Group invariant solutions and conservation laws for jet flow models of non-Newtonian power-law fluids

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Offered unto the divine Lotus Feet of the Lord

Abstract

The non-Newtonian incompressible power-law fluid in jet flow models is investigated. An important feature of the model is the definition of a suitable Reynolds number, and this is achieved using the standard definition of a Reynolds number and ascertaining the magnitude of the effective viscosity.

The jets under examination are the two-dimensional free, liquid and wall jets. The two-dimensional free and wall jets satisfy a different partial differential equation to the two-dimensional liquid jet. Further, the jets are reformulated in terms of a third order partial differential equation for the stream function. The boundary conditions for each jet are unique, but more significantly these boundary conditions are homogeneous. Due to this homogeneity the conserved quantities are critical in the solution process.

The conserved quantities for the two-dimensional free and liquid jet are constructed by first deriving the conservation laws using the multiplier approach. The conserved quantity for the two-dimensional free jet is also derived in terms of the stream function. For a Newtonian fluid with $n = 1$ the two-dimensional wall jet gives a conservation law. However, this is not the case for the two-dimensional wall jet for a non-Newtonian power-law fluid.

The various approaches that have been applied in an attempt to derive a conservation law for the two-dimensional wall jet for a power-law fluid with $n \neq 1$ are discussed. In conjunction with the attempt at obtaining conservation laws for the two-dimensional wall jet we present tenable reasons for its failure, and a feasible way forward.

Similarity solutions for the two-dimensional free jet have been derived for both the velocity components as well as for the stream function. The associated Lie point symmetry approach is also presented for the stream function. A parametric solution has been obtained for shear thinning fluid free jets for $0 < n < 1$ and shear thickening fluid free jets for $n > 1$. It is observed that for values of $n > 1$ in the range $1/2 < n < \infty$, the velocity profile extends over a finite range.

For the two-dimensional liquid jet, along with a similarity solution the complete Lie point symmetries have been obtained. By associating the Lie point symmetry with the elementary conserved vector an invariant solution is found. A parametric solution for the two-dimensional liquid jet is derived for $1/2 < n < \infty$. The solution does not exist for $n = 1/2$ and the range

$0 < n < 1/2$ requires further investigation.

Declaration

I declare that this is my own, unaided work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature. It is being submitted for the Degree of Master of Science by Dissertation to the University of the Witwatersrand. It has not been submitted before for any degree or examination to any other University.

28/05/2014

Avnish Bhowan Magan

Dedication

To my incredible parents and wonderful sister Arisha

Acknowledgements

I would firstly like to express my gratitude to the Almighty for blessing me with the opportunity, courage and strength to pursue my dreams.

To my parents and my sister, your unyielding support, love and commitment towards me has been tremendous, and without whom this work would not have been possible. I would like to say thank you!

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Chapter 1

General introduction

1.1 Introduction

Flow phenomena form an integral branch of a vast and varying range of applications. Industry and engineering are heavily reliant on the fundamental concepts of fluid mechanics and fluid dynamics. This theory extends to astrophysics, and even to medicine. As a result, the mathematical rigour of its theory cannot be underestimated.

Many renowned scientists have been fascinated with flow problems since Archimedes (285-212 B.C.). Despite the tremendous contributions towards the science of fluid flow by great scientists such as Sir Isaac Newton (1642-1727) and Leonard Euler (1707-1783), it was Ludwig Prandtl (1875-1953) who revolutionised fluid mechanics with his concept of the boundary layer. Essentially, the idea of a boundary layer details the case where the effects of friction result in the no slip condition at the surface or the boundary. Further these frictional effects are only observed in a thin layer adjacent to the boundary. Hence the term boundary layer. Once the fictional effects become negligible, we resort back to inviscid flow. It can be noted that this region is a region of large velocity gradients [1]. The Navier-Stokes equations, undoubtedly the most important equations in fluid mechanics, can be simplified to pertain to the boundary layer. The boundary layer equations differ from the Navier-Stokes equations in that despite being a set of coupled, non-linear partial differential equations they are parabolic in nature. The Navier-Stokes equations are elliptic. However, this difference has allowed for the boundary layer equations to be solved more concisely.

A fluid can be defined as a substance "unable to withstand any tendency by applied forces to deform it without change of volume" [2]. In other words, when an applied shear stress acts on the fluid it will deform. The rate at which the fluid deforms is of critical importance. Fluids can be classified into Newtonian and non-Newtonian fluids according to the properties which characterize them. Newtonian fluids can be typified by the linear proportionality of the fluid's viscous stresses to its shear rate. Examples of Newtonian fluids include water, air and other gases. However, non-Newtonian fluids are characterized differently in that the viscosity of the fluid depends on the rate at which the material substance deforms. This dependence is generally non-linear.

Many of the flow problems in industry in particular demonstrate non-Newtonian behaviour. These non-Newtonian fluids exhibit properties such as shear thinning and shear thickening. The former describes the case in which the viscosity of the fluid decreases as the shear rate (rate at which a material substance deforms) increases. Examples of shear thinning fluids include blood, paint, lava and polymer solutions. When the viscosity of the fluid increases with the shear rate, we term this property shear thickening. Silly putty and cornstarch are prime examples which display this property.

This dissertation focuses primarily on the non-Newtonian power-law fluid. Due to the relative simplicity of the power-law model, its extensive use in many engineering applications has proved worthwhile. However, the strength of this model is based on empirical data and as such has its pitfalls. The approximations of this model are only valid over a certain range. Furthermore, for very large and very small shear rates the model breaks down, and evidence can be found in the linear relationship between the strain-rate and shear stress. This is typical Newtonian behaviour. Further reading on the various models which are used to deal with these pitfalls can be found in [3]. Consider the diagram below, adapted from [4].

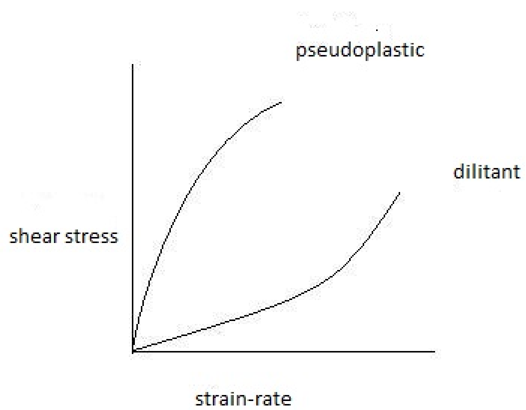

Figure 1.1.1: Shear stress and strain-rate relationship of pseudoplastic (shear thickening) and dilitant (shear thinning) fluids

As is evident, when the shear rate is close to zero or approaching infinity the model fails due to the linear relationship between the strain-rate and shear stress.

The ascendency of Newtonian fluids to a significant science has its roots in the concept of the boundary layer, and the swift development of the aircraft industry. The underlying reason for this progress was the need for engineers to understand and administer effective procedures when working with exterior air flows past rigid objects. Similarly, non-Newtonian fluid mechanics flourished as a result of the polymer industry as scientists were deeply interested in understanding how these polymeric systems behaved under strain [4].

One cannot model all the non-Newtonian fluids with a single constitutive equation, as is done with Newtonian fluids. For example, the constitutive equations describing the power-law fluid and Sisko fluid are different, due to the complex physical nature of the fluids and the stresses acting on the fluid. At present only the fourth grade fluid seems to encompass the characteristics of a non-Newtonian fluid most generally. These partial differential equations prove extremely difficult to solve due to the high order non-linearity. However, Sophus Lie (1842-1899) developed his theory of Lie group analysis and presented an algorithmic approach of obtaining the symmetries of a given partial differential equation and using these point symmetries to obtain an invariant solution. Since the emphasis of this dissertation considers jet flow models of non-Newtonian power-law fluids we derive the conservation laws for the respective jets. These conservation laws are central to solving many differential equations of jet flow problems in fluid mechanics due to the homogeneity of the boundary conditions. A conserved quantity is required to complete the solution. The two-dimensional free, liquid and wall jets will be studied.

Schlichting [5] and Glauert [6] integrated Prandtl's momentum boundary layer equation across the jet and derived conserved quantities for the two-dimensional free and liquid jet respectively. The conserved quantity for the two-dimensional liquid jet was obtained by Watson [7] through physical arguments. A more structured approach was developed by Naz et al. [8] to obtain these conserved quantities using the multiplier method.

1.2 Principal objectives of Dissertation

Ideally, the aim of the dissertation is to obtain group invariant solutions for jet flow models of non-Newtonian power-law fluids.

First and foremost the mathematical model describing a power-law fluid will be derived, where attention is drawn to the constitutive equations for a power-law fluid. The conservation laws will be obtained using the systematic methodology introduced by Naz et al. [8].

Three analytical approaches will be exploited to derive the group invariant solutions. Similarity solutions constitute the first of these analytical approaches. Secondly, we derive the Lie point symmetry associated with the conserved vector, and use this symmetry to deduce an invariant solution [9, 10]. This method, first used by Kara and Mahomed [9], greatly reduces the computation in obtaining invariant solutions, however with regard to the two-dimensional liquid jet it fails to provide sufficient information. Subsequently for the final approach, we compute the full Lie point symmetry and simply determine its constants by associating it with a conserved vector. The solutions are parametric in form.

Finally numerical results will be presented and further analysis will be carried out on the properties of the solution.

1.3 Outline of Dissertation

An outline of the dissertation is as follows. Chapter 2 discusses the basic concepts and definitions that will be used and applied throughout the Dissertation. In Chapter 3, we present a concise mathematical model describing the flow behaviour of a non-Newtonian power-law fluid for the various jet flows. The derivation of the conservation laws for the two-dimensional free and liquid jets will be given in Chapter 4. Chapter 5 and Chapter 6 detail the parametric solutions for the two-dimensional free and liquid jets respectively using the methods outlined in Section 1.2. In conjunction with the parametric solutions for these jets, numerical analysis will also be carried out. The two-dimensional wall jet will be considered separately in Chapter 7. The reason for this is that we are unable to derive a conserved quantity for the two-dimensional wall jet for a non-Newtonian power-law fluid. A discussion on the analysis we have carried out on the two-dimensional wall jet will thus be presented. Further a possible way forward to obtaining the conservation laws will be addressed. The conclusions will be summarized in Chapter 8.

Chapter 2

Basic definitions and concepts

2.1 Introduction

This chapter will outline the definitions and concepts to be used throughout this dissertation. In particular, we focus on Lie group analysis and conservation laws for partial differential equations. Further, this chapter will also introduce the notation to be adopted.

2.2 Elementary concepts and operators

2.2.1 Lie operators

The Einstein summation convention will be adopted for the purposes of this dissertation. Let x^i, $i = 1, 2, ..., n$ be n independent variables and u^α, $\alpha = 1, 2, ..., N$ be N dependent variables [11]. Further the partial derivatives are denoted by a subscript. Then we have the following definitions:

Definition 2.2.1. The Euler operator is given by [11]

$$E_{u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} ... D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1...i_s}}, \quad \alpha = 1, 2, ..., N \tag{2.2.1}$$

where the total derivative operator is

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + ..., \quad i = 1, 2, ...n \tag{2.2.2}$$

Definition 2.2.2. The Lie-Bäcklund operator is defined by [11]

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1...i_s} \frac{\partial}{\partial u^\alpha_{i_1...i_s}}, \tag{2.2.3}$$

where the extended infinitesimals are

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_s} - u_{ji_1 \dots i_{s-1}}^\alpha D_{i_s}(\xi^j), \quad s > 1. \end{aligned} \tag{2.2.4}$$

2.2.2 Conservation laws

Consider a k-th-order system of differential equations of n independent and N dependent variables [11]:

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0, \quad \alpha = 1, 2, ...N \tag{2.2.5}$$

where $u_{(i)}$ denotes the collection of partial derivatives of order i.

Definition 2.2.3. A conserved vector for (2.2.5) is an n-tuple $T = (T^1, T^2, ..., T^n)$, $T^i \in \mathcal{A}$, i=1,2,...n, such that [11]:

$$D_i T^i = 0 \tag{2.2.6}$$

holds for all solutions of (2.2.5). Equation (2.2.6) is called a local conservation law. Here $\mathcal{A}$ is the space of differential functions.

Chapter 3

Mathematical model for a power-law fluid

3.1 Introduction

For the purposes of this chapter, I intend to construct the mathematical model for non-Newtonian power-law fluids in various jet flow models. Power law fluids can be described as generalized Newtonian fluids. Such fluids are, "one of a number of classes of non-Newtonian fluids that feature variable viscosity" [12]. A property of such fluids is that the shear stress is a function of the shear rate. In particular, the effective viscosity can be described in terms of the shear rate as shown by (3.2.5)

The parameter n plays a vital part in the classification of the fluids. For values $0 < n < 1$ we have pseudoplastic fluids (shear thinning). When $n > 1$ we have dilitant or shear thickening fluids and for $n = 1$ Newtonian fluids.

The flow behaviour forms an integral part of the modelling process. We consider incompressible power-law fluids where the flow is two-dimensional and steady. Finally the jet models under consideration are the free, liquid and wall jets.

An outline of this chapter is as follows. In Section 3.2, we will define the characteristic quantities and establish an appropriate definition of the Reynolds number for a non-Newtonian power-law fluid. In Section 3.3 the dimensionless equations are derived. Further, in Section 3.4 we make use of the boundary layer approximation to deduce the governing equations and finally in Section 3.5 the conclusions from the chapter are drawn.

3.2 Characteristic quantities and Reynolds number

In order to define a Reynolds number for a power-law fluid we need to know the magnitude of the effective viscosity. The effective viscosity will give an approximate indication of the manner in which the non-Newtonian fluid behaves.

For an incompressible Newtonian fluid, the Cauchy stress tensor is given by

$$\tau_{ik} = -p\delta_{ik} + \mu A_{ik}, \tag{3.2.1}$$

where μ is the viscosity of the fluid and

$$A_{ik} = \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \tag{3.2.2}$$

is the rate-of-strain tensor. For a power-law fluid, the Cauchy stress tensor is given by

$$\tau_{ik} = -p\delta_{ik} + S_{ik} \tag{3.2.3}$$

$$= -p\delta_{ik} + b\left|\sqrt{\frac{1}{2}\mathrm{tr}(A^2)}\right|^{n-1} A_{ik}, \tag{3.2.4}$$

where b is a constant. Comparing (3.2.1) and (3.2.4), the effective viscosity is given by

$$\mu_e = b\left|\sqrt{\frac{1}{2}\mathrm{tr}(A^2)}\right|^{n-1}. \tag{3.2.5}$$

The Reynolds number Re is the ratio of the inertial force to the viscous force. As shown later in boundary layer theory we consider Reynolds numbers for which $\sqrt{Re} \gg 1$. The Reynolds number is

$$Re = \frac{\rho U L}{\mu_e} = \frac{UL}{\nu_e}, \tag{3.2.6}$$

where L is the characteristic length in the flow, U is the magnitude of the velocity at a large distance from the body and $\nu_e = \mu_e/\rho$ is the effective kinematic viscosity.

Let L be the characteristic length in the x-direction which is along the boundary and δ be the characteristic length in the y-direction which is perpendicular to the boundary. In boundary layer theory $\delta \ll L$. This means that the variation of the flow rate parallel to the boundary is far more insignificant than the flow rate normal to the boundary. In the boundary layer the viscosity is important regardless of how high the Reynolds number may be.

Let U be the characteristic velocity in the x-direction. We use the continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3.2.7}$$

to derive the characteristic velocity, V, in the y-direction as follows. In terms of the characteristic variables, (3.2.7) gives

$$\frac{U}{L} + \frac{V}{\delta} \sim 0. \tag{3.2.8}$$

The characteristic velocity in the y-direction is thus $V = \delta U/L$.

We now calculate the magnitude of the effective viscosity in order to define a suitable Reynolds number. Consider first the expansion of μ_e. We note that $A_{ik} = A_{ki}$ and that $(A^2)_{ik} = A_{in}A_{nk}$ where there is summation over the repeated index n from 1 to 2. This gives

$$\begin{aligned} \mathrm{tr}(A^2) &= (A^2)_{kk} = A_{kn}A_{nk} \\ &= A_{11}^2 + 2A_{12}^2 + A_{22}^2 > 0. \end{aligned} \tag{3.2.9}$$

The components of the rate of strain tensor (3.2.2) are given by
$A_{11} = 2\dfrac{\partial v_x}{\partial x}, \quad A_{12} = \dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}, \quad A_{22} = 2\dfrac{\partial v_y}{\partial y}.$
Hence

$$\begin{aligned} \mathrm{tr}(A^2) &= \left(2\frac{\partial v_x}{\partial x}\right)^2 + 2\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(2\frac{\partial v_y}{\partial y}\right)^2 \\ &= 4\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v_x}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial v_y}{\partial x}\right)^2 + \frac{\partial v_x}{\partial y}\frac{\partial v_y}{\partial x} + \left(\frac{\partial v_y}{\partial y}\right)^2\right]. \end{aligned} \tag{3.2.10}$$

We consider the order of magnitude of the terms and use the boundary layer approximation to identify which terms contribute and which do not. Then the order of magnitude of $\mathrm{tr}(A^2)$ is given by

$$\begin{aligned} [\mathrm{tr}(A^2)] &= 4\left(\frac{U}{L}\right)^2 + 2\left(\frac{U}{\delta}\right)^2 + 2\left(\frac{\delta U}{L^2}\right)^2 + 4\left(\frac{U}{\delta}\frac{\delta U}{L^2}\right)^2 + 4\left(\frac{\delta U}{\delta L}\right)^2 \\ &= 2\left(\frac{U}{\delta}\right)^2\left[2\left(\frac{\delta}{L}\right)^2 + 1 + \left(\frac{\delta}{L}\right)^4 + 2\left(\frac{\delta}{L}\right)^2 + 2\left(\frac{\delta}{L}\right)^2\right] \\ &= 2\left(\frac{U}{\delta}\right)^2\left[1 + O\left(\frac{\delta}{L}\right)^2\right]. \end{aligned} \tag{3.2.11}$$

In boundary layer theory $\delta \ll L$ and hence $\delta/L \ll 1$. Therefore the magnitude of $\mathrm{tr}(A^2)$ is approximately $2\,(U/\delta)^2$.

We can now calculate the magnitude of the effective viscosity. We have

$$\mu_e = b \left| \sqrt{\frac{1}{2}\mathrm{tr}(A^2)} \right|^{n-1} = b \left(\frac{U}{\delta} \right)^{n-1}. \tag{3.2.12}$$

The Reynolds number can be defined using (3.2.6) and (3.2.12) as

$$Re = \frac{\rho U L}{\mu_e} = \frac{\rho L U^{2-n} \delta^{n-1}}{b}. \tag{3.2.13}$$

Now the distance of diffusion of vorticity from the boundary in time t is $O\left((\mu_e t/\rho)^{\frac{1}{2}} \right)$. Therefore

$$\delta \;=\; O\left((\mu_e t/\rho)^{\frac{1}{2}} \right) = \left(\frac{LU}{Re} \frac{L}{U} \right)^{\frac{1}{2}} = \frac{L}{\sqrt{Re}}. \tag{3.2.14}$$

The characteristic pressure, p, is obtained by balancing the pressure gradient with the inertia. Then for the x-direction

$$\rho \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) \sim -\frac{\partial p}{\partial x}. \tag{3.2.15}$$

Since the flow is steady and U is the characteristic velocity in the x-direction we have

$$\frac{\rho U^2}{L} \sim \frac{p}{L} \tag{3.2.16}$$

and thus

$$p = \rho U^2. \tag{3.2.17}$$

We define the following dimensionless variables:

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{\delta}, \quad \bar{p} = \frac{p}{\rho U^2}, \quad \bar{v_x} = \frac{v_x}{U}, \quad \bar{v_y} = \frac{v_y L}{U \delta}. \tag{3.2.18}$$

The conservation of mass equation and the x and y components of the momentum balance are now written in dimensionless form.

The conservation of mass equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3.2.19}$$

which, expressed in dimensionless variables, becomes

$$\frac{\partial \bar{v}_x}{\partial \bar{x}} + \frac{\partial \bar{v}_y}{\partial \bar{y}} = 0. \tag{3.2.20}$$

The x-component of the steady state momentum balance equation is given by

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial x}, \tag{3.2.21}$$

where

$$S_{xx} = b \left| \sqrt{\frac{1}{2}\mathrm{tr}(A^2)} \right|^{n-1} A_{xx} = 2b \left| \sqrt{\frac{1}{2}\mathrm{tr}(A^2)} \right|^{n-1} \frac{\partial v_x}{\partial x}, \tag{3.2.22}$$

and

$$S_{xy} = b \left| \sqrt{\frac{1}{2}\mathrm{tr}(A^2)} \right|^{n-1} A_{xy} = b \left| \sqrt{\frac{1}{2}\mathrm{tr}(A^2)} \right|^{n-1} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right). \tag{3.2.23}$$

But $\mathrm{tr}(A^2)$ is given by equation (3.2.10). In dimensionless form

$$\begin{aligned} \mathrm{tr}(A^2) &= 4\left(\frac{U}{L}\right)^2 \left(\frac{\partial \bar{v}_x}{\partial \bar{x}}\right)^2 + 2\left(\frac{U}{\delta}\right)^2 \left(\frac{\partial \bar{v}_x}{\partial \bar{y}}\right)^2 + 2\left(\frac{\delta U}{L^2}\right)^2 \left(\frac{\partial \bar{v}_y}{\partial \bar{x}}\right)^2 \\ &+ 4\left(\frac{U^2}{L^2}\right) \frac{\partial \bar{v}_x}{\partial \bar{y}} \frac{\partial \bar{v}_y}{\partial \bar{x}} + 4\left(\frac{U}{L}\right)^2 \left(\frac{\partial \bar{v}_y}{\partial \bar{y}}\right) \\ &= 2\left(\frac{U}{\delta}\right)^2 \left[\left(\frac{\partial \bar{v}_x}{\partial \bar{y}}\right)^2 + O\left(\frac{\delta}{L}\right)^2 \right]. \end{aligned} \tag{3.2.24}$$

We neglect terms $O\left((\delta/L)^2\right)$. Thus we have

$$\mathrm{tr}(A^2) = 2\left(\frac{U}{\delta}\right)^2 \left(\frac{\partial \bar{v}_x}{\partial \bar{y}}\right)^2. \tag{3.2.25}$$

Hence

$$b \left| \sqrt{\frac{1}{2}\mathrm{tr}(A^2)} \right|^{n-1} = b \left(\frac{U}{\delta}\right)^{n-1} \left| \frac{\partial \bar{v}_x}{\partial \bar{y}} \right|^{n-1}. \tag{3.2.26}$$

With this approximation, equation (3.2.22) in dimensionless variables becomes

$$S_{xx} = 2\frac{bU^n}{\delta^{n-1}L} \left| \frac{\partial \bar{v}_x}{\partial \bar{y}} \right|^{n-1} \frac{\partial \bar{v}_x}{\partial \bar{x}}. \tag{3.2.27}$$

Furthermore

$$\frac{\partial}{\partial x}S_{xx} = \frac{2bU^n}{L^2\delta^{n-1}}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{x}}\right). \tag{3.2.28}$$

Similarly from equation (3.2.23) and using approximation (3.2.26),

$$\begin{aligned} S_{xy} &= \frac{bU^{n-1}}{\delta^{n-1}}\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{U}{\delta}\left(\frac{\partial \bar{v}_x}{\partial \bar{y}} + \frac{\delta^2}{L^2}\frac{\partial \bar{v}_y}{\partial \bar{x}}\right) \\ &= \frac{bU^n}{\delta^n}\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}\left(1 + O\left(\left(\frac{\delta}{L}\right)^2\right)\right). \end{aligned} \tag{3.2.29}$$

Neglecting terms $O\left((\delta/L)^2\right)$ we have

$$S_{xy} = \frac{bU^n}{\delta^n}\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}. \tag{3.2.30}$$

Differentiating (3.2.30) with respect to y gives

$$\frac{\partial}{\partial y}S_{xy} = \frac{bU^n}{\delta^{n+1}}\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}\right). \tag{3.2.31}$$

The x-component of the steady state momentum balance equation (3.2.21) in dimensionless form is

$$\begin{aligned} \bar{v}_x\frac{\partial \bar{v}_x}{\partial \bar{x}} + \bar{v}_y\frac{\partial \bar{v}_x}{\partial \bar{y}} &= -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{2bU^{n-2}}{\rho L\delta^{n-1}}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{x}}\right) \\ &+ \frac{bLU^{n-2}}{\rho\delta^{n+1}}\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}\right). \end{aligned} \tag{3.2.32}$$

We can now express b in terms of the Reynolds number, Re, defined by (3.2.13),

$$b = \frac{\rho LU^{2-n}\delta^{n-1}}{Re}. \tag{3.2.33}$$

Using (3.2.33),the x-component of momentum (3.2.32) becomes

$$\begin{aligned} \bar{v}_x\frac{\partial \bar{v}_x}{\partial \bar{x}} + \bar{v}_y\frac{\partial \bar{v}_x}{\partial \bar{y}} &= -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{2}{Re}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{x}}\right) \\ &+ \frac{1}{Re}\left(\frac{L}{\delta}\right)^2\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}\right). \end{aligned} \tag{3.2.34}$$

But from (3.2.14)

$$\left(\frac{L}{\delta}\right)^2 = Re \tag{3.2.35}$$

and hence (3.2.34) reduces to

$$\bar{v_x}\frac{\partial \bar{v_x}}{\partial \bar{x}} + \bar{v_y}\frac{\partial \bar{v_x}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{2}{Re}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_x}}{\partial \bar{x}}\right) + \frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_x}}{\partial \bar{y}}\right). \tag{3.2.36}$$

We note that when $n = 1$, the x-component of the momentum balance equation is the x-component of the Navier-Stokes equation for a Newtonian fluid.

The y-component of the steady state momentum balance equation is given by

$$\rho\left(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}, \tag{3.2.37}$$

Differentiating equation (3.2.30) with respect to x gives

$$\frac{\partial}{\partial x}S_{xy} = \frac{bU^n}{\delta^n L}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_x}}{\partial \bar{y}}\right). \tag{3.2.38}$$

Also using the approximation (3.2.26) we obtain

$$\begin{aligned} S_{yy} &= b\left|\sqrt{\frac{1}{2}tr(A^2)}\right|^{n-1} A_{yy} = 2b\left|\sqrt{\frac{1}{2}tr(A^2)}\right|^{n-1}\frac{\partial v_y}{\partial y} \\ &= 2\frac{bU^n}{L\delta^{n-1}}\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_y}}{\partial \bar{y}}. \end{aligned} \tag{3.2.39}$$

Thus we have

$$\frac{\partial}{\partial y}S_{yy} = \frac{2bU^n}{\delta^n L}\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_y}}{\partial \bar{y}}\right). \tag{3.2.40}$$

The y-component of the steady state momentum balance equation (3.2.37) in dimensionless form is

$$\begin{aligned} \left(\frac{\delta}{L}\right)^2\left(\bar{v_x}\frac{\partial \bar{v_y}}{\partial \bar{x}} + \bar{v_y}\frac{\partial \bar{v_y}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\delta^{1-n}bU^{n-2}}{\rho L}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_x}}{\partial \bar{y}}\right) \\ &+ \frac{2\delta^{1-n}bU^{n-2}}{\rho L}\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v_x}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v_y}}{\partial \bar{y}}\right). \end{aligned} \tag{3.2.41}$$

We can now express b in terms of the Reynolds number using (3.2.33) and using also (3.2.35), the y-component of the momentum balance equation,(3.2.41), becomes

$$\begin{aligned}\frac{1}{Re}\left(\bar{v}_x\frac{\partial\bar{v}_y}{\partial\bar{x}}+\bar{v}_y\frac{\partial\bar{v}_y}{\partial\bar{y}}\right) &= -\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{1}{Re}\frac{\partial}{\partial\bar{x}}\left(\left|\frac{\partial\bar{v}_x}{\partial\bar{y}}\right|^{n-1}\frac{\partial\bar{v}_x}{\partial\bar{y}}\right)\\ &+ \frac{2}{Re}\frac{\partial}{\partial\bar{y}}\left(\left|\frac{\partial\bar{v}_x}{\partial\bar{y}}\right|^{n-1}\frac{\partial\bar{v}_y}{\partial\bar{y}}\right).\end{aligned} \tag{3.2.42}$$

We can compare (3.2.42) with the corresponding equation for a Newtonian fluid. The y-component of the momentum balance equation for a Newtonian fluid is given by

$$\frac{1}{Re}\left(\bar{v}_x\frac{\partial\bar{v}_y}{\partial\bar{x}}+\bar{v}_y\frac{\partial\bar{v}_y}{\partial\bar{y}}\right)=-\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{1}{Re^2}\frac{\partial^2\bar{v}_y}{\partial\bar{x}^2}+\frac{1}{Re}\frac{\partial^2\bar{v}_y}{\partial\bar{y}^2}. \tag{3.2.43}$$

The apparent difference between the two momentum balance equations can be explained by considering what happens to (3.2.42) when $n=1$. The last two terms on the right hand side of (3.2.42), with $n=1$, are

$$\begin{aligned}\frac{1}{Re}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}_x}{\partial\bar{y}}\right)+\frac{2}{Re}\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\bar{v}_y}{\partial\bar{y}}\right) &= \frac{1}{Re}\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\bar{v}_x}{\partial\bar{x}}+\frac{\partial\bar{v}_y}{\partial\bar{y}}\right)+\frac{1}{Re}\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\bar{v}_y}{\partial\bar{y}}\right)\\ &= \frac{1}{Re}\frac{\partial^2\bar{v}_y}{\partial\bar{y}^2},\end{aligned} \tag{3.2.44}$$

using the continuity equation (3.2.20). The next term in (3.2.43) is smaller, of order $\frac{1}{Re^2}$.

In summary we have the following dimensionless equations.

Continuity equation:

$$\frac{\partial\bar{v}_x}{\partial\bar{x}}+\frac{\partial\bar{v}_y}{\partial\bar{y}}=0. \tag{3.2.45}$$

x-component of momentum balance equation:

$$\bar{v}_x\frac{\partial\bar{v}_x}{\partial\bar{x}}+\bar{v}_y\frac{\partial\bar{v}_x}{\partial\bar{y}}=-\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{2}{Re}\frac{\partial}{\partial\bar{x}}\left(\left|\frac{\partial\bar{v}_x}{\partial\bar{y}}\right|^{n-1}\frac{\partial\bar{v}_x}{\partial\bar{x}}\right)+\frac{\partial}{\partial\bar{y}}\left(\left|\frac{\partial\bar{v}_x}{\partial\bar{y}}\right|^{n-1}\frac{\partial\bar{v}_x}{\partial\bar{y}}\right). \tag{3.2.46}$$

y-component of momentum balance equation:

$$\begin{aligned}\frac{1}{Re}\left(\bar{v}_x\frac{\partial \bar{v}_y}{\partial \bar{x}}+\bar{v}_y\frac{\partial \bar{v}_y}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{y}}+\frac{1}{Re}\frac{\partial}{\partial \bar{x}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}\right)\\ &+ \frac{2}{Re}\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_y}{\partial \bar{y}}\right).\end{aligned} \tag{3.2.47}$$

3.3 Boundary layer approximation

From (3.2.14), for a boundary layer to exist $\sqrt{Re} \gg 1$. We make the boundary layer approximation in which terms of order $\dfrac{1}{Re}$ are neglected. Equations (3.2.45),(3.2.46) and (3.2.47) reduce to the following equations.

Continuity equation:

$$\frac{\partial \bar{v}_x}{\partial \bar{x}}+\frac{\partial \bar{v}_y}{\partial \bar{y}}=0, \tag{3.3.1}$$

x-component of momentum balance equation:

$$\bar{v}_x\frac{\partial \bar{v}_x}{\partial \bar{x}}+\bar{v}_y\frac{\partial \bar{v}_x}{\partial \bar{y}}=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial}{\partial \bar{y}}\left(\left|\frac{\partial \bar{v}_x}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{v}_x}{\partial \bar{y}}\right), \tag{3.3.2}$$

y-component of momentum balance equation:

$$\frac{\partial \bar{p}}{\partial \bar{y}}=0. \tag{3.3.3}$$

In the remainder of the dissertation the overhead bar will be suppressed to simplify the notation, it being understood that dimensionless variables are being used.

Consider the y-component of momentum balance equation which states that the pressure is constant across a boundary layer. In other words the pressure is independent of y, that is $p = p(x)$. Thus the pressure can be determined by the mainstream conditions. We consider inviscid flow everywhere except in the boundary layer close to the boundary where viscosity is important.

Consider $\partial p/\partial x$ in (3.3.2). Suppose that $v_x(x, y) \to U(x)$ as $y \to \infty$, where $U(x)$ is the mainstream velocity. Outside the boundary layer Euler's equation of motion for inviscid flow is approximately satisfied. Euler's equation in dimensional variables is

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)\mathbf{U}+\nabla p=0, \tag{3.3.4}$$

where $\mathbf{U}=(U_x,0,0)$ is the mainstream velocity. Expressed in dimensionless form (3.3.4) is

$$\frac{\partial U_x}{\partial t}+U_x\frac{\partial U_x}{\partial x}=-\frac{\partial p}{\partial x}, \tag{3.3.5}$$

Since we are considering steady flow outside the boundary layer, $\partial U/\partial t=0$ and hence (3.3.5) reduces to

$$U_x\frac{\partial U_x}{\partial x}=-\frac{\partial p}{\partial x}. \tag{3.3.6}$$

Equations (3.3.1)-(3.3.2) become

Continuity equation:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0. \tag{3.3.7}$$

x-component of momentum balance equation:

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=U_x\frac{\partial U_x}{\partial x}+\frac{\partial}{\partial y}\left(\left|\frac{\partial v_x}{\partial y}\right|^{n-1}\frac{\partial v_x}{\partial y}\right). \tag{3.3.8}$$

The boundary layer equations also apply to the fluid flow in a jet although there is no solid boundary. This is due to there being a region of sharp change perpendicular to the axis of the jet. We consider flow behaviour close to the axis of the jet where the boundary layer approximation applies .For jet flows the mainstream velocity $U_x(x)$ is generally zero. As a result (3.3.8) reduces to

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=\frac{\partial}{\partial y}\left(\left|\frac{\partial v_x}{\partial y}\right|^{n-1}\frac{\partial v_x}{\partial y}\right). \tag{3.3.9}$$

The term $\left|\frac{\partial v_x}{\partial y}\right|^{n-1}$ is of particular importance. For the free jet, which is illustrated in Figure 3.3.1

$$\frac{\partial v_x}{\partial y}<0 \text{ for } y>0, \quad \frac{\partial v_x}{\partial y}>0 \text{ for } y<0. \tag{3.3.10}$$

We will consider $0 \leq y < \infty$. Then

$$\left|\frac{\partial v_x}{\partial y}\right|^{n-1} = \left(-\frac{\partial v_x}{\partial y}\right)^{n-1}. \tag{3.3.11}$$

For the liquid jet

$$0 \leq y < \phi(x) : \ \frac{\partial v_x}{\partial y} > 0, \tag{3.3.12}$$

where $y = \phi(x)$ is the surface of the jet. This is illustrated in Figure 3.3.2. As a result

$$0 \leq y < \phi(x) : \ \left|\frac{\partial v_x}{\partial y}\right|^{n-1} = \left(\frac{\partial v_x}{\partial y}\right)^{n-1}. \tag{3.3.13}$$

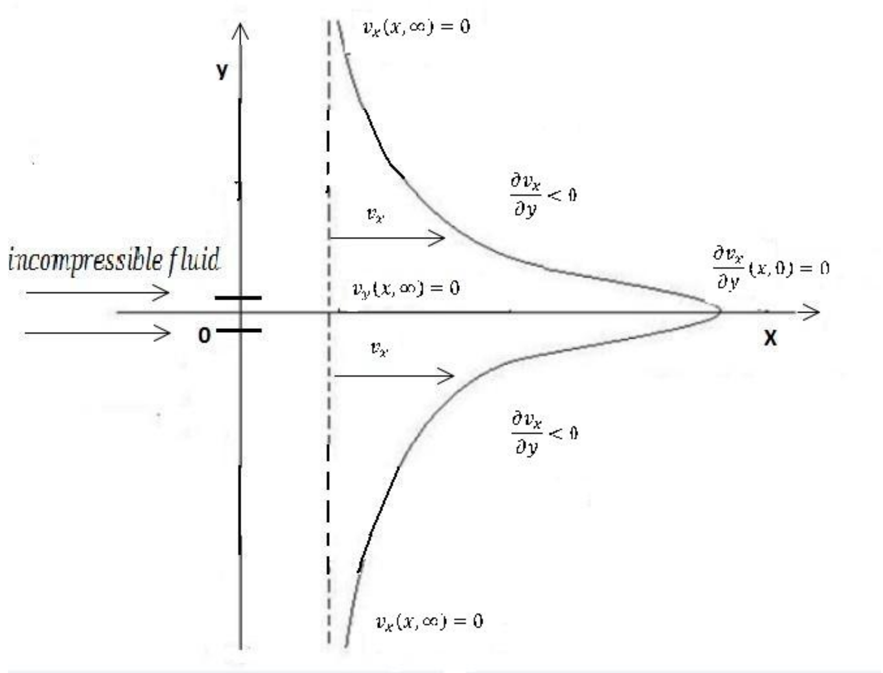

Figure 3.3.1: Velocity profile for a two-dimensional free jet

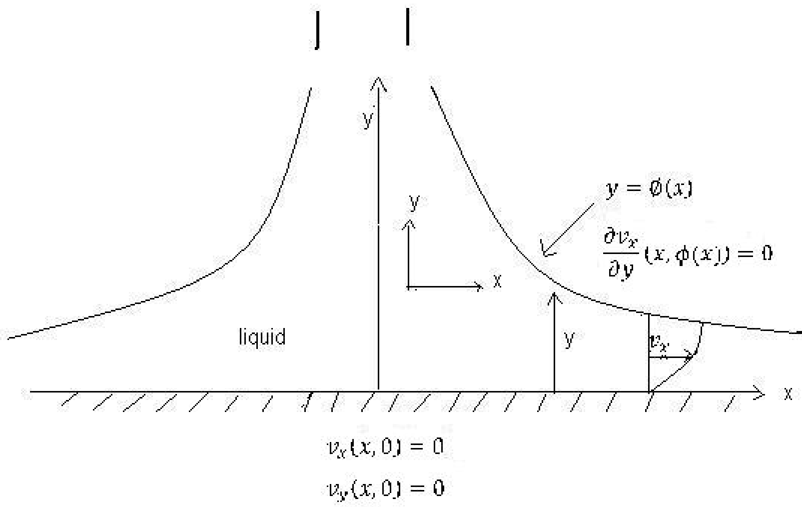

Figure 3.3.2: Velocity profile for a two-dimensional liquid jet

Let $u = v_x$ and $v = v_y$. The complete set of governing equations reduce to

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.3.14}$$

x-component of momentum balance equation - upper half of free jet

$$0 \leq y < \infty : \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = n\left(-\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}. \tag{3.3.15}$$

x-component of momentum balance equation - liquid jet

$$0 \leq y < \phi(x) : \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = n\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}. \tag{3.3.16}$$

3.4 Conclusions

In this chapter we have derived the governing equations for a non-Newtonian power-law fluid in the free and liquid jets. In order to derive these equations we defined a suitable Reynolds number for a non-Newtonian power-law fluid. This was done by using the usual definition of a Reynolds number and defining an effective viscosity. With this definition the Reynolds number the condition for a boundary layer to exist, $\sqrt{Re} \gg 1$ is the same for n=1 and $n \neq 1$. Further we made use of the boundary layer approximation close to the axis of a free jet. Of particular interest in this chapter was the free jet. For the free jet the sign of $\partial v_x/\partial y$ is different in the upper and lower half-plane. We will consider only the upper half-plane and impose boundary conditions on the axis of the jet, $y = 0$.

Chapter 4

Conservation laws and conserved quantities for jet flow models

4.1 Introduction

Fundamental to solving the differential equations of jet flow problems in fluid mechanics is the derivation of conservation laws. Mathematically we can formulate these conservation laws by considering how the mass, momentum or energy fluctuate within a specified volume in space.

Due to the homogeneity of the boundary conditions in jet flow problems the conserved quantity for the jet is essential in deriving the unknown exponent in the similarity solution. For flow problems where the boundary conditions are not homogeneous the mainstream matching boundary condition can be used to determine this exponent. Further, the conserved quantity will give insight into the strength of the jet.

Schlichting [5] derived the conserved quantity for the two-dimensional free jet by integrating Prandtl's momentum boundary layer equation across the jet. Further Watson [7] argued that the volume flux is constant in an incompressible fluid, which gave the conserved quantity for a two-dimensional liquid jet. Glauret [6] applied the same technique as Schlichting to deduce the conserved quantity for the two-dimensional wall jet.

Naz et al. [8] presented a new, more systematic, method of obtaining the conserved quantities for jet flow problems by first deriving the conservation laws for the governing partial differential equations. The multiplier approach will be used in this chapter to deduce a basis of conserved vectors for the power-law fluid in the various jet flow models. Further, once the conservation laws have been obtained the conserved quantities will be derived using the conservation laws and boundary conditions.

An outline of the chapter is as follows. In Section 4.2 the multiplier method will be used to derive the conservation laws for the system of partial differential equations for the velocity components as well as for the single third order partial differential equation for the stream function. In Section 4.3 the derivation of the conserved quantities will be presented. Finally concluding remarks will be given in Section 4.4.

4.2 Conservation laws for the two-dimensional free, liquid and wall jets

The fluid in the jet flows which we are considering is viscous and incompressible.

A liquid jet is formed when a two-dimensional jet of liquid strikes a plane boundary at right angles and spreads over its surface [13]. In terms of the geometry of the problem, the x- and y-axes are along and perpendicular to the boundary which is at y=0. We have a further condition along the free surface given by $y = \phi(x)$. This is the kinematic condition, which states that a fluid particle on the free surface must remain of the free surface [8].

A free jet is formed when fluid is projected into the same fluid which is at rest. A physical example of this would be fluid projected from a narrow orifice in a wall into surrounding fluid. The x-axis is along the axis of symmetry of the jet and perpendicular to that is the y-axis [8].

A wall jet is formed when a sluice gate separating two sections of a canal is slightly raised [6]. Fluid will flow into the section of the canal with a lower fluid level to allow both sections to have the same fluid level. The wall jet is responsible for raising this fluid level. The x-axis is along the wall and the y-axis perpendicular to the wall of the jet. A diagram of the two-dimensional wall jet can be found in Chapter 7.

When finding conservation laws or deriving Lie point symmetries from determining equations, we denote partial derivatives with a subscript. These derivatives are regardless independent variables.

4.2.1 Conservation laws for velocity components

We present only the derivation of the conservation laws for the system of partial differential equations for the free jet because the derivation of the conservation laws for the liquid jet is the same. The continuity and momentum balance equations are written as

$$u_x + v_y = 0, \tag{4.2.1}$$

$$uu_x + vu_y = n(-u_y)^{n-1}u_{yy}. \tag{4.2.2}$$

For equations (4.2.1) and (4.2.2) we consider multipliers Λ_1 and Λ_2. By the property of multipliers we have,

$$\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - n(-u_y)^{n-1}u_{yy}) = D_xT^1 + D_yT^2, \quad (4.2.3)$$

where $T = (T^1, T^2)$ are the components of the conserved vector and the total derivative operators D_x and D_y are defined as

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xy}\frac{\partial}{\partial u_y} + v_{xy}\frac{\partial}{\partial v_y} + ..(4.2.4)$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + u_{yx}\frac{\partial}{\partial u_x} + v_{yx}\frac{\partial}{\partial v_x} + ..(4.2.5)$$

We consider multipliers of the form

$$\Lambda_1 = \Lambda_1(x, y, u, v), \quad \Lambda_2 = \Lambda_2(x, y, u, v). \quad (4.2.6)$$

In order to derive the determining equations we utilise the Euler operator. The Euler operator will annihilate the divergence of the conservation law.

The determining equations can be found by expanding

$$E_u[\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - n(-u_y)^{n-1}u_{yy})] = 0, \quad (4.2.7)$$

$$E_v[\Lambda_1(u_x + v_y) + \Lambda_2(uu_x + vu_y - n(-u_y)^{n-1}u_{yy})] = 0, \quad (4.2.8)$$

where the Euler operators are defined by

$$E_u = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_xD_y\frac{\partial}{\partial u_{xy}} - ..,(4.2.9)$$

$$E_v = \frac{\partial}{\partial v} - D_x\frac{\partial}{\partial v_x} - D_y\frac{\partial}{\partial v_y} + D_x^2\frac{\partial}{\partial v_{xx}} + D_y^2\frac{\partial}{\partial v_{yy}} + D_xD_y\frac{\partial}{\partial v_{xy}} - ..(4.2.10)$$

Equation (4.2.7) can be expanded using equation (4.2.9),

$$\begin{aligned} &\frac{\partial}{\partial u}\; [\Lambda_1u_x + \Lambda_1v_y + \Lambda_2uu_x + \Lambda_2vu_y - \Lambda_2n(-u_y)^{n-1}u_{yy}] \\ -\;& D_x\frac{\partial}{\partial u_x}[\Lambda_1u_x + \Lambda_1v_y + \Lambda_2uu_x + \Lambda_2vu_y - \Lambda_2n(-u_y)^{n-1}u_{yy}] \\ -\;& D_y\frac{\partial}{\partial u_y}[\Lambda_1u_x + \Lambda_1v_y + \Lambda_2uu_x + \Lambda_2vu_y - \Lambda_2n(-u_y)^{n-1}u_{yy}] \\ +\;& D_y^2\frac{\partial}{\partial u_{yy}}[\Lambda_1u_x + \Lambda_1v_y + \Lambda_2uu_x + \Lambda_2vu_y - \Lambda_2n(-u_y)^{n-1}u_{yy}] = 0. \end{aligned} \quad (4.2.11)$$

Equation (4.2.11) can be simplified further using the definitions of the total derivative operators given by equations (4.2.4) and (4.2.5),

$$\begin{aligned} & u_x\frac{\partial\Lambda_1}{\partial u}+v_y\frac{\partial\Lambda_1}{\partial u}+uu_x\frac{\partial\Lambda_2}{\partial u}+\Lambda_2u_x+vu_y\frac{\partial\Lambda_2}{\partial u}-n(-u_y)^{n-1}u_{yy}\frac{\partial\Lambda_2}{\partial u}\\ - & D_x(\Lambda_1+\Lambda_2u)-D_y(\Lambda_2v+\Lambda_2n(n-1)(-u_y)^{n-2}u_{yy})\\ + & D_y^2(-\Lambda_2n(-u_y)^{n-1})=0. \end{aligned} \tag{4.2.12}$$

Expanding equation (4.2.12) completely gives the determining equation

$$\begin{aligned} & u_x\frac{\partial\Lambda_1}{\partial u}+v_y\frac{\partial\Lambda_1}{\partial u}+uu_x\frac{\partial\Lambda_2}{\partial u}+\Lambda_2u_x+vu_y\frac{\partial\Lambda_2}{\partial u}-n(-u_y)^{n-1}u_{yy}\frac{\partial\Lambda_2}{\partial u}\\ - & \frac{\partial\Lambda_1}{\partial x}-u\frac{\partial\Lambda_2}{\partial x}-u_x\frac{\partial\Lambda_1}{\partial u}-uu_x\frac{\partial\Lambda_2}{\partial u}-u_x\Lambda_2-v_x\frac{\partial\Lambda_1}{\partial v}-uv_x\frac{\partial\Lambda_2}{\partial v}-v\frac{\partial\Lambda_2}{\partial y}\\ - & u_{yy}(-u_y)^{n-2}n(n-1)\frac{\partial\Lambda_2}{\partial y}-u_yv\frac{\partial\Lambda_2}{\partial u}+u_{yy}(-u_y)^{n-1}n(n-1)\frac{\partial\Lambda_2}{\partial u}\\ - & v_yv\frac{\partial\Lambda_2}{\partial v}-v_y\Lambda_2-v_y(-u_y)^{n-2}u_{yy}n(n-1)\frac{\partial\Lambda_2}{\partial v}+u_{yy}^2(-u_y)^{n-3}n(n-1)(n-2)\Lambda_2\\ - & u_{yyy}(-u_y)^{n-2}n(n-1)\Lambda_2-(-u_y)^{n-1}n\frac{\partial^2\Lambda_2}{\partial y^2}+(-u_y)^{n}n\frac{\partial^2\Lambda_2}{\partial y\partial u}\\ - & v_y(-u_y)^{n-1}n\frac{\partial^2\Lambda_2}{\partial y\partial v}+u_{yy}(-u_y)^{n-2}n(n-1)\frac{\partial\Lambda_2}{\partial y}+(-u_y)^{n}n\frac{\partial^2\Lambda_2}{\partial y\partial u}\\ - & (-u_y)^{n+1}n\frac{\partial^2\Lambda_2}{\partial y^2}+v_y(-u_y)^{n}n\frac{\partial^2\Lambda_2}{\partial u\partial v}-u_{yy}(-u_y)^{n-1}n(n-1)\frac{\partial\Lambda_2}{\partial u}\\ - & v_y(-u_y)^{n-1}n\frac{\partial^2\Lambda_2}{\partial v\partial y}+v_y(-u_y)^{n}n\frac{\partial^2\Lambda_2}{\partial v\partial u}-v_y^2(-u_y)^{n-1}n\frac{\partial^2\Lambda_2}{\partial v^2}\\ + & v_yu_{yy}(-u_y)^{n-2}n(n-1)\frac{\partial\Lambda_2}{\partial v}-u_{yy}(-u_y)^{n-2}n(n-1)\frac{\partial\Lambda_2}{\partial y}-u_{yy}(-u_y)^{n-1}n^2\frac{\partial\Lambda_2}{\partial u}\\ + & u_{yy}v_y(-u_y)^{n-2}n(n-1)\frac{\partial\Lambda_2}{\partial v}+u_{yy}^2(u_y)^{n-3}n(n-1)(n-2)\Lambda_2\\ - & v_{yy}(-u_y)^{n-1}n\frac{\partial\Lambda_2}{\partial v}+u_{yyy}(-u_y)^{n-2}n(n-1)\Lambda_2=0. \end{aligned} \tag{4.2.13}$$

We are considering $n > 0$. The case $n = 1$ describes a Newtonian fluid. The multipliers for $n = 1$ have been derived by Naz et al. [8]. We will consider $n \neq 1$ and then state the results for $n = 1$. We now split the determining equation (4.2.13) in powers and products for the derivatives of u and v.

Case 1: $n > 0$, $n \neq 1$

$$
\begin{aligned}
v_y &: \quad \frac{\partial \Lambda_1}{\partial u} - v\frac{\partial \Lambda_2}{\partial v} - \Lambda_2 = 0, && (4.2.14)\\
u_{yy}(-u_y)^{n-1} &: \quad n(n+1)\frac{\partial \Lambda_2}{\partial u} = 0, && (4.2.15)\\
v_x &: \quad \frac{\partial \Lambda_1}{\partial v} + u\frac{\partial \Lambda_2}{\partial v} = 0, && (4.2.16)\\
u_{yy}(-u_y)^{n-2} &: \quad n(n-1)\frac{\partial \Lambda_2}{\partial y} = 0 && (4.2.17)\\
v_y(-u_y)^{n-2}u_{yy} &: \quad n(n-1)\frac{\partial \Lambda_2}{\partial v} = 0, && (4.2.18)\\
(-u_y)^{n-1} &: \quad n\frac{\partial^2 \Lambda_2}{\partial y^2} = 0, && (4.2.19)\\
(-u_y)^{n+1} &: \quad n\frac{\partial^2 \Lambda_2}{\partial u^2} = 0, && (4.2.20)\\
v_y(-u_y)^{n} &: \quad n\frac{\partial^2 \Lambda_2}{\partial u \partial v} = 0, && (4.2.21)\\
v_y^2(-u_y)^{n-1} &: \quad n\frac{\partial^2 \Lambda_2}{\partial v^2} = 0, && (4.2.22)\\
v_{yy}(-u_y)^{n-1} &: \quad n\frac{\partial \Lambda_2}{\partial v} = 0, && (4.2.23)\\
\textit{Remainder} &: \quad \frac{\partial \Lambda_1}{\partial x} + u\frac{\partial \Lambda_2}{\partial x} + v\frac{\partial \Lambda_2}{\partial y} = 0, && (4.2.24)
\end{aligned}
$$

From equations (4.2.15), (4.2.17) and (4.2.18), we have $\Lambda_2 = A(x)$, where $A(x)$ is an arbitrary function of x. This result leaves us with three remaining equations,

$$
\begin{aligned}
v_y &: \quad \frac{\partial \Lambda_1}{\partial u} - A(x) = 0, && (4.2.25)\\
v_x &: \quad \frac{\partial \Lambda_1}{\partial v} = 0, && (4.2.26)\\
1 &: \quad \frac{\partial \Lambda_1}{\partial x} + u\frac{dA(x)}{dx} = 0. && (4.2.27)
\end{aligned}
$$

From equation (4.2.26), $\Lambda_1 = \Lambda_1(x, y, u)$. Further by solving equation (4.2.25) we have

$$\Lambda_1(x, y, u) = A(x)u + B(x, y), \qquad (4.2.28)$$

where $B(x, y)$ is an arbitrary function of x and y. Substituting the results for Λ_1 and Λ_2 into equation (4.2.27) gives

$$2u\frac{dA(x)}{dx} + \frac{\partial B}{\partial x}(x, y) = 0. \qquad (4.2.29)$$

We can split (4.2.29) by u to give

$$u \quad : \quad \frac{dA(x)}{dx} = 0, \tag{4.2.30}$$

$$1 \quad : \quad \frac{\partial B}{\partial x}(x, y) = 0. \tag{4.2.31}$$

Hence $A(x) = c_1$ where c_1 is a constant and $B(x, y) = B(y)$ where $B(y)$ is an arbitrary function of y. As a result, the multipliers are

$$\Lambda_1 = c_1 u + B(y), \quad \Lambda_2 = c_1. \tag{4.2.32}$$

In order to determine the arbitrary function $B(y)$ we substitute equation (4.2.32) into the second determining equation (4.2.8). That is

$$E_v[2c_1 uu_x + B(y)u_x + c_1 uv_y + B(y)v_y + c_1 vu_y - c_1 n(-u_y)^{n-1}u_{yy}] = 0. \tag{4.2.33}$$

We can expand equation (4.2.33) further by using equation (4.2.10), which gives

$$\begin{aligned} &\frac{\partial}{\partial v}[2c_1 uu_x + B(y)u_x + c_1 uv_y + B(y)v_y + c_1 vu_y - c_1 n(-u_y)^{n-1}u_{yy}] \\ - \; &D_y \frac{\partial}{\partial v_y}[2c_1 uu_x + B(y)u_x + c_1 uv_y + B(y)v_y + c_1 vu_y - c_1 n(-u_y)^{n-1}u_{yy}] = 0 \end{aligned} \tag{4.2.34}$$

Equation (4.2.34) can be simplified to give

$$c_1 u_y - D_y(c_1 u + B(y)) = 0. \tag{4.2.35}$$

Expanding equation (4.2.35) using equation (4.2.5), we obtain $dB(y)/dy = 0$ and therefore $B(y) = c_2$ where c_2 is a constant. Hence, for $n \neq 1$, the multipliers are given by

$$\Lambda_1 = c_1 u + c_2, \quad \Lambda_2 = c_1. \tag{4.2.36}$$

Case 2: $n = 1$

For $n = 1$ equation (4.2.2) reduces to Prandtl's momentum boundary layer equation for a two-dimensional laminar jet of a Newtonian fluid. That is, the momentum equation is given in dimensionless form by

$$uu_x + vu_y - u_{yy} = 0. \tag{4.2.37}$$

Naz et al. [8] found the multipliers to be

$$\Lambda_1 = c_1 u + c_2, \quad \Lambda_2 = c_1, \tag{4.2.38}$$

which are identical to the case when $n \neq 1$. In paper [8] the equation had been transformed back to dimensional form. However, the multipliers are unaffected.

We proceed to write the system in conserved form in order to obtain the conserved vectors.
Case 1: $n \neq 1$

From equations (4.2.3) and (4.2.36),

$$(c_1 u + c_2)(u_x + v_y) + c_1(uu_x + vu_y - n(-u_y)^{n-1}u_{yy}) = D_x T^1 + D_y T^2. \quad (4.2.39)$$

Thus in conserved form we have

$$\begin{aligned}(c_1 u + c_2)(u_x + v_y) + c_1(uu_x + vu_y - n(-u_y)^{n-1}u_{yy}) \\ = D_x[c_1 u^2 + c_2 u] + D_y[c_1(uv + (-u_y)^n) + c_2 v]. \quad (4.2.40)\end{aligned}$$

When $u(x, y)$ and $v(x, y)$ are solutions of equations (4.2.1) and (4.2.2), equation (4.2.40) reduces to

$$D_x[c_1 u^2 + c_2 u] + D_y[c_1(uv + (-u_y)^n) + c_2 v] = 0. \quad (4.2.41)$$

We can therefore write the conserved vector as a linear combination of the two conserved vectors

$$T^1 = u^2, \quad T^2 = uv + (-u_y)^n, \quad (4.2.42)$$
$$T^1 = u, \quad T^2 = v. \quad (4.2.43)$$

Case 2: $n = 1$

Naz et al. [8] showed that when $n = 1$ the conserved vectors in dimensional form are,

$$T^1 = u^2, \quad T^2 = uv - \nu u_y, \quad (4.2.44)$$
$$T^1 = u, \quad T^2 = v. \quad (4.2.45)$$

Expressed in dimensionless form the conserved vectors for $n = 1$ are given by

$$T^1 = u^2, \quad T^2 = uv - u_y, \quad (4.2.46)$$
$$T^1 = u, \quad T^2 = v. \quad (4.2.47)$$

Using multipliers of the form $\Lambda_1 = \Lambda_1(x, y, u, v)$, $\Lambda_2 = \Lambda_2(x, y, u, v)$, we have derived the conserved vectors for a power-law fluid with $n > 0$ given by equations (4.2.42) and (4.2.43). The conserved vectors for a Newtonian fluid are obtained by putting $n = 1$ in the conserved vectors for a power-law fluid.

4.2.2 Conservation laws for the stream function

The system of equations (4.2.1) and (4.2.2) can be written as a third order partial differential equation using the transformation

$$u(x,y) = \psi_y, \quad v(x,y) = -\psi_x. \tag{4.2.48}$$

The continuity equation is identically satisfied using the transformation (4.2.48) and the momentum balance equation reduces to a single third order partial differential equation,

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy} = 0. \tag{4.2.49}$$

For equation (4.2.49) we consider a multiplier of the form $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$. By the property of multipliers, we have

$$\Lambda(\psi_y\psi_{xy} - \psi_x\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy}) = D_xT^1 + D_yT^2, \tag{4.2.50}$$

where $T = (T^1, T^2)$ are the components of the conserved vector and the derivative operators D_x and D_y are defined as

$$D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial\psi} + \psi_{xx}\frac{\partial}{\partial\psi_x} + \psi_{xy}\frac{\partial}{\partial\psi_y} + ..., \tag{4.2.51}$$

$$D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial\psi} + \psi_{yy}\frac{\partial}{\partial\psi_y} + \psi_{yx}\frac{\partial}{\partial\psi_x} + ... \tag{4.2.52}$$

The determining equation is given by

$$E_\psi[\Lambda(\psi_y\psi_{xy} - \psi_x\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy})] = 0, \tag{4.2.53}$$

where the Euler operator is defined as

$$E_\psi = \frac{\partial}{\partial\psi} - D_x\frac{\partial}{\partial\psi_x} - D_y\frac{\partial}{\partial\psi_y} + D_x^2\frac{\partial}{\partial\psi_{xx}} + D_xD_y\frac{\partial}{\partial\psi_{xy}} + D_y^2\frac{\partial}{\partial\psi_{yy}} - ... \tag{4.2.54}$$

Equation (4.2.53) can be expanded using equation (4.2.54):

$$\begin{aligned}
& \frac{\partial}{\partial\psi}[\Lambda\psi_y\psi_{xy} - \Lambda\psi_x\psi_{yy} - \Lambda n(-\psi_{yy})^{n-1}\psi_{yyy}] \\
- \quad & D_x\frac{\partial}{\partial\psi_x}[\Lambda\psi_y\psi_{xy} - \Lambda\psi_x\psi_{yy} - \Lambda n(-\psi_{yy})^{n-1}\psi_{yyy}] \\
- \quad & D_y\frac{\partial}{\partial\psi_y}[\Lambda\psi_y\psi_{xy} - \Lambda\psi_x\psi_{yy} - \Lambda n(-\psi_{yy})^{n-1}\psi_{yyy}] \\
+ \quad & D_xD_y\frac{\partial}{\partial\psi_{xy}}[\Lambda\psi_y\psi_{xy} - \Lambda\psi_x\psi_{yy} - \Lambda n(-\psi_{yy})^{n-1}\psi_{yyy}] \\
+ \quad & D_y^2\frac{\partial}{\partial\psi_{yy}}[\Lambda\psi_y\psi_{xy} - \Lambda\psi_x\psi_{yy} - \Lambda n(-\psi_{yy})^{n-1}\psi_{yyy}] \\
- \quad & D_y^3\frac{\partial}{\partial\psi_{yyy}}[\Lambda\psi_y\psi_{xy} - \Lambda\psi_x\psi_{yy} - \Lambda n(-\psi_{yy})^{n-1}\psi_{yyy}] = 0.
\end{aligned} \tag{4.2.55}$$

Equation (4.2.55) can be simplified further,

$$\begin{aligned} & \psi_y\psi_{xy}\frac{\partial\Lambda}{\partial\psi} - \psi_x\psi_{yy}\frac{\partial\Lambda}{\partial\psi} - n(-\psi_{yy})^{n-1}\psi_{yyy}\frac{\partial\Lambda}{\partial\psi} \\ - & D_x\left[\psi_y\psi_{xy}\frac{\partial\Lambda}{\partial\psi_x} - \psi_x\psi_{yy}\frac{\partial\Lambda}{\partial\psi_x} - \Lambda\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy}\frac{\partial\Lambda}{\partial\psi_x}\right] \\ - & D_y\left[\psi_y\psi_{xy}\frac{\partial\Lambda}{\partial\psi_y} + \Lambda\psi_{xy} - \psi_x\psi_{yy}\frac{\partial\Lambda}{\partial\psi_y} - n(-\psi_{yy})^{n-1}\psi_{yyy}\frac{\partial\Lambda}{\partial\psi_y}\right] + D_xD_y[\Lambda\psi_y] \\ + & D_y^2[-\Lambda\psi_x + \Lambda n(n-1)(-\psi_{yy})^{n-2}\psi_{yyy}] - D_y^3[-\Lambda n(-\psi_{yy})^{n-1}] = 0. \end{aligned} \tag{4.2.56}$$

Equating the higher order derivative terms $(-\psi_{yy})^{n-1}\psi_{yyyy}$ and $(-\psi_{yy})^{n-1}\psi_{yyyx}$ to zero gives

$$\Lambda_{\psi_y} = 0, \quad \Lambda_{\psi_x} = 0, \tag{4.2.57}$$

which reduces the multiplier to $\Lambda = \Lambda(x, y, \psi)$. Substituting this result into (4.2.56) we obtain a further simplification,

$$\begin{aligned} & \psi_y\psi_{xy}\frac{\partial\Lambda}{\partial\psi} - \psi_x\psi_{yy}\frac{\partial\Lambda}{\partial\psi} - n(-\psi_{yy})^{n-1}\psi_{yyy}\frac{\partial\Lambda}{\partial\psi} + D_x[+\Lambda\psi_{yy}] + D_y[+\Lambda\psi_{xy}] \\ + & D_xD_y[+\Lambda\psi_y] + D_y^2[-\Lambda\psi_x + \Lambda n(n-1)(-\psi_{yy})^{n-2}\psi_{yyy}] \\ + & D_y^3[+\Lambda n(-\psi_{yy})^{n-1}] = 0. \end{aligned} \tag{4.2.58}$$

Equation (4.2.58) can be expanded further using equations (4.2.51) and (4.2.52). The resulting determining equation can be separated according to products and powers of the partial derivatives of ψ. As stated earlier the parameter n plays a vital role in the classification of these fluids. Since we are primarily concerned with the non-Newtonian power-law fluid the case for $n \neq 1$ is our main interest. We do note however, that special cases do arise. In particular we will consider the case for $n = 1$ and $n = 2$. We first consider the general case.

Case 1: $n \neq 1$, $n \neq 2$.

$$\psi_{yyy}^2\psi_y(-\psi_{yy})^{n-3} \; : \; (n-2)(n-1)n\Lambda_\psi = 0, \tag{4.2.59}$$

$$\psi_{yyy}^2(-\psi_{yy})^{n-3} \; : \; (n-2)(n-1)n\Lambda_y = 0, \tag{4.2.60}$$

$$
\begin{aligned}
\psi_{yyyy}\psi_y(-\psi_{yy})^{n-2} &: (n-1)n\Lambda_\psi = 0, & (4.2.61)\\
\psi_{yyyy}(-\psi_{yy})^{n-2} &: (n-1)n\Lambda_y = 0, & (4.2.62)\\
\psi_{xy} &: \Lambda_y = 0, & (4.2.63)\\
\psi_{yyy}(-\psi_{yy})^{n-1} &: (n-1)n\Lambda_\psi = 0, & (4.2.64)\\
\psi_{yyy}(-\psi_{yy})^{n-2}\psi_y &: (n-1)n\Lambda_{y\psi} = 0, & (4.2.65)\\
\psi_{yyy}(-\psi_{yy})^{n-2}\psi_y^2 &: (n-1)n\Lambda_{\psi\psi} = 0, & (4.2.66)\\
\psi_{yyy}(-\psi_{yy})^{n-2} &: (n-1)n\Lambda_{yy} = 0, & (4.2.67)\\
\psi_x\psi_y &: \Lambda_{y\psi} = 0, & (4.2.68)\\
\psi_x &: \Lambda_{yy} = 0, & (4.2.69)\\
(-\psi_{yy})^n &: n\Lambda_{y\psi} = 0, & (4.2.70)\\
\psi_y(-\psi_{yy})^n &: n\Lambda_{\psi\psi} = 0, & (4.2.71)\\
\psi_y(-\psi_{yy})^{n-1} &: n\Lambda_{yy\psi} = 0, & (4.2.72)\\
\psi_y^2(-\psi_{yy})^{n-1} &: \Lambda_{y\psi\psi} = 0, & (4.2.73)\\
\psi_y^3(-\psi_{yy})^{n-1} &: n\Lambda_{\psi\psi\psi} = 0, & (4.2.74)\\
(-\psi_{yy})^{n-1} &: n\Lambda_{yyy} = 0, & (4.2.75)\\
\psi_{yy} &: \Lambda_x = 0, & (4.2.76)\\
\psi_y^2 &: \Lambda_{x\psi} = 0, & (4.2.77)\\
\psi_y &: \Lambda_{xy} = 0. & (4.2.78)
\end{aligned}
$$

From equations (4.2.59), (4.2.63) and (4.2.76) we have

$$\Lambda = c_1. \tag{4.2.79}$$

Case 2: $n = 2$

When $n = 2$ the separation of the determining equation given by (4.2.59) to (4.2.78) remains valid except for equations (4.2.75) and (4.2.76) because when $n = 2$, $(-\psi_{yy})^{n-1} = -\psi_{yy}$. Separating the determining equation for $n = 2$ with respect to the derivative ψ_{yy} gives

$$\frac{\partial \Lambda}{\partial x} - \frac{\partial^3 \Lambda}{\partial y^3} = 0 \tag{4.2.80}$$

But from (4.2.63), $\Lambda_y = 0$ and therefore (4.2.80) reduces to $\Lambda_x = 0$. Also from (4.2.61), $\Lambda_\psi = 0$ and therefore the multiplier is again given by (4.2.79).

Case 3: $n = 1$

When $n = 1$ the separation of the determining equation given by (4.2.59) to (4.2.78) remains valid except for the following changes. Equations (4.2.70) and

(4.2.76) are replaced by one equation obtained by separating by the derivative ψ_{yy},

$$2\frac{\partial \Lambda}{\partial x} + 3\frac{\partial^2 \Lambda}{\partial y \partial \psi} = 0 \tag{4.2.81}$$

Equations (4.2.72) and (4.2.78) are replaced by one equation obtained by separating by the derivative ψ_y,

$$\frac{\partial^2 \Lambda}{\partial x \partial y} + 3\frac{\partial^3 \Lambda}{\partial \psi \partial y^2} = 0. \tag{4.2.82}$$

Equations (4.2.73) and (4.2.77) are replaced by one equation obtained by separating the derivative ψ_y^2,

$$\frac{\partial^2 \Lambda}{\partial x \partial \psi} + 4\frac{\partial^3 \Lambda}{\partial y \partial \psi^2} = 0. \tag{4.2.83}$$

Equation (4.2.75) now becomes the remainder independent of the partial derivatives of ψ,

$$\frac{\partial^3 \Lambda}{\partial y^3} = 0 \tag{4.2.84}$$

From (4.2.63), $\Lambda_y = 0$ and therefore from (4.2.81), $\Lambda_x = 0$. Thus $\Lambda = \Lambda(\psi)$ and the system of equations derived from the determining equations reduce to (4.2.71),

$$\frac{\partial^2 \Lambda}{\partial \psi^2} = 0. \tag{4.2.85}$$

Hence

$$\Lambda(\psi) = c_1 + c_2\psi \tag{4.2.86}$$

where c_1 and c_2 are constants.

We now write the third order partial differential equation in conserved form in order to obtain the conserved vectors.

Case 1: $n \neq 1$

From equations (4.2.50) and (4.2.79),

$$c_1(\psi_y\psi_{xy} - \psi_x\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy}) = D_xT^1 + D_yT^2. \tag{4.2.87}$$

Thus in conserved form have by elementary manipulations

$$c_1(\psi_y\psi_{xy} - \psi_x\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy}) = D_x[c_1\psi_y^2] + D_y[c_1(-\psi_x\psi_y + (-\psi_{yy})^n)]. \tag{4.2.88}$$

Equation (4.2.88) is satisfied for arbitrary functions $\psi(x,y)$. When $\psi(x,y)$ is a solution of (4.2.49),

$$D_x[c_1\psi_y^2] + D_y[c_1(-\psi_x\psi_y + (-\psi_{yy})^n)] = 0. \tag{4.2.89}$$

The conserved vector therefore is

$$T^1 = \psi_y^2, \quad T^2 = -\psi_x\psi_y + (-\psi_{yy})^n. \tag{4.2.90}$$

We note that for the case $n = 2$ we will obtain the same conserved vector.

Case 2: $n = 1$

From equations (4.2.50) and (4.2.86)

$$(c_1 + c_2\psi)(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \psi_{yyy}) = D_xT^1 + D_yT^2. \tag{4.2.91}$$

Thus we have in conserved form by elementary manipulations

$$\begin{aligned}&(c_1 + c_2\psi)(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \psi_{yyy})\\ &\quad = D_x[c_2(\psi\psi_y^2) + c_1(\psi_y^2)] + D_y[c_2(-\psi\psi_x\psi_y - \psi\psi_{yy} + \frac{1}{2}\psi_y^2) + c_1(-\psi_x\psi_y - \psi_{yy})].\end{aligned} \tag{4.2.92}$$

Equation (4.2.92) is satisfied for arbitrary functions $\psi(x, y)$. When $\psi(x, y)$ is a solution of equation (4.2.49)

$$D_x[c_2(\psi\psi_y^2) + c_1(\psi_y^2)] + D_y[c_2(-\psi\psi_x\psi_y - \psi\psi_{yy} + \frac{1}{2}\psi_y^2) + c_1(-\psi_x\psi_y - \psi_{yy})] = 0. \tag{4.2.93}$$

The conserved vectors for $n = 1$ are thus given by

$$T^1 = \psi_y^2, \quad T^2 = -\psi_x\psi_y - \psi_{yy}, \tag{4.2.94}$$

$$T^1 = \psi\psi_y^2, \quad T^2 = -\psi\psi_x\psi_y + \frac{1}{2}\psi_y^2 - \psi\psi_{yy}. \tag{4.2.95}$$

The conserved vectors, (4.2.94) and (4.2.95) are derived using the dimensionless momentum equation. Naz et al.[8] showed them to be in dimensional form,

$$T^1 = \psi_y^2, \quad T^2 = -\psi_x\psi_y - \nu\psi_{yy}, \tag{4.2.96}$$

$$T^1 = \psi\psi_y^2, \quad T^2 = -\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}. \tag{4.2.97}$$

We have derived a basis of conserved vectors using the multiplier of the form $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$. The set of conserved vectors for $n = 1$ gives not only the elementary conserved vector (4.2.94), which gives the conserved quantity for the two-dimensional free jet, but also provides another conserved vector (4.2.95), which will give the conserved quantity for the two-dimensional wall jet. For the case $n \neq 1$ we only have the elementary conserved vector. The consequence of this is that we will only be able to derive the conserved quantity for a two-dimensional free jet in a power-law fluid.

4.3 Conserved quantities for the two-dimensional liquid and free jets

The systematic approach presented by Naz et al.[8] will be adopted in this section in order to derive the conserved quantities for the two-dimensional liquid and free jets for a non-Newtonian power-law fluid. We do note that for a two-dimensional liquid jet the conserved quantity can only be derived for the velocity components and not for the stream function. However, the conserved quantity for the two-dimensional free jet can be derived using both the velocity components and the stream function. These conserved vectors are chosen such that the integral of the flux component across the jet vanishes when the boundary conditions are imposed.

4.3.1 Two-dimensional liquid jet

The boundary conditions for the liquid jet which is illustrated in Figure 3.3.2 are given by

$$y = 0 \quad : \quad u(x,0) = 0, \quad v(x,0) = 0, \tag{4.3.1}$$

$$y = \phi(x) \quad : \quad \frac{\partial u}{\partial y}(x, \phi(x)) = 0. \tag{4.3.2}$$

Equation (4.3.1) represents the no slip boundary condition and the fact that the normal component of velocity at the boundary is zero. The condition that there is no tangential stress on the free surface is given by equation (4.3.2).

We proceed to calculate the y-component of velocity on the free surface which is $y = \phi(x)$. This is the kinematic condition, which states that a fluid particle on the surface should remain on the surface. Taking the material time derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla), \tag{4.3.3}$$

the y-component of velocity becomes

$$v(x, \phi(x)) = u(x, \phi(x)) \frac{d\phi(x)}{dx}. \tag{4.3.4}$$

The conserved vector that we will use to derive the conserved quantity for the liquid jet is (4.2.43). Now

$$\int_0^{\phi(x)} \left(\frac{\partial T^1(x,y)}{\partial x} + \frac{\partial T^2(x,y)}{\partial y} \right) dy = 0, \tag{4.3.5}$$

that is

$$\int_0^{\phi(x)} \left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} \right) dy = 0. \tag{4.3.6}$$

Integrating (4.3.6) gives

$$v(x,y)|_0^{\phi(x)} = -\int_0^{\phi(x)} \frac{\partial u(x,y)}{\partial x} dy \tag{4.3.7}$$

Making use of the boundary condition given by equations (4.3.1) we have

$$v(x,\phi(x)) = -\int_0^{\phi(x)} \frac{\partial u(x,y)}{\partial x} dy \tag{4.3.8}$$

Now the formula for differentiation under an integral sign is [14]

$$\frac{d}{dx}\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}(x,y)dy + f(x,\phi_2(x))\frac{d\phi_2}{dx} - f(x,\phi_1(x))\frac{d\phi_1}{dx}. \tag{4.3.9}$$

Hence

$$\frac{d}{dx}\int_0^{\phi(x)} u(x,y)dy = \int_0^{\phi(x)} \frac{\partial u(x,y)}{\partial x} dy + u(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.3.10}$$

and therefore

$$\int_0^{\phi(x)} \frac{\partial u(x,y)}{\partial x} dy = \frac{d}{dx}\int_0^{\phi(x)} u(x,y)dy - u(x,\phi(x))\frac{d\phi(x)}{dx} \tag{4.3.11}$$

Thus (4.3.8) becomes

$$v(x,\phi(x)) = -\frac{d}{dx}\int_0^{\phi(x)} u(x,y)dy + u(x,\phi(x))\frac{d\phi(x)}{dx}. \tag{4.3.12}$$

Using (4.3.4) this simplifies to

$$\frac{d}{dx}\int_0^{\phi(x)} u(x,y)dy = 0 \tag{4.3.13}$$

and therefore

$$\int_0^{\phi(x)} u(x,y)dy = \text{constant independent of } x. \tag{4.3.14}$$

Thus the conserved quantity for the two-dimensional liquid jet is

$$J = \int_0^{\phi(x)} u(x,y)dy. \tag{4.3.15}$$

This can be interpreted as a constant volume flux along the liquid jet Naz et al. [8].

4.3.2 Two-dimensional free jet

The boundary conditions for the free jet which is illustrated in Figure 3.3.1 are specified by:

$$y = \infty \quad : \quad u(x,\infty) = 0, \quad \frac{\partial u}{\partial y}(x,\infty) = 0, \tag{4.3.16}$$

$$y = 0 \quad : \quad v(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = 0. \tag{4.3.17}$$

The normal component of the velocity on the axis of symmetry, $y = 0$, is zero. This can be attributed to symmetry. Since $v(x,y) = -v(x,-y)$, at $y = 0$ the velocity $v(x,0)$ must be zero. Further we have a mathematical boundary condition that $u(x,y)$ has a maximum value at $y = 0$. These results are given by the boundary conditions (4.3.17).

The conserved vector which will be used to derive the conserved quantity is given by equation (4.2.42). Now

$$\int_0^\infty \left(\frac{\partial T^1(x,y)}{\partial x} + \frac{\partial T^2(x,y)}{\partial y} \right) dy = 0, \tag{4.3.18}$$

that is

$$\int_0^\infty \left[\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv + (-u_y)^n) \right] dy = 0. \tag{4.3.19}$$

Integrating (4.3.19), we obtain

$$uv + (-u_y)^n|_0^\infty = -\int_0^\infty \frac{\partial}{\partial x}(u^2)dy. \tag{4.3.20}$$

Using the boundary conditions (4.3.16)-(4.3.17), and the fact that $v(x,\infty)$ is finite gives

$$0 = -\int_{-\infty}^\infty \frac{\partial}{\partial x}(u^2(x,y)). \tag{4.3.21}$$

Using the formula for differentiation under an integral sign, (4.3.9),we obtain

$$\frac{d}{dx}\int_0^\infty u^2(x,y)dy \quad = \quad 0. \tag{4.3.22}$$

Hence

$$\int_0^\infty u^2(x,y)dy = \text{constant independent of } x. \tag{4.3.23}$$

We use as the conserved quantity

$$J = 2\rho \int_0^\infty u^2(x,y)dy, \tag{4.3.24}$$

where ρ is the constant density of the fluid. Physically this can be interpreted as a constant momentum flux in the direction of the jet along the jet [8]. It is a measure of the strength of the jet.

A conserved quantity for the two-dimensional free jet can also be formulated in terms of the stream function. Using the transformations given by equation (4.2.48) the boundary conditions can be written as

$$y = \infty \quad : \quad \psi_y(x, \infty) = 0, \quad \psi_{yy}(x, \infty) = 0, \tag{4.3.25}$$

$$y = 0 \quad : \quad \psi_y(x, 0) = 0, \quad \psi_{yy}(x, 0) = 0. \tag{4.3.26}$$

The conserved vector that will be used to derive the conserved quantity is given by equation (4.2.90). Now

$$\int_0^\infty \left(\frac{\partial T^1(x,y)}{\partial x} + \frac{\partial T^2(x,y)}{\partial y} \right) dy = 0, \tag{4.3.27}$$

that is

$$\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x}(\psi_y^2) + \frac{\partial}{\partial y}(-\psi_x\psi_y + (-\psi_{yy})^n) \right] dy \quad = \quad 0. \tag{4.3.28}$$

Integrating equation (4.3.28) gives

$$-\psi_x\psi_y + (-\psi_{yy})^n|_0^\infty = -\int_0^\infty \frac{\partial}{\partial x}(\psi_y^2). \tag{4.3.29}$$

Using the boundary conditions (4.3.25)-(4.3.26), and the fact that $\psi_x(x, \infty)$ is finite gives

$$0 = -\int_0^\infty \frac{\partial}{\partial x}(\psi_y^2(x,y)). \tag{4.3.30}$$

Using the formula for differentiation under an integral sign yields

$$\frac{d}{dx}\int_0^\infty \psi_y^2(x,y)dy. = 0 \tag{4.3.31}$$

and therefore

$$\int_0^\infty \psi_y^2(x,y)dy = \text{constant independent of } x. \tag{4.3.32}$$

The conserved quantity which we will use is given by,

$$J = 2\rho \int_0^\infty \psi_y^2(x,y)dy. \tag{4.3.33}$$

4.4 Conclusion

In this chapter we derived the conservation laws for the two-dimensional liquid, free and wall jets using the multiplier approach. The resulting determining equations presented us with different cases to consider. Firstly we examined the primary case for $n \neq 1$. The other special cases that arose were for $n = 0$ and $n = 1$ when using velocity components. The case $n = 0$ applies only under very special physical conditions and was not considered. For $n = 1$ the multipliers were identical to those for $n \neq 1$. The conserved vectors could be expressed in dimensionless form or in dimensional form as in Naz et al.[8].

Further we could reformulate the system of two equations for the velocity components into a single third order partial differential equation for the stream function. The determining equation gave one more special case, that being $n = 2$. However, it was found that for all values of $n \neq 1$ the multipliers were the same. For $n = 1$ the multiplier was different. As a result the set of conserved vectors for $n = 1$ proved sufficient to derive the conserved quantities for both the free and wall jets [8], whereas for $n \neq 1$ the elementary conserved vector was obtained which only derived the conserved quantity for the free jet.

Naz et al.[8] presented a new, more systematic approach to deriving conserved quantities and this method was adopted to finding the conserved quantities for the two-dimensional liquid and free jets for a non-Newtonian power-law fluid. Interestingly the conserved quantity for the free jet could be derived using both the velocity components and the stream function. The conserved quantity for the liquid jet however, could only be derived using the velocity components. Further the physical significance of the conserved quantities was stated.

Chapter 5

Parametric solutions for the two-dimensional free jet

5.1 Introduction

Despite the vast and powerful computing tools available, solving many boundary layer equations proves difficult. In many instances singularities present themselves in the solution process. The value of analytical solutions therefore cannot be underestimated. These analytical solutions prove to be relatively efficient due to the transparency of the solutions.

Similarity solutions are prime examples of analytical solutions. The primary aim of this method is to transform the partial differential equation to an ordinary differential equation by reducing the number of independent variables. The underlying feature of this method is that it identifies solutions with a similarity structure.

For the purposes of this chapter, we will derive the ordinary differential equation of the non-Newtonian power-law fluid for both the stream function and the velocity components. Another analytic approach that will be used is the associated Lie point symmetries method which associates a Lie point symmetry with a conserved vector for the partial differential equation [9, 10]. This method was first derived by Kara and Mahomed [9]. It proves to be more efficient than computing the full group of Lie point symmetries of the partial differential equation as introduced by Sophus Lie (1870). The resulting ordinary differential equation will be identical to the ordinary differential equation found when using similarity solutions for both the stream function and velocity components. The associated Lie point symmetries approach will only be applied to the stream function. The difficulty when applying this approach to the velocity components is that the momentum balance equation fails to provide us with enough information.

A complete solution for the free jet will be provided in parametric form. Historically numerical analysts have found difficulties for $n = 1/2$. However, we have found a parametric solution for the values $0 < n < 1/2$, $n = 1/2$ and $1/2 < n < \infty$.

An outline of this chapter is as follows. Section 5.2 will outline the similarity solution for the two-dimensional free jet in terms of the stream function. The similarity solution in terms of the velocity components will be presented in Section 5.3. Section 5.4 will detail the more efficient approach based on the symmetry and conservation law relation. The parametric solutions for $0 < n < 1/2$, $n = 1/2$ and $1/2 < n < \infty$ will be presented in Section 5.5 and the analysis of the solution will be given in Section 5.6. The conclusions will be drawn in Section 5.7.

5.2 Similarity solution for the free jet using the stream function

The momentum balance equation (4.2.49) for the stream function is

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - n\left(-\frac{\partial^2 \psi}{\partial y^2}\right)^{n-1}\frac{\partial^3 \psi}{\partial y^3} = 0. \tag{5.2.1}$$

Consider the scaling transformation

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{\psi} = \lambda^c \psi. \tag{5.2.2}$$

Substituting the transformation (5.2.2) into equation (5.2.1) we obtain

$$\lambda^{b-c}\frac{\partial \bar{\psi}}{\partial \bar{y}}\lambda^{a+b-c}\frac{\partial^2 \bar{\psi}}{\partial \bar{x}\partial \bar{y}} - \lambda^{a-c}\frac{\partial \bar{\psi}}{\partial \bar{x}}\lambda^{2b-c}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - n\left(-\lambda^{2b-c}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^{n-1}\lambda^{3b-c}\frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} = 0. \tag{5.2.3}$$

This can be simplified to give

$$\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial^2 \bar{\psi}}{\partial \bar{x}\partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - n\lambda^{b(2n-1)+c(2-n)-a}\left(-\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^{n-1}\frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} = 0. \tag{5.2.4}$$

Equation (5.2.1) will be invariant under the transformation (5.2.2) provided

$$a = b(2n-1) + c(2-n) \tag{5.2.5}$$

The transformed equation becomes, under the invariant condition (5.2.5),

$$\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial^2 \bar{\psi}}{\partial \bar{x}\partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - n\left(-\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^{n-1}\frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} = 0. \tag{5.2.6}$$

Suppose $\psi = f(x, y)$ is a solution of the original partial differential equation (5.2.1). Then since the form of the partial differential equation (5.2.4) is the same as that of (5.2.1), $\bar{\psi}$ will be the same function of $\bar{x}$ and $\bar{y}$ as ψ is of x and y. Hence the solution of (5.2.4) is of the form $\bar{\psi} = f(\bar{x}, \bar{y})$. Consider first $\bar{\psi} = f(\bar{x}, \bar{y})$. This can be written as

$$\lambda^c \psi = f(\lambda^a x, \lambda^b y) \tag{5.2.7}$$

using the scaling transformation (5.2.2). Since $\psi = f(x, y)$, equation (5.2.7) can be written as

$$\lambda^c f(x, y) = f(\lambda^a x, \lambda^b y). \tag{5.2.8}$$

We differentiate (5.2.8) with respect to λ,

$$c\lambda^{c-1} f(x, y) = \frac{\partial f}{\partial \bar{x}} a\lambda^{a-1} x + \frac{\partial f}{\partial \bar{y}} b\lambda^{b-1} y. \tag{5.2.9}$$

Since λ is an arbitrary parameter, we can set $\lambda = 1$. This in turn reduces the scaling variables to $\bar{x} = x$, $\bar{y} = y$ and $\bar{\psi} = \psi$. As a result equation (5.2.9) becomes

$$ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} = cf. \tag{5.2.10}$$

There are two cases to consider, $n \neq 2$ and $n = 2$. When $n = 2$, c cannot be determined from (5.2.5).

Case 1: $n \neq 2$

The characteristic curves are given by

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{df}{cf}. \tag{5.2.11}$$

But from (5.2.5)

$$c = \frac{a + b(1 - 2n)}{2 - n}. \tag{5.2.12}$$

The characteristic curves of differential equations can thus be written as

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{(2 - n)df}{[a + b(1 - 2n)]f}. \tag{5.2.13}$$

By considering the first combination,

$$\frac{dx}{ax} = \frac{dy}{by} \tag{5.2.14}$$

and integrating both sides, the first integral is given by

$$\frac{y}{x^{\frac{b}{a}}} = c_1, \tag{5.2.15}$$

where c_1 is a constant. Finally we consider the combination

$$\frac{dx}{ax} = \frac{(2-n)df}{[a+b(1-2n)]f} \tag{5.2.16}$$

which can be written as

$$\frac{1+\frac{b}{a}(1-2n)}{(2-n)}\frac{dx}{x} = \frac{df}{f}. \tag{5.2.17}$$

This is a variables separable differential equation which can be solved to give

$$\frac{1+\frac{b}{a}(1-2n)}{(2-n)}\ln x = \ln f + c_2, \tag{5.2.18}$$

where c_2 is a constant. Simplifying further yields

$$\frac{f}{x^{\frac{1}{2-n}(1+\frac{b}{a}(1-2n))}} = c_2. \tag{5.2.19}$$

The general solution of equation (5.2.13) is

$$c_2 = F(c_1), \tag{5.2.20}$$

for an arbitrary function F. Since $\psi = f(x, y)$, the general solution is

$$\psi(x, y) = x^{\frac{1}{2-n}(1+\frac{b}{a}(1-2n))}F(\xi), \tag{5.2.21}$$

where the similarity variable ξ is

$$\xi = \frac{y}{x^{\frac{b}{a}}}. \tag{5.2.22}$$

We define

$$\frac{b}{a} = \alpha. \tag{5.2.23}$$

Thus the general solution is terms of α is

$$\psi(x, y) = x^{\frac{1}{2-n}(1+\alpha(1-2n))}F(\xi), \tag{5.2.24}$$

where

$$\xi = \frac{y}{x^{\alpha}}. \tag{5.2.25}$$

Substituting equation (5.2.24) into the third order partial differential equation (5.2.1) and simplifying we obtain,

$$\begin{aligned}&\left(\frac{1-\alpha(n+1)}{2-n}\right)\left(\frac{dF}{d\xi}\right)^2 - \left(\frac{1+\alpha(1-2n)}{2-n}\right)\frac{d^2F}{d\xi^2}F(\xi)\\ &-n\left(-\frac{d^2F}{d\xi^2}\right)^{n-1}\frac{d^3F}{d\xi^3} = 0.\end{aligned} \tag{5.2.26}$$

We now make use of the conserved quantity for a two-dimensional free jet in terms of the stream function, given by equation (4.3.33), to determine the value of α. In (4.3.33) we transform from y to ξ using $y = x^\alpha \xi$. Since the integration in the conserved quantity is with respect to y at a fixed point x, $dy = x^\alpha d\xi$. Thus substituting (5.2.24) for ψ into the conserved quantity we obtain

$$J = 2\rho x^{\frac{2-3n\alpha}{2-n}} \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi = \text{constant independent of } x, \tag{5.2.27}$$

thus giving

$$\alpha = \frac{2}{3n}. \tag{5.2.28}$$

As a result the transformed conserved quantity, which is essential in the solution of the two-dimensional free jet, becomes

$$J = 2\rho \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{5.2.29}$$

The similarity solution for ψ and the ordinary differential equation, after substituting the value of α, become respectively

$$\psi(x,y) = x^{\frac{1}{3n}} F(\xi), \quad \xi = \frac{y}{x^{\frac{2}{3n}}}, \tag{5.2.30}$$

and

$$3n^2 \left(-\frac{d^2F}{d\xi^2}\right)^{n-1} \frac{d^3F}{d\xi^3} + F\frac{d^2F}{d\xi^2} + \left(\frac{dF}{d\xi}\right)^2 = 0. \tag{5.2.31}$$

Since we are considering only the upper half plane for the two-dimensional free jet, the boundary conditions are

$$y = \infty \quad : \quad \frac{\partial \psi}{\partial y}(x, \infty) = 0, \tag{5.2.32}$$

$$y = 0 \quad : \quad \frac{\partial \psi}{\partial x}(x, 0) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x, 0) = 0. \tag{5.2.33}$$

Now from (5.2.30)

$$\begin{aligned} \frac{\partial \psi(x,y)}{\partial y} &= x^{-\frac{1}{3n}} \frac{dF}{d\xi}, \quad \frac{\partial^2 \psi(x,y)}{\partial y^2} = x^{-\frac{1}{n}} \frac{d^2F}{d\xi^2}, \\ \frac{\partial \psi(x,y)}{\partial x} &= \frac{1}{3n} x^{\frac{1-3n}{3n}} \left[F(\xi) - 2\xi \frac{dF}{d\xi}\right]. \end{aligned} \tag{5.2.34}$$

Hence the boundary conditions expressed in terms of $F(\xi)$ are

$$\xi = \infty : \quad \frac{dF}{d\xi}(\infty) = 0, \tag{5.2.35}$$

$$\xi = 0 : \quad F(0) = 0, \tag{5.2.36}$$

$$\xi = 0 : \quad \frac{d^2F}{d\xi^2}(0) = 0 \tag{5.2.37}$$

Case 2: $n = 2$

When $n = 2$ the invariance condition (5.2.5) reduces to

$$a = 3b \tag{5.2.38}$$

and there is no condition on c. The partial differential equation (5.2.10) for f remains valid with $b/a = 1/3$ and c unspecified. The general solution is

$$f(x, y) = x^{\frac{c}{a}} F\left(\frac{y}{x^{\frac{1}{3}}}\right). \tag{5.2.39}$$

Since $\psi(x, y) = f(x, y)$ the general solution for ψ is

$$\psi(x, y) = x^{\beta} F(\xi), \quad \xi = \frac{y}{x^{\frac{1}{3}}} \tag{5.2.40}$$

where $\beta = c/a$. The constant β is determined from the conserved quantity (4.3.33). Substituting (5.2.40) into (4.3.33) gives

$$J = 2\rho x^{2\beta - \frac{1}{3}} \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi = \text{constant independent of } x, \tag{5.2.41}$$

which gives

$$\beta = \frac{1}{6}. \tag{5.2.42}$$

Hence (5.2.40) becomes

$$\psi(x, y) = x^{\frac{1}{6}} F(\xi), \quad \xi = \frac{y}{x^{\frac{1}{3}}}. \tag{5.2.43}$$

The solution (5.2.43) is the same as the general solution (5.2.30) with $n = 2$. The remainder of the derivation for $n = 2$ is therefore the same as for $n \neq 2$. The two cases are treated together.

In this section we reduced the third order partial differential equation (5.2.1) to an ordinary differential equation by means of a scaling transformation. This ordinary differential equation (5.2.31) will be solved parametrically subject to the boundary conditions (5.2.35) to (5.2.37) in Section 5.5.

5.3 Similarity solution for free jet using velocity components

In this section, we consider the similarity solution for the system of partial differential equations for the velocity components u and v. The approach will involve firstly determining an ordinary differential equation for the conservation of mass, and secondly, determining an ordinary differential equation for the momentum balance. Information obtained in the derivation of the continuity equation will be used when deriving the momentum balance equation.

The system of partial differential equations describing the two-dimensional free jet are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5.3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - n\left(-\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2} = 0. \tag{5.3.2}$$

We consider equation (5.3.1) first. Introduce the scaling transformation,

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{u} = \lambda^c u, \quad \bar{v} = \lambda^d v \tag{5.3.3}$$

Substituting the scaling transformation (5.3.3) into equation (5.3.1) yields,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \lambda^{b-d-a+c}\frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{5.3.4}$$

For equation (5.3.1) to remain invariant, we require

$$d = b - a + c. \tag{5.3.5}$$

The transformed equation is thus

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{5.3.6}$$

Suppose $u = f(x, y)$, $v = g(x, y)$ is a solution of the original partial differential equation (5.3.1). Then $\bar{u} = f(\bar{x}, \bar{y})$, $\bar{v} = g(\bar{x}, \bar{y})$ is a solution of the transformed partial differential equation (5.3.6).

Consider $\bar{u} = f(\bar{x}, \bar{y})$ first. That is, using $u = f$,

$$\lambda^c f = f(\lambda^a x, \lambda^b y). \tag{5.3.7}$$

Differentiating (5.3.7) with respect to λ we have

$$c\lambda^{c-1} f = a\lambda^{a-1} x\frac{\partial f}{\partial \bar{x}} + b\lambda^{b-1} y\frac{\partial f}{\partial \bar{y}}. \tag{5.3.8}$$

Since λ is an arbitrary parameter we let $\lambda = 1$. This in turns gives $\bar{x} = x$, $\bar{y} = y$. Equation (5.3.8) reduces to

$$ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} = cf. \tag{5.3.9}$$

We consider solutions with $a \neq 0$. The differential equations of characteristic curves of (5.3.9) are given by

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{df}{cf}. \tag{5.3.10}$$

By considering the first pair and integrating both sides, the first integral is given by

$$\frac{y}{x^{\frac{b}{a}}} = c_1, \tag{5.3.11}$$

where c_1 is a constant. Solving the next pair of equations, the first and last terms of equation (5.3.10), gives

$$\frac{f}{x^{\frac{c}{a}}} = c_2, \tag{5.3.12}$$

where c_2 is a constant. The general solution of equation (5.3.9) is

$$c_2 = G(c_1), \tag{5.3.13}$$

where G is an arbitrary function. That is, using $f = u$,

$$u(x, y) = x^{\frac{c}{a}}G(\xi), \tag{5.3.14}$$

where

$$\xi = \frac{y}{x^{\frac{b}{a}}}. \tag{5.3.15}$$

Next we consider $\bar{v} = g(\bar{x}, \bar{y})$. That is, since $v = g$,

$$\lambda^d g = g(\lambda^a x, \lambda^b y). \tag{5.3.16}$$

By differentiating (5.3.16) with respect to λ and using (5.3.5) for d, we have

$$(b - a + c)\lambda^{b-a+c-1}g = a\lambda^{a-1}\frac{\partial g}{\partial \bar{x}} + b\lambda^{b-1}\frac{\partial g}{\partial \bar{y}}. \tag{5.3.17}$$

Since λ is an arbitrary parameter we let $\lambda = 1$ and therefore $\bar{x} = x$, $\bar{y} = y$. As a result equation (5.3.17) reduces to

$$ax\frac{\partial g}{\partial x} + by\frac{\partial g}{\partial y} = (b - a + c)g. \tag{5.3.18}$$

The differential equations of characteristic curves of (5.3.18) are given by

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{dg}{(b-a+c)g}. \tag{5.3.19}$$

By considering the first pair of terms in equation (5.3.19) and solving, the first integral is given by

$$\frac{y}{x^{\frac{b}{a}}} = k_1, \tag{5.3.20}$$

where k_1 is a constant. Considering the first and the last terms in equation (5.3.19) and solving we obtain

$$\frac{g}{x^{-1+\frac{b}{a}+\frac{c}{a}}} = k_2, \tag{5.3.21}$$

where k_2 is a constant. The general solution of equation (5.3.18) is

$$k_2 = H(k_1), \tag{5.3.22}$$

where H is an arbitrary function. Thus, since $g = v$,

$$v(x,y) = x^{-1+\frac{b}{a}+\frac{c}{a}} H(\xi), \tag{5.3.23}$$

where ξ is defined by (5.3.15).

By substituting equations (5.3.14) and (5.3.23) into the continuity equation (5.3.1) we obtain the ordinary differential equation

$$\frac{dH}{d\xi} - \frac{b}{a}\xi\frac{dG}{d\xi} + \frac{c}{a}G(\xi) = 0 \tag{5.3.24}$$

which depends on the ratios c/a and b/a.

Next we consider the momentum balance equation (5.3.2). Consider again the scaling transformation (5.3.3). Substituting the scaling transformation (5.3.3) into the momentum balance equation (5.3.2) gives

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \lambda^{-a+b+c-d}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} - n\lambda^{-a+(n+1)b-(n-2)c}\left(-\frac{\partial \bar{u}}{\partial \bar{y}}\right)^{n-1}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = 0. \tag{5.3.25}$$

For equation (5.3.2) to remain invariant under the scaling transformation (5.3.3) we require

$$b - d - a + c = 0, \tag{5.3.26}$$

and

$$a = b(n+1) - (n-2)c. \tag{5.3.27}$$

We note that equation (5.3.26) is the same as (5.3.5).

Since we are looking for an invariant solution for the system (5.3.1) and (5.3.2) for u and v, it follows that u, v and ξ are given by (5.3.14), (5.3.23) and (5.3.15) where $G(\xi)$ and $H(\xi)$ are related by (5.3.24). Now (5.3.14), (5.3.15), (5.3.23) and (5.3.24) depend only on the ratios b/a and c/a. We define

$$\alpha = \frac{b}{a}, \quad \beta = \frac{c}{a} \tag{5.3.28}$$

and we can write

$$\xi = \frac{y}{x^{\alpha}}, \tag{5.3.29}$$

$$u(x, y) = x^{\beta} G(\xi), \tag{5.3.30}$$

$$v(x, y) = x^{\alpha+\beta-1} H(\xi), \tag{5.3.31}$$

where

$$\frac{dH}{d\xi} - \alpha\xi \frac{dG}{d\xi} + \beta G = 0 \tag{5.3.32}$$

and from (5.3.27),

$$(n+1)\alpha - (n-2)\beta = 1. \tag{5.3.33}$$

Substituting u and v given by (5.3.30) and (5.3.31) into the momentum balance equation (5.3.2) and using (5.3.33) gives the ordinary differential equation

$$n\left(-\frac{dG}{d\xi}\right)^{n-1} \frac{d^2G}{d\xi^2} - H(\xi)\frac{dG}{d\xi} - \left(\beta G(\xi) - \alpha\xi \frac{dG}{d\xi}\right) G(\xi) = 0. \tag{5.3.34}$$

In order to obtain the second relation between α and β consider the conserved quantity for the two-dimensional free jet. It was shown in Chapter 4 Section 4.3 that for a two-dimensional free jet in terms of velocity components

$$J = 2\rho \int_0^{\infty} u^2(x, y)dy = \text{constant independent of } x \tag{5.3.35}$$

By using (5.3.29) and (5.3.30), J can be rewritten at a given point x as

$$J = 2\rho x^{\alpha+2\beta} \int_0^{\infty} G^2(\xi)d\xi. \tag{5.3.36}$$

For J to be independent of x we require

$$\alpha + 2\beta = 0. \tag{5.3.37}$$

The conserved quantity J becomes

$$J = \int_0^{\infty} G^2(\xi)d\xi. \tag{5.3.38}$$

It follows from (5.3.33) and (5.3.37) that

$$\alpha = \frac{2}{3n} \quad \text{and} \quad \beta = -\frac{1}{3n}. \tag{5.3.39}$$

We now collect the results.

$$u(x,y) = x^{-\frac{1}{3n}} G(\xi), \tag{5.3.40}$$

$$v(x,y) = x^{\frac{1}{3n}-1} H(\xi), \tag{5.3.41}$$

$$\xi = \frac{y}{x^{\frac{2}{3n}}} \tag{5.3.42}$$

$$3n\frac{dH}{d\xi} - 2\xi\frac{dF}{d\xi} - F(\xi) = 0, \tag{5.3.43}$$

$$n\left(-\frac{dG}{d\xi}\right)^{n-1}\frac{d^2G}{d\xi^2} - H\frac{dG}{d\xi} + \frac{1}{3n}\left[G(\xi) + 2\xi\frac{dG}{d\xi}\right]G = 0. \tag{5.3.44}$$

$$J = \int_0^\infty G^2(\xi)d\xi. \tag{5.3.45}$$

We are considering only the upper half of the two-dimensional free jet for which $\partial u/\partial y < 0$. The relevant boundary conditions for the two-dimensional free jet are given by

$$y = \infty \quad : \quad u(x,\infty) = 0, \tag{5.3.46}$$

$$y = 0 \quad : \quad \frac{\partial u}{\partial y}(x,0) = 0, \quad v(x,0) = 0. \tag{5.3.47}$$

Substituting equations (5.3.40) and (5.3.41) into boundary conditions (5.3.46) and (5.3.47) we obtain the following transformed boundary conditions

$$G(\infty) = 0, \quad \frac{dG}{d\xi}(0) = 0, \quad H(0) = 0. \tag{5.3.48}$$

Equations (5.3.43) and (5.3.44) are a set of two coupled ordinary differential equations. We replace them by one third order differential equation. The continuity equation (5.3.43) is

$$\frac{dH}{d\xi} = \frac{1}{3n}G(\xi) + \frac{2}{3n}\xi\frac{dG}{d\xi}. \tag{5.3.49}$$

Integrating (5.3.49) with respect to ξ from 0 to ξ gives

$$H(\xi) - H(0) = \frac{1}{3n}\int_0^\xi G(\xi)d\xi + \frac{2}{3n}\int_0^\xi \xi\frac{dG}{d\xi}d\xi, \tag{5.3.50}$$

By using the boundary condition $H(0) = 0$ and integrating by parts we obtain

$$H(\xi) = \frac{1}{3n}\int_0^\xi G(\xi)d\xi + \frac{2}{3n}\left[\xi G(\xi) - \int_0^\xi G(\xi)d\xi\right]. \tag{5.3.51}$$

Let

$$\int_0^\xi G(\xi)d\xi = F(\xi). \tag{5.3.52}$$

As a result (5.3.51) becomes

$$H(\xi) = \frac{2}{3n}\xi G(\xi) - \frac{1}{3n}F(\xi). \tag{5.3.53}$$

Now from definition (5.3.52)

$$\frac{dF}{d\xi} = G(\xi), \quad \frac{d^2F}{d\xi^2} = \frac{dG}{d\xi}, \quad \frac{d^3F}{d\xi^3} = \frac{d^2G}{d\xi^2}. \tag{5.3.54}$$

Substituting equations (5.3.53) and (5.3.54) into the momentum balance equation (5.3.44) we obtain

$$3n^2\left(-\frac{d^2F}{d\xi^2}\right)^{n-1}\frac{d^3F}{d\xi^3} + F\frac{d^2F}{d\xi^2} + \left(\frac{dF}{d\xi}\right)^2 = 0, \tag{5.3.55}$$

The first two boundary conditions in (5.3.48) becomes

$$F'(\infty) = 0, \quad F''(0) = 0. \tag{5.3.56}$$

The third boundary condition on $F(\xi)$ follows directly from the definition (5.3.52) for $F(\xi)$:

$$F(0) = 0. \tag{5.3.57}$$

Further the conserved quantity (5.3.45) becomes

$$J = 2\rho\int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{5.3.58}$$

The third order ordinary differential equation (5.3.55) is the same as the differential equation (5.2.31) for the stream function and the boundary conditions (5.3.56) and (5.3.57) are the same as the boundary conditions (5.3.35)-(5.3.37) for the stream function. The similarity solution for the two dimensional free jet in terms of the velocity components is therefore the same as the similarity solution for the stream function.

5.4 Associated Lie point symmetry: free jet

An alternative way to derive an invariant solution for the two-dimensional free jet is to obtain the Lie point symmetry associated with one of the conserved vectors for the partial differential equation for the stream function. The conserved vector to choose is determined by the boundary conditions on the two-dimensional free jet. The boundary conditions are used when deriving the conserved quantity from the conservation law. A similar analysis will not be done on the system of equation for the velocity components because we have seen that the two treatments are equivalent.

The partial differential equation for the stream function for the upper half $y > 0$ of the two-dimensional free jet is

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} - n(-\psi_{yy})^{n-1}\psi_{yyy} = 0. \qquad (5.4.1)$$

The Lie point symmetry

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi} \qquad (5.4.2)$$

of (5.4.1) is said to be associated with the conserved vector $T = T(T^1, T^2)$ if

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2 \qquad (5.4.3)$$

where there is summation over the repeated index k from 1 to 2. The generator X is prolonged to as many derivatives as required. This was first introduced by Kara and Mahomed [9] and was applied by Mason and Hill [15] and Mason and Anthonyrajah [16]. Equation (5.4.3) consists of two components

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \qquad (5.4.4)$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0 \qquad (5.4.5)$$

which form the determining equations.

The conserved vector we will choose is the elementary conserved vector which has components

$$T^1 = \psi_y^2, \quad T^2 = -\psi_x\psi_y + (-\psi_{yy})^n. \qquad (5.4.6)$$

We take the second prolongation of X since T is dependent on the first and second partial derivatives of ψ. The second prolongation is given by

$$\begin{aligned} X^{[2]} &= \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} + \zeta_x\frac{\partial}{\partial \psi_x} + \zeta_y\frac{\partial}{\partial \psi_y} + \zeta_{xx}\frac{\partial}{\partial \psi_{xx}} \\ &+ \zeta_{xy}\frac{\partial}{\partial \psi_{xy}} + \zeta_{yy}\frac{\partial}{\partial \psi_{yy}}, \end{aligned} \qquad (5.4.7)$$

where

$$\begin{aligned} \zeta_i &= D_i(\eta) - \psi_k D_i(\xi^k), & (5.4.8) \\ \zeta_{ij} &= D_j(\zeta_i) - \psi_{ik} D_j(\xi^k). & (5.4.9) \end{aligned}$$

The total derivative operators are given by

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{xy} \frac{\partial}{\partial \psi_y} + ..., \qquad (5.4.10)$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \psi_{yx} \frac{\partial}{\partial \psi_x} + ... \qquad (5.4.11)$$

Substituting equation (5.4.6) into equation (5.4.4) and using the definitions (5.4.8)-(5.4.11) the first determining equation becomes

$$\begin{aligned} & 2\psi_y \frac{\partial \eta}{\partial y} + 2\psi_y^2 \frac{\partial \eta}{\partial \psi} - 2\psi_y \psi_x \frac{\partial \xi^1}{\partial y} - 2\psi_x \psi_y^2 \frac{\partial \xi^1}{\partial \psi} \\ & - 2\psi_y^2 \frac{\partial \xi^2}{\partial y} - 2\psi_y^3 \frac{\partial \xi^2}{\partial \psi} + \psi_y^2 \frac{\partial \xi^2}{\partial \psi} + \psi_y^3 \frac{\partial \xi^2}{\partial \psi} + \psi_x \psi_y \frac{\partial \xi^1}{\partial y} \\ & - (-\psi_{yy})^n \frac{\partial \xi^1}{\partial y} + \psi_x \psi_y^2 \frac{\partial \xi^1}{\partial \psi} - (-\psi_{yy})^n \psi_y \frac{\partial \xi^1}{\partial \psi} = 0. \end{aligned} \qquad (5.4.12)$$

Similarly substituting (5.4.6) into (5.4.5) and using the definitions (5.4.8)-(5.4.11) the second determining equation becomes

$$\begin{aligned} & -\psi_y \zeta_x - \psi_x \zeta_y - n(-\psi_{yy})^{n-1} \zeta_{yy} - \psi_x \psi_y \frac{\partial \xi^1}{\partial x} + (-\psi_{yy})^n \frac{\partial \xi^1}{\partial x} \\ & -\psi_x^2 \psi_y \frac{\partial \xi^1}{\partial \psi} + (-\psi_{yy})^n \psi_x \frac{\partial \xi^1}{\partial \psi} - \psi_y^2 \frac{\partial \xi^2}{\partial x} - \psi_y^2 \psi_x \frac{\partial \xi^2}{\partial \psi} = 0, \end{aligned} \qquad (5.4.13)$$

where

$$\zeta_x = \frac{\partial \eta}{\partial x} + \psi_x \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial x} - \psi_x^2 \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial x} - \psi_x \psi_y \frac{\partial \xi^2}{\partial \psi}, \qquad (5.4.14)$$

$$\zeta_y = \frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_y^2 \frac{\partial \xi^2}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi}, \qquad (5.4.15)$$

and

$$\begin{aligned} \zeta_{yy} &= \frac{\partial^2 \eta}{\partial y^2} + 2\psi_y \frac{\partial^2 \eta}{\partial y \partial \psi} - \psi_x \frac{\partial^2 \xi^1}{\partial y^2} - 2\psi_x \psi_y \frac{\partial^2 \xi^1}{\partial y \partial \psi} \\ &- \psi_y \frac{\partial^2 \xi^2}{\partial y^2} - 2(\psi_y)^2 \frac{\partial^2 \xi^2}{\partial y \partial \psi} + (\psi_y)^2 \frac{\partial^2 \eta}{\partial \psi^2} - \psi_x (\psi_y)^2 \frac{\partial^2 \xi^1}{\partial \psi^2} \\ &- (\psi_y)^3 \frac{\partial^2 \xi^2}{\partial \psi^2} - 2\psi_{xy} \frac{\partial \xi^1}{\partial y} - 2\psi_y \psi_{xy} \frac{\partial \xi^1}{\partial \psi} + \psi_{yy} \frac{\partial \eta}{\partial \psi} \\ &- \psi_x \psi_{yy} \frac{\partial \xi^1}{\partial \psi} - 2\psi_{yy} \frac{\partial \xi^2}{\partial y} - 3\psi_y \psi_{yy} \frac{\partial \xi^2}{\partial \psi}. \end{aligned} \qquad (5.4.16)$$

We can separate the first determining equation (5.4.12) by products and powers of the partial derivatives of ψ:

$$\psi_y \quad : \quad \frac{\partial \eta}{\partial y} = 0, \tag{5.4.17}$$

$$\psi_y^2 \quad : \quad 2\frac{\partial \eta}{\partial \psi} - \frac{\partial \xi^2}{\partial y} = 0, \tag{5.4.18}$$

$$\psi_y \psi_x \quad : \quad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.4.19}$$

$$\psi_x \psi_y^2 \quad : \quad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.4.20}$$

$$\psi_y^3 \quad : \quad \frac{\partial \xi^2}{\partial \psi} = 0, \tag{5.4.21}$$

$$(-\psi_{yy})^n \quad : \quad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.4.22}$$

$$(-\psi_{yy})^n \psi_y \quad : \quad \frac{\partial \xi^1}{\partial \psi} = 0. \tag{5.4.23}$$

From equations (5.4.17)-(5.4.23) that we have the following results

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x, y), \quad \eta = \eta(x, \psi). \tag{5.4.24}$$

We can differentiate the remaining equation (5.4.18) with respect to y to obtain

$$\frac{\partial^2 \xi^2}{\partial y^2} = 0 \tag{5.4.25}$$

which can be integrated to give

$$\xi^2(x, y) = A(x)y + B(x), \tag{5.4.26}$$

where $A(x)$ and $B(x)$ are arbitrary functions of x. Further substituting this result back into (5.4.18) and solving for η we obtain

$$\eta(x, \psi) = \frac{1}{2}\psi A(x) + C(x), \tag{5.4.27}$$

where $C(x)$ is an arbitrary function of x. Thus from the first determining equation we have

$$\xi^1 = \xi^1(x), \quad \xi^2 = A(x)y + B(x), \quad \eta = \frac{1}{2}\psi A(x) + C(x). \tag{5.4.28}$$

Using the results (5.4.28) from the first determining equation in the second determining equation (5.4.13) yields

$$\begin{aligned} & - \psi_y \frac{1}{2}\psi\frac{dA(x)}{dx} - \psi_y\frac{dC(x)}{dx} - \psi_y\psi_x\frac{1}{2}A(x) + \psi_y\psi_x\frac{d\xi^1}{dx} \\ & + \psi_y^2\frac{dA(x)}{dx}y + \psi_y^2\frac{dB(x)}{dx} + \frac{1}{2}\psi_y\psi_x A(x) - (-\psi_{yy})^n\frac{3}{2}nA(x) \\ & - \psi_x\psi_y\frac{d\xi^1}{dx} + (-\psi_{yy})^n\frac{d\xi^1}{dx} - \psi_y^2\left(\frac{dA}{dx}y + \frac{dB}{dx}\right) = 0. \end{aligned} \tag{5.4.29}$$

The determining equation (5.4.29) can be separated by products and powers of the partial derivatives of ψ:

$$\psi_y \ : \ \frac{1}{2}\psi\frac{dA(x)}{dx} + \frac{dC(x)}{dx} = 0, \tag{5.4.30}$$

$$\psi_{yy}^n \ : \ \frac{d\xi^1}{dx} - \frac{3}{2}nA(x) = 0. \tag{5.4.31}$$

If we consider equation (5.4.30) we see that neither $A(x)$ nor $C(x)$ depend on ψ. Thus we can split (5.4.30) into powers of ψ:

$$\psi \ : \ \frac{dA(x)}{dx} = 0 \tag{5.4.32}$$

$$1 \ : \ \frac{dC(x)}{dx} = 0. \tag{5.4.33}$$

Integrating equations (5.4.32) and (5.4.33) we obtain

$$A(x) = c_1, \quad C(x) = c_2, \tag{5.4.34}$$

where c_1 and c_2 are constants. Further solving equation (5.4.31) using the results (5.4.34) we have for ξ^1

$$\xi^1(x) = \frac{3}{2}nc_1x + c_3, \tag{5.4.35}$$

where c_3 is a constant. Thus in summary the infinitesimals are given by

$$\xi^1 = \frac{3}{2}nc_1x + c_3, \quad \xi^2 = c_1y + B(x), \quad \eta = \frac{1}{2}\psi c_1 + c_2. \tag{5.4.36}$$

The infinitesimal generator is given by

$$X = \left(\frac{3}{2}nc_1x + c_3\right)\frac{\partial}{\partial x} + (c_1y + B(x))\frac{\partial}{\partial y} + \left(\frac{1}{2}\psi c_1 + c_2\right)\frac{\partial}{\partial\psi}. \tag{5.4.37}$$

The linearly independent Lie point symmetry generators are

$$\begin{aligned} X_1 &= \frac{3}{2}nx\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial\psi}, \\ X_2 &= \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial\psi}, \quad X_B = B(x)\frac{\partial}{\partial y}. \end{aligned} \tag{5.4.38}$$

Now $\psi = \Psi(x, y)$ is an invariant solution generated by the associated Lie point symmetry X provided

$$X(\psi - \Psi(x, y))|_{\psi=\Psi(x,y)} = 0. \tag{5.4.39}$$

That is

$$\left[\frac{3}{2}nc_1x + c_3\right]\frac{\partial \Psi}{\partial x} + [c_1y + B(x)]\frac{\partial \Psi}{\partial y} = \frac{1}{2}\Psi c_1 + c_2. \tag{5.4.40}$$

The differential equations of the characteristic curves (5.4.40) are given by

$$\frac{dx}{\left[\frac{3}{2}nc_1x + c_3\right]} = \frac{dy}{[c_1y + B(x)]} = \frac{d\Psi}{\left[\frac{1}{2}\Psi c_1 + c_2\right]}. \tag{5.4.41}$$

The first pair of terms can be written as

$$\frac{dx}{\frac{3}{2}nc_1\left(x + \frac{2c_3}{3nc_1}\right)} = \frac{dy}{[c_1y + B(x)]} \tag{5.4.42}$$

Further this differential equation can be written as a first order linear differential equation

$$\frac{dy}{dx} - \frac{y}{\frac{3}{2}n\left(x + \frac{2c_3}{3nc_1}\right)} = \frac{B(x)}{\frac{3}{2}nc_1\left(x + \frac{2c_3}{3nc_1}\right)} \tag{5.4.43}$$

The integrating factor is given by

$$\exp\left[-\frac{2}{3n}\int\frac{dx}{\left(x + \frac{2c_3}{3nc_1}\right)}\right] = \left(x + \frac{2c_3}{3nc_1}\right)^{-\frac{2}{3n}}. \tag{5.4.44}$$

Therefore

$$\frac{d}{dx}\left[y\left(x + \frac{2c_3}{3nc_1}\right)^{-\frac{2}{3n}}\right] = \frac{B(x)}{\frac{3}{2}nc_1\left(x + \frac{2c_3}{3nc_1}\right)^{1+\frac{2}{3n}}}. \tag{5.4.45}$$

Integrating equation (5.4.45) with respect to x gives

$$\frac{y}{\left(x + \frac{2c_3}{3nc_1}\right)^{\frac{2}{3n}}} = \frac{2}{3nc_1}\int^x \frac{B(x)dx}{\left(x + \frac{2c_3}{3nc_1}\right)^{1+\frac{2}{3n}}} + k_1, \tag{5.4.46}$$

where k_1 is a constant. We let

$$\frac{2}{3nc_1}\int^x \frac{B(x)dx}{\left(x + \frac{2c_3}{3nc_1}\right)^{1+\frac{2}{3n}}} = B^*(x). \tag{5.4.47}$$

The first integral (5.4.46) becomes

$$k_1 = \frac{y}{\left(x + \frac{2c_3}{3nc_1}\right)^{\frac{2}{3n}}} - B^*(x). \tag{5.4.48}$$

Considering the first and last terms of equation (5.4.41) and solving gives

$$k_2 = \frac{\Psi + \frac{2c_2}{c_1}}{\left(x + \frac{2c_3}{3nc_1}\right)^{\frac{1}{3n}}}, \tag{5.4.49}$$

where k_2 is a constant. The general solution is

$$k_2 = F(k_1). \tag{5.4.50}$$

Since $\psi = \Psi(x, y)$, the general solution is

$$\psi(x, y) = \left(x + \frac{2c_3}{3nc_1}\right)^{\frac{1}{3n}} F(\xi) - \frac{2c_2}{c_1}, \tag{5.4.51}$$

where

$$\xi = \frac{y}{\left(x + \frac{2c_3}{3nc_1}\right)^{\frac{2}{3n}}} - B^*(x). \tag{5.4.52}$$

We now substitute equations (5.4.51) and (5.4.52) into the third order partial differential equation (5.4.1) and simplify to obtain

$$3n^2 \left(-\frac{d^2F}{d\xi^2}\right)^{n-1} \frac{d^3F}{d\xi^3} + F(\xi)\frac{d^2F}{d\xi^2} + \left(\frac{dF}{d\xi}\right)^2 = 0 \tag{5.4.53}$$

The ordinary differential equation (5.4.53) is identical to the ordinary differential equations (5.2.31) derived using a scaling transformation and (5.3.55) obtained by starting from the system of the two equations for the velocity components. Equation (5.4.53) is independent of the arbitrary function $B^*(x)$ in (5.4.52). Now $B(x)$ in the Lie point symmetry (5.4.40) is arbitrary. We choose $B(x) = 0$. Thus $B^*(x) = 0$ and we have the useful property that $\xi = 0$ when $y = 0$.

It can be verified that the boundary conditions are again given by (5.2.35)-(5.2.37):

$$F(0) = 0, \quad \frac{d^2F}{d\xi^2}(0) = 0, \quad \frac{dF}{d\xi}(\infty) = 0. \tag{5.4.54}$$

The conserved quantity for the two-dimensional free jet is

$$J = 2\rho \int_0^\infty \left(\frac{\partial \psi}{\partial y}\right)^2 dy = \text{constant independent of } x, \tag{5.4.55}$$

where the integration is done at a fixed point x. Now from (5.4.51) and (5.4.52),

$$\frac{\partial\psi}{\partial y} = \left(x + \frac{2c_3}{3nc_1}\right)^{-\frac{1}{3n}} \frac{dF}{d\xi}, \quad dy = \left(x + \frac{2c_3}{3nc_1}\right)^{\frac{2}{3n}} d\xi \tag{5.4.56}$$

and (5.4.55) becomes

$$J = 2\rho \int_0^\infty \left(\frac{\partial F}{\partial \xi}\right)^2 d\xi. \tag{5.4.57}$$

The condition that J is independent of x is automatically satisfied. This is because the Lie point symmetry (5.4.40) is associated with the conserved vector used to derive the conserved quantity J.

Consider now the constants c_2 and c_3. Since an arbitrary additive constant in the stream function (5.4.51) does not contribute to the velocity components, u and v, we choose $c_2 = 0$. Also

$$u = \frac{\partial\psi}{\partial y} = \left(x + \frac{2c_3}{3nc_1}\right)^{-\frac{1}{3n}} \frac{dF}{d\xi} \tag{5.4.58}$$

which has a singularity at

$$x = -\frac{2c_3}{3nc_1}. \tag{5.4.59}$$

But the only position at which $u(x, y)$ could have a singularity is at the origin $x = 0$, $y = 0$. Thus $c_3 = 0$. Equations (5.4.51) and (5.4.52) therefore reduce to (5.2.30) and

$$u = \frac{\partial\psi}{\partial y} = x^{-\frac{1}{3n}} \frac{dF}{d\xi}, \tag{5.4.60}$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{1}{3n} x^{\frac{1-3n}{3n}} \left[2\xi \frac{dF}{d\xi} - F\right]. \tag{5.4.61}$$

The equations for the formulation using the associated Lie point symmetry are the same as those obtained in Section 5.2 using the scaling transformation.

5.5 Solution of the two-dimensional free jet

In the this section parametric solutions for the two-dimensional free jet are presented. Three cases arise during the solution process.

Equations (5.2.31), (5.3.55) and (5.4.53) can be written more compactly as

$$3n \frac{d}{d\xi} \left(-\frac{d^2F}{d\xi^2}\right)^n - \frac{d}{d\xi}\left(F \frac{dF}{d\xi}\right) = 0. \tag{5.5.1}$$

Integrating with respect to ξ gives

$$3n \left(-\frac{d^2F}{d\xi^2}\right)^n - F\frac{dF}{d\xi} = c_1, \tag{5.5.2}$$

where c_1 is a constant. We impose the boundary conditions (5.4.54) at $\xi = 0$. Now $dF/d\xi$ at $x = 0$ is finite because

$$u(x,0) = \frac{\partial \psi}{\partial y}(x,0) = x^{-\frac{1}{3n}} \frac{dF}{d\xi}(0) \tag{5.5.3}$$

and $u(x,0)$ is finite for $x > 0$. Thus $c_1 = 0$ and equation (5.5.2) becomes

$$\left(-\frac{d^2F}{d\xi^2}\right)^n = \frac{1}{3n} F \frac{dF}{d\xi} \tag{5.5.4}$$

and hence

$$\frac{d^2F}{d\xi^2} + \left(\frac{1}{3n} F \frac{dF}{d\xi}\right)^{\frac{1}{n}} = 0. \tag{5.5.5}$$

Now the differential equation (5.5.5) does not depend on ξ explicitly. Using

$$\frac{d^2F}{d\xi^2} = \frac{d}{d\xi}\left(\frac{dF}{d\xi}\right) = \frac{d}{dF}\left(\frac{dF}{d\xi}\right)\frac{dF}{d\xi} = F' \frac{dF'}{dF}, \tag{5.5.6}$$

we can write it as

$$F' \frac{dF'}{dF} + \left(\frac{1}{3n} F F'\right)^{\frac{1}{n}} = 0. \tag{5.5.7}$$

Equation (5.5.7) is variable separable and can be written as

$$(F')^{\frac{n-1}{n}} dF' = -\left(\frac{1}{3n}\right)^{\frac{1}{n}} F^{\frac{1}{n}} dF. \tag{5.5.8}$$

Further analysis depends on whether $0 < n < 1/2$, $n = 1/2$ or $1/2 < n < \infty$.

5.5.1 Case 1: $0 < n < \frac{1}{2}$

In this subsection we present the parametric solution for the case $0 < n < 1/2$. Integrating each side of (5.5.8) we obtain

$$\frac{(F')^{\frac{-(1-2n)}{n}}}{(1-2n)} = \left(\frac{1}{3n}\right)^{\frac{1}{n}} \frac{F^{\frac{n+1}{n}}}{n+1} + k. \tag{5.5.9}$$

where k is a positive constant. To show that k is positive consider (5.5.9) at $\xi = 0$. Then from the boundary condition (5.4.54), $F(0) = 0$ and since $v_x(x,0) > 0$ it follows that $\frac{dF}{d\xi}(0) > 0$. Thus $k > 0$. Equation (5.5.9) can be written as

$$\left(\frac{dF}{d\xi}\right)^{\frac{-(1-2n)}{n}} = \left(\frac{1-2n}{n+1}\right)\left(\frac{1}{3n}\right)^{\frac{1}{n}} \left[F^{\frac{n+1}{n}} + b^{\frac{n+1}{n}}\right], \tag{5.5.10}$$

where the positive constant b is defined by

$$b^{\frac{n+1}{n}} = (1+n)(3n)^{\frac{1}{n}} k. \tag{5.5.11}$$

Thus from (5.5.11)

$$\frac{dF}{d\xi} = \left(\frac{1+n}{1-2n}\right)^{\frac{n}{1-2n}} (3n)^{\frac{1}{1-2n}} \frac{1}{\left[F^{\frac{n+1}{n}} + b^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}}}. \tag{5.5.12}$$

Equation (5.5.12) is variables separable. Thus

$$d\xi = \left(\frac{1-2n}{1+n}\right)^{\frac{n}{1-2n}} \left(\frac{1}{3n}\right)^{\frac{1}{1-2n}} \left[F^{\frac{n+1}{n}} + b^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}} dF. \tag{5.5.13}$$

Integrating both sides from $\xi = 0$ to ξ and using again the boundary condition $F(0) = 0$ gives

$$\xi = \left(\frac{1-2n}{1+n}\right)^{\frac{n}{1-2n}} \left(\frac{1}{3n}\right)^{\frac{1}{1-2n}} \int_0^F \left[f^{\frac{n+1}{n}} + b^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}} df \tag{5.5.14}$$

where f is a dummy variable of integration. We make the transformation

$$\frac{f}{b} = r, \quad \frac{F}{b} = R. \tag{5.5.15}$$

Equation (5.5.14) becomes

$$\xi = \left(\frac{1-2n}{1+n}\right)^{\frac{n}{1-2n}} \left(\frac{1}{3n}\right)^{\frac{1}{1-2n}} b^{\frac{2-n}{1-2n}} \int_0^R \left[1 + r^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}} dr. \tag{5.5.16}$$

The constant b is determined from the conserved quantity (5.4.57) which can be written as

$$J = 2\rho \int_0^\infty \frac{dF}{d\xi} \left(\frac{dF}{d\xi} d\xi\right) = 2\rho \int_0^\infty \frac{dF}{d\xi} dF. \tag{5.5.17}$$

Using (5.5.12), equation (5.5.17) becomes

$$J = 2\rho \left(\frac{1+n}{1-2n}\right)^{\frac{n}{1-2n}} (3n)^{\frac{1}{1-2n}} \int_0^\infty \frac{dF}{\left[F^{\frac{n+1}{n}} + b^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}}} \tag{5.5.18}$$

Making the transformation (5.5.15) the conserved quantity becomes

$$J = 2\rho \left(\frac{1+n}{1-2n}\right)^{\frac{n}{1-2n}} (3n)^{\frac{1}{1-2n}} \frac{A(n)}{b^{\frac{3n}{1-2n}}} \tag{5.5.19}$$

where

$$A(n) = \int_0^\infty \frac{dr}{\left[1 + r^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}}}, \quad 0 < n < \frac{1}{2}. \tag{5.5.20}$$

Equation (5.5.19) can be solved for b to give

$$b = \left[\frac{2\rho A(n)}{J}\right]^{\frac{1-2n}{3n}} \left(\frac{1+n}{1-2n}\right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}}. \tag{5.5.21}$$

Substituting (5.5.21) for b in (5.5.16) we obtain

$$\xi = \left[\frac{2\rho A(n)}{J}\right]^{\frac{2-n}{3n}} (3n)^{\frac{2}{3n}} \left(\frac{1+n}{1-2n}\right)^{\frac{2}{3}} \int_0^R \left[1 + r^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}} dr. \tag{5.5.22}$$

Also using the transformation (5.5.15), equation (5.5.12) can be written as

$$\frac{dF}{d\xi} = \left(\frac{1+n}{1-2n}\right)^{\frac{n}{1-2n}} (3n)^{\frac{1}{1-2n}} b^{-\left(\frac{n+1}{1-2n}\right)} \frac{1}{\left[1 + R^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}}} \tag{5.5.23}$$

and substituting (5.2.21) for b into (5.5.23) gives

$$\frac{dF}{d\xi} = \left[\frac{J}{2\rho A(n)}\right]^{\frac{n+1}{3n}} + \left(\frac{1}{3n}\right)^{\frac{1}{3n}} \left(\frac{1-2n}{1+n}\right)^{\frac{1}{3}} \frac{1}{\left[1 + R^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}}}. \tag{5.5.24}$$

These results give an analytical solution for $u(x, y)$ in parametric form. Now

$$u(x, y) = x^{-\frac{1}{3n}} \frac{dF}{d\xi} \tag{5.5.25}$$

and using (5.5.24) we obtain

$$u(x, y) = x^{-\frac{1}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{n+1}{3n}} + \left(\frac{1}{3n}\right)^{\frac{1}{3n}} \left(\frac{1-2n}{1+n}\right)^{\frac{1}{3}} \frac{1}{\left[1 + R^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}}}. \tag{5.5.26}$$

Also

$$y = x^{\frac{2}{3n}} \xi \tag{5.5.27}$$

and therefore using (5.5.22),

$$y = x^{\frac{2}{3n}} \left[\frac{2\rho A(n)}{J}\right]^{\frac{2-n}{3n}} (3n)^{\frac{2}{3n}} \left(\frac{1+n}{1-2n}\right)^{\frac{2}{3}} \int_0^R \left[1 + r^{\frac{n+1}{n}}\right]^{\frac{n}{1-2n}} dr. \tag{5.5.28}$$

The parametric solution $u(x, y)$ is given by (5.5.26) and (5.5.28). The parameter is R with range $0 \leq R \leq \infty$.

We also obtain an analytical solution for $v(x, y)$ in parametric form. From (5.4.61)

$$v_y(x, y) = \frac{1}{3n} x^{\frac{1-3n}{3n}} \left[F(\xi) - 2\xi \frac{dF}{d\xi} \right] \tag{5.5.29}$$

and by using (5.5.15) for $F(\xi)$ and (5.5.24) for $dF/d\xi$ it can be shown that

$$\begin{aligned} v(x, y) &= \frac{1}{3n} x^{\frac{1-3n}{3n}} \left[\frac{2\rho A(n)}{J} \right]^{\frac{1-2n}{3n}} \left(\frac{1+n}{1-2n} \right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}} \\ & \left[2(1 + r^{\frac{n+1}{n}}) \int_0^R \left(1 + r^{\frac{n+1}{n}}\right)^{\frac{n}{1-2n}} dr - R \right]. \end{aligned} \tag{5.5.30}$$

The coordinate y is again given by (5.5.28). The parametric solution for $v(x, y)$ is given by (5.5.30) and (5.5.28) where the parameter R lies in the range $0 \leq R \leq \infty$.

5.5.2 Case 2: $\frac{1}{2} < n < \infty$

We now derive the parametric solution for $1/2 < n < \infty$. Integrating each side of (5.5.8) gives

$$\frac{1}{(2n-1)} F'^{\frac{2n-1}{n}} = k - \frac{1}{(n+1)} \left(\frac{1}{3n} \right)^{\frac{1}{n}} F^{\frac{n+1}{n}}, \tag{5.5.31}$$

where k is a constant. To determine if k is positive or negative consider $\xi = 0$. Since $F(0) = 0$ and $F'(0) > 0$ because $u(x, 0) > 0$, it follows that k is a positive constant. Equation (5.5.31) can be written as

$$\left(\frac{dF}{d\xi} \right)^{\frac{2n-1}{n}} = \left(\frac{2n-1}{n+1} \right) \left(\frac{1}{3n} \right)^{\frac{1}{n}} \left[b^{\frac{n+1}{n}} - F^{\frac{n+1}{n}} \right], \tag{5.5.32}$$

where b is the positive constant defined by

$$b^{\frac{n+1}{n}} = (1+n)(3n)^{\frac{1}{n}} k. \tag{5.5.33}$$

Hence

$$\frac{dF}{d\xi} = \left(\frac{2n-1}{n+1} \right)^{\frac{n}{2n-1}} \left(\frac{1}{3n} \right)^{\frac{1}{2n-1}} \left[b^{\frac{n+1}{n}} - F^{\frac{n+1}{n}} \right]^{\frac{n}{2n-1}}. \tag{5.5.34}$$

Equation (5.5.34) is valid for $0 \leq F \leq b$. It is variables separable and therefore

$$d\xi = \left(\frac{n+1}{2n-1} \right)^{\frac{n}{2n-1}} (3n)^{\frac{1}{2n-1}} \frac{dF}{\left[b^{\frac{n+1}{n}} - F^{\frac{n+1}{n}} \right]^{\frac{n}{2n-1}}}. \tag{5.5.35}$$

We integrate both sides from $\xi = 0$ to ξ and use the boundary condition $F(0) = 0$. Then

$$\xi = \left(\frac{n+1}{2n-1}\right)^{\frac{n}{2n-1}} (3n)^{\frac{1}{2n-1}} \int_0^F \frac{df}{\left[b^{\frac{n+1}{n}} - f^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.36}$$

for $0 \leq F \leq b$. Making again the transformation (5.5.15), equation (5.5.36) becomes

$$\xi = \left(\frac{n+1}{2n-1}\right)^{\frac{n}{2n-1}} (3n)^{\frac{1}{2n-1}} b^{\frac{n-2}{2n-1}} \int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.37}$$

where $0 \leq R \leq 1$.

The constant b is determined from the conserved quantity (5.4.57) which can be expressed as

$$J = 2\rho \int_0^b \frac{dF}{d\xi} dF. \tag{5.5.38}$$

By substituting (5.5.34) into (5.5.38) we obtain

$$J = 2\rho \left(\frac{2n-1}{n+1}\right)^{\frac{n}{2n-1}} \left(\frac{1}{3n}\right)^{\frac{1}{2n-1}} b^{\frac{3n}{2n-1}} A(n) \tag{5.5.39}$$

where

$$A(n) = \int_0^1 \left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}} dr, \quad \frac{1}{2} < n < \infty. \tag{5.5.40}$$

Solving (5.5.39) for b gives

$$b = \left[\frac{J}{2\rho A(n)}\right]^{\frac{2n-1}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}}. \tag{5.5.41}$$

Substituting (5.5.41) for b into (5.5.37) gives

$$\xi = \left[\frac{J}{2\rho A(n)}\right]^{\frac{n-2}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{2}{3}} (3n)^{\frac{2}{3n}} \int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.42}$$

for $0 \leq R \leq 1$. Also making the transformation (5.5.15), equation (5.5.34) becomes

$$\frac{dF}{d\xi} = \left(\frac{2n-1}{n+1}\right)^{\frac{n}{2n-1}} \left(\frac{1}{3n}\right)^{\frac{1}{2n-1}} b^{\frac{n+1}{2n-1}} \left[1 - R^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}} \tag{5.5.43}$$

and using (5.5.41) for b we obtain

$$\frac{dF}{d\xi} = \left[\frac{J}{2\rho A(n)}\right]^{\frac{n+1}{3n}} \left(\frac{2n-1}{n+1}\right)^{\frac{1}{3}} \left(\frac{1}{3n}\right)^{\frac{1}{3n}} \left[1 - R^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}} \tag{5.5.44}$$

where $0 \leq R \leq 1$.

Consider now the range of ξ. The range of ξ depends on whether the integral

$$\int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.45}$$

is convergent or divergent as $R \to 1$. Consider first $n = 1$. Then

$$\int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} = \int_0^R \frac{dr}{1 - r^2} = \frac{1}{2} \ln\left(\frac{1+R}{1-R}\right) \tag{5.5.46}$$

and

$$\ln\left(\frac{1+R}{1-R}\right) \to +\infty \quad \text{as } R \to 1. \tag{5.5.47}$$

Thus when $n = 1$ the range of x is $0 \leq \xi \leq \infty$. Consider now $1/2 < n \leq \infty$ with $n \neq 1$. To investigate the behaviour of the integral as $R \to 1$ let $r = 1 - s$. Then

$$\int^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} = -\int^{1-R} \frac{ds}{\left[1 - (1-s)^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.48}$$

and expanding for small s,

$$\int^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} = -\left(\frac{n}{n+1}\right)^{\frac{n}{n+1}} \int^{1-R} \frac{ds}{s^{\frac{n}{2n-1}}(1 + O(s))} \tag{5.5.49}$$

and therefore for $n \neq 1$,

$$\int^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} = -\left(\frac{n}{n+1}\right)^{\frac{n}{n+1}} \left(\frac{2n-1}{n-1}\right)(1-R)^{\frac{n-1}{2n-1}} \quad \text{as } R \to 1. \tag{5.5.50}$$

Thus for $n > 1$ the integral is convergent as $R \to 1$ while for $1/2 < n < 1$ the integral diverges to $+\infty$ as $R \to 1$. In summary, for a shear thickening fluid with $n > 1$ the range of ξ is finite, $0 \leq \xi \leq \xi_{max}$, where

$$\xi_{max} = \left[\frac{J}{2\rho A(n)}\right]^{\frac{n-2}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{2}{3}} (3n)^{\frac{2}{3n}} \int_0^1 \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.51}$$

For a Newtonian fluid with $n = 1$ and a shear thinning fluid with $1/2 < n < 1$ the range of ξ is infinite, $0 \leq \xi \leq \infty$.

Analytical solutions in parametric form are obtained for $u(x,y)$ and $v(x,y)$. From (5.5.25) with (5.5.44),

$$u(x,y) = x^{-\frac{1}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{n+1}{3n}} \left(\frac{2n-1}{n+1}\right)^{\frac{1}{3}} \left(\frac{1}{3n}\right)^{\frac{1}{3n}} \left[1 - R^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}} \tag{5.5.52}$$

and from (5.5.27) and (5.5.42),

$$y = x^{\frac{2}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{n-2}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{2}{3}} (3n)^{\frac{2}{3n}} \int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \tag{5.5.53}$$

where for $n > 1$,

$$y_{max} = x^{\frac{2}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{n-2}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{2}{3}} (3n)^{\frac{2}{3n}} \int_0^1 \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}}. \tag{5.5.54}$$

The parameter is R and the range of the parameter is $0 \leq R \leq 1$.

From (5.5.29), (5.5.15), (5.5.42) and (5.5.44) we obtain

$$\begin{aligned} v(x,y) &= \frac{1}{3n} x^{\frac{1-3n}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{2n-1}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}} \\ & \left[2\left(1 - R^{\frac{n+1}{n}}\right)^{\frac{n}{2n-1}} \int_0^R \frac{dr}{\left(1 - r^{\frac{n+1}{n}}\right)^{\frac{n}{2n-1}}} - R\right] \end{aligned} \tag{5.5.55}$$

where y is again given by (5.5.53). The parameter is again R with range $0 \leq R \leq 1$.

When $1 < n \leq \infty$, the integral (5.5.45) is convergent as $R \to 1$ and therefore

$$v(x, y_{max}) = -\frac{1}{3n} x^{\frac{1-3n}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{2n-1}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}}, \tag{5.5.56}$$

which describes fluid inflow at $y = y_{max}$ to maintain conservation of mass. When $n = 1$,

$$\begin{aligned} \left[1 - R^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}} \int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} &= (1 - R^2) \int_0^R \frac{dr}{1 - r^2} \\ &= \frac{1}{2}(1 - R^2) \ln\left(\frac{1+R}{1-R}\right) \end{aligned} \tag{5.5.57}$$

and

$$(1 - R^2) \ln\left(\frac{1+R}{1-R}\right) \to 0 \text{ as } R \to 1. \tag{5.5.58}$$

When $n = 1$, $y_{max} = \infty$ and we obtain the Newtonian result

$$v(x, \infty) = x^{-\frac{2}{3}} \left[\frac{J}{6\rho}\right]^{\frac{1}{3}}. \tag{5.5.59}$$

Finally, for $1/2 < n < 1$, by using the asymptotic result (5.5.50) we see that

$$\left[1 - R^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}} \int_0^R \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}} \to 0 \ \text{ as } R \to 1 \tag{5.5.60}$$

and therefore when $1/2 < n < 1$,

$$v(x, \infty) = -\frac{1}{3n} x^{\frac{1-3n}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{2n-1}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}}. \tag{5.5.61}$$

5.5.3 Case 3: $n = \frac{1}{2}$

The approach for deriving the parametric solution for $n = 1/2$ is slightly different in that the value of n is substituted into equation (5.5.7). The resulting ordinary differential equation is then solved as for the first two cases.

Substituting $n = 1/2$ reduces equation (5.5.7) to

$$F' \frac{dF'}{dF} + \left(\frac{2}{3} F F'\right)^2 = 0. \tag{5.5.62}$$

Equation (5.5.62) is a variables separable differential equation and may be written as

$$\frac{dF'}{F'} = -\frac{4}{9} F^2 dF. \tag{5.5.63}$$

Integrating both sides gives

$$\ln(F') = -\frac{4}{27} F^3 + k, \tag{5.5.64}$$

where k is a constant and therefore

$$\frac{dF}{d\xi} = K \exp\left(-\frac{4}{27} F^3\right) \tag{5.5.65}$$

where $K = \exp(k)$. Equation (5.5.65) can be written as

$$d\xi = \frac{1}{K} \exp\left(\frac{4}{27} F^3\right) dF. \tag{5.5.66}$$

In order to determine the limits of integration we use the boundary condition $F(0) = 0$. Thus

$$\int_0^\xi d\xi = \frac{1}{K} \int_0^F \exp\left(\frac{4}{27} f^3\right) df \tag{5.5.67}$$

and hence

$$\xi = \frac{1}{K}\int_0^F \exp\left(\frac{4}{27}f^3\right) df. \tag{5.5.68}$$

The constant K is determined from the conserved quantity (5.4.57) which can be expressed as

$$J = 2\rho\int_0^\infty \frac{dF}{d\xi}dF. \tag{5.5.69}$$

Using (5.5.65) this becomes

$$J = 2\rho K\int_0^\infty \exp\left(-\frac{4}{27}f^3\right) df. \tag{5.5.70}$$

Letting

$$\frac{4}{27}f^3 = g, \tag{5.5.71}$$

equation (5.5.70) becomes

$$J = 2^{\frac{1}{3}}\rho K\int_0^\infty g^{-\frac{2}{3}} \exp(-g)dg. \tag{5.5.72}$$

But the Gamma Function $\Gamma(n)$ is defined by [17]

$$\Gamma(n) = \int_0^\infty x^{n-1}\exp(-x)dx. \tag{5.5.73}$$

Thus (5.5.72) becomes

$$J = 2^{\frac{1}{3}}\rho K\Gamma\left(\frac{1}{3}\right) \tag{5.5.74}$$

and therefore

$$K = \frac{J}{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)} \tag{5.5.75}$$

Expressed in terms of J, equation (5.5.65) and (5.5.68) are

$$\frac{dF}{d\xi} = \frac{J}{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)}\exp\left(-\frac{4}{27}F^3\right), \tag{5.5.76}$$

$$\xi = \frac{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)}{J}\int_0^F \exp\left(\frac{4}{27}f^3\right) df. \tag{5.5.77}$$

Analytical solutions for $u(x, y)$ and $v(x, y)$ can be obtained in parametric form. From (5.4.60) with $n = 1/2$,

$$u(x, y) = x^{-\frac{2}{3}}\frac{dF}{d\xi} \tag{5.5.78}$$

and using (5.5.76) this gives

$$u(x,y) = \frac{J}{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)} x^{-\frac{2}{3}} \exp\left(-\frac{4}{27}F^3\right). \tag{5.5.79}$$

Also, from (5.2.30) with $n = 1/2$,

$$y = x^{\frac{2}{3n}}\xi = x^{\frac{4}{3}}\xi \tag{5.5.80}$$

and using (5.5.77) we obtain

$$y = \frac{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)}{J} x^{\frac{4}{3}} \int_0^F \exp\left(\frac{4}{27}f^3\right) df. \tag{5.5.81}$$

From (5.4.61) with $n = 1/2$,

$$v(x,y) = \frac{2}{3}x^{-\frac{1}{3}}\left[2\xi\frac{dF}{d\xi} - F(\xi)\right] \tag{5.5.82}$$

and therefore

$$v(x,y) = \frac{2}{3}x^{-\frac{1}{3}}\left[2\exp\left(-\frac{4}{27}F^3\right)\int_0^F \exp\left(\frac{4}{27}f^3\right) df - F\right]. \tag{5.5.83}$$

The parametric solution for $u(x,y)$ is given by (5.5.79) and (5.5.81) with parameter F which takes the values $0 \le F \le \infty$. The parametric solution for $v(x,y)$ is given by (5.5.83) and (5.5.81) with parameter F again taking the values $0 \le F \le \infty$.

In Cases 1 and 2 the parameter was denoted by R. Expressed in terms of R the parametric solution is

$$u(x,y) = \frac{J}{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)} x^{-\frac{2}{3}} \exp\left(-\frac{4}{27}R^3\right) \tag{5.5.84}$$

$$v(x,y) = \frac{2}{3}x^{-\frac{1}{3}}\left[2\exp\left(-\frac{4}{27}R^3\right)\int_0^R \exp\left(\frac{4}{27}r^3\right) dr - R\right] \tag{5.5.85}$$

where

$$y = \frac{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)}{J} x^{\frac{4}{3}} \int_0^R \exp\left(\frac{4}{27}r^3\right) dr. \tag{5.5.86}$$

We can make a change of variables

$$\frac{4}{27}R^3 = S, \quad \frac{4}{27}r^3 = s. \tag{5.5.87}$$

The parametric solution becomes

$$u(x,y) = \frac{J}{2^{\frac{1}{3}}\rho\Gamma\left(\frac{1}{3}\right)} x^{-\frac{2}{3}} \exp(-S), \tag{5.5.88}$$

$$v(x,y) = \frac{2}{3}x^{-\frac{1}{3}}\left[2\exp(-S)\int_0^S s^{-\frac{2}{3}}\exp(s)ds - S^{\frac{1}{3}}\right], \tag{5.5.89}$$

where

$$y = \frac{\rho\Gamma\left(\frac{1}{3}\right)}{2^{\frac{1}{3}}J}x^{\frac{4}{3}}\int_0^S s^{-\frac{2}{3}}\exp(s)ds. \tag{5.5.90}$$

We note that in all three cases the conserved quantity (4.3.24) for the velocity components is used to complete the derivation of the solution.

5.6 Results and discussion

5.6.1 Case 1: $0 < n < \frac{1}{2}$

This case only allows us to analyse the shear thinning property of the power-law fluid. The graphical results are presented in Figures 5.6.1 and 5.6.2. An estimate of the half-width of the jet at a fixed x is the value of y at $\xi = 1$. The width of the two-dimensional free jet for $0 < n < 1/2$ increases as $x^{\frac{2}{3n}}$. Figures 5.6.1 and 5.6.2 demonstrate the results for varying jet strengths J, in particular for $J < 1$ and $J > 1$.

For lower jet strength J the two-dimensional free jets are broader and as we increase the jet strength J, the maximum velocity $u(x, 0)$ of the jet increases. Furthermore, for smaller values of J, a larger proportion of the velocity profile consists of a flat central region. The primary reason for this flat region is that for small values of the parameter R, the value of y is small. In addition, in can be shown that $\partial v_x/\partial y \sim (JR)^{\frac{1}{n}}$ and therefore for higher jet strengths J the rate of change of v_x with y is more rapid. This is also confirmed by the flatter regions for values of $J < 1$.

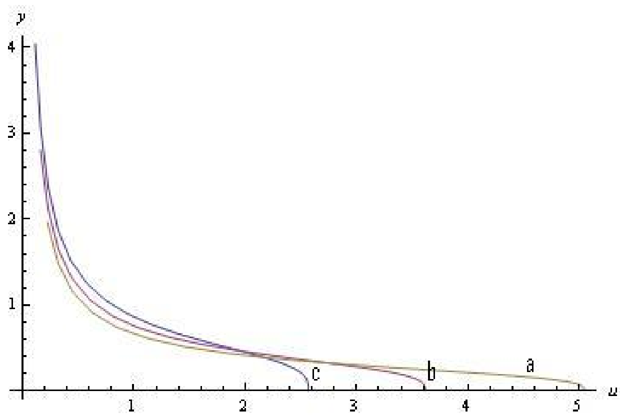

Figure 5.6.1: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 0.45$ for the range of values of $J > 1$ given by (a) $J = 15$, (b) $J = 11$ and (c) $J = 8$. Only the range $0 \leq R \leq 1$ of the full range $0 \leq R \leq \infty$ is plotted

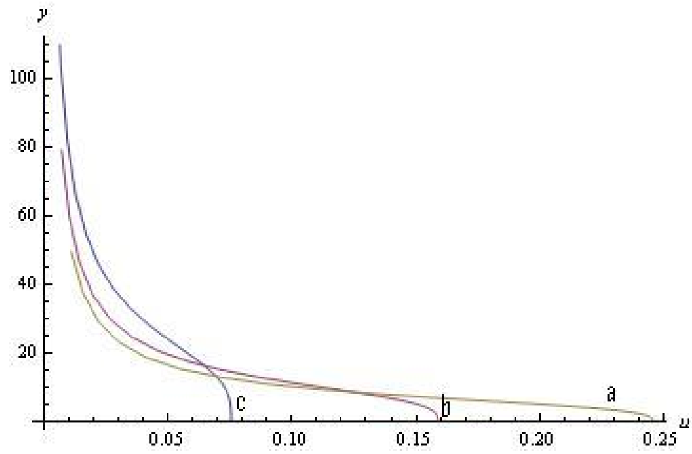

Figure 5.6.2: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 0.45$ for the range of values of $0 < J < 1$ given by (a) $J = 0.9$, (b) $J = 0.6$ and (c) $J = 0.3$. Only the range $0 \leq R \leq 1$ of the full range $0 \leq R \leq \infty$ is plotted

5.6.2 Case 2: $\frac{1}{2} < n < \infty$

When the two-dimensional free-jet is defined over this range, we have more scope in terms of describing the properties of the non-Newtonian power-law fluid. We can establish the shear thinning as well as the shear thickening properties and provide a comparison between the two. Moreover, we can recover the Newtonian case for $n = 1$.

It was found that for both $J > 1$ and $J < 1$, the shear thinning properties of the jet behave in a similar fashion to Case 1, in that the width of the jet increases as $x^{\frac{2}{3n}}$ and a stronger jet strength J results in a greater maximum velocity of the jet. Further for weaker jet strengths, the jets are much broader. Since $\partial v_x/\partial y \sim (JR)^{\frac{1}{n}}$, the rate of change of v_x with y is greater for a larger jet strength. This is shown in Figure 5.6.3. For $1/2 < n \leq 1$ the range of the parameter R is $0 \leq R \leq \infty$ and the jet extends to $y = \infty$. The velocity $v(x, \infty)$ is finite and negative and given by (5.5.61) for $1/2 < n < 1$ and by (5.5.59) for $n = 1$. There is inflow of fluid at $y = \infty$ to maintain conservation of mass. The fluid inflow at $y = \infty$ is a consequence of the boundary layer approximation.

From equation (5.5.53) we see that y and therefore the half-width of the jet satisfies

$$y \propto J^{\frac{n-2}{3n}}. \tag{5.6.1}$$

Thus for $1/2 < n < 2$ the half-width will decrease as J increases, for $n = 2$ the half-width will be independent of J and for $n > 2$ the half width will increase as J increases. This is illustrated in Figure 5.6.3 for $n = 0.75$, Figure 5.6.7 for $n = 1$, Figure 5.6.4 for $n = 2$ and Figure 5.6.5 for $n = 3$.

From (5.5.52),

$$u(x,0) \propto J^{\frac{n+1}{3n}} \tag{5.6.2}$$

and the maximum velocity of the jet always increases as J increases.

It was shown in Section 5.5.2 that for $n > 1$, $u(x,y) = 0$ at $y = y_{max}$ where from (5.5.54),

$$y_{max} = x^{\frac{2}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{n-2}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{2}{3}} (3n)^{\frac{2}{3n}} \int_0^1 \frac{dr}{\left[1 - r^{\frac{n+1}{n}}\right]^{\frac{n}{2n-1}}}. \tag{5.6.3}$$

where $A(n)$ is defined by (5.5.40). The curve of y_{max} against x in the (x,y)-plane gives the boundary of the jet in the upper half of the (x,y)-plane. It is illustrated in Figure 5.6.6 for $n = 2$. The y-component of the fluid velocity on the boundary is given by (5.5.56):

$$v(x, y_{max}) = -\frac{1}{3n} x^{\frac{1-3n}{3n}} \left[\frac{J}{2\rho A(n)}\right]^{\frac{2n-1}{3n}} \left(\frac{n+1}{2n-1}\right)^{\frac{1}{3}} (3n)^{\frac{1}{3n}}. \tag{5.6.4}$$

The velocity $v(x, y_{max})$ is finite and negative and describes fluid inflow to maintain conservation of mass in the jet. The fluid inflow at $y = y_{max}$ is a consequence of boundary layer approximation. The velocity component v increases from a minimum negative value at $y = y_{max}$ to zero at $y = 0$. The velocity components u and v are not defined beyond the boundary $y = y_{max}$. For $y > y_{max}$ the shear thickening fluid is not disturbed by the jet.

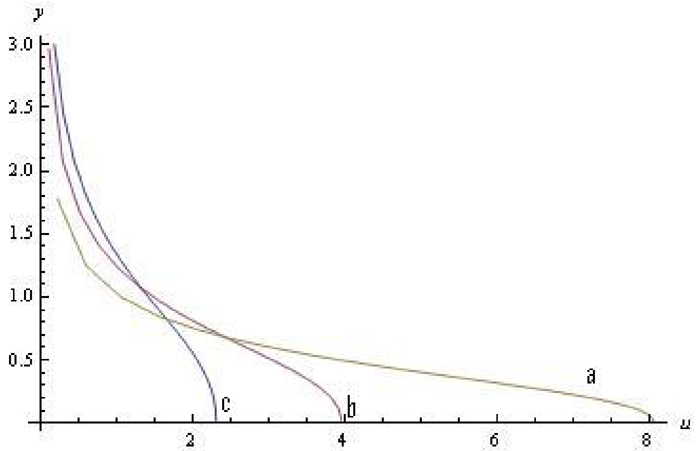

Figure 5.6.3: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 0.75$ for the range of values of $J > 1$ given by (a) $J = 50$, (b) $J = 20$ and (c) $J = 10$. Only the range $0 \le R \le 1$ of the full range $0 \le R \le \infty$ is plotted.

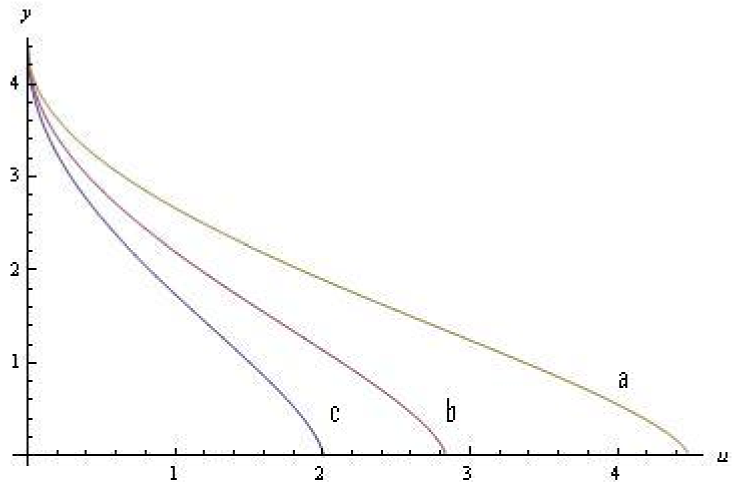

Figure 5.6.4: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 2$ for the range of values of $J > 1$ given by (a) $J = 50$, (b) $J = 20$ and (c) $J = 10$ over the full range $0 \leq R \leq 1$. When $n = 2$, y_{max} is independent of J.

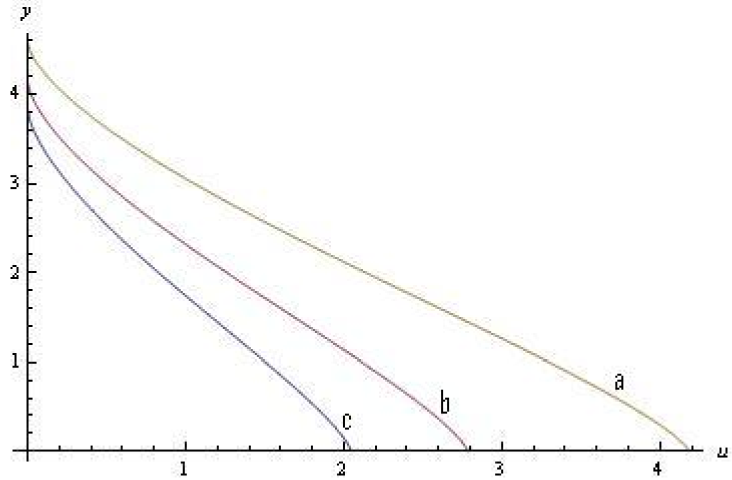

Figure 5.6.5: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 3$ for the range of values of $J > 1$ given by (a) $J = 50$, (b) $J = 20$ and (c) $J = 10$ over the full range $0 \leq R \leq 1$. When $n = 2$, y_{max} is independent of J.

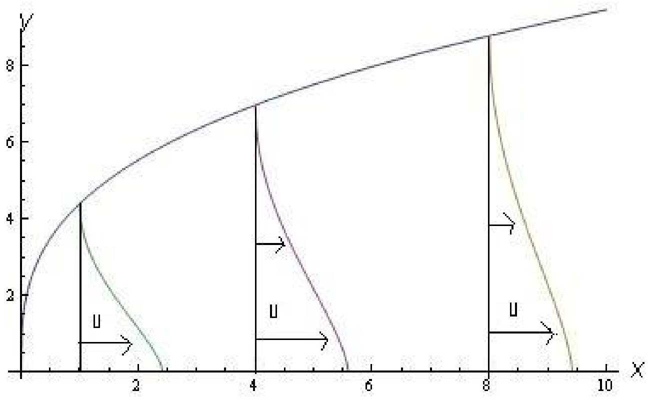

Figure 5.6.6: Velocity profile of x against y for $x = 1$, $x = 4$ and $x = 8$ for a two-dimensional free jet with $n = 2$ and $J = 10$.

For comparison we include the graph for $n = 1$.

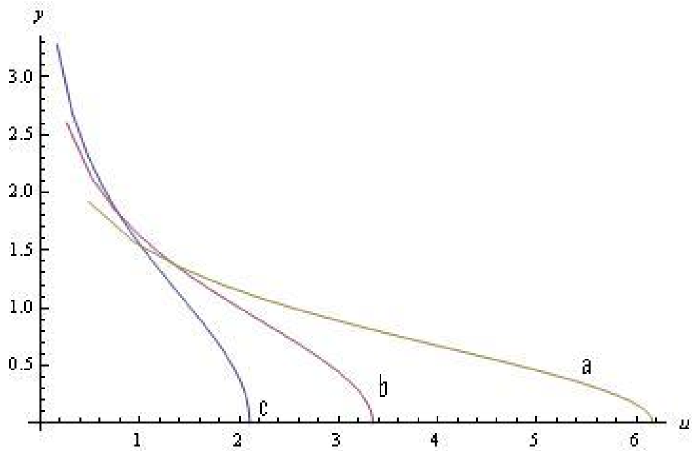

Figure 5.6.7: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 1$ for the range of values of $J > 1$ given by (a) $J = 50$, (b) $J = 20$ and (c) $J = 10$ over the full range $0 \leq R \leq 1$. When $n = 1$ the range of y is $0 \leq y \leq \infty$.

5.6.3 Case 3: $n = \frac{1}{2}$

The behaviour of the two-dimensional free jet for $n = 1/2$ is consistent with the shear thinning properties described for $0 < n < 1/2$ and $1/2 < n < 1$. From (5.5.81), y and therefore the half-width of the jet, satisfies

$$y \propto \frac{1}{J} \tag{5.6.5}$$

while from (5.5.79) the maximum velocity of her jet, $u(x,0)$, satisfies

$$u(x,0) \propto J. \tag{5.6.6}$$

Thus the width of the jet decreases as J increases and the maximum velocity increases as J increases. This is clearly illustrated in Figure 5.6.8. The central flat regions of the graphs can be explained through the rate-of-change of the velocity $u(x,y)$ with y. For greater values of J the rate of change is more rapid and this is confirmed by the relation $\partial u/\partial y \sim (JR)^2$.

The velocity $v(x,y)$ no longer remains finite as $y \to \infty$. From (5.5.85) we found that $v(x,y) \to -\infty$ as $y \to \infty$ which describes fluid inflow at $y = \infty$ to maintain conservation of mass in the jet.

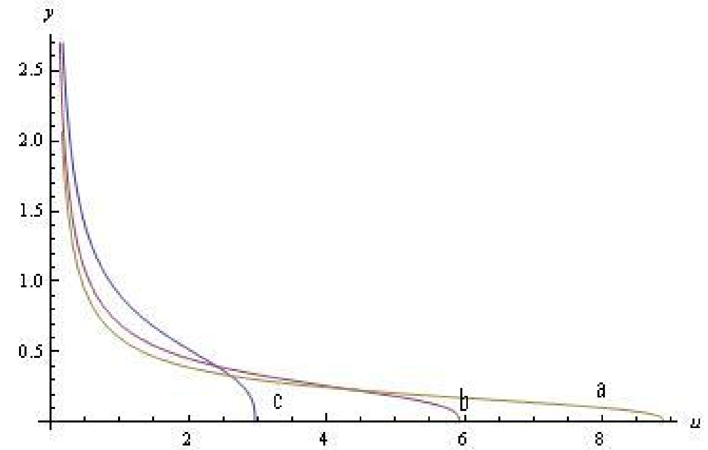

Figure 5.6.8: Velocity profile of u against y at $x = 1$ for a two-dimensional free jet with $n = 0.5$ for the range of values of $J > 1$ given by (a) $J = 30$, (b) $J = 20$ and (c) $J = 10$. Only the range $0 \leq R \leq 1$ of the full range $0 \leq R \leq \infty$ is plotted.

5.7 Conclusions

Lemieux and Unny [18] considered the two-dimensional jet for a power-law fluid and reduced the problem by a similarity transformation to the ordinary differential equation (5.5.2). These authors the solved the problem numerically. In this chapter we derived a new analytical solution in parametric form for $0 < n < 1/2$, $n = 1/2$ and $1/2 < n \leq \infty$.

In order to derive the ordinary differential equation that was solved, three different approaches were used. Firstly a similarity solution was considered for the third order partial differential equation for the stream function. Secondly a similarity solution for the velocity components was considered. The final approach that was used was through the symmetry and conservation law relation, where we derived the Lie point symmetry associated with the conserved vector and used this Lie point symmetry to deduce an invariant solution. The associated Lie point symmetry method was the most powerful because it can be used in problems for which the transformation is not a scaling symmetry.

In some problems with power-law fluids it can be difficult to investigate numerically the shear thinning region, $0 \leq n \lesssim 1/2$. The numerical method can sometimes breakdown in this region. The change in the form of the solution at $n = 1/2$ in this problem explains why numerical methods which are successful for shear thickening fluids can breakdown for shear thinning fluids with small values of n.

Our analytical solution in parametric form allowed all the properties of the power-law jet in the regions $0 < n < 1/2$, $n = 1/2$ and $1/2 < n < \infty$ to be investigated analytically. The change in the type of the solution at $n = 1/2$ could be followed and the actual solution at $n = 1/2$ was derived. The behaviour of the two-dimensional free-jet in the range $0 < n < 1/2$ and $n = 1/2$ has not been well studied and from the obtained results we were able to discuss the shear thinning properties of the power-law fluid in this range.

We found that for all values of n $(0 < n < \infty)$, the maximum velocity of the jet, $u(x, 0)$, increased as the strength J of the jet increased. We also found that for $0 < n < 2$ the width of the jet decreased as the strength of the jet, J, increased. For $n = 2$ the width is independent of J while for $2 < n < \infty$ the width of the jet increased as J increased. For $0 < n \leq 2$ the flat region in the velocity profile at the axis of the jet increased in width as J decreased while for $2 < n < \infty$ the flat region increased as J increased.

The main feature of the shear thickening jet $(n > 1)$ was the finite width of the jet described by the boundary curve $y = y_{max}$ between fluid which is carried along by the jet and fluid which remains at rest. Since

$$y_{max} \propto x^{\frac{2}{3n}} \tag{5.7.1}$$

this region of entrained fluid becomes narrower as n increases and the shear thickening becomes stronger. For a Newtonian fluid $(n = 1)$ and a shear thinning fluid $(0 < n < 1)$ this boundary curve does not exist and the whole region $x > 0$ is set in motion by the jet. In the numerical solution of Lemieux and Unny [18] the boundary curve $y = y_{max}$ exists for all $0 < n < \infty$ with a different shape for $0 < n < 2/3$, $n = 2/3$ and $2/3 < n < \infty$. Our analytical parametric solution shows that the boundary exists only for a shear thickening fluid with $n > 1$.

For $n > 1$, the y-component of the fluid velocity at the boundary, $v(x, y_{max})$, is finite and negative and describes fluid inflow to maintain conservation of mass in the jet. For $1/2 < n \leq 1$, the jet extends to $y = \infty$ and $v(x, \infty)$ is finite and negative. This is a well established result for a Newtonian fluid for $n = 1$. For $n = 1/2$ and $0 < n < 1/2$ the jet also extends to $y = \infty$ but we found that $v(x, \infty) = -\infty$. These results for v at $y = y_{max}$ and $y = \infty$ are a consequence of the boundary layer approximation.

Chapter 6

Parametric solution of the two-dimensional liquid jet

6.1 Introduction

Many analytical approaches have been developed to solve differential equations. However, a drawback of many of these approaches is that they do not succeed in deriving exact solutions for large classes of differential equations. This problem was remedied by the Norwegian mathematician Sophus Lie, who put forward his theory of symmetry analysis. He provided an algorithmic technique to obtain the symmetries of a given differential equation and one can utilise these symmetries to obtain exact solutions.

Central to this chapter will be obtaining the complete Lie point symmetries of the two-dimensional liquid jet for a non-Newtonian power law fluid for the system of equations for the velocity components. When obtaining these symmetries for an arbitrary n, special cases do arise. The special cases for $n = 1$ and $n = 2$ will also be solved. Once these Lie point symmetries have been obtained, the constants of the linearly independent operators will be associated with a conserved vector to give the associated Lie point symmetry. The choice of conserved vector depends on the boundary condition. The reason that we do not calculate the associated Lie point symmetry directly is that when considering the velocity components, the momentum balance equation fails to provide sufficient information in the derivation of the associated symmetry. This is because the conserved quantity is obtained from the continuity equation. The associated Lie point symmetry will then be utilised to deduce an invariant solution.

Similarity solutions, which are fundamental in solving fluid mechanics problems, will also be adopted. This analysis is similar to that carried out on the free jet when using velocity components. A parametric solution will be derived for $1/2 < n < \infty$. We note that no solution exists for fluid outflow in the range $0 < n < 1/2$, however fluid inflow may require further attention. Further no solution exists at $n = 1/2$.

An outline of this chapter is as follows. Section 6.2 will present the similarity solution for the two-dimensional liquid jet. The complete group of Lie point symmetries and the associated symmetry for an arbitrary $n > 0$ as well as for the special cases will be outlined in Section 6.3. In Section 6.4 we present the derivation of the parametric solution for a liquid jet of a non-Newtonian power-law fluid. In Section 6.5 we give a thorough analysis of the solution and the concluding remarks will be drawn in Section 6.6.

6.2 Similarity solution for the liquid jet using velocity components

This section will detail the derivation of the similarity solution for the two-dimensional liquid jet. Since we are dealing with a system of partial differential equations, we derive a set of coupled ordinary differential equations which will be solved parametrically. We follow a similar procedure to deriving these equations as for the two-dimensional free jet using velocity components. The system of partial differential equations describing the two-dimensional liquid jet are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6.2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - n\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2} = 0. \tag{6.2.2}$$

We consider equation (6.2.1) first. We introduce the scaling transformation

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{u} = \lambda^c u, \quad \bar{v} = \lambda^d v. \tag{6.2.3}$$

Substituting the scaling transformation (6.2.3) into the conservation of mass equation (6.2.1) yields,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \lambda^{b-d-a+c}\frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{6.2.4}$$

To leave equation (6.2.4) invariant we require.

$$d = b - a + c. \tag{6.2.5}$$

The transformed equation is thus

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{6.2.6}$$

Suppose $u = f(x, y)$, $v = g(x, y)$ is a solution of the original partial differential equation (6.2.1). Then since the form of (6.2.6) is the same as (6.2.1) $\bar{u} =$

$f(\bar{x}, \bar{y})$, $\bar{v} = g(\bar{x}, \bar{y})$ is a solution of the transformed partial differential equation (6.2.6).

Consider $\bar{u} = f(\bar{x}, \bar{y})$ first. That is, using $u = f$,

$$\lambda^c f = f(\lambda^a x, \lambda^b y). \tag{6.2.7}$$

Differentiate with respect to λ to obtain

$$c\lambda^{c-1} f = a\lambda^{a-1} x \frac{\partial f}{\partial \bar{x}} + b\lambda^{b-1} y \frac{\partial f}{\partial \bar{y}}. \tag{6.2.8}$$

Since λ is an arbitrary parameter we let $\lambda = 1$. This in turns gives $\bar{x} = x$, $\bar{y} = y$. Equation (6.2.8) reduces to

$$ax \frac{\partial f}{\partial x} + by \frac{\partial f}{\partial y} = cf. \tag{6.2.9}$$

We consider solutions with $a \neq 0$. The corresponding differential equations of the characteristic curves are given by

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{df}{cf}. \tag{6.2.10}$$

By considering the first pair of terms in (6.2.10) and integrating both sides gives the first integral

$$\frac{y}{x^{\frac{b}{a}}} = c_1, \tag{6.2.11}$$

where c_1 is a constant. Similarly solving the first and last terms of equation (6.2.10), gives

$$\frac{f}{x^{\frac{c}{a}}} = c_2, \tag{6.2.12}$$

where c_2 is a constant. The general solution of equation (6.2.9) is

$$c_2 = F(c_1), \tag{6.2.13}$$

where F is an arbitrary function. As a result, using $f = u$,

$$u(x, y) = x^{\frac{c}{a}} F(\xi), \tag{6.2.14}$$

where

$$\xi = \frac{y}{x^{\frac{b}{a}}}. \tag{6.2.15}$$

Next we consider $\bar{v} = g(\bar{x}, \bar{y})$. That is, since $v = g$,

$$\lambda^d g = g(\lambda^a x, \lambda^b y). \tag{6.2.16}$$

We differentiate equation (6.2.16) with respect to λ and use (6.2.5) for d. This gives

$$(b-a+c)\lambda^{b-a+c-1}g = a\lambda^{a-1}x\frac{\partial g}{\partial \bar{x}} + b\lambda^{b-1}y\frac{\partial g}{\partial \bar{y}}. \tag{6.2.17}$$

Since λ is an arbitrary parameter we let $\lambda = 1$ and therefore $\bar{x} = x$, $\bar{y} = y$. Hence equation (6.2.17) reduces to

$$ax\frac{\partial g}{\partial x} + by\frac{\partial g}{\partial y} = (b-a+c)g. \tag{6.2.18}$$

The corresponding differential equations of the characteristic curves are given by

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{dg}{(b-a+c)g}. \tag{6.2.19}$$

By integrating both sides of the first pair of terms in equation (6.2.19) we have again the first integral

$$\frac{y}{x^{\frac{b}{a}}} = k_1, \tag{6.2.20}$$

where k_1 is a constant. By considering the first and last terms in equation (6.2.19) and solving we obtain

$$\frac{g}{x^{\frac{b}{a}+\frac{c}{a}-1}} = k_2, \tag{6.2.21}$$

where k_2 is a constant. The general solution of equation (6.2.18) is

$$k_2 = H(k_1), \tag{6.2.22}$$

where H is an arbitrary function. Thus since $g = v$,

$$v(x,y) = x^{\frac{b}{a}+\frac{c}{a}-1}H(\xi), \tag{6.2.23}$$

where ξ is defined by (6.2.15).

The ordinary differential equation satisfied by $F(\xi)$ and $H(\xi)$ is obtained by substituting equations (6.2.14) and (6.2.23) into the continuity equation (6.2.1):

$$\frac{dH}{d\xi} - \frac{b}{a}\xi\frac{dF}{d\xi} + \frac{c}{a}F(\xi) = 0, \tag{6.2.24}$$

which depends on the ratios c/a and b/a.

We now consider the momentum balance equation (6.2.2). Consider the scaling transformation

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{u} = \lambda^c u, \quad \bar{v} = \lambda^d v. \tag{6.2.25}$$

Substituting the scaling transformation (6.2.25) into the momentum balance equation (6.2.2) gives

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \lambda^{-a+b+c-d}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} - n\lambda^{-a+(n+1)b-(n-2)c}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^{n-1}\frac{\partial^2 \bar{u}}{\partial \bar{y}} = 0. \qquad (6.2.26)$$

For equation (6.2.26) to remain invariant we require

$$d = b - a + c \qquad (6.2.27)$$

and

$$a = b(n+1) - (n-2)c. \qquad (6.2.28)$$

We note that equation (6.2.27) is the same as (6.2.5).

Since we are looking for an invariant solution for the system (6.2.1) and (6.2.2) for u and v, it follows that u, v and ξ are given by (6.2.14), (6.2.23) and (6.2.15) where $F(\xi)$ and $H(\xi)$ are related by (6.2.24). Now (6.2.14), (6.2.15), (6.2.23) and (6.2.24) depend only on the ratios b/a and c/a. We define

$$\alpha = \frac{b}{a}, \quad \beta = \frac{c}{a} \qquad (6.2.29)$$

and we can write

$$\xi = \frac{y}{x^{\alpha}}, \qquad (6.2.30)$$

$$u(x,y) = x^{\beta}F(\xi), \qquad (6.2.31)$$

$$v(x,y) = x^{\alpha+\beta-1}H(\xi), \qquad (6.2.32)$$

where

$$\frac{dH}{d\xi} - \alpha\xi\frac{dF}{d\xi} + \beta F = 0 \qquad (6.2.33)$$

and from (6.2.28),

$$(n+1)\alpha - (n-2)\beta = 1. \qquad (6.2.34)$$

Substituting u and v given by (6.2.31) and (6.2.32) into the momentum balance equation (6.2.2) and using (6.2.34) gives the ordinary differential equation

$$n\left(\frac{dF}{d\xi}\right)^{n-1}\frac{d^2F}{d\xi^2} - H(\xi)\frac{dF}{d\xi} - \left(\beta F(\xi) - \alpha\xi\frac{dF}{d\xi}\right)F(\xi) = 0. \qquad (6.2.35)$$

In order to obtain the second relation between α and β consider the conserved quantity for the two-dimensional liquid jet. It was shown in Chapter 4 Section 4.3 that for a two-dimensional liquid jet,

$$J = \int_0^{\phi(x)} u(x,y)dy = \text{constant independent of } x, \qquad (6.2.36)$$

where $y = \phi(x)$ is the free surface of the liquid jet. By using (6.2.30) and (6.2.31), J can be rewritten at a given point x as

$$J = x^{\alpha+\beta} \int_0^{\phi(x)/x^\alpha} F(\xi) d\xi. \tag{6.2.37}$$

For J to be independent of x we require

$$\alpha + \beta = 0 \tag{6.2.38}$$

and

$$\frac{\phi(x)}{x^\alpha} = k \tag{6.2.39}$$

where k is a constant. The conserved quantity J becomes

$$J = \int_0^k F(\xi) d\xi. \tag{6.2.40}$$

It follows from (6.2.34) and (6.2.38) that

$$\beta = -\alpha \quad \text{and} \quad (2n-1)\alpha = 1. \tag{6.2.41}$$

Thus

$$\alpha = \frac{1}{2n-1} \quad \text{provided} \quad \alpha \neq \frac{1}{2}. \tag{6.2.42}$$

Unlike the free jet, there is no solution when $\alpha = 1/2$.

We now collect the results. The form of the invariant solution is as follows:

$$u(x,y) = x^{-\frac{1}{2n-1}} F(\xi), \tag{6.2.43}$$

$$v(x,y) = x^{-1} H(\xi), \tag{6.2.44}$$

$$\xi = \frac{y}{x^{\frac{1}{2n-1}}}, \tag{6.2.45}$$

$$\phi(x) = k x^{\frac{1}{2n-1}}, \tag{6.2.46}$$

$$\frac{dH}{d\xi} - \frac{1}{(2n-1)} \left[\xi \frac{dF}{d\xi} + F(\xi) \right] = 0, \tag{6.2.47}$$

$$n \left(\frac{dF}{d\xi} \right)^{n-1} \frac{d^2F}{d\xi^2} - H(\xi) \frac{dF}{d\xi} + \frac{1}{(2n-1)} \left[F(\xi) + \xi \frac{dF}{d\xi} \right] F = 0. \tag{6.2.48}$$

$$J = \int_0^k F(\xi) d\xi. \tag{6.2.49}$$

where k is a constant.

Finally, consider the boundary conditions for the two-dimensional liquid jet. There is no-slip at the base $y = 0$ and the base is not porous. Thus

$$y = 0: \quad u(x,0) = 0, \quad v(x,0) = 0. \tag{6.2.50}$$

There is no tangential stress on the free surface $y = \phi(x)$. Thus

$$y = kx^\alpha : \quad \frac{\partial u}{\partial y}(x, kx^\alpha) = 0. \tag{6.2.51}$$

Expressed in terms of the similarity variables the boundary conditions become

$$F(0) = 0, \quad H(0) = 0, \quad \frac{dF}{d\xi}(k) = 0. \tag{6.2.52}$$

A parametric solution for the two-dimensional liquid jet will be derived in Section 6.4.

6.3 Lie point symmetries for the liquid jet

This section will employ Lie's algorithmic approach for determining the Lie point symmetries of a system of two partial differential equations to obtain the symmetries for the two-dimensional liquid jet. Once the Lie point symmetries have been acquired, we determine the symmetry associated with a conserved vector, and use the associated Lie point symmetry to deduce an invariant solution. This modified approach was applied by Naz and Naeem [19].

The system of partial differential equations under investigation are

$$u_x + v_y = 0, \tag{6.3.1}$$

$$uu_x + vu_y - n(u_y)^{n-1} u_{yy} = 0. \tag{6.3.2}$$

We consider first the continuity equation (6.3.1). The Lie point symmetry is given by

$$X = \xi^1(x, y, u, v)\frac{\partial}{\partial x} + \xi^2(x, y, u, v)\frac{\partial}{\partial y} + \eta^1(x, y, u, v)\frac{\partial}{\partial u} + \eta^2(x, y, u, v)\frac{\partial}{\partial v}. \tag{6.3.3}$$

Since the continuity equation depends on first derivatives, we take the first prolongation. That is

$$\begin{aligned} X^{[1]} &= \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} \\ &+ \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_y^1\frac{\partial}{\partial u_y} + \zeta_x^2\frac{\partial}{\partial v_x} + \zeta_y^2\frac{\partial}{\partial v_y}, \end{aligned} \tag{6.3.4}$$

where we can define explicitly the extended infinitesimals as

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) \tag{6.3.5}$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l) \tag{6.3.6}$$

where $u^1 = u$ and $u^2 = v$. The derivative operators are given by

$$D_1 = D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + ..., \tag{6.3.7}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + ... \tag{6.3.8}$$

The invariance criterion is

$$X^{[1]}(u_x + v_y)|_{\Delta=0} = 0, \tag{6.3.9}$$

where Δ defines the system (6.3.1)-(6.3.2). That is

$$\zeta_x^1 + \zeta_y^2 = 0, \quad \text{when} \quad u_x = -v_y. \tag{6.3.10}$$

The extended infinitesimals for equation (6.3.10) are given by

$$\begin{aligned}
\zeta_x^1 &= D_x(\zeta^1) - u_x D_x(\xi^1) - u_y D_x(\xi^2) \\
&= \frac{\partial\eta^1}{\partial x} + u_x\frac{\partial\eta^1}{\partial u} + v_x\frac{\partial\eta^1}{\partial v} - u_x\frac{\partial\xi^1}{\partial x} - u_x^2\frac{\partial\xi^1}{\partial u} \\
&- u_x v_x\frac{\partial\xi^1}{\partial v} - u_y\frac{\partial\xi^2}{\partial x} - u_x u_y\frac{\partial\xi^2}{\partial u} - u_y v_x\frac{\partial\xi^2}{\partial v}
\end{aligned} \tag{6.3.11}$$

and similarly

$$\begin{aligned}
\zeta_y^2 &= D_y(\zeta^2) - v_x D_y(\xi^1) - v_y D_y(\xi^2) \\
&= \frac{\partial\eta^2}{\partial y} + u_y\frac{\partial\eta^2}{\partial u} + v_y\frac{\partial\eta^2}{\partial v} - v_x\frac{\partial\xi^1}{\partial y} - v_x u_y\frac{\partial\xi^1}{\partial u} \\
&- v_x v_y\frac{\partial\xi^1}{\partial v} - v_y\frac{\partial\xi^2}{\partial y} - v_y u_y\frac{\partial\xi^2}{\partial u} - v_y^2\frac{\partial\xi^2}{\partial v}.
\end{aligned} \tag{6.3.12}$$

We substitute the extended infinitesimals (6.3.11), (6.3.12) and the relation $u_x = -v_y$ into equation (6.3.10) to give the first determining equation

$$\begin{aligned}
&\frac{\partial\eta^1}{\partial x} - v_y\frac{\partial\eta^1}{\partial u} + v_x\frac{\partial\eta^1}{\partial u} + v_y\frac{\partial\xi^1}{\partial x} - v_y^2\frac{\partial\xi^1}{\partial u} + v_y v_x\frac{\partial\xi^1}{\partial v} - u_y\frac{\partial\xi^2}{\partial x} \\
+ \;& v_y u_y\frac{\partial\xi^2}{\partial u} - u_y v_x\frac{\partial\xi^2}{\partial v} + \frac{\partial\eta^2}{\partial y} - u_y\frac{\partial\eta^2}{\partial u} + v_y\frac{\partial\eta^2}{\partial v} + v_x\frac{\partial\xi^1}{\partial y} - v_x u_y\frac{\partial\xi^1}{\partial u} \\
- \;& v_x v_y\frac{\partial\xi^1}{\partial v} - v_y\frac{\partial\xi^2}{\partial y} - v_y u_y\frac{\partial\xi^2}{\partial u} - v_y^2\frac{\partial\xi^2}{\partial v} = 0.
\end{aligned} \tag{6.3.13}$$

We now proceed to ascertain the second determining equation. A similar procedure as above will be followed. Consider the momentum balance equation (6.3.2). The Lie point symmetry is given by (6.3.3).

$$X = \xi^1(x,y,u,v)\frac{\partial}{\partial x} + \xi^2(x,y,u,v)\frac{\partial}{\partial y} + \eta^1(x,y,u,v)\frac{\partial}{\partial u} + \eta^2(x,y,u,v)\frac{\partial}{\partial v}. \tag{6.3.14}$$

Since the momentum equation depends on second derivatives, we take the second prolongation. That is

$$\begin{aligned} X^{[2]} &= \xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial y}+\eta^1\frac{\partial}{\partial u}+\eta^2\frac{\partial}{\partial v}+\zeta^1_x\frac{\partial}{\partial u_x}+\zeta^1_y\frac{\partial}{\partial u_y}\\ &+ \zeta^1_{xx}\frac{\partial}{\partial u_{xx}}+\zeta^1_{xy}\frac{\partial}{\partial u_{xy}}+\zeta^1_{yy}\frac{\partial}{\partial u_{yy}}++\zeta^2_x\frac{\partial}{\partial v_x}\\ &+ \zeta^2_y\frac{\partial}{\partial v_y}+\zeta^2_{xx}\frac{\partial}{\partial v_{xx}}+\zeta^2_{xy}\frac{\partial}{\partial v_{xy}}+\zeta^2_{yy}\frac{\partial}{\partial v_{yy}}, \end{aligned} \tag{6.3.15}$$

where the extended infinitesimals and the derivative operators are defined explicitly in equations (6.3.5)-(6.3.8). The invariance criterion is given by

$$X^{[2]}(uu_x+vu_y-n(u_y)^{n-1}u_{yy})|_{\Delta=0}=0, \tag{6.3.16}$$

where Δ defines the system (6.31)-(6.3.2). Substituting the extended infinitesimals and derivative operators given by (6.3.5)-(6.3.8) along with the relations $u_{yy}=(uu_x+vu_y)/(n(u_y)^{n-1})$ and $u_x=-v_y$ into equation (6.3.16) yields the second determining equation:

$$\begin{aligned}
&- \eta^1 v_y+\eta^2 u_y+u\frac{\partial\eta^1}{\partial x}-uv_y\frac{\partial\eta^1}{\partial u}+uv_x\frac{\partial\eta^1}{\partial v}+uv_y\frac{\partial\xi^1}{\partial x}-uv_y^2\frac{\partial\xi^1}{\partial u}+uv_yv_x\frac{\partial\xi^1}{\partial v}\\
&- uu_y\frac{\partial\xi^2}{\partial x}+uu_yv_y\frac{\partial\xi^2}{\partial u}-uu_yv_x\frac{\partial\xi^2}{\partial v}+v\frac{\partial\eta^1}{\partial y}+vu_y\frac{\partial\eta^1}{\partial u}+vv_y\frac{\partial\eta^1}{\partial v}+vv_y\frac{\partial\xi^1}{\partial y}\\
&+ vv_yu_y\frac{\partial\xi^1}{\partial u}+vv_y^2\frac{\partial\xi^1}{\partial v}-vu_y\frac{\partial\xi^2}{\partial y}-vu_y^2\frac{\partial\xi^2}{\partial u}-vu_yv_y\frac{\partial\xi^2}{\partial v}+(n-1)u_y^{-1}uv_y\frac{\partial\eta^1}{\partial y}\\
&- (n-1)v\frac{\partial\eta^1}{\partial y}+(n-1)uv_y\frac{\partial\eta^1}{\partial u}-(n-1)vu_y\frac{\partial\eta^1}{\partial u}+(n-1)u_y^{-1}uv_y^2\frac{\partial\eta^1}{\partial v}\\
&- (n-1)vv_y\frac{\partial\eta^1}{\partial v}+(n-1)u_y^{-1}uv_y^2\frac{\partial\xi^1}{\partial y}-(n-1)vv_y\frac{\partial\xi^1}{\partial y}+(n-1)uv_y^2\frac{\partial\xi^1}{\partial u}\\
&- (n-1)vu_yv_y\frac{\partial\xi^1}{\partial u}+(n-1)u_y^{-1}uv_y^3\frac{\partial\xi^1}{\partial v}-(n-1)vv_y^2\frac{\partial\xi^1}{\partial v}-(n-1)uv_y\frac{\partial\xi^2}{\partial y}\\
&+ (n-1)vu_y\frac{\partial\xi^2}{\partial y}-(n-1)u_yuv_y\frac{\partial\xi^2}{\partial u}+(n-1)u_y^2v\frac{\partial\xi^2}{\partial u}-(n-1)uv_y^2\frac{\partial\xi^2}{\partial v}\\
&+ (n-1)vu_yv_y\frac{\partial\xi^2}{\partial v}-nu_y^{n-1}\frac{\partial^2\eta^1}{\partial y^2}-nu_y^n\frac{\partial^2\eta^1}{\partial y\partial u}-nu_y^{n-1}v_y\frac{\partial^2\eta^1}{\partial y\partial v}\\
&- nu_y^{n-1}v_y\frac{\partial^2\xi^1}{\partial y^2}-nu_y^nv_y\frac{\partial^2\xi^1}{\partial y\partial u}-nu_y^{n-1}v_y^2\frac{\partial^2\xi^1}{\partial y\partial v}+nu_y^n\frac{\partial^2\xi^2}{\partial y^2}\\
&+ nu_y^{n+1}\frac{\partial^2\xi^2}{\partial y\partial u}+nu_y^nv_y\frac{\partial^2\xi^2}{\partial y\partial v}-nu_y^n\frac{\partial^2\eta^1}{\partial u\partial y}-nu_y^{n+1}\frac{\partial^2\eta^1}{\partial u^2}-nu_y^nv_y\frac{\partial^2\eta^1}{\partial u\partial v}\\
&- nu_y^nv_y\frac{\partial^2\xi^1}{\partial u\partial y}-nu_y^{n+1}v_y\frac{\partial^2\xi^1}{\partial u^2}-nu_y^nv_y^2\frac{\partial^2\xi^1}{\partial u\partial v}+nu_y^{n+1}\frac{\partial^2\xi^2}{\partial u\partial y}+nu_y^{n+2}\frac{\partial^2\xi^2}{\partial u^2}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad nu_y^{n+1}v_y\frac{\partial^2\xi^2}{\partial u\partial v} - nu_y^{n-1}v_y\frac{\partial^2\eta^1}{\partial v\partial y} - nu_y^n v_y\frac{\partial^2\eta^1}{\partial v\partial u} - nu_y^{n-1}v_y^2\frac{\partial^2\eta^1}{\partial v^2}\\
&- \quad nu_y^{n-1}v_y^2\frac{\partial^2\xi^1}{\partial v\partial y} - nu_y^n v_y^2\frac{\partial^2\xi^1}{\partial v\partial u} - nu_y^{n-1}v_y^3\frac{\partial^2\xi^1}{\partial v^2} + nu_y^n v_y\frac{\partial^2\xi^2}{\partial v\partial y}\\
&+ \quad nu_y^{n+1}v_y\frac{\partial^2\xi^2}{\partial v\partial u} + nu_y^n v_y^2\frac{\partial^2\xi^2}{\partial v^2} + uv_y\frac{\partial\eta^1}{\partial u} - vu_y\frac{\partial\eta^1}{\partial u} + uv_y^2\frac{\partial\xi^1}{\partial u}\\
&- \quad vu_y v_y\frac{\partial\xi^1}{\partial u} - uv_y\frac{\partial\xi^2}{\partial y} + vu_y\frac{\partial\xi^2}{\partial y} - 2u_y uv_y\frac{\partial\xi^2}{\partial u} + 2u_y^2 v\frac{\partial\xi^2}{\partial u}\\
&- \quad uv_y^2\frac{\partial\xi^2}{\partial v} + vu_y v_y\frac{\partial\xi^2}{\partial v} - nu_y^{n-1}v_{yy}\frac{\partial\eta^1}{\partial v} - nu_y^{n-1}v_{yy}v_y\frac{\partial\xi^1}{\partial v}\\
&+ \quad nu_y^n v_{yy}\frac{\partial\xi^2}{\partial v} + nu_y^{n-1}u_{yx}\frac{\partial\xi^1}{\partial y} + nu_y^n u_{yx}\frac{\partial\xi^1}{\partial u} + nu_y^{n-1}u_{yx}v_y\frac{\partial\xi^1}{\partial v}\\
&+ \quad nu_y^{n-1}u_{yx}\frac{\partial\xi^1}{\partial y} + nu_y^n u_{yx}\frac{\partial\xi^1}{\partial u} + nu_y^{n-1}u_{yx}v_y\frac{\partial\xi^1}{\partial v} - uv_y\frac{\partial\xi^2}{\partial y}\\
&+ \quad vu_y\frac{\partial\xi^2}{\partial y} - uu_y v_y\frac{\partial\xi^2}{\partial u} + vu_y^2\frac{\partial\xi^2}{\partial u} - uv_y^2\frac{\partial\xi^2}{\partial v} + vu_y v_y\frac{\partial\xi^2}{\partial v} = 0. \quad (6.3.17)
\end{aligned}
$$

The determining equation (6.3.13) can be split into derivative products and powers of u and v as follows

$$v_y \quad : \quad -\frac{\partial\eta^1}{\partial u} + \frac{\partial\xi^1}{\partial x} + \frac{\partial\eta^2}{\partial v} - \frac{\partial\xi^2}{\partial y} = 0, \tag{6.3.18}$$

$$1 \quad : \quad \frac{\partial\eta^1}{\partial x} + \frac{\partial\eta^2}{\partial y} = 0, \tag{6.3.19}$$

$$v_x \quad : \quad \frac{\partial\eta^1}{\partial v} - \frac{\partial\xi^1}{\partial y} = 0, \tag{6.3.20}$$

$$v_y^2 \quad : \quad \frac{\partial\xi^1}{\partial u} + \frac{\partial\xi^2}{\partial v} = 0, \tag{6.3.21}$$

$$u_y \quad : \quad -\frac{\partial\xi^2}{\partial x} + \frac{\partial\eta^2}{\partial u} = 0, \tag{6.3.22}$$

$$u_y v_x \quad : \quad \frac{\partial\xi^2}{\partial v} + \frac{\partial\xi^1}{\partial u} = 0. \tag{6.3.23}$$

Similarly, by splitting equation (6.3.17) into derivative products and powers of u and v, we have

$$
\begin{aligned}
v_y \quad &: \quad -\eta^1 + u(n-1)\frac{\partial\eta^1}{\partial u} + u\frac{\partial\xi^1}{\partial x} + v(2-n)\frac{\partial\eta^1}{\partial v}\\
&+ \quad v(2-n)\frac{\partial\xi^1}{\partial y} - u(n+1)\frac{\partial\xi^2}{\partial y} = 0, \quad (6.3.24)
\end{aligned}
$$

$$u_y \;:\; \eta^2 - u\frac{\partial \xi^2}{\partial x} + v(1-n)\frac{\partial \eta^1}{\partial u} + nv\frac{\partial \xi^2}{\partial y} = 0 \tag{6.3.25}$$

$$1 \;:\; u\frac{\partial \eta^1}{\partial x} + (2-n)v\frac{\partial \eta^1}{\partial y} = 0, \tag{6.3.26}$$

$$v_x \;:\; u\frac{\partial \eta^1}{\partial v} = 0, \tag{6.3.27}$$

$$v_y^2 \;:\; u(n-1)\frac{\partial \xi^1}{\partial u} + (2-n)v\frac{\partial \xi^1}{\partial v} - u(n+1)\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.28}$$

$$v_y v_x \;:\; u\frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.29}$$

$$u_y v_y \;:\; -u(n+1)\frac{\partial \xi^2}{\partial u} - v(n-1)\frac{\partial \xi^1}{\partial u} + nv\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.30}$$

$$u_y v_x \;:\; u\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.31}$$

$$u_y^2 \;:\; v(1+n)\frac{\partial \xi^2}{\partial u} = 0, \tag{6.3.32}$$

$$u_y^{-1}v_y \;:\; (n-1)u\frac{\partial \eta^1}{\partial y} = 0, \tag{6.3.33}$$

$$u_y^{-1}v_y^2 \;:\; (n-1)u\frac{\partial \eta^1}{\partial v} + (n-1)u\frac{\partial \xi^1}{\partial y} = 0, \tag{6.3.34}$$

$$u_y^{-1}v_y^3 \;:\; (n-1)u\frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.35}$$

$$u_y^{n-1} \;:\; n\frac{\partial^2 \eta^1}{\partial y^2} = 0, \tag{6.3.36}$$

$$u_y^{n} \;:\; n\frac{\partial^2 \xi^2}{\partial y^2} - 2n\frac{\partial^2 \eta^1}{\partial u \partial y} = 0, \tag{6.3.37}$$

$$u_y^{n-1}v_y \;:\; 2n\frac{\partial^2 \eta^1}{\partial y \partial v} + n\frac{\partial^2 \xi^1}{\partial y^2} = 0, \tag{6.3.38}$$

$$u_y^{n}v_y \;:\; -2n\frac{\partial^2 \xi^1}{\partial y \partial u} + 2n\frac{\partial^2 \xi^2}{\partial y \partial v} - 2n\frac{\partial^2 \eta^1}{\partial u \partial v} = 0, \tag{6.3.39}$$

$$u_y^{n-1}v_y^2 \;:\; 2n\frac{\partial^2 \xi^1}{\partial y \partial v} + n\frac{\partial^2 \eta^1}{\partial v^2} = 0, \tag{6.3.40}$$

$$u_y^{n+1} \;:\; 2n\frac{\partial^2 \xi^2}{\partial y \partial u} - n\frac{\partial^2 \eta^1}{\partial u^2} = 0, \tag{6.3.41}$$

$$u_y^{n+1}v_y \;:\; -n\frac{\partial^2 \xi^1}{\partial u^2} + 2n\frac{\partial^2 \xi^2}{\partial u \partial v} = 0, \tag{6.3.42}$$

$$u_y^{n}v_y^2 \;:\; -2n\frac{\partial^2 \xi^1}{\partial u \partial v} + n\frac{\partial^2 \xi^2}{\partial v^2} = 0, \tag{6.3.43}$$

$$u_y^{n+2} \quad : \quad n\frac{\partial^2 \xi^2}{\partial u^2} = 0, \tag{6.3.44}$$

$$u_y^{n-1}v_y^3 \quad : \quad n\frac{\partial^2 \xi^1}{\partial v^2} = 0, \tag{6.3.45}$$

$$u_y^{n-1}v_{yy} \quad : \quad n\frac{\partial \eta^1}{\partial v} = 0, \tag{6.3.46}$$

$$u_y^{n-1}v_{yy}v_y \quad : \quad n\frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.47}$$

$$u_y^{n}v_{yy} \quad : \quad n\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.48}$$

$$u_y^{n-1}u_{yx} \quad : \quad n\frac{\partial \xi^1}{\partial y} = 0, \tag{6.3.49}$$

$$u_y^{n}u_{yx} \quad : \quad n\frac{\partial \xi^1}{\partial u} = 0, \tag{6.3.50}$$

$$u_y^{n-1}u_{yx}v_y \quad : \quad n\frac{\partial \xi^1}{\partial v} = 0. \tag{6.3.51}$$

From the determining equations it is evident that special cases for the two-dimensional liquid jet arise. We will firstly obtain the Lie point symmetries and deduce an invariant solution for $n \neq 1, 2$, before proceeding to do the same for the special cases $n = 1$ and $n = 2$. The special cases for $n = 0$ and $n = -1$ will not be considered.

Case 1: $n \neq 1, 2$

From equations (6.3.49)-(6.3.51) we have $\xi^1 = \xi^1(x)$. Further, using equations (6.3.31) and (6.3.32) it follows that $\xi^2 = \xi^2(x, y)$. Equations (6.3.33) and (6.3.46) gives $\eta^1 = \eta^1(x, u)$. Moreover, equation (6.3.26) yields $\eta^1 = \eta^1(u)$. In summary, the infinitesimals are

$$\xi^1 = \xi(x), \quad \xi^2 = \xi^2(x, y), \quad \eta^1 = \eta^1(u), \quad \eta^2 = \eta^2(x, y, u, v). \tag{6.3.52}$$

Substituting these results into equations (6.3.41) and (6.3.37) gives $\eta^1 = c_1 u + c_2$ and $\xi^2 = A(x)y + B(x)$ respectively, where c_1 and c_2 are constants and $A(x)$ and $B(x)$ are arbitrary functions of x. Thus we have so far,

$$\xi^1 = \xi(x), \quad \xi^2 = A(x)y + B(x), \quad \eta^1 = c_1 u + c_2, \quad \eta^2 = \eta^2(x, y, u, v). \tag{6.3.53}$$

The infinitesimals (6.3.53)is substituted into equation (6.3.24). The resulting equation is separated in terms of u, and it is found that $c_2 = 0$ and

$$\frac{d\xi^1}{dx} = c_1(2 - n) + A(x)(n + 1). \tag{6.3.54}$$

Equation (6.3.54) can be solved to give

$$\xi^1 = c_1 x(2 - n) + I(x)(n + 1) + c_3, \tag{6.3.55}$$

where c_3 is a constant and $I(x) = \int^x A(x)dx$. Thus it follows that $dI(x)/dx = A(x)$. Consequently, the summarized infinitesimals are now

$$\begin{aligned} \xi^1 &= c_1x(2-n) + I(x)(n+1) + c_3, \quad \xi^2 = \frac{dI}{dx}y + B(x), \quad \eta^1 = c_1 u, \\ \eta^2 &= \eta^2(x,y,u,v). \end{aligned} \tag{6.3.56}$$

We substitute (6.3.56) into (6.3.25) to determine the value of η^2, which is $\eta^2 = uyd^2I/dx^2 + udB/dx - vc_1(1-n) - nvdI/dx$. Thus we have

$$\begin{aligned} \xi^1 &= c_1x(2-n) + I(x)(n+1) + c_3, \quad \xi^2 = \frac{dI}{dx}y + B(x), \\ \eta^1 &= c_1u, \quad \eta^2 = uy\frac{d^2I}{dx^2} + u\frac{dB}{dx} - vc_1(1-n) - nv\frac{dI}{dx}. \end{aligned} \tag{6.3.57}$$

Substituting equation (6.3.57) into equation (6.3.19) and solving the resulting ordinary differential equation gives $I(x) = c_4x + c_5$, where c_4 and c_5 are constants. The remaining determining equations are identically satisfied. Thus the infinitesimals are given by

$$\begin{aligned} \xi^1 &= x[c_1(2-n) + c_4(n+1)] + c_5(n+1) + c_3, \quad \xi^2 = c_4y + B(x), \\ \eta^1 &= c_1u, \quad \eta^2 = u\frac{dB}{dx} - [c_1(1-n) + nc_4]v. \end{aligned} \tag{6.3.58}$$

The Lie point symmetry generator is

$$\begin{aligned} X &= (x[c_1(2-n) + c_4(n+1)] + c_5(n+1) + c_3)\frac{\partial}{\partial x} + [c_4y + B(x)]\frac{\partial}{\partial y} \\ &+ c_1u\frac{\partial}{\partial u} + \left[u\frac{dB}{dx} - [c_1(1-n) + nc_4]v\right]\frac{\partial}{\partial v}. \end{aligned} \tag{6.3.59}$$

Obtaining the associated Lie point symmetry directly, as was done for the two-dimensional free jet, proved difficult for the two-dimensional liquid jet. Therefore we associate the constants of the known Lie point symmetry with the conserved vector to obtain the associated Lie point symmetry. In order to determine the Lie point symmetry associated with the conserved vector

$$T^1 = u, \quad T^2 = v, \tag{6.3.60}$$

we make use of the explicit formula [9]

$$X(T^i) + T^iD_k(\xi^k) - T^kD_k(\xi^i) = 0, \tag{6.3.61}$$

For the conserved vector (6.3.60), equation (6.3.61) gives

$$u[c_1 + c_4] = 0, \quad v[c_1 + c_4] = 0. \tag{6.3.62}$$

Thus equation (6.3.61) is satisfied if and only of $c_1 = -c_4$. Hence, the Lie point symmetry generator associated with the conserved vector (6.3.60) is

$$\begin{aligned} X &= [c_4(2n-1)x + c_5(n+1) + c_3]\frac{\partial}{\partial x} + [c_4 y + B(x)]\frac{\partial}{\partial y} \\ &- c_4 u\frac{\partial}{\partial u} + \left[c_4(1-2n)v + u\frac{dB}{dx}\right]\frac{\partial}{\partial v}. \end{aligned} \tag{6.3.63}$$

We proceed to deduce the group invariant solution. We have that $u = U(x, y)$ and $v = V(x, y)$ are invariant solutions generated by the Lie point symmetry (6.3.63) provided

$$X(u - U(x, y))|_{u=U(x,y)} = 0 \tag{6.3.64}$$

and

$$X(v - V(x, y))|_{v=V(x,y)} = 0. \tag{6.3.65}$$

We consider equation (6.3.64) first. This gives the first order linear partial differential equation

$$[c_4(2n-1)x + c_5(n+1) + c_3]\frac{\partial U}{\partial x} + [c_4 y + B(x)]\frac{\partial U}{\partial y} = -c_4 U. \tag{6.3.66}$$

The corresponding differential equations of the characteristic curves are

$$\frac{dx}{c_4(2n-1)\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]} = \frac{dy}{[c_4 y + B(x)]} = -\frac{dU}{c_4 U}. \tag{6.3.67}$$

We solve the first pair of terms which is a linear ordinary differential equation to give the first integral

$$k_1 = \frac{y}{\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}}} - B^*(x) \tag{6.3.68}$$

where k_1 is a constant and

$$B^*(x) = \frac{1}{(2n-1)c_4}\int^x \frac{B(x)}{\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{\frac{2n}{2n-1}}}dx. \tag{6.3.69}$$

We choose $B(x) = 0$ so that $B^*(x) = 0$ and therefore $y = 0$ when $k_1 = 0$. Thus we obtain the first integral

$$k_1 = \frac{y}{\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}}} \tag{6.3.70}$$

The solution of the first and last terms of equation (6.3.67) yields

$$k_2 = \left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}} U \tag{6.3.71}$$

where k_2 is a constant. The general solution for the partial differential equation (6.3.66) is $k_2 = F(k_1)$ where F is an arbitrary function. Therefore, with $u = U(x, y)$ the general solution is

$$u(x, y) = \left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{-\frac{1}{2n-1}} F(\xi), \tag{6.3.72}$$

where

$$\xi = \frac{y}{\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}}}. \tag{6.3.73}$$

Similarly for equation (6.3.65) and using the fact that $B(x) = 0$ we have the first order linear partial differential equation

$$X = [(2n-1)c_4x + (n+1)c_5 + c_3]\,\frac{\partial V}{\partial x} + c_4y\frac{\partial V}{\partial y} = (1-2n)c_4V. \tag{6.3.74}$$

The corresponding differential equations of the characteristic curves are

$$\frac{dx}{c_4(2n-1)\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]} = \frac{dy}{c_4y} = \frac{dV}{c_4(1-2n)V}. \tag{6.3.75}$$

By considering the first pair of terms in equation (6.3.75) and solving the differential equation we obtain the first integral

$$k_3 = \frac{y}{\left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}}} = k_1 \tag{6.3.76}$$

where k_3 is a constant. Finally, solving the variables separable differential equation given by the first and the last terms of equation (6.3.75) yields

$$k_4 = \left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right] V \tag{6.3.77}$$

where k_4 is a constant. The general solution of (6.3.74) is $k_4 = H(k_1)$ where H is an arbitrary function. Using this and the fact that $v = V(x, y)$, we have

$$v(x, y) = \left[x + \frac{c_5(n+1)+c_3}{(2n-1)c_4}\right]^{-1} H(\xi) \tag{6.3.78}$$

where ξ is given by (6.3.73).

We now substitute equations (6.3.72) and (6.3.78) where ξ is given by (6.3.73) into the continuity equation (6.3.1) and the momentum balance equation (6.3.2) to obtain the set of coupled differential equations

$$\frac{dH}{d\xi} - \frac{1}{(2n-1)}\left(\xi\frac{dF}{d\xi} + F(\xi)\right) = 0, \tag{6.3.79}$$

$$n\left(\frac{dF}{d\xi}\right)^{n-1}\frac{d^2F}{d\xi^2} - H(\xi)\frac{dF}{d\xi} + \frac{1}{(2n-1)}\left(\xi\frac{dF}{d\xi} + F(\xi)\right)F = 0. \tag{6.3.80}$$

Equations (6.3.79) and (6.3.80) are the same as (6.2.47) and (6.2.48) derived using the scaling transformation.

Consider now the conserved quantity (6.2.36):

$$J = \int_0^{\phi(x)} u(x,y)dy = \text{independent of } x. \tag{6.3.81}$$

Transform from variables x, y to the similarity variable ξ given by (6.3.73). Since integration in (6.3.81) is performed at a fixed point x, and using (6.3.72) for $u(x,y)$, we obtain

$$J = \int_0^k F(\xi)d\xi, \tag{6.3.82}$$

where

$$k = \frac{\phi(x)}{\left[x + \dfrac{(n+1)c_5 + c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}}}. \tag{6.3.83}$$

Since J is independent of x, k must be constant and therefore the free surface of the liquid jet satisfies

$$\phi(x) = k\left[x + \frac{(n+1)c_5 + c_3}{(2n-1)c_4}\right]^{\frac{1}{2n-1}}. \tag{6.3.84}$$

The boundary conditions are (6.2.50) and (6.2.51). Using (6.3.72) for u, and (6.3.73) for ξ and (6.3.78) for v the boundary conditions (6.2.52) are again obtained:

$$F(0) = 0, \quad H(0) = 0, \quad \frac{dF}{d\xi}(k) = 0. \tag{6.3.85}$$

For the next two cases, which are the special cases for $n = 1$ and $n = 2$, the determining equations for the momentum balance equation alter slightly. I will present these determining equations in both cases for completeness. We will however, refer to the original determining equations for the continuity equation in my analysis as these are unchanged.

Case 2: $n = 1$

The determining equations for the momentum balance equation are

$$v_y : -\eta^1 + u\frac{\partial \xi^1}{\partial x} + v\frac{\partial \eta^1}{\partial v} + v\frac{\partial \xi^1}{\partial y} - 2u\frac{\partial \xi^2}{\partial y} - 2\frac{\partial^2 \eta^1}{\partial y \partial v} - \frac{\partial^2 \xi^1}{\partial y^2} = 0, \tag{6.3.86}$$

$$u_y \ : \ \eta^2 - u\frac{\partial \xi^2}{\partial x} + v\frac{\partial \xi^2}{\partial y} - 2\frac{\partial^2 \eta^1}{\partial y \partial u} + \frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{6.3.87}$$

$$1 \ : \ u\frac{\partial \eta^1}{\partial x} + v\frac{\partial \eta^1}{\partial y} - \frac{\partial^2 \eta^1}{\partial y^2} = 0, \tag{6.3.88}$$

$$v_x \ : \ u\frac{\partial \eta^1}{\partial v} = 0, \tag{6.3.89}$$

$$v_y^2 \ : \ v\frac{\partial \xi^1}{\partial v} - 2u\frac{\partial \xi^2}{\partial v} - \frac{\partial^2 \xi^1}{\partial y \partial v} - \frac{\partial^2 \eta^1}{\partial v^2} - \frac{\partial^2 \xi^1}{\partial v \partial y} = 0, \tag{6.3.90}$$

$$v_y v_x \ : \ u\frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.91}$$

$$u_y v_y \ : \ -2u\frac{\partial \xi^2}{\partial u} + v\frac{\partial \xi^2}{\partial v} - 2\frac{\partial^2 \xi^1}{\partial y \partial u} + 2\frac{\partial^2 \xi^2}{\partial y \partial v} - 2\frac{\partial^2 \eta^1}{\partial u \partial v} = 0, \tag{6.3.92}$$

$$u_y v_x \ : \ u\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.93}$$

$$u_y^2 \ : \ 2v\frac{\partial \xi^2}{\partial u} + 2\frac{\partial^2 \xi^2}{\partial y \partial u} - \frac{\partial^2 \eta^1}{\partial u^2} = 0, \tag{6.3.94}$$

$$u_y^2 v_y \ : \ -\frac{\partial^2 \xi^1}{\partial u^2} + 2\frac{\partial^2 \xi^2}{\partial u \partial v} = 0, \tag{6.3.95}$$

$$u_y v_y^2 \ : \ -2\frac{\partial^2 \xi^1}{\partial u \partial v} + \frac{\partial^2 \xi^2}{\partial v^2} = 0, \tag{6.3.96}$$

$$u_y^3 \ : \ \frac{\partial^2 \xi^2}{\partial u^2} = 0, \tag{6.3.97}$$

$$v_y^3 \ : \ \frac{\partial^2 \xi^1}{\partial v^2} = 0, \tag{6.3.98}$$

$$v_{yy} \ : \ \frac{\partial \eta^1}{\partial v} = 0, \tag{6.3.99}$$

$$v_{yy} v_y \ : \ \frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.100}$$

$$u_y v_{yy} \ : \ \frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.101}$$

$$u_{yx} \ : \ \frac{\partial \xi^1}{\partial y} = 0, \tag{6.3.102}$$

$$u_y u_{yx} \quad : \quad \frac{\partial \xi^1}{\partial u} = 0, \tag{6.3.103}$$

$$u_{yx} v_y \quad : \quad \frac{\partial \xi^1}{\partial v} = 0. \tag{6.3.104}$$

From equations (6.3.102)-(6.3.104) we have $\xi^1 = \xi^1(x)$. Equations (6.3.101) and (6.3.99) give $\xi^2 = \xi^2(x, y, u)$ and $\eta^1 = \eta^1(x, y, u)$, respectively. Using these results and substituting them into equation (6.3.92) yields $\xi^2 = \xi^2(x, y)$. Further, using these results in equation (6.3.94), we obtain $\eta^1 = A(x, y)u + B(x, y)$, where $A(x, y)$ and $B(x, y)$ are arbitrary functions of x and y. In summary the infinitesimals are

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x, y), \quad \eta^1 = A(x, y)u + B(x, y), \quad \eta^2 = \eta^2(x, y, u, v). \tag{6.3.105}$$

Substituting equation (6.3.105) into (6.3.88) results in the following equation

$$u\left[\frac{\partial A}{\partial x}u + \frac{\partial B}{\partial x}\right] + v\left[\frac{\partial A}{\partial y}u + \frac{\partial B}{\partial y}\right] - \frac{\partial^2 A}{\partial y^2}u - \frac{\partial^2 B}{\partial y^2} = 0. \tag{6.3.106}$$

Separating equation (6.3.106) in terms of v gives

$$v \quad : \quad \frac{\partial B}{\partial y} + u\frac{\partial A}{\partial y} = 0 \tag{6.3.107}$$

$$1 \quad : \quad u^2\frac{\partial A}{\partial x} + u\frac{\partial B}{\partial x} - u\frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 B}{\partial y^2} = 0. \tag{6.3.108}$$

Separating equation (6.3.107) in terms of u and solving results in $A = A(x)$ and $B = B(x)$. Substituting these results for A and B into equation (6.3.108) yields $A(x) = c_1$ and $B(x) = c_2$ where c_1 and c_2 are arbitrary constants. Consequently, the summarized infinitesimals are

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x, y), \quad \eta^1 = c_1 u + c_2, \quad \eta^2 = \eta^2(x, y, u, v). \tag{6.3.109}$$

Using the results from equation (6.3.109) and substituting them into equation (6.3.87) yields

$$\eta^2 = u\frac{\partial \xi^2}{\partial x} - v\frac{\partial \xi^2}{\partial y} - \frac{\partial^2 \xi^2}{\partial y^2}. \tag{6.3.110}$$

Substituting the value of η^2, given by (6.3.110) along with the results from (6.3.109) into equation (6.3.19) and separating in terms of u and v, and solving the resulting equations gives that $\xi^2 = c_3 y + D(x)$, where c_3 is a constant and $D(x)$ is an arbitrary function of x. Consequently, the infinitesimals are

$$\xi^1 = \xi^1(x), \quad \xi^2 = c_3 y + D(x), \quad \eta^1 = c_1 u + c_2, \quad \eta^2 = u\frac{dD}{dx} - c_3 v \tag{6.3.111}$$

We further substitute the infinitesimals (6.3.111) into equation (6.3.18) and determine the value of ξ^1 to be $\xi^1 = (c_1 + 2c_3)x + c_4$, where c_4 is a constant. Invoking this result for ξ^1 along with (6.3.111) and applying it to equation (6.3.86) we find that $c_2 = 0$. Thus we have

$$\xi^1 = (c_1 + 2c_3)x + c_4, \quad \xi^2 = c_3 y + D(x), \tag{6.3.112}$$

$$\eta^1 = c_1 u, \quad \eta^2 = u\frac{dD}{dx} - c_3 v. \tag{6.3.113}$$

The remaining determining equations are identically satisfied. The Lie point symmetry generator is

$$X = [(c_1 + 2c_3)x + c_4]\frac{\partial}{\partial x} + [c_3 y + D(x)]\frac{\partial}{\partial y} + c_1 u\frac{\partial}{\partial u} + \left[u\frac{dD}{dx} - c_3 v\right]\frac{\partial}{\partial v}. \tag{6.3.114}$$

We will apply the same procedure to obtain the associated Lie point symmetries as was done for Case 1. The constants of the obtained Lie point symmetry will be associated with the conserved vector (6.3.60). For the conserved vector (6.3.60), equation (6.3.61) gives

$$u[c_1 + c_3] = 0, \quad v[c_1 + c_3] = 0. \tag{6.3.115}$$

Thus equation (6.3.61) is satisfied if and only if $c_1 = -c_3$. Hence, the Lie point symmetry generator associated with the conserved vector (6.3.60) for $n = 1$ is given by

$$X = [c_3 x + c_4]\frac{\partial}{\partial x} + [c_3 y + D(x)]\frac{\partial}{\partial y} - c_3 u\frac{\partial}{\partial u} + \left[u\frac{dD}{dx} - c_3 v\right]\frac{\partial}{\partial v}. \tag{6.3.116}$$

The invariant solution is obtained in a similar fashion as done for the general case. We use equations (6.3.64) and (6.3.65). Applying equation (6.3.64) first, we find that

$$(c_3 x + c_4)\frac{\partial U}{\partial x} + (c_3 y + D(x))\frac{\partial U}{\partial y} = -c_3 U. \tag{6.3.117}$$

A first integral of the differential equations of the characteristic curves is

$$k_1 = \frac{y}{[x + \frac{c_4}{c_3}]} - D^*(x), \tag{6.3.118}$$

where k_1 is a constant and

$$D^*(x)dx = \frac{1}{c_3}\int^x \frac{D(x)dx}{\left(x + \dfrac{c_4}{c_3}\right)^2}. \tag{6.3.119}$$

We choose $D(x) = 0$ and therefore $D^*(x) = 0$ so that $y = 0$ when $\xi = 0$. Thus

$$k_1 = \frac{y}{[x + \frac{c_4}{c_3}]}. \tag{6.3.120}$$

A second integral is

$$k_2 = \left(x + \frac{c_4}{c_3}\right) U \tag{6.3.121}$$

where k_2 is a constant. Since $u = U(x, y)$, we find that

$$u(x, y) = \left[x + \frac{c_4}{c_3}\right]^{-1} F(\xi), \tag{6.3.122}$$

where

$$\xi = \frac{y}{[x + \frac{c_4}{c_3}]} \tag{6.3.123}$$

and $F(\xi)$ is an arbitrary function of ξ. Equation (6.3.65) for V has the same form as (6.3.64) for U and therefore

$$v(x, y) = \left[x + \frac{c_4}{c_3}\right]^{-1} H(\xi) \tag{6.3.124}$$

We substitute equations (6.3.122)-(6.3.124) into the continuity (6.3.1) and momentum balance equation (6.3.2) to obtain the coupled set of differential equations

$$\frac{dH}{d\xi} - \left(\xi \frac{dF}{d\xi} + F(\xi)\right) = 0, \tag{6.3.125}$$

$$\frac{d^2F}{d\xi^2} - H(\xi)\frac{dF}{d\xi} + \left(\xi \frac{dF}{d\xi} + F(\xi)\right) F = 0. \tag{6.3.126}$$

Equations (6.3.125) and (6.3.126) agree with the general equations (6.3.79) and (6.3.80) with $n = 1$.

Finally consider the conserved quantity (6.3.81) which when $n = 1$ can be expressed

$$J = \int_0^k F(\xi) d\xi \tag{6.3.127}$$

where

$$k = \frac{\phi(x)}{\left(x + \frac{c_4}{c_3}\right)} \tag{6.3.128}$$

For J to be independent of x, k must be a constant and therefore the equation of the free surface $y = \phi(x)$ satisfies

$$\phi(x) = k\left(x + \frac{c_4}{c_3}\right). \tag{6.3.129}$$

This agrees with the general result (6.3.84) with $n = 1$.

The boundary conditions when $n = 1$ are again given by (6.3.85).

Case 3: $n = 2$

The determining equations for the momentum balance equation are

$$v_y \quad : \quad -\eta^1 + u\frac{\partial \eta^1}{\partial u} + u\frac{\partial \xi^1}{\partial x} - 3u\frac{\partial \xi^2}{\partial y} = 0, \tag{6.3.130}$$

$$u_y \quad : \quad \eta^2 - u\frac{\partial \xi^2}{\partial x} - v\frac{\partial \eta^1}{\partial u} + 2v\frac{\partial \xi^2}{\partial y} - 2\frac{\partial^2 \eta^1}{\partial y^2} = 0, \tag{6.3.131}$$

$$1 \quad : \quad u\frac{\partial \eta^1}{\partial x} = 0, \tag{6.3.132}$$

$$v_x \quad : \quad u\frac{\partial \eta^1}{\partial v} = 0, \tag{6.3.133}$$

$$v_y^2 \quad : \quad u\frac{\partial \xi^1}{\partial u} - 3u\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.134}$$

$$v_y v_x \quad : \quad u\frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.135}$$

$$u_y v_y \quad : \quad -3u\frac{\partial \xi^2}{\partial u} - v\frac{\partial \xi^1}{\partial u} + 2v\frac{\partial \xi^2}{\partial v} - 4\frac{\partial^2 \eta^1}{\partial y \partial v} - 2\frac{\partial^2 \xi^1}{\partial y^2} = 0, \tag{6.3.136}$$

$$u_y v_x \quad : \quad u\frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.137}$$

$$u_y^2 \quad : \quad 3v\frac{\partial \xi^2}{\partial u} - 4\frac{\partial^2 \eta^1}{\partial y \partial u} + 2\frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{6.3.138}$$

$$u_y^{-1} v_y \quad : \quad u\frac{\partial \eta^1}{\partial y} = 0, \tag{6.3.139}$$

$$u_y^{-1} v_y^2 \quad : \quad u\frac{\partial \eta^1}{\partial v} + u\frac{\partial \xi^1}{\partial y} = 0, \tag{6.3.140}$$

$$u_y^{-1} v_y^3 \quad : \quad u\frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.141}$$

$$u_y^2 v_y \quad : \quad \frac{\partial^2 \xi^1}{\partial y \partial u} - \frac{\partial^2 \xi^2}{\partial y \partial v} + \frac{\partial^2 \eta^1}{\partial u \partial v} = 0, \tag{6.3.142}$$

$$u_y v_y^2 \quad : \quad 2\frac{\partial^2 \xi^1}{\partial y \partial v} + \frac{\partial^2 \eta^1}{\partial v^2} = 0, \tag{6.3.143}$$

$$u_y^3 \quad : \quad 2\frac{\partial^2 \xi^2}{\partial y \partial u} - \frac{\partial^2 \eta^1}{\partial u^2} = 0, \tag{6.3.144}$$

$$u_y^3 v_y \quad : \quad -\frac{\partial^2 \xi^1}{\partial u^2} + 2\frac{\partial^2 \xi^2}{\partial u \partial v} = 0, \tag{6.3.145}$$

$$u_y^2 v_y^2 \quad : \quad -2\frac{\partial^2 \xi^1}{\partial u \partial v} + \frac{\partial^2 \xi^2}{\partial v^2} = 0, \tag{6.3.146}$$

$$u_y^4 \quad : \quad \frac{\partial^2 \xi^2}{\partial u^2} = 0, \tag{6.3.147}$$

$$u_y v_y^3 \quad : \quad \frac{\partial^2 \xi^1}{\partial v^2} = 0, \tag{6.3.148}$$

$$u_y v_{yy} \quad : \quad \frac{\partial \eta^1}{\partial v} = 0, \tag{6.3.149}$$

$$u_y v_{yy} v_y \quad : \quad \frac{\partial \xi^1}{\partial v} = 0, \tag{6.3.150}$$

$$u_y^2 v_{yy} \quad : \quad \frac{\partial \xi^2}{\partial v} = 0, \tag{6.3.151}$$

$$u_y u_{yx} \quad : \quad \frac{\partial \xi^1}{\partial y} = 0, \tag{6.3.152}$$

$$u_y^2 u_{yx} \quad : \quad \frac{\partial \xi^1}{\partial u} = 0, \tag{6.3.153}$$

$$u_y u_{yx} v_y \quad : \quad \frac{\partial \xi^1}{\partial v} = 0. \tag{6.3.154}$$

From equations (6.3.152)-(6.3.154) we find that $\xi^1 = \xi^1(x)$. Equations (6.3.132), (6.3.133) and (6.3.139) give $\eta^1 = \eta^1(u)$. While equations (6.3.151) and (6.3.136) give $\xi^2 = \xi^2(x, y)$. Thus we have

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x, y), \quad \eta^1 = \eta^1(u), \quad \eta^2 = \eta^2(x, y, u, v). \tag{6.3.155}$$

The results (6.3.155), can be substituted into equations (6.3.144) and (6.3.138) to obtain $\eta^1 = c_1 u + c_2$ and $\xi^2 = A(x)y + B(x)$ respectively, where c_1 and c_2 are constants and A and B are arbitrary functions of x. In summary we have

$$\xi^1 = \xi^1(x), \quad \xi^2 = A(x)y + B(x), \quad \eta^1 = c_1 u + c_2, \quad \eta^2 = \eta^2(x, y, u, v). \tag{6.3.156}$$

Substituting equation (6.3.156) into (6.3.130) and separating the resulting differential equation in terms of u, we find that $c_2 = 0$ and $\xi^1 = 3I(x) + c_3$, where c_3 is a constant and $I(x) = \int^x A(x)dx$. That is

$$\begin{aligned} \xi^1 &= 3I(x) + c_3, \quad \xi^2 = \frac{dI}{dx}y + B(x), \quad \eta^1 = c_1 u, \\ \eta^2 &= \eta^2(x, y, u, v). \end{aligned} \tag{6.3.157}$$

Furthermore, we determine η^2 by substituting equation (6.3.157) into (6.3.131). Thus we have

$$\begin{aligned} \xi^1 &= 3I(x) + c_3, \quad \xi^2 = \frac{dI}{dx}y + B(x), \quad \eta^1 = c_1 u, \\ \eta^2 &= u\left[\frac{d^2 I}{dx^2}y + \frac{dB}{dx}\right] + \left[c_1 - 2\frac{dI}{dx}\right]v. \end{aligned} \tag{6.3.158}$$

Finally, by substituting the results (6.3.158) into equation (6.3.19), we find that $I = c_4x + c_5$, where c_4 and c_5 are constants. Therefore

$$\begin{aligned} \xi^1 &= 3(c_4x + c_5) + c_3, \quad \xi^2 = c_4y + B(x), \quad \eta^1 = c_1u \\ \eta^2 &= u\frac{dB}{dx} + (c_1 - 2c_4)v.. \end{aligned} \tag{6.3.159}$$

The remaining determining equations are identically satisfied. The Lie point symmetry generator is given by

$$\begin{aligned} X &= [3c_4x + 3c_5 + c_3]\frac{\partial}{\partial x} + [c_4y + B(x)]\frac{\partial}{\partial y} + c_1u\frac{\partial}{\partial u} \\ &+ \left[u\frac{dB}{dx} + (c_1 - 2c_4)v\right]\frac{\partial}{\partial v}. \end{aligned} \tag{6.3.160}$$

We proceed to deduce the invariant solution by associating the constants of the derived Lie point symmetry with the conserved vector (6.3.60). For the conserved vector (6.3.60), equation (6.3.61) gives

$$u[c_1 + c_4] = 0, \quad v[c_1 + c_4] = 0. \tag{6.3.161}$$

Thus equation (6.3.61) is satisfied if and only if $c_1 = -c_4$. Hence, the Lie point symmetry generator associated with the conserved vector (6.3.60), for $n = 2$, is given by

$$\begin{aligned} X &= [3c_4x + 3c_5 + c_3]\frac{\partial}{\partial x} + [c_4y + B(x)]\frac{\partial}{\partial y} - c_4u\frac{\partial}{\partial u} \\ &+ \left[u\frac{dB}{dx} - 3c_4v\right]\frac{\partial}{\partial v}. \end{aligned} \tag{6.3.162}$$

The invariant solution is obtained in a similar way as done for Cases 1 and 2. We use equations (6.3.64) and (6.3.65). Applying equation (6.3.64) first, we find that U satisfies

$$(3c_4x + 3c_5 + c_3)\frac{\partial U}{\partial x} + (c_4y + B(x))\frac{\partial U}{\partial y} = -c_4U. \tag{6.3.163}$$

A first integral of the differential equations of the characteristic curves is

$$k_1 = \frac{y}{[x + \frac{3c_5+c_3}{3c_4}]^{\frac{1}{3}}} - B^*(x), \tag{6.3.164}$$

where k_1 is a constant and

$$B^*(x) = \frac{1}{3c_4}\int^x \frac{B(x)dx}{\left[x + \dfrac{3c_5 + c_3}{3c_4}\right]^{\frac{4}{3}}} \tag{6.3.165}$$

We choose $B(x) = 0$ so that $B^*(x) = 0$ and $y = 0$ when $\xi = 0$. Thus

$$k_1 = \frac{y}{[x + \frac{3c_5+c_3}{3c_4}]^{\frac{1}{3}}}. \tag{6.3.166}$$

A second integral is

$$k_2 = \left[x + \frac{(3c_5 + c_3)}{3c_4}\right] U \tag{6.3.167}$$

and since $u = U(x, y)$, we find that

$$u(x, y) = \left[x + \frac{3c_5 + c_3}{3c_4}\right]^{-\frac{1}{3}} F(\xi), \tag{6.3.168}$$

where $F(\xi)$ is an arbitrary function and

$$\xi = \frac{y}{[x + \frac{3c_5+c_3}{3c_4}]^{\frac{1}{3}}}. \tag{6.3.169}$$

Equation (6.3.65) for V is, since $B(x) = 0$,

$$(3c_4 x + 3c_5 + c_3)\frac{\partial V}{\partial x} + c_4 y \frac{\partial V}{\partial y} = -3c_4 V. \tag{6.3.170}$$

The first integrals are (6.3.166) and

$$k_2 = \left[x + \frac{3c_5 + c_3}{3c_4}\right] V \tag{6.3.171}$$

and since $v = V(x, y)$ the general solution is

$$v(x, y) = \left[x + \frac{3c_5 + c_3}{3c_4}\right]^{-1} H(\xi), \tag{6.3.172}$$

where $H(\xi)$ is an arbitrary function of ξ.

We substitute equations (6.3.168), (6.3.169) and (6.3.172) into the continuity (6.3.1) and momentum balance equation (6.3.2) to obtain the coupled set of differential equations

$$\frac{dH}{d\xi} - \frac{1}{3}\left(\xi\frac{dF}{d\xi} + F(\xi)\right) = 0, \tag{6.3.173}$$

$$2\frac{dF}{d\xi}\frac{d^2F}{d\xi^2} - H(\xi)\frac{dF}{d\xi} + \frac{1}{3}\left(\xi\frac{dF}{d\xi} + F(\xi)\right) F = 0. \tag{6.3.174}$$

Equations (6.3.173) and (6.3.174) agree with the general result (6.3.79) and (6.3.80) with $n = 2$.

The conserved quantity when $n = 2$ is, from (6.2.36),

$$J = \int_0^k F(\xi) d\xi \tag{6.3.175}$$

where

$$k = \frac{\phi(x)}{\left(x + \dfrac{3c_5 + c_3}{3c_4}\right)^{\frac{1}{3}}} = \text{constant.} \tag{6.3.176}$$

Thus

$$\phi(x) = k\left(x + \frac{3c_5 + c_3}{3c_4}\right)^{\frac{1}{3}} \tag{6.3.177}$$

which agrees with (6.3.84) with $n = 2$.

The boundary conditions for $n = 2$ are again given by (6.3.85).

In this section we derived the complete Lie point symmetries of the two-dimensional liquid jet. Further, we associated the constants of the derived Lie point symmetry with a conserved vector, and used these associated Lie point symmetries to deduce an invariant solution. Special cases for n arose in this investigation. The two ordinary differential equations, the equation of the free surface of the liquid jet and the boundary conditions that were derived for $n \neq 1$ and $n = 1$ were consistent with the special cases for $n = 1$ and $n = 2$. As with the two-dimensional free jet, the set of coupled ordinary differential equations derived for the two-dimensional liquid jet when using a scaling transformation are identical to equations (6.3.79) and (6.3.80).

It remains to determine $(n+1)c_5 + c_3$ for the general case $(n \neq 1, n \neq 2)$, c_4/c_3 for $n = 1$ and $3c_5 + c_3$ for $n = 2$. In each case the constants are obtained by observing that if a singularity exists in $v(x, y)$ it can only exist at the source, $x = 0$. Thus from (6.3.78), (6.3.124) and (6.3.172) for $v(x, y)$ it follows that

$$n \neq 1, \quad n \neq 2 \quad : \quad (n+1)c_5 + c_3 = 0, \tag{6.3.178}$$

$$n = 1 \quad : \quad c_4 = 0, \tag{6.3.179}$$

$$n = 2 \quad : \quad 3c_5 + c_3 = 0. \tag{6.3.180}$$

In each case, $u(x, y)$, $v(x, y)$, ξ and ϕ reduce to the results derived for using the scaling transformation.

The problem can therefore be stated for $0 < n < \infty$, excluding $n = 1/2$, by (6.2.43) to (6.2.49) subject to the boundary conditions (6.2.52).

6.4 Solution of two-dimensional liquid jet

This section will outline the parametric solution for the two-dimensional liquid jet. From the coupled set of differential equations, it is apparent that in the range $0 < n < 1/2$ and at $n = 1/2$ problems might arise. Consequently, we present the parametric solutions for $1/2 < n < \infty$. The values $n = 1$ and $n = 2$ do not need to be treated separately.

6.4.1 Case 1: $\frac{1}{2} < n < \infty$

We consider the first equation (6.2.47) which can be written more compactly as

$$\frac{dH}{d\xi} = \frac{1}{(2n-1)}\frac{d}{d\xi}(\xi F(\xi)). \tag{6.4.1}$$

Integrating with respect to ξ we obtain

$$H(\xi) = \frac{1}{2n-1}\xi F(\xi) + \alpha, \tag{6.4.2}$$

where α is a constant. Using the boundary condition $H(0) = 0$, we have that $\alpha = 0$. Thus equation (6.4.2) becomes

$$H(\xi) = \frac{1}{(2n-1)}\xi F(\xi). \tag{6.4.3}$$

We substitute $H(\xi)$, given by (6.4.3) into the momentum balance equation (6.2.48) and simplify to obtain

$$n\left(\frac{dF}{d\xi}\right)^{n-1}\frac{d^2F}{d\xi^2} + \frac{1}{(2n-1)}(F(\xi))^2 = 0. \tag{6.4.4}$$

Equation (6.4.4) does not depend explicitly on ξ. Using

$$\frac{d^2F}{d\xi^2} = \frac{d}{d\xi}\left(\frac{dF}{d\xi}\right) = \frac{d}{dF}\left(\frac{dF}{d\xi}\right)\frac{dF}{d\xi} = F'\frac{dF'}{dF}, \tag{6.4.5}$$

where the dash denotes differentiation with respect to ξ, equation (6.4.4) can be written as

$$n(F')^n\frac{dF'}{dF} = -\frac{1}{(2n-1)}F^2. \tag{6.4.6}$$

Equation (6.4.6) is variables separable,

$$n(F')^n dF' = -\frac{1}{2n-1}F^2 dF. \tag{6.4.7}$$

Integrating both sides gives

$$\left(\frac{n}{n+1}\right)(F')^{n+1} = -\frac{1}{3(2n-1)}F^3 + A, \tag{6.4.8}$$

where A is a constant. But from the boundary condition (6.2.52)

$$\frac{dF}{d\xi}(k) = 0 \tag{6.4.9}$$

and therefore

$$A = \frac{1}{3(2n-1)} F^3(k). \tag{6.4.10}$$

Equation (6.4.8) becomes

$$\frac{dF}{d\xi} = \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{n+1}} [F^3(k) - F^3]^{\frac{1}{n+1}}, \tag{6.4.11}$$

which is variables separable

$$d\xi = \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{n+1}} \frac{dF}{[F^3(k) - F^3]^{\frac{1}{n+1}}}. \tag{6.4.12}$$

In order to find the limits of integration we make use of the boundary condition $F(0) = 0$. We integrate the left hand side from 0 to ξ and therefore the right hand side from 0 to F.

$$\int_0^\xi d\xi = \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{n+1}} \int_0^F \frac{df}{[F^3(k) - f^3]^{\frac{1}{n+1}}} \tag{6.4.13}$$

where f is a dummy variable of integration.

We make the change of variables

$$g = \frac{f}{F(k)}, \quad G(\xi) = \frac{F(\xi)}{F(k)}. \tag{6.4.14}$$

Equation (6.4.13) becomes

$$\xi = \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{n+1}} F(k)^{\frac{n-2}{n+1}} \int_0^G \frac{dg}{[1 - g^3]^{\frac{1}{n+1}}}. \tag{6.4.15}$$

In order to determine $F(k)$ we make use of the conserved quantity given by

$$J = \int_0^k F(\xi) d\xi = \int_0^{F(k)} F \frac{d\xi}{dF} = \int_0^{F(k)} \frac{F}{F'} dF, \tag{6.4.16}$$

where we used the boundary condition $F(0) = 0$. The conserved quantity, using equation (6.4.11) becomes

$$J = \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{n+1}} \int_0^{F(k)} \frac{F\, dF}{[F^3(k) - F^3]^{\frac{1}{n+1}}} \tag{6.4.17}$$

Using the transformation (6.4.14), equation (6.4.17) becomes

$$J = \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{n+1}} F(k)^{\frac{2n-1}{n+1}} \int_0^1 \frac{G\,dG}{[1-G^3]^{\frac{1}{n+1}}}. \tag{6.4.18}$$

We rename the dummy variable G in (6.4.18) by g as in (6.4.15) and to distinguish it from the similarity variable G used later. Let

$$I(n) = \int_0^1 \frac{g\,dg}{[1-g^3]^{\frac{1}{n+1}}}. \tag{6.4.19}$$

We express $F(k)$ in terms of the conserved quantity as

$$F(k) = \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n+1}{2n-1}}. \tag{6.4.20}$$

Using (6.4.20) we can now express ξ given by (6.4.15) in terms of J as

$$\xi = \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n-2}{2n-1}} \int_0^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}} \tag{6.4.21}$$

Using (6.4.14) and (6.4.20) we can express $F(\xi)$ in terms of $G(\xi)$ as

$$F(\xi) = \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n+1}{2n-1}} G. \tag{6.4.22}$$

Finally using (6.4.3), (6.4.21) and (6.4.22) we have

$$H(\xi) = \frac{J}{(2n-1)I(n)} G \int_0^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}. \tag{6.4.23}$$

We can now express the solution in parametric form with G as parameter. From (6.2.43),

$$u(x,y) = x^{-\frac{1}{2n-1}} F(\xi). \tag{6.4.24}$$

Thus using equation (6.4.22)

$$u(x,y) = x^{-\frac{1}{2n-1}} \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n+1}{2n-1}} G. \tag{6.4.25}$$

Further from (6.2.44) the y-component of velocity is given by

$$v(x,y) = x^{-1} H(\xi). \tag{6.4.26}$$

Therefore, using equation (6.4.23),

$$v(x,y) = x^{-1}\frac{J}{(2n-1)I(n)}G\int_0^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}. \tag{6.4.27}$$

Also from (6.2.45)

$$y = x^{\frac{1}{2n-1}}\xi \tag{6.4.28}$$

and using (6.4.21) we obtain

$$y = x^{\frac{1}{2n-1}}\left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{2n-1}}\left[\frac{J}{I(n)}\right]^{\frac{n-2}{2n-1}}\int_0^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}} \tag{6.4.29}$$

In the equations for $u(x,y)$, $v(x,y)$ and y,

$$I(n) = \int_0^1 \frac{g\,dg}{[1-g^3]^{\frac{1}{n+1}}} \tag{6.4.30}$$

and the range of the parameter G is

$$0 \le G \le 1. \tag{6.4.31}$$

We now verify that the range of ξ and therefore of y is finite. To do that we investigate the convergence of the integral

$$\int^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}, \tag{6.4.32}$$

as $G \to 1$. This analysis will also verify that $I(n)$ is finite. Let $g = 1-h$. Then

$$\int^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}} = -\int^{1-G} \frac{dh}{[1-(1-h)^3]^{\frac{1}{n+1}}}. \tag{6.4.33}$$

Expanding for small h we obtain

$$\int^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}} = -3^{-\frac{1}{n+1}}\int^{1-G} \frac{dh}{h^{\frac{1}{n+1}}(1+O(h))} \tag{6.4.34}$$

and therefore

$$\int^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}} = -3^{-\frac{1}{n+1}}\frac{(n+1)}{n}(1-G)^{\frac{n}{n+1}} \quad \text{as } \to 1. \tag{6.4.35}$$

The integral is therefore convergent as $G \to 1$. The range of ξ and y is finite. Thus

$$0 \le \xi \le \xi_{max} \tag{6.4.36}$$

where

$$\xi_{max} = \xi = \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n-2}{2n-1}} \int_0^1 \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}. \qquad (6.4.37)$$

Since

$$\xi_{max} = \frac{\phi(x)}{x^{\frac{1}{2n-1}}} \qquad (6.4.38)$$

it follows that $\xi_{max} = k$ and that the equation of the free surface of the liquid jet is

$$y = \phi(x) = \xi_{max} x^{\frac{1}{2n-1}} \qquad (6.4.39)$$

and therefore that

$$\phi(x) = x^{\frac{1}{2n-1}} \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n-2}{2n-1}} \int_0^1 \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}. \qquad (6.4.40)$$

6.4.2 Case 2: $0 < n < \frac{1}{2}$

Firstly equation (6.2.47) can be written as

$$\frac{dH}{d\xi} = -\frac{1}{(2n-1)}\frac{d}{d\xi}(\xi F(\xi)). \qquad (6.4.41)$$

Integrating with respect to ξ and using the boundary condition $H(0) = 0$ we obtain

$$H(\xi) = -\frac{1}{(1-2n)}\xi F(\xi). \qquad (6.4.42)$$

The momentum balance equation (6.2.48) becomes

$$n\left(\frac{dF}{d\xi}\right)^{n-1}\frac{d^2F}{d\xi^2} - \frac{1}{(1-2n)}F^2 = 0. \qquad (6.4.43)$$

Using again the transformation

$$\frac{d^2F}{d\xi^2} = F'\frac{dF'}{dF}, \qquad (6.4.44)$$

equation (6.4.43) becomes the variables separable differential equation

$$n(F')^n dF' = \frac{1}{(1-2n)}F^2 dF. \qquad (6.4.45)$$

Integrating both sides yields

$$\left(\frac{n}{n+1}\right)(F')^{n+1} = \frac{1}{3(1-2n)}F^3 + A, \qquad (6.4.46)$$

where A is a constant. The boundary condition (6.2.52),

$$\frac{dF}{d\xi}(k) = 0, \tag{6.4.47}$$

gives

$$A = -\frac{1}{3(1-2n)}F^3(k) \tag{6.4.48}$$

and therefore

$$(F')^{n+1} = -\frac{(n+1)}{3n(1-2n)}(F^3(k) - F^3). \tag{6.4.49}$$

In the momentum balance equation (6.2.2) we assumed that

$$\frac{\partial u}{\partial y} > 0, \quad 0 \leq y \leq y_{max}, \tag{6.4.50}$$

and the modulus sign on $\partial u/\partial y$ for the power-law fluid was removed. From (6.2.43) and (6.2.45)

$$u(x,y) = x^{\frac{1}{1-2n}}F(\xi), \quad \frac{\partial u(x,y)}{\partial y} = x^{\frac{2}{1-2n}}\frac{dF}{d\xi}, \tag{6.4.51}$$

and therefore the analysis assumes that

$$F'(\xi) > 0, \quad 0 \leq \xi \leq k. \tag{6.4.52}$$

Consider first a liquid jet with outflow of fluid from $x = 0$. Then $F(\xi) > 0$ and $F(\xi)$ increases from $F = 0$ at $\xi = 0$ to $F(k)$ at $\xi = k$. Hence $F'(\xi) > 0$ and the equations apply. But from equation (6.4.49),

$$(F')^{n+1} < 0. \tag{6.4.53}$$

The solution for a liquid jet with outflow from $x = 0$ therefore does not exist when $0 < n < 1/2$.

Consider next a liquid jet with inflow towards $x = 0$. The $F(\xi) < 0$ for $0 < \xi \leq k$ and $F(\xi)$ decreases from $F = 0$ at $\xi = 0$ to $F(k)$ (< 0) when $\xi = k$ on the free surface. Thus $F'(\xi) < 0$ for $0 \leq \xi \leq k$ and the equations do not apply.

We conclude that for $0 < n < 1/2$ there is no solution of the liquid jet with fluid outflow but a liquid jet with fluid inflow requires further investigation.

6.5 Results and discussion

Case 1: $\frac{1}{2} < n < \infty$

The two-dimensional liquid jet defined over this range allows for a comparison between the differing properties of the non-Newtonian power-law fluid. It includes shear thinning ($1/2 < n < 1$), Newtonian ($n = 1$) and shear thickening ($n > 1$) fluids. We start by summarizing the parametric solution for the fluid velocity components and the equation of the free surface:

$$u(x,y) = x^{-\frac{1}{2n-1}} \left[\frac{n+1}{3n(2n-1)}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n+1}{2n-1}} G, \tag{6.5.1}$$

$$v(x,y) = x^{-1} \frac{J}{(2n-1)I(n)} G \int_0^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}, \tag{6.5.2}$$

$$\phi(x) = x^{\frac{1}{2n-1}} \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n-2}{2n-1}} \int_0^1 \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}, \tag{6.5.3}$$

$$y = x^{\frac{1}{2n-1}} \left[\frac{3n(2n-1)}{n+1}\right]^{\frac{1}{2n-1}} \left[\frac{J}{I(n)}\right]^{\frac{n-2}{2n-1}} \int_0^G \frac{dg}{[1-g^3]^{\frac{1}{n+1}}}. \tag{6.5.4}$$

Consider first how the width of the liquid jet, $\phi(x)$, depends on the jet strength J at a fixed point x. From (6.5.3) at a fixed x,

$$\phi(x) \propto J^{\frac{n-2}{2n-1}}. \tag{6.5.5}$$

Hence the width decreases as the strength J increases for $1/2 < n < 2$, it does not depend on J for $n = 2$ and the width increases as J increases for $n > 2$. This is the same behaviour as found for a free jet.

Consider now the velocity profile of $u(x,y)$ against y in the liquid jet. In Figures 6.5.1 to 6.5.3 the velocity profile of $u(x,y)$ is plotted against y at $x = 1$ for a range of values of J and for $n = 3/4$, $n = 1$, $n = 1.25$. From (6.5.1) and (6.5.4),

$$\frac{\partial u(x,y)}{\partial y} \propto J^{\frac{3}{2n-1}} \tag{6.5.6}$$

and thus a greater jet strength J results in a larger rate of change of $u(x,y)$ with y in the velocity profiles which is consistent with the graphs in Figures 6.5.1 to 6.5.3.

From (6.5.2) we see that the y-component of velocity $v(x,y)$ is always positive and that it decreases like $1/x$ as x increases and increases linearly with the jet strength J.

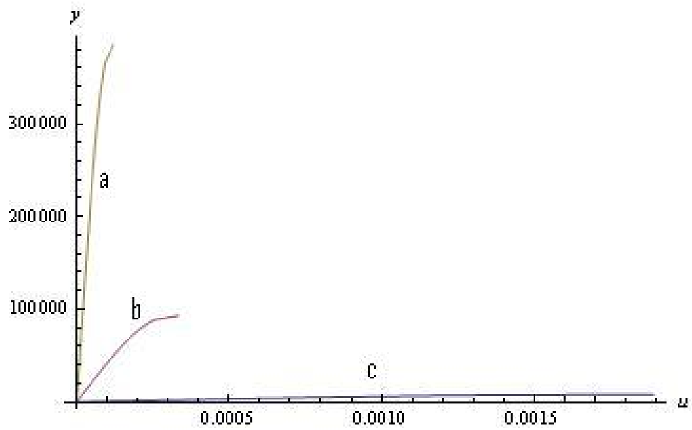

Figure 6.5.1: Velocity profile of u against y at $x = 1$ for a two-dimensional liquid jet with $n = 0.75$ for (a) $J = 30$, (b) $J = 20$ and (c) $J = 10$ plotted over the full parameter range $0 \leq G \leq 1$.

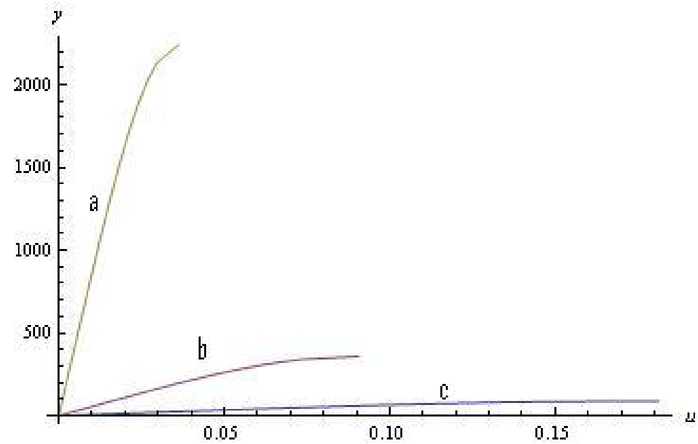

Figure 6.5.2: Velocity profile of u against y at $x = 1$ for a two-dimensional liquid jet with $n = 1$ for (a) $J = 50$, (b) $J = 20$ and (c) $J = 10$ over the full parameter range $0 \leq G \leq 1$.

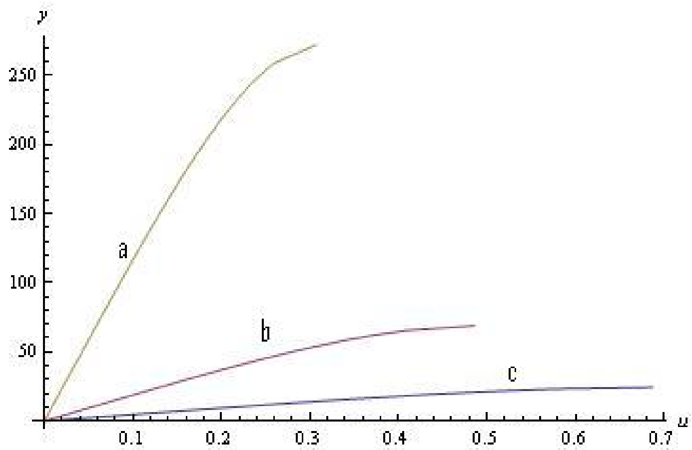

Figure 6.5.3: Velocity profile of u against y at $x = 1$ for a two-dimensional liquid jet with $n = 1.25$ for (a) $J = 50$, (b) $J = 20$ and (c) $J = 10$ over the full parameter range $0 \leq G \leq 1$.

6.6 Conclusions

This chapter presented a descriptive outline of the similarity solutions and Lie point symmetry analysis of the two-dimensional liquid jet. The coupled differential equations that were derived from the similarity solution approach and the Lie point symmetry approach were identical.

We considered the derived Lie point symmetries of the system of partial differential equations for the velocity components, and associated these symmetries with a conserved vector to deduce an invariant solution. We were not able to derive the associated Lie point symmetry directly from condition (6.3.61), as was done in Chapter 5 for the stream function because the momentum balance equation fails to provide sufficient information. In the calculation of the Lie point symmetries, special cases for n arose but these special cases for n were consistent with the primary case for an arbitrary n.

A parametric solution for the liquid jet for the case $1/2 < n < \infty$ was derived. This range contained shear thinning, Newtonian and shear thickening fluids. There was no solution for $n = 1/2$. The case $0 < n < 1/2$ requires further investigation. It may describe inflow of fluid towards the origin $x = 0$ and the equations for a power-law fluid that we were using did not represent these flows. The parametric solution was a convenient form of solution to interpret the physical properties of the liquid jet. The dependence of the fluid flow on J, x and n could be readily deduced.

Chapter 7

Two-dimensional wall jet

7.1 Introduction

From a physical perspective there are various ways in which a wall jet can be defined. One of the most common descriptions of a wall jet can be found by considering two sections of a canal with dissimilar water levels separated by a sluice gate. A plane wall jet is formed when the sluice gate is opened and water flows into the section with the lower level [8]. Other descriptions of wall jets include fluids falling into a partly full tank and spreading out over the bottom [6]. This case represents a radial wall jet. Of particular importance in such an instance is the condition at the free surface. For the free and wall jets, the radial velocity diminishes as we move outward, whereas for a liquid jet the pressure remains constant at the free surface [6].

Naz et al. [8] considered the conservation laws for the two-dimensional liquid, free and wall jets for a Newtonian fluid. Using a multiplier of the form $\Lambda(x, y, \psi)$, Naz et al. [8] found, for the third order partial differential equation for the stream function, a multiplier $\Lambda = c_3+c_4\psi$, where c_3 and c_4 are constants. This multiplier produced two sets of conserved vectors (4.2.96) and (4.2.97). The elementary conserved vector (4.2.96) produced the conserved quantity for the two-dimensional free jet and (4.2.97) gave the conserved quantity for the two-dimensional wall jet.

This Chapter will be structured as follows. In Section 7.2 we describe the multiplier obtained for a two-dimensional wall jet. The difficulties and a possible way forward to obtaining conservation laws for the two-dimensional wall jet for a power-law fluid will be discussed in Section 7.3. The conclusions will be drawn in Section 7.4.

7.2 Conservation laws for the two-dimensional wall jet

In Chapter 4 the system of partial differential equations (4.2.1) and (4.2.2) were transformed to a third order partial differential equation for the stream function (4.2.49). Firstly a multiplier of the form $\Lambda = \Lambda(x, y, \psi)$ was used. This was fruitless in deriving a conserved quantity for the two-dimensional wall jet for $n \neq 1$. The multiplier was simply a constant. We then extended our multiplier to include the higher derivatives, $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$. However it was found that for the case $n \neq 1$ and $n \neq 2$, $\Lambda = c_1$, where c_1 is a constant as for the first multiplier.

This multiplier will guarantee the elementary conserved vector. This conserved vector however is only sufficient to derive the conserved quantity for the two-dimensional free jet. We have been unable to derive a further conserved vector which would allow for a conserved quantity for the two-dimensional wall jet for a power-law fluid.

A plausible rationale for the failure of the multiplier method to obtain a conservation law for the wall jet for $n \neq 1$ could be due to the behaviour of the velocity gradient of the fluid. The modulus of the velocity gradient occurs in the momentum balance equation of a power-law fluid. The modulus sign must be correctly interpreted. Consider Figure 7.2.1 below.

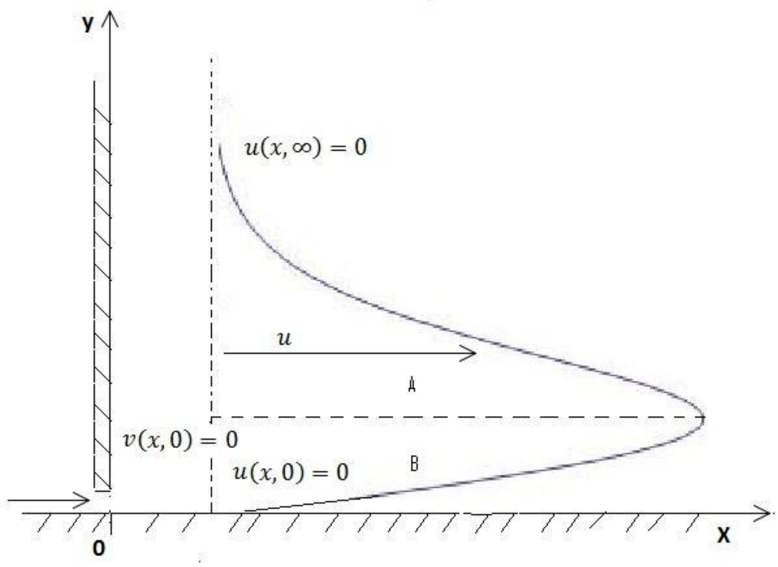

Figure 7.2.1: Velocity profile for a two-dimensional wall jet.

In Figure 7.2.1 in segment B, we note that as the value of y increases the velocity in the x-direction ,u, increases and therefore $\partial u/\partial y > 0$. However, at a specific value of y the velocity in the x-direction begins to decrease with an increase in y and $\partial u/\partial y < 0$ as shown in segment A. A possible approach to obtaining the conservation laws for the two-dimensional wall jet, would be to calculate separately the conservation laws for each segment A and B. The equation governing the flow in segment B is identical the third order partial differential equation for a liquid jet, whereas segment A is modelled as the two-dimensional free jet. The major problem with the approach is determining the turning point. This point can only be established once the two-dimensional wall jet has been solved completely.

However, the equations are the same across the jet for odd positive integer values of n. We considered the case for $n = 3$ and found that the multiplier still resulted in $\Lambda = c_1$. Consequently, the change of the momentum equation across each segment is not a contributing factor for the failure to obtain a second conservation laws for the two-dimensional wall jet.

7.3 A possible way forward

There are various other methods to obtain conservation laws apart from the multiplier method. A reasonable alternative would be to apply the Partial Noether approach developed by Kara and Mahomed [9], where the conservation laws can be derived once a partial Lagrangian for the partial differential equation has been constructed.

Further, more powerful computer algebra software might be useful in that we can extend the multipliers to be functions of even higher derivatives than what was used above.

7.4 Conclusion

The derivation of conservation laws for the two-dimensional wall jet for a non-Newtonian power-law fluid is an open question. The multiplier approach, using a multiplier of the form $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$, has failed to provide a conservation law. Speculation arises as to the reason a conserved vector is lost when moving from a Newtonian fluid to a non-Newtonian power-law fluid in the two-dimensional wall jet. Initially we postulated that the change in the velocity profile might be the foremost reason. However, for odd positive integer values of the parameter n, the governing equations for both segments in Figure 7.2.1 are identical and yet a non-constant multiplier was not found. Subsequently we deduced that the change in the sign of the velocity gradient is insignificant in not deriving the second conservation law.

Naz et al. [11] have outlined eight other different approaches for determining conservation laws, and these other approaches might prove beneficial. Further, more efficient computing power could prove useful, as the calculations by hand are extremely expensive.

Chapter 8

Conclusions

8.1 Mathematical model and conservation laws

This dissertation was concerned with the incompressible flow of non-Newtonian power-law fluids in various jet flow models. The two-dimensional free, liquid and wall jets were the jets under study. The development of the mathematical model governing the power-law fluid in these jet models depended on defining a suitable Reynolds number and calculating the magnitude of an effective viscosity. It was found that the form of the Reynolds number for a Newtonian fluid and for non-Newtonian fluids were the same. Further, we considered the boundary layer approximation where the rate of change normal to the boundary is much greater than the rate of change parallel to the boundary.

The partial differential equations governing the jet flows differed due to the sign of the velocity gradient. The free jet had different equations in the upper and lower half planes, but for this dissertation only the upper half plane was considered. The essential difference between the two-dimensional jets was the boundary conditions, and these boundary conditions are fundamental in deriving conserved quantities.

The conservation laws were derived using the multiplier approach. This method proved to be efficient in obtaining multipliers due to its systematic construct. The multiplier approach was applied to the two-dimensional free jet for both the velocity components and the stream function. The multiplier method gave two conserved vectors for the velocity components and a single conserved vector for the stream function. The two conserved vectors derived when using the velocity components allowed us to derive the conserved quantities for the two-dimensional free and liquid jets in terms of the velocity components. However, since the multiplier method only yielded the elementary conserved vector when implemented on the stream function formulation, we were only able to derive the conserved vector for the two-dimensional free jet in terms of the stream function. For a Newtonian fluid, the multiplier approach gave two conserved vectors in terms of the stream function. We derived the conserved quantity for the two-dimensional free jet using the elementary conserved vector and the conserved quantity for the two-dimensional wall jet using the remaining conserved vector.

In this dissertation, we explored plausible avenues to determine the reason for the failure to derive a second conservation law for the two-dimensional wall jet. We discussed how the change in the velocity profile might be the reason for not obtaining a second conserved vector and thus a conserved quantity. For odd positive integer values of n the momentum equation is the same regardless of the sign of the velocity gradient, yet we failed to derive the conserved vector, and concluded that the velocity profile is not important. Alternately, we endeavoured to extend the multiplier to be a function of even higher derivatives and again we failed to find a second conservation law for the two-dimensional wall jet. There are other approaches one could utilise to derive conservation laws, or even use more powerful algebra software.

8.2 Symmetry analysis and parametric solutions

The equations governing non-Newtonian fluids in general are extremely difficult to solve due to their high-order nonlinearity. However, Lie group analysis has proved efficient in obtaining closed form solutions to many of these complicated equations.

For this dissertation, various approaches to obtain group invariant solutions were used. We solved the two-dimensional free jet using similarity solutions for both the velocity components and the stream function. Furthermore, the same problem was solved by associating a Lie point symmetry with a conserved vector and using this associated Lie point symmetry to derive an invariant solution. This method proved to be more efficient than calculating the Lie point symmetries of the partial differential equation. However, this method was only applied to the third order partial differential equation for the stream function, since the system of equations for the velocity components fails to provide sufficient information.

The analytical solution was derived in parametric form. The two-dimensional free jet was solved completely over the three ranges of n, $0 < n < 1/2$, $n = 1/2$ and $1/2 < n < \infty$. The free jet had distinctive features which distinguished a shear thickening fluid from a shear thinning fluid. For a shear thickening fluid the whole of the fluid region is not set in motion by the free jet. The region of entrained fluid about the axis of symmetry becomes narrower as the value of n for the shear thickening fluid is increased.

A similarity reduction for the two-dimensional liquid jet was obtained. In addition we calculated the Lie point symmetries of the system of equations for the velocity components. The method of associating a Lie point symmetry with a conserved vector proved inefficient. Instead, we used the derived Lie point symmetry and associated the constants of the linearly independent operators with a conserved vector to deduce an invariant solution. As for the two-dimensional free jet, the two-dimensional liquid jet was solved parametrically. The liquid jet was solved only for $1/2 < n < \infty$ and unlike the free jet the solution does not exist for $n = 1/2$. The range $0 < n < 1/2$ requires further investigation. The dependence of the thickness of the jet could easily be determined from the parametric solution. The parametric solution was a very convenient form to investigate the physical properties of both the free jet and the liquid jet.

The solution of the two-dimensional wall jet for a non-Newtonian power law fluid remains an open question and requires further attention.

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