\epsilon^3). \tag{5.42}$$

The tangential stress vanishes on the upper surface. This gives the boundary condition

$$z = H + \frac{1}{2}h : \qquad \frac{\partial v_1}{\partial z} = -\frac{\partial W_0}{\partial y} \left(x, y, H + \frac{1}{2}h \right)$$

$$+ \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right) + 2 \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y} \right). \tag{5.43}$$

Consider now the lower surface $z = H - \frac{1}{2}h$. We first calculate $T^{(1)}$ correct to first order in ϵ. The unit normal and tangential vectors are (2.93) and (2.101):

$$n_1 = \epsilon \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x} \right) + O(\epsilon^3), \quad n_2 = \epsilon \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y} \right) + O(\epsilon^3),$$

$$n_3 = -1 + O(\epsilon^2) \tag{5.44}$$

and

$$t_1{}^{(1)} = 1 + O(\epsilon^2), \quad t_2{}^{(1)} \equiv 0, \quad t_3{}^{(1)} = \epsilon \left(\frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial h}{\partial x} \right) + O(\epsilon^3). \tag{5.45}$$

Since $t_2{}^{(1)} \equiv 0$, $T^{(1)}$ is again given by (5.25). The zero order quantities P_0 and W_0 and their partial derivatives are evaluated at $z = H - \frac{1}{2}h$. Now

$z = H - \frac{1}{2}h$:

$$n_1 t_1{}^{(1)} \tau_{11} = \epsilon \frac{\rho_G}{\rho_T} \left(\frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial h}{\partial x} \right) \left[-h + 2\frac{\mu_0}{A} \left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} \right) \right] + O(\epsilon^3), \tag{5.46}$$

$$n_3 t_3{}^{(1)} \tau_{33} = \epsilon \frac{\rho_G}{\rho_T} \left(\frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial h}{\partial x} \right) h + O(\epsilon^3), \tag{5.47}$$

$$n_2 t_1{}^{(1)} \tau_{12} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial H}{\partial y} - \frac{1}{2} \frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right) + O(\epsilon^3), \tag{5.48}$$

$$n_1 t_3{}^{(1)} \tau_{13} = O(\epsilon^3), \tag{5.49}$$

$$n_3 t_1{}^{(1)} \tau_{13} = -\epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial u_1}{\partial z} + \frac{\partial W_0}{\partial x}(x, y, H - \frac{1}{2}h) \right) + O(\epsilon^3), \tag{5.50}$$

$$n_2 t_3{}^{(1)} \tau_{23} = O(\epsilon^3). \tag{5.51}$$

Thus on the lower surface $z = H - \frac{1}{2}h$,

$$T^{(1)} = \epsilon \frac{\rho_G}{\rho_T}\frac{\mu_0}{A}\left[-\frac{\partial u_1}{\partial z} - \frac{\partial W_0}{\partial x}(x, y, H - \frac{1}{2}h) + 2\left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right] + O(\epsilon^3). \qquad (5.52)$$

Since the viscosity of the molten tin is much less than the viscosity of the molten glass we have made the approximation that the tangential stress exerted by the molten tin on the lower surface of the glass ribbon is zero. We therefore have the boundary condition,

$$z = H - \frac{1}{2}h: \qquad \frac{\partial u_1}{\partial z} = -\frac{\partial W_0}{\partial x}(x, y, H - \frac{1}{2}h) + 2\left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right). \qquad (5.53)$$

Consider now $T^{(2)}$ on the lower surface of the glass to first order in ϵ. From (2.109),

$$t_1^{(2)} \equiv 0, \quad t_2^{(2)} = 1 + O(\epsilon^2), \quad t_3^{(2)} = \epsilon\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right) + O(\epsilon^3). \qquad (5.54)$$

Since $t_1^{(2)} \equiv 0$, $T^{(2)}$ on the lower surface is given by (5.35). Now

$z = H - \frac{1}{2}h$:

$$n_2 t_2^{(2)} \tau_{22} = \epsilon\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\left[-\frac{\rho_G}{\rho_T}h + 2\frac{\rho_G}{\rho_T}\frac{\mu_0}{A}\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\right] + O(\epsilon^3), \qquad (5.55)$$

$$n_3 {t_3}^{(2)} \tau_{33} = \epsilon \frac{\rho_G}{\rho_T} \left(\frac{\partial H}{\partial y} - \frac{1}{2} \frac{\partial h}{\partial y} \right) h + O(\epsilon^3), \tag{5.56}$$

$$n_1 {t_2}^{(2)} \tau_{12} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial h}{\partial x} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right) + O(\epsilon^3), \tag{5.57}$$

$$n_1 {t_3}^{(2)} \tau_{13} = O(\epsilon^3), \tag{5.58}$$

$$n_2 {t_3}^{(2)} \tau_{23} = O(\epsilon^3), \tag{5.59}$$

$$n_3 {t_2}^{(2)} \tau_{23} = -\epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \left(\frac{\partial v_1}{\partial z} + \frac{\partial W_0}{\partial y} (x, y, H - \frac{1}{2} h) \right) + O(\epsilon^3), \tag{5.60}$$

Thus on the lower surface $z = H - \frac{1}{2} h$,

$$T^{(2)} = \epsilon \frac{\rho_G}{\rho_T} \frac{\mu_0}{A} \Bigg[- \frac{\partial v_1}{\partial z} - \frac{\partial W_0}{\partial y} (x, y, H - \frac{1}{2} h) + \left(\frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial h}{\partial x} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right)$$
$$+ 2 \left(\frac{\partial H}{\partial y} - \frac{1}{2} \frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial x} + 2 \frac{\partial V_0}{\partial y} \right) \Bigg] + O(\epsilon^3). \tag{5.61}$$

Since we make the approximation that the tangential stress is zero on the lower surface we obtain the boundary condition

$$z = H - \frac{1}{2} h : \qquad \frac{\partial v_1}{\partial z} = - \frac{\partial W_0}{\partial y} (x, y, H - \frac{1}{2} h) + \left(\frac{\partial H}{\partial x} - \frac{1}{2} \frac{\partial h}{\partial x} \right) \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} \right)$$
$$+ 2 \left(\frac{\partial H}{\partial y} - \frac{1}{2} \frac{\partial h}{\partial y} \right) \left(\frac{\partial U_0}{\partial x} + 2 \frac{\partial V_0}{\partial y} \right). \tag{5.62}$$

Finally, consider the thermal boundary condition on the upper and lower surfaces of the molten glass ribbon. The thermal boundary condition on the upper surface is given by (3.24). Substitute the perturbation expansion (5.5) into (3.24) . Since S_0 does not depend on z this gives

$$z = H + \frac{1}{2}h : \qquad \left[-\epsilon^2 \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x} \right) + O(\epsilon^4) \right] \left[\frac{\partial S_0}{\partial x} + \epsilon^2 \frac{\partial S_1}{\partial x} + O(\epsilon^4) \right]$$
$$+ \left[-\epsilon^2 \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y} \right) + O(\epsilon^4) \right] \left[\frac{\partial S_0}{\partial y} + \epsilon^2 \frac{\partial S_1}{\partial y} + O(\epsilon^4) \right]$$
$$+ \left[1 + O(\epsilon^2) \right] \left[\epsilon^2 \frac{\partial S_1}{\partial z} + O(\epsilon^4) \right] = 0. \qquad (5.63)$$

By equating terms of order ϵ^2 to zero we obtain the thermal boundary condition on the upper surface.

$$z = H + \frac{1}{2}h : \qquad \frac{\partial S_1}{\partial z} = \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x} \right) \frac{\partial S_0}{\partial x} + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y} \right) \frac{\partial S_0}{\partial y}. \qquad (5.64)$$

The thermal boundary condition on the lower surface is given by (3.35). Substituting the perturbation expansion (5.5) into (3.35) gives

$$z = H - \frac{1}{2}h : \qquad \left[\epsilon^2 \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x} \right) + O(\epsilon^4) \right] \left[\frac{\partial S_0}{\partial x} + \epsilon^2 \frac{\partial S_1}{\partial x} + O(\epsilon^4) \right]$$
$$+ \left[\epsilon^2 \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y} \right) + O(\epsilon^4) \right] \left[\frac{\partial S_0}{\partial y} + \epsilon^2 \frac{\partial S_1}{\partial y} + O(\epsilon^4) \right]$$
$$+ \left[-1 + O(\epsilon^2) \right] \left[\epsilon^2 \frac{\partial S_1}{\partial z} + O(\epsilon^4) \right] = 0. \qquad (5.65)$$

The thermal boundary condition on the lower surface is obtained by equating to zero the terms of order ϵ^2.

$$z = H - \frac{1}{2}h : \qquad \frac{\partial S_1}{\partial z} = \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\frac{\partial S_0}{\partial x} + \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\frac{\partial S_0}{\partial y}. \quad (5.66)$$

This completes the derivation of the boundary conditions for $\frac{\partial u_1}{\partial z}$, $\frac{\partial v_1}{\partial z}$ and $\frac{\partial S_1}{\partial z}$ on the upper and lower surfaces of the molten glass ribbon. We summarize here for convenience of reference the boundary conditions obtained.

Upper surface $z = H + \frac{1}{2}h$:

$$\begin{aligned}\frac{\partial u_1}{\partial z} &= -\frac{\partial W_0}{\partial x}(x, y, H + \frac{1}{2}h)\\ &+2\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\\ &+\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right),\end{aligned} \quad (5.67)$$

$$\begin{aligned}\frac{\partial v_1}{\partial z} &= -\frac{\partial W_0}{\partial y}(x, y, H + \frac{1}{2}h)\\ &+2\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right)+\\ &+\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right),\end{aligned} \quad (5.68)$$

$$\frac{\partial S_1}{\partial z} = \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\frac{\partial S_0}{\partial x} + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\frac{\partial S_0}{\partial y}. \quad (5.69)$$

Lower surface $z = H - \frac{1}{2}h$:

$$\frac{\partial u_1}{\partial z} = -\frac{\partial W_0}{\partial x}\left(x, y, H - \frac{1}{2}h\right) + 2\left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)$$
$$+\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right), \tag{5.70}$$

$$\frac{\partial v_1}{\partial z} = -\frac{\partial W_0}{\partial y}(x, y, H - \frac{1}{2}h) + 2\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right)$$
$$+\left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right), \tag{5.71}$$

$$\frac{\partial S_1}{\partial z} = \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\frac{\partial S_0}{\partial x} + \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\frac{\partial S_0}{\partial y}. \tag{5.72}$$

5.4 System of equations for zero order quantities

We now derive the remaining three equations for the four zero order quantities ,U_0, V_0, S_0 and h by integrating the three equations (5.20) to (5.22) with respect to z and imposing the six boundary conditions (5.67) to (5.72) on the upper and lower surfaces of the molten glass ribbon.

Consider first equation (5.20). Integrating (5.20) with respect to z gives

$$\mu_0\frac{\partial u_1}{\partial z} = \left[A\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right) - 3\frac{\partial}{\partial x}\left(\mu_0\frac{\partial U_0}{\partial x}\right)\right.$$
$$\left.-\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial y}\right) - 2\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial y}\right)\right]z + E(x, y, t), \tag{5.73}$$

where $E(x, y, t)$ is an arbitrary function of integration. By imposing the boundary condition (5.67) on the upper surface we obtain

$$\mu_0\Bigg[-\frac{\partial W_0}{\partial x}(x,y,H+\frac{1}{2}h)+2\left(\frac{\partial H}{\partial x}+\frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)$$
$$+\left(\frac{\partial H}{\partial y}+\frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\Bigg]$$
$$=\Bigg[A\left(\frac{\partial H}{\partial x}+\frac{1}{2}\frac{\partial h}{\partial x}\right)-3\frac{\partial}{\partial x}\left(\mu_0\frac{\partial U_0}{\partial x}\right)$$
$$-\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial y}\right)-2\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial y}\right)\Bigg]\left(H+\frac{1}{2}h\right)+E(x,y,t). \tag{5.74}$$

We next impose the boundary condition (5.70) on the lower surface, $z = H-\frac{1}{2}h$, which gives

$$\mu_0\Bigg[-\frac{\partial W_0}{\partial x}(x,y,H-\frac{1}{2}h)+2\left(\frac{\partial H}{\partial x}-\frac{1}{2}\frac{\partial h}{\partial x}\right)\left(2\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)$$
$$+\left(\frac{\partial H}{\partial y}-\frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\Bigg]$$
$$=\Bigg[A\left(\frac{\partial H}{\partial x}+\frac{1}{2}\frac{\partial h}{\partial x}\right)-3\frac{\partial}{\partial x}\left(\mu_0\frac{\partial U_0}{\partial x}\right)$$
$$-\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial y}\right)-2\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial y}\right)\Bigg]\left(H-\frac{1}{2}h\right)+E(x,y,t). \tag{5.75}$$

In order to eliminate $E(x,y,t)$ we subtract (5.75) from (5.74) . This gives

$$\mu_0\Bigg[\frac{\partial W_0}{\partial x}(x,y,H-\frac{1}{2}h)-\frac{\partial W_0}{\partial x}(x,y,H+\frac{1}{2}h)$$
$$+2\frac{\partial h}{\partial x}\left(2\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)+\frac{\partial h}{\partial y}\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\Bigg]$$
$$=\Bigg[A\left(\frac{\partial H}{\partial x}+\frac{1}{2}\frac{\partial h}{\partial x}\right)-3\frac{\partial}{\partial x}\left(\mu_0\frac{\partial U_0}{\partial x}\right)-\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial y}\right)-2\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial y}\right)\Bigg]h. \tag{5.76}$$

But from (5.7),

$$\frac{\partial W_0}{\partial x}(x, y, H - \frac{1}{2}h) - \frac{\partial W_0}{\partial x}(x, y, H + \frac{1}{2}h) = h\frac{\partial}{\partial x}\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) \tag{5.77}$$

and from (4.55),

$$\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x} = \left(1 - \frac{\rho_G}{\rho_T}\right)\frac{\partial h}{\partial x} \tag{5.78}$$

Thus (5.76) becomes

$$\mu_0\left[h\frac{\partial}{\partial x}\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + 2\frac{\partial h}{\partial x}\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right]$$
$$+h\left[3\frac{\partial}{\partial x}\left(\mu_0\frac{\partial U_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial y}\right) + 2\frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial y}\right)\right] = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial x}. \tag{5.79}$$

Equation (5.79) can be rewritten in the form

$$\frac{\partial}{\partial x}\left[2\mu_0 h\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right]$$
$$-h\left(\frac{\partial \mu_0}{\partial x}\frac{\partial U_0}{\partial x} + \frac{\partial \mu_0}{\partial y}\frac{\partial V_0}{\partial x}\right) = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial x}. \tag{5.80}$$

Equation (5.80) reduces to the equation derived by Howell [2], page 99, when μ is constant.

Consider next equation (5.21) . By integrating (5.21) with respect to z we obtain

$$\mu_0\frac{\partial v_1}{\partial z} = \left[A\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial x}\right) - 3\frac{\partial}{\partial y}\left(\mu_0\frac{\partial V_0}{\partial y}\right)\right.$$
$$\left.-2\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial x}\right)\right]z + F(x, y, t), \tag{5.81}$$

where $F(x, y, t)$ is an arbitrary function of integration. Imposing the boundary condition (5.68) on the upper surface gives

$$\begin{aligned}\mu_0 \Big[&- \frac{\partial W_0}{\partial y}(x, y, H + \frac{1}{2}h) + 2\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right) \\ &+ \left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\Big] \\ &= \Big[A\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial x}\left(\mu_0 \frac{\partial V_0}{\partial x}\right) - 3\frac{\partial}{\partial y}\left(\mu_0\frac{\partial V_0}{\partial y}\right) \\ &-2\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial x}\right)\Big]\left(H + \frac{1}{2}h\right) + F(x, y, t), \end{aligned} \tag{5.82}$$

while imposing the boundary condition (5.71) on the lower surface we obtain

$$\begin{aligned}\mu_0 \Big[&- \frac{\partial W_0}{\partial y}(x, y, H - \frac{1}{2}h) + 2\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right) \\ &+ \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\Big] \\ &= \Big[A\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial x}\left(\mu_0 \frac{\partial V_0}{\partial x}\right) - 3\frac{\partial}{\partial y}\left(\mu_0\frac{\partial V_0}{\partial y}\right) \\ &-2\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial x}\right)\Big]\left(H - \frac{1}{2}h\right) + F(x, y, t). \end{aligned} \tag{5.83}$$

The function $F(x, y, t)$ is eliminated by subtracting (5.83) from (5.82) . We obtain

$$\begin{aligned}\mu_0 \Big[&\frac{\partial W_0}{\partial y}(x, y, H - \frac{1}{2}h) - \frac{\partial W_0}{\partial y}(x, y, H + \frac{1}{2}h) + 2\frac{\partial h}{\partial y}\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right) \\ &+\frac{\partial h}{\partial x}\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\Big] = \Big[A\left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial x}\right) \\ &-3\frac{\partial}{\partial y}\left(\mu_0\frac{\partial V_0}{\partial y}\right) - 2\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial x}\right)\Big]h. \end{aligned} \tag{5.84}$$

But from (5.7)

$$\frac{\partial W_0}{\partial y}(x, y, H - \frac{1}{2}h) - \frac{\partial W_0}{\partial y}(x, y, H + \frac{1}{2}h) = h\frac{\partial}{\partial y}\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) \tag{5.85}$$

and from (4.55)

$$\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y} = \left(1 - \frac{\rho_G}{\rho_T}\right)\frac{\partial h}{\partial y}. \tag{5.86}$$

Hence (5.84) becomes

$$\mu_0\left[h\frac{\partial}{\partial y}\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) + 2\frac{\partial h}{\partial y}\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right]$$
$$+h\left[3\frac{\partial}{\partial y}\left(\mu_0\frac{\partial V_0}{\partial y}\right) + \frac{\partial}{\partial x}\left(\mu_0\frac{\partial V_0}{\partial x}\right) + 2\frac{\partial}{\partial y}\left(\mu_0\frac{\partial U_0}{\partial x}\right)\right] = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial y}. \tag{5.87}$$

Equation (5.87) can be expressed in the form

$$\frac{\partial}{\partial x}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right)\right]$$
$$-h\left(\frac{\partial \mu_0}{\partial x}\frac{\partial U_0}{\partial y} + \frac{\partial \mu_0}{\partial y}\frac{\partial V_0}{\partial y}\right) = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial y}. \tag{5.88}$$

Equation (5.88) reduces to the second equation derived by Howell [2],page 99, when μ is a constant.

Finally, consider the energy equation (5.22). By integrating (5.22) with respect to z we obtain

$$\frac{\partial S_1}{\partial z} = -\left(\frac{\partial^2 S_0}{\partial x^2} + \frac{\partial^2 S_0}{\partial y^2}\right) z - E_c P_r \mu_0 \left[2\left(\frac{\partial U_0}{\partial x}\right)^2 + 2\left(\frac{\partial V_0}{\partial y}\right)^2 \right.$$
$$\left. +2\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)^2 + \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)^2 \right] z + G(x,t), \qquad (5.89)$$

where $G(x,t)$ is an arbitrary function of integration. Imposing the boundary condition (5.69) on the upper surface gives

$$\left(\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right)\frac{\partial S_0}{\partial x} + \left(\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\right)\frac{\partial S_0}{\partial y}$$
$$= -\left(\frac{\partial^2 S_0}{\partial x^2} + \frac{\partial^2 S_0}{\partial y^2}\right)\left(H + \frac{1}{2}h\right)$$
$$-E_c P_r \mu_0 \left[2\left(\frac{\partial U_0}{\partial x}\right)^2 + 2\left(\frac{\partial V_0}{\partial y}\right)^2 \right.$$
$$\left. +2\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)^2 + \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)^2 \right]\left(H + \frac{1}{2}h\right) + G(x,y,t). \qquad (5.90)$$

Imposing the boundary condition (5.72) on the lower surface we obtain

$$\left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)\frac{\partial S_0}{\partial x} + \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right)\frac{\partial S_0}{\partial y}$$
$$= -\left(\frac{\partial^2 S_0}{\partial x^2} + \frac{\partial^2 S_0}{\partial y^2}\right)\left(H - \frac{1}{2}h\right)$$
$$-E_c P_r \mu_0 \left[2\left(\frac{\partial U_0}{\partial x}\right)^2 + 2\left(\frac{\partial V_0}{\partial y}\right)^2 \right.$$
$$\left. +2\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)^2 + \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)^2 \right]\left(H - \frac{1}{2}h\right) + G(x,y,t). \qquad (5.91)$$

By subtracting (5.91) from (5.90) we obtain

$$h\left(\frac{\partial^2 S_0}{\partial x^2}+\frac{\partial^2 S_0}{\partial y^2}\right)+\frac{\partial h}{\partial x}\frac{\partial S_0}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial S_0}{\partial y}$$
$$+E_cP_r\mu_0\left[2\left(\frac{\partial U_0}{\partial x}\right)^2+2\left(\frac{\partial V_0}{\partial y}\right)^2+2\left(\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)^2+\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)^2\right]h=0. \tag{5.92}$$

Equations (4.50), (5.80), (5.88) and (5.92) depend only on $U_0(x,y,t)$, $V_0(x,y,t)$, $S_0(x,y,t)$ and $h(x,y,t)$. They form the system of four equations for the four zero order quantities U_0, V_0, S_0 and h.The four equations are:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hU_0)+\frac{\partial}{\partial y}(hV_0)=0, \tag{5.93}$$

$$2\frac{\partial}{\partial x}\left[\mu_0 h\left(2\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\right]$$
$$-h\left(\frac{\partial \mu_0}{\partial x}\frac{\partial U_0}{\partial x}+\frac{\partial \mu_0}{\partial y}\frac{\partial V_0}{\partial x}\right)=A\left(1-\frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial x}, \tag{5.94}$$

$$\frac{\partial}{\partial x}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\right]+2\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial x}+2\frac{\partial V_0}{\partial y}\right)\right]$$
$$-h\left(\frac{\partial \mu_0}{\partial x}\frac{\partial U_0}{\partial y}+\frac{\partial \mu_0}{\partial y}\frac{\partial V_0}{\partial y}\right)=A\left(1-\frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial y}, \tag{5.95}$$

$$h\left(\frac{\partial^2 S_0}{\partial x^2}+\frac{\partial^2 S_0}{\partial y^2}\right)+\frac{\partial h}{\partial x}\frac{\partial S_0}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial S_0}{\partial y}$$
$$+E_cP_r\mu_0\left[2\left(\frac{\partial U_0}{\partial x}\right)^2+2\left(\frac{\partial V_0}{\partial y}\right)^2\right.$$
$$\left.+2\left(\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)^2+\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)^2\right]h=0, \tag{5.96}$$

where

$$\mu_0(x,y,t) = Kexp\left[\frac{E_0}{S_0(x,y,t)}\right]. \tag{5.97}$$

Once U_0, V_0, S_0 and h have been calculated we can obtain H, W_0 and P_0 from (4.55), (5.7) and (5.8):

$$H = \left(\frac{1}{2} - \frac{\rho_G}{\rho_T}\right) h, \tag{5.98}$$

$$W_0(x,y,z,t) = \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)(H - z) + \frac{\partial H}{\partial t} + U_0\frac{\partial H}{\partial x} + V_0\frac{\partial H}{\partial y}, \tag{5.99}$$

$$P_0(x,y,z,t) = \frac{\rho_G}{\rho_T}\left(H + \frac{1}{2}h - z\right) - \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right). \tag{5.100}$$

The partial differential equations (5.94), (5.95) and (5.96) depend explicitly on μ_0 and apply for arbitrary $\mu_0 = \mu_0(x,y,t)$. When μ_0 is given by (5.97),

$$\frac{\partial \mu_0}{\partial x} = -E_0\frac{\mu_0}{{S_0}^2}\frac{\partial S_0}{\partial x}, \qquad \frac{\partial \mu_0}{\partial y} = -E_0\frac{\mu_0}{{S_0}^2}\frac{\partial S_0}{\partial y} \tag{5.101}$$

and equations (5.94) and (5.95) become

$$\begin{aligned} &2\frac{\partial}{\partial x}\left[\mu_0 h\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right] \\ &+E_0\frac{\mu_0}{{S_0}^2}\frac{h}{}\left(\frac{\partial S_0}{\partial x}\frac{\partial U_0}{\partial x} + \frac{\partial S_0}{\partial y}\frac{\partial V_0}{\partial x}\right) = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial x}, \end{aligned} \tag{5.102}$$

$$\begin{aligned} &\frac{\partial}{\partial x}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right)\right] \\ &+E_0\frac{\mu_0}{{S_0}^2}\frac{h}{}\left(\frac{\partial S_0}{\partial x}\frac{\partial U_0}{\partial y} + \frac{\partial S_0}{\partial y}\frac{\partial V_0}{\partial y}\right) = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial y}. \end{aligned} \tag{5.103}$$

5.5 Conclusions

The boundary and initial conditions for the system of four partial differential equations,(5.93) to (5.96), have still to be specified. The edge conditions at the boundary between the glass ribbon, the molten tin and the air remain to be derived. These conditions will be considered in the next chapter.

Chapter 6

The edge conditions and boundary and initial conditions

6.1 Introduction

In this chapter the conditions at the edge of the molten glass ribbon will be considered. They determine the boundary of the glass ribbon. Four edge conditions will be derived. A free glass ribbon without edge rollers will be considered.The edge rollers are used in the float glass process to spread the molten glass ribbon and make it either thicker or thinner. Edge rollers will not be considered in this dissertation. The derivation of the edge conditions will follow the treatment of Howell [2], pages $99 - 102$, filling in details where necessary. In addition to the kinematic condition and tangential and normal stress conditions at the edge a fourth condition at the edge for the normal component of the heat flux vector will be derived.

The boundary and initial conditions required for the numerical or analytical solution of the system of four partial differential equations for the zero order quantities will be discussed.

The edge conditions and the boundary and initial conditions will be formulated in terms of the zero order quantities which apply for the lubrication (thin film) approximation.

6.2 Kinematic condition at the edge of the glass ribbon

Consider a free glass ribbon without edge rollers as shown in Figure 6.1. The glass ribbon is symmetrical about the $x-$axis. The equation of the edge of the glass ribbon for $y > 0$, in dimensionless form, is

$$y = b(x, t). \tag{6.1}$$

At the edge, $H(x, b, t) = 0$ and $h(x, b, t) = 0$.The characteristic length for both y and b is the characteristic length L in the $xy-$plane, used in (2.9). The kinematic condition at the free boundary of the glass ribbon is that a fluid particle of the molten glass on the boundary remains on the boundary as the glass ribbon evolves. Thus

$$\frac{D}{Dt}(y - b(x, t))\bigg|_{y=b(x,t)} = 0, \tag{6.2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_0(x, y, t)\frac{\partial}{\partial x} + V_0(x, y, t)\frac{\partial}{\partial y} + W_0(x, y, t)\frac{\partial}{\partial z} \tag{6.3}$$

is the total time derivative. Hence

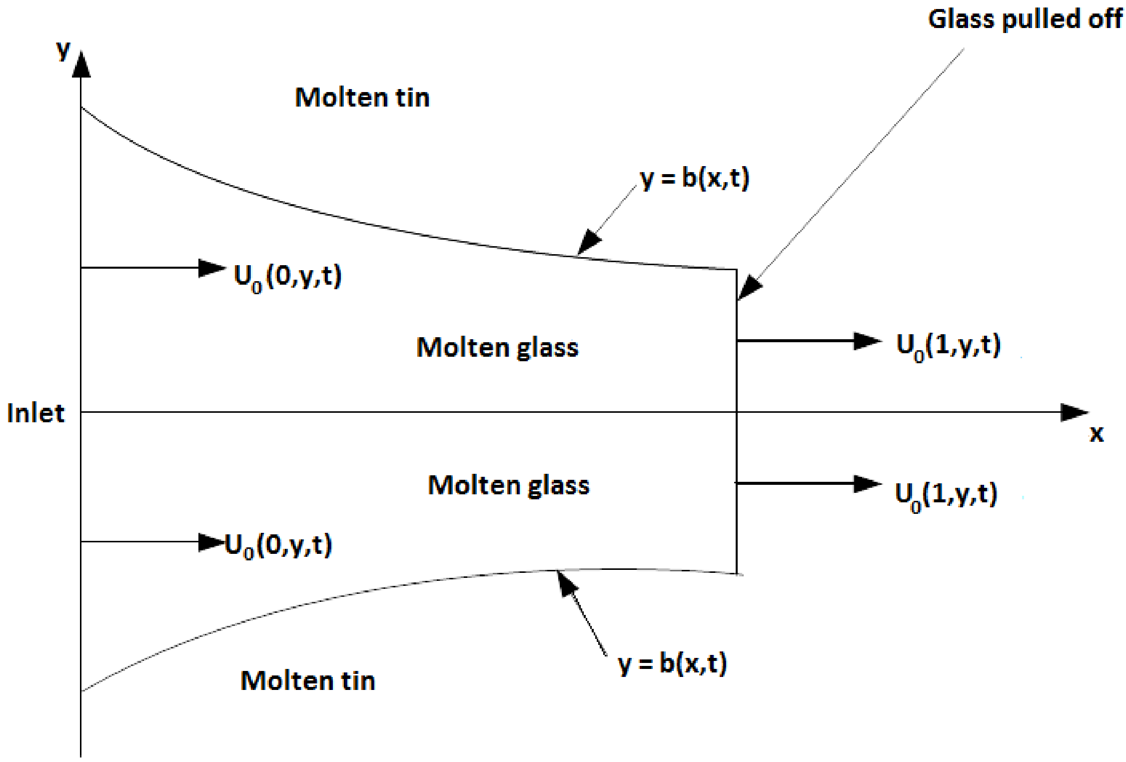

Figure 6.1: Molten glass ribbon on bath of molten tin. At the edges $y = \pm\, b(t)$, $H(x, \pm\, b(t), t) = 0$ and $h(x, \pm\, b(t), t) = 0$.

$$\frac{D}{Dt}yy = b(x,t) = \frac{\partial b}{\partial t} + U_0(x,b,t)\frac{\partial b}{\partial x}. \tag{6.4}$$

But

$$\left.\frac{D}{Dt}y\right|_{y=b(x,t)} = V_0(x,b,t) \tag{6.5}$$

and therefore

$$V_0(x,b,t) = \frac{\partial b}{\partial t} + U_0(x,b,t)\frac{\partial b}{\partial x}. \tag{6.6}$$

Equation (6.6) is the kinematic condition at the edge of the glass ribbon.

6.3 Tangential stress at the edge of the glass ribbon

Consider now the tangential stress at the edge of the glass ribbon. The dimensionless components of the Cauchy stress tensor on the (x, y)- plane are given by (2.55),(2.56) and (2.58). For zero order in ϵ^2 these components are

$$\tau_{11} = -P_0 + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\frac{\partial U_0}{\partial x}, \tag{6.7}$$

$$\tau_{22} = -P_0 + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\frac{\partial V_0}{\partial y}, \tag{6.8}$$

$$\tau_{12} = \tau_{21} = \frac{1}{A}\frac{\rho_G}{\rho_T}\mu_0\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right). \tag{6.9}$$

Using (5.98), equation (5.100) for P_0 becomes

$$P_0(x, y, z, t) = \frac{\rho_G}{\rho_T}\left[\left(1 - \frac{\rho_G}{\rho_T}\right)h - z\right] - \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right). \tag{6.10}$$

Using (6.10) the components of the Cauchy stress tensor in the (x, y)- plane become

$$\tau_{11} = -\frac{\rho_G}{\rho_T}\left[\left(1 - \frac{\rho_G}{\rho_T}\right)h - z\right] + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right), \tag{6.11}$$

$$\tau_{22} = -\frac{\rho_G}{\rho_T}\left[\left(1 - \frac{\rho_G}{\rho_T}\right)h - z\right] + \frac{2}{A}\frac{\rho_G}{\rho_T}\mu_0\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right), \tag{6.12}$$

$$\tau_{12} = \tau_{21} = \frac{1}{A}\frac{\rho_G}{\rho_T}\mu_0\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right), \tag{6.13}$$

Let $\underline{s}$ be the unit tangent vector and $\underline{n}$ the unit normal vector to the edge $y = b(x,t)$ in the $(x,y)-$ plane as shown in Figure 6.2. Let θ be the angle the unit tangent vector $\underline{s}$ makes with the $x-$ axis. Then

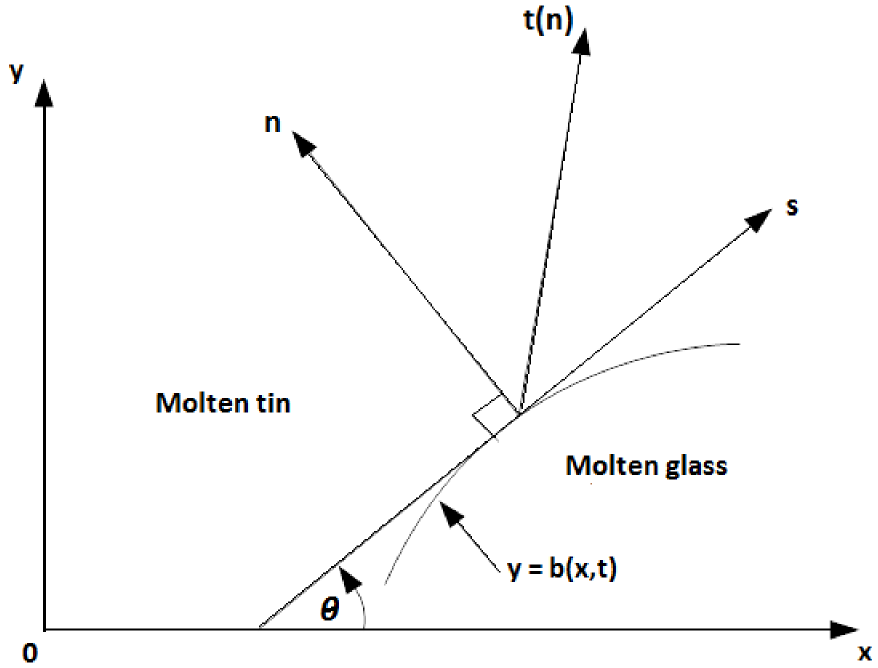

Figure 6.2:
Unit normal vector **n** and unit tangent vector **s** to edge curve $y = b(x,t)$.

$$\underline{s} = (cos\theta, sin\theta), \quad \underline{n} = (-sin\theta, cos\theta) \tag{6.14}$$

where

$$tan\theta = \frac{\partial b}{\partial x}, \quad cos\theta = \frac{1}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]^{\frac{1}{2}}}, \quad sin\theta = \frac{\frac{\partial b}{\partial x}}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]^{\frac{1}{2}}}. \tag{6.15}$$

By Cauchy's formula

$$t_i(\mathbf{n}) = n_k \tau_{ki}, \tag{6.16}$$

where $t_i(n)$ is the stress vector in the (x, y)- plane acting on the edge. Then

$$\textit{tangential component of stress vector at edge} = \mathbf{s} \cdot \mathbf{t}(\mathbf{n}) = s_i n_k \tau_{ki}. \tag{6.17}$$

where i and k run over the values 1 and 2. But the viscosity of the molten tin is much less than the viscosity of the molten glass. As in Chapter 2, the approximation is made that the tangential stress exerted by the molten tin on the glass ribbon is zero. Thus

$$y = b(x,t): \qquad s_i n_k \tau_{ki} = 0 \tag{6.18}$$

and therefore

$$y = b(x,t): \qquad s_1 n_1 \tau_{11} + (s_1 n_2 + s_2 n_1)\tau_{12} + s_2 n_2 \tau_{22} = 0. \tag{6.19}$$

Using (6.14), equation (6.19) becomes

$$y = b(x,t): \qquad (\tau_{22} - \tau_{11}) sin\theta \ cos\theta + (cos^2\theta - sin^2\theta)\tau_{12} = 0 \tag{6.20}$$

and therefore by (6.15),

$$y = b(x,t): \qquad (\tau_{22} - \tau_{11})\frac{\partial b}{\partial x} + \tau_{12}\left[1 - \left(\frac{\partial b}{\partial x}\right)^2\right] = 0. \tag{6.21}$$

Using the components of the Cauchy stress tensor, (6.11) to (6.13), equation (6.21) becomes

$$y = b(x,t): \qquad 2\frac{\partial b}{\partial x}\left(\frac{\partial U_0}{\partial x} - \frac{\partial V_0}{\partial y}\right) = \left[1 - \left(\frac{\partial b}{\partial x}\right)^2\right]\left[\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right]. \tag{6.22}$$

Equation (6.22) is the condition that the tangential stress vanishes at the edge of the ribbon.

6.4 Normal stress at the edge of the glass ribbon

There is a normal stress on the edge of the glass ribbon due to surface tension. Surface tension has the dimensions of a force per unit length. Consider a small region of molten glass between the edge $y = b(x,t)$ and a neighboring line $y = c(x,t)$ which is sufficiently far from the edge that the spatial gradients $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are small at $y = c(x,t)$. Later it will be assumed that $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are order ϵ.

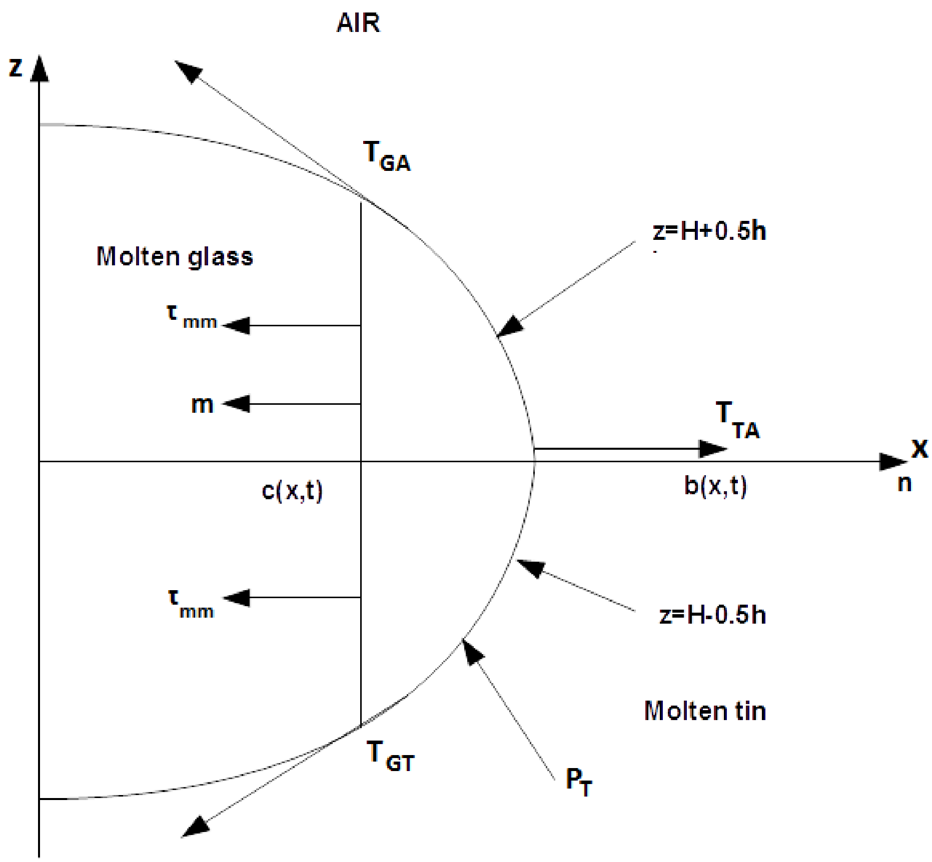

Figure 6.3: Normal force balance at the edge of the molten glass ribbon.

We consider the balance of normal forces on this small region at the edge. The normal forces per unit breadth are the components in the direction of $\underline{\mathrm{n}}$ of the following forces.

- Internal stress τ_{mm} on the surface

$$y = c(x,t), \qquad H(x,c,t) - \frac{1}{2}h(x,c,t) < z < H(x,c,t) + \frac{1}{2}h(x,c,t), \quad (6.23)$$

with unit outward normal $\mathbf{m} = -\mathbf{n}$

- Surface tension on the glass − air interface T_{GA}
- surface tension on the glass − tin interface T_{GT}
- surface tension on the tin − air interface T_{TA}
- hydrostatic pressure

$$P_T = \rho_T g \left(H - \frac{1}{2}h \right) \quad (6.24)$$

on the lower surface $z = H - \frac{1}{2}h$.

The forces are illustrated in Figure 6.3. The inertia of the molten glass is neglected because the Reynolds number $Re \ll 1$. We first use dimensional variables. The force balance in the direction of the unit normal $\underline{\mathrm{n}}$ is :

$$ {}_{TA} - \left[\frac{T_{GT}}{\left[1 + \left(\frac{\partial H}{\partial n} - \frac{1}{2}\frac{\partial h}{\partial n}\right)^2\right]^{\frac{1}{2}}} + \frac{T_{GA}}{\left[1 + \left(\frac{\partial H}{\partial n} + \frac{1}{2}\frac{\partial h}{\partial n}\right)^2\right]^{\frac{1}{2}}} \right]_{y=c(x,t)} $$

$$ - \int_{H(x,c,t)-\frac{1}{2}h(x,c,t)}^{H(x,c,t)+\frac{1}{2}h(x,c,t)} \tau_{mm} \, \mathrm{d}z $$

$$ +\rho_{TG} \int_c^b \left(H - \frac{1}{2}h \right) \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y} \right) \left[1 + \left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y} \right)^2 \right]^{\frac{1}{2}} \mathrm{d}y = 0. \quad (6.25)$$

The second term in (6.25) is the component of the surface tension T_{GT} on the lower surface in the direction **n** while the third term is the component of the surface tensions T_{GA} on the upper surface in the direction **n**. Differentiation is with respect to the coordinate n in the direction of the unit vector **n**. The fourth term is the total normal force per unit breadth on the internal surface (6.23) with unit outward normal $\mathbf{m} = -\mathbf{n}$. The fifth term is the total force on the lower surface in the direction of **n** due to the hydrostatic pressure P_T on the lower surface. A positive sign is used because from (5.98),

$$\left(H - \frac{1}{2}h\right)\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right) = \left(\frac{\rho_G}{\rho_T}\right)^2 h\frac{\partial h}{\partial y} < 0 \tag{6.26}$$

since h is a decreasing function of y.

Equation (6.25) is made dimensionless. Now *characteristic surface tension is*

$$\mu^* W = \mu^* \epsilon U, \tag{6.27}$$

which has the dimensions of force per unit length. The dimensionless variables defined in (2.9) are used. We also define

$$T_{TA} = \mu^* \epsilon U \bar{T}_{TA}, \quad T_{GT} = \mu^* \epsilon U \bar{T}_{GT}, \quad T_{GA} = \mu^* \epsilon U \bar{T}_{GA},$$

$$\tau_{mm} = \rho_T g \epsilon L \bar{\tau}_{mm}, \quad c = L\bar{c}, \quad n = L\bar{n}. \tag{6.28}$$

Equation (6.25) expressed in dimensionless variables becomes

$$\bar{T}_{TA} - \left[\frac{\bar{T}_{GT}}{\left[1+\epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{n}} - \frac{1}{2}\frac{\partial \bar{n}}{\partial \bar{n}}\right)^2\right]^{\frac{1}{2}}} + \frac{\bar{T}_{GA}}{\left[1+\epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{n}} + \frac{1}{2}\frac{\partial \bar{n}}{\partial \bar{y}}\right)^2\right]^{\frac{1}{2}}} \right]_{y=\bar{c}(\bar{x},\bar{t})}$$

$$-\frac{\rho_T}{\rho_G} A \int_{\bar{H}(\bar{x},\bar{c},\bar{t})-\frac{1}{2}\bar{h}(\bar{x},\bar{c},\bar{t})}^{\bar{H}(\bar{x},\bar{c},\bar{t})+\frac{1}{2}\bar{h}(\bar{x},\bar{c},\bar{t})} \bar{\tau}_{mm}\, \mathrm{d}\bar{z}$$

$$+\frac{\rho_T}{\rho_G} A \int_{\bar{c}}^{\bar{b}} \left(\bar{H} - \frac{1}{2}\bar{h}\right)\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right) \mathrm{d}y \left[1+\epsilon^2\left(\frac{\partial \bar{H}}{\partial \bar{y}} - \frac{1}{2}\frac{\partial \bar{h}}{\partial \bar{y}}\right)^2\right]^{-\frac{1}{2}} = 0 \tag{6.29}$$

where A is defined by (2.15). We neglect terms of order ϵ^2 and suppress the overhead bar on dimensionless variables. Equation (6.29) reduces to

$$T_{TA} - T_{GT} - T_{GA} - \frac{\rho_T}{\rho_G} A \int_{H(x,c,t)-\frac{1}{2}h(x,c,t)}^{H(x,c,t)+\frac{1}{2}h(x,c,t)} \tau_{mm}\, \mathrm{d}z$$

$$+\frac{\rho_T}{\rho_G} A \int_{c}^{b} \left(H - \frac{1}{2}h\right)\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right) \mathrm{d}y = 0. \tag{6.30}$$

In order to evaluate (6.30) consider first τ_{mm}. Since $\mathbf{m} = -\mathbf{n}$, it follows from (6.14) that

$$\mathbf{m} = (sin\theta, -cos\theta, 0) \tag{6.31}$$

and therefore from (6.15)

$$m_1 = \frac{\frac{\partial b}{\partial x}}{\left[1+\left(\frac{\partial b}{\partial x}\right)^2\right]^{\frac{1}{2}}},\ m_2 = -\frac{1}{\left[1+\left(\frac{\partial b}{\partial x}\right)^2\right]^{\frac{1}{2}}},\ m_3 \equiv 0. \tag{6.32}$$

From Cauchy's formula,

$$t_i(\mathbf{m}) = m_k \tau_{ki} \tag{6.33}$$

and hence

$$\tau_{mm} = \mathbf{m} \cdot \mathbf{t}(\mathbf{m}) = \mathbf{m_i m_k} \tau_{\mathbf{ki}} \tag{6.34}$$

Since $m_3 = 0$,

$$\tau_{mm} = {m_1}^2 \tau_{11} + 2 m_1 m_2 \tau_{12} + m^2 \tau_{22} \tag{6.35}$$

and therefore

$$\tau_{mm} = \frac{1}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]} \left[\left(\frac{\partial b}{\partial x}\right)^2 \tau_{11} - 2\frac{\partial b}{\partial x}\tau_{12} + \tau_{22} \right]. \tag{6.36}$$

The components of the stress tensor τ_{11}, τ_{12} and τ_{22} are given by (6.11), (6.12) and (6.13) . Substituting (6.11) to (6.13) into (6.36) gives

$$\begin{aligned} \tau_{mm} = &-\frac{\rho_G}{\rho_T}\left[\left(1 - \frac{\rho_G}{\rho_T}\right) h - z\right] \\ &+2\frac{\mu_0}{A}\frac{\rho_G}{\rho_T}\frac{1}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]}\left[\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\left(\frac{\partial b}{\partial x}\right)^2 - \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\frac{\partial b}{\partial x} \right. \\ &\left. + \frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right]. \end{aligned} \tag{6.37}$$

Hence

$$\begin{aligned} \frac{\rho_T}{\rho_G} A \int_{H - \frac{1}{2}h}^{H + \frac{1}{2}h} \tau_{mm}\, \mathrm{d}z = &-A\left[\left(1 - \frac{\rho_G}{\rho_T}\right) h - H\right] h \\ &+ \frac{2\mu_0}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]}\left[\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\left(\frac{\partial b}{\partial x}\right)^2 - \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\frac{\partial b}{\partial x} \right. \\ &\left. + \frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right] h \end{aligned} \tag{6.38}$$

and using (5.98) to express H in terms of h we obtain

$$\frac{\rho_T}{\rho_G} A \int_{H(x,c,t)-\frac{1}{2}h(x,c,t)}^{H(x,c,t)+\frac{1}{2}h(x,c,t)} \tau_{mm} \, \mathrm{d}z = -\frac{1}{2} A h^2(x,c,t)$$
$$+ \frac{2\mu_0}{\left[1+\left(\frac{\partial b}{\partial x}\right)^2\right]} \left[\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\left(\frac{\partial b}{\partial x}\right)^2 - \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\frac{\partial b}{\partial x} \right.$$
$$\left. + \frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right] h(x,c,t). \tag{6.39}$$

Consider the last term in (6.30). Now

$$\int_c^b \left(H - \frac{1}{2}h\right)\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right) \mathrm{d}y = \frac{1}{2}\int_c^b \frac{\partial}{\partial y}\left[\left(H - \frac{1}{2}h\right)^2\right] \mathrm{d}y. \tag{6.40}$$

Integrating (6.40), noting that

$$\left(H - \frac{1}{2}h\right)_{y=b} = 0 \tag{6.41}$$

because $H - \frac{1}{2}h = 0$ at the edge of the glass ribbon and using (5.98) to express H in terms of h we obtain

$$\int_c^b \left(H - \frac{1}{2}h\right)\left(\frac{\partial H}{\partial y} - \frac{1}{2}\frac{\partial h}{\partial y}\right) \mathrm{d}y = -\frac{1}{2}\left(\frac{\rho_G}{\rho_T}\right)^2 h^2(x,c,t). \tag{6.42}$$

Finally, substituting (6.39) and (6.42) into (6.30) we obtain

$$\frac{2\mu_0}{\left[1+\left(\frac{\partial b}{\partial x}\right)^2\right]} \left[\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\left(\frac{\partial b}{\partial x}\right)^2 - \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\frac{\partial b}{\partial x} + \frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right]$$
$$= \frac{1}{2} A \left(1 - \frac{\rho_G}{\rho_T}\right) h(x,c,t) - \frac{T_{GT} + T_{GA} - T_{TA}}{h(x,c,t)}. \tag{6.43}$$

Equation (6.42) is the condition on the normal stress at the edge of the glass ribbon. The quantity $h(x, c, t)$ is the value of h sufficiently far from the edge that $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are small so that the glass ribbon is approximately flat.

6.5 Normal component of the heat flux at the edge of the glass ribbon

The thermal boundary condition at the edge of the glass ribbon is that the normal component of the heat flux is continuous. We make the approximation, as in Section 3.4, that the heat flux in the air and molten tin can be neglected even although the thermal conductivity of the molten tin is greater than the thermal conductivity of the molten glass. The condition at the edge of the glass ribbon is therefore

$$y = b(x,t): \qquad \mathbf{n} \cdot \mathbf{q} = \mathbf{0}. \tag{6.44}$$

Using Fourier's law of heat conduction, (3.15), and expressing T in terms of S using (3.6) we obtain for the thin film approximation with $S = S_0(x, y, t)$,

$$y = b(x,t): \qquad n \cdot \boldsymbol{\nabla} S_0 = 0, \tag{6.45}$$

where from (6.14) and (6.15),

$$n_1 = -\frac{\frac{\partial b}{\partial x}}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]^{\frac{1}{2}}}, \qquad n_2 = \frac{1}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]^{\frac{1}{2}}}. \tag{6.46}$$

The heat flux condition at the edge of the glass ribbon is therefore

$$y = b(x,t): \qquad -\frac{\partial b}{\partial x}\frac{\partial S_0}{\partial x}(x,b,t) + \frac{\partial S_0}{\partial y}(x,b,t) = 0. \tag{6.47}$$

6.6 Boundary and initial conditions

The glass ribbon on the bath of molten tin is illustrated in Figure 6.1. The partial differential equations for $U_0(x,y,t)$, $V_0(x,y,t)$, $S_0(x,y,t)$ and $h(x,y,t)$ have to be solved subject to given boundary and initial conditions. Since the glass ribbon is symmetric about the $x-$ axis there are also symmetry conditions. Typical conditions could be as follows (Howell [2], page 102).

Boundary conditions at inlet

$$x = 0: \qquad U_0(0,y,t) \quad \textit{given}, \tag{6.48}$$

$$x = 0: \qquad V_0(0,y,t) = 0, \tag{6.49}$$

$$x = 0: \qquad S_0(0,y,t) \quad \textit{given}, \tag{6.50}$$

$$x = 0: \qquad h(0,y,t) \quad \textit{given}, \tag{6.51}$$

$$x = 0: \qquad b(0,t) \quad \textit{given}. \tag{6.52}$$

Boundary conditions at outlet

$$x = 1: \qquad U_0(1,y,t) \quad \textit{given}, \tag{6.53}$$

$$x = 1 : \qquad V_0(1, y, t) = 0, \tag{6.54}$$

$$x = 1 : \qquad S_0(1, y, t) \quad given. \tag{6.55}$$

Symmetry conditions

$$y = 0 : \qquad \frac{\partial U_0}{\partial y}(x, 0, t) = 0, \tag{6.56}$$

$$y = 0 : \qquad \frac{\partial S_0}{\partial y}(x, 0, t) = 0, \tag{6.57}$$

$$y = 0 : \quad V_0(x, 0, t) = 0. \tag{6.58}$$

Initial conditions

$$t = 0 : \qquad h(x, y, 0) \quad given, \tag{6.59}$$

$$t = 0 : \qquad b(x, 0) \quad given. \tag{6.60}$$

The precise boundary and initial conditions would have to be specified by the glass manufacturer.

6.7 Conclusions

The conditions at the edge of the glass ribbon are given by (6.6), (6.22), (6.43) and (6.47). In (6.43) the surface tension is a given quantity. The first three edge conditions are the same as when viscosity does not depend on temperature [2]. The four edge conditions have to be combined with the four partial differential equations for U_0, V_0, S_0 and h. The complete system of equations and boundary and initial conditions is presented in the next chapter.

Chapter 7

Conclusions

7.1 Complete system of zero order equations

We first bring together and summarize the partial differential equations, edge conditions and boundary and initial conditions which have been derived in several different sections. The unknown functions are $U_0(x, y, t)$, $V_0(x, y, t)$, $S_0(x, y, t)$, $h(x, y, t)$ and the edge curve $b(x, t)$.

Partial differential equations:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU_0) + \frac{\partial}{\partial y}(hV_0) = 0, \tag{7.1}$$

$$2\frac{\partial}{\partial x}\left[\mu_0 h\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right]$$
$$+E_0\frac{\mu_0 h}{{S_0}^2}\left(\frac{\partial S_0}{\partial x}\frac{\partial U_0}{\partial x} + \frac{\partial S_0}{\partial y}\frac{\partial V_0}{\partial x}\right) = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial x}, \tag{7.2}$$

$$\frac{\partial}{\partial x}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right)\right]$$
$$+E_0\frac{\mu_0 h}{{S_0}^2}\left(\frac{\partial S_0}{\partial x}\frac{\partial U_0}{\partial y} + \frac{\partial S_0}{\partial y}\frac{\partial V_0}{\partial y}\right) = A\left(1 - \frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial y}, \tag{7.3}$$

$$\frac{\partial^2 S_0}{\partial x^2} + \frac{\partial^2 S_0}{\partial y^2} + \frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial S_0}{\partial x} + \frac{1}{h}\frac{\partial h}{\partial y}\frac{\partial S_0}{\partial y}$$

$$+E_c P_r \mu_0 \left[2\left(\frac{\partial U_0}{\partial x}\right)^2 + 2\left(\frac{\partial V_0}{\partial y}\right)^2 + 2\left(\frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right)^2 + \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)^2 \right] = 0, \tag{7.4}$$

where

$$\mu_0(x,y,t) = Kexp\left[\frac{E_0}{S_0(x,y,t)}\right]. \tag{7.5}$$

Edge conditions

$$y = b(x,t): \qquad V_0(x,b,t) = \frac{\partial b}{\partial t} + U_0(x,b,t)\frac{\partial b}{\partial x}, \tag{7.6}$$

$$y = b(x,t): \qquad 2\frac{\partial b}{\partial x}\left[\frac{\partial U_0}{\partial x} - \frac{\partial V_0}{\partial y}\right] = \left[1 - \left(\frac{\partial b}{\partial x}\right)^2\right]\left[\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right], \tag{7.7}$$

$$y = b(x,t): \quad \frac{2\mu_0}{\left[1 + \left(\frac{\partial b}{\partial x}\right)^2\right]}\left[\left(2\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)\left(\frac{\partial b}{\partial x}\right)^2 - \left(\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x}\right)\frac{\partial b}{\partial x} + \frac{\partial U_0}{\partial x} + 2\frac{\partial V_0}{\partial y}\right]$$

$$= \frac{1}{2}A\left(1 - \frac{\rho_G}{\rho_T}\right)h(x,c,t) - \frac{T_{GT} + T_{GA} - T_{TA}}{h(x,c,t)}, \tag{7.8}$$

$$y = b(x,t): \qquad -\frac{\partial b}{\partial x}\frac{\partial S_0}{\partial x}(x,b,t) + \frac{\partial S_0}{\partial y}(x,b,t) = 0, \tag{7.9}$$

where $y = c(x,t)$ is sufficiently far from the edge that $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are small.

Boundary conditions at inlet

$$x = 0: \quad U_0(0, y, t), \quad S_0(0, y, t), \quad h(0, y, t),\ b(0, t)\ \textit{are prescribed}, \tag{7.10}$$

$$x = 0: \quad V_0(0, y, t) = 0. \tag{7.11}$$

Boundary conditions at outlet

$$x = 1: \quad U_0(1, y, t), \quad S_0(1, y, t)\ \textit{are prescribed}, \tag{7.12}$$

$$x = 1: \quad V_0(1, y, t) = 0. \tag{7.13}$$

Symmetry conditions

$$y = 0: \quad \frac{\partial U_0}{\partial y}(x, 0, t) = 0, \quad \frac{\partial S_0}{\partial y}(x, 0, t) = 0, \quad V_0(x, 0, t) = 0. \tag{7.14}$$

Initial conditions

$$t = 0: \quad h(x, y, 0) \quad \textit{and} \quad b(x, 0)\ \textit{are prescribed}. \tag{7.15}$$

Once U_0, V_0, S_0 and h have been calculated, H, W_0 and P_0 are obtained from :

$$H(x, y, t) = \left(\frac{1}{2} - \frac{\rho_G}{\rho_T}\right) h, \tag{7.16}$$

$$W_0(x, y, z, t) = \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right)(H - z) + \frac{\partial H}{\partial t} + U_0 \frac{\partial H}{\partial x} + V_0 \frac{\partial H}{\partial y}, \tag{7.17}$$

$$P_0(x, y, z, t) = \frac{\rho_G}{\rho_T}\left(H + \frac{1}{2}h - z\right) - \frac{2}{A}\frac{\rho_G}{\rho_T}\frac{\mu_0}{A}\left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right). \tag{7.18}$$

When $\mu(x, y, t)$ is not specified as in (7.5), equations (7.2) and (7.3) are of the form

$$2\frac{\partial}{\partial x}\left[\mu_0 h\left(2\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\right]$$
$$-h\left(\frac{\partial \mu_0}{\partial x}\frac{\partial U_0}{\partial x}+\frac{\partial \mu_0}{\partial y}\frac{\partial V_0}{\partial x}\right)=A\left(1-\frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial x}, \tag{7.19}$$

$$\frac{\partial}{\partial x}\left[\mu_0 h\left(\frac{\partial U_0}{\partial y}+\frac{\partial V_0}{\partial x}\right)\right]+2\frac{\partial}{\partial y}\left[\mu_0 h\left(\frac{\partial U_0}{\partial x}+2\frac{\partial V_0}{\partial y}\right)\right]$$
$$-h\left(\frac{\partial \mu_0}{\partial x}\frac{\partial U_0}{\partial y}+\frac{\partial \mu_0}{\partial y}\frac{\partial V_0}{\partial y}\right)=A\left(1-\frac{\rho_G}{\rho_T}\right)h\frac{\partial h}{\partial y}. \tag{7.20}$$

7.2 Conclusions

Equations (7.1) to (7.18) describe the evolution of the molten glass ribbon on a bath of molten thin in the lubrication (thin film) approximation taking into account temperature dependence of the viscosity. The temperature can decrease from $1100°C$ at the inlet to $600°C$ at the outlet and the viscosity can increase from 10^3 $Pa\ s$ to 10^{10} $Pa\ s$ in this range.

It was the objective of this dissertation only to derive the thin film equations and not to solve them. Howell [2] has outlined a numerical procedure for constant μ_0 which could be extended to a temperature dependent viscosity. When the initial values $h(x, y, 0)$ and $b(x, 0)$ are given, the edge conditions (7.7), (7.8) and (7.9) and the end conditions on U_0, V_0, and S_0 at $x = 0$ and $x = 1$ could serve as boundary conditions in the solution of the partial differential equations (7.2), (7.3) and (7.4) for U_0, V_0 and S_0. Equations (7.1) and (7.6) could then be used to obtain h and b at a later time.

In order to obtain the complete set of equations for the thin fluid film approximation it was necessary to consider terms in the next order, order ϵ^2, is the perturbation expansion. This occurs frequently in perturbation methods, for example, in the method of multiple scales. Otherwise terms of order ϵ^2 are not useful because $\epsilon = 3 \times 10^{-3}$ and they are very small.

The heat flux from the molten glass into the molten tin at the lower surface and at the edge of the glass ribbon was neglected even although the thermal conductivity of the molten tin is greater than the thermal conductivity of the molten glass. This approximation was made to keep the model simple. The effect of heat flow into the molten tin on the spreading of the glass ribbon could be investigated by imposing the exact boundary condition that the normal component of the heat flux is at the interface continuous
Further work would be to consider the numerical solution of equations (7.1) to (7.18). The approximation of a narrow glass ribbon could also be considered. The width, W^*, of the glass ribbon is approximately 4.2 m and the length, L, is about 35 m and therefore

$$\delta = \frac{W^*}{L} = 0.12. \tag{7.21}$$

The length scale in the $y-$ direction is smaller than in the $x-$ direction. The partial differential equations and edge conditions could be rescaled by defining

$$y = \delta y^*, \quad V_0 = \delta {V_0}^*, \quad b = \delta b^* \tag{7.22}$$

and a perturbation expansion of U_0, ${V_0}^*$, S_0 and b^* in powers of δ^2 could be considered.

Appendix A

A.1 The Energy Balance Equation

We present an outline of derivation of the energy balance equation for a viscous incompressible Newtonian fluid [9, 10]. Consider a material volume V with a closed material surface S which consists at all times of the same fluid particles as shown in Figure A.1. The energy balance equation states:

$$\begin{aligned}
&\textit{rate of change of kinetic energy of } V + \textit{rate of change of internal energy of } V \\
&\quad = \textit{rate of heat inflow in } V + \textit{rate of working of surface stresses } \mathbf{T}(\mathbf{n}) \\
&\qquad + \textit{rate of working of body force per unit mass } \mathbf{F}. \qquad \text{(A.1)}
\end{aligned}$$

Consider first the rate of heat inflow into the material volume V. The heat flow vector is $\mathbf{q}$. It is defined by :

$$\begin{aligned}
&\textit{rate of flow of heat into material volume } V \\
&= -\int_S \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S \\
&= -\int_V \boldsymbol{\nabla} \cdot \mathbf{q} \, \mathrm{d}V, \qquad \text{(A.2)}
\end{aligned}$$

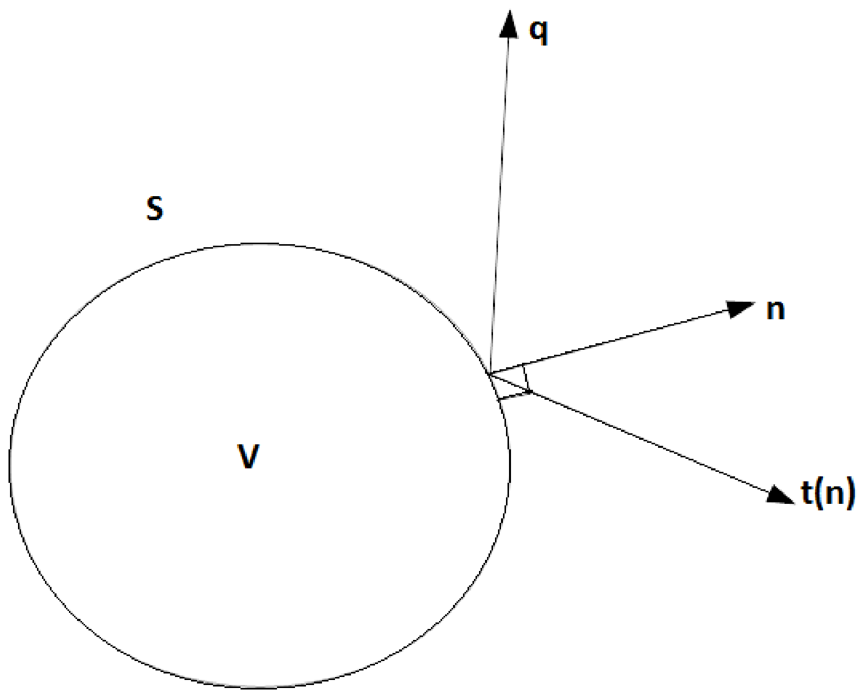

Figure $A.1$ Material volume V with closed material surface S. The unit normal vector is **n**, the stress vector is **T** (**n**) and the heat flux is **q**

.

where the divergence theorem was applied to change from a surface to a volume integral. We assume that Fourier's law of heat induction is satisfied:

$$\mathbf{q} = -k\boldsymbol{\nabla}T, \tag{A.3}$$

where k is the coefficient of heat conduction. We assume that k is a constant. Equation $(A.2)$ becomes:

$$\textit{rate of flow of heat into material volume } V = k\int_V \nabla^2 T \,\mathrm{d}V. \tag{A.4}$$

Consider now a small material volume δV. Then

$$\textit{rate of flow of heat into material volume} \quad \delta V = k\nabla^2 T\delta V. \tag{A.5}$$

Consider next the rate of working of the surface stresses $\mathbf{T}(\mathbf{n})$. We apply Cauchy's formula relating the stress vector $T_i(\mathbf{n})$ to the Cauchy stress tensor τ_{ik} :

$$T_i(\mathbf{n}) = n_k\tau_{ki}. \tag{A.6}$$

We assume that the Cauchy stress tensor is symmetric so that

$$\tau_{ik} = \tau_{ki}. \tag{A.7}$$

Now,

$$\textit{rate of working of the surface stresses} = \int_S T_i(\mathbf{n})v_i \,\mathrm{d}S$$

$$= \int_S n_k \tau_{ki} v_i \,\mathrm{d}S$$
$$= \int_V (\tau_{ki} v_i)_{,k} \,\mathrm{d}V$$
$$= \int_V (\tau_{ki,k} v_i + \tau_{ki} v_{i,k}) \,\mathrm{d}V, \tag{A.8}$$

where a comma followed by the index k denotes partial differentiation with respect to x_k and the divergence theorem in index notation was used to change from a surface to a volume integral. But since τ_{ik} is symmetric

$$\tau_{ki} v_{i,k} = \frac{1}{2} \tau_{ik} (v_{i,k} + v_{k,i}) = \tau_{ik} D_{ik}, \tag{A.9}$$

where the rate-of-strain tensor is defined as

$$D_{ik} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right). \tag{A.10}$$

We now assume that the fluid is a Newtonian fluid. The Navier- Poisson law for an incompressible Newtonian fluid is

$$\tau_{ik} = -p\delta_{ik} + 2\mu D_{ik} \tag{A.11}$$

and therefore from $(A.9)$,

$$\tau_{ki} v_{i,k} = -p D_{kk} + 2\mu D_{ik} D_{ik} = 2\mu D_{ik} D_{ik} \tag{A.12}$$

because

$$D_{kk} = \nabla \cdot \mathbf{v} = 0 \tag{A.13}$$

for an incompressible fluid. Using $(A.13)$, equation $(A.8)$ becomes

$$\textit{rate of working of the surface stresses} = \int_V (\tau_{ki,k} v_i + 2\mu D_{ik} D_{ik})\,\mathrm{d}V. \quad (A.14)$$

Consider a small material volume δV. Then

$$\textit{rate of working of the surface stresses} = (\tau_{ki,k} v_i + 2\mu D_{ik} D_{ik})\delta V. \quad (A.15)$$

Consider now the rate-of-change of the kinetic energy of a small material volume δV moving with the fluid. Let δM be the mass of the material volume δV. Then

$$\delta M = \rho \delta V \quad (A.16)$$

and since δV consists at all times of the same fluid particles as the fluid evolves

$$\frac{D}{Dt}(\rho \delta V) = \frac{D}{Dt}(\delta M) = 0. \quad (A.17)$$

Now,

$$\begin{aligned} &\textit{rate of change of the kinetic energy of the material volume } \delta V \\ &= \frac{D}{Dt}\left(\frac{1}{2}\delta M \mathbf{v}\cdot\mathbf{v}\right) \\ &= \frac{1}{2}\delta M \frac{D}{Dt}(v_i v_i) \\ &= \rho v_i \frac{Dv_i}{Dt}\delta V. \end{aligned} \quad (A.18)$$

Consider now the rate-of-change of initial energy. The internal energy consists of the kinetic energy of random thermal motions of the fluid particles and other forms of energy such as energy of vibration if the fluid particles are in the form of molecules. Let

$$\textit{internal energy per unit mass} = e = c_v T, \tag{A.19}$$

where T is the absolute temperature and c_v is the specific heat at constant volume which we assume is a constant. Consider again a small material volume δV moving with the fluid. Then

$$\begin{aligned}\textit{rate of change of internal energy of material volume } \delta V &= \\ &= \frac{D}{Dt}(e\delta M) \\ &= \delta M \frac{D}{Dt}(c_v T) \\ &= \rho c_v \frac{DT}{Dt}\delta V. \end{aligned} \tag{A.20}$$

Finally consider the rate of working of the body force. Let

$$\textit{body force per unit mass} = \mathbf{F}. \tag{A.21}$$

and consider again a small material volume δV moving with the fluid. Then

$$\begin{aligned}\textit{rate of working of body force acting on } \delta V &= \delta M \mathbf{F}\cdot\mathbf{v} \\ &= \rho F_i v_i \delta V. \end{aligned} \tag{A.22}$$

We substitute $(A.5)$, $(A.15)$, $(A.18)$, $(A.20)$ and $(A.22)$ into the energy balance equation $(A.1)$. This gives

$$v_i\left[\rho\frac{Dv_i}{Dt} - \tau_{ki,k} - \rho F_i\right]\delta V + \rho c_v \frac{DT}{Dt}\delta V$$

$$= k\nabla^2 T\delta V + 2\mu D_{ik} D_{ik}\delta V. \tag{A.23}$$

But by Cauchy's first law of motion

$$\rho\frac{Dv_i}{D_t} = \tau_{ki,k} + \rho F_i. \tag{A.24}$$

Equation $(A.23)$ reduces to

$$c_v\rho\frac{DT}{Dt} = k\nabla^2 T + 2\mu D_{ik} D_{ik}. \tag{A.25}$$

Equation $(A.25)$ is the energy balance equation for an incompressible viscous Newtonian fluid.

The term $2\mu D_{ik} D_{ik}$ is positive because it is the sum of squares. It is the rate per unit volume at which viscous stresses are converting mechanical energy into heat. In expanded form

$$\begin{aligned} D_{ik}D_{ik} &= {D_{11}}^2 + {D_{22}}^2 + {D_{33}}^2 + 2{D_{12}}^2 + 2{D_{13}}^2 + 2{D_{23}}^2 \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} \\ &+ \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]. \end{aligned} \tag{A.26}$$

Expanded in full the energy balances equation $(A.25)$ becomes

$$\begin{aligned} \rho c_v &\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] \\ &= k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] \\ &+\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right. \end{aligned}$$

$$
\begin{aligned}
&+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}+2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} \\
&+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\Bigg]. \qquad (A.27)
\end{aligned}
$$

Appendix B

Parameter Values

B.1 Introduction

Estimates of the physical parameters and dimensionless numbers in a typical float glass process are presented. Most glass produced in the float glass process is soda-lime glass. The thermal conductivity of glasses at high temperatures are given by Kiyohashi at al [11]. The thermal conductivity of liquid tin is given by Peralta-Martinez and Wakeham [12].

B.2 Physical parameters

Some values of the physical parameters are given at a specific temperature and some values are mean values over a range of temperatures. The following standard notation for units is used:

$$m = meter, \quad kg = kilogram, \quad s = second, \quad K = degree\ Kelvin,$$

$$C = degree\ centigrade, \quad Pa = Pascal, \quad J = Joule, \quad W = Watt.$$

density of molten glass at $1100°C$ $\rho_G = 2.4 \times 10^3 kg\ m^{-1}$

density of molten glass at $600°C$ $\rho_G = 2.8 \times 10^3 kgm^{-1}$

mean density of molten glass $\rho_G = 2.6 \times 10^3 kg\ m^{-3}$

density of molten tin $\rho_T = 7 \times 10^3 kg\ m^{-3}$

viscosity of molten glass at $1100°C$ $\mu = 10^3 Pa\ s$

viscosity of molten glass at $850°C$ $\mu = 10^5 Pa\ s$

viscosity of molten glass at $700°C$ $\mu = 10^7 Pa\ s$

viscosity of molten glass at $600°C$ $\mu = 10^{10} Pa\ s$

length of glass ribbon $L = 35\ m$

thickness of glass ribbon $H^* = 10^{-2}\ m$

width of glass ribbon $W^* = 4.2\ m$

velocity of glass ribbon $U = 10^{-1} ms^{-1}$

mean specific heat of molten glass (25° to 850°) $c_v = 1162\ J\ kg^{-1}K^{-1}$

thermal conductivity of molten glass at $1100°C$ $k_G = 1.5\ Wm^{-1}K^{-1}$

thermal conductivity of molten glass at $600°C$ $k_G = 1.2\ Wm^{-1}K^{-1}$

mean thermal conductivity of molten glass at ($600°C$ to $1100°C$)
$k_G = 1.35\ Wm^{-1}K^{-1}$

thermal conductivity of molten tin at $600°C$ $k_T = 34\ Wm^{-1}K^{-1}$

thermal conductivity of molten tin at $750°C$ $k_T = 35\ Wm^{-1}K^{-1}$

thermal conductivity of molten tin at $25°C$ $k_A = 2.4W\ m^{-1}K^{-1}$

$$mean\ density\ ratio\ \frac{\rho_G}{\rho_T} = 0.37$$

$$thin\ fluid\ film\ parameter\ \epsilon = \frac{H^*}{L} = 3 \times 10^{-4}$$

$$narrow\ ribbon\ parameter\ \delta = \frac{W^*}{L} = 1.2 \times 10^{-1}$$

B.3 Dimensionless numbers

$$Reynolds\ number\ Re = \frac{\rho_G U L}{\mu}$$

$$Ratio\ of\ gravity\ force\ to\ viscous\ force\ A = \frac{\epsilon \rho_G g L^2}{\mu U}$$

$$Prandtl\ number\ P_r = \frac{\mu c_v}{k_G}$$

$$Eckert\ number\ E_c = \frac{U^2}{c_v T_0}$$

The dependence of the dimensionless numbers on the temperature and viscosity is illustrated on Table $B.1$. The molten glass enters at the inlet at 1100°C and is pulled off at the outlet at 600°C.

T °C	μ $Pa\ s$	Re	$Re\left(\frac{H^*}{L}\right)^2$	A	E_c	P_r	E_cP_r
1100	10^3	9	8.1×10^{-7}	94	7.8×10^{-9}	8.6×10^5	6.7×10^{-3}
850	10^5	9×10^{-2}	8.1×10^{-9}	9.4×10^{-1}	10^{-8}	8.6×10^7	8.6×10^{-1}
700	10^7	9×10^{-4}	8.1×10^{-11}	9.4×10^{-3}	1.2×10^{-8}	8.6×10^9	10^2
600	10^{10}	9×10^{-7}	8.1×10^{-14}	9.4×10^{-6}	1.4×10^{-8}	8.6×10^{12}	1.2×10^5

Table $B.1$ Dependence of the dimensionless numbers on the temperature and the viscosity of the molten glass.

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