On the Rogers-Ramanujan identities and partition congruences

by

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

In this dissertation, we study the Rogers-Ramanujan identities and partition congruences. The original Rogers-Ramanujan identities are proved analytically via Bailey's construction. To the related Rogers-Ramanujan identities, our approach to the proof is via partition analysis and Bailey's construction. The summation part of these Rogers-Ramanujan identities is established using partition analysis.

Our work on congruences starts with a revisit of the popular Ramanujan's congruences for the unrestricted partition function. The Atkin-Swinnerton-Dyer congruences for moduli 5 and 7 are obtained and the Ramanujan's most beautiful identity is proved. The proof techniques for Ramanujan's most beautiful identity are extended to another version in modulo 7.

As part of the contribution to knowledge, a recurrence formula for the parity of the number of 2-color partitions of $2n$ in which one of the colors appears only in parts that are even is derived.

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Chapter 1

Introduction

1.1 q-Series and Ramanujan theta functions

Throughout this dissertation, the following standard notation for q-series shall be followed and also assume $q \leq 1$.

$$(a)_n := (a;q)_n := \prod_{i=0}^{n-1}(1-aq^i) = \prod_{i=0}^{\infty}\frac{1-aq^i}{1-aq^{i+n}} = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1.1}$$

where $(a;q)_0 = 1$.

In a case where the base q of the series is understood, we shall use the notation $(a)_n$ and in a case where more than one base occurs, then the full notation $(a;q)_n$ shall be used in order to distinguish base q expressions from others. We shall also use frequently the following most useful identities in this section:

$$(-a^{-1}q^{1-n};q)_n = a^{-n}q^{-\binom{n}{2}}(-a)_n, \tag{1.2}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \tag{1.3}$$

$$(a;q)_{n-k} = \frac{(a;q)_n(-qa^{-1})^k q^{\binom{k}{2}-nk}}{(a^{-1}q^{1-n};q)_k}, \tag{1.4}$$

$$(q;q)_{2n} = (q^2;q^2)_n(q;q^2)_n, \tag{1.5}$$

$$(q^2;q^2)_n = (q;q)_n(-q;q)_n, \tag{1.6}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2}}{(q^2;q^2)_m} = (q;q^2)_\infty, \tag{1.7}$$

$$(tq^{2m};q)_\infty = \frac{(t,q)_\infty}{(t;q^2)_m(tq;q^2)_m}, \tag{1.8}$$

$$\frac{(t)_\infty}{(tq;q^2)_\infty} = (t;q^2)_\infty, \tag{1.9}$$

$$(q^{-n})_k = \frac{(q)_n}{(q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}. \tag{1.10}$$

Next, we prove the q-binomial theorem which provides one of the most significant formulae in the theory of q-series. It was discovered independently by Cauchy, Hein, Gauss as well as a number of other mathematicians. We will also find a special case of the q-binomial theorem which was given by Euler and Rothe.

Theorem 1 (q-Binomial theorem)**.** *For $|q| < 1$, $|t| < 1$, we have*

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(at)_\infty}{(t)_\infty}. \tag{1.11}$$

The series in (1.11) is an example of a basic hypergeometric series.

Proof. We first define the following function

$$f_a(t) := \sum_{j=0}^{\infty} \frac{(a)_j}{(q)_j} t^j \tag{1.12}$$

and find that

$$\begin{aligned}
\frac{f_a(t)-f_a(qt)}{t} &= \sum_{j=0}^{\infty}\left(\frac{(a)_j}{(q)_j}t^{j-1}-\frac{(a)_j}{(q)_j}q^j t^{j-1}\right)\\
&= \sum_{j=0}^{\infty}\frac{(a)_j}{(q)_j}(1-q^j)t^{j-1}\\
&= \sum_{j=1}^{\infty}\frac{(1-a)(aq)_{j-1}(1-q^j)}{(q)_{j-1}(1-q^j)}t^{j-1}\\
&= \sum_{j=1}^{\infty}\frac{(1-a)(aq)_{j-1}}{(q)_{j-1}}t^{j-1}\\
&= (1-a)f_{aq}(t). \qquad (1.13)
\end{aligned}$$

If we start rearranging (1.13), we have the following

$$f_a(t)-f_a(qt)=t(1-a)f_{aq}(t). \qquad (1.14)$$

Therefore, we show that

$$\begin{aligned}
f_a(t)-f_{aq}(t) &= \sum_{j=0}^{\infty}\frac{(a)_j}{(q)_j}t^j-\sum_{j=0}^{\infty}\frac{(aq)_j}{(q)_j}t^j\\
&= \sum_{j=1}^{\infty}\frac{(1-a)(aq)_{j-1}}{(q)_j}t^j-\sum_{j=1}^{\infty}\frac{(aq)_{j-1}(1-aq^j)}{(q)_j}t^j\\
&= \sum_{j=1}^{\infty}\frac{(aq)_{j-1}}{(q)_j}t^j[(1-a)-(1-aq^j)]\\
&= \sum_{j=1}^{\infty}\frac{(aq)_{j-1}}{(q)_{j-1}(1-q^j)}t^j[-a(1-q^j)]\\
&= -a\sum_{j=1}^{\infty}\frac{(aq)_{j-1}}{(q)_{j-1}}t^j\cdot t^{-1}t\\
&= -at\sum_{j=1}^{\infty}\frac{(aq)_{j-1}}{(q)_{j-1}}t^j\cdot t^{-1}\\
&= -atf_{aq}(t).
\end{aligned}$$

Thus,

$$f_a(t) - f_{aq}(t) = -atf_{aq}(t)$$

which gives

$$\begin{aligned} f_a(t) &= f_{aq}(t) - atf_{aq}(t) \\ &= (1-at)f_{aq}(t). \end{aligned} \tag{1.15}$$

From (1.15), we have

$$f_{aq}(t) = \frac{f_a(t)}{(1-at)}$$

which, when substituted in (1.14), yields

$$f_a(t) - f_a(qt) = t(1-a)\frac{f_a(t)}{(1-at)},$$

i.e.

$$f_a(t)(1-t) = f_a(qt)(1-at)$$

so that

$$f_a(t) = \frac{f_a(qt)(1-at)}{(1-t)}. \tag{1.16}$$

Finally, iterating (1.16) n times gives

$$f_a(t) = \frac{(at)_n}{(t)_n} f_a(q^n t)$$

and if we let $n \to \infty$, we obtain

$$\begin{aligned} f_a(t) &= \frac{(at)_\infty}{(t)_\infty} f_a(0) \\ &= \frac{(at)_\infty}{(t)_\infty}. \end{aligned}$$

Since $f_a(0) = 1$ from (1.12), this concludes the proof.

□

The following corollary, known as the Euler's identity, is a special case of (1.11).

Corollary 1. *For $|t| < 1$ and $|q| < 1$ we have,*

$$\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{(q)_n} = \prod_{n=0}^{\infty}(1 + tq^n) = (-t; q)_\infty. \tag{1.17}$$

Proof. To obtain (1.17) we replace a by $\frac{a}{b}$ and t by bz in (1.11); hence by $|bz| < 1$,

$$\sum_{n=0}^{\infty} \frac{(1 - \frac{a}{b})(1 - \frac{a}{b}q) \cdots (1 - \frac{a}{b}q^{n-1})(bz)^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{n=0}^{\infty} \frac{1 - \frac{a}{b}bzq^n}{1 - bzq^n},$$

i.e.

$$\sum_{n=0}^{\infty} \frac{(\frac{b-a}{b})(\frac{b-aq}{b}) \cdots (\frac{b-aq^{n-1}}{b})(bz)^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{n=0}^{\infty} \frac{1 - azq^n}{1 - bzq^n}$$

so that

$$\sum_{n=0}^{\infty} \frac{(b - a)(b - aq) \cdots (b - aq^n)}{(1 - q)(1 - q^2) \cdots (1 - q^n)}(z)^n = \prod_{n=0}^{\infty} \frac{1 - azq^n}{1 - bzq^n},$$

and if we substitute $b = 0$ and $a = -1$, we get

$$\sum_{n=0}^{\infty} \frac{qq^2 \cdots q^{n-1}}{(1 - q)(1 - q^2) \cdots (1 - q^n)} z^n = \prod_{n=0}^{\infty}(1 + zq^n),$$

i.e.

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(q)_n} = \prod_{n=0}^{\infty}(1 + zq^n).$$

□

E. Heine [8], is said to be the first person to generalize Gauss's hypergeometric series to q-hypergeometric series. Heine's fundamental transformation theorem is another application of the q-binomial theorem and can be directly proven from the q-binomial theorem as follows.

Theorem 2 (Heine's transformation)**.** *For* $|q| < 1$, $|t| < 1$ *and* $|b| < 1$ *we have*

$$\sum_{j=0}^{\infty} \frac{(a)_j (b)_j t^j}{(c)_j (q)_j} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(\frac{c}{b})_m (t)_m b^m}{(q)_m (at)_m}. \tag{1.18}$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{(a)_j (b)_j t^j}{(c)_j (q)_j} &= \sum_{j=0}^{\infty} (a)_j \cdot \frac{(cq^j)_\infty}{(c)_\infty} \cdot \frac{(b)_\infty}{(bq^j)_\infty} \cdot \frac{1}{(q)_j} t^j \quad \text{(by (1.1))} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{j=0}^{\infty} \frac{(a)_j t^j}{(q)_j} \cdot \frac{(cq^j)_\infty}{(bq^j)_\infty} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{j=0}^{\infty} \frac{(a)_j t^j}{(q)_j} \cdot \sum_{k=0}^{\infty} \frac{(\frac{c}{b})_k (bq^j)^k}{(q)_k} \quad \text{(by (1.11))} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{k=0}^{\infty} \frac{(\frac{c}{b})_k b^k}{(q)_k} \sum_{j=0}^{\infty} \frac{(a)_j t^j q^{kj}}{(q)_j} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{k=0}^{\infty} \frac{(\frac{c}{b})_k b^k}{(q)_k} \frac{(atq^k)_\infty}{(tq^k)_\infty} \quad \text{(by (1.11))} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{k=0}^{\infty} \frac{(\frac{c}{b})_k b^k}{(q)_k} \frac{(at)_\infty}{(at)_k} \frac{(t)_k}{(t)_\infty} \quad \text{(by (1.1))} \\
&= \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{k=0}^{\infty} \frac{(\frac{c}{b})_k (t)_k b^k}{(q)_k (at)_k}.
\end{aligned}$$

□

The following example demonstrates an application of q-series and Heine's transformation.

Example 1.

$$\sum_{n=0}^{\infty} \frac{(a)_n (b;q^2)_n t^n}{(q)_n (atb;q^2)_n} = \frac{(at;q^2)_\infty (bt;q^2)_\infty}{(t;q^2)_\infty (abt;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(a;q^2)_m (b;q^2)_m (tq)^m}{(q^2;q^2)_m (bt;q^2)_m}. \tag{1.19}$$

Proof.

$$\sum_{n=0}^{\infty} \frac{(a)_n (b;q^2)_n t^n}{(q)_n (atb;q^2)_n} = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \cdot \frac{(b;q^2)_\infty}{(bq^{2n};q^2)_\infty} \cdot \frac{(atbq^{2n};q^2)_\infty}{(abt;q^2)_\infty} t^n \quad \text{(by (1.1))}$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \frac{(atbq^{2n};q^2)_\infty}{(bq^{2n};q^2)_\infty} t^n$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n \sum_{j=0}^{\infty} \frac{(at;q^2)_j}{(q^2;q^2)_j} (bq^{2n})^j \quad \text{(by (1.11))}$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(at;q^2)_j}{(q^2;q^2)_j} b^j \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n q^{2jn}$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(at;q^2)_j}{(q^2;q^2)_j} b^j \frac{(atq^{2j})_\infty}{(tq^{2j})_\infty} \quad \text{(by (1.11))}$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(at;q^2)_j}{(q^2;q^2)_j} b^j \cdot \frac{(at)_\infty}{(at;q^2)_j (atq;q^2)_j} \cdot \frac{(t;q^2)_j (tq;q^2)_j}{(t)_\infty} \quad \text{(by (1.8))}$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \frac{(at)_\infty}{(t)_\infty} \sum_{j=0}^{\infty} \frac{(t;q^2)_j (tq;q^2)_j}{(q^2;q^2)_j (atq;q^2)_j} b^j$$

$$= \frac{(b;q^2)_\infty}{(abt;q^2)_\infty} \frac{(at)_\infty}{(t)_\infty} \frac{(tq;q^2)_\infty (bt;q^2)_\infty}{(atq;q^2)_\infty (b;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(a;q^2)_j (b;q^2)_j}{(q^2;q^2)_j (bt;q^2)_j} (tq)^j \quad \text{(by (1.18))}$$

$$\left(\text{by (1.9) } \frac{(at)_\infty}{(atq;q^2)_\infty} = (at;q^2)_\infty \text{ and } \frac{(tq;q^2)_\infty}{(t)_\infty} = \frac{1}{(t;q^2)_\infty}\right)$$

$$= \frac{(at;q^2)_\infty (bt;q^2)_\infty}{(t;q^2)_\infty (abt;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(a;q^2)_m (b;q^2)_m (tq)^m}{(q^2;q^2)_m (bt;q^2)_m}.$$

□

The following proposition will be relevant when proving some Rogers-Ramanujan's identities.

Proposition 1.

$$\sum_{n=0}^{\infty} \frac{(b)_{2n} t^{2n}}{(q^2;q^2)_n} = \frac{(-tb)_\infty}{(-t)_\infty} \sum_{m=0}^{\infty} \frac{(b)_m t^m}{(q)_m (-tb)_m}. \tag{1.20}$$

Proof.

$$\sum_{n=0}^{\infty} \frac{(b)_{2n} t^{2n}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(b)_\infty}{(bq^{2n})_\infty} \frac{t^{2n}}{(q^2;q^2)_n} \quad \text{(by (1.1))}$$

$$= (b)_\infty \sum_{n=0}^{\infty} \frac{t^{2n}}{(bq^{2n})_\infty (q^2;q^2)_n}$$

$$
\begin{aligned}
&= (b)_\infty \sum_{n=0}^{\infty} \frac{t^{2n}}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{(bq^{2n})^m}{(q;q)_m} \quad \text{(by (1.11))} \\
&= (b)_\infty \sum_{m=0}^{\infty} \frac{b^m}{(q;q)_m} \sum_{n=0}^{\infty} \frac{t^{2n} q^{2mn}}{(q^2;q^2)_n} \\
&= (b)_\infty \sum_{m=0}^{\infty} \frac{b^m}{(q;q)_m} \frac{1}{(t^2 q^{2m};q^2)_\infty} \text{ (by (1.11))} \\
&= (b)_\infty \sum_{m=0}^{\infty} \frac{b^m}{(q;q)_m} \frac{1}{(tq^m;q)_\infty (-tq^m;q)_\infty} \text{ (by (1.6))} \\
&= (b)_\infty \sum_{m=0}^{\infty} \frac{b^m}{(q;q)_m} \frac{(t)_m}{(t)_\infty} \frac{(-t)_m}{(-t)_\infty} \quad \text{(by (1.1))} \\
&= \frac{(b)_\infty}{(-t)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(t)_m (-t)_m}{(q)_m} b^m \\
&= \frac{(b)_\infty}{(-t)_\infty (t)_\infty} \frac{(t)_\infty (-tb)_\infty}{(b)_\infty} \sum_{m=0}^{\infty} \frac{(b)_m}{(q)_m (-tb)_m} t^m \quad \text{(by (1.18))} \\
&= \frac{(-tb)_\infty}{(-t)_\infty} \sum_{m=0}^{\infty} \frac{(b)_m t^m}{(q)_m (-tb)_m}.
\end{aligned}
$$

□

We now derive a q-analogue of the Chu-Vandermonde theorem and record a special case that will be useful when proving Rogers-Ramanujan's identities using Bailey chains.

Theorem 3 (q-analogue of Chu-Vandermonde Theorem)**.** *For each nonnegative integer n,*

$$
\sum_{j=0}^{\infty} \frac{(q^{-n})_j (b)_j}{(c)_j (q)_j} \left(\frac{cq^n}{b} \right)^j = \frac{(c/b)_n}{(c)_n} \tag{1.21}
$$

and

$$
\sum_{j=0}^{\infty} \frac{(q^{-n})_j (b)_j}{(c)_j (q)_j} q^j = \frac{(c/b)_n b^n}{(c)_n}. \tag{1.22}
$$

Proof. To prove (1.21),

$$
\sum_{j=0}^{\infty} \frac{(q^{-n})_j (b)_j}{(c)_j (q)_j} \left(\frac{cq^n}{b} \right)^j = \frac{(b)_\infty (\frac{c}{b})_\infty}{(c)_\infty (\frac{cq^n}{b})_\infty} \sum_{j=0}^{\infty} \frac{(\frac{c}{b})_j (\frac{cq^n}{b})_j}{(q)_j (\frac{c}{b})_j} b^j \text{ (by (1.18))}
$$

$$= \frac{(b)_\infty(\frac{c}{b})_\infty}{(c)_\infty(\frac{cq^n}{b})_\infty}\sum_{j=0}^{\infty}\frac{(\frac{cq^n}{b})_j}{(q)_j}b^j$$

$$= \frac{(b)_\infty(\frac{c}{b})_\infty}{(c)_\infty(\frac{cq^n}{b})_\infty}\frac{(cq^n)_\infty}{(b)_\infty} \text{ (by (1.11))}$$

$$= \frac{(\frac{c}{b})_\infty(cq^n)_\infty}{(c)_\infty(\frac{cq^n}{b})_\infty}$$

$$= \frac{(\frac{c}{b})_n}{(c)_n}.$$

To prove (1.22), we simply reverse the order of summation on the left-hand side of (1.21).

□

Theta functions are an integral part of partition theory. In the following definition, we define the Ramanujan's theta function.

Definition 1. *The Ramanujan's theta function is given by*

$$\begin{aligned} f(a,b) &:= 1+\sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}(a^n+b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}} \quad \textit{where } |ab|<1. \end{aligned} \tag{1.23}$$

Some useful properties of the theta function are given in the following proposition.

Proposition 2. *We have*

$$f(a,b) = f(b,a), \tag{1.24}$$

$$f(1,a) = 2f(a,a^3), \tag{1.25}$$

$$f(-1,a) = 0. \tag{1.26}$$

Proof. For (1.24),

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2}+\frac{2n}{2}} b^{\frac{n(n-1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} (ab)^{\frac{n(n-1)}{2}} a^{n} \\
&= \sum_{n=-\infty}^{\infty} (ab)^{\frac{n(n+1)}{2}} a^{-n} \\
&= \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}} \\
&= f(b,a).
\end{aligned}
$$

For (1.25),

$$
\begin{aligned}
f(1,a) &= \sum_{n=-\infty}^{\infty} 1^{\frac{n(n+1)}{2}} a^{\frac{n(n-1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} a^{\frac{n(n-1)}{2}} \\
&= \sum_{n=1}^{\infty} a^{\frac{n(n-1)}{2}} + \sum_{n=-\infty}^{0} a^{\frac{n(n-1)}{2}} \\
&= \sum_{n=1}^{\infty} a^{\frac{n(n-1)}{2}} + \sum_{n=0}^{\infty} a^{\frac{n(n+1)}{2}} \\
&\quad \text{(in the first summation, let } n = n+1\text{)} \\
&= 2\sum_{n=0}^{\infty} a^{\frac{n(n+1)}{2}} \\
&\quad \text{(let } n = 2n\text{)} \\
&= 2\sum_{n=0}^{\infty} a^{2n^2+n} \\
&= 2f(a,a^3).
\end{aligned}
$$

For (1.26),

$$
\begin{aligned}
f(-1,a) &= \sum_{n=-\infty}^{\infty} (-1)^{\frac{n(n+1)}{2}} a^{\frac{n(n-1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} (-1)^{\frac{n(n-1)}{2}+\frac{2n}{2}} a^{\frac{n(n-1)}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} (-1)^n (-a)^{\frac{n(n-1)}{2}} \\
&= \sum_{n=0}^{\infty} (-1)^n (-a)^{\frac{n(n-1)}{2}} + \sum_{n=-\infty}^{-1} (-1)^n (-a)^{\frac{n(n-1)}{2}} \\
&= \sum_{n=0}^{\infty} (-1)^n (-a)^{\frac{n(n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^{-n} (-a)^{\frac{n(n+1)}{2}} \\
&= \sum_{n=0}^{\infty} (-1)^n (-a)^{\frac{n(n-1)}{2}} + \sum_{n-1=1}^{\infty} (-1)^{-(n-1)} (-a)^{\frac{(n-1)(n-1+1)}{2}} \\
&= \sum_{n=0}^{\infty} (-1)^n (-a)^{\frac{n(n-1)}{2}} + \sum_{n=2}^{\infty} (-1)^{-(n-1)} (-a)^{\frac{n(n-1)}{2}} \\
&= \sum_{n=0}^{\infty} (-1)^n (-a)^{\frac{n(n-1)}{2}} + \sum_{n=0}^{\infty} (-1)^{-(n-1)} (-a)^{\frac{n(n-1)}{2}} \\
&= 0.
\end{aligned}
$$

□

We then move to establish one of the most important powerful identities, the Jacobi's triple product which is arguably the most useful and celebrated identity as it can be used to prove theta function formulae and identities of Rogers-Ramanujan type. It was introduced by Carl Gustav Jacob Jacobi, who proved it in 1829 in his work [18].

Theorem 4. *For $z \neq 0$ and $|q| < 1$,*

$$
\begin{aligned}
\sum_{n=\infty}^{\infty} z^n q^{n^2} &= \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) \\
&= (q^2; q^2)_\infty (-zq; q^2)_\infty (-z^{-1}q; q^2)_\infty. \qquad (1.27)
\end{aligned}
$$

Proof. In (1.17), if we substitute q^2 for q and zq for t, we get

$$
\sum_{k=0}^{\infty} \frac{(zq)^k q^{2(\frac{k(k-1)}{2})}}{(q^2; q^2)_k} = \prod_{k=0}^{\infty} (1 + (zq)q^{2k}),
$$

i.e.

$$
\sum_{k=0}^{\infty} \frac{z^k q^k q^{k^2-k}}{(q^2; q^2)_k} = \prod_{k=0}^{\infty} (1 + zq^{2k+1}),
$$

which finally gives

$$\sum_{k=0}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k} = \prod_{k=0}^{\infty}(1+zq^{2k+1}). \tag{1.28}$$

If we expand the right-hand side of (1.28), we get

$$(1+zq)(1+zq^3)(1+zq^5)(1+zq^7)\cdots = \sum_{k=0}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k}. \tag{1.29}$$

Suppose that $q \neq 0$ in (1.29) and replace z by zq^{-2n} with $n > 0$, we then get

$$(1+zq^{-2n+1})(1+zq^{-2n+3})(1+zq^{-2n+5})\cdots(1+zq^{-1})(1+zq)(1+zq^3)\cdots = \sum_{k\geq 0} \frac{z^k q^{k^2-2nk}}{(q^2;q^2)_k}. \tag{1.30}$$

Next, we multiply (1.30) by q^{n^2} and using the fact that $1+3+5+\cdots+(2n-1) = n^2$, we have

$$(q^{2n-1}+z)(q^{2n-3}+z)\cdots(q+z)(1+zq)(1+zq^3)\cdots = \sum_{k\geq 0} \frac{z^k q^{k^2-2nk+n^2}}{(q^2;q^2)_k},$$

which can be written as

$$(z+q)(z+q^3)\cdots(z+q^{2n-1})(1+zq)(1+zq^3)\cdots = \sum_{k\geq 0} \frac{z^k q^{(k-n)^2}}{(q^2;q^2)_k}. \tag{1.31}$$

Suppose that $z \neq 0$. Dividing (1.31) by z^n, we obtain

$$\begin{aligned}(1+z^{-1}q)(1+z^{-1}q^3)\cdots(1+z^{-1}q^{2n-1})(1+zq)(1+zq^3)\cdots &= \sum_{k\geq 0} \frac{z^{k-n} q^{(k-n)^2}}{(q^2;q^2)_k} \\ &= \sum_{k=-n}^{\infty} \frac{z^n q^{k^2}}{(q^2;q^2)_{n+k}}\end{aligned} \tag{1.32}$$

which can be written as

$$(-z^{-1}q;q^2)_n(-zq;q^2)_\infty = \sum_{k=-n}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k}. \tag{1.33}$$

We earlier assumed that $q \neq 0$ in (1.29) until we got to (1.33), incidentally, we also note that this is true for $q = 0$, reason being;

$$\begin{aligned}\sum_{k=-n}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k} &= \sum_{k=-n}^{-1} \frac{z^k q^{k^2}}{(q^2;q^2)_k} + \frac{z^0 q^{0^2}}{(q^2;q^2)_0} + \sum_{k=1}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k} \\ &= \sum_{k=1}^{n} \frac{z^{-k} q^{(-k)^2}}{(q^2;q^2)_{-k}} + 1 + \sum_{k=1}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k},\end{aligned}$$

and using the identity $(a;q)_{-k} = \frac{(-\frac{q}{a})^k q^{\binom{k}{2}}}{(\frac{q}{a};q)_k}$ for a non-negative integer k, we get

$$\begin{aligned}\sum_{k=-n}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k} &= \sum_{k=1}^{n} \frac{z^{-k} q^{k^2}(1,q)_k}{(-1)^k q^{\binom{k}{2}}} + 1 + \sum_{k=1}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k} \\ &= 0 + 1 + \sum_{k=1}^{\infty} \frac{z^k q^{k^2}}{(q^2;q^2)_k}.\end{aligned}$$

This equals 1 when $q = 0$. The left-hand side of (1.33) also equals 1 when $q = 0$.

If we let $n \to \infty$ in (1.33), then for fixed k, we obtain,

$$(q^2;q^2)_{n+k} \to (q^2;q^2)_\infty,$$

and so

$$(-z^{-1}q;q^2)_\infty(-zq;q^2)_\infty = \sum_{k=-\infty}^{\infty} \frac{z^k q^{(k)^2}}{(q^2;q^2)_\infty},$$

which can be written as

$$(-z^{-1}q;q^2)_\infty(-zq;q^2)_\infty(q^2;q^2)_\infty = \sum_{k=-\infty}^{\infty} z^k q^{k^2}.$$

Hence, (1.27) is proven.

□

In Ramanujan's notation, the three most important special cases of (1.23) and (1.17) are $f(q,q)$, $f(q,q^3)$ and $f(-q;-q^2)$. Observe that

$$\begin{aligned} f(q,q) &= \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}+\frac{n}{2}+\frac{n^2}{2}-\frac{n}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{\frac{2n^2}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{n^2}, \end{aligned}$$

$$\begin{aligned} f(q,q^3) &= \sum_{n=\infty}^{\infty} (q)^{\frac{n(n+1)}{2}} (q)^{3\frac{n(n-1)}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}+\frac{n}{2}+\frac{3n^2}{2}-\frac{3n}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{\frac{4n^2}{2}-\frac{2n}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{2n^2-n} \end{aligned}$$

and

$$\begin{aligned} f(-q,-q^2) &= \sum_{n=-\infty}^{\infty} (-q)^{\frac{n(n+1)}{2}} (-q^2)^{\frac{n(n-1)}{2}} \\ &= \sum_{n=-\infty}^{\infty} (-1)^{\frac{n(n+1)}{2}+\frac{n(n-1)}{2}} q^{\frac{n(n+1)}{2}+\frac{2n(n-1)}{2}} \\ &= \sum_{n=-\infty}^{\infty} (-1)^{\frac{n^2+n^2}{2}} q^{\frac{n^2+n+2n^2-2n}{2}} \\ &= \sum_{n=-\infty}^{\infty} (-1)^{n^2} (q)^{\frac{n(3n-1)}{2}} \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}. \qquad (1.34)$$

Using (1.27), the three special functions above can be expressed as infinite products. More explicitly,

$$\begin{aligned} f(q,q) &= (-q;qq)_\infty(-q;qq)_\infty(qq;qq)_\infty \\ &= (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty \\ &= (-q;q^2)_\infty{}^2(q^2;q^2)_\infty, \end{aligned}$$

$$\begin{aligned} f(q;q^3) &= (-q;qq^3)_\infty(-q^3;qq^3)_\infty(qq^3;qq^3)_\infty \\ &= (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \end{aligned}$$

and

$$\begin{aligned} f(-q;-q^2) &= (-(-q);(-q)(-q^2))_\infty \times (-(-q^2);(-q)(-q^2))_\infty \\ &\quad \times (-(-q)(-q^2);(-q)(-q^2))_\infty \\ &= (q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty \\ &= (q,q)_\infty. \qquad (1.35) \end{aligned}$$

The pentagonal number theorem, originally due to Euler, relates the product and a power series in which the powers are pentagonal numbers. It links (1.34) and (1.35) and below we formally state the theorem with its proof.

Theorem 5 (Euler's Pentagonal Number Theorem)**.** *For* $|q| < 1$*, we have*

$$(q;q)_\infty = \sum_{-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}}. \qquad (1.36)$$

Proof. If we replace q by $q^{\frac{3}{2}}$ in (1.27), we find that

$$\sum_{n=\infty}^{\infty} z^n (q^{\frac{3}{2}})^{n^2} = ((q^{\frac{3}{2}})^2;(q^{\frac{3}{2}})^2)_\infty(-z(q^{\frac{3}{2}});(q^{\frac{3}{2}})^2)_\infty(-z^{-1}(q^{\frac{3}{2}});(q^{\frac{3}{2}})^2)_\infty$$

which simplifies to

$$\sum_{n=\infty}^{\infty} z^n (q^{\frac{3}{2}})^{n^2} = (q^3;q^3)_\infty(-z(q^{\frac{3}{2}});q^3)_\infty(-z^{-1}(q^{\frac{3}{2}});q^3)_\infty.$$

If we now set $z = -q^{\frac{1}{2}}$, we get

$$\sum_{n=\infty}^{\infty}(-q^{\frac{1}{2}})^n(q^{\frac{3}{2}})^{n^2} = (q^3;q^3)_\infty(-(-q^{\frac{1}{2}})(q^{\frac{3}{2}});q^3)_\infty(-(-q^{\frac{1}{2}})^{-1}(q^{\frac{3}{2}});q^3)_\infty,$$

i.e.

$$\begin{aligned}\sum_{n=\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} &= (q^3;q^3)_\infty(q^2;q^3)_\infty(q;q^3)_\infty \\ &= (q;q)_\infty.\end{aligned}$$

□

Another consequence of Jacobi triple product is the following identity, also due to Jacobi, which gives the expansion of the cube of Euler's product.

Theorem 6. *For $|q| < 1$, we have*

$$(q;q)^3{}_\infty = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k^2+k}{2}} \tag{1.37}$$

Proof. From (1.27), we have

$$(-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty(q^2;q^2)_\infty = \sum_{-\infty}^{\infty} a^k q^{k^2}.$$

If we replace q by $q^{\frac{1}{2}}$ and replace a by $-aq^{\frac{1}{2}}$, we obtain

$$(a^{-1};q)_\infty(aq;q)_\infty(q;q)_\infty = \sum_{-\infty}^{\infty}(-1)^k a^k q^{\frac{k^2+k}{2}} \tag{1.38}$$

which can be written as

$$(1-a^{-1})(a^{-1}q;q)_\infty(aq;q)_\infty(q;q)_\infty = \sum_{k=0}^{\infty}(-1)^k(a^k - a^{-k-1})q^{\frac{k^2+k}{2}}. \qquad (1.39)$$

Next, we multiply by $a^{\frac{1}{2}}$ to obtain the identity, invariant under $a \to -a^{-1}$,

$$(a^{\frac{1}{2}} - a^{-\frac{1}{2}})(a^{-1}q;q)_\infty(aq;q)_\infty(q^2;q^2)_\infty = \sum_{k=0}^{\infty}(-1)^k(a^{k+\frac{1}{2}} - a^{-k-\frac{1}{2}})q^{\frac{k^2+k}{2}}. \qquad (1.40)$$

If we suppose $a \neq 1$ and divide by $(a^{\frac{1}{2}} - a^{-\frac{1}{2}})$, we obtain

$$\begin{aligned}(a^{-1}q;q)_\infty(aq;q)_\infty(q;q)_\infty &= 1 + \sum_{k=1}^{\infty}(-1)^k\left(\frac{a^{k+\frac{1}{2}} - a^{-k-\frac{1}{2}}}{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}\right)q^{\frac{k^2+k}{2}} \\ &= 1 + \sum_{k=1}^{\infty}(-1)^k(a^k + a^{k-1} + a^{k-2} + \cdots a^{-k})q^{\frac{k^2+k}{2}}. \qquad (1.41)\end{aligned}$$

If we let $a \to 1$, we obtain (1.37). $\square$

1.2 Weak form of Bailey's lemma

The concept of Bailey chains was first introduced by George Andrews to describe the iterative nature of Bailey's lemma. This iteration mechanism allows one to derive many q-series identities by reducing them to more elementary ones.

We now proceed to study a small variation on Bailey chains which concerns pairs of sequeces $\{\alpha_j\}_{j=0}^{\infty}$ and $\{\beta_j\}_{j=0}^{\infty}$ of rational functions of the variables a and q. They are said to form a Bailey pair, provided that for all $n \geq 0$,

$$\beta_j = \sum_{k=0}^{j} \frac{\alpha_k}{(q)_{j-k}(aq)_{j+k}}. \tag{1.42}$$

A limiting (weak form) of Bailey's lemma asserts that if (1.42) holds, then the following theorem follows:

Theorem 7 (Weak form of Bailey's lemma). *Let (α_j, β_j) be a Bailey pair. Then (subject to convergence conditions),*

$$\sum_{j=0}^{\infty} q^{j^2} a^j \beta_j = \frac{1}{(aq;q)_\infty} \sum_{j=0}^{\infty} q^{j^2} a^j \alpha_j. \tag{1.43}$$

Proof. See [11] for a proof.

□

The next lemma shows how $\{\alpha_j\}_{j=0}^{\infty}$ can be calculated directly in terms of $\{\beta_j\}_{j=0}^{\infty}$.

Lemma 1. *If (α_n, β_n) is a Bailey pair, then*

$$\alpha_n = \frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{(a;q)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q;q)_{n-j}}. \tag{1.44}$$

Proof. Recall (1.42) and note that the right-hand side of (1.44) is

$$\frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{(a;q)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q;q)_{n-j}}$$

$$= \frac{1-aq^{2n}}{1-a}\sum_{j=0}^{n}\frac{(a;q)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q;q)_{n-j}}\sum_{r=0}^{j}\frac{\alpha_r}{(q)_{j-r}(aq)_{j+r}}$$

$$= \frac{1-aq^{2n}}{1-a}\sum_{r=0}^{n}\alpha_r\sum_{j=r}^{n}\frac{(a)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}(q)_{j-r}(aq)_{j+r}}$$

(set $j=j+r$)

$$= \frac{1-aq^{2n}}{1-a}\sum_{r=0}^{n}\alpha_r\sum_{j=0}^{n-r}\frac{(a)_{n+j+r}(-1)^{n-j-r}q^{\binom{n-j-r}{2}}}{(q)_{n-j-r}(q)_j(aq)_{j+2r}}$$

We note that $(q)_{n-r}=(q)_{n-r-j}(q^{n-r-j+1})_j$ using $(a;q)_{n+k}=(a;q)_n(aq^n;q)_k)$

so that $(q)_{n-j-r}=\frac{(q)_{n-r}}{(q^{n-r-j+1})_j}$ and $(a)_{n+j+r}=(a)_{n+r}(aq^{n+r})_j$.

Similarly $(aq)_{j+2r}=(aq)_{2r}(aq^{2r+1})_j$. Also note that $\binom{n-j-r}{2}=\binom{n-r}{2}+\binom{j}{2}+j-(n-r)j$ which is equal to $\binom{n-r}{2}+\binom{j+1}{2}-(n-r)j$.

Then, we have

$$\begin{aligned}
&\frac{1-aq^{2n}}{1-a}\sum_{j=0}^{n}\frac{(a;q)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j}{(q;q)_{n-j}}\\
=&\frac{1-aq^{2n}}{1-a}\sum_{r=0}^{n}\alpha_r\sum_{j=0}^{n-r}\frac{(a)_{n+r}(aq^{n+r})_j(-1)^{n-r}(-1)^jq^{\binom{n-j-r}{2}}(q^{n-r-j+1})_j}{(q)_j(q)_{n-r}(aq)_{2r}(aq^{2r+1})_j}\\
=&\frac{1-aq^{2n}}{1-a}\sum_{r=0}^{n}\alpha_r\frac{(a)_{n+r}(-1)^{n-r}q^{\binom{n-r}{2}}}{(aq)_{2r}(q)_{n-r}}\\
&\times\sum_{j=0}^{n-r}\frac{(aq^{n+r})_j(-1)^j(q^{n-r-j+1})_jq^{\binom{j+1}{2}-(n-r)j}}{(q)_j(aq^{2r+1})_j}.
\end{aligned}\tag{1.45}$$

Using the fact that $(q^{-n})_k=\frac{(q)_n}{(q)_{n-k}}(-1)^kq^{\binom{k}{2}-nk}$ ((1.10), we observe that

$$\begin{aligned}
(q^{-(n-r)})_j &= \frac{(q)_{n-r}}{(q)_{n-r-j}}(-1)^jq^{\binom{j}{2}-(n-r)k}\\
&= \frac{(q)_{n-r-j}(1-q^{n-r-j+1})(1-q^{n-r+j+2})\dots(1-q^{n-r})}{(q)_{n-r-j}}(-1)^jq^{\binom{j}{2}-(n-r)k}\\
&= \frac{(q)_{n-r-j}(q^{n-r-j+1})_j}{(q)_{n-r-j}}(-1)^jq^{\binom{j}{2}-(n-r)k}
\end{aligned}$$

$$= (q^{n-r-j+1})_j(-1)^j q^{\binom{j}{2}-(n-r)k}$$

so that

$$(q^{n-r-j+1})_j(-1)^j q^{\binom{j+1}{2}-(n-r)k} = (q^{-(n-r)})_j q^j. \tag{1.46}$$

Denote the inner sum of (1.45) by $A(r,q)$. Thus

$$\begin{aligned} A(r,q) &= \sum_{j=0}^{n-r} \frac{(aq^{n+r})_j(-1)^j(q^{n-r-j+1})_j q^{\binom{j+1}{2}-(n-r)j}}{(q)_j(aq^{2r+1})_j} \\ &= \sum_{j=0}^{n-r} \frac{(q^{-(n-r)})_j(aq^{n+r})_j}{(q)_j(aq^{2r+1})_j} q^j \quad \text{(by (1.46))} \\ &= \frac{(q^{-n+r+1})_{n-r}}{(q^{2r+1})_{n-r}} \quad \text{(by (1.21))}. \end{aligned}$$

Note that $A(r,q) = 1$ if $r = n$ and $A(r,n) = 0$ if $r < n$. Thus, the right-hand side of (1.44) is

$$\frac{1-aq^{2n}}{1-a} \sum_{r=0}^{n} \alpha_r \frac{(a)_{n+r}(-1)^{n-r} q^{\binom{n-r}{2}}}{(aq)_{2r}(q)_{n-r}} \frac{(q^{-n+r+1})_{n-r}}{(q^{2r+1})_{n-r}} = \frac{1-aq^{2n}}{1-a}\alpha_n \frac{(a)_{2n}}{(aq)_{2n}} = \alpha_n.$$

□

1.3 Partitions of Integers

Partition theory is an essential branch of number theory and combinatorics relating to enumerative properties of integer partitions. Partition theory's mathematical origin goes back to the 17^{th} century. It has evolved through contributions made by influential mathematicians like S. Ramanujan, L. Euler, G. H. Hardy, A. M. Legengre, F. Dyson and A. Selberg. It continues to be an active area of study till today.

Partitions were first studied by Euler. For many years, one of the most interesting and tough questions about integer partitions was determining the exact formula of $p(n)$, where $p(n)$ denotes the number of partitions of n. Hardy, Ramanujan and Rademacher finally answered this question quite completely [15, 19]. An example that remains unsolved in the integer partition theory, despite a good deal of effort having been expanded on it, is finding an easier criterion to determine whether $p(n)$ is odd or even. There is no pattern discovered to date even though values of $p(n)$ have been computed for n into billions [21]. There are many other intriguing problems in partition theory that remains unsolved today.

As a research area, Theory of Partitions has some trouble fitting in with other fields, maybe because of its multidisciplinary nature. It originated as part of Analysis, and then became part of Number Theory when numerous applications had emerged. It was then later considered as part of Combinatorial Analysis, which is a subject that evolved into modern day Combinatorial and Discrete Mathematics. For some further studies in partition theory, see [1, 2, 3, 4, 5]. In the next section we discuss some elementary aspects of partition theory.

A partition of an integer n, where $n \geq 0$, is a non-increasing sequence of positive intergers, called parts, who's sum equals n. This generally means the number of ways in which a given number can be expressed as a sum of unordered positive integers. As stated in the previous section, the function $p(n)$ shall represent the number of unrestricted partitions of n. For example $p(5) = 7$, the following are all partiions of 5.

$$4 + 1$$

$$3 + 2$$

$$3 + 1 + 1$$

$$2 + 2 + 1$$

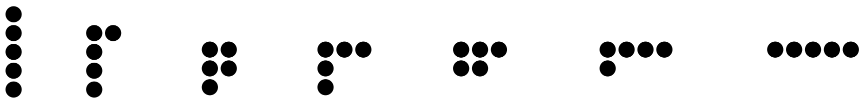
$$2 + 1 + 1 + 1$$

$$1 + 1 + 1 + 1 + 1$$

When explicitly listing the partitions of n, the simplest form is the so-called natural representation which simply gives the sequence of numbers in the representation (e.g. (3, 1, 1)) for the number $5 = 3 + 1 + 1$. The multiplicity representation gives the number of times a number occurs e.g. $5 = 3 + 1 + 1$ can be written as $(3, 1^2)$.

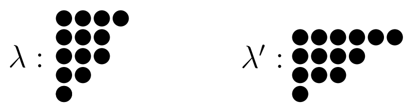
Many results on partitions can be obtained by the use of Ferrers' diagram. The Ferrers' diagram of partition is obtained by putting down a row of dots equal in number to the largest part, the immediately below it a row of dots equal in number to the next largest part, and so on. Such diagrams for partitions of 5 are shown below:

5 4+1 3+2 3+1+1 2+2+1 2+1+1+1 1+1+1+1+1



By rotating the Ferrers' diagram of a partition $x_1+x_2+\cdots+x_k$ about its diagonal,we obtain the conjugate partition ${x_1}^* + {x_2}^* + {x_3}^* + \cdots {x_n}^*$, in which ${x_i}^*$ is the number of parts in the original partition of size i or more. For geometric clarity, Ferrers' diagram is obtained by exchanging rows and columns.
For example,

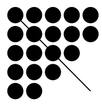
4+3+3+2+1+1 6+4+3+1



From the example above, it is easy to see that the conjugate λ' of λ is the partition whose Ferrers' diagram is the transpose of A. It is also important to note that the conjugate of a Ferrers diagram has the same number of dots as the original diagram. Therefore, they both represent partitions of the same number. In the above example, the number is 14.

Some Ferrers' diagrams have the property of being identical to their conjugate. In this case, they are called self-conjugate. For example:

$$5 + 5 + 4 + 3 + 2$$



The diagram above has a mirror symmetry with respect to the line. Ferrers' diagrams are useful as they can be used to prove many non-trivial relations between partition functions. There are also other ways to prove partition identities and one of them is to use generating functions.

A generating function for a sequence of numbers $b_0, b_1, b_2, b_3, ...$ is defined as:

$$B(x) = \sum_{j=0}^{\infty} b_j x^j.$$

Euler discovered that one could define a generating function for $p(n)$ and that the generating function is given as

$$\begin{aligned} P(q) &= \sum_{n=0}^{\infty} p(n) q^n \\ &= \prod_{j=1}^{\infty} \frac{1}{1-q^j}. \end{aligned}$$

Definition 2. *The number of partitions of n into parts not more than m parts is to be denoted by $p_m(n)$ in the sequel.*

$$
\begin{aligned}
\sum_{n\geq 0} p_m(n)q(n) &= \sum_{n_1\geq n_2\geq\cdots n_m\geq 0} q^{n_1+n_2+\cdots+n_m} \\
&= \sum_{n_1\geq n_2} \sum_{n_2\geq\cdots n_m\geq 0} q^{n_1+n_2+\cdots+n_m} \\
&\qquad\text{(set } k_1 = n_1 - n_2 \text{ so that } n_1 = k_1 + n_2\text{)} \\
&= \sum_{k_1\geq 0} \sum_{n_2\geq\cdots n_m\geq 0} q^{k_1+n_2+n_2+\cdots+n_m} \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{n_2\geq\cdots n_m\geq 0} q^{2n_2+\cdots+n_m} \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{n_2\geq n_3} \sum_{n_3\geq n_4\geq\cdots n_m\geq 0} q^{2n_2+n_3\cdots+n_m} \\
&\qquad\text{(set } k_2 = n_2 - n_3 \text{ so that } n_2 = k_2 + n_3\text{)} \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{k_2\geq 0} \sum_{n_3\geq n_4\geq\cdots n_m\geq 0} q^{2(k_2+n_3)+n_3\cdots+n_m} \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{k_2\geq 0} q^{2k_2} \sum_{n_3\geq n_4} \sum_{n_4\geq n_5\geq\cdots n_m\geq 0} q^{3n_3+n_4\cdots+n_m} \\
&\qquad\text{(set } k_3 = n_3 - n_4 \text{ so that } n_3 = k_3 + n_4 \text{)} \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{k_2\geq 0} q^{2k_2} \sum_{k_3\geq 0} \sum_{n_4\geq n_5\geq\cdots n_m\geq 0} q^{3(k_3+n_4)+n_4\cdots+n_m} \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{k_2\geq 0} q^{2k_2} \sum_{k_3\geq 0} q^{3k_3} \sum_{n_4\geq n_5\geq\cdots n_m\geq 0} q^{4n_4+n_5\cdots+n_m} \\
&\qquad\vdots \\
&= \sum_{k_1\geq 0} q^{k_1} \sum_{k_2\geq 0} q^{2k_2} \sum_{k_3\geq 0} q^{3k_3} \cdots \sum_{k_{m-1}\geq 0} q^{(m-1)k_{m-1}} \sum_{k_m\geq 0} q^{mk_m} \\
&= \frac{1}{1-q^1}\frac{1}{1-q^2}\frac{1}{1-q^3}\cdots\frac{1}{1-q^{m-1}}\frac{1}{1-q^m} \\
&= \prod_{k=1}^{m} \frac{1}{1-q^k}
\end{aligned}
$$

$$= \frac{1}{(q;q)_m}.$$

We discuss a uniform approach for finding generating functions for certain partition functions. This approach is called partition analysis (or omega operator). It was introduced by P.A. MacMahon.

1.4 Partition Analysis

MacMahon introduced Partition Analysis in his famous book 'Combinatory Analysis ' as a computational method that is used for solving problems in connection with linear homogenous diaphantine inequalities and equations.

Definition 3. *Given an absolutely convergent multiple Laurent series*

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_r=-\infty}^{\infty} C_{n_1,n_2,\cdots,n_r}{\lambda_1}^{n_1}{\lambda_2}^{n_2}\cdots{\lambda_r}^{n_r}$$

where $C_{n_1,\ldots,n_r}$ is some rational function in several complex variables and each λ_i is in some region $1-t<|\lambda_i|<1+t$ for $t>0$, we define the omega operator $\Omega_{\geq}$ on the series as:

$$\Omega_{\geq}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_r=-\infty}^{\infty} C_{n_1,n_2,\cdots,n_r}{\lambda_1}^{n_1}{\lambda_2}^{n_2}\cdots{\lambda_r}^{n_r} := \sum_{n_1\geq 0}\sum_{n_2\geq 0}\cdots\sum_{n_r\geq 0} C_{n_1,n_2,\cdots,n_r}.$$

Remark 1. *If an exponent of λ_i is negative, then any term with λ_i is removed. If an exponent of λ_i is nonnegative, then λ_i is set to 1.*

An example demonstrating the above definition is shown below:

Example 2. *For $|q|<1$,*

$$\Omega_{\geq}\sum_{n_2=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} q^{n_1+n_2}{\lambda_1}^{n_1}{\lambda_2}^{n_2}$$
$$= \Omega_{\geq}\sum_{n_2=-\infty}^{\infty}\left(\sum_{n_1=-\infty}^{-1} q^{n_1+n_2}{\lambda_1}^{n_1}{\lambda_2}^{n_2} + \sum_{n_1=0}^{\infty} q^{n_1+n_2}{\lambda_1}^{n_1}{\lambda_2}^{n_2}\right)$$

$$
\begin{aligned}
&= \Omega_{\geq}\Big[\sum_{n_2=-\infty}^{-1} \Big(\sum_{n_1=-\infty}^{-1} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} + \sum_{n_1=0}^{\infty} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} \Big) \\
&\quad + \sum_{n_2=0}^{\infty} \Big(\sum_{n_1=-\infty}^{-1} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} + \sum_{n_1=0}^{\infty} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} \Big) \Big] \\
&= \Omega_{\geq}\Big[\sum_{n_2=-\infty}^{-1} \sum_{n_1=-\infty}^{-1} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} + \sum_{n_2=-\infty}^{-1} \sum_{n_1=0}^{\infty} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} \\
&\quad + \sum_{n_2=0}^{\infty} \sum_{n_1=-\infty}^{-1} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} + \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} q^{n_1+n_2} {\lambda_1}^{n_1} {\lambda_2}^{n_2} \Big] \\
&= \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} q^{n_1+n_2} \\
&= \sum_{n_2=0}^{\infty} q^{n_2} \sum_{n_1=0}^{\infty} q^{n_1}.
\end{aligned}
$$

Some of the properties of the omega operator are summarized in the following proposition.

Proposition 3. *For integers $0 \geq \alpha > s$ and $b \geq 0$ we have*

$$\Omega_{\geq} \frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^s})} = \frac{1}{(1-x)(1-x^s y)}, \tag{1.47}$$

$$\Omega_{\geq} \frac{1}{(1-\lambda^s x)(1-\frac{y}{\lambda})} = \frac{1+xy\frac{1-y^{s-1}}{1-y}}{(1-x)(1-xy^s)}, \tag{1.48}$$

$$\Omega_{\geq} \frac{\lambda^\alpha}{(1-\lambda^s A)(1-\frac{B}{\lambda^{s+b}})} = \Omega_{\geq} \frac{\lambda^\alpha}{(1-\lambda^s A)(1-\frac{AB}{\lambda^b})}. \tag{1.49}$$

We will not prove all the identities in the proposition but showcase the proof of (1.47). Observe that

$$
\begin{aligned}
\Omega_{\geq} \frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^s})} &= \Omega_{\geq} \sum_{n,m\geq 0} (\lambda x)^n (\frac{y}{\lambda^s})^m \\
&= \Omega_{\geq} \sum_{n,m\geq 0} \lambda^{n-sm} x^n y^m
\end{aligned}
$$

$$
\begin{aligned}
&\text{(set } n - sm = k \text{ so that } n = k + sm)\\
&= \sum_{k\geq 0}\sum_{m\geq 0} x^{k+sm}y^m\\
&= \sum_{k\geq 0} x^k \sum_{m\geq 0} x^{sm}y^m\\
&= \frac{1}{(1-x)(1-x^s y)}.
\end{aligned}
$$

We demonstrate partition analysis in the following example.

Example 3. *The number of partitions of n of the form $b_3 + b_2 + b_1$ wherein*

$$\frac{b_3}{3} \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0$$

equals the number of partitions of n into odd parts each ≤ 5.

Proof. We have $b_1 \geq 0$, $b_2 \geq 2b_1$ and $2b_3 \geq 3b_2$.
Let $l(n)$ be the number of partitions of n of the form $b_3 + b_2 + b_1$ wherein

$$\frac{b_3}{3} \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0.$$

Thus, the generating function for $l(n)$, denoted by $L(q)$, can be written as

$$
\begin{aligned}
L(q) &= \Omega_{\geq} \sum_{\frac{b_3}{3}\geq\frac{b_2}{2}\geq\frac{b_1}{1}\geq 0} q^{b_1+b_2+b_3} {\lambda_3}^{2b_3-3b_2} {\lambda_2}^{b_2-2b_1} {\lambda_1}^{b_1}\\
&= \Omega_{\geq} \sum_{b_1=0}^{\infty} (q{\lambda_1}^2)^{b_1} \sum_{b_2=0}^{\infty} \left(q\frac{\lambda_2}{{\lambda_1}^3}\right)^{b_2} \sum_{b_3=0}^{\infty} \left(q\frac{\lambda_3}{{\lambda_2}^2}\right)^{b_3}\\
&= \Omega_{\geq} \frac{1}{(1-q{\lambda_1}^2)} \frac{1}{\left(1-q\frac{\lambda_2}{{\lambda_1}^3}\right)} \frac{1}{\left(1-q\frac{\lambda_3}{{\lambda_2}^2}\right)}\\
&\quad \text{(by (1.49), where } \alpha = 0,\ \lambda = \lambda_1,\ s = 2,\ b = 1,\ A = q,\ B = q\lambda_2, \text{ we have:)}\\
&= \Omega_{\geq} \frac{1}{(1-q{\lambda_1}^2)} \frac{1}{\left(1-q^2\frac{\lambda_2}{\lambda_1}\right)} \frac{1}{\left(1-q\frac{\lambda_3}{{\lambda_2}^2}\right)}\\
&\quad \text{(by (1.48), where } \lambda = \lambda_1,\ s = 2,\ x = q,\ y = q^2\lambda_2, \text{ we have:)}\\
&= \Omega_{\geq} \frac{1+q^3\lambda_2}{(1-q)(1-q(q^2\lambda_2)^2)\left(1-q\frac{\lambda_3}{{\lambda_2}^2}\right)}
\end{aligned}
$$

$$\begin{aligned}
&= \Omega_{\geq} \frac{1+q^3\lambda_2}{(1-q)(1-q^5{\lambda_2}^2)\left(1-q\frac{\lambda_3}{{\lambda_2}^2}\right)} \\
&\quad \text{(by (1.49), where } \alpha=1,\ \lambda=\lambda_2,\ s=2,\ b=0,\ A=q^5,\ B=q\lambda_3, \text{ we have:)} \\
&= \Omega_{\geq} \frac{1+q^3\lambda_2}{(1-q)(1-q^5{\lambda_2}^2)\,(1-q^6\lambda_3)} \\
&= \frac{1+q^3}{(1-q)(1-q^5)(1-q^6)} \\
&= \frac{1}{(1-q)(1-q^3)(1-q^5)}.
\end{aligned}$$

□

Chapter 2

The Rogers-Ramanujan identities

In this chapter, we provide proofs of the original Rogers-Ramanujan identities. These identities were first discovered by L. J. Rogers in 1894, but were almost not noticed around that time. S. Ramanujan rediscovered them but had no proof. Fortunately, in 1917, Ramanujan accidentally found Rogers paper. They finally worked together and published a new proof in a joint paper in 1919.

The two famous Rogers-Ramanujan identities are given as

$$F(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{2.1}$$

and

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}. \tag{2.2}$$

The identities (2.1) and (2.2) are perhaps the most mysterious and ceebrated results in partition theory.

We now show how Rogers-Ramanujan identities in (2.1) and (2.2) can be proved using Baliey chains.

From (1.44), we see that

$$\alpha_n = \frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{(a;q)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q;q)_{n-j}}$$

$$
\begin{aligned}
&= \frac{1-aq^{2n}}{1-a}\sum_{j=0}^{n}\frac{(a)_n(aq^n)_j(-1)^{n-j}q^{\binom{n-j}{2}}(q^{-1}q^{1-n})_j\beta_j}{(q)_n(-qq^{-1})^j q^{\binom{j}{2}-nj}} \quad \text{(by (1.3) and (1.4))}\\
&= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n}{(q)_n}\sum_{j=0}^{n}\frac{(aq^n)_j(-1)^{-j}q^{\binom{n-j}{2}}(q^{-n})_j\beta_j}{(-1)^j q^{\binom{j}{2}-nj}}\\
&= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n}{(q)_n}\sum_{j=0}^{n}(aq^n)_j(q^{-n})_j q^{\binom{n-j}{2}-\binom{j}{2}+nj}\beta_j\\
&\qquad\qquad \text{(but } q^{\binom{n-j}{2}-\binom{j}{2}+nj} = q^{\binom{n}{2}}q^j\text{)}\\
&= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n q^{\binom{n}{2}}}{(q)_n}\sum_{j=0}^{n}(aq^n)_j(q^{-n})_j q^j\beta_j\\
&\qquad\qquad \text{(now let } \beta_j = \frac{1}{(q;q)_j}\text{)}\\
\alpha_n &= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n q^{\binom{n}{2}}}{(q)_n}\sum_{j=0}^{n}\frac{(aq^n)_j(q^{-n})_j q^j}{(q;q)_j}\\
&= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n q^{\binom{n}{2}}(aq^n)^n}{(q)_n} \quad \text{(by (1.22))}\\
&= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n q^{\frac{n(n-1)}{2}+n^2}a^n}{(q)_n}\\
&= \frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n q^{\frac{3n^2-n}{2}}a^n}{(q)_n}. \qquad (2.3)
\end{aligned}
$$

Therefore, using (1.2) and (2.3), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} q^{n^2}a^n\frac{1}{(q;q)_n} &= \frac{1}{(a;q)_\infty}\sum_{n=0}^{\infty} q^{n^2}a^n\frac{1-aq^{2n}}{1-a}\frac{(a)_n(-1)^n q^{\frac{3n^2-n}{2}}a^n}{(q)_n}\\
&= \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a)_n(-1)^n a^{2n}q^{\frac{3n^2-n+2n^2}{2}}}{(1-a)(q)_n}\\
&= \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a)_n(-1)^n a^{2n}q^{\frac{5n^2-n}{2}}}{(1-a)(q)_n}\\
&= \frac{1}{(aq;q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(1-a)(aq)_{n-1}(-1)^n a^{2n}q^{\frac{5n^2-n}{2}}}{(1-a)(q)_n}\right]
\end{aligned}
$$

$$= \frac{1}{(aq;q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(aq)_{n-1}(-1)^n a^{2n} q^{\frac{5n^2-n}{2}}}{(q)_n}\right] \tag{2.4}$$

To obtain (2.1), we let $a = 1$ in (2.4). That is,

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(1-q^{2n})(q)_{n-1}(-1)^n q^{\frac{5n^2-n}{2}}}{(q)_n}\right]\\
&= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(1-q^{n})(1+q^n)(q)_{n-1}(-1)^n q^{\frac{5n^2-n}{2}}}{(q)_n}\right]\\
&= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(1+q^n)(q)_{n}(-1)^n q^{\frac{5n^2-n}{2}}}{(q)_n}\right]\\
&= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}(1+q^n)(-1)^n q^{\frac{5n^2-n}{2}}\right]\\
&= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}}+\sum_{n=1}^{\infty}(q^n)(-1)^n q^{\frac{5n^2-n}{2}}\right]\\
&= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}}+\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}}\right]\\
&= \frac{1}{(q;q)_\infty}\left[1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}}+\sum_{n=-\infty}^{-1}(-1)^n q^{\frac{5n^2-n}{2}}\right]\\
&= \frac{1}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}}+\sum_{n=-\infty}^{-1}(-1)^n q^{\frac{5n^2-n}{2}}\right]\\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}}\\
&= f(-q^2;-q^3)\\
&= (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty\\
&= \prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.
\end{aligned}$$

Finally, to obtain (2.2), we let $a = q$ in (2.4). That is,

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(q)_n(-1)^n q^{2n} q^{\frac{5n^2-n}{2}}}{(1-q)(q)_n}$$

$$
\begin{aligned}
&= \frac{1}{(q^2;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(-1)^n q^{\frac{5n^2+3n}{2}}}{1-q} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^n q^{\frac{5n^2+3n}{2}} \\
&= \frac{1}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}} - \sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2+7n+2}{2}}\right] \\
&\qquad\qquad\text{(set } n=-m-1 \text{ in the second summation)} \\
&= \frac{1}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}} - \sum_{-m-1=0}^{-m-1=\infty}(-1)^{-m-1} q^{\frac{5(-m-1)^2+7(-m-1)+2}{2}}\right] \\
&= \frac{1}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}} + \sum_{m=-\infty}^{m=-1}(-1)^{m} q^{\frac{5(-m-1)^2+7(-m-1)+2}{2}}\right] \\
&= \frac{1}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}} + \sum_{m=-\infty}^{-1}(-1)^{m} q^{\frac{5m^2+3m}{2}}\right] \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}} \\
&= f(-q^4;-q) \\
&= (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty \\
&= \prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})}.
\end{aligned}
$$

The original Rogers-Ramanujan identities have given rise to several other identities. These identities are still called Rogers-Ramanujan identities. We discuss a few of them in the sequel.

The next two theorems are applications of partition analysis and q-series to prove Rogers-Ramanujan identities.

Theorem 8. *The number of partitions of m of the form $b_1 + b_2 + \cdots$ where $b_1 \geq b_2 \geq b_3 \cdots$ and each b_i is odd or $\equiv \pm\, 4 \pmod{20}$ equals the number of even-length partitions of m of the form $c_1 + c_2 + \cdots$ where $c_1 > c_2 \geq c_3 > c_4 \geq c_5 > \cdots$,* *i.e.*

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}=\frac{1}{(q;q^2)_\infty(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}. \tag{2.5}$$

Proof. Let $A(q)$ denote the generating function for the number of partitions of m of the form $b_1+b_2+\cdots$ where $b_1\geq b_2\geq b_3\cdots$ and each b_i is odd or $\equiv\pm 4 \pmod{20}$. Let $B(q)$ denote the generating function for the number of partitions of m of the form $c_1+c_2+\cdots$ where $c_1>c_2\geq c_3>c_4\geq c_5>\cdots$.
Let $B(q,n)$ denote the generating function for the number of partitions of m of the form $c_1+c_2+\cdots+c_{2n}$ where $c_1>c_2\geq c_3>c_4\geq c_5>\cdots$. Then we have

$$\begin{aligned}
B(q,n) &= \Omega_{\geq}\sum_{c_1,c_2,c_3,\cdots,c_{2n}\geq 0} q^{c_1+c_2+c_3+\cdots c_{2n}}\lambda_1^{c_1-c_2-1}\lambda_2^{c_2-c_3}\lambda_3^{c_3-c_4-1}\lambda_4^{c_4-c_5}\cdots\lambda_{2n}^{c_{2n}}\\
&= \Omega_{\geq}\sum_{c_1\geq 0}(q\lambda_1)^{c_1}\sum_{c_2\geq 0}(q\frac{\lambda_2}{\lambda_1})^{c_2}\sum_{c_3\geq 0}(q\frac{\lambda_3}{\lambda_2})^{c_3}\cdots\sum_{c_{2n}\geq 0}(q\frac{\lambda_{2n}}{\lambda_{2n-1}})^{c_{2n}}\lambda_1{}^{-1}\lambda_3{}^{-1}\lambda_5{}^{-1}\lambda_7{}^{-1}\cdots\lambda_{2n-1}^{-1}\\
&= \Omega_{\geq}\frac{\lambda_1{}^{-1}\lambda_3{}^{-1}\lambda_5{}^{-1}\lambda_7{}^{-1}\cdots\lambda_{2n-1}{}^{-1}}{(1-q\lambda_1)(1-q\frac{\lambda_2}{\lambda_1})(1-q\frac{\lambda_3}{\lambda_2})(1-q\frac{\lambda_4}{\lambda_3})\cdots(1-q\frac{\lambda_{2n}}{\lambda_{2n-1}})}\\
&\qquad (\text{let } x=q,\ y=q\lambda_2 \text{ and } \lambda=\lambda_1)\\
&= \Omega_{\geq}\frac{q\lambda_3{}^{-1}\lambda_5{}^{-1}\lambda_7{}^{-1}\cdots\lambda_{2n-1}{}^{-1}}{(1-q)(1-q\lambda_2)(1-q\frac{\lambda_3}{\lambda_2})(1-q\frac{\lambda_4}{\lambda_3})\cdots(1-q\frac{\lambda_{2n}}{\lambda_{2n-1}})}\\
&\qquad (\text{let } x=q^2,\ y=q\lambda_3 \text{ and } \lambda=\lambda_2)\\
&= \Omega_{\geq}\frac{q\lambda_3{}^{-1}\lambda_5{}^{-1}\lambda_7{}^{-1}\cdots\lambda_{2n-1}{}^{-1}}{(1-q)(1-q^2)(1-q\lambda_3)(1-q\frac{\lambda_4}{\lambda_3})\cdots(1-q\frac{\lambda_{2n}}{\lambda_{2n-1}})}\\
&\qquad (\text{let } x=q^3,\ y=q\lambda_4 \text{ and } \lambda=\lambda_3)\\
&= \Omega_{\geq}\frac{qq^3\lambda_5{}^{-1}\lambda_7{}^{-1}\cdots\lambda_{2n-1}{}^{-1}}{(1-q)(1-q^2)(1-q^3)(1-q\lambda_4)\cdots(1-q\frac{\lambda_{2n}}{\lambda_{2n-1}})}\\
&\qquad\vdots\\
&= \frac{qq^3q^5q^7\cdots q^{2n-1}}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots(1-q^{2n})}\\
&= \frac{q^{n^2}}{(q;q)_{2n}}.
\end{aligned}$$

Thus, we have

$$B(q)=\sum_{n=0}^{\infty}B(q,n)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}.$$

Observe that

$$A(q) = \prod_{i=0}^{\infty} \frac{1}{(1-q^{2i+1})(1-q^{20i+16})(1-q^{20i+4})}$$
$$= \frac{1}{(q^1;q^2)_\infty (q^{16};q^{20})_\infty (q^4;q^{20})_\infty}.$$

We now prove (2.5). Replace q by q^2 in (1.20) and set $b = \frac{-aq}{t}$ and let $t \to 0$. The left-hand side of (1.20) becomes

$$\sum_{n=0}^{\infty} \frac{(-\frac{aq}{t};q^2)_{2n} t^{2n}}{(q^4;q^4)_n} = \sum_{n=0}^{\infty} \frac{(1+\frac{aq}{t})(1+\frac{aq^3}{t})(1+\frac{aq^5}{t})(1+\frac{aq^7}{t})\cdots(1+\frac{aq^{4n-1}}{t})t^{2n}}{(q^4;q^4)_n}$$
$$= \sum_{n=0}^{\infty} \frac{(t+aq)(t+aq^3)(t+aq^5)(t+aq^7)\cdots(t+aq^{4n-1})}{(q^4;q^4)_n}.$$

Let $t \to 0$. Then

$$\sum_{n=0}^{\infty} \frac{(aq)(aq^3)(aq^5)(aq^7)\cdots(aq^{4n-1})}{(q^4;q^4)_n} = \sum_{n=0}^{\infty} \frac{a^{2n}q^{4n^2}}{(q^4;q^4)_n}. \tag{2.6}$$

The right-hand side of (1.20) is

$$\frac{(-tb;q^2)_\infty}{(-t;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(b;q^2)_n}{(q^2;q^2)_n(-tb;q^2)_n} t^n = \frac{(aq;q^2)_\infty}{(-t;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(t+aq)(t+aq^3)(t+aq^5)\cdots(t+aq^{2n-1})t^n}{t^n(q^2;q^2)_n(aq;q^2)_n}$$

and letting $t \to 0$ results in

$$(aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2;q^2)_n(aq;q^2)_n}. \tag{2.7}$$

Thus, combing (2.6) and (2.7) gives

$$(aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2;q^2)_n(aq;q^2)_n} = \sum_{n=0}^{\infty} \frac{a^{2n}q^{4n^2}}{(q^4;q^4)_n} \tag{2.8}$$

and setting $a = 1$ gives

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^4;q^4)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n (q;q^2)_n},$$

i.e.

$$\frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^4;q^4)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}}, \tag{2.9}$$

and from (2.1)

$$\frac{1}{(q;q^2)_\infty} F(q^4) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}}. \tag{2.10}$$

But

$$F(q^4) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{20n-4})(1-q^{20n-16})}$$

so that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{1}{(q;q^2)_\infty (q^4;q^{20})_\infty (q^{16};q^{20})_\infty}. \tag{2.11}$$

□

Theorem 9. *The number of partitions of n of the form $b_1 + b_2 + \cdots$ where $b_1 \geq b_2 \geq b_3 \cdots$ and each b_i is odd or $\equiv \pm\, 8 (mod\ 20)$ equals the number of partitions of n of the form $c_1 + c_2 + \cdots$ into an odd number of parts where $c_1 \geq c_2 \geq c_3 > c_4 \geq c_5 > c_6 \geq c_7 > \cdots$,*

i.e.

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \frac{1}{(q;q^2)_\infty (q^8;q^{20})_\infty (q^{12};q^{20})_\infty}. \tag{2.12}$$

Proof. Let $D(q)$ denote the generating function for the number of partitions of n of the form $b_1 + b_2 + \cdots$ where $b_1 \geq b_2 \geq b_3 \cdots$ and each b_i is odd or $\equiv \pm\, 8 \pmod{20}$.
Let $C(q)$ denote the generating function for the number of partitions of n of the form $c_1 + c_2 + \cdots$ into an odd number of parts where $c_1 \geq c_2 \geq c_3 > c_4 \geq c_5 > c_6 \geq c_7 > \cdots$.
Let $C(q,k)$ denote the generating function for the number of partitions of n of the form $c_1 + c_2 + \cdots c_{2k+1}$ into an odd number of parts where $c_1 \geq c_2 \geq c_3 > c_4 \geq c_5 > c_6 \geq c_7 > \cdots$.

$$
\begin{aligned}
C(q,k) &= \Omega_{\geq} \sum_{c_1,c_2,c_3,\cdots,c_{2k+1}\geq 0} q^{c_1+c_2+c_3+\cdots c_{2k+1}} \lambda_1^{c_1-c_2} \lambda_2^{c_2-c_3} \lambda_3^{c_3-c_4-1} \lambda_4^{c_4-c_5} \lambda_4^{c_5-c_6-1} \lambda_6^{c_6-c_7} \cdots \lambda_{2k+1}^{c_{2k+1}-1} \\
&= \Omega_{\geq} \sum_{c_1\geq 0} (q\lambda_1)^{c_1} \sum_{c_2\geq 0} (q\frac{\lambda_2}{\lambda_1})^{c_2} \sum_{c_3\geq 0} (q\frac{\lambda_3}{\lambda_2})^{c_3} \cdots \sum_{c_{2k+1}\geq 0} (q\frac{\lambda_{2k+1}}{\lambda_{2k}})^{c_{2k+1}} \cdot {\lambda_3}^{-1}{\lambda_5}^{-1}{\lambda_7}^{-1}\cdots{\lambda_{2k+1}}^{-1} \\
&= \Omega_{\geq} \frac{{\lambda_3}^{-1}{\lambda_5}^{-1}{\lambda_7}^{-1}\cdots{\lambda_{2k+1}}^{-1}}{(1-q\lambda_1)(1-q\frac{\lambda_2}{\lambda_1})(1-q\frac{\lambda_3}{\lambda_2})(1-q\frac{\lambda_4}{\lambda_3})\cdots(1-q\frac{\lambda_{2k+1}}{\lambda_{2k}})} \\
&\qquad\qquad (\text{let } x = q,\ y = q\lambda_2 \text{ and } \lambda = \lambda_1) \\
&= \Omega_{\geq} \frac{{\lambda_3}^{-1}{\lambda_5}^{-1}{\lambda_7}^{-1}\cdots{\lambda_{2k+1}}^{-1}}{(1-q)(1-q\lambda_2)(1-q\frac{\lambda_3}{\lambda_2})(1-q\frac{\lambda_4}{\lambda_3})\cdots(1-q\frac{\lambda_{2k+1}}{\lambda_{2k}})} \\
&\qquad\qquad (\text{let } x = q^2,\ y = q\lambda_3 \text{ and } \lambda = \lambda_2) \\
&= \Omega_{\geq} \frac{{\lambda_3}^{-1}{\lambda_5}^{-1}{\lambda_7}^{-1}\cdots{\lambda_{2k+1}}^{-1}}{(1-q)(1-q^2)(1-q\lambda_3)(1-q\frac{\lambda_4}{\lambda_3})\cdots(1-q\frac{\lambda_{2k+1}}{\lambda_{2k}})} \\
&\qquad\qquad (\text{let } x = q^3,\ y = q\lambda_4 \text{ and } \lambda = \lambda_3) \\
&= \Omega_{\geq} \frac{q^3{\lambda_5}^{-1}{\lambda_7}^{-1}\cdots{\lambda_{2k+1}}^{-1}}{(1-q)(1-q^2)(1-q^3)(1-q\lambda_4)\cdots(1-q\frac{\lambda_{2k+1}}{\lambda_{2k}})} \\
&\qquad\qquad \vdots \\
&= \frac{q^3q^5q^7\cdots q^{2k+1}}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots(1-q^{2k+1})} \\
&= \frac{q^{k(k+2)}}{(q;q)_{2k+1}}
\end{aligned}
$$

Thus, we have

$$
C(q) = \sum_{k=0}^{\infty} C(q,k) = \sum_{k=0}^{\infty} \frac{q^{k(k+2)}}{(q;q)_{2k+1}}.
$$

Observe that

$$\begin{aligned} D(q) &= \prod_{i=0}^{\infty} \frac{1}{(1-q^{2i+1})(1-q^{20i+12})(1-q^{20i+8})} \\ &= \frac{1}{(q^1;q^2)_\infty (q^{12};q^{20})_\infty (q^8;q^{20})_\infty}. \end{aligned}$$

We shall also show that (2.12) is true similar to (2.5).

Similar to (2.10), we continue from (2.8) and set $a = q^2$ to get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{4n^2} q^{4n}}{(q^4;q^4)_n} &= (q^3;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} q^{2n}}{(q^2;q^2)_n (q^3;q^2)_n} \\ &= (q^3;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} q^{2n}}{(q^2;q^2)_n (q^3;q^2)_n} \frac{(1-q)}{(1-q)} \\ &= (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} q^{2n}}{(q;q)_{2n+1}}. \end{aligned}$$

We also observe from (2.2) that

$$G(q^4) := \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{20n-8})(1-q^{20n-12})}.$$

Hence,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \frac{1}{(q;q^2)_\infty (q^8;q^{20})_\infty (q^{12};q^{20})_\infty}.$$

□

Chapter 3

Ramanujan's partition congruences

Much of the current work involving the arithmetic properties of the partition function find their seed in some keen observations of Ramanujan. In particular, he discovered what are referred to as the Ramanujan congruences of $p(n)$. These are appropriately named because Ramanujan was the first [7] to notice these interesting properties of the partition function. He found that, for all $n \in \mathbb{Z}$,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7}$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

In addition to noticing these peculiar relations, he conjectured that the relations above were the only congruences of this form. In particular, these Ramanujan congruences are the only congruences of the form

$$p(ln+\beta) \equiv 0 \pmod{l}$$

for all $n \in \mathbb{Z}$, l prime and some fixed $\beta \in \mathbb{Z}$.

These Ramanujan congruences are an example of congruence properties for partition function.

In this section, we shall only give proofs of Ramanujan's congruences for $p(n)$ modulo 5 and 7 as well as the most beautiful identities and Atkin-Swinnerton-Dyer Congruences.

The following observations are true for a prime p. Let r be a non-negative integer. For $1 \leq j \leq p^r - 1$,

$$\binom{p^r}{j} \equiv 0 \pmod{p}. \tag{3.1}$$

The equation (3.1) implies that

$$(x+y)^{p^r} \equiv (x^{p^r} + y^{p^r}) \pmod{p}. \tag{3.2}$$

Proposition 4. *For a prime p and non-negative integer r, we have*

$$(q^{p^r}; q^{p^r}) \equiv (q;q)^{p^r} \pmod{p}. \tag{3.3}$$

3.1 The identity $p(5n+4) \equiv 0 \pmod 5$

We want to show that

$$p(5n+4) \equiv 0 \pmod{5}.$$

By proposition (3.3), it is clear that

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty$$

and thus

$$\frac{1}{(q;q)_\infty^5} \equiv \frac{1}{(q^5;q^5)_\infty}.$$

By using Euler's result, we recall that in (1.36)

$$A := (q,q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}}.$$

and note that

$$\frac{3k^2-k}{2} \equiv 0, 1 \text{ or } 2 \pmod 5.$$

So,

$$\begin{aligned}(q;q)_\infty &= \sum_{-\infty}^{\infty}(-1)^k q^{\frac{3k^2-k}{2}} \\ &= E_0 + E_1 + E_2 \qquad (3.4)\end{aligned}$$

where E_i consists of thos terms of E in which the power of $q \equiv i \pmod 5$.

This is another example of an m-dissection, where we have expressed a series as a sum of series, in each of which the powers fall in just one residue class modulo m. Here, $m = 5$.

Also, from Jacobi's result, observe that,

$$\frac{k^2+k}{2} \equiv 0, 1 \text{ or } 3 \pmod 5. \qquad (3.5)$$

Most importantly that $\frac{k^2+k}{2} \equiv 3 \pmod 5$ if and only if $k \equiv 2 \pmod 5$ and then the coefficient $2k+1 \equiv 0 \pmod 5$. We find that by (4.2), for modulus 5,

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k^2+k}{2}}.$$

But for $k = 2$, we have $2k+1 \equiv 0 \pmod 5$. Thus

$$(q;q)_\infty^3 \equiv \widetilde{C_0} + \widetilde{C_1} \qquad (3.6)$$

where $\widetilde{C_i}$ consists of terms in which the power of q is congruent to i modulo 5. So,

$$\begin{aligned}\sum_{n\geq 0} p(n)q^n &= \frac{1}{(q;q)_\infty} \\ &= \frac{(q;q)_\infty^4}{(q;q)_\infty^5} \\ &= \frac{(q;q)_\infty(q;q)_\infty^3}{(q;q)_\infty^5}\end{aligned}$$

(applying (3.2) and substituting (3.4) and (3.6), we obtain:)

$$\equiv \frac{1}{(q^5;q^5)_\infty}(E_0+E_1+E_3)(\widetilde{C_0}+\widetilde{C_1}) \pmod 5$$
$$= \frac{1}{(q^5;q^5)_\infty}[E_0\widetilde{C_0}+(E_0\widetilde{C_1}+E_1\widetilde{C_0})+(E_1\widetilde{C_1}+E_2\widetilde{C_0})+E_2\widetilde{C_1}].$$

There are no terms on the right-hand side in which the exponents on q is congruent to $4 \pmod 5$ and so

$$\sum_{n\geq 0} p(5n+4)q^{5n+4} \equiv 0 \pmod 5,$$

i.e.

$$p(5n+4) \equiv 0 \pmod 5.$$

3.2 The identity $p(7n+5) \equiv 0 \pmod 7$

It is clear that

$$(q;q)_\infty^7 \equiv (q^7;q^7)_\infty \pmod 7.$$

Recall that

$$E := (q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k^2+k}{2}},$$

it is not difficult to see that

$$\frac{k^2+k}{2} \equiv 0, 1, 3 \text{ or } 6 \pmod 7.$$

So,

$$(q;q)_\infty^3 = E_0+E_1+E_3+E_6 \tag{3.7}$$

where the E_i consists of those terms of E in which the power of $q \equiv i \pmod 7$. But for $k=3$, we have $2k+1 \equiv 0 \pmod 7$, thus

$$(q;q)_\infty^3 \equiv \widetilde{C_0}+\widetilde{C_1}+\widetilde{C_3} \tag{3.8}$$

where $\widetilde{C_i}$ consists of terms in which the power of q is congruent to i modulo 7.

$$\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= \frac{1}{(q;q)_\infty} \\
&= \frac{(q;q)_\infty^6}{(q;q)_\infty^7} \\
&= \frac{(q;q)_\infty^3 (q;q)_\infty^3}{(q;q)_\infty^7} \\
&\qquad \text{(applying (3.2) and substituting (3.8), we obtain)} \\
&\equiv \frac{1}{(q^7;q^7)_\infty}(\widetilde{C_0} + \widetilde{C_1} + \widetilde{C_3})^2 \pmod 7 \\
&= \frac{1}{(q^7;q^7)_\infty}(\widetilde{C_0}\widetilde{C_0} + 2\widetilde{C_0}\widetilde{C_1} + \widetilde{C_1}\widetilde{C_1} + 2\widetilde{C_0}\widetilde{C_3} + 2\widetilde{C_1}\widetilde{C_3} + \widetilde{C_3}\widetilde{C_3})
\end{aligned}$$

it can easily be noted that there are no terms on the right-hand side in which the exponent on q is congruent to 5 modulo 7, hence

$$p(7n+5) \equiv 0 \pmod 7.$$

3.3 Atkin-Swinnerton-Dyer congruences for modulus 5

Recall from (1.27), that

$$\sum_{k=-\infty}^{\infty} a^k q^{k^2} = (-a^{-1}q, -aq, q^2; q^2)_\infty$$

and that (2.1) and (2.2) can be written as

$$(q^2, q^3, q^5; q^5)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{5k^2-k}{2}}$$

and

$$(q, q^4, q^5; q^5)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{5k^2-3k}{2}},$$

respectively.

We've already seen in (3.6) that, for modulus 5, we have

$$\begin{aligned}(q;q)_\infty^3 &= \sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k^2+k}{2}} \\ &\equiv \widetilde{C_0}+\widetilde{C_1} \\ &= Y(q^5)+qX(q^5)\end{aligned}$$

where $\widetilde{C_1}=qX(q^5)$ and $\widetilde{C_0}=Y(q^5)$.

The aim is to express $X(q^5)$ and $Y(q^5)$ as a triple product. We proceed as follows:

$$\begin{aligned}X(q^5) &= q^{-1}\widetilde{C_1} \\ &= q^{-1}\sum_{k\equiv 1,3 \pmod{5, k\geq 0}}^{\infty}(-1)^k(2k+1)q^{\frac{k^2+k}{2}} \\ &= \sum_{i=0}^{\infty}(-1)^{5i+1}(2(5i+1)+1)q^{\frac{(5i+1)^2+(5i+1)}{2}-1}+\sum_{i=0}^{\infty}(-1)^{5i+3}(2(5i+3)+1)q^{\frac{(5i+3)^2+(5i+3)}{2}-1} \\ &= \sum_{i=0}^{\infty}(-1)^{5i+1}(10i+3)q^{\frac{(5i+1)^2+(5i+1)}{2}-1}+\sum_{i=0}^{\infty}(-1)^{5i+3}(10i+7)q^{\frac{(5i+3)^2+(5i+3)}{2}-1} \\ &= \sum_{i=0}^{\infty}(-1)^{5i}(-10i-3)q^{\frac{(5i+1)^2+(5i+1)-2}{2}}+\sum_{i=0}^{\infty}(-1)^{5i+3}(10i+7)q^{\frac{(5i+3)^2+(5i+3)-2}{2}} \\ &\equiv \sum_{i=0}^{\infty}(-1)^{i}(-3)q^{\frac{(5i+1)^2+(5i+1)-2}{2}}-\sum_{i=0}^{\infty}(-1)^{i}(-3)q^{\frac{(5i+3)^2+(5i+3)-2}{2}} \pmod 5 \\ &= -3\left(\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{(5i+1)^2+(5i+1)-2}{2}}-\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{(5i+3)^2+(5i+3)-2}{2}}\right) \pmod 5 \\ &= -3[(q^0-q^5)-(q^{20}-q^{35})+(q^{65}-q^{90})-\cdots] \\ &= -3\widetilde{X}(q^5)\end{aligned}$$

where

$$\widetilde{X}(q^5) = \sum_{i=0}^{\infty}(-1)^{i}q^{\frac{(5i+1)^2+(5i+1)-2}{2}}-\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{(5i+3)^2+(5i+3)-2}{2}}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+10i+1+15+i-2}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+30i+9+5i+3-2}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+15i}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+35i+10}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+15i}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{25(i+1)^2-15(i+1)}{2}} \\
&\quad \text{(replace } i \text{ by } i-1 \text{ in the second summation)} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+15i}{2}} - \sum_{i=1}^{\infty}(-1)^{i-1} q^{\frac{25i^2-15i}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+15i}{2}} + \sum_{i=1}^{\infty}(-1)^i q^{\frac{25i^2-15i}{2}} \\
&= \sum_{i=-\infty}^{0}(-1)^i q^{\frac{25i^2-15i}{2}} + \sum_{i=1}^{\infty}(-1)^i q^{\frac{25i^2-15i}{2}} \\
&= \sum_{i=-\infty}^{\infty}(-1)^i q^{\frac{25i^2-15i}{2}} \\
&= (q^5, q^{20}, q^{25}, q^{25})_{\infty}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Y(q^5) &= q^{-1} \sum_{k\equiv 0,4 \pmod 5, k\geq 0}^{\infty} (-1)^k(2k+1)q^{\frac{k^2+k}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{5i}(2(5i)+1)q^{\frac{(5i)^2+(5i)}{2}} + \sum_{i=0}^{\infty}(-1)^{5i+4}(2(5i+4)+1)q^{\frac{(5i+4)^2+(5i+4)}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^i(10i+1)q^{\frac{25i^2+5i}{2}} + \sum_{i=0}^{\infty}(-1)^i(10i+9)q^{\frac{25i^2+40i+16+5i+4}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^i(10i+1)q^{\frac{25i^2+5i}{2}} + \sum_{i=0}^{\infty}(-1)^i(10i+9)q^{\frac{25i^2+45i+20}{2}} \\
&\equiv \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+5i}{2}} + \sum_{i=0}^{\infty}(-1)^i(-1)q^{\frac{25i^2+45i+20}{2}} \pmod 5 \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+5i}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+45i+20}{2}} \\
&= q^0 - q^{10} - q^{15} + q^{45} + q^{55} - q^{100} - \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+5i}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{25(i+1)^2-5(i+1)}{2}} \\
&\quad \text{(replace } i \text{ by } i-1 \text{ in the second summation)} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{25i^2+5i}{2}} - \sum_{i=1}^{\infty}(-1)^{i-1} q^{\frac{25i^2-5i}{2}} \\
&= \sum_{i=-\infty}^{0}(-1)^i q^{\frac{25i^2-5i}{2}} + \sum_{i=1}^{\infty}(-1)^i q^{\frac{25i^2-5i}{2}} \\
&= \sum_{i=-\infty}^{\infty}(-1)^i q^{\frac{25i^2-5i}{2}} \\
&= (q^{10}, q^{15}, q^{25}; q^{25})_\infty.
\end{aligned}
$$

Thus,

$$
(q;q)_\infty^3 \equiv (q^{10}, q^{15}, q^{25}; q^{25})_\infty - 3(q^5, q^{20}, q^{25}; q^{25})_\infty. \tag{3.9}
$$

By making use of (3.2) and (3.9), we have, for modulus 5,

$$
\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= \frac{1}{(q;q)_\infty} \\
&= \frac{(q;q)_\infty^9}{(q;q)_\infty^{10}} \\
&= \frac{((q;q)_\infty^3)^3}{((q;q)_\infty^5)^2} \\
&= \frac{(Y(q^5) - 3qX(q^5))^3}{((q;q)_\infty^5)^2} \\
&\equiv \frac{Y(q^5)^3 - 9qY(q^5)^2X(q^5) + 27q^2Y(q^5)X(q^5)^2 - 27q^3X(q^5)^3}{(q^5;q^5)_\infty^2} \pmod 5 \\
&\equiv \frac{Y(q^5)^3 + qY(q^5)^2X(q^5) + 2q^2Y(q^5)X(q^5)^2 + 3q^3X(q^5)^3}{(q^5;q^5)_\infty^2} \pmod 5
\end{aligned} \tag{3.10}
$$

and finally, using (3.9) and (3.10), we find the following congruences. They are known as the Atkin-Swinnerton-Dyer congruences.

$$\sum_{n=0}^{\infty} p(5n)q^{5n} \equiv \frac{Y(q^5)^3}{(q^5;q^5)_\infty^2}$$
$$\equiv \frac{(q^{10},q^{15},q^{25};q^{25})_\infty^3}{(q^5;q^5)_\infty^2},$$

i.e.

$$\sum_{n=0}^{\infty} p(5n)q^{n} \equiv \frac{(q^2,q^3,q^5;q^5)_\infty^3}{(q;q)_\infty^2}.$$

$$\sum_{n=0}^{\infty} p(5n+1)q^{5n} \equiv \frac{Y(q^5)^2X(q^5)}{(q^5;q^5)_\infty^2}$$
$$\equiv \frac{(q^{10},q^{15},q^{25};q^{25})_\infty^2(q^5,q^{20},q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2},$$

i.e.

$$\sum_{n=0}^{\infty} p(5n+1)q^{n} \equiv \frac{(q^2,q^3,q^5;q^5)_\infty^2(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty^2}.$$

$$\sum_{n=0}^{\infty} p(5n+2)q^{5n} \equiv 2\frac{Y(q^5)X(q^5)^2}{(q^5;q^5)_\infty^2}$$
$$\equiv 2\frac{(q^5,q^{20},q^{25};q^{25})_\infty^2(q^{10},q^{15},q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2},$$

i.e.

$$\sum_{n=0}^{\infty} p(5n+2)q^{n} \equiv 2\frac{(q,q^4,q^5;q^5)_\infty^2(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty^2}.$$

$$\sum_{n=0}^{\infty} p(5n+3)q^{5n} \equiv 3\frac{X(q^5)^3}{(q^5;q^5)_\infty^2}$$

$$\equiv 3\frac{(q^5, q^{20}, q^{25}; q^{25})^3_\infty}{(q^5; q^5)^2_\infty},$$

i.e.

$$\sum_{n=0}^{\infty} p(5n+3)q^n \equiv 3\frac{(q, q^4, q^5; q^5)^3_\infty}{(q; q)^2_\infty}.$$

and

$$\sum_{n=0}^{\infty} p(5n+4)q^n \equiv 0 \pmod 5$$

In the next section, we will show that similar congruences with respect to modulus 7 will hold.

3.4 Atkin-Swinnerton-Dyer congruences for modulus 7

We recall from (3.8) that

$$\begin{aligned}(q;q)^3_\infty &= \widetilde{C_0} + \widetilde{C_1} + \widetilde{C_3} \\ &= L(q^7) + qM(q^7) + q^3 N(q^7)\end{aligned}$$

where $\widetilde{C_0} = L(q^7)$, $\widetilde{C_1} = qM(q^7)$ and $\widetilde{C_3} = q^3 N(q^7)$.

We want to write $L(q^5)$, $M(q^5)$ and $N(q^5)$ as a triple products. Observe that

$$\begin{aligned}\widetilde{C_0} &= L(q^7) \\ &= \sum_{k\equiv 0,6 \pmod 7, k\geq 0}^{\infty} (-1)^k (2k+1) q^{\frac{k^2+k}{2}} \\ &= \sum_{i=0}^{\infty} (-1)^{7i}(2(7i)+1) q^{\frac{(7i)^2+(7i)}{2}} + \sum_{i=0}^{\infty} (-1)^{7i+6}(2(7i+6)+1) q^{\frac{(7i+6)^2+(7i+6)}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}(-1)^{7i}(14i+1)q^{\frac{49i^2+7i}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+6}(14i+13)q^{\frac{49i^2+84i+36+7i+6}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{7i}(14i+1)q^{\frac{49i^2+7i}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+6}(14i+13)q^{\frac{49i^2+91i+42}{2}} \\
&\equiv \sum_{i=0}^{\infty}(-1)^{i}(1)q^{\frac{49i^2+7i}{2}} + \sum_{i=0}^{\infty}(-1)^{i}(-1)q^{\frac{49i^2+91i+42}{2}} \pmod 7 \\
&= \sum_{i=0}^{\infty}(-1)^{i}(1)q^{\frac{49i^2+7i}{2}} - \sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+91i+42}{2}} \\
&= q^0 - q^{21} - q^{28} + q^{91} + q^{105} - q^{210} - \cdots \\
&= \sum_{i=0}^{\infty}(-1)^{i}(1)q^{\frac{49i^2+7i}{2}} - \sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49(i+1)^2-7(i+1)}{2}} \\
&\quad \text{(replace } i \text{ with } i-1 \text{ in the second summation)} \\
&= \sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+7i}{2}} - \sum_{i=1}^{\infty}(-1)^{i-1}q^{\frac{49i^2-7i}{2}} \\
&= \sum_{i=-\infty}^{0}(-1)^{i}q^{\frac{49i^2-7i}{2}} + \sum_{i=1}^{\infty}(-1)^{i}q^{\frac{49i^2-7i}{2}} \\
&= \sum_{i=-\infty}^{\infty}(-1)^{i}q^{\frac{49i^2-7i}{2}} \\
&= (q^{21}, q^{28}, q^{49}; q^{49})_{\infty}.
\end{aligned}
$$

$\widetilde{C_1} = qM(q^7)$,
i.e.

$$
\begin{aligned}
M(q^7) &= q^{-1}\widetilde{B_1} \\
&= q^{-1} \sum_{k\equiv 1,5 \pmod 7, k\geq 0}^{\infty} (-1)^k(2k+1)q^{\frac{k^2+k}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{7i+1}(2(7i+1)+1)q^{\frac{(7i+1)^2+(7i+1)-2}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+5}(2(7i+5)+1)q^{\frac{(7i+5)^2+(7i+5)-2}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{7i+1}(14i+3)q^{\frac{49i^2+14i+1+7i+1-2}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+5}(14i+11)q^{\frac{49i^2+70i+25+7i+5-2}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{7i+1}(14i+3)q^{\frac{49i^2+14i+1+7i+1-2}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+5}(14i+11)q^{\frac{49i^2+70i+25+7i+5-2}{2}}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty}(-1)^{7i+1}(14i+3)q^{\frac{49i^2+21i}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+5}(14i+11)q^{\frac{49i^2+77i+28}{2}} \\
&= -3\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+21i}{2}} - \sum_{i=0}^{\infty}(-1)^{i}(-3)q^{\frac{49i^2+77i+28}{2}} \\
&= -3\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+21i}{2}} + 3\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+77i+28}{2}} \\
&= -3(\sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+21i}{2}} - \sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+77i+28}{2}}) \\
&= -3\widetilde{M(q^7)},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{M(q^7)} &= \sum_{i=0}^{\infty}(-1)^{7i}q^{\frac{49i^2+21i}{2}} - \sum_{i=0}^{\infty}(-1)^{7i}q^{\frac{49(i+1)^2-21(i+1)}{2}} \\
&\quad \text{(replace } i \text{ with } i-1 \text{ in the second summation)} \\
&= \sum_{i=0}^{\infty}(-1)^{i}q^{\frac{49i^2+21i}{2}} - \sum_{i=1}^{\infty}(-1)^{i-1}q^{\frac{49i^2-21i}{2}} \\
&= \sum_{i=-\infty}^{0}(-1)^{i}q^{\frac{49i^2-21i}{2}} - \sum_{i=1}^{\infty}(-1)^{i-1}q^{\frac{49i^2-21i}{2}} \\
&= \sum_{i=-\infty}^{\infty}(-1)^{i}q^{\frac{49i^2-21i}{2}} \\
&= (q^{14}, q^{35}, q^{49}; q^{49})_\infty.
\end{aligned}$$

$\widetilde{C_3} = q^3N(q^7)$, i.e.

$$\begin{aligned}
N(q^7) &= q^{-3}\widetilde{C_3} \\
&= q^{-3}\sum_{k\equiv 2,4 \pmod 7}^{\infty}(-1)^{k}(2k+1)q^{\frac{k^2+k}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{7i+2}(2(7i+2)+1)q^{\frac{(7i+2)^2+(7i+2)-6}{2}} + \sum_{i=0}^{\infty}(-1)^{7i+4}(2(7i+4)+1)q^{\frac{(7i+4)^2+(7i+4)-6}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{7i}(14i+5)q^{\frac{49i^2+28i+4+7i+2-6}{2}} + \sum_{i=0}^{\infty}(-1)^{7i}(14i+9)q^{\frac{49i^2+56i+16+7i+4-6}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^{i}(14i+5)q^{\frac{49i^2+35i}{2}} + \sum_{i=0}^{\infty}(-1)^{i}(14i+9)q^{\frac{49i^2+63i+14}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= 5\sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+35i}{2}} - 5\sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+63i+14}{2}} \\
&= 5[\sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+35i}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+63i+14}{2}}] \\
&= 5\widetilde{N(q^7)},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{N(q^7)} &= \sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+35i}{2}} - \sum_{i=0}^{\infty}(-1)^i q^{\frac{49(i+1)^2-35(i+1)}{2}} \\
&\quad \text{(for the second summation, let } i = i-1\text{)} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+35i}{2}} - \sum_{i=0}^{\infty}(-1)^{i-1} q^{\frac{49i^2-35i}{2}} \\
&= \sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2+35i}{2}} + \sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2-35i}{2}} \\
&= \sum_{i=-\infty}^{0}(-1)^i q^{\frac{49i^2-35i}{2}} + \sum_{i=0}^{\infty}(-1)^i q^{\frac{49i^2-35i}{2}} \\
&= \sum_{i=-\infty}^{\infty}(-1)^i q^{\frac{49i^2-35i}{2}} \\
&= (q^7, q^{42}, q^{49}; q^{49})_\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(q;q)_\infty^3 &= \widetilde{C_0} + \widetilde{C_1} + \widetilde{C_3} \\
&= L(q^7) + qM(q^7) + q^3N(q^7) \\
&= (q^{21}, q^{28}, q^{49}; q^{49})_\infty - 3q(q^{14}, q^{35}, q^{49}; q^{49})_\infty + 5q^3(q^7, q^{42}, q^{49}; q^{49})_\infty
\end{aligned}
\tag{3.11}
$$

Using (3.2) and (3.11), it follows that

$$
\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= \frac{1}{(q;q)_\infty} \\
&= \frac{(q;q)_\infty^6}{(q;q)_\infty^7}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{((q;q)_\infty^3)^2}{(q;q)_\infty^7} \\
&= \frac{(L(q^7) - qM(q^7) + 5q^3N(q^7))^2}{(q;q)_\infty^7} \\
&\equiv \frac{L(q^7)^2 - 6qL(q^7)M(q^7) + 9q^2M(q^7)^2 + 10q^3L(q^7)N(q^7)}{(q^7;q^7)_\infty} \\
&\quad + \frac{-30q^4M(q^7)N(q^7) + 25q^6N(q^7)^2}{(q^7;q^7)_\infty} \\
&\equiv \frac{L(q^7)^2 + qL(q^7)M(q^7) + 2q^2M(q^7)^2 + 3q^3L(q^7)N(q^7)}{(q^7;q^7)_\infty} \\
&\quad + \frac{5q^4M(q^7)N(q^7) + 11q^6N(q^7)^2}{(q^7;q^7)_\infty} \qquad (3.12)
\end{aligned}
$$

From (3.12), modulo 7, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(7n)q^{7n} &\equiv \frac{L(q^7)^2}{(q^7;q^7)_\infty} \\
&\equiv \frac{(q^{21}, q^{28}, q^{49}; q^{49})_\infty^2}{(q^7;q^7)_\infty},
\end{aligned}
$$

i.e.

$$
\sum_{n=0}^{\infty} p(7n)q^{n} \equiv \frac{(q^3, q^4, q^7; q^7)_\infty^2}{(q;q)_\infty}.
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(7n+1)q^{7n} &\equiv \frac{L(q^7)M(q^7)}{(q^7;q^7)_\infty} \\
&\equiv \frac{(q^{21}, q^{28}, q^{49}; q^{49})_\infty (q^{14}, q^{35}, q^{49}; q^{49})_\infty}{(q^7;q^7)_\infty},
\end{aligned}
$$

i.e.

$$
\sum_{n=0}^{\infty} p(7n+1)q^{n} \equiv \frac{(q^3, q^4, q^7; q^7)_\infty (q^2, q^5, q^7; q^7)_\infty}{(q;q)_\infty}.
$$

$$\begin{aligned}\sum_{n=0}^{\infty} p(7n+2)q^{7n} &\equiv 2\frac{M(q^7)^2}{(q^7;q^7)_\infty} \\ &\equiv 2\frac{(q^{14},q^{35},q^{49};q^{49})_\infty^2}{(q^7;q^7)_\infty},\end{aligned}$$

i.e.

$$\sum_{n=0}^{\infty} p(7n+2)q^{n} \equiv 2\frac{(q^2,q^5,q^7;q^7)_\infty^2}{(q;q)_\infty}.$$

$$\begin{aligned}\sum_{n=0}^{\infty} p(7n+3)q^{7n} &\equiv 5\frac{L(q^7)N(q^7)}{(q^7;q^7)_\infty} \\ &\equiv 3\frac{(q^{21},q^{28},q^{49};q^{49})_\infty(q^7,q^{42},q^{49};q^{49})_\infty}{(q^7;q^7)_\infty},\end{aligned}$$

i.e.

$$\sum_{n=0}^{\infty} p(7n+3)q^{n} \equiv 3\frac{(q^3,q^4,q^7;q^7)_\infty(q,q^6,q^7;q^7)_\infty}{(q;q)_\infty}.$$

$$\begin{aligned}\sum_{n=0}^{\infty} p(7n+4)q^{7n} &\equiv 5\frac{M(q^7)N(q^7)}{(q^7;q^7)_\infty} \\ &\equiv 5\frac{(q^{14},q^{35},q^{49};q^{49})_\infty(q^7,q^{42},q^{49};q^{49})_\infty}{(q^7;q^7)_\infty},\end{aligned}$$

i.e.

$$\sum_{n=0}^{\infty} p(7n+4)q^{n} \equiv 5\frac{(q^2,q^5,q^7;q^7)_\infty(q,q^6,q^7;q^7)_\infty}{(q;q)_\infty}.$$

We also have

$$\sum_{n=0}^{\infty} p(7n+5)q^{n} \equiv 0$$

and

$$\sum_{n=0}^{\infty} p(7n+6)q^{7n} \equiv 4\frac{N(q^7)^2}{(q^7;q^7)_\infty}$$
$$\equiv 4\frac{(q^7,q^{42},q^{49};q^{49})_\infty^2}{(q^7;q^7)_\infty},$$

i.e.

$$\sum_{n=0}^{\infty} p(7n+6)q^{n} \equiv 4\frac{(q,q^6,q^7;q^7)_\infty^2}{(q;q)_\infty}.$$

3.5 Ramanujan's most beautiful identity

In this section, our goal is to prove the identity:

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6}. \tag{3.13}$$

We know from (3.4) that

$$\begin{aligned} E(q) &= (q;q)_\infty \\ &= \sum_{-\infty}^{\infty}(-1)^k q^{\frac{3k^2-k}{2}} \\ &\equiv E_0+E_1+E_2. \end{aligned}$$

We can write, with $\zeta \neq 1$ a fifth root of unity,

$$\begin{aligned} \frac{1}{(q;q)_\infty} &= \frac{E(\zeta q)E(\zeta^2 q)E(\zeta^3 q)E(\zeta^4 q)}{E(q)E(\zeta q)E(\zeta^2 q)E(\zeta^3 q)E(\zeta^4 q)} \\ &= \frac{(\zeta q;\zeta q)_\infty(\zeta^2 q;\zeta^2 q)_\infty(\zeta^3 q;\zeta^3 q)_\infty(\zeta^4 q;\zeta^4 q)_\infty}{(q;q)_\infty(\zeta q;\zeta q)_\infty(\zeta^2 q;\zeta^2 q)_\infty(\zeta^3 q;\zeta^3 q)_\infty(\zeta^4 q;\zeta^4 q)_\infty}. \end{aligned} \tag{3.14}$$

Let $E(q)$ be the denominator of (3.14). Then

$$E(q) = (q;q)_\infty(\zeta q;\zeta q)_\infty(\zeta^2 q;\zeta^2 q)_\infty(\zeta^3 q;\zeta^3 q)_\infty(\zeta^4 q;\zeta^4 q)_\infty$$

$$
\begin{aligned}
&= \prod_{n\geq 1}(1-q^n)(1-\zeta^n q^n)(1-\zeta^{2n}q^n)(1-\zeta^{3n}q^n)(1-\zeta^{4n}q^n) \\
&= \prod_{n\equiv 0 \pmod 5\ n\geq 1}(1-q^n)(1-\zeta^n q^n)(1-\zeta^{2n}q^n)(1-\zeta^{3n}q^n)(1-\zeta^{4n}q^n) \\
&\quad \times \prod_{n\not\equiv 0 \pmod 5\ n\geq 1}(1-q^n)(1-\zeta^n q^n)(1-\zeta^{2n}q^n)(1-\zeta^{3n}q^n)(1-\zeta^{4n}q^n) \\
&= a_1(q)\times a_2(q)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1(q) &= \prod_{n\equiv 0 \pmod 5\ n\geq 1}(1-q^n)(1-\zeta^n q^n)(1-\zeta^{2n}q^n)(1-\zeta^{3n}q^n)(1-\zeta^{4n}q^n) \\
&= \prod_{n\geq 1}(1-q^{5n})(1-\zeta^{5n}q^{5n})(1-\zeta^{10n}q^{5n})(1-\zeta^{15n}q^{5n})(1-\zeta^{20n}q^{5n}) \\
&= \prod_{n\geq 1}(1-q^{5n})(1-q^{5n})(1-q^{5n})(1-q^{5n})(1-q^{5n}) \\
&= \prod_{n\geq 1}(1-q^{5n})^5
\end{aligned}
$$

and

$$
\begin{aligned}
a_2(q) &= \prod_{n\not\equiv 0 \pmod 5,\ n\geq 1}(1-q^n)(1-\zeta^n q^n)(1-\zeta^{2n}q^n)(1-\zeta^{3n}q^n)(1-\zeta^{4n}q^n) \\
&= \prod_{n\not\equiv 0 \pmod 5} -\zeta^3q^n+\zeta^7q^{2n}-\zeta^2q^n+\zeta^6q^{2n}+\zeta^5q^{2n}-\zeta^9q^{3n}-q^n\zeta+q^{2n}\zeta^5+2q^{2n}\zeta^4 \\
&\quad -q^{3n}\zeta^8-2q^{3n}\zeta^6+q^{4n}\zeta^{10}+q^{2n}\zeta^2-2q^{3n}\zeta^5+q^{4n}\zeta^9+q^{2n}\zeta-3q^{3n}\zeta^4+q^{4n}\zeta^8-q^{3n}\zeta^3 \\
&\quad +q^{4n}\zeta^7+q^{4n}\zeta^6-q^{5n}\zeta^{10}-q^n\zeta^4+2q^{2n}\zeta^3-2q^{3n}\zeta^7-q^n+1 \\
&= \prod_{n\not\equiv 0 \pmod 5} -\zeta^3q^n+\zeta^2q^{2n}-\zeta^2q^n+\zeta^1q^{2n}+q^{2n}-\zeta^4q^{3n}-q^n\zeta+q^{2n}+2q^{2n}\zeta^4 \\
&\quad -q^{3n}\zeta^3-2q^{3n}\zeta+q^{4n}+q^{2n}\zeta^2-2q^{3n}+q^{4n}\zeta^4+q^{2n}\zeta-3q^{3n}\zeta^4+q^{4n}\zeta^3-q^{3n}\zeta^3 \\
&\quad +q^{4n}\zeta^2+q^{4n}\zeta-q^{5n}-q^n\zeta^4+2q^{2n}\zeta^3-2q^{3n}\zeta^2-q^n+1 \\
&= \prod_{n\not\equiv 0 \pmod 5} q^n(-\zeta^3-\zeta^2-\zeta-\zeta^4-1) \\
&\quad +q^{2n}(\zeta^2+\zeta+1+1+2\zeta^4+\zeta^2+\zeta+2\zeta^3) \\
&\quad +q^{3n}(-\zeta^4-\zeta^3-2\zeta-2-\zeta^4-\zeta^3-2\zeta^2)
\end{aligned}
$$

$$+q^{4n}(1+\zeta^4+\zeta^3+\zeta^2+\zeta)$$
$$+q^{5n}(-1)+1.$$

Note that

$$\begin{aligned}
-\zeta^3-\zeta^2-\zeta-\zeta^4-1 &= -(1+\zeta+\zeta^2+\zeta^3+\zeta^4) \\
&= -\frac{1-\zeta^5}{1-\zeta} \\
&= 0, \text{since } \zeta^5=1,
\end{aligned}$$

$$\begin{aligned}
\zeta^2+\zeta+1+1+2\zeta^4+\zeta^2+\zeta+2\zeta^3 &= 2(1+\zeta+\zeta^2+\zeta^3+\zeta^4) \\
&= 2\frac{1-\zeta^5}{1-\zeta} \\
&= 0, \text{since } \zeta^5=1,
\end{aligned}$$

$$\begin{aligned}
-\zeta^4-\zeta^3-2\zeta-2-\zeta^4-\zeta^3-2\zeta^2 &= -2(1+\zeta+\zeta^2+\zeta^3+\zeta^4) \\
&= -2\frac{1-\zeta^5}{1-\zeta} \\
&= 0, \text{since } \zeta^5=1.
\end{aligned}$$

Thus

$$a_2(q) = \prod_{n\not\equiv 0 \pmod 5} (1-q^{5n}).$$

So we have

$$\begin{aligned}
E(q) &= a_1(q)\cdot a_2(q) \\
&= \prod_{n=1}^{\infty}(1-q^{5n})^5 \cdot \prod_{n\not\equiv 0 \pmod 5}(1-q^{5n}) \times \prod_{n\equiv 0 \pmod 5}\frac{(1-q^{5n})}{(1-q^{5n})} \\
&= \frac{\prod_{n\geq 1}(1-q^{5n})^5 \cdot \prod_{n\geq 1}(1-q^{5n})}{\prod_{n\equiv 0 \pmod 5}(1-q^{5n})}
\end{aligned}$$

$$= \frac{\prod_{n\geq 1}(1-q^{5n})^6}{\prod_{n\geq 1}(1-q^{25n})}$$
$$= \frac{(q^5;q^5)_\infty{}^6}{(q^{25};q^{25})_\infty}$$
$$= \frac{E(q^5)^6}{E(q^{25})}. \qquad (3.15)$$

We note that the numerator in (3.14) is

$$\begin{aligned}
N(q) &= E(\zeta q)E(\zeta^2 q)E(\zeta^3 q)E(\zeta^4 q) \\
&= (E_0+\zeta E_1+\zeta^2 E_2)(E_0+\zeta^2 E_1+\zeta^4 E_2)(E_0+\zeta^3 E_1+\zeta^6 E_2)(E_0+\zeta^4 E_1+\zeta^8 E_2) \\
&= ({E_0}^2+\zeta^2 E_0E_1+\zeta^4 E_0E_2+\zeta E_0E_1+\zeta^3 {E_1}^2+\zeta^5 E_1E_2+\zeta^2 E_0E_2+\zeta^4 E_1E_2+\zeta^6 {E_2}^2) \\
&\quad \times({E_0}^2+\zeta^4 E_0E_1+\zeta^8 E_0E_2+\zeta^3 E_0E_1+\zeta^7 {E_1}^2+\zeta^{11} E_1E_2+\zeta^6 E_0E_2+\zeta^{10} E_1E_2+\zeta^{14} {E_2}^2) \\
&= ({E_0}^4+\zeta^4 {E_0}^3E_1+\zeta^8 {E_0}^3E_2+\zeta^3 {E_0}^3E_1+\zeta^7 {E_0}^2{E_1}^2+\zeta^{11} {E_0}^2E_1E_2+\zeta^6 {E_0}^3E_2 \\
&\quad +\zeta^{10} {E_0}^2E_1E_2+\zeta^{14} {E_0}^2{E_2}^2 \\
&\quad +\zeta^2 {E_0}^3E_1+\zeta^6 {E_0}^2{E_1}^2+\zeta^{10} {E_0}^2E_1E_2+\zeta^5 {E_0}^2{E_1}^2+\zeta^9 E_0{E_1}^3+\zeta^{13} E_0{E_1}^2E_2+\zeta^8 {E_0}^2E_1E_2 \\
&\quad +\zeta^{12} E_0{E_1}^2E_2+\zeta^{16} E_0E_1{E_2}^2 \\
&\quad +\zeta^4 {E_0}^3E_2+\zeta^8 {E_0}^2E_1E_2+\zeta^{12} {E_0}^2{E_2}^2+\zeta^7 {E_0}^2E_1E_2+\zeta^{11} E_0{E_1}^1E_2+\zeta^{15} E_0E_1{E_2}^2 \\
&\quad +\zeta^{10} {E_0}^2{E_2}^2+\zeta^{12} E_0{E_1}^2E_2+\zeta^{16} E_0E_1{E_2}^2 \\
&\quad +\zeta {E_0}^3E_1+\zeta^5 {E_0}^2{E_1}^2+\zeta^9 {E_0}^2E_1E_2+\zeta^4 {E_0}^2{E_1}^2+\zeta^8 E_0{E_1}^3+\zeta^{12} E_0{E_1}^2E_2+\zeta^7 {E_0}^2E_1E_2 \\
&\quad +\zeta^{11} E_0{E_1}^2E_2+\zeta^{15} E_0E_1{E_2}^2 \\
&\quad +\zeta^3 {E_0}^2{E_1}^2+\zeta^7 E_0{E_1}^3+\zeta^{11} E_0{E_1}^2E_2+\zeta^6 E_0{E_1}^3+\zeta^{10} {E_1}^4+\zeta^{14} {E_1}^3E_2+\zeta^9 E_0{E_1}^2E_2 \\
&\quad +\zeta^{13} {E_1}^3E_2+\zeta^{17} {E_1}^2{E_2}^2 \\
&\quad +\zeta^5 {E_0}^2E_1E_2+\zeta^9 E_0{E_1}^2E_2+\zeta^{13} E_0E_1{E_2}^2+\zeta^8 E_0{E_1}^2E_2+\zeta^{12} {E_1}^3E_2+\zeta^{16} {E_1}^2{E_2}^2 \\
&\quad +\zeta^{11} E_0E_1{E_2}^2+\zeta^{15} {E_1}^2{E_2}^2+\zeta^{19} E_1{E_2}^3 \\
&\quad +\zeta^2 {E_0}^3E_2+\zeta^6 {E_0}^2E_1E_2+\zeta^{10} {E_0}^2{E_2}^2+\zeta^5 {E_0}^2E_1E_2+\zeta^9 E_0{E_1}^2E_2+\zeta^{13} E_0E_1{E_2}^2 \\
&\quad +\zeta^8 {E_0}^2{E_2}^2+\zeta^{12} E_0E_1{E_2}^2+\zeta^{16} E_0{E_2}^3 \\
&\quad +\zeta^4 {E_0}^2E_1E_2+\zeta^8 E_0{E_1}^2E_2+\zeta^{12} E_0E_1{E_2}^2+\zeta^7 E_0{E_1}^2E_2+\zeta^{11} {E_1}^3E_2+\zeta^{15} {E_1}^2{E_2}^2 \\
&\quad +\zeta^{10} E_0E_1{E_2}^2+\zeta^{14} {E_1}^2{E_2}^2+\zeta^{18} E_1{E_2}^3 \\
&\quad +\zeta^6 {E_0}^2{E_2}^2+\zeta^{10} E_0E_1{E_2}^2+\zeta^{14} E_0{E_2}^3+\zeta^9 E_0E_1{E_2}^2+\zeta^{13} {E_1}^2{E_2}^2+\zeta^{17} E_1{E_2}^3+\zeta^{12} E_0{E_2}^3 \\
&\quad +\zeta^{16} E_1{E_2}^3+\zeta^{20} E_0{E_2}^4
\end{aligned}$$

$$
\begin{aligned}
= \;& {E_0}^4 + \zeta^4 {E_0}^3 E_1 + \zeta^3 {E_0}^3 E_2 + \zeta^3 {E_0}^3 E_1 + \zeta^2 {E_0}^2 {E_1}^2 + \zeta {E_0}^2 E_1 E_2 + \zeta {E_0}^3 E_2 + {E_0}^2 E_1 E_2 \\
& + \zeta^4 {E_0}^2 {E_2}^2 \\
& + \zeta^2 {E_0}^3 E_1 + \zeta {E_0}^2 {E_1}^2 + {E_0}^2 E_1 E_2 + {E_0}^2 {E_1}^2 + \zeta^4 E_0 {E_1}^3 + \zeta^3 E_0 {E_1}^2 E_2 + \zeta^3 {E_0}^2 E_1 E_2 \\
& + \zeta^2 E_0 {E_1}^2 E_2 + \zeta E_0 E_1 {E_2}^2 \\
& + \zeta^4 {E_0}^3 E_2 + \zeta^3 {E_0}^2 E_1 E_2 + \zeta^2 {E_0}^2 {E_2}^2 + \zeta^2 {E_0}^2 E_1 E_2 + \zeta E_0 {E_1}^1 E_2 + E_0 E_1 {E_2}^2 \\
& + {E_0}^2 {E_2}^2 + \zeta^2 E_0 {E_1}^2 E_2 + \zeta E_0 E_1 {E_2}^2 \\
& + \zeta {E_0}^3 E_1 + {E_0}^2 {E_1}^2 + \zeta^4 {E_0}^2 E_1 E_2 + \zeta^4 {E_0}^2 {E_1}^2 + \zeta^3 E_0 {E_1}^3 + \zeta^2 E_0 {E_1}^2 E_2 + \zeta^2 {E_0}^2 E_1 E_2 \\
& + \zeta E_0 {E_1}^2 E_2 + E_0 E_1 {E_2}^2 \\
& + \zeta^3 {E_0}^2 {E_1}^2 + \zeta^2 E_0 {E_1}^3 + \zeta E_0 {E_1}^2 E_2 + \zeta E_0 {E_1}^3 + {E_1}^4 + \zeta^{14} {E_1}^3 E_2 + \zeta^4 E_0 {E_1}^2 E_2 \\
& + \zeta^3 {E_1}^3 E_2 + \zeta^2 {E_1}^2 {E_2}^2 \\
& + {E_0}^2 E_1 E_2 + \zeta^4 E_0 {E_1}^2 E_2 + \zeta^3 E_0 E_1 {E_2}^2 + \zeta^3 E_0 {E_1}^2 E_2 + \zeta^2 {E_1}^3 E_2 + \zeta {E_1}^2 {E_2}^2 \\
& + \zeta E_0 E_1 {E_2}^2 + {E_1}^2 {E_2}^2 + \zeta^4 E_1 {E_2}^3 \\
& + \zeta^2 {E_0}^3 E_2 + \zeta {E_0}^2 E_1 E_2 + {E_0}^2 {E_2}^2 + {E_0}^2 E_1 E_2 + \zeta^4 E_0 {E_1}^2 E_2 + \zeta^4 E_0 E_1 {E_2}^2 + \zeta^3 {E_0}^2 {E_2}^2 \\
& + \zeta^2 E_0 E_1 {E_2}^2 + \zeta E_0 {E_2}^3 \\
& + \zeta^4 {E_0}^2 E_1 E_2 + \zeta^3 E_0 {E_1}^2 E_2 + \zeta^2 E_0 E_1 {E_2}^2 + \zeta^2 E_0 {E_1}^2 E_2 + \zeta {E_1}^3 E_2 + {E_1}^2 {E_2}^2 \\
& + E_0 E_1 {E_2}^2 + \zeta^4 {E_1}^2 {E_2}^2 + \zeta^3 E_1 {E_2}^3 \\
& + \zeta {E_0}^2 {E_2}^2 + E_0 E_1 {E_2}^2 + \zeta^4 E_0 {E_2}^3 + \zeta^4 E_0 E_1 {E_2}^2 + \zeta^3 {E_1}^2 {E_2}^2 + \zeta^2 E_1 {E_2}^3 + \zeta^2 E_0 {E_2}^3 \\
& + \zeta E_1 {E_2}^3 + E_0 {E_2}^4 \\
= \;& {E_0}^4 + {E_0}^3 E_1 (\zeta^4 + \zeta^3 + \zeta^2 + \zeta) + {E_0}^3 E_2 (\zeta^4 + \zeta^3 + \zeta^2 + \zeta) \\
& + {E_0}^2 {E_1}^2 (\zeta^2 + \zeta + 1 + 1 + \zeta^3 + \zeta^4) \\
& + {E_0}^2 E_1 E_2 (\zeta + 1 + 1 + \zeta^3 + \zeta^3 + \zeta^4 + \zeta^2 + 1 + \zeta + 1 + \zeta^4 + \zeta^2) \\
& + {E_0}^2 {E_2}^2 (\zeta^4 + \zeta^2 + 1 + 1 + \zeta^3 + \zeta) + E_0 {E_1}^3 (\zeta^4 + \zeta^3 + \zeta^2 + \zeta) \\
& + E_0 {E_1}^2 E_2 (\zeta^3 + \zeta^2 + \zeta + \zeta^2 + \zeta + \zeta + \zeta^4 + \zeta^4 + \zeta^3 + \zeta^4 + \zeta^3 + \zeta^2) \\
& + E_0 E_1 {E_2}^2 (\zeta + \zeta^2 + 1 + \zeta^3 + \zeta + \zeta^3 + \zeta^2 + 1 + 1 + \zeta^4 + 1 + \zeta) \\
& + E_0 {E_2}^2 (\zeta^3 + \zeta + \zeta^4 + \zeta^2) + {E_1}^4 + {E_1}^3 E_2 (\zeta^4 + \zeta^3 + \zeta^2 + \zeta) \\
& + {E_1}^2 {E_2}^2 (\zeta^2 + \zeta + 1 + 1 + \zeta^4 + \zeta^3) + E_1 {E_2}^3 (\zeta^4 + \zeta^3 + \zeta^2 + \zeta) + {E_2}^4.
\end{aligned}
$$

Since

$$\zeta^4 + \zeta^3 + \zeta^2 + \zeta = \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 - 1$$

$$= \frac{1-\zeta^5}{1-\zeta} - 1$$
$$= -1,$$

we have

$$\begin{aligned} N(q) &= {E_0}^4 + {E_0}^3 E_1(-1) + {E_0}^3 E_2(-1) + {E_0}^2 {E_1}^2 \\ &\quad + {E_0}^2 E_1 E_2(2) + {E_0}^2 {E_2}^2(1) \\ &\quad + E_0 {E_1}^3(-1) + E_0 {E_1}^2 E_2(-3) \\ &\quad + E_0 E_1 {E_2}^2(2) + E_0 {E_2}^3(-1) \\ &\quad + {E_1}^4 + {E_1}^3 E_2(-1) + {E_1}^2 {E_2}^2(1) + E_1 {E_2}^3(-1) + {E_2}^4 \\ &= ({E_0}^4 + 2E_0 E_1 {E_2}^2 - {E_1}^3 E_2) + (-{E_0}^3 E_1 - E_0 {E_2}^3 + {E_1}^2 {E_2}^2) \\ &\quad + (-{E_0}^3 E_2 + {E_0}^2 {E_1}^2 - E_1 {E_2}^3) + (2{E_0}^2 E_1 E_2 - E_0 {E_1}^3 + {E_2}^4) \\ &\quad + ({E_0}^2 {E_2}^2 - 3E_0 {E_1}^2 E_2 + {E_1}^4). \end{aligned} \tag{3.16}$$

It now follows from (3.15) and (3.16) that

$$\sum_{n=0}^{\infty} p(5n) q^{5n} = \frac{E(q^{25})}{E(q^5)^6}({E_0}^4 + 2E_0 E_1 {E_2}^2 - {E_1}^3 E_2),$$
$$\sum_{n=0}^{\infty} p(5n+1) q^{5n+1} = \frac{E(q^{25})}{E(q^5)^6}(-{E_0}^3 E_1 - E_0 {E_2}^3 + {E_1}^2 {E_2}^2),$$
$$\sum_{n=0}^{\infty} p(5n+2) q^{5n+2} = \frac{E(q^{25})}{E(q^5)^6}(-{E_0}^3 E_2 + {E_0}^2 {E_1}^2 - E_1 {E_2}^3),$$
$$\sum_{n=0}^{\infty} p(5n+3) q^{5n+3} = \frac{E(q^{25})}{E(q^5)^6}(2{E_0}^2 E_1 E_2 - E_0 {E_1}^3 + {E_2}^4)$$

and

$$\sum_{n=0}^{\infty} p(5n+4) q^{5n+4} = \frac{E(q^{25})}{E(q^5)^6}({E_0}^2 {E_2}^2 - 3E_0 {E_1}^2 E_2 + {E_1}^4). \tag{3.17}$$

To simplify (3.17), we first need to prove that

$$E_0 E_2 = -{E_1}^2 \tag{3.18}$$

and

$$E_1 = -qE(q^{25}). \tag{3.19}$$

We recall from (3.4) and (4.2) that

$$(E_0 + E_1 + E_2)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}. \tag{3.20}$$

The left-hand side of (3.20) is

$$\begin{aligned}(E_0 + E_1 + E_2)^3 &= (E_0 + E_1 + E_2)({E_0}^2 + E_0E_1 + E_0E_2 + E_0E_1 + {E_1}^2 + E_1E_2 \\ &\quad +E_2E_0 + E_2E_1 + {E_2}^2) \\ &= (E_0 + E_1 + E_2)({E_0}^2 + 2E_0E_1 + 2E_0E_2 + 2E_1E_2 + {E_1}^2 + {E_2}^2) \\ &= {E_0}^3 + 2{E_0}^2E_1 + 2{E_0}^2E_2 + 2E_0E_1E_2 + 2E_0{E_1}^2 + 2E_0{E_2}^2 \\ &\quad +{E_0}^2E_1 + 2E_0{E_1}^2 + 2E_0E_1E_2 + 2{E_1}^2E_2 + {E_1}^3 + E_1{E_2}^2 \\ &\quad +{E_0}^2E_2 + 2E_0E_1E_2 + 2E_0{E_2}^2 + 2E_1{E_2}^2 + {E_1}^2E_2 + {E_2}^3 \\ &= {E_0}^3 + {E_1}^3 + {E_2}^3 + 3{E_0}^2E_1 + 3E_0{E_2}^2 + 3{E_0}^2E_2 + 3E_0{E_2}^2 + 3{E_1}^2E_2 \\ &\quad +3E_1{E_2}^2 + 6E_0E_1E_2 \\ &= ({E_0}^3 + 3E_1{E_2}^2) + (3{E_0}^2E_1 + {E_2}^3) + 3E_0(E_0E_2 + {E_1}^2) \\ &\quad +(6E_0E_1E_2 + {E_1}^3) + 3E_2(E_0E_2 + {E_1}^2).\end{aligned}$$

Thus

$$\begin{aligned}&({E_0}^3 + 3E_1{E_2}^2) + (3{E_0}^2E_1 + {E_2}^3) + 3E_0(E_0E_2 + {E_1}^2) + (6E_0E_1E_2 + {E_1}^3) + 3E_2(E_0E_2 + {E_1}^2) \\ &\qquad = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}.\end{aligned} \tag{3.21}$$

We've already seen in (3.5), that

$$\frac{k^2+k}{2} \equiv 0, 1 \text{ or } 3 \pmod 5$$

and so if we only only extract the terms in which the exponent on q is 2 or 4 modulo 5, we obtain

$$3E_0(E_0E_2 + {E_1}^2) + 3E_2(E_0E_2 + {E_1}^2) = 0,$$

i.e.

$$(E_0E_2 + {E_1}^2)(3E_0 + 3E_2) = 0$$

which implies

$$E_0E_2 = -{E_1}^2$$

and this proves (3.18).
We have also observed in (3.4) that

$$E = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}} \tag{3.22}$$

and that

$$\frac{3k^2-k}{2} \equiv 1 \pmod 5$$

if and only if

$$k \equiv 1 \pmod 5.$$

Thus

$$\begin{aligned} E_1 &= \sum_{k=-\infty}^{\infty} (-1)^{5k+1} q^{\frac{3(5k+1)^2-(5k+1)}{2}} \\ &= \sum_{-\infty}^{\infty} (-1)^{5k+1} q^{\frac{75k^2+25k+2}{2}} \\ &= -q\sum_{-\infty}^{\infty} (-1)^k q^{\frac{75k^2+25k}{2}} \\ &= -qE(q^{25}) \end{aligned}$$

and this proves (3.19).

We have also seen from (3.17) that

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = \frac{E(q^{25})}{E(q^5)^6}({E_0}^2{E_2}^2 - 3E_0{E_1}^2E_2 + {E_1}^4)$$

and through the application of (3.18) and (3.19), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} &= \frac{E(q^{25})}{E(q^5)^6}\left[(-{E_1}^2)^2 - 3(-{E_1}^2)({E_1}^2) + {E_1}^4)\right] \\ &= \frac{E(q^{25})}{E(q^5)^6}(5{E_1}^4) \\ &= \frac{E(q^{25})}{E(q^5)^6}(5(qE(q^{25})^4) \\ &= 5q^4\frac{E(q^{25})^5}{E(q^5)^6}.\end{aligned}$$

Finally, if we divide both sides by q^4 and replace q^5 by q, we obtain

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6} \tag{3.23}$$

which is (3.13), the Ramanujan's most beautiful identity.

3.6 A mod 7 version of Ramanujan's most beautiful identity

In this section, our goal iis to prove that,

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)^3{}_\infty}{(q;q)^4{}_\infty} + 49q\frac{(q^7;q^7)^7{}_\infty}{(q;q)^8{}_\infty}. \tag{3.24}$$

Note that $\frac{3k^2-k}{2} \equiv 2 \pmod 7$ if and only if $k \equiv 1 \pmod 7$.

and so

$$E = E_0 + E_1 + E_2 + E_5 \tag{3.25}$$

where E_i consists of those terms in which the exponent on q is congrurnt to i modulo 7.

Thus

$$\begin{aligned} E_2 &= \sum_{k=-\infty}^{\infty} (-1)^{7k+1} q^{\frac{3(7k+1)^2+(7k+1)}{2}} \\ &= \sum_{-\infty}^{\infty} (-1)^{7k+1} q^{\frac{147k^2+49k+4}{2}} \\ &= -q^2 \sum_{-\infty}^{\infty} (-1)^k q^{\frac{147k^2+49k}{2}} \\ &= -q^2 (q^{49}, q^{98}, q^{147}; q^{147})_\infty \\ &= -q^2 (q^{49}; q^{49})_\infty. \end{aligned} \tag{3.26}$$

By (3.26), and (3.25), we can write

$$\begin{aligned} E &= E_0 + E_1 + E_2 + E_5 \\ &= E_0 + E_1 - q^2 E(q^{49}) + E_5 \\ &= q^2 E(q^{49})(\alpha + \beta - 1 + \gamma) \end{aligned} \tag{3.27}$$

where

$$\alpha = -\frac{E_0}{E_2},$$

$$\beta = -\frac{E_1}{E_2}$$

and

$$\gamma = -\frac{E_5}{E_2}.$$

Hence,

$$q^6 E(q^{49})^3 (\alpha + \beta - 1 + \gamma)^3 = \sum_{k \geq 0} (-1)^k (2k+1) q^{\frac{k^2+k}{2}}. \tag{3.28}$$

The last product on the left-hand side of (3.28) becomes

$$
\begin{aligned}
(\alpha+\beta-1+\gamma)^3 &= [(\alpha+\beta)+(\gamma-1)]^3 \\
&= (\alpha+\beta)^3+3(\alpha+\beta)^2(\gamma-1)+3(\gamma-1)^2(\alpha+\beta)+(\gamma-1)^3 \\
&= \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3+3\gamma(\alpha^2+2\alpha\beta+\beta^2)-3(\alpha^2+2\alpha\beta+\beta^2) \\
&\quad +3\alpha(\gamma^2-2\gamma+1)+3\beta(\gamma^2-2\gamma+1)+\gamma^3-3\gamma^2+3\gamma-1 \\
&= \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3+3\alpha^2\gamma+2\alpha\beta\gamma+3\beta^2\gamma-3\alpha^2-6\alpha\beta-3\beta^2 \\
&\quad +3\alpha\gamma^2-6\alpha\gamma+3\alpha+3\beta\gamma^2-6\beta\gamma_3\beta+\gamma^3-3\gamma^2+3\gamma-1 \\
&= (\alpha^3-6\alpha\gamma+3\beta^\gamma)+(3\alpha^2\beta-6\beta\gamma+\gamma^3)+3(-\alpha^2+\alpha\beta^2+\gamma)+(-6\alpha\beta \\
&\quad +3\alpha\gamma^2+\beta^3)+3(\alpha-\beta^2+\beta\gamma^2)+3(\alpha^2\gamma+\beta-\gamma^2).
\end{aligned}
$$

Then substituting (3.29) in (3.28), we get

$$
\begin{aligned}
q^6E(q^{49})^3[(\alpha^3-6\alpha\gamma+3\beta^\gamma)+(3\alpha^2\beta-6\beta\gamma+\gamma^3)+3(-\alpha^2+\alpha\beta^2+\gamma)+(-6\alpha\beta+3\alpha\gamma^2+\beta^3) \\
+3(\alpha-\beta^2+\beta\gamma^2)+3(\alpha^2\gamma+\beta-\gamma^2)+(6\alpha\beta\gamma-1)] = \sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}.
\end{aligned}
$$

Since $\frac{n^2+n}{2} \equiv 6 \pmod 7$ if and only if $n \equiv 3 \pmod 7$ and if we consider and extract terms in which exponent on q is 6 modulo 7, we have

$$
\begin{aligned}
q^6E(q^{49})^3(6\alpha\beta\gamma-1) &= \sum_{k\geq 0}(-1)^{7k+3}[2(7k+3)+1]q^{\frac{(7k+3)^2+(7k+3)}{2}} \\
&= \sum_{k\geq 0}(-1)^{7k+3}[14k+7]q^{\frac{49k^2+49k}{2}}q^6 \\
&= (-1)^3(7)(q^6)\sum_{k\geq 0}(-1)^{7k}[2k+1]q^{\frac{49k^2+49k}{2}} \\
&= -7q^6E(q^{49})^3. \qquad (3.29)
\end{aligned}
$$

It then follows that

$$
q^6E(q^{49})^3(6\alpha\beta\gamma-1) = -7q^6E(q^{49})^3
$$

and

$$
\alpha\beta\gamma = -1. \qquad (3.30)
$$

We have also noted that $\frac{n^2+n}{2}$ is not congruent to 2, 4 and 5 modulo 7, hence,

$$-\alpha^2+\alpha\beta^2+\gamma=0, \tag{3.31}$$

$$\alpha-\beta^2+\beta\gamma^2=0, \tag{3.32}$$

$$\alpha^2\gamma+\beta-\gamma^2=0. \tag{3.33}$$

If we multiply the three equations (3.31), (3.32) and (3.33), we obtain

$$\begin{aligned} 0 &= (-\alpha^2+\alpha\beta^2+\gamma)(\alpha-\beta^2+\beta\gamma^2)(\alpha^2\gamma+\beta-\gamma^2) \\ &= (-\alpha^2+\alpha\beta^2+\gamma)(\alpha^3\gamma+\alpha\beta-\alpha\gamma^2-\beta^2\alpha^2\gamma-\beta^3+\beta^2\gamma^2+\alpha^2\beta\gamma^3+\beta^2\gamma^2-\beta\gamma^4) \\ &= -\alpha^5\gamma-\alpha^3\beta+\alpha^3\gamma^2+\alpha^4\beta^2\gamma+\alpha^2\beta^3-2\alpha^2\beta^2\gamma^2-\alpha^4\beta\gamma^3+\alpha^2\beta\gamma^4 \\ &\quad +\alpha^4\beta^2\gamma+\alpha^2\beta^3-\alpha^2\beta^2\gamma^2-\alpha^3\beta^4\gamma-\alpha\beta^5+2\alpha\beta^4\gamma^2+\alpha^3\beta^3\gamma^3-\alpha\beta^3\gamma^4 \\ &\quad +\alpha^3\gamma^2+\alpha\beta\gamma-\alpha\gamma^3-\alpha^2\beta^2\gamma^2-\gamma\beta^3+2\beta^2\gamma^3+\alpha^2\beta\gamma^4-\beta\gamma^5. \end{aligned} \tag{3.34}$$

From (3.30), $\alpha\beta\gamma=-1$, (3.34) becomes

$$\begin{aligned} 0 &= -\alpha^5\gamma-\alpha^3\beta+\alpha^3\gamma^2+(-1)\alpha^3\beta+\alpha^2\beta^3-2(-1)(-1)-(-1)\alpha^3\gamma^2+(-1)\alpha\gamma^3 \\ &\quad +(-1)\alpha^3\beta+\alpha^2\beta^3-(-1)(-1)-(-1)\alpha^2\beta^3-\alpha\beta^5+2(-1)\beta^3\gamma+(-1)(-1)(-1)-(-1)\beta^2\gamma^3 \\ &\quad +\alpha^3\gamma^2+(-1)-\alpha\gamma^3-(-1)(-1)-\gamma\beta^3+2\beta^2\gamma^3+(-1)\alpha\gamma^3-\beta\gamma^5 \\ &= -\alpha^5\gamma-\alpha^3\beta+\alpha^3\gamma^2-\alpha^3\beta+\alpha^2\beta^3-2+\alpha^3\gamma^2-\alpha\gamma^3 \\ &\quad -\alpha^3\beta+\alpha^2\beta^3-1+\alpha^2\beta^3-\alpha\beta^5-2\beta^3\gamma-1+\beta^2\gamma^3 \\ &\quad +\alpha^3\gamma^2-1-\alpha\gamma^3-1-\gamma\beta^3+2\beta^2\gamma^3-\alpha\gamma^3-\beta\gamma^5 \\ &= -\alpha^5\gamma-3\alpha^3\beta+3\alpha^3\gamma^2+3\alpha^2\beta^3-3\alpha\gamma^3-6+3\beta^2\gamma^3-3\beta^3\gamma-\alpha\beta^5-\beta\gamma^5 \\ &= 3(\alpha^2\beta^3+\beta^2\gamma^3+\gamma^3\alpha^3)-3(\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha)-(\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5)-6. \end{aligned} \tag{3.35}$$

We now set

$$\sigma=\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha. \tag{3.36}$$

Then by (3.31), (3.32) and (3.33), we see that the first term of (3.35) can be written as

$$\begin{aligned}\alpha^2\beta^3+\beta^2\gamma^3+\gamma^3\alpha^3 &= \alpha\beta(\alpha^2-\gamma)+\beta\gamma(\beta^2-\alpha)+\gamma\alpha(\gamma^2-\beta)\\ &= (\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha)-3\alpha\beta\gamma\\ &= \sigma+3. \qquad (3.37)\end{aligned}$$

Then (3.35) becomes

$$3(\sigma+3)-3\sigma-(\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5)-6=0,$$

i.e.

$$\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5=3. \qquad (3.38)$$

Observe that

$$\begin{aligned}\alpha^7+\beta^7+\gamma^7 &= \alpha^5(\alpha\beta^2+\gamma)+\beta^5(\beta\gamma^2+\alpha)+\gamma^5(\gamma\alpha^2+\beta) \text{ (by (3.31), (3.32) and (3.33))}\\ &= \alpha^6\beta^2+\beta^6\gamma^2+\beta^6\gamma^2+\alpha\beta^5+\alpha^2\gamma^6+\beta\gamma^5\\ &= (\alpha^6\beta^2+\beta^6\gamma^2+\gamma^6\alpha^2)+(\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5).\end{aligned}$$

Since

$$\begin{aligned}(\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha)^2 &= \alpha^6\beta^2+\alpha^3\beta^4\gamma+\alpha^4\beta\gamma^3+\beta^6\gamma^2+\alpha^3\beta^4\gamma+\alpha\beta^3\gamma^4+\alpha^2\gamma^6+\alpha^4\beta\gamma^3+\alpha\beta^3\gamma^4\\ &= \alpha^6\beta^2+\beta^6\gamma^2+\alpha^2\gamma^6+2\alpha^3\beta^4\gamma+2\alpha^4\beta\gamma^3+2\alpha\beta^3\gamma^4,\end{aligned}$$

then

$$\alpha^6\beta^2+\beta^6\gamma^2+\alpha^2\gamma^6=(\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha)^2-2\alpha^3\beta^4\gamma-2\alpha^4\beta\gamma^3-2\alpha\beta^3\gamma^4. \quad (3.39)$$

So,

$$\alpha^7+\beta^7+\gamma^7 = (\alpha^3\beta+\beta^3\gamma+\gamma^3\alpha)^2-2\alpha^3\beta^4\gamma-2\alpha^4\beta\gamma^3-2\alpha\beta^3\gamma^4+3$$

$$
\begin{aligned}
&= \sigma^2 - 2\alpha\beta\gamma(\alpha^2\beta^3 + \beta^2\gamma^3 + \gamma^2\alpha^3) + 3 \\
&= \sigma^2 - 2(-1)(\sigma + 3) + 3 \\
&= \sigma^2 + 2\sigma + 6 + 3 \\
&= \sigma^2 + 2\sigma + 9. \qquad (3.40)
\end{aligned}
$$

We know that

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{E(q)} \qquad (3.41)$$

where

$$E(q) = \prod_{i=1}^{\infty}(1 - q^i).$$

Now, with η being a seventh root of unity other than 1, we have

$$
\begin{aligned}
\frac{1}{(q;q)_\infty} &= \frac{E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)E(\eta^5 q)E(\eta^6 q)}{E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)E(\eta^5 q)E(\eta^6 q)} \\
&= \frac{(\eta q;\eta q)_\infty(\eta^2 q;\eta^2 q)_\infty(\eta^3 q;\eta^3 q)_\infty(\eta^4 q;\eta^4 q)_\infty(\eta^5 q;\eta^5 q)_\infty(\eta^6 q;\eta^6 q)_\infty}{(q;q)_\infty(\eta q;\eta q)_\infty(\eta^2 q;\eta^2 q)_\infty(\eta^3 q;\eta^3 q)_\infty(\eta^4 q;\eta^4 q)_\infty(\eta^5 q;\eta^5 q)_\infty(\eta^6 q;\eta^6 q)_\infty}.
\end{aligned}
\qquad (3.42)
$$

Similar to the manipulations that yielded Ramanujan's most beautiful identity, the denominator in (3.42) can be written as

$$
\begin{aligned}
D(q) &= \frac{E(q^7)^8}{E(q^{49})} \qquad (3.43) \\
&= \frac{{(q^7;q^7)_\infty}^8}{(q^{49};q^{49})_\infty}. \qquad (3.44)
\end{aligned}
$$

By (3.27), $E(q) = q^2E(q^49)(\alpha + \beta - 1 + \gamma)$ and so

$$
\begin{aligned}
D(q) &= E(q)E(\eta q)E(\eta^q)E(\eta^3 q)E(\eta^4 q)E(\eta^5 q)E(\eta^6 q) \\
&= q^2E(q^{49})(\alpha + \beta - 1 + \gamma)q^2E(q^{49})(\alpha + \eta\beta - \eta + \eta\gamma)q^2E(q^{49})(\alpha + \eta^2\beta - \eta^4 + \eta^{10}\eta) \\
&\quad +\eta^{25}\gamma)q^2E(q^{49})(\alpha + \eta^3\beta - \eta^6 + \eta^{15}\gamma) + q^2E(q^{49})(\alpha + \eta^4\beta - \eta^8 + \eta^{20}\gamma)
\end{aligned}
$$

$$\begin{aligned}
& q^2E(q^{49})(\alpha+\eta^5\beta-\eta^{10}q^2E(q^{49})(\alpha+\eta^6\beta-\eta^{12}+\eta^{30}\gamma) \\
= \;& q^{14}E(q^{49})^7(\alpha+\beta-1+\gamma)(\alpha+\eta\beta-\eta+\eta\gamma)(\alpha+\eta^2\beta-\eta^4+\eta^{10}\gamma)(\alpha+\eta^3\beta-\eta^6+\eta^{15}\gamma) \\
& +(\alpha+\eta^4\beta-\eta^8+\eta^{20}\gamma)(\alpha+\eta^5\beta-\eta^{10}+\eta^{25}\gamma)(\alpha+\eta^6\beta-\eta^{12}+\eta^{30}\gamma) \\
= \;& q^{14}E(q^{49})^7[(\alpha^7+\beta^7+\gamma^7)+7(\alpha^5\beta+\beta^5\gamma+\gamma^5\alpha)+14(\alpha^2\beta+\beta^2\gamma^3+\gamma^2\alpha^3)-8] \\
= \;& q^{14}E(q^{49})^7[(\sigma^2+2\sigma+9)+7\cdot 3+14(\sigma+3)-8] \\
= \;& q^{14}E(q^{49})^7(\sigma^2+16\sigma+64) \\
= \;& q^{14}E(q^{49})^7(\sigma+8)^2. \qquad (3.45)
\end{aligned}$$

Hence, by (3.43) and (3.45), we have

$$q^{14}E(q^{49})^7(\sigma+8)^2=\frac{E(q^7)^8}{E(q^{49})}$$

which gives

$$(\sigma+8)^2=\left(\frac{E(q^7)^4}{q^7E(q^{49})^4}\right)^2. \qquad (3.46)$$

It then follows that

$$\sigma+8=-\frac{E(q^7)^4}{q^7E(q^{49})^4}. \qquad (3.47)$$

The numerator in (3.42) can be writen as

$$\begin{aligned}
N(q) \;=\;& E(q)E(\eta q)E(\eta^2q)E(\eta^3q)E(\eta^4q)E(\eta^5q)E(\eta^6q) \\
=\;& q^{12}E(q^{49})^6(\alpha+\eta\beta-\eta+\eta\gamma)(\alpha+\eta^2\beta-\eta^4+\eta^{10}\gamma)(\alpha+\eta^3\beta-\eta^6+\eta^{15}\gamma) \\
& \times(\alpha+\eta^4\beta-\eta^8+\eta^{20}\gamma)(\alpha+\eta^5\beta-\eta^{10}+\eta^{25}\gamma)(\alpha+\eta^6\beta-\eta^{12}+\eta^{30}\gamma) \\
=\;& q^{12}E(q^{49})^6(\alpha^6-\alpha^5\beta+(\alpha^5+\alpha^4\beta^2)+(-2\alpha^4\beta-\alpha^3\beta^3)+(\alpha^4+3\alpha^3\beta^2+\alpha^2\beta^4) \\
& +(-\alpha^5\gamma-3\alpha^3\beta-4\alpha^2\beta^3-\alpha\beta^5)+(-\alpha^3+6\alpha^2\beta^2+5\alpha\beta^4+\beta^6) \\
& +(5\alpha^4\gamma-\alpha^2\beta+4\alpha\beta^3+\beta^5) \\
& +(2\alpha^2+6\alpha\beta^2+\beta^4)+(4\alpha^3\gamma-\beta^3)+(\alpha^4\gamma^2+3\alpha=\beta^5\gamma+2\beta^2)(6\alpha^2\gamma-2\beta^4\gamma+3\beta) \\
& +(-4\alpha^3\gamma^2-3\beta^3\gamma+8)-\beta^2\gamma+(6\alpha^2\gamma+\beta^4\gamma^2)+(-\alpha^3\gamma^3+3\beta^3\gamma^2+3\gamma)
\end{aligned}$$

$$+(-\alpha\gamma^2+6\beta^2\gamma^2)+(3\alpha^2\gamma^3+6\beta\gamma^2)+(-\beta^3\gamma^3+2\gamma^2)+(-3\alpha\gamma^3-4\beta^2\gamma^3)+(\alpha^2\gamma^4)-\gamma^3$$
$$+(-2\alpha\gamma^4+\beta^2\gamma^4)+5\beta\gamma^4+\gamma^4-\alpha\gamma^5-\beta\gamma^5+\gamma^5+\gamma^6).$$

Using (3.43) and extracting terms in which the exponent is 5 modulo 7 from (3.48), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} p(7n+5)q^{7n+5} &= q^{12}\frac{E(q^{49})^7}{E(q^7)^8}(-4(\alpha^2\beta^3+\beta^2\gamma^3+\gamma^2\alpha^3)-(\alpha\beta^5+\beta\gamma^5+\gamma\alpha^5) \\
&\quad -3(\alpha^3\beta+\beta^2\gamma+\gamma^3\alpha)+8) \\
&= q^{12}\frac{E(q^{49})^7}{E(q^7)^8}(-4(\sigma+3)-3-3\sigma+8) \\
&= q^{12}\frac{E(q^{49})^7}{E(q^7)^8}(-7\sigma-7) \\
&= q^{12}\frac{E(q^{49})^7}{E(q^7)^8}\left(7\frac{E(q^7)^4}{q^7E(q^{49})^4}+49\right) \\
&= 7q^5\frac{E(q^{49})^3}{E(q^7)^4}+49q^{12}\frac{E(q^{49})^7}{E(q^7)^8}
\end{aligned}$$

which leads to (3.24).

Chapter 4

Application

In this chapter, we present our result on parity of some 2-color partition function.

Recall the following partition function $p_k(n)$ (see [14]), where $p_k(n)$ is the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of k. The generating function is given by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty (q^k;q^k)_\infty}.$$

Congruences modulo 3 have been given for $k = 2$ (for example, by Chern [14]). We examine parity, instead. We will first find an expression for $\sum\limits_{n=0}^{\infty} p_2(2n)q^n$ and then do some computations to find the recurrence. We use the notation $f_j = (q^j;q^j)_\infty$ and by binomial theorem, recall that

$$f_{p^i} \equiv f_1^{p^i} \pmod{p}$$

where p is prime. We also recall the following result from Hirschorn [16]:

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2}\left(\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty}\right). \quad (4.1)$$

From the previous work observe that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = f_1^3. \quad (4.2)$$

Lemma 2. *We have the following result*

$$\sum_{n=0}^{\infty} p_2(2n)q^n = \frac{f_2}{f_1^4 f_{16}}(q, q^7, q^8; q^8)_\infty (q^6, q^{10}, q^{16}; q^{16})_\infty.$$

Proof.

$$\sum_{n=0}^{\infty} p_2(n)q^n = \frac{1}{(q^2;q^2)_\infty}\frac{1}{(q;q)_\infty}$$

$$= \frac{1}{(q^2;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty^2}\left(\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty}\right) \quad \text{(by (4.1))}$$

$$= \frac{1}{f_2^3}\left(\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty} + q\frac{(q^4,q^{16},q^{28},q^{32};q^{32})_\infty}{(q^2,q^{14};q^{16})_\infty}\right)$$

so that

$$\sum_{n=0}^{\infty} p_2(2n)q^{2n} = \frac{1}{f_2^3}\frac{(q^{12},q^{16},q^{20},q^{32};q^{32})_\infty}{(q^6,q^{10};q^{16})_\infty}.$$

Replacing q with $q^{1/2}$ yields

$$\begin{aligned}
\sum_{n=0}^{\infty} p_2(2n)q^n &= \frac{1}{f_1^3}\frac{(q^6,q^8,q^{10},q^{16};q^{16})_\infty}{(q^3,q^5;q^8)_\infty}\\
&= \frac{1}{f_1^3}\frac{(q^2,q^4,q^{12},q^{14};q^{16})_\infty(q^6,q^8,q^{10},q^{16};q^{16})_\infty}{(q^2,q^4,q^{12},q^{14};q^{16})_\infty(q^3,q^5;q^8)_\infty}\\
&= \frac{1}{f_1^3}\frac{(q^2;q^2)_\infty}{(q^4,q^{12};q^{16})_\infty(q^2,q^{14};q^{16})_\infty(q^3,q^5;q^8)_\infty}\\
&= \frac{f_2}{f_1^3}\frac{1}{(q^4;q^8)_\infty(q^2,q^{14};q^{16})_\infty(q^3,q^5;q^8)_\infty}\\
&= \frac{f_2}{f_1^3}\frac{1}{(q^2,q^{14};q^{16})_\infty(q^3,q^4,q^5;q^8)_\infty}\\
&= \frac{f_2}{f_1^3}\frac{1}{(q^2,q^{14};q^{16})_\infty}\frac{(q,q^2,q^6,q^7,q^8;q^8)_\infty}{(q;q)_\infty}\\
&= \frac{f_2(q,q^7,q^8;q^8)_\infty}{f_1^4}\frac{(q^2,q^6;q^8)_\infty}{(q^2,q^{14};q^{16})_\infty}\\
&= \frac{f_2(q,q^7,q^8;q^8)_\infty}{f_1^4}\frac{(q^2,q^{10},q^6,q^{14};q^{16})_\infty}{(q^2,q^{14};q^{16})_\infty}
\end{aligned}$$

$$= \frac{f_2(q,q^7,q^8;q^8)_\infty}{f_1^4} \frac{(q^{10},q^6;q^{16})_\infty (q^{16};q^{16})_\infty}{(q^{16};q^{16})_\infty}$$
$$= \frac{f_2(q,q^7,q^8;q^8)_\infty (q^6,q^{10},q^{16};q^{16})_\infty}{f_1^4 f_{16}}$$

□

Theorem 10. *For all $n \geq 0$, we have*

$$p_2(2n) \equiv p_2(n) + \sum_{j=1}^{\lfloor \frac{\sqrt{64n+1}+1}{8} \rfloor} \left(p_2(n-4j^2-j) + p_2(n-4j^2+j) \right) \pmod 2.$$

where $p_2(0) = 1$ and $p_2(n) = 0$ for all $n < 0$.

Proof. From Lemma 2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_2(2n) q^n &= \frac{f_2(q,q^7,q^8;q^8)_\infty (q^6,q^{10},q^{16};q^{16})_\infty}{f_1^4 f_{16}} \\
&\equiv \frac{f_2(q,q^7,q^8;q^8)_\infty (q^3,q^5,q^8;q^8)_\infty^2}{f_1^4 f_{16}} \pmod 2 \\
&\equiv \frac{f_2(q,q^3,q^5,q^7;q^8)_\infty (q^3,q^5,q^8;q^8)_\infty f_8^2}{f_1^4 f_{16}} \pmod 2 \\
&\equiv \frac{f_2(q,q^2)_\infty (q^3,q^5,q^8;q^8)_\infty f_8^2}{f_1^4 f_{16}} \pmod 2 \\
&\equiv \frac{f_2 f_8^2 f_1}{f_1^4 f_{16} f_2} (q^3,q^5,q^8;q^8)_\infty \pmod 2 \\
&\equiv \frac{1}{f_1^3} (q^3,q^5,q^8;q^8)_\infty \pmod 2 \\
&\equiv \frac{1}{f_1^3} \sum_{n=-\infty}^{\infty} q^{4n^2+n} \pmod 2 \\
&\equiv \frac{1}{f_1^3} \sum_{n=-\infty}^{\infty} q^{4n^2+n} \pmod 2.
\end{aligned}$$

But

$$\frac{1}{f_1^3} \equiv \sum_{n=0}^{\infty} p_2(n) q^n \pmod 2$$

so that

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_2(2n)q^n &\equiv \sum_{n=0}^{\infty} p_2(n)q^n \sum_{n=-\infty}^{\infty} q^{4n^2+n} \pmod 2 \\
&= \sum_{n=0}^{\infty} p_2(n)q^n \left(\sum_{n=0}^{\infty} q^{4n^2+n} + \sum_{n=0}^{\infty} q^{4n^2-n} - 1 \right) \pmod 2 \\
&= \sum_{n=0}^{\infty} p_2(n)q^n \sum_{n=0}^{\infty} q^{4n^2+n} + \sum_{n=0}^{\infty} p_2(n)q^n \sum_{n=0}^{\infty} q^{4n^2-n} - \sum_{n=0}^{\infty} p_2(n)q^n \pmod 2 \\
&= F(n) + G(n) - \sum_{n=0}^{\infty} p_2(n)q^n \pmod 2
\end{aligned}
$$

where

$$
F(n) = \sum_{n=0}^{\infty} p_2(n)q^n \sum_{n=0}^{\infty} q^{4n^2+n} \pmod 2 \tag{4.3}
$$

and

$$
G(n) = \sum_{n=0}^{\infty} p_2(n)q^n \sum_{n=0}^{\infty} q^{4n^2-n} \pmod 2. \tag{4.4}
$$

Let $f(j) = 4j^2 + j$. Then

$$
\sum_{n=0}^{\infty} q^{4n^2+n} = \sum_{n=0}^{\infty} t_n q^n \text{ where } t_n = \begin{cases} 1, & \text{if } n = f(j)\ j \geq 0 \\ 0, & \text{otherwise.} \end{cases}
$$

By Cauchy product, we have

$$
F(n) = \sum_{n=0}^{\infty} \left(\sum_{k=0 \text{ and } k=4j^2+j}^{\infty} t_k p_2(n-k) + \sum_{k=0 \text{ and } k\neq 4j^2+j}^{\infty} t_k p_2(n-k) \right) q^n.
$$

To find a more precise upper bound for j, note that

$$
0 \leq k \leq n \implies 0 \leq 4j^2 + j \leq n \implies 0 \leq j \leq \lfloor \frac{\sqrt{64n+1}-1}{8} \rfloor.
$$

Hence,

$$\sum_{k=0}^{\infty} t_k p_2(n-k) = \sum_{j=0}^{\lfloor \frac{\sqrt{64n+1}-1}{8} \rfloor} p_2(n-j(4j+1)).$$

Similarly, we have

$$G(n) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} p_2(n-k) s_k \right) q^n,$$

where

$$s_k = \begin{cases} 1, & \text{if } k = f(j),\ j \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

An upper bound for j is computed as follows:

$$0 \leq k \leq n \implies 0 \leq 4j^2 - j \leq n \implies 0 \leq j \leq \lfloor \frac{\sqrt{64n+1}+1}{8} \rfloor$$

Hence,

$$\sum_{k=0}^{\infty} s_k p_2(n-k) = \sum_{j=0}^{\lfloor \frac{\sqrt{64n+1}+1}{8} \rfloor} p_2(n-j(4j-1)).$$

Combining everything together, we have

$$\sum_{n=0}^{\infty} p_2(2n) q^n \equiv \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor \frac{\sqrt{64n+1}-1}{8} \rfloor} p_2(n-4j^2-j) \right) q^n + \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor \frac{\sqrt{64n+1}+1}{8} \rfloor} p_2(n-4j^2+j) \right) q^n - \sum_{n=0}^{\infty} p_2(n) q^n,$$

i.e.

$$p_2(2n) \equiv \sum_{j=0}^{\lfloor \frac{\sqrt{64n+1}-1}{8} \rfloor} p_2(n-4j^2-j) + \sum_{j=0}^{\lfloor \frac{\sqrt{64n+1}+1}{8} \rfloor} p_2(n-4j^2+j) + p_2(n) \pmod{2},$$

which impies that

$$p_2(2n) \equiv p_2(n) + \sum_{j=1}^{\lfloor \frac{\sqrt{64n+1}+1}{8} \rfloor} \left(p_2(n - 4j^2 - j) + p_2(n - 4j^2 + j) \right) \pmod{2}.$$

□

Chapter 5

Conclusion

This dissertation comprises of four chapters. In the first chapter, we studied the theory of q-series and the general Ramanujan's theta function. We gave the proof of the q-binomial theorem, Heine's transformation formula as well as some applications. We also introduced Bailey's lemma which was used later in the thesis to prove Roger-Ramanujan's two famous identities. We then touched on integer partitions and concluded by providing a link to partition analysis. We also proved some further Rogers-Ramanujan identities using partition analysis.

The second chapter looked at Ramanujan's congruences specifically for moduli 5 and 7. This then extended to Atkin-Swinnerton-Dyer congruences as well as Ramanujan's most beatiful identities. Finally, in the last chapter, we gave a new result on the parity of the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of 2.

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