ANTICIPATIVE STOCHASTIC CALCULUS WITH APPLICATIONS TO FINANCIAL MARKETS

by

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To my mother Therese

ABSTRACT

In this thesis, we study both local time and Malliavin calculus and their application to stochastic calculus and finance. In the first part, we analyze three aspects of applications of local time. We first focus on the existence of the generalized covariation process and give an approximation when it exists. Thereafter, we study the decomposition of ranked semimartingales. Lastly, we investigate an application of ranked semimartingales to finance and particularly pricing using Bid-Ask. The second part considers three problems of optimal control under asymmetry of information and also the uniqueness of decomposition of "Skorohod-semimartingales". First we look at the problem of optimal control under partial information, and then we investigate the uniqueness of decomposition of "Skorohod-semimartingales" in order to study both problems of optimal control and stochastic differential games for an insider.

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DECLARATION

This is to certify that

(i) the thesis comprises only my original work toward the PhD, except where indicated in the Preface,

(ii) due acknowledgment has been made in the text to all other material used,

I hereby certify that this paper/thesis was independently written by me. No material was used other than that referred to. Sources directly quoted and ideas used, including figures, tables, sketches, drawings and photos, have been correctly denoted. Those not otherwise indicated belong to the author.

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Structure of the thesis

This thesis is constituted of two parts, the first part contains three chapters in collaboration with Raouf Ghomrasni, Paul Kettler and Frank Proske. The second Chapter has led us to an article accepted for publication in Stochastic Analysis and Applications (see [56]) while the two other chapters have led us to articles in submission. (See [57, 75].) The second part contains four chapters and, as in the first part, each chapter conduct us to article in the process of editing or in submission. (See [34, 35, 89, 107].) This second part is a work performed in collaboration with Giulia Di Nunno, Thilo Meyer Brandis, Bernt Øksendal, Frank Proske and Hassilah Binti Salleh.

These works are preceded by a general introduction intended to put them in their context, and not to give an exhaustive review of the subject. Between the introduction and the body of the thesis, the mains results obtained are resumed and the plan of the thesis is specified.

General introduction

Introduction

The purpose of this thesis is to study both local time and Malliavin calculus and their application to finance. In the first part of this thesis, we analyze three aspects of the applications of local time. We first focus on the existence of the generalized covariation process and give an approximations of this process (when it exists). There after, we study the decomposition of ranked semimartingales (not necessarily continuous). Lastly, we investigate an application of local time to finance and in particular to pricing using bid-ask. The second part of this thesis considers Malliavin calculus and its application to problems of optimal control under asymmetry of information. First, we examine the problem of optimal control under incomplete information (partial information in this thesis). We then investigate the uniqueness of decompositions of "*Skorohod-semimartingales*" processes. This is of great importance in the study of optimal control for an insider. We begin by introducing some notions in stochastic analysis and finance, in order to explain the context and motivation of this work, before presenting the main results.

0.1 Local time and its applications

In classical stochastic integration, the usual Itô formula, first established by Itô for a standard Brownian motion, and later extended to continuous semimartingales by Kunita and Watanabe, states that if X is a $\mathbb{R}$-valued semimartingale, and F is a function in C^2, then $F(X)$ is a semimartingale. The decomposition of $F(X)$ is specific and can be given though the first and second derivatives of F and the quadratic variation $[X, X]$. There after, various extensions of the Itô formula have been established for functions $F \notin C^2$. The most well known of these extensions is the Itô-Tanaka formula first, derived by Tanaka for $F(x) = |x|$, to which the local time was beautifully linked with the quadratic variation term. An extension to an absolutely continuous function F with locally bounded F' is due to Bouleau and Yor (see [18]). Here the quadratic variation term in the formula was expressed using an integral with respect to local time, in a manner which suggests formal integration by parts. Since then, a lot of research extending the Itô formula, by means of local time-space integration, has been done.

Eisenbaum in [37] made a fundamental contribution in the case of standard Brownian motion by deriving an extension of the Itô formula. The quadratic variation term in her result is expressed as an area integral, with respect to both the time variable s, and the space variable x, of the local time L_s^x. The arguments of Eisenbaum rely on combining the Bouleau-Yor extension in [18] with the Föllmer-Protter-Shiryaev extension in [49] and thus depend strongly on the time-reversal property of standard Brownian motion. Furthermore she extended in [38] the results to Lévy processes, and later, in [39], to reversible semimartingales. Ghomrasni in [54] established a generalized occupation time formula for continuous semimartingales. See also [58].

Russo and Vallois in a series of papers [121, 122] introduced a technique of stochastic integration via regularization. In [121], they defined forward, backward and symmetric integrals by a limit procedure. These integrals are respectively extensions of the Itô, backward and Stratonovich integrals. In [122], Russo and Vallois introduced a notion of a generalized covariation process $[X, Y]$, in a general setting, concerning essentially continuous processes X and Y. It is an extension of the usual approach if we consider a continuous semimartingale, and is in general, defined by a limit procedure.

In her thesis, Bergery [12], through a regularization procedure, gave schemes for approximations of the local time of a large class of continuous semimartingales and reversible diffusions.

The convergence in those approximations holds uniformly in compact in probability (ucp) sense. The limit, in her work, is taken with respect to time. Using the generalized occupation time formula, given in [54], do we have such approximation with respect to space by regularization? If so, do these approximations coincide?

From a financial point of view, if we are interested in looking at the capital distribution and size effect in stochastic portfolio theory, then we have to consider stocks identified by rank, as opposed to by name. The question of decomposition or stochastic differential equation of such processes follows. Chitashvili and Mania in [24] introduced the problem of decomposition for the maximum of n semimartingales. They showed that the maximum process can be expressed in terms of the original processes, adjusted by local times. Fernholz in [45], defined the more general notion of ranked processes (i.e. order statistics) of n continuous Itô processes and gave the decomposition of such processes. However, the main drawback of the latter result, is that triple points do not exist, i.e not more than two processes coincide at the same time, almost surely. Motivated by the question of extending this decomposition to triple points (and higher orders of incidence), Banner and Ghomrasni recently [8] developed some general formulae for ranked processes of *continuous* semimartingales. They showed that the ranked processes can be expressed in terms of original processes, adjusted by the local times of ranked processes. However, is it possible to have such a decomposition for more general semimartingales (not necessarily continuous)?

The theory of asset pricing and its fundamental theorem was initiated in the Arrow-Debreu model, the Black and Scholes formula, and the Cox and Ross model. They have now been formalized in a general framework by Harisson and Kreps [60], Harrison and Pliska [61], and Kreps [78] according to the principle of no arbitrage. In the classical setting, the market is assumed to be frictionless, i.e a no arbitrage dynamic price process is a martingale under a probability measure equivalent to the reference probability measure. However, since real financial markets are not frictionless, important literature on pricing under transaction costs and liquidity risk has appeared. (See [15, 69] and the references therein). The bid-ask spread in this setting can be interpreted as the transaction cost, or as the result of entering buy and sell orders.

In the past, in real financial markets, the load of providing liquidity was given to market makers, specialists, and brokers, who trade only when they expect to make profits. Such profits are the price that investors and other traders pay, in order to execute their orders when they want to trade. To ensure steady trading, the market makers sell to buyers and buy from sellers, and get compensated by the so-called bid-ask spread. The most common price for referencing stocks is the last trade price. However, the last price is not necessarily the price at which one can subsequently trade. At any given moment, in a sufficiently liquid market, there is a best or highest "bid" price, from someone who wants to buy the stock, and there is a best or lowest "ask" price, from someone who wants to sell the stock. We consider models of financial markets in which all parties involved (buyers, sellers) find incentives to participate, and we assume that the dynamic of the different bid and ask prices are given. The question we address is how to determine an SDE for the "best bid" (respectively, "best ask") price process so as to obtain an SDE for the stock price. Is such a market arbitrage free? Is the market complete?

0.2 Malliavin calculus applied to optimal control under asymmetry of information

The mathematical theory known as *Malliavin calculus* was first introduced by Paul Malliavin in [83], as an infinite-dimensional calculus. This calculus was designed to study the smoothness of the densities of the solutions of stochastic differential equations. In 1991, Karatzas and Ocone in [71], showed that the representation theorem formulated by Clark in [26] and, latter by Haussmann in [62] and Ocone in [98] could be used in finance. This result is often cited as the CHO (Clark-Haussmann-Ocone) Theorem and it provides a technique of computing hedging portfolios in complete markets driven by Brownian motion. This discovery leads to a huge growth in the direction of Malliavin calculus, both among mathematicians, and finance researchers. Recently, the theory has been generalized and new applications have been found, e.g partial information optimal control, insider trading and more generally, anticipative stochastic calculus. Malliavin calculus has also been expanded to Lévy processes. Therefore, there has also been an increased interest in anticipative integration with respect to a Lévy process, partly owing to its application to insider trading in finance (see e.g. [33, 99] and [103]). In finance, one of the objectives of the investor is to characterize an optimal portfolio to maximize his utility. In this thesis, we will focus on the application of Malliavin calculus to an optimal portfolio, under asymmetry of information.

Starting from Louis Bachelier's thesis [5] in 1900 on "Theorie de la speculation" up until the Black, Scholes and Merton model in 1972 [16, 17], and further, in most problems of stochastic analysis applied to finance, one of the fundamental hypotheses is the homogeneity of information that market participants have. This homogeneity does not reflect reality. In fact, there exist many types of agents in the market, who have different levels of information. We shall investigate this asymmetry of information.

Back observed in [6] that, the term asymmetry of information can be understood in two ways: as incomplete information (partial information in this thesis) and as supplementary information (insider information).

The term incomplete information means that we have been given a filtration in which the processes are adapted, and we assume that investors only have access to a part of that information. The study in an incomplete information framework can be seen as an application of filtering theory. Much research has been done in this setting, to solve the problem of optimal control, using either dynamic programming or the stochastic maximum principle. We note that the authors in [7, 10, 11, 52, 74, 102, 112, 128, 136], studied partially observed optimal control problems for diffusions, i.e, the controls under consideration are based on noisy observations described by the state process. If we consider a general subfiltration as in [7] (for example the delay information case), and allow our control to be adapted to this general subfiltration, the problem is not of Markovian type and hence cannot be solved by dynamic programming and Hamilton-Jacobi-Bellman (HJB) equations. In this framework, Barghery and Øksendal in [7] studied the problem of optimal control of a jump diffusion, i.e a process which is the solution of a stochastic differential equation (SDE) driven by Lévy processes, and employed the stochastic maximum principle. However, these papers assume the existence of a solution of the adjoint equations. This is an assumption which often fails in the partial information case. Meyer-Brandis, Øksendal and Zhou in [88] used Malliavin calculus to obtain a maximum principle for this general non-Markovian partial information stochastic control problem. If the controlled process follows a stochastic partial differential equation (SPDE) (rather than an SDE) driven by a Lévy process, do we have similar results obtained in [88]?

The study of supplementary information, often used to model insider trading, is an application of the theory of enlargement of filtration. In this case, we start with a given filtration in which the processes are adapted, and we assume that the traders has an additional information, i.e we enlarge the filtration representing this available information. Karatzas and Pikovsky in [72] considered the special case of initial enlargement of filtration. This means at the initial time, the trader has an information concerning the future, i.e he knows the value of the stock price in the future. They studied a maximization of expected logarithmic utility from terminal wealth and/or consumption. The results of finiteness of the value of the control problem were obtained in various setups. One of the main assumptions in this paper is that the Brownian motion is a semimartingale in the enlarged filtration. What happens if we consider a more general insider filtration or a more general utility function? These questions were answered in [14] by Biagini and Øksendal. They presented a more general approach to insider trading which does not assume that the Brownian motion is a semimartingale in the bigger filtration. They used techniques of forward integration introduced by Russo and Vallois [121] to model insider trading. There are many papers on insider trading and optimal control. We are interested in a more general setting, where we work with a general insider filtration, a more general utility function, and we allow the financial market model for prices to have jumps. In such a framework, can we use Malliavin calculus to solve an optimal control problem for an insider?

Asymmetry of information can also be applied on game theory. Ewald and Xiao in [44] considered a continuous time market model, and used a stochastic differential games with anticipative strategy to model a competition of two heterogeneously informed agents in a financial market. In their model, the agents share the same utility function but are allowed to possess different levels of information. They derived necessary and sufficient criteria for the existence of Nash-equilibria and characterize them for various levels of information asymmetry. Furthermore, they had a look at, how far the asymmetry in the level of information influences Nash-equilibria and general welfare. What happens, if we consider a discontinuous time market model? if the agents do not share the same utility functions? and if the giving filtration are more general?

In the setting of enlargement of filtration, since the integrator need not to be adapted to the filtration generated by the integrands (Brownian motion and the compensated Poisson process), we have to consider *anticipative stochastic integrals*. Nualart and Pardoux in [95] studied the stochastic integral defined by Skorohod of a possibly anticipating integrand, as a function of its upper limit, and established an extended Itô formula. Another result in that paper is the uniqueness of decomposition of *Skorohod-semimartingales* in continuous case. Does this uniqueness hold in the jumps case under mild conditions on the integrators?

0.3 Motivation and results

We believe that the questions asked in the preceding sections require further investigations. In what follows, we shall revisit these questions and provide some answers. We hope that these results will motivate others to develop even better solutions.

1. Using the generalized occupation time formula given in [54], do we have such an approximation with respect to space by regularization? In this case, do these approximations coincide?

   The answers to these questions are given in the first chapter, where we focus our attention on the existence of the generalized covariation process $[F(X), X]$ of a wide class of function F when the process $[X, X]$ exists. The results we introduce here extend previous works by Russo and Vallois [121, 122, 123], where they prove the existence of the generalized covariation and give an extension of the Itô formula, when X is a process admitting a generalized quadratic variation and F is a function in C^2. We also give a new approximation, in terms of space, for the generalized covariation process. The proof of the existence of the generalized covariation process is based on the Lebesgue differentiation theorem. The approximation function used here, is the same as that used by Ghomrasni in [55]. We also generalize the result to time-dependent functions, and consider an application to the transformation of semimartingales. The case of n-dimensional continuous processes, when all mutual brackets exist, is also explored. We give in Theorem 1.6.1 and in Remark 1.6.2 an example which illustrates that our approximation of generalized covariation does not hold in the random case. Furthermore, the different time and space approximations do coincide in the deterministic case, but not in the random case.

2. Is it possible to have such a decomposition for a more general semimartingale (not necessarily continuous)?

   We give a new decomposition of order statistics of semimartingales (not necessarily continuous) in the same setting as in [8]. The result obtained is slightly different to the one in [8], in the sense that we express the order statistics of semimartingales firstly in terms of order statistics processes adjusted by their local times, and secondly in terms of original processes adjusted by their local times. The proof of this result is a modified and shorter version of the proof given in [8], and is based on the homogeneity property. As a consequence of this result, we are independently able to derive an extension of Ouknine's formula in the case of general semimartingales. The desired generalization, which is essential in the demonstration of Theorem 2.3 in [8], is not used here to prove our decomposition.

3. How do we determine the dynamics of the "best bid" (respectively,"best ask") price process with the intention of obtaining the stock price process? Is such a market arbitrage free and/or complete?

   In order to answer this question, we introduce the notion of semimartingale local time and derive the dynamics of the best bid, best ask and thus, the price process. An important consequence is that the price process possesses the Markov property, if the bid and ask, are Brownian motion or Ornstein-Uhlenbeck type, and more generally Feller processes. We conclude, from the evolution of these prices, that they are all continuous semimartingles. The latter remains valid, when the bid-ask prices are given by general diffusion processes. We define the notion of completeness in the same way as Jarrow and Protter in [68], and study the possibility for arbitrage in such a market. We also discuss (insider) hedging for contingents claims with respect to the stock price process.

4. In an optimal control under partial information, if the controlled process follows a stochastic partial differential equation (SDPE) rather than a SDE driven by a Lévy process, do we have similar results obtained in [88]?

   Note first of all that, in this thesis, we cover the partial observation case in [10, 11, 128], since we deal with controls being adapted to a general subfiltration of the underlying reference filtration. We use Malliavin calculus to prove a general stochastic maximum principle for stochastic partial differential equations (SPDE's) with jumps under partial information. More precisely, the controlled process is given by a quasilinear stochastic heat equation driven by a Wiener process and a Poisson random measure. Further, our Malliavin calculus approach to stochastic control of SPDE's allows for optimization of very general performance functionals. Thus our method is useful for examine control problems of non-Markovian type, which cannot be solved by stochastic dynamic programming. Another important advantage of our technique is that we may relax considerably the assumptions on our Hamiltonian. For example, we do not need to impose concavity on the Hamiltonian. (See e.g. [102, 7].) We apply the previous results to solve a partial information optimal harvesting problem (Theorem 4.4.1). Furthermore, we investigate into an portfolio optimization problem under partial observation. Note that the last example cannot be treated within the framework of [88], since the random measure $N_\lambda(dt, d\xi)$ is not necessarily a functional of a Lévy process. Let us also mention that the SPDE maximum principle studied in [102] does not apply to Example 4.4.3. This is due to the fact that the corresponding Hamiltonian in [102] fails to be concave.

5. Does the uniqueness of "*Skorohod-Semimartingale*" hold in the mixed case? and if we consider mild conditions on our integrators?

   The answers of these two questions are obtained in Theorem 5.3.5 as a special case of a more general decomposition uniqueness theorem for an extended class of Skorohod integral processes with values in in the space of generalized random variables. (See Theorem 5.3.3.) Our proof uses white noise theory of Lévy processes. Our decomposition uniqueness is motivated by applications in anticipative stochastic control theory, including insider trading in finance asked in the previous Section.

6. In an optimal control under general insider information, can we use Malliavin calculus to solve an optimal control problem for an insider?

   We supply a partial answer of this question and hope that further research will be done in a more general setting. As in question 4, we use Malliavin calculus to prove a general stochastic maximum principle for stochastic differential equations (SDE's) with jumps under insider information. The main result here is difficult to apply because of the appearance of some terms, which all depend on the control. We then consider the special case when the coefficients of the controlled process X do not depend on X; we call such processes *controlled Itô-Lévy processes*. In this case, we give a necessary and sufficient conditions for the existence of optimal control. Using white noise theory, and uniqueness of decomposition of a Skorohod-semimartingale, we derive more precise results when our enlarged filtration is first chaos generated (the class of such filtrations contains the class of initially enlarged filtrations and also advanced information filtrations). We applied our results maximize the expected utility of terminal wealth for the insider. We show that there do not exist an optimal portfolio for the insider. For the advanced information case, this conclusion is in accordance with the results in [14] and [33], since the Brownian motion is not a semimartingale with respect to the advanced information filtration. It follows that the stock price is not a semimartingale with respect to that filtration either. Hence, we can deduce that the market has an arbitrage for the insider in this case, by Theorem 7.2 in [29]. In the initial enlargement of filtration case, knowing the terminal value of the stock price, we also prove that there does not exist an optimal portfolio for the insider. This result is a generalization of a result in [72], where the same conclusion was obtained in the special case when the utility function was the logarithm function and there were no jumps in the stock price. The other application is to optimal insider consumption. We show that there exists an optimal insider consumption, and in some special cases the optimal consumption can be expressed explicitly.

7. In a stochastic differential games with anticipative strategy, what happens, if we consider a discontinuous time market model? if the agents do not share the same utility functions? and if the giving filtration are more general?

   We shall use again Malliavin calculus to derive a general maximum principle for stochastic differential games under insider information. This maximum principle covers the insider case in [44], since we deal with controls being adapted to general sup-filtrations of the underlying reference filtration. Moreover, our Malliavin calculus approach to stochastic differential games with insider information for *Itô-Lévy processes* allows for optimization of very general performance functionals. We apply our results to solve a worst case scenario portfolio problem in finance under additional information. We show that there does not exist a Nash-equilibrium for the insider. We prove that there exists a Nash-equilibrium insider consumption, and in some special cases the optimal solution can be expressed explicitly.

0.4 Outline

Chapter 1 is devoted to the existence of the generalized covariation process $[F(X), X]$ of a wide class of function F when the process $[X, X]$ exists. We also give a new approximation of the generalized covariation process. The chapter contains a generalization of the result to time-dependent functions, and an application to the transformation of semimartingales.

In Chapter 2, we examine the decomposition of ranked (order-statistics) processes for semimartingales (not necessarily continuous) using a simple approach. We also give a generalization of Ouknine [105, 106] and Yan's [132] formula for local times of ranked processes.

In Chapter 3, we derive the evolution of a stock price from the dynamics of the "best bid" and "best ask". Under the assumption that the bid and ask prices are described by semimartingales, we study the completeness and the possibility for arbitrage on such a market. Further, we discuss (insider) hedging for contingent claims with respect to the stock price process.

In Chapter 4, we employ Malliavin calculus to derive a general stochastic maximum principle for stochastic partial differential equations with jumps, under partial information. We apply this result to solve an optimal harvesting problem in the presence of partial information. Another application pertaining to portfolio optimization under partial observation, is examined.

In Chapter 5, we introduce Skorohod-semimartingales as an expanded concept of classical semimartingales in the setting of Lévy processes. We show under mild conditions that Skorohod-semimartingales similarly to semimartingales admit a unique decomposition.

In Chapter 6, we gather the results obtain in Chapter 5 to suggest a general stochastic maximum principle for anticipating stochastic differential equations, driven by a Lévy type noise. We use techniques of Malliavin calculus and forward integration. We apply our results to study a general optimal portfolio problem for an insider.

In Chapter 7, we consider a general insider information stochastic differential games where the state process is driven by a Lévy type of noise. We use techniques of Malliavin calculus and forward integration to derive a general stochastic maximum principle for anticipating stochastic differential games.

Part I

Local time and its applications

Chapter 1

An approximation of the generalized covariation process

1.1 Introduction

In classical stochastic integration, integrands are practically bounded predictable processes and integrators are semimartingales. Many authors have examined extensions of stochastic integrals to a certain class of anticipating integrands. One of the most popular extensions has been Skorohod integration [95].

Since early 1990's, Russo and Vallois in a series of papers [121, 122] introduced a technique of stochastic integration via regularization. In [121], they defined forward, backward and symmetric integrals by a limit procedure. These integrals are respectively extensions of Itô, backward and Stratonovich integrals. This approach constitutes a counterpart of a discretization approach initiated by Föllmer [47] and continued by many authors, see for instance [37, 43, 49, 58].

In the usual stochastic integration, Itô formula says that if X is a $\mathbb{R}$-valued semimartingale, and F is a function in C^2, then $F(X)$ is a semimartingale. The decomposition of $F(X)$ is specific and can be given though the first and second derivatives of F and the quadratic variation $[X, X]$. In [122], the authors introduced a notion of generalized covariation process $[X, Y]$, in a general setting concerning essentially continuous processes X and Y. It is an extension of the usual one if we consider a continuous semimartingale and it is defined, in general, by a limit procedure.

In the present chapter, we focus our attention on the existence of the generalized covariation process $[F(X), X]$ of a wide class of function F when the process $[X, X]$ exists. The results we introduce here extend previous works by Russo and Vallois [121, 122, 123] where they prove the existence of the generalized covariation and give an extension of Itô's formula when X is a process admitting a generalized quadratic variation and F is a function in C^2. We also give a new approximation of the generalized covariation process. The motivation for this latest point comes from the desire to connect the results of Eisenbaum [37] and Ghomrasni [55] results with those of Russo and Vallois [121, 122, 123]. The proof of the existence of the generalized covariation process is based on the Lebesgue differentiation theorem. The approximation function we use here is the same as that used by Ghomrasni in [55]. The chapter also contains a generalization of the result to time-dependent functions, and an application to the transformation of semimartingales. The case of n-dimensional continuous processes when all mutual brackets exist is also explored. We give in Theorem 1.6.1 and in Remark 1.6.2 an example which illustrates that our approximation of generalized covariation does not hold in the random case.

The chapter is organized as follows. In Section 1.2, we recall the basic definitions and properties of forward, backward, symmetric integrals and covariation. In Section 1.3, we present our result for time independent continuous functions and then we deal with the time dependent case. In Section 1.4, we extend the results for functions in L^2_{loc} both in the time independent and dependent setting. Section 1.5 deals with transformation of processes and we also concentrate on the case where the continuous process X is a multidimensional process. In Section 1.6 the random case is visited. The Conclusion gives results on an equivalence between existence of the generalized covariation process and the new approximation.

1.2 Notations and preliminaries

For the convenience of the reader, we recall some basic definitions and fundamental results about stochastic calculus with respect to finite quadratic variation processes which have been introduced in [121, 122]. In the whole chapter $(\Omega, \mathcal{F}, \mathbb{P})$ will be a fixed probability space, $X = (X_t, 0 \le t \le T)$, $Y = (Y_t, 0 \le t \le T)$ be two continuous processes. We will assume that all filtrations fulfill the usual conditions. The following definitions are from [121, 122, 125].

Definition 1.2.1 *Let $X = (X(t), 0 \le t \le T)$ denote a continuous stochastic process and $Y = (Y(t), t \in [0,T])$ a process with path in $L^\infty([0,T])$. The ϵ-forward integral (respectively ϵ-backward, ϵ-symmetric integrals and the ϵ-covariation) is defined as follow:*

$$\begin{aligned}
I^-(\epsilon, Y, dX)(t) &:= \int_0^t Y(s)\frac{X(s+\epsilon)-X(s)}{\epsilon}ds,\\
I^+(\epsilon, Y, dX)(t) &:= \int_0^t Y(s)\frac{X(s)-X(s-\epsilon)}{\epsilon}ds,\\
I^0(\epsilon, Y, dX)(t) &:= \int_0^t Y(s)\frac{X(s+\epsilon)-X(s-\epsilon)}{2\epsilon}ds,\\
C_\epsilon(X,Y)(t) &:= \frac{1}{\epsilon}\int_0^t (X(s+\epsilon)-X(s))(Y(s+\epsilon)-Y(s))\,ds.
\end{aligned}$$

Observe that these four processes are continuous.

Definition 1.2.2 *1. A family of processes $\left(H_t^{(\epsilon)}\right)_{t\in[0,T]}$ is said to converge to a process $(H_t)_{t\in[0,T]}$ uniformly on compacts in probability (abbreviated ucp), if $\sup_{0\le t\le T}\left|H_t^{(\epsilon)} - H_t\right| \to 0$ in probability, as $\epsilon \to 0$.*

2. The forward, backward, symmetric integrals and the covariation process are defined *by the following limits in the ucp sense whenever they exist:*

$$
\begin{aligned}
\int_0^t Y(s)d^-X(s) &:= \lim_{\epsilon\downarrow 0} I^-(\epsilon, Y, dX)(t), & (1.2.1)\\
\int_0^t Y(s)d^+X(s) &:= \lim_{\epsilon\downarrow 0} I^+(\epsilon, Y, dX)(t), & (1.2.2)\\
\int_0^t Y(s)d^0X(s) &:= \lim_{\epsilon\downarrow 0} I^0(\epsilon, Y, dX)(t), & (1.2.3)\\
[X, Y](t) &:= \lim_{\epsilon\downarrow 0} C_\epsilon(X, Y)(t), & (1.2.4)
\end{aligned}
$$

When $X = Y$ we often put $[X, X] = [X]$.

Definition 1.2.3

1) If $[X]$ exists then it is always increasing and X is said to be a finite quadratic variation process and $[X]$ is called the quadratic variation of X.

2) If $[X] = 0$, X is called a zero quadratic variation process (or a zero-energy process).

3) We will say that an m-dimensional process $X = \left(X^1, \cdots, X^m\right)$ has all the mutual brackets if $\left[X^i, X^j\right]$ exists for every $i, j = 1, \cdots m$.

In the following, we recall some definitions and facts which are introduced in [27, 121, 122, 125]. The notations we use are those of [27, 121].

Remark 1.2.4

1) If X, Y are two continuous semimartingales, then $[X, Y] = \langle X, Y\rangle$.

2) If $X = Y$ is a continuous semimartingale then $\langle X, X\rangle$ is the quadratic variation of X and it is an increasing process. In the rest of the chapter, we will note $\langle X, X\rangle = \langle X\rangle$.

3) If A is a zero quadratic variation process and X is a finite quadratic variation process, then $[X, A] \equiv 0$.

4) A continuous bounded variation process is a zero quadratic variation process.

5) We have $[X, V] \equiv 0$ if V is a bounded variation process.

6) As a consequence of 5), if X, Y are two continuous process such that $[X, Y]$ exists and is of bounded variation, then $[[X, Y], Z] \equiv 0$ for every continuous process Z.

Definition 1.2.5 *Let $X = (X_t, 0 \le t \le T)$, $Y = (Y_t, 0 \le t \le T)$ be processes with paths respectively in $C^0([0,T])$ and $L^1_{loc}([0,T])$ i.e. $\int_0^t |Y(s)|\, ds < \infty$ for all $t < T$.*

1. *if $Y I_{[0,t]}$ is X-forward integrable for every $0 \le t < T$, Y is said to be locally X-forward integrable on $[0,T)$. In this case there exists a continuous process, which coincides, on every compact interval $[0,t]$ of $[0,T)$, with the forward integral of $Y_{[0,t]}$ with respect to X. That process will be denoted by $I(\cdot, Y, dX) = \int_0^{\cdot} Y d^- X$.*

2. *If Y is locally X-forward integrable and $\lim_{t\to T} I(t, Y, dX)$ exists almost surely, Y is said to be X-improperly forward integrable on $[0,T]$.*

3. *If the covariation process $\left[X, Y I_{[0,t]}\right]$ exists, for every $0 \le t < T$, we say that the covariation process $[X,Y]$ exists locally on $[0,T)$ and it is denoted by $[X,Y]$. In this case there exists a continuous process, which coincides, on every compact interval $[0,t]$ of $[0,T)$, with the covariation process $\left[X, Y I_{[0,t]}\right]$. That process will be denoted by $[X,Y]$. If $X = Y$, we will say that the quadratic variation $[X,X]$ of X exists locally on $[0,T]$.*

4. *If the covariation process $[X,Y]$ exists locally on $[0,T)$ and $\lim_{t\to T}[X,Y]_t$ exists, the limit will be called the improper covariation process between X and Y and it will be denoted by $[X,Y]$. If $X = Y$, we will say that the quadratic variation $[X,X]$ of X exists improperly on $[0,T]$.*

The existence of the generalized covariation process of transformation of continuous process by functions in C^1 and $C^{1,1}$ is given in the following propositions.

Proposition 1.2.6 *If X, Y are continuous processes such that $[X,Y]$, $[X]$, $[Y]$ exist and $F, G \in C^1(\mathbb{R})$, then $[F(X), G(Y)]$ exists and*

$$[F(X), G(Y)]_t = \int_0^t F'(X(s)) G'(Y(s))\, d\, [X,Y]_s\,, \tag{1.2.5}$$

in particular

1. *if* $X = Y$ *and* $G(X) = X$, *we have*

$$[F(X), X]_t = \int_0^t F'(X(s))d[X]_s, \tag{1.2.6}$$

2. *if* $G(Y) = Y$, *we have*

$$[F(X), Y]_t = \int_0^t F'(X(s))d[X, Y]_s. \tag{1.2.7}$$

Proposition 1.2.7 *If* X, Y *are continuous processes such that* $[X, Y]$, $[X]$, $[Y]$ *exist and* $F, G : [0, T] \times \mathbb{R} \to \mathbb{R}$ *two functions in* $C^{1,1}(\mathbb{R})$, *then* $[F(\cdot, X), G(\cdot, Y)]$ *exists and*

$$[F(\cdot, X), G(\cdot, Y)]_t = \int_0^t \frac{\partial F}{\partial x}(s, X(s))\frac{\partial G}{\partial y}(s, Y(s))d[X, Y]_s, \tag{1.2.8}$$

in particular if $X = Y$ *and* $G(\cdot, X) = X$, *we have*

$$[F(\cdot, X), X]_t = \int_0^t \frac{\partial F}{\partial x}(s, X(s))d[X]_s. \tag{1.2.9}$$

Proof. See Appendix A, Section A.1. ■

1.3 Main results

In this section we prove our main results for time dependent and independent cases.

1.3.1 The time independent case

Theorem 1.3.1 *Let* X *be a continuous process such that* $[X]$ *exists and* $F \in C^0(\mathbb{R})$, *then* $[F(X), X]$ *exists and we have the following:*

$$\begin{aligned} [F(X), X]_t &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{F(X(s) + \varepsilon) - F(X(s))\Big\} d[X]_s, \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{F(X(s)) - F(X(s) - \varepsilon)\Big\} d[X]_s, \end{aligned} \tag{1.3.1}$$

or

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int_0^t \frac{F(X(s+\varepsilon)) - F(X(s))}{\varepsilon} (X(s+\varepsilon) - X(s))\, ds \\ =&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{F(X(s)) - F(X(s) - \varepsilon)\Big\} d[X]_s. \end{aligned} \tag{1.3.2}$$

Proof. Let us first prove the existence of the generalized covariation. For this, we associate to F the following function:

$$H_n(x) := n \int_x^{x+\frac{1}{n}} F(y)\, dy. \tag{1.3.3}$$

On the one hand we have

$$H_n(x) = n \int_x^{x+\frac{1}{n}} F(y)\, dy \to F(x) \quad \text{for} \quad n \to \infty. \tag{1.3.4}$$

On the other hand

$$H_n'(x) = n \left\{ F(x + \frac{1}{n}) - F(x) \right\}. \tag{1.3.5}$$

We note that the function $H_n(x)$ in Equation (1.3.3) is a C^1 function. Then by Proposition 2.1 [122], the generalized covariation process $[H_n(X), X]$ exists. Using the definition of forward integral and generalized covariation, introduced by Russo and Vallois [121], we have

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_0^t \frac{H_n(X(s+\varepsilon)) - H_n(X(s))}{\varepsilon} (X(s+\varepsilon) - X(s))\, ds \\
=& [H_n(X), X]_t \\
=& \int_0^t H_n'(X_s) d\,[X]_s \\
=& n \int_0^t \left\{ F(X(s) + \frac{1}{n}) - F(X(s)) \right\} d\,[X]_s .
\end{aligned}$$

Since F is continuous, and by (1.3.4) H_n converges uniformly on each compact to F, it

follows that $H_n(X.)$ converges *ucp* to $F(X.)$. Moreover, the continuity of the processes $I^\sigma = \lim_{\epsilon \downarrow 0} I^\sigma_\epsilon$, for $\sigma = +, -, 0$ imply that $I^\sigma(H_n(X), dX)(t)$ converges *ucp* to $I^\sigma(F(X), dX)(t)$, for $\sigma = +, -, 0$. Then by the definition of the generalized covariation, it follows that

$$[H_n(X), X]_s \ \text{converges } ucp \text{ to } \ [F(X), X]_s . \tag{1.3.6}$$

Equation (1.3.6) means that $[F(X), X]$ exists as the limit in the *ucp* sense of $[H_n(X), X]$ when $n \to \infty$. Thus the first part of theorem is proved. Moreover, since

$$n \int_0^t \left\{ F(X(s) + \frac{1}{n}) - F(X(s)) \right\} d\,[X]_s = [H_n(X), X]_t ,$$

by the uniqueness of the limit, the equality still holds if we take the limit on both sides in the *ucp* sense when $n \to +\infty$. Equation (1.3.1) follows by (1.3.6).

Concerning the second equality of Equation (1.3.1), it suffices to define I_n as $I_n(x) = n\int_{x-\frac{1}{n}}^{x} F(y)\,dy$. Then I_n and H_n have the same properties and the result follows. ■

Corollary 1.3.2 *Let X be a continuous semimartingale and $F \in C^0(\mathbb{R})$. Then $[F(X), X]$ exists and*

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \Big\{F(X(s)+\varepsilon)\ -\ F(X(s))\Big\}\,d\left\langle X\right\rangle_s\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t \frac{F(X(s+\varepsilon))-F(X(s))}{\varepsilon}\,(X(s+\varepsilon)-X(s))\,ds\\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \Big\{F(X(s))\ -\ F(X(s)-\varepsilon)\Big\}\,d\left\langle X\right\rangle_s.
\end{aligned}\tag{1.3.7}$$

Proof. Since X is a continuous semimartingale, it is known that $[X]$ exists and $[X] = \langle X\rangle$. Thus the result follows by the preceding Theorem. ■

Corollary 1.3.3 *Let $B = (B_t, 0 \le t \le T)$ be a one-dimensional standard Brownian motion and $F \in C^0(\mathbb{R})$. Then $[F(B), B]$ exists and we have the following:*

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \Big\{F(B(s)+\varepsilon)\ -\ F(B(s))\Big\}\,ds\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t \frac{F(B(s+\varepsilon))-F(B(s))}{\varepsilon}\,(B(s+\varepsilon)-B(s))\,ds\\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \Big\{F(B(s))\ -\ F(B(s)-\varepsilon)\Big\}\,ds.
\end{aligned}\tag{1.3.8}$$

Remark 1.3.4 *Let X be a continuous process such that $[X]$ exists. If F is a function in $C^1(\mathbb{R})$, then Equation (1.3.1) becomes Equation (1.2.6).*

1.3.2 The time-dependent case

Theorem 1.3.5 *Let X be a continuous process such that $[X]$ exists and $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function in x uniformly in s, and continuous in (s,x). Then $[F(\cdot, X), X]$ exists and*

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\Big\{F(s,X(s)+\varepsilon)\;-\;F(s,X(s))\Big\}\,d\,[X]_s\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t\frac{F(s+\varepsilon,X(s+\varepsilon))-F(s,X(s))}{\varepsilon}\,(X(s+\varepsilon)-X(s))\,ds\\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\Big\{F(s,X(s))\;-\;F(s,X(s)-\varepsilon)\Big\}\,d\,[X]_s. \qquad (1.3.9)
\end{aligned}
$$

Proof. Let us associate to F the following function:

$$
H_n(t,x):=\frac{1}{\varepsilon}\int_x^{x+\frac{1}{n}}F(t,y)\,dy. \qquad (1.3.10)
$$

As before, on the one hand we have

$$
H_n(t,x)=n\int_x^{x+\frac{1}{n}}F(t,y)\,dy\to F(t,x)\quad\text{for}\quad n\to\infty. \qquad (1.3.11)
$$

On the other hand

$$
\frac{\partial}{\partial x}H_n(t,x)=n\left\{F(t,x+\frac{1}{n})-F(t,x)\right\}. \qquad (1.3.12)
$$

We note that the function $H_n(t,x)$ in Equation (1.3.10) admits a derivative with respect to x and which is continuous. Using Proposition 1.2.7, we have

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_0^t\frac{H_n(s+\varepsilon,X(s+\varepsilon))-H_n(s,X(s))}{\varepsilon}\,(X(s+\varepsilon)-X(s))\,ds\\
=&\,[H_n(\cdot,X),X]_t\\
=&\int_0^t\frac{\partial H_n}{\partial x}(s,X(s))d\,[X]_s\\
=&\,n\int_0^t\left\{F(s,X(s)+\frac{1}{n})-F(s,X(s))\right\}d\,[X]_s\,.
\end{aligned}
$$

Since H_n converges uniformly on each compact to F, it follows that $H_n(s,X.)$ converges *ucp* to $F(s,X.)$. Moreover, as in the Proof of Theorem 1.3.1, the continuity of the forward and the backward integral imply that

$$
[H_n(\cdot,X),X]_s\ \text{converges } ucp \text{ to }\ [F(\cdot,X),X]_s\,. \qquad (1.3.13)
$$

Thus $[F(\cdot,X),X]$ exists and the first part of the Theorem is proved.

Since

$$n\int_0^t \left\{F(s,X(s)+\frac{1}{n})-F(s,X(s))\right\}d\,[X]_s=[H_n(\cdot,X),X]_t\,,$$

the equality holds if we take the limit on both sides in the *ucp* sense when $n\to+\infty$. The result follows by Equation (1.3.13).

The second equality follows from the same argument as in the proof of Theorem 1.3.1. ■

Corollary 1.3.6 *Let X be a continuous semimartingale and $F\in C^0(\mathbb{R})$, and $F:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous in x uniformly in s, and continuous in (s,x), then $[F(\cdot,X),X]$ exists and*

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{F(s,X(s)+\varepsilon)\;-\;F(s,X(s))\right\}d\,\langle X\rangle_s\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t\frac{F(s+\varepsilon,X(s+\varepsilon))-F(s,X(s))}{\varepsilon}\,(X(s+\varepsilon)-X(s))\,ds\\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{F(s,X(s))\;-\;F(s,X(s)-\varepsilon)\right\}d\,\langle X\rangle_s.\end{aligned}\tag{1.3.14}$$

Proof. Since X is a continuous semimartingale, it is known that $\langle X\rangle$ exists and the result follows by applying the preceding Theorem. ■

Corollary 1.3.7 *Let $B=(B(t),0\le t\le T)$ be a one-dimensional standard Brownian motion and $F:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous in x uniformly in s, and continuous in (s,x), then $[F(\cdot,B),B]$ exists and we have the following*

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{F(s,B(s)+\varepsilon)\;-\;F(s,B(s))\right\}ds\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t\frac{F(s+\varepsilon,B(s+\varepsilon))-F(s,B(s))}{\varepsilon}\,(B(s+\varepsilon)-B(s))\,ds\\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{F(s,B(s))\;-\;F(s,B(s)-\varepsilon)\right\}ds.\end{aligned}\tag{1.3.15}$$

Remark 1.3.8 *Let X be a continuous process such that $[X]$ exists, if F is a function in $C^{1,1}(\mathbb{R})$, then Equation (1.3.9) becomes Equation (1.2.9).*

1.4 Extension to the case $F \in L^2_{\mathbf{loc}}(\mathbb{R})$

Our approach developed in Section 1.3 allows us to improve our generalized covariation process to F belonging to $L^2_{\text{loc}}(\mathbb{R})$ and X is a Brownian martingale.

1.4.1 The time independent case

Under the assumption of Theorem 5 in [91], we have the following result

Theorem 1.4.1 *Let $u = \{u(t), t \in [0,T]\}$ be an adapted stochastic process such that $\int_0^T u(s)^2\, ds < \infty$ a.s. Set $X(t) = \int_0^t u(s) dB(s)$. Suppose that for all $\delta > 0$, there exist constants c_i^δ, $i = 1, 2$, such that we have,*

$$P\left(\int_0^T F(X(s))^2 (u_s)^2\, ds > \delta\right) \leq c_1^\delta \left\|F\right\|_2^2, \qquad (1.4.1)$$

$$P\left(\sup_{0\leq u\leq t}\left|\int_0^u \frac{F(X(s+\varepsilon)) - F(X(s))}{\varepsilon}\left(X(s+\varepsilon) - X(s)\right) ds\right| > \delta\right) \leq c_2^\delta \left\|F\right\|_2, \qquad (1.4.2)$$

for all $t \in [0,T]$, and for any F in $C_K^\infty(\mathbb{R})$ (infinitely differentiable with compact support). Then, the generalized covariation process $[F(X), X]$ exists, and we have

$$[F(X), X]_t \;=\; \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \Big\{F(X(s)+\varepsilon) \;-\; F(X(s))\Big\}\, d\,[X]_s. \qquad (1.4.3)$$

Proof. We follow the idea of the proof of Theorem 5 in [91]. Notice that by an approximation argument, the inequalities (1.4.1) and (1.4.2) hold for any function $F \in L^2(\mathbb{R})$. Fix $t \in [0,T]$, and set

$$V_t^q\,(F) = q\int_0^t \left[F(X(s+\frac{1}{q})) - F(X(s))\right]\left(X(s+\frac{1}{q}) - X(s)\right) ds.$$

For any $K > 0$ we define the stopping time $T_K = \inf\{t : |X(t)| > K\}$. Let $\delta > 0$ and take $K > 0$ in such a way that $P\left(T_K \leq t\right) \leq \delta$. In order to show the equality, we can assume, by a localization argument, that the process X_t takes values in a compact interval $[-K, K]$ and that F has support in this interval. Consider a sequence of infinitely differentiable functions φ_n with support included in $[-K, K]$ such that $\left\|F - \varphi_n\right\|_{L^2}$ converges to zero.

Since $F \in L^2_{loc}(\mathbb{R})$, the choice of such a sequence guarantees that, for any $K > 0$,

$$\int_{-K}^{K} (\varphi_n(x) - F(x))^2 \, dx \to 0 \quad \text{as } n \to \infty. \tag{1.4.4}$$

This means that φ_n converges uniformly on each compact set of $\mathbb{R}$ to F in $L^2(\mathbb{R})$. Consequently, for any $K > 0$,

$$\begin{aligned} & E\left[\int_0^t (\varphi_n(X(s)) - F(X(s)))^2 \, I(|X(s)| \le K) \, ds\right] \\ & = E\left[\int_0^t (\varphi_n(X(s)) - F(X(s)))^2 \, I(T_K > t) \, ds\right] \underset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{1.4.5}$$

For a process $(H_s)_{0\le s\le t}$, define

$$H(t)^* = \sup_{0\le s\le t} |H(s)| .$$

We have that

$$\begin{aligned} P\left[(V^p(F) - V^q(F))^* > \eta\right] &\le P(T_K \le t) + P\left[T_K > t, (V^p(F-\varphi_n))_t^* > \frac{\eta}{3}\right] \\ &\quad + P\left[T_K > t, (V^q(F-\varphi_n))_t^* > \frac{\eta}{3}\right] \\ &\quad + P\left[T_K > t, (V^p(\varphi_n) - V^q(\varphi_n))_t^* > \frac{\eta}{3}\right] \\ &\le \delta + 2c_2^{\eta}\delta + P\left[T_K > t, (V^p(\varphi_n) - V^q(\varphi_n))_t^* > \frac{\eta}{3}\right]. \end{aligned}$$

We have that $\lim_{p,q} P\left[(V^p(\varphi_n) - V^q(\varphi_n))_t^* > \frac{\eta}{3}\right] = 0$. As a consequence, the generalized covariation $[F(X), X]_t$ exists for any function F that satisfies the conditions of the theorem. Let

$$\begin{aligned} H(x) &= F(0) + \int_0^x F(y)dy, \\ H_n(x) &= F(0) + \int_0^x \varphi_n(y)dy. \end{aligned}$$

Then by Theorem 2.3 of [123] we have

$$\begin{aligned} H_n(X(t)) &= H_n(X(0)) + \int_0^t H_n'(X(s))d^{\pm}X(s) \pm \frac{1}{2}\left[H_n'(X), X\right]_t, \\ &= H_n(X(0)) + \int_0^t \varphi_n(X(s))d^{\pm}X(s) \pm \frac{1}{2}\left[\varphi_n(X), X\right]_t, \end{aligned}$$

where the existence of the last term of the equality follows since H_n is a function in C^1 and X is a reversible semimartingale. Equations (1.4.4) and (1.4.5) imply that φ_n converges uniformly on each compact to F. It follows that $\varphi_n(X.)$ converges *ucp* to $F(X.)$. On the other hand, since H_n converges uniformly on each compact to H, it follows that $H_n(X.)$ converges *ucp* to $H(X.)$. Moreover, the continuity of the processes $I^\sigma = \lim_{\epsilon \downarrow 0} I^\sigma_\epsilon$, for $\sigma = +, -, 0$ implies that $I^\sigma(\varphi_n, dX)(t)$ converges *ucp* to $I^\sigma(F, dX)(t)$, for $\sigma = +, -$ or more simply,

$$\int_0^t \varphi_n(X(s))d^\pm X(s) \text{ converges } ucp \text{ to } \int_0^t F(X(s))d^\pm X(s).$$

Then by the definition of the generalized covariation, it follows that $[H_n'(X), X] = [\varphi_n(X), X]$ converges *ucp* to $[H'(X), X] = [F(X), X]$. We have

$$\begin{aligned} \left[H_n'(X), X\right] &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ H_n'(X(s)+\varepsilon) - H_n'(X(s)) \right\} d\left[X\right]_s \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ \varphi_n(X(s)+\varepsilon) - \varphi_n(X(s)) \right\} d\left[X\right]_s. \end{aligned}$$

Taking the limit in both sides of the equality, we have the result. ■

Corollary 1.4.2 *Under the same argument on X, suppose that Relations (1.4.1) and (1.4.2) hold. Let F be absolutely continuous with a locally bounded derivative f and $F(0) = 0$ so that*

$$F(x) = \int_0^x f(y)dy.$$

Then, $[f(X), X]$ exists and is given by

$$[f(X), X] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ f(X(s)+\varepsilon) - f(X(s)) \right\} d\left[X\right]_s. \tag{1.4.6}$$

Proof. As before, we can assume, by a localization argument, that the process X_t takes values in a compact interval $[-K, K]$ and that f has support in this interval. Consider a sequence of infinitely differentiable functions f_n with support included in $[-K, K]$ such that f_n converges uniformly to f in $L^2(\mathbb{R})$ as $p \to \infty$. It follows that $f_n(X\cdot)$ converges *ucp* to

$f(X\cdot)$. Let

$$\begin{aligned} H_p(x) &= \int_x^{x+\frac{1}{p}} f(y)dy, \\ H_{p,n}(x) &= \int_x^{x+\frac{1}{p}} f_n(y)dy, \end{aligned}$$

Then by the previous arguments, since $\left[H'_{p,n}(X), X\right]$ converges *ucp* to $[f_n(X), X]$, it follows that $[f_n(X), X]$ exists and

$$[f_n(X), X] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ f_n(X(s) + \varepsilon) \; - \; f_n(X(s)) \right\} d\,[X]_s \, .$$

The result follows from the convergence *ucp* of $f_n(X\cdot)$ to $f(X\cdot)$. ■

1.4.2 The time dependent case

Theorem 1.4.3 *Let $u = \{u(t), t \in [0,T]\}$ be an adapted stochastic process such that $\int_0^T u(s)^2\, ds < \infty$ a.s. Set $X(t) = \int_0^t u(s)\, dB(s)$. Suppose that for all $\delta > 0$, there exist constants c_i^δ, $i = 1, 2$, such that we have*

$$P\left(\int_0^T F(s, X(s))^2 (u_s)^2 ds > \delta \right) \leq c_1^\delta \, \|F\|_2^2 \, , \tag{1.4.7}$$

$$P\left(\sup_{0 \leq u \leq t} \left| \int_0^u \frac{F(s+\varepsilon, X(s+\varepsilon)) - F(s, X(s))}{\varepsilon} \left(X(s+\varepsilon) - X(s) \right) ds \right| > \delta \right) \leq c_2^\delta \, \|F\|_2 \, , \tag{1.4.8}$$

for all $t \in [0,T]$, and for any $F(t,\cdot)$ in $C_K^\infty(\mathbb{R})$ (infinitely differentiable with compact support) for all t and continuously differentiable in t. Let $F(t,x)$ be $L^2_{\text{loc}}(\mathbb{R})$ in x on $\mathbb{R} \backslash \{0\}$ and continuously differentiable in t.

Then, the generalized covariation process $[F(\cdot, X), X]$ exists, and we have

$$[F(\cdot, X), X]_t \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ F(s, X(s) + \varepsilon) \; - \; F(s, X(s)) \right\} d\,[X]_s. \tag{1.4.9}$$

Proof. Fix $t \in [0,T]$, and set

$$V_t^q\,(F) = q \int_0^t \left[F(s, X(s + \frac{1}{q})) - F(s, X(s)) \right] \left(X(s + \frac{1}{q}) - X(s) \right) ds.$$

For any $K > 0$ we define the stopping time $T_K = \inf\{t : |X(t)| > K\}$. Let $\delta > 0$ and take $K > 0$ in such a way that $P(T_K \leq t) \leq \delta$. As before, consider a sequence of infinitely differentiable functions φ_n with support included in $[-K, K]$ such that $\|F(t,\cdot) - \varphi_n(t,\cdot)\|_{L^2}$ converges to zero.

The choice of such a sequence guarantees that, for any $K > 0$,

$$\int_{-K}^{K} (\varphi_n(t,x) - F(t,x))^2\, dx \to 0 \quad \text{as } n \to \infty. \tag{1.4.10}$$

This means that $\varphi_n(t,\cdot)$ converges uniformly on each compact to $F(t,\cdot)$ in $L^2(\mathbb{R})$ for all t. Consequently, for any $K > 0$,

$$\begin{aligned} & E\left[\int_0^t (\varphi_n(s,X(s)) - F(s,X(s)))^2\, I(|X(s)| \leq K)\, ds\right] \\ =& E\left[\int_0^t (\varphi_n(s,X(s)) - F(s,X(s)))^2\, I(T_K > t)\, ds\right] \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Using the same arguments as before, we can show the generalized covariation $[F(\cdot,X),X]_t$ exists for any function F satisfying the conditions of the Theorem.

Let

$$\begin{aligned} H(t,x) &= \int_0^x F(t,y)dy, \\ H_n(t,x) &= F(t,0) + \int_0^x \varphi_n(t,y)dy, \end{aligned}$$

The result follows by the same arguments as in the proof of the previous Theorem. ■

Under conditions of Theorem 2 in [1], we derive the following

Theorem 1.4.4 *Let $B = (B(t), 0 \leq t \leq T)$ be a one dimensional standard Brownian motion. Suppose that $F(t,x)$ is continuously differentiable in t and absolutely continuous in x with locally bounded derivative $\dfrac{\partial F}{\partial x}$. Furthermore, suppose that*

1. *$F(t,0) = 0$ so that for all $t \geq 0$*

$$F(t,x) = \int_0^x \frac{\partial F}{\partial x}(t,y)dy,$$

2. *for all* $t \geq 0$
$$\frac{\partial F}{\partial t}(t,x) = \int_0^x \frac{\partial^2 F}{\partial x \partial t}(t,y)dy,$$
where $\dfrac{\partial^2 F}{\partial x \partial t}$ *is locally bounded,*

3. $F^*(x) = \sup_{t\leq T} |F(t,x)| \in L^2_{\text{loc}}(\mathbb{R})$*, for some* $T > 0$*,*

4. $\dfrac{\partial F^*}{\partial t}(x) = \sup_{t\leq T} \left|\dfrac{\partial F}{\partial t}(t,x)\right| \in L^1_{\text{loc}}(\mathbb{R})$*, for some* $T > 0$*,*
where $\dfrac{\partial F}{\partial x}(t,x) = f(t,x)$*.*

Then $[F(\cdot,B),B]$ *exists and*

$$\begin{aligned}&[F(\cdot,B),B]_t\\ =&\lim_{\varepsilon\downarrow 0}\left(\int_0^t f(s,B(s))1_{\{|B(s)|\geq\varepsilon\}}ds + \int_0^t F(s,\varepsilon)d_sL_s^{\varepsilon} - \int_0^t F(s,-\varepsilon)d_sL_s^{-\varepsilon}\right),\end{aligned} \tag{1.4.11}$$

where the limit holds in the ucp sense and L^a_s *is the local time at* a *of the process* B *given by*
$$L^a_s(Z) := |B(t)-a| - |a| + \int_0^t \operatorname{sgn}(B(s)-a)dB(s).$$

Proof. Since F is absolutely continuous in x, it follows that F is continuous in x. Then applying Theorem 1.3.5, it follows that $[F(\cdot,B),B]$ exists.

In order to show the equality, we adapt here the proof given by AlHussaini and Elliot in [1]. Define
$$G(t,x) = \int_0^x F(t,y)\,dy \quad \text{and} \quad \frac{\partial G}{\partial t}(t,x) = \int_0^x \frac{\partial F}{\partial t}(t,y)\,dy.$$
Write $F_\varepsilon(t,x) = F(t,x)1_{|x|\geq\varepsilon}$ and
$$G_\varepsilon(t,x) = \int_0^x F_\varepsilon(t,y)\,dy, \quad \text{then} \quad \frac{\partial G_\varepsilon}{\partial t}(t,x) = \int_0^x \frac{\partial F_\varepsilon}{\partial t}(t,y)\,dy.$$
Applying Corollary 8 in [1] to G_ε with a standard Brownian motion B
$$\begin{aligned}G_\varepsilon(t,B(t)) &= \int_0^t F_\varepsilon(s,B(s))\,dB(s) + \int_0^t \frac{\partial G_\varepsilon}{\partial t}(s,B(s))\,ds\\ &\quad -\frac{1}{2}\int_{-\infty}^{\infty} F_\varepsilon(s,a)\,d_aL_t^a + \frac{1}{2}\int_0^t\int_{-\infty}^{\infty}\frac{\partial F_\varepsilon}{\partial t}(s,a)\,d_aL_s^a\,ds.\end{aligned}$$

Writing

$$A_t^{F_\varepsilon} = -\int_{-\infty}^{\infty} F_\varepsilon(s,a)\, d_a L_t^a + \int_0^t \int_{-\infty}^{\infty} \frac{\partial F_\varepsilon}{\partial t}(s,a)\, d_a L_s^a\, ds,$$

we have

$$\begin{aligned} A_t^{F_\varepsilon} &= -\left(\int_\varepsilon^\infty F(s,a)\, d_a L_t^a + \int_{-\infty}^{-\varepsilon} F(s,a)\, d_a L_t^a\right) \\ &\quad + \int_0^t \left(\int_\varepsilon^\infty \frac{\partial F}{\partial s}(s,a)\, d_a L_s^a + \int_{-\infty}^{-\varepsilon} \frac{\partial F}{\partial s}(s,a)\, d_a L_s^a\right) ds, \end{aligned}$$

and integrating by parts in a, we have

$$\begin{aligned} A_t^{F_\varepsilon} &= L_t^\varepsilon F(t,\varepsilon) - L_t^{-\varepsilon} F(t,-\varepsilon) + \left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} f(t,a)\, L_t^a\, da\right) \\ &\quad - \int_0^t \left(L_t^\varepsilon \frac{\partial F}{\partial s}(s,\varepsilon) - L_t^{-\varepsilon}\frac{\partial F}{\partial s}(s,-\varepsilon)\right) ds - \int_0^t \left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} \frac{\partial^2 F}{\partial x \partial s}(s,a) L_s^a\, da\right) ds. \end{aligned}$$

Applying Fubini's Theorem to the final term and integrating by parts in s

$$\left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon}\right)\left(\int_0^t \frac{\partial^2 F}{\partial x \partial s}(s,a)\, L_s^a\, ds\right) da = \left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon}\right)\left(L_t^a f(t,a) - \int_0^t f(s,a)\, d_s L_s^a\right) da.$$

Therefore,

$$\begin{aligned} A_t^{F_\varepsilon} &= -\int_{-\infty}^{\infty} F_\varepsilon(s,a)\, d_a L_t^a + \int_0^t \int_{-\infty}^{\infty} \frac{\partial F_\varepsilon}{\partial t}(s,a)\, d_a L_s^a\, ds \\ &= L_t^\varepsilon F(t,\varepsilon) - L_t^{-\varepsilon} F(t,-\varepsilon) - \int_0^t \left(L_t^\varepsilon \frac{\partial F}{\partial s}(s,\varepsilon) - L_t^{-\varepsilon}\frac{\partial F}{\partial s}(s,-\varepsilon)\right) ds \\ &\quad + \int_0^t f(s,B(s)) 1_{|B(s)|\geq \varepsilon}\, ds. \end{aligned}$$

For the function $G(t,x)$ since the hypotheses of Theorem 2.3 [123] are satisfied the process $[F(\cdot,B),B]$ is also defined by

$$[F(\cdot,B),B] = 2\left(G(t,x) - \int_0^t F(s,B(s))\, dB(s) + \int_0^t \frac{\partial G}{\partial t}(s,B(s))\, ds\right).$$

Therefore,

$$\begin{aligned} [F(\cdot,B),B] - A_t^{F_\varepsilon} &= 2\left(\int_0^{B(t)} F(t,y) 1_{|y|\leq\varepsilon}\, dy - \int_0^t F(s,B(s)) 1_{|B(s)|\leq\varepsilon}\, dB(s) \right. \\ &\qquad \left. - \int_0^t \int_0^{B(s)} \frac{\partial F}{\partial t}(s,y) 1_{|y|\leq\varepsilon}\, dy\, ds\right), \end{aligned}$$

and for $T > 0$,

$$\begin{aligned} E\left[\sup_{t\leq T}\left|[F(\cdot,B),B] - A_t^{F_\varepsilon}\right|\right] &\leq KE\left[\sup_{t\leq T}\left|\int_0^{B(t)} F(t,y)1_{|y|\leq\varepsilon}\,dy\right|\right. \\ &+\sup_{t\leq T}\left|\int_0^t F(s,B(s))1_{|B(s)|\leq\varepsilon}\,dB(s)\right| \\ &\left.+\sup_{t\leq T}\left|\int_0^t\left(\int_0^{B(s)}\frac{\partial F}{\partial t}(s,y)1_{|y|\leq\varepsilon}dy\right)\,ds\right|\right]. \end{aligned}$$

Denote the three terms in the expectation by I^1, I^2 and I^3, respectively. Then

$$E\left[I^1\right] \leq E\left[\sup_{t\leq T}\left(\int_{-\varepsilon}^{\varepsilon}|F(t,y)|\right)\right] \leq \int_{-\varepsilon}^{\varepsilon} F^*(y)\,dy,$$

and this converges to 0 as $\varepsilon \to 0$.

$$\begin{aligned} E\left[I^2\right] &\leq C_pE\left[\left|\int_0^T F(s,B(s))1_{|B(s)|\leq\varepsilon}\,dB(s)\right|\right] \leq CE\left(\int_0^T F^2(s,B(s))1_{|B(s)|\leq\varepsilon}\,ds\right)^{\frac{1}{2}} \\ &= CE\left(\int_{-\varepsilon}^{\varepsilon}\left(\int_0^T F^2(s,a)1_{|a|\leq\varepsilon}\,d_sL_T^a\right)da\right)^{\frac{1}{2}} \\ &\leq CE\left(\int_{-\varepsilon}^{\varepsilon} F^*(a)\,L_T^a\,da\right)^{\frac{1}{2}} \leq C\left(E(L_T^a)^{\frac{1}{2}}\right)\left(\int_{-\varepsilon}^{\varepsilon} F^*(a)\,da\right)^{\frac{1}{2}} \end{aligned}$$

which again converges to 0 as $\varepsilon \to 0$. Finally,

$$E\left[I^3\right] \leq E\left[\sup_{t\leq T}\left|\int_0^t\int_{-\varepsilon}^{\varepsilon}\left|\frac{\partial F}{\partial t}(s,y)\right|\,dy\,ds\right|\right] \leq T^p\int_{-\varepsilon}^{\varepsilon}\frac{\partial F^*}{\partial t}(y)\,dy,$$

which converges to 0 as $\varepsilon \to 0$. This means that $[F(\cdot,B),B]$ converges in mean to $A_t^{F_\varepsilon}$, which implies uniform convergence on compacts in probability, and then the result follows.

■

Definition 1.4.5 *The principal value of the integral* $\int_0^t F(s,B(s))\,ds$, *where* $B = (B(t), 0 \leq t \leq T)$ *is a one dimensional standard Brownian motion, is defined by*

$$\text{vp.}\int_0^t F(s,B(s))\,ds = \lim_{\varepsilon\downarrow 0}\int_0^t F(s,B(s))1_{\{|B(s)|\geq 0\}}\,ds,$$

where F *is a function such that the right hand side of above equality converges in probability.*

Corollary 1.4.6 *Under the same conditions of Theorem 1.4.4, if we assume moreover that either the principal value of $\int_0^t f(s, B(s))\, ds$ exists, or the following limit $\lim_{\varepsilon \downarrow 0} \int_0^t F(s,\varepsilon) d_s L_s^{\varepsilon} - \int_0^t F(s,-\varepsilon) d_s L_s^{-\varepsilon}$ exists, then the equality holds in the ucp sense,*

$$[F(\cdot, B), B]_t = \mathrm{vp.} \int_0^t f(s, B(s))\, ds + \lim_{\varepsilon \downarrow 0} \left(\int_0^t F(s,\varepsilon)\, d_s L_s^{\varepsilon} - \int_0^t F(s,-\varepsilon)\, d_s L_s^{-\varepsilon} \right).$$

Proof. It follows directly from the proof of Theorem 1.4.4. ■

As a corollary we have the following, which is proved in [22].

Corollary 1.4.7 *Assume that F is time independent, under the same condition of Corollary 1.4.6, we have*

$$[F(B), B]_t = \mathrm{vp.} \int_0^t F(B(s))\, ds + \lim_{\varepsilon \downarrow 0} \left(F(\varepsilon) - F(-\varepsilon)\right) L_s^0,$$

Proof. It follows from Corollary 1.4.6 that

$$[F(\cdot, B), B]_t = \mathrm{vp.} \int_0^t f(s, B(s))\, ds + \lim_{\varepsilon \downarrow 0} \left(\int_0^t F(s,\varepsilon)\, d_s L_s^{\varepsilon} - \int_0^t F(s,-\varepsilon)\, d_s L_s^{-\varepsilon} \right).$$

If F is time independent, then the equality becomes

$$\begin{aligned} [F(B), B]_t &= \mathrm{vp.} \int_0^t F(B(s))\, ds + \lim_{\varepsilon \downarrow 0} \left(F(\varepsilon)\, L_s^{\varepsilon} - F(-\varepsilon)\, L_s^{-\varepsilon}\right) \\ &= \mathrm{vp.} \int_0^t F(B(s))\, ds + \lim_{\varepsilon \downarrow 0} \left(F(\varepsilon) - F(-\varepsilon)\right) L_s^0. \end{aligned}$$

The last inequality follows from the fact that L_t^x admits a continuous modification in x. ■

1.5 Application to transformation of semimartingales

In this section, we derive a generalized covariation for transformation of continuous process by continuous functions.

Theorem 1.5.1 *Let X, Y be two continuous processes such that $[X, Y]$ exists and $F \in C^0(\mathbb{R})$, then $[F(X), Y]$ exists and we have the following:*

$$\begin{aligned} [F(X), Y]_t &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{ F(X(s)+\varepsilon) - F(X(s)) \Big\}\, d\,[X, Y]_s \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{ F(X(s)) - F(X(s)-\varepsilon) \Big\}\, d\,[X, Y]_s. \end{aligned} \tag{1.5.1}$$

Proof. We will only show the first equality. The second follows using the same arguments. Associate to F the following function:

$$H_n(x) := n \int_x^{x+\frac{1}{n}} F(y)\, dy. \tag{1.5.2}$$

On the one hand we have

$$H_n(x) = n \int_x^{x+\frac{1}{n}} F(y)\, dy \to F(x) \quad \text{for} \quad n \to \infty. \tag{1.5.3}$$

On the other hand

$$H_n'(x) = n \left\{ F(x + \frac{1}{n}) - F(x) \right\}. \tag{1.5.4}$$

We note that the function $H_n(x)$ is a C^1 function. Then by Proposition 2.1 [122], the generalized covariation process $[H_n(X), Y]$ exists. As before, we have

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_0^t \frac{H_n(X(s+\varepsilon)) - H_n(X(s))}{\varepsilon} \left(Y(s+\varepsilon) - Y(s)\right) ds \\
=& \left[H_n(X), Y\right]_s \\
=& \int_0^t H_n'(X(s)) d\left[X, Y\right]_s \\
=& n \int_0^t \left\{ F(X(s) + \frac{1}{n}) - F(X(s)) \right\} d\left[X, Y\right]_s.
\end{aligned}$$

Since F is continuous, H_n converges uniformly on each compact to F, it follows that $H_n(X.)$ converges *ucp* to $F(X.)$. Moreover, the continuity of the processes $I^\sigma = \lim_{\epsilon \downarrow 0} I^\sigma_\epsilon$, for $\sigma = +, -, 0$ implies that $I^\sigma(H_n, dX)(t)$ converges *ucp* to $I^\sigma(F, dX)(t)$, for $\sigma = +, -, 0$, then by the definition of the generalized covariation, it follows that

$$\left[H_n(X), Y\right]_s \text{ converges } ucp \text{ to } \left[F(X), Y\right]_s. \tag{1.5.5}$$

Equation (1.5.5) means that $[F(X), Y]$ exists as the limit in the *ucp* sense of $[H_n(X), Y]$ when $n \to \infty$. Thus the first part of Theorem is proved. Moreover, since

$$n \int_0^t \left\{ F(X(s) + \frac{1}{n}) - F(X(s)) \right\} d\left[X, Y\right]_s = \left[H_n(X), Y\right]_s,$$

by the uniqueness of the limit, the equality still holds if we take the limit on both sides in the *ucp* sense when $n \to +\infty$. Equation (1.5.1) follows by Equation (1.5.5). ■

Remark 1.5.2 *Let X, Y be two continuous processes such that $[X,Y]$ exists. If F is a function in $C^1(\mathbb{R})$, then Equation (1.5.1) becomes Equation (1.2.7).*

Proposition 1.5.3 *Let X, Y be two continuous processes such that $[X,Y]$ exists and F, G be two functions in $C^0(\mathbb{R})$. Then $[F(X), G(Y)]$ exists and the following equality holds in the ucp sense:*

$$\begin{aligned}
&[F(X), G(Y)]_t \\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t (F(X(s+\varepsilon)) - F(X(s)))\,(G(Y(s+\varepsilon)) - G(Y(s)))\,ds \\
=&\lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)+\varepsilon) - F(X(s))}{\varepsilon}\right)\left(\frac{G(Y(s)+\varepsilon) - G(Y(s))}{\varepsilon}\right) d\,[X,Y]_s \qquad (1.5.6)\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)) - F(X(s)-\varepsilon)}{\varepsilon}\right)\left(\frac{G(Y(s)) - G(Y(s)-\varepsilon)}{\varepsilon}\right) d\,[X,Y]_s\,. \qquad (1.5.7)
\end{aligned}$$

Proof. As in the previous sections, we define H_n and J_n by

$$H_n(x) = n\int_x^{x+\frac{1}{n}} F(y)\,dy,\;\; J_n(x) = n\int_x^{x+\frac{1}{n}} G(y)\,dy.$$

We have

$$H_n(x) = n\int_x^{x+\frac{1}{n}} F(y)\,dy \to\; F(x) \quad\text{and}\quad J_n(x) = n\int_x^{x+\frac{1}{n}} G(y)\,dy \to\; G(x) \quad\text{for}\quad n\to\infty,$$

and

$$H_n'(x) = n\left\{F(x+\frac{1}{n}) - F(x)\right\},\;\; J_n'(x) = n\left\{G(x+\frac{1}{n}) - G(x)\right\}. \qquad (1.5.8)$$

It follows by the properties of the functions H_n and J_n that the generalized covariation process $[H_n(X), J_n(Y)]$ exists and we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_0^t \frac{H_n(X(s+\varepsilon)) - H_n(X(s))}{\varepsilon}\cdot (J_n(X(s+\varepsilon)) - J_n(X(s)))\;ds \\
=\;& [H_n(X), J_n(Y)]_t \\
=\;& \int_0^t H_n'(X(s))J_n'(Y(s))d\,[X,Y]_s \\
=\;& n^2\int_0^t \left\{F(X(s)+\frac{1}{n}) - F(X(s))\right\}\left\{G(Y(s)+\frac{1}{n}) - G(Y(s))\right\} d\,[X,Y]\,.
\end{aligned}$$

The continuity of the generalized covariation process and the convergence of H_n to F (respectively J_n to G) imply that

$$[H_n(X), J_n(Y)] \text{ converges } ucp \text{ to } [F(X), G(Y)]$$

and the result follows by taking the limit *ucp* when $n \to \infty$ in the last term of the equality.

■

Corollary 1.5.4 *Let X be a continuous process such that $[X]$ exists and F be a function in $C^0(\mathbb{R})$. Then $[F(X)]$ exists and the following equalities hold in the ucp sense:*

$$\begin{aligned}
[F(X)]_t &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t (F(X(s+\varepsilon)) - F(X(s)))^2\, ds \\
&= \lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)+\varepsilon) - F(X(s))}{\varepsilon}\right)^2 d\,[X]_s \\
&= \lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)) - F(X(s)-\varepsilon)}{\varepsilon}\right)^2 d\,[X]_s\,. \qquad (1.5.9)
\end{aligned}$$

Proof. It suffices here to chose $X = Y$ and $G = F$ and the result follows by the preceding Theorem. ■

In the case that the process X is a continuous semimartingale, we have

Remark 1.5.5 *Let X, Y two continuous semimartingales, admitting mutual bracket and F, G two functions in $C^0(\mathbb{R})$. Then the following equalities hold.*

$$\begin{aligned}
\langle F(X), G(Y)\rangle_t &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t (F(X(s+\varepsilon)) - F(X(s)))\,(G(Y(s+\varepsilon)) - G(Y(s)))\, ds \\
&= \lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)+\varepsilon) - F(X(s))}{\varepsilon}\right)\left(\frac{G(Y(s)+\varepsilon) - G(Y(s))}{\varepsilon}\right) d\,\langle X, Y\rangle_s\,, \\
\langle F(X)\rangle_t &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t (F(X(s+\varepsilon)) - F(X(s)))^2\, ds \\
&= \lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s+\varepsilon)) - F(X(s))}{\varepsilon}\right)^2 d\,\langle X\rangle_s\,.
\end{aligned}$$

We derive the following result which is proved in [131].

Corollary 1.5.6 *Let X be a continuous semimartingale and f be a function in $L^2_{loc}(\mathbb{R})$, $F(x) = \int_0^x f(y)dy$, $x \in \mathbb{R}$ and $Z = F(X)$. Then*

$$[Z]_t = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t (F(X(s+\varepsilon)) - F(X(s)))^2\, ds = \int_{\mathbb{R}} f^2(x)\, L_t^x(X)\, dx. \qquad (1.5.10)$$

where $L_t^x(X)$ is the local time of the process X at point x.

Proof. Using the previous Theorem, we have

$$[Z]_t = \lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)+\varepsilon)-F(X(s))}{\varepsilon}\right)^2 d\langle X\rangle_s.$$

Using, the fact that $F'(x)=f(x)$, the right hand side of the equality gives $\int_0^t f^2(X(s))\,d\langle X\rangle_s$ and the result follows by the occupation formula. ■

Corollary 1.5.7 *Let X,Y be two continuous semimartingales and F,G be two functions in $C^0(\mathbb{R})$. Then $\langle F(X),G(Y)\rangle$ exists and the following equalities hold in the ucp sense:*

$$\begin{aligned}
&[F(X),G(Y)]_t\\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t (F(X(s+\varepsilon))-F(X(s)))\,(G(Y(s+\varepsilon))-G(Y(s)))\,ds\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s)+\varepsilon)-F(X(s))}{\varepsilon}\right)\left(\frac{G(Y(s)+\varepsilon)-G(Y(s))}{\varepsilon}\right) d\langle X,Y\rangle_s.\\
=&\lim_{\varepsilon\downarrow 0}\int_0^t \left(\frac{F(X(s))-F(X(s)-\varepsilon)}{\varepsilon}\right)\left(\frac{G(Y(s))-G(Y(s)-\varepsilon)}{\varepsilon}\right) d\langle X,Y\rangle_s.
\end{aligned} \tag{1.5.11}$$

A multi-dimensional and useful extension of Proposition 1.5.3 is given by the following result.

Proposition 1.5.8 *Let $X=\left(X^1,\cdots,X^m\right)$, $Y=\left(Y^1,\cdots,Y^m\right)$ be continuous $\mathbb{R}^m$-valued processes such that $\left\{X^1,\cdots,X^m,Y^1,\cdots,Y^m\right\}$ have all mutual brackets. Let F,G be two functions in $C^0(\mathbb{R}^m)$, then $\{F(X),G(Y)\}$ have all the mutual brackets and*

$$\begin{aligned}
&[F(X),G(Y)]_t\\
=&\sum_{i,j=1}^m \lim_{\varepsilon\downarrow 0}\int_0^t \left\{\frac{F(X^1(s),\cdots,X^i(s)+\varepsilon,\cdots,X^m(s))\ -\ F(X^1(s),\cdots,X^i(s),\cdots,X^m(s))}{\varepsilon}\right\}\\
&\left\{\frac{G(Y^1(s),\cdots,Y^j(s)+\varepsilon,\cdots,Y^m(s))\ -\ G(Y^1(s),\cdots,Y^j(s),\cdots,Y^m(s))}{\varepsilon}\right\} d\left[X^i,X^j\right]_s\\
=&\sum_{i,j=1}^m \lim_{\varepsilon\downarrow 0}\int_0^t \left\{\frac{F(X^1(s),\cdots,X^i(s),\cdots,X^m(s))\ -\ F(X^1(s),\cdots,X^i(s)-\varepsilon,\cdots,X^m(s))}{\varepsilon}\right\}\\
&\left\{\frac{G(Y^1(s),\cdots,Y^j(s),\cdots,Y^m(s))\ -\ G(Y^1(s),\cdots,Y^j(s)-\varepsilon,\cdots,Y^m(s))}{\varepsilon}\right\} d\left[X^i,X^j\right]_s.
\end{aligned} \tag{1.5.12}$$

Proof. We define for all $i = 1, \cdots, n,\ H_n$ and J_n by

$$H_n(x_1, \cdots, x_n) = n \int_{x_i}^{x_i + \frac{1}{n}} F(x_1, \cdots, x_{i-1}, \overset{i}{y}, x_{i+1}, \cdots, x_m)\, dy,$$

$$J_n(x_1, \cdots, x_n) = n \int_{x_i}^{x_i + \frac{1}{n}} G(x_1, \cdots, x_{i-1}, \overset{i}{y}, x_{i+1}, \cdots, x_m)\, dy,$$

We have

$$H_n(x_1, \cdots, x_n) \to F(x_1, \cdots, x_n) \quad \text{and} \quad J_n(x_1, \cdots, x_n) \to G(x_1, \cdots, x_n) \quad \text{for} \quad n \to \infty,$$

and

$$\frac{\partial}{\partial x_i} H_n(x_1, \cdots, x_n) = n \left\{ F(x_1, \cdots, x_i + \frac{1}{n}, \cdots, x_n) - F(x_1, \cdots, x_i, \cdots, x_n) \right\},$$

$$\frac{\partial}{\partial x_i} J_n(x_1, \cdots, x_n) = n \left\{ G(x_1, \cdots, x_i + \frac{1}{n}, \cdots, x_n) - G(x_1, \cdots, x_i, \cdots, x_n) \right\}.$$

The result follows by applying Proposition 1.1 in [123] on H_n and J_n and using the same argument as in proof of the preceding Proposition 1.5.3. ■

Corollary 1.5.9 *Let $X = \left(X^1, \cdots, X^m\right)$ be a continuous $\mathbb{R}^m$-valued process with all mutual brackets. Let F be a function in $C^0(\mathbb{R}^m)$. Then $\left\{F(X), X^i\right\}$ have all the mutual brackets for $i = 1, \cdots, m$ and*

$$\begin{aligned} &\left[F(X), X^i\right]_t \\ =&\sum_{j=1}^m \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ F(X^1(s), \cdots, X^j(s), \cdots, X^m(s)) - F(X^1(s), \cdots, X^j(s) + \varepsilon, \cdots, X^m(s)) \right\} \\ &\times d\left[X^i, X^j\right]_s, \end{aligned} \tag{1.5.13}$$

or

$$\begin{aligned} &\sum_{j=1}^m \lim_{\varepsilon \downarrow 0} \int_0^t \frac{F(X^1(s), \cdots, X^j(s+\varepsilon), \cdots, X^m(s)) - F(X^1(s), \cdots, X^j(s), \cdots, X^m(s))}{\varepsilon} \\ &\times (X^i(s+\varepsilon) - X^i(s))\, ds \\ =&\sum_{j=1}^m \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ F(X^1(s), \cdots, X^j(s), \cdots, X^m(s)) - F(X^1(s), \cdots, X^j(s) + \varepsilon, \cdots, X^m(s)) \right\} \\ &\times d\left[X^i, X^j\right]_s. \end{aligned} \tag{1.5.14}$$

1.6 The random function case

In this section we shall give an answer of the following question: Do the equalities we have in our Theorems always hold for any class of functions? We shall give here an illustrative example where the equalities fail, thanks to the result obtained by Walsh in [129].

Let $B(t)$ denotes a standard Brownian motion on $\mathbb{R}$, $B(0) = 0$, with jointly continuous local time $\{L(t,x) : t \geq 0,\, x \in \mathbb{R}\}$. For each $x \in \mathbb{R}$ we define

$$\begin{aligned} A(t,x) &= \int_0^t 1_{\{B(s)\leq x\}}\, ds \\ &= \int_{-\infty}^x L(t,y)\, dy. \end{aligned} \tag{1.6.1}$$

Let us first give some facts about function A. The following come from (1.6.1).

1. $A(t,x)$ is jointly continuous in (t,x).
2. For fixed x, A is an increasing Lipschitz continuous function of t.
3. For fixed t, A is an increasing C^1 function of x with

$$\frac{\partial A}{\partial x} = L(t,x).$$

Let us now recall Walsh's theorem about the decomposition of $A(t,x)$.

Theorem 1.6.1 *$A(t, B(t))$ has the following decomposition*

$$A(t,B(t)) = \int_0^t L(s,B(s))\, dB(s) + X(t),$$

where

$$\begin{aligned} X(t) &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \Big\{L(s,B(s)) \,-\, L(s,B(s)-\varepsilon)\Big\}\, ds \\ &= t + \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \Big\{L(s,B(s)+\varepsilon) \,-\, L(s,B(s))\Big\}\, ds. \end{aligned}$$

The limits exist in probability, uniformly for t in compact sets.

As a consequence of this theorem, we can make the following remark.

Remark 1.6.2 *The preceding Theorem simply means that we have*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{L_s^{B(s)} - L_s^{B(s)-\varepsilon}\right\}ds \;\neq\; \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{L_s^{B(s)+\varepsilon} - L_s^{B(s)}\right\}ds.$$

In other words

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{L(s,B(s)) - L(s,B(s)-\varepsilon)\right\}ds \;\neq\; \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{L(s,B(s)+\varepsilon) - L(s,B(s))\right\}ds.$$

Note that since $L(t,y)$ is continuous in y it follows by Theorem 1.3.5 that $[L(t,B),B]$ exists, but we do not have the equalities (1.3.9).

1.7 Conclusion

Theorem 1.7.1 *Let X be a continuous process with finite quadratic variation $[X]$ and $F\in C^0(\mathbb{R})$. The following are equivalent*

1. $\lim_{\varepsilon\downarrow 0}\int_0^t \frac{F(X(s+\varepsilon))-F(X(s))}{\varepsilon}\,(X(s+\varepsilon)-X(s))\,ds$ *exists,*
2. $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\left\{F(X(s)+\varepsilon) - F(X(s))\right\}d\,[X]_s$ *exists.*

Proof. (1) $\Longrightarrow$ (2) has already been done. Let us show that under some conditions on F, (2) $\Longrightarrow$ (1).

(a) We will first show the implication for a function $F\in C^1(\mathbb{R})$. Using Taylor's formula, one can write

$$F(X(s)+\epsilon)-F(X(s)) \;=\; F'(X(s))\epsilon + R_\epsilon(s)\epsilon,\quad s\geq 0,\,\epsilon\geq 0, \tag{1.7.1}$$

where $R_\epsilon(s)$ denotes a process which converges in the *ucp* sense to 0 when $\epsilon\to 0$. Multiplying both sides of Equation (1.7.1) by $d\,[X]_s$, integrating from 0 to t and dividing by ϵ, we have

$$\frac{1}{\epsilon}\int_0^t (F(X(s)+\epsilon)-F(X(s)))\;d\,[X]_s = \int_0^t F'(X(s))\,d\,[X]_s + \int_0^t R_\epsilon(s)\,d\,[X]_s\,. \tag{1.7.2}$$

The second term in the right hand side converges to 0 in the sense *ucp*. In fact, putting $I_\epsilon(t) = \int_0^t R_\epsilon(s)\, d\,[X]_s$ one can write

$$\sup_{t\leq T} |I_\epsilon(t)| \leq \sup_{t\leq T} \left|R^1_\epsilon(s)\right| [X]_T \,.$$

Then, the existence of $[X]$ and the convergence of $R_\epsilon(\cdot)$ to 0 in the *ucp* sense imply that $I_\epsilon(\cdot)$ converges *ucp* to 0. Taking the limit in the sense *ucp* in both sides of Equation (1.7.2), we have that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{ F(X(s)) \;-\; F(X(s)+\varepsilon) \Big\}\, d\,[X]_s = \int_0^t F'(X(s))\, d\,[X]_s \,.$$

We also know that the term on the right exists since the quadratic covariation $[X]$ exists and $F \in C^1(\mathbb{R})$. Then by Proposition 2.1 of [122] the generalized covariation $[F(X), X]$ exists and we have

$$[F(X), X]_t = \int_0^t F'(X(s))\, d\,[X]_s \,.$$

It follows by the definition of the generalized covariation that the limit in (1) exists in the *ucp* term and is equal to the limit in (2).

(b) We now prove the implication for a function F in $C^0(\mathbb{R})$ As before, let H_n the function defined by

$$H_n(x) := n \int_x^{x+\frac{1}{n}} F(y)\, dy. \tag{1.7.3}$$

Then

$$H_n(x) = n \int_x^{x+\frac{1}{n}} F(y)\, dy \to F(x) \quad \text{for} \quad n \to \infty, \tag{1.7.4}$$

and

$$H_n'(x) = n \left\{ F(x+\frac{1}{n}) - F(x) \right\}. \tag{1.7.5}$$

Since we note that the function $H_n(x)$ is a C^1 function, by **(a)** the existence of the limit $\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \Big\{ H_n(X(s)) \;-\; H_n(X(s)+\varepsilon) \Big\}\, d\,[X]_s$ in the *ucp* sense implies that the generalized covariation process $[H_n(X), X]$ exists. Then the following limit exists in the *ucp* term

$$\lim_{\varepsilon\downarrow 0} \int_0^t \frac{H_n(X(s+\varepsilon)) - H_n(X(s))}{\varepsilon} \left(X(s+\varepsilon) - X(s)\right) ds,$$

and we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \Big\{H_n(X(s)) \;-\; H_n(X(s)+\varepsilon)\Big\}\,d\,[X]_s = [H_n(X),X]_t\,.$$

By hypothesis, we have the existence in the *ucp* sense of

$$\lim_{n\to\infty} n\int_0^t \left\{F(X(s)+\frac{1}{n}) - F(X(s))\right\}d\,[X]_s\,,$$

Using the definition of H_n, we have in the *ucp* sense

$$\begin{aligned}\lim_{n\to\infty} n\int_0^t \left\{F(X(s)+\frac{1}{n}) - F(X(s))\right\}d\,[X]_s &= \lim_{n\to\infty}\int_0^t H_n'(X(s))\,d\,[X]_s\\ &= \lim_{n\to\infty}\,[H_n(X),X]_t\\ &= [F(X),X]_t\,,\end{aligned}$$

where the second equality follows from **(a)** and the third one by the continuity of the generalized covariation process (since it exists by **(a)**) and the fact that H_n converges *ucp* to F. By the definition of the last term, the existence of the limit in *ucp* sense of (1) is proved. ■

Chapter 2

Decomposition of order statistics of semimartingales using local times

2.1 Introduction

Some recent developments in mathematical finance and particularly the distribution of capital in stochastic portfolio theory have led to the necessity of understanding dynamics of the kth-ranked stock amongst n given stocks, at all levels $k = 1, \cdots, n$. For example, $k = 1$ and $k = n$ correspond to the maximum and minimum processes of the collection, respectively. The problem of decomposition for the maximum of n semimartingales was introduced by Chitashvili and Mania in [24]. The authors showed that the maximum process can be expressed in terms of the original processes, adjusted by local times. In [45], Fernholz defined the more general notion of ranked processes (i.e. order statistics) of n continuous Itô processes and gave the decomposition of such processes. However, the main drawback of the latter result is that triple points do not exist, i.e., not more than two processes coincide at the same time, almost surely. Closely related results also appeared earlier in the paper by Nagasawa and Tanaka [92]. Motivated by the question of extending this decomposition to triple points (and higher orders of incidence) as was posed by Fernholz in Problem 4.1.13 of [46], Banner and Ghomrasni recently [8] developed some general formulas for ranked processes of *continuous* semimartingales. In the setting of problem 4.1.13 in [46], they showed that the ranked processes can be expressed in terms of original processes adjusted by the local times of ranked processes. The proofs of those results are based on the generalization of Ouknine's formula [105, 106, 132].

In the present Chapter, we give a new decomposition of order statistics of semimartingales (i.e., not necessarily continuous) in the same setting as in [8]. The result obtained is slightly different to the one in [8] in the sense that we express the order statistics of semimartingales firstly in terms of order statistics processes adjusted by their local times and secondly in terms of original processes adjusted by their local times. The proof of this result is a modified and shorter version of the proof given in [8] based on the homogeneity property. Furthermore, we use the theory of predictable random open sets, introduced by Zheng in [135] and the idea of the proof of Theorem 2.2 in [8] to show that

$$\sum_{i=1}^{n} 1_{\{X^{(i)}(t-)=0\}}\, dX^{(i)^+}(t) = \sum_{i=1}^{n} 1_{\{X_i(t-)=0\}}\, dX_i^+(t)\,,$$

where X_i, $i = 1, \cdots, n$ represent the original processes and $X^{(i)}$ represent the ranked processes. As a consequence of this result, we are independently able to derive an extension of Ouknine's formula in the case of general semimartingales. The desired generalization which is essential in the demonstration of Theorem 2.3 in [8] is not used here to prove our decomposition.

The Chapter is organized as follows. In Section 2.2, we prove the two different decompositions of ranked processes for general semimartingales. In Section 2.3, after showing the above equality, we derive a generalization of Ouknine and Yan's formula.

2.2 Decomposition of Ranked Semimartingales

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions. In our study, any given semimartingale X is supposed to satisfy the following condition (A):

$$\sum_{0<s\leq t} |\Delta X(s)| < \infty \quad a.s. \quad \text{for all} \quad t > 0\,, \tag{A}$$

where $\Delta X(s) = X(s) - X(s-)$.

We begin by giving the definition of the local time of a semimartingale X.

Definition 2.2.1 *Let $X = (X(t))_{t\geq 0}$ be a semimartingale and $a \in \mathbb{R}$. The local time $L_t^a(X)$ of X at a is defined by the following Tanaka-Meyer formula*

$$\begin{aligned} |X(t) - a| &= |X(0) - a| + \int_0^t \operatorname{sgn}(X(s-) - a)\, dX(s) + L_t^a(X) \\ &\quad + \sum_{s\leq t} \left(|X(s) - a| - |X(s-) - a| - \operatorname{sgn}(X(s-) - a)\Delta X(s)\right), \end{aligned}$$

where $\operatorname{sgn}(x) = -1_{(-\infty,0]}(x) + 1_{(0,\infty)}(x)$.

As has been proved by Yor [133], under the condition (A), a measurable version of $(a, t, \omega) \mapsto L_t^a(X)(\omega)$ exists which is continuous in t and right continuous with left limits (i.e. càdlàg) in a. We shall deal exclusively with this version.

Let us recall the definition of the k-th rank process of a family of n semimartingales.

Definition 2.2.2 *Let $X_1, \cdots, X_n$ be semimartingales. For $1 \leq k \leq n$, the k-th rank process of $X_1, \cdots, X_n$ is defined by*

$$X^{(k)} = \max_{1\leq i_1<\cdots<i_k\leq n} \min(X_{i_1}, \cdots, X_{i_k}), \tag{2.2.1}$$

where $1 \leq i_1$ and $i_k \leq n$.

Note that, according to Definition 2.2.2, for $t \in \mathbb{R}^+$,

$$\max_{1\leq i\leq n} X_i(t) = X^{(1)}(t) \geq X^{(2)}(t) \geq \cdots \geq X^{(n)}(t) = \min_{1\leq i\leq n} X_i(t), \tag{2.2.2}$$

so that at any given time, the values of the ranked processes represent the values of the original processes arranged in descending order (i.e. the (reverse) order statistics).

The following theorem shows that the ranked processes derived from semimartingales can be expressed in terms of stochastic integral with respect to the original process adjusted by local times and expressed in terms of stochastic integral with respect to the ranked process adjusted by local times. We shall need the following definitions

$$S_{t-}(k) = \{i : X_i(t-) = X^{(k)}(t-)\} \text{ and } N_{t-}(k) = |S_{t-}(k)|,$$

for $t > 0$.

Then $N_{t-}(k)$ is the number of subscripts i such that $X_i(t-) = X^{(k)}(t-)$. It is a predictable process and we have the following explicit decomposition.

Theorem 2.2.3 *Let $X_1, \cdots, X_n$ be semimartingales. Then the k-th ranked processes $X^{(k)}, k = 1, \cdots, n$, are semimartingales and we have*

$$\begin{aligned} dX^{(k)}(t) &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} 1_{\{X^{(k)}(t-)=X^{(i)}(t-)\}}\, dX^{(i)}(t) + \sum_{i=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0(X^{(k)} - X^{(i)}) \\ &\quad - \sum_{i=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0(X^{(i)} - X^{(k)}), \qquad (2.2.3) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} 1_{\{X^{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0((X^{(k)} - X_i)^+) \\ &\quad - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0((X^{(k)} - X_i)^-), \qquad (2.2.4) \end{aligned}$$

where $\mathcal{L}_t^0(X) = \frac{1}{2} L_t^0(X) + \sum_{s \le t} 1_{\{X_{s-}=0\}} \Delta X_s$ and $L_t^0(X)$ is the local time of the semimartingale X at zero.

Proof. For all $t > 0$, using the fact that we can define $N_t(k)$ as

$$\begin{aligned} N_{t-}(k) &= \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X_i(t-)\}} \\ &= \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X^{(i)}(t-)\}},\ \forall \omega \end{aligned}$$

we have the following equalities

$$\begin{aligned} N_{t-}(k) dX^{(k)}(t) &= \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X_i(t-)\}} dX^{(k)}(t) \\ &= \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X^{(i)}(t-)\}} dX^{(k)}(t). \qquad (2.2.5) \end{aligned}$$

Then, by homogeneity, to show (2.2.3), it suffices to show that

$$\begin{aligned} N_{t-}(k)dX^{(k)}(t) &= \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X^{(i)}(t-)\}}\, dX^{(i)}(t) + \sum_{i=k+1}^{n} d\,\mathcal{L}_t^0(X^{(k)} - X^{(i)}) \\ &\quad - \sum_{i=1}^{k-1} d\,\mathcal{L}_t^0(X^{(i)} - X^{(k)}). \end{aligned} \tag{2.2.6}$$

We have, by the second equality of (2.2.5)

$$\begin{aligned} N_{t-}(k)dX^{(k)}(t) &= \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} dX^{(k)}(t) \\ &= \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} dX^{(i)}(t) \\ &\quad + \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} d\left(X^{(k)}(t) - X^{(i)}(t)\right). \end{aligned}$$

We use the formula

$$\mathcal{L}_t^0(Z) = \int_0^t 1_{\{Z_{s-}=0\}} dZ_s, \tag{2.2.7}$$

which is valid for nonnegative semimartingales Z and we apply (2.2.7) to $N_{t-}(k)dX^{(k)}(t),\ t > 0$, to obtain

$$\begin{aligned} N_{t-}(k)dX^{(k)}(t) &= \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} dX^{(i)}(t) \\ &\quad + \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} d\left((X^{(k)}(t) - X^{(i)}(t))^+\right) \\ &\quad - \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} d\left((X^{(k)}(t) - X^{(i)}(t))^-\right) \\ &= \sum_{i=1}^{n} 1_{\left\{X^{(k)}(t-)=X^{(i)}(t-)\right\}} dX^{(i)}(t) + \sum_{i=1}^{n} d\,\mathcal{L}_t^0\left((X^{(k)} - X^{(i)})^+\right) \\ &\quad - \sum_{i=1}^{n} d\,\mathcal{L}_t^0\left((X^{(k)} - X^{(i)})^-\right). \end{aligned} \tag{2.2.8}$$

Noting that

$$(X^{(k)} - X^{(j)})^+ = \begin{cases} X^{(k)} - X^{(j)}, & \text{if } j > k \\ 0, & \text{if } j \le k \end{cases}$$

and that

$$(X^{(k)} - X^{(j)})^- = \begin{cases} X^{(j)} - X^{(k)}, & \text{if } j < k \\ 0, & \text{if } j \geq k, \end{cases}$$

Equation (2.2.3) follows. In the same way, we prove (2.2.4) by applying the first equality of (2.2.5), and (2.2.7). ■

2.2.1 Local time and Norms

The next result is proved in [25].

Lemma 2.2.4 *Let $X = (X_1, \cdots, X_n)$ be a $n-$dimensional semimartingale, N_1 and N_2 be norms on $\mathbb{R}^n$ such that $N_1 \leq N_2$. Then $L_t^0(N_1(X)) \leq L_t^0(N_2(X))$.*

For example

$$L_t^0(\max_{1\leq i\leq n} |X_i|) \leq L_t^0(\sum_{i=1}^{n} |X_i|) \leq n\, L_t^0(\max_{1\leq i\leq n} |X_i|).$$

For positive continuous semimartingales, we have the following result.

Corollary 2.2.5 *Let $X_1, \cdots, X_n$ be positive continuous semimartingales. Then we have*

$$L_t^0(\sum_{i=1}^{n} X_i) \leq n \sum_{i=1}^{n} L_t^0(X_i).$$

Proof. It is known that the equality

$$\sum_{i=1}^{n} L_t^0(X^{(i)}) = \sum_{i=1}^{n} L_t^0(X_i), \quad (*)$$

holds for continuous semimartigales (see [8]). Putting $L_t^0(X^{(1)}) = L_t^0(\max_{1\leq i\leq n} X_i)$, we have by the preceding lemma

$$\begin{aligned} L_t^0(\sum_{i=1}^{n} X_i) &\leq n\, L_t^0(\max_{1\leq i\leq n} X_i) = n\, L_t^0(X^{(1)}) \\ &\leq n \sum_{i=1}^{n} L_t^0(X_i) \quad (\text{ by } (*)). \end{aligned}$$

■

Remark 2.2.6 *The preceding corollary means that if we have a collection of n positive continuous semimartingales such that the local time at the origin of each semimartingale is zero then the local time of their sum is also zero at this point.*

2.3 Generalization of Ouknine and Yan's Formula for Semimartingales

In this section we derive a generalization of Ouknine and Yan's formula for semimartingales. Such a result was proved in [8] in the case of continuous semimartingales. In order to give such an extension, we need first to prove the next theorem.

Theorem 2.3.1 *Let $X_1, \cdots, X_n$ be semimartingales. Then the following equality holds:*

$$\sum_{i=1}^{n} 1_{\{X^{(i)}(t-)=0\}}\, dX^{(i)^+}(t) = \sum_{i=1}^{n} 1_{\{X_i(t-)=0\}}\, dX_i^+(t). \tag{2.3.1}$$

Proof. We will proceed by induction. The case $n = 1$ is trivial. For $n = 2$, let us show that

$$\begin{aligned} & 1_{\{X^{(1)}(t-)=0\}}\, dX^{(1)^+}(t) \,+\, 1_{\{X^{(2)}(t-)=0\}}\, dX^{(2)^+}(t) \\ = \;\; & 1_{\{X_1(t-)=0\}}\, dX_1^+(t) \,+\, 1_{\{X_2(t-)=0\}}\, dX_2^+(t), \end{aligned} \tag{2.3.2}$$

where $X^{(1)} = X_1 \vee X_2$ and $X^{(2)} = X_1 \wedge X_2$. At this point we follow the same idea as in the proof of the second theorem in [106]. Since

$$\begin{aligned} \{X_1(t-) \vee X_2(t-) = 0\} \;=\; & \{X_1(t-) < X_2(t-) = 0\} \cup \{X_2(t-) < X_1(t-) = 0\} \\ & \cup \{X_1(t-) = X_2(t-) = 0\}\,, \end{aligned}$$

and

$$\begin{aligned} \{X_1(t-) \wedge X_2(t-) = 0\} \;=\; & \{X_1(t-) > X_2(t-) = 0\} \cup \{X_2(t-) > X_1(t-) = 0\} \\ & \cup \{X_1(t-) = X_2(t-) = 0\}\,, \end{aligned}$$

we can write

$$\begin{aligned} & 1_{\{X^{(1)}(t-)=0\}}\, dX^{(1)^+}(t) \\ = \;\; & 1_{\{X_1(t-)<X_2(t^-)=0\}} dX^{(1)^+}(t) \,+\, 1_{\{X_2(t-)<X_1(t-)=0\}} dX^{(1)^+}(t) \\ & + 1_{\{X_1(t-)=X_2(t-)=0\}} dX^{(1)^+}(t), \end{aligned} \tag{2.3.3}$$

and

$$\begin{aligned}
& 1_{\{X^{(2)}(t-)=0\}}\, dX^{(2)^+}(t) \\
= \;& 1_{\{X_1(t-)>X_2(t-)=0\}} dX^{(2)^+}(t) \;+\; 1_{\{X_2(t-)>X_1(t-)=0\}} dX^{(2)^+}(t) \\
& + 1_{\{X_1(t-)=X_2(t-)=0\}} dX^{(2)^+}(t). \qquad (2.3.4)
\end{aligned}$$

As remarked by Ouknine in [106], the predictable set $H = \{t > 0 : X_1(t-) < X_2(t-)\}$ is not a random open set so the theory developed in [135] cannot be directly applied to replace the semimartingale $X^{(1)^+}$ by the semimartingale X_2^+ which are equal in H. However, the semimartingale $Z = X^{(1)^+} - X_2^+$ is such that $Z_- = 0$ in H, thus $1_H dZ = 1_H(1_{\{Z_-=0\}} dZ)$, and the latter is of finite variation, and null in any open interval where Z is constant, thus in the interior of H. Then the continuous part of $1_H dZ$ is equal to zero and the replacement is permitted. Therefore, the first term of the right hand side of (2.3.3) is $1_{\{X_1(t-)<0\}} 1_{\{X_2(t-)=0\}} dX_2^+(t)$. Applying the same reasoning, the second term is $1_{\{X_2(t-)<0\}} 1_{\{X_1(t-)=0\}} dX_1^+(t)$. For the third term, we can write $X^{(1)^+} = (X_1 \vee X_2)^+ = X_1^+ \vee X_2^+ = X_2^+ + (X_1^+ - X_2^+)^+$ which then becomes

$$1_{\{X_1(t-)=X_2(t-)=0\}} dX_2^+(t) \;+\; 1_{\{X_1(t-)=X_2(t-)=0\}} d(X_1^+(t) - X_2^+(t))^+.$$

These remarks allow us to write (2.3.3) as

$$\begin{aligned}
& 1_{\{X^{(1)}(t-)=0\}}\, dX^{(1)^+}(t) \\
&= 1_{\{X_1(t-)<0\}} 1_{\{X_2(t-)=0\}} dX_2^+(t) \;+\; 1_{\{X_2(t-)<0\}} 1_{\{X_1(t-)=0\}} dX_1^+(t) \\
&\quad + 1_{\{X_1(t-)=X_2(t)=0\}} dX_2^+ \;+\; 1_{\{X_1(t-)=X_2(t-)=0\}} d(X_1^+ - X_2^+)^+ \\
&= 1_{\{X_1(t-)<0\}} 1_{\{X_2(t-)=0\}} dX_2^+(t) \;+\; 1_{\{X_2(t-)<0\}} 1_{\{X_1(t-)=0\}} dX_1^+(t) \\
&\quad + 1_{\{X_1(t-)=0\}} 1_{\{X_2(t-)=0\}} dX_2^+(t) \;+\; 1_{\{X_1(t-)=X_2(t-)=0\}} d(X_1^+(t) - X_2(t)^+)^+. \qquad (2.3.5)
\end{aligned}$$

Following the argument for the process $X^{(2)}$, we obtain

$$\begin{aligned}
&1_{\{X^{(2)}(t-)=0\}}\, dX^{(2)^+}(t) \\
&= 1_{\{X_1(t-)>0\}}1_{\{X_2(t-)=0\}}dX_2^+(t) \;+\; 1_{\{X_2(t-)>0\}}1_{\{X_1(t-)=0\}}dX_1^+(t) \\
&\quad +\; 1_{\{X_1(t-)=0\}}1_{\{X_2(t-)=0\}}dX_1^+(t) \;-\; 1_{\{X_1(t-)=X_2(t-)=0\}}d(X_1^+(t)-X_2^+(t))^+, \qquad (2.3.6)
\end{aligned}$$

where we have used the fact that $X^{(2)^+} = (X_1 \wedge X_2)^+ = X_1^+ \wedge X_2^+ = X_1^+ - (X_1^+ - X_2^+)^+$. Summing (2.3.5) and (2.3.6) we obtain the desired result for $n = 2$.

Now assume the result holds for some n. We adjust here the proof given by Banner and Ghomrasni in [8]. Given semimartingales $X_1, \cdots, X_n, X_{n+1}$, we define $X^{(k)}$, $k = 1, \cdots, n$, as above and also set

$$X^{[k]}(\cdot) = \max_{1\le i_1<\cdots<i_k\le n+1} \min(X_{i_1}(\cdot), \cdots, X_{i_k}(\cdot)).$$

The process $X^{[k]}(\cdot)$ is the kth-ranked process with respect to all $n+1$ semimartingales $X_1, \cdots, X_n, X_{n+1}$. It will be convenient to set $X^{(0)}(\cdot) :\equiv \infty$. In order to show the equality for $n+1$ we start by showing that

$$\begin{aligned}
&1_{\left\{X^{(k-1)}(t-)\wedge X_{n+1}(t-)=0\right\}}d(X^{(k-1)^+}(t)\wedge X_{n+1}^+(t)) + 1_{\left\{X^{(k)}(t-)=0\right\}}dX^{(k)^+}(t) \\
&\quad = 1_{\left\{X^{[k]}(t-)=0\right\}}dX^{[k]^+}(t) + 1_{\left\{X^{(k)}(t-)\wedge X_{n+1}(t-)=0\right\}}d(X^{(k)^+}(t)\wedge X_{n+1}^+(t)) \qquad (2.3.7)
\end{aligned}$$

for $k = 1, \cdots, n$ and $t > 0$. Suppose first that $k > 1$. By (2.3.2), we have

$$\begin{aligned}
&1_{\left\{X^{(k-1)}(t-)\wedge X_{n+1}(t-)=0\right\}}d(X^{(k-1)^+}(t)\wedge X_{n+1}^+(t)) + 1_{\left\{X^{(k)}(t-)=0\right\}}dX^{(k)^+}(t) \\
&= 1_{\left\{(X^{(k-1)}(t-)\wedge X_{n+1}(t-))\vee X^{(k)}(t-)=0\right\}}d\left((X^{(k-1)^+}(t)\wedge X_{n+1}^+(t))\vee X^{(k)^+}(t)\right) \\
&+1_{\left\{(X^{(k-1)}(t-)\wedge X_{n+1}(t-))\wedge X^{(k)}(t-)=0\right\}}d\left((X^{(k-1)^+}(t)\wedge X_{n+1}^+(t))\wedge X^{(k)^+}(t)\right).
\end{aligned}$$

Since $X^{(k)}(t) \le X^{(k-1)}(t)$ for all $t > 0$, the second term of the right hand side of the above equation is simply $1_{\left\{X_{n+1}(t-)\wedge X^{(k)}(t-)=0\right\}}d(X_{n+1}^+(t)\wedge X^{(k)^+}(t))$. On the other hand, we have

$$(X^{(k-1)}\wedge X_{n+1})\vee X^{(k)}(t) = \begin{cases} X^{(k-1)}(t), & \text{if } X_{n+1}(t) \ge X^{(k-1)}(t) \ge X^{(k)}(t) \\ X_{n+1}(t), & \text{if } X^{(k-1)}(t) \ge X_{n+1}(t) \ge X^{(k)}(t) \\ X^{(k)}(t), & \text{if } X^{(k-1)}(t) \ge X^{(k)}(t) \ge X_{n+1}(t). \end{cases}$$

In each case it can be checked that $(X^{(k-1)} \wedge X_{n+1}) \vee X^{(k)}(t)$ is the kth smallest of the numbers $X_1, \cdots, X_{n+1}$; that is, $(X^{(k-1)} \wedge X_{n+1}) \vee X^{(k)}(\cdot) \equiv X^{[k]}(\cdot)$. It follows that $X^{[k]}$ is a semimartingale for $k = 1, \cdots, n$. Equation (2.3.7) follows for $k = 2, \cdots, n$. If $k = 1$, then $X^{(0)}(\cdot) \equiv \infty$ and applying (2.3.2), Equation (2.3.7) reduces to

$$\begin{aligned}
& 1_{\{X_{n+1}(t-)=0\}} dX^+_{n+1}(t) + 1_{\{X^{(1)}(t-)=0\}} dX^{(1)^+}(t) \\
= \;& 1_{\{X^{(1)}(t-)\vee X_{n+1}(t-)=0\}} d(X^{(1)^+}(t) \vee X^+_{n+1}(t)) \\
+ \;& 1_{\{X^{(1)}(t-)\wedge X_{n+1}(t-)=0\}} d(X^{(1)^+}(t) \wedge X^+_{n+1}(t)) \\
= \;& 1_{\{X^{(1)}(t-)\wedge X_{n+1}(t-)=0\}} d(X^{(1)^+}(t) \wedge X^+_{n+1}(t)) \\
+ \;& 1_{\{X^{[1]}(t-)=0\}} dX^{[1]^+}(t),
\end{aligned} \tag{2.3.8}$$

where we observe that $(X^{(1)} \vee X_{n+1})(\cdot) \equiv X^{[1]}(\cdot)$.

Finally, by the induction hypothesis and Equation(2.3.7), we have

$$\begin{aligned}
\sum_{i=1}^{n+1} 1_{\{X_i(t-)=0\}}\, dX^+_i(t) &= \sum_{i=1}^{n} 1_{\{X_i(t-)=0\}}\, dX^+_i(t) \;+\; 1_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t) \\
&= \sum_{i=1}^{n} 1_{\{X^{(i)}(t-)=0\}}\, dX^{(i)^+}(t) \;+\; 1_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t) \\
&= \sum_{i=1}^{n} 1_{\{X^{[i]}(t-)=0\}}\, dX^{[i]^+}(t) \;+\; 1_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t) \\
&\quad - \sum_{i=1}^{n} 1_{\{X^{(i-1)}(t-)\wedge X_{n+1}(t-)=0\}} d(X^{(i-1)^+}(t) \wedge X^+_{n+1}(t)) \\
&\quad + \sum_{i=1}^{n} 1_{\{X^{(i)}(t-)\wedge X_{n+1}(t-)=0\}} d(X^{(i)^+}(t) \wedge X^+_{n+1}(t)) \\
&= \sum_{i=1}^{n} 1_{\{X^{[i]}(t-)=0\}}\, dX^{[i]^+}(t) \;+\; 1_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t) \\
&\quad - 1_{\{X^{(0)}(t-)\wedge X_{n+1}(t-)=0\}} d(X^{(0)^+}(t) \wedge X^+_{n+1}(t)) \\
&\quad + 1_{\{X^{(n)}(t-)\wedge X_{n+1}(t-)=0\}} d(X^{(n)^+}(t) \wedge X^+_{n+1}(t)) \\
&= \sum_{i=1}^{n+1} 1_{\{X^{[i]}(t-)=0\}}\, dX^{[i]^+}(t).
\end{aligned}$$

The third equality follows from Equation (2.3.7) while the last comes from the fact that

$$X^{(0)}(t) \wedge X_{n+1}(t) = X_{n+1}(t) \text{ and } (X^{(n)} \wedge X_{n+1})(\cdot) \equiv X^{[n+1]}(\cdot) \text{ for all } t > 0\ ;$$

then the result follows by induction. ■

It follows that

Corollary 2.3.2 *Let $X_1, \cdots, X_n$ be semimartingales. Then the k-th ranked processes $X^{(k)}, k = 1, \cdots, n$, are semimartingales and we have*

$$\sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X^{(i)}(t-)\}} d\Big(X^{(i)}(t) - X^{(k)}(t)\Big)^{+} = \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X_i(t-)\}} d\Big(X_i(t) - X^{(k)}(t)\Big)^{+} \tag{2.3.9}$$

Proof. Fix $X^{(k)}$, for $k = 1, \cdots, n$ and define the processes $Y_1, \cdots, Y_n$ by $Y_i(t) = X_i(t) - X^{(k)}(t),\ i = 1, \cdots, n$. Then $Y_1, \cdots, Y_n$ are semimartingales and the processes $Y^{(1)}, \cdots, Y^{(n)}$ defined by $Y^{(i)}(t) = X^{(i)}(t) - X^{(k)}(t),\ i = 1, \cdots, n$ are the *i-th* ranked processes of $Y_i(t),\ i = 1, \cdots, n$ with the property $Y^{(1)} \geq Y^{(2)} \geq \cdots \geq Y^{(n)}$, and, they are semimartingales. By Theorem 2.3.1, we have

$$\sum_{i=1}^{n} 1_{\{Y^{(i)}(t-)=0\}}\, dY^{(i)+}(t) = \sum_{i=1}^{n} 1_{\{Y_i(t-)=0\}}\, dY_i^{+}(t), \tag{2.3.10}$$

and the result follows. ■

In the case of positive semimartingales, the preceding theorem becomes

Corollary 2.3.3 *Let $X_1, \cdots, X_n$ be positive semimartingales. Then the following equality holds*

$$\sum_{i=1}^{n} 1_{\{X^{(i)}(t-)=0\}}\, dX^{(i)}(t) = \sum_{i=1}^{n} 1_{\{X_i(t-)=0\}}\, dX_i(t). \tag{2.3.11}$$

A consequence of Theorem 2.3.1 is the following theorem, which is a generalization of Yan [132] and Ouknine's [105, 106] formula.

Theorem 2.3.4 *Let $X_1, \cdots, X_n$ be semimartingales. Then we have*

$$\sum_{i=1}^{n} L_t^0(X^{(i)}) = \sum_{i=1}^{n} L_t^0(X_i), \tag{2.3.12}$$

where $L_t^0(X)$ is the local time of the semimartingale X at 0.

Proof. We recall first that $L_t^0(Z) = L_t^0(Z^+)$ for every semimartingale Z. By Theorem 2.3.1 the following equality holds:

$$\sum_{i=1}^{n} 1_{\{X^{(i)}(t-)=0\}}\, dX^{(i)^+}(t) = \sum_{i=1}^{n} 1_{\{X_i(t-)=0\}}\, dX_i^+(t). \tag{2.3.13}$$

We know that

$$\mathcal{L}_t^0(Z^+) = \int_0^t 1_{\{Z_{s-}=0\}} dZ_s^+,$$

for all semimartingales. Then the preceding equation becomes

$$\sum_{i=1}^{n} \mathcal{L}_t^0(X^{(i)^+}) = \sum_{i=1}^{n} \mathcal{L}_t^0(X_i^+). \tag{2.3.14}$$

Putting

$$A(t) = \sum_{i=1}^{n} \mathcal{L}_t^0(X^{(i)^+}) \text{ and } B(t) = \sum_{i=1}^{n} \mathcal{L}_t^0(X_i^+),$$

then

$$\begin{aligned} A(t) &= \sum_{i=1}^{n} \left(\frac{1}{2} L_t^0(X^{(i)^+}) + \sum_{s\le t} 1_{\{X^{(i)}(s-)=0\}} \Delta X^{(i)^+}(s) \right), \\ B(t) &= \sum_{i=1}^{n} \left(\frac{1}{2} L_t^0(X_i^+) + \sum_{s\le t} 1_{\{X_i(s-)=0\}} \Delta X_i^+(s) \right). \end{aligned}$$

Since $A(t) = B(t)$ for all $t > 0$, we have $A^c(t) = B^c(t)$ where A^c (resp. B^c) is the continuous part of A (resp. B). The desired result follows from the continuity of local time and the fact $L_t^0(Z) = L_t^0(Z^+)$. ■

In particular,

Corollary 2.3.5 (**Yan** [**132**], **Ouknine** [**105**, **106**])

Let X and Y be semimartingales. We have the following

$$L_t^0(X \vee Y) + L_t^0(X \wedge Y) = L_t^0(X) + L_t^0(Y), \tag{2.3.15}$$

where $L_t^0(X)$ denotes the local time at 0 of X.

Remark 2.3.6 *Assume that $X_1, \cdots, X_n$ are continuous semimartingales. Banner and Ghomrasni have shown that (see [8], Theorem 2.3)*

$$dX^{(k)}(t) = \sum_{i=1}^{n} \frac{1}{N_t(k)} 1_{\{X^{(k)}(t)=X_i(t)\}}\, dX^{(i)}(t) + \sum_{i=k+1}^{n} \frac{1}{2\,N_t(k)}\, d\,L_t^0(X^{(k)} - X^{(i)}) - \sum_{i=1}^{k-1} \frac{1}{2\,N_t(k)}\, d\,L_t^0(X^{(i)} - X^{(k)}). \tag{2.3.16}$$

Identifying Equations (2.2.3) and (2.3.16), it follows that

$$\sum_{i=1}^{n} 1_{\{X^{(k)}(t)=X^{(i)}(t)\}}\, dX^{(i)}(t) = \sum_{i=1}^{n} 1_{\{X^{(k)}(t)=X_i(t)\}}\, dX_i(t). \tag{2.3.17}$$

We extend below Equation (2.3.17) for general semimartingales.

Proposition 2.3.7 *Let $X_1, \cdots, X_n$ be semimartingales and $k \in \{1, 2, \cdots, n\}$. Then we have*

$$dX^{(k)}(t) = \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} 1_{\{X^{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0(X^{(k)} - X^{(i)}) - \sum_{i=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0(X^{(i)} - X^{(k)}), \tag{2.3.18}$$

and

$$\sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X^{(i)}(t-)\}}\, dX^{(i)}(t) = \sum_{i=1}^{n} 1_{\{X^{(k)}(t-)=X_i(t-)\}}\, dX_i(t). \tag{2.3.19}$$

Proof. Consider the following family of semimartingales $\{Z_i := -X_i\}_{i=1,\cdots,n}$, the rank processes are then given by

$$Z^{(n)} = -X^{(1)} \le \cdots \le Z^{(n+1-i)} = -X^{(i)} \le \cdots \le Z^{(1)} = -X^{(n)}$$

and from Theorem 2.3.1 we have that

$$\sum_{i=1}^{n} 1_{\{Z^{(n+1-i)}(t-)=0\}}\, dZ^{(n+1-i)^+}(t) = \sum_{i=1}^{n} 1_{\{Z_{n+1-i}(t-)=0\}}\, dZ^+_{n+1-i}(t).$$

from which it follows that

$$\sum_{i=1}^{n} 1_{\{Z^{(i)}(t-)=0\}}\, dZ^{(i)^+}(t) = \sum_{i=1}^{n} 1_{\{Z_i(t-)=0\}}\, dZ_i^+(t).$$

Using Relation (**??**), we have

$$\sum_{i=1}^{n} \mathcal{L}_t^0(Z^{(i)+}) = \sum_{i=1}^{n} \mathcal{L}_t^0(Z_i^+).$$

Since $Z_i = -X_i$, we have $Z_i^+ = X_i^-$ and $Z^{(i)+} = X^{(n+1-i)-}$, from which we obtain that

$$\sum_{i=1}^{n} \mathcal{L}_t^0(X^{(n+1-i)-}) = \sum_{i=1}^{n} \mathcal{L}_t^0(X^{(i)-}) = \sum_{i=1}^{n} \mathcal{L}_t^0(X_i^-). \tag{2.3.20}$$

Equations (2.3.14) and (2.3.20) imply that Equation (2.2.4) may be rewritten as,

$$\begin{aligned}
dX^{(k)}(t) &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} 1_{\{X^{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0((X^{(k)} - X_i)^+) \\
&\quad - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0((X^{(k)} - X_i)^-) \\
&= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} 1_{\{X^{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0((X^{(k)} - X^{(i)})^+) \\
&\quad - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0((X^{(k)} - X^{(i)})^-)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
dX^{(k)}(t) &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} 1_{\{X^{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0(X^{(k)} - X^{(i)}) \\
&\quad - \sum_{i=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\,\mathcal{L}_t^0(X^{(i)} - X^{(k)})\,.
\end{aligned} \tag{2.3.21}$$

We obtain the desire result by identifying both Equations (2.2.3) and (2.3.21). ■

Chapter 3

On local times: application to pricing using bid-ask

3.1 Introduction

The theory of asset pricing and its fundamental theorem were initiated in the Arrow-Debreu model, the Black and Scholes formula, and the Cox and Ross model. They have now been formalized in a general framework by Harrison and Kreps [60], Harrison and Pliska [61], and Kreps [78] according to the no arbitrage principle. In the classical setting, the market is assumed to be frictionless i.e a no arbitrage dynamic price process is a martingale under a probability measure equivalent to the reference probability measure.

However, real financial markets are not frictionless, and so an important literature on pricing under transaction costs and liquidity risk has appeared. See [15, 69] and references therein. In these papers the bid-ask spreads are explained by transaction costs. Jouini and Kallal in [69] in an axiomatic approach in continuous time assigned to financial assets a dynamic ask price process (respectively, a dynamic bid price process.) They proved that the absence of arbitrage opportunities is equivalent to the existence of a frictionless arbitrage-free process lying between the bid and the ask processes, *i.e.*, a process which could be transformed into a martingale under a well-chosen probability measure. The bid-ask spread in this setting can be interpreted as transaction costs or as the result of entering buy and sell orders.

Taking into account both transaction costs and liquidity risk Bion-Nadal in [15] changed the assumption of sublinearity of ask price (respectively, superlinearity of bid price) made in [69] to that of convexity (respectively, concavity) of the ask (respectively, bid) price. This assumption combined with the time-consistency property for dynamic prices allowed her to generalize the result of Jouini and Kallal. She proved that the "no free lunch" condition for a time-consistent dynamic pricing procedure [TCPP] is equivalent to the existence of an equivalent probability measure Q that transforms a process between the bid and ask processes of any financial instrument into a martingale.

In recent years, a pricing theory has also appeared taking inspiration from the theory of risk measures. First to investigate in a static setting were Carr, Geman, and Madan [23] and Föllmer and Schied [51]. The point of view of pricing via risk measures was also considered in a dynamic way using backward stochastic differential equations [BSDE] by El Karoui and Quenez [40], El Karoui, Peng, and Quenez [41], and Peng [109, 110]. This theory soon became a useful tool for formulating many problems in mathematical finance, in particular for the study of pricing and hedging contingent claims [41]. Moreover, the BSDE point of view gave a simple formulation of more general recursive utilities and their properties, as initiated by Duffie and Epstein (1992) in their [stochastic differential] formulation of recursive utility [41].

In the past, in real financial markets, the load of providing liquidity was given to market makers, specialists, and brokers, who trade only when they expect to make profits. Such profits are the price that investors and other traders pay, in order to execute their orders when they want to trade. To ensure steady trading, the market makers sell to buyers and buy from sellers, and get compensated by the so-called bid-ask spread. The most common price for referencing stocks is the last trade price. At any given moment, in a sufficiently liquid market there is a best or highest "bid" price, from someone who wants to buy the stock and there is a best or lowest "ask" price, from someone who wants to sell the stock. The best bid price $R(t)$ and best ask (or best offer) price $T(t)$ are the highest buying price and the lowest selling price at any time t of trading.

In the present work, we consider models of financial markets in which all parties involved (buyers, sellers) find incentives to participate. Our framework is different from the existing approach (see [15, 69] and references therein) where the authors assume some properties (sublinearity, convexity, *etc*) on the ask (respectively, bid) price function in order to define a dynamic ask (respectively, bid.) Rather, we assume that the different bid and ask prices are given. Then the question we address is how to model the "best bid" (respectively, the "best ask") price process with the intention to obtain the stock price dynamics.

The assumption that the bid and ask processes are described by (continuous) semimartingales in our special setting entails that the stock price admits arbitrage opportunities. Further, it turns out that the price process possesses the Markov property, if the bid and ask are Brownian motion or Ornstein-Uhlenbeck type, or more generally Feller processes. Note that our results are obtained without assuming arbitrage opportunities.

This chapter is also related with [68] where the authors explore market situations where a large trader causes the existence of arbitrage opportunities for small traders in complete markets. The arbitrage opportunities considered are "hidden" which means that they are almost not observable to the small traders, or to scientists studying markets because they occur on time sets of Lebesgue measure zero.

The Chapter is organized as follows: Section 3.2 presents the model. Section 3.3 studies the Markovian property of the processes, while Sections 3.4 and 3.5 are devoted to the study of completeness, arbitrage and (insider) hedging on a market driven by such processes.

3.2 The model

Let $B_s = (B(s)^1, \cdots, B(s)^n)^T$ (where $(\cdot)^T$ denotes transpose) be a n-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

Suppose bid and ask price processes $X_i(t) \in \mathbb{R},\ 1 \leq i \leq n$, which are modeled by continuous semimartingales

$$X_i(t) \ = \ X_i(0) + \int_0^t a_i(s, X_s, \omega)\, ds + \int_0^t \sigma(s, X_s, \omega)\, dB_i(s). \tag{3.2.1}$$

Here we consider the following model for bid and ask prices.

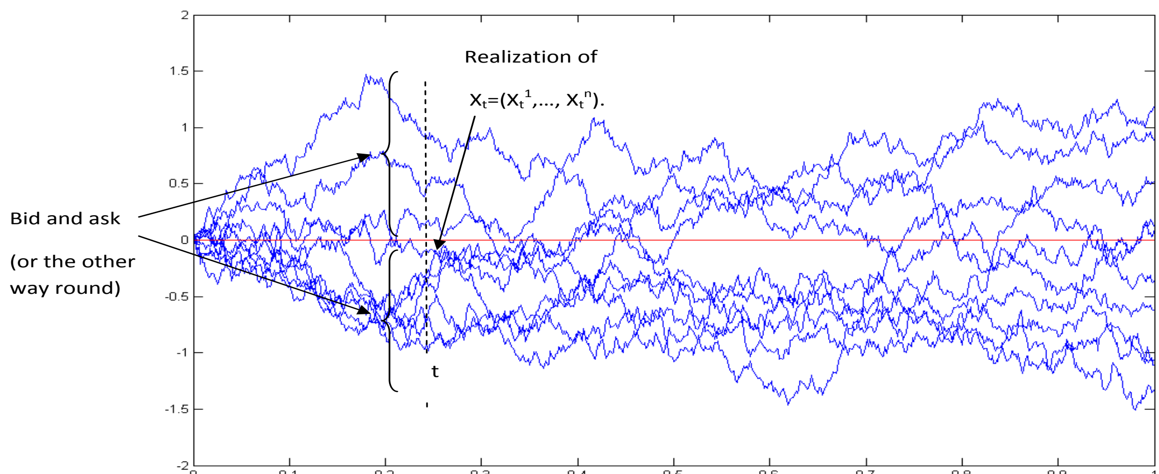

Figure 3.1 Realization of bid and ask

The evolution of the stock price process $S(t)$ is based on $X_i(t),\ i = 1, ..., n$. Denote by $Bid(t)$ the Best Bid and $Ask(t)$ the Best Ask at time t. Then $Bid(t)$ is the lowest price that a day trader seller is willing to accept for a stock at that time and $Ask(t)$ is the highest price that a day trader buyer is willing to pay for that stock at any particular point in time.

Let us define the processes $X^+(t) = \max(X(t), 0)$ and $X^*(t) = \min(X(t), 0)$. Further set

$$\begin{aligned} R(t) &:= \min_{1 \le i \le n} X_i^+(t) \\ T(t) &:= \max_{1 \le i \le n} X_i^*(t). \end{aligned}$$

where we use the convention that $\min\{\emptyset\} = 0$ and $\max\{\emptyset\} = 0$. Then $Bid(t)$ and $Ask(t)$ can be modeled as

$$Bid(t) := \min\{R(t),\ -T(t)\}, \tag{3.2.2}$$

and

$$Ask(t) := \max\{R(t),\ -T(t)\}. \tag{3.2.3}$$

Given $Bid(t)$ and $Ask(t)$, the market makers will agree on a stock price within the Bid/Ask spread, that is

$$S(t) = \alpha(t)Bid(t) + (1 - \alpha(t))Ask(t), \tag{3.2.4}$$

where $\alpha(t)$ is a stochastic process such that $0 \le \alpha(t) \le 1$. One could choose *e.g.*,

$$\alpha(t) = \sigma(t)$$

for a function $\sigma : [0, T] \to [0, 1]$ or

$$\alpha(t) = f(R(t), T(t))$$

for a function $f : \mathbb{R} \times \mathbb{R} \to [0, 1]$.

For convenience, we will from now on assume that $\alpha(t) \equiv 1/2$, that is

$$\begin{aligned} S_t &= \frac{Bid(t) + Ask(t)}{2} \\ &= \frac{R(t) - T(t)}{2}. \end{aligned} \tag{3.2.5}$$

3.3 Markovian property of processes R, T and S

For convenience, let us briefly discuss the Markovian property of the processes $R(t)$, $T(t)$ and $S(t)$ in some particular cases. The two cases considered here are the cases when the process $\left\{X^i(t)\right\}_{t\geq 0}$ are Brownian motions or Ornstein-Uhlenbeck processes. Let us first have on the definition of semimartingales rank processes.

Definition 3.3.1 *Let $X_1, \cdots, X_n$ be continuous semimartingales. For $1 \leq k \leq n$, the k-th rank process of $X_1, \cdots, X_n$ is defined by*

$$X^{(k)} = \max_{i_1 < \cdots < i_k} \min(X_{i_1}, \cdots, X_{i_k}), \tag{3.3.1}$$

where $1 \leq i_1$ and $i_k \leq n$.

Note that, according to Definition 3.3.1, for $t \in \mathbb{R}_+$,

$$\max_{1\leq i\leq n} X_i(t) = X^{(1)}(t) \geq X^{(2)}(t) \geq \cdots \geq X^{(n)}(t) = \min_{1\leq i\leq n} X_i(t), \tag{3.3.2}$$

so that at any given time, the values of the rank processes represent the values of the original processes arranged in descending order (i.e. the (reverse) order statistics).

Using Definition 3.3.1, we get

$$\begin{aligned} R(t) &:= X^{(n)+}(t) \\ T(t) &:= X^{(1)*}(t). \end{aligned} \tag{3.3.3}$$

3.3.1 The Brownian motion case

Here we assume that the processes $\left\{X^i(t)\right\}_{t\geq 0}$, $1 \leq i \leq n$ are independent Brownian motions.

Proposition 3.3.2 *The process R possesses the Markov property with respect to the filtration $\mathcal{F}_t := \mathcal{F}_t^B \cap \sigma(R(t); 0 \leq t \leq T)$.*

Proof. : We first prove that $B^+ = \max(B, 0)$ is a Markov process. define the process

$$Y(t) = \begin{pmatrix} |B(t)| \\ B(t) \end{pmatrix} \in \mathbb{R}^2.$$

Then $(Y(t))_{t\geq 0}$ is a two dimensional Feller process.

Let $g(x_1, x_2) = \frac{1}{2}(x_1 + x_2)$. One observes that $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous and open map. Thus is follows from Remark 1 p. 327, in [36] that $B^+(t) = Y^+(t) = g(Y(t))$ is a Feller process, too.

The latter argument also applies to the n-dimensional case, that is

$$\widetilde{Y} := \left(B_1^+(t), \cdots, B_n^+(t)\right)$$

is a Feller process. Since

$$\begin{array}{rlcl} f: & \mathbb{R}^n & \to & \mathbb{R} \\ & (x_1, \cdots, x_n) & \mapsto & \min(x_1, \cdots, x_n) \end{array}$$

is a continuous and open map we conclude that $R(t) = f(\widetilde{Y})$ is a Feller process. ■

Proposition 3.3.3 *The process T possesses Markov property with respect to the filtration $\mathcal{F}_t := \mathcal{F}_t^B \cap \sigma(T(t); 0 \leq t \leq T)$.*

Proof. See the proof of Proposition 3.3.2. ■

Corollary 3.3.4 *The process S possesses Markov property with respect to the filtration $\mathcal{F}_t := \mathcal{F}_t^B \cap \sigma(S(t); 0 \leq t \leq T)$..*

Proof. The process Z defined by $Z_t = R_t + T_t$ for all $t \geq 0$ is a Markov process as sum of two Markov processes. ■

3.3.2 The Ornstein-Uhlenbeck case

Here we assume that the process $X(t) = (X_1(t), \cdots, X_n(t))$ is an n-dimensional Ornstein-Uhlenbeck, that is

$$dX_i(t) = -\alpha_i X_i(t)dt + \sigma_i dB_i(t), \quad 1 \leq i \leq n, \tag{3.3.4}$$

where α_i and σ_i are parameters. It is clear that an Ornstein-Uhlenbeck process is a Feller process. So we obtain

Proposition 3.3.5 *The process R, T and S defined by (3.3.3) and (3.2.5) possess Markov property.*

Proof. The conclusion follows from the proof of Proposition 3.3.2. ■

Remark 3.3.6 *Using continuous and open transformations of Markov processes, the above results can be generalized to the case, when the bid and ask processes are Feller processes. See [36].*

3.4 Further properties of $S(t)$

In this Section, we want to use the semimartingale decomposition of our price process S_t to analyze completeness and arbitrage on market driven by such a process.

We need the following result. See Proposition 4.1.11 in [46].

Theorem 3.4.1 *Let $X_1, \cdots, X_n$ be continuous semimartingales of the form (3.2.1). For $k \in \{1, 2, \cdots, n\}$, let $u(k) = (u_t(k),\, t \geq 0) : \Omega \times [0, \infty[\to \{1, 2, \cdots, n\}$ be any predictable process with the property:*

$$X^{(k)}(t) = X_{u_t(k)}(t). \tag{3.4.1}$$

Then the k-th rank processes $X^{(k)}, k = 1, \cdots, n$, are semimartingales and we have:

$$\begin{aligned} X^{(k)}(t) \quad &= \quad X^{(k)}(0) + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, dX_i(s) \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L_s^0((X^{(k)} - X_i)^+) \\ &\quad - \frac{1}{2} \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L_s^0((X^{(k)} - X_i)^-), \end{aligned} \tag{3.4.2}$$

where $L^0_t(X)$ is the local time of the semimartingale X at zero, defined by

$$|X_t| = |X_0| + \int_0^t \operatorname{sgn}(X_{s-})\, dX_s + L^0_t(X),$$

where $\operatorname{sgn}(x) = -1_{(-\infty,0]}(x) + 1_{(0,\infty)}(x)$.

Proof. We find that

$$X_t^{(k)} - X_0^{(k)} = \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, dX_s^i + \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, d\,(X_s^{(k)} - X_s^i), \tag{3.4.3}$$

where we used the property $\sum_{i=1}^n 1_{\{u_s(k)=i\}} = 1$. It follows,

$$\begin{aligned} X_t^{(k)} - X_0^{(k)} = &\sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, dX_s^i \\ &+ \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, d\,(X_s^{(k)} - X_s^i)^+ \\ &- \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, d\,(X_s^{(k)} - X_s^i)^-. \end{aligned}$$

We note the fact

$$\{u_s(k) = i\} \subset \{X_s^{(k)} = X_i(s)\}. \tag{3.4.4}$$

We now use the following formula:

$$\frac{1}{2}\, L^0_t(X) = \int_0^t 1_{\{X_s=0\}}\, d\,X_s, \tag{3.4.5}$$

which is valid for non-negative semimartingales X. See *e.g.*, [24, 46]

Then, by applying (3.4.5) to $(X^{(k)}(t) - X_i(t))^{\pm},\ t \geq 0$, Equation (3.4.3) becomes:

$$\begin{aligned} X^{(k)}(t)\ -\ X^{(k)}(0) = &\sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, dX_i(s) \\ &+ \frac{1}{2} \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L^0_s((X^{(k)} - X_i)^+) \\ &- \frac{1}{2} \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L^0_s((X^{(k)} - X_i)^-). \end{aligned}$$

Then the above result follows. ■

3.4.1 The Brownian motion case

If $X_i(t) = B_i^+(t)$ or $B_i^*(t)$, $i = 1, ..., n$ are n independent Brownian motions, the evolution of $R(t)$ and $T(t)$ follows from Theorem 3.4.1.

Corollary 3.4.2 *Let the processes $\{R(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ be given by Equation (3.3.3). Then $R(t) = B^{(n)+}(t)$ and $T(t) = B^{(1)*}(t)$ and we have:*

$$\begin{aligned} R(t) &= R(0) + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(n)=i\}} \left\{ dB_i^+(s) - \frac{1}{2} d_s L_s^0 \left(B_i^+ - R\right) \right\} \\ &= R(0) + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(n)=i\}} \left\{ 1_{\{B_i(s)>0\}} dB_i(s) + \frac{1}{2} \left[d_s L_s^0(B_i) - d_s L_s^0 \left(B_i^+ - R\right) \right] \right\}, \end{aligned} \tag{3.4.6}$$

and

$$\begin{aligned} T(t) &= T(0) + \sum_{i=1}^{n} \int_0^t 1_{\{v_s(n)=i\}} \left\{ dB_i^*(s) + \frac{1}{2} d_s L_s^0 \left(T - B_i^*\right) \right\} \\ &= T(0) + \sum_{i=1}^{n} \int_0^t 1_{\{v_s(n)=i\}} \left\{ 1_{\{B_i(s)\leq 0\}} dB_i(s) + \frac{1}{2} \left[d_s L_s^0 \left(T - B_i^*\right) - d_s L_s^0(B_i) \right] \right\}. \end{aligned} \tag{3.4.7}$$

We can rewrite $R(t)$ and $T(t)$ as follows:

$$\begin{aligned} R(t) &= R(0) + M^R(t) + V^R(t), \\ T(t) &= T(0) + M^T(t) + V^T(t), \end{aligned}$$

where $M^R(t)$, $M^T(t)$ are continuous local martingales and $V^R(t)$, $V^T(t)$ are continuous processes of locally bounded variation given by:

$$V^R(t) = \sum_{i=1}^{n} \int_0^t 1_{\{u_s(n)=i\}} \frac{1}{2} \left[d_s L_s^0(B_i) - d_s L_s^0 \left(B_i^+ - R\right) \right], \tag{3.4.8}$$

$$M^R(t) = \sum_{i=1}^{n} \int_0^t 1_{\{u_s(n)=i\}} 1_{\{B_i(s)>0\}} dB_i(s), \tag{3.4.9}$$

$$V^T(t) = \sum_{i=1}^{n} \int_0^t 1_{\{v_s(1)=i\}} \frac{1}{2} \left[d_s L_s^0 \left(T - B_i^*\right) - d_s L_s^0(B_i) \right], \tag{3.4.10}$$

$$M^T(t) = \sum_{i=1}^{n} \int_0^t 1_{\{v_s(n)=i\}} 1_{\{B_i(s)\leq 0\}} dB_i(s). \tag{3.4.11}$$

The following corollary gives the semimartingale decomposition satisfied by the process S_t.

Corollary 3.4.3 *Assume that the process $S(\cdot)$ is given by Equation (3.2.5). Then one can write $S(t) = f(A(t))$ where $A(t) = (R(t), T(t))$ and $f(x_1, x_2) = \frac{1}{2}(x_1 - x_2)$, and we have:*

$$\begin{aligned} S(t) &= S(0) + \frac{1}{2}\sum_{i=1}^{n}\int_0^t \left(1_{\{u_s(n)=i\}}\, 1_{\{B_i(s)>0\}} - 1_{\{v_s(n)=i\}} 1_{\{B_i(s)\le 0\}}\right) dB_i(s) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\int_0^t \left(1_{\{u_s(n)=i\}} + 1_{\{v_s(n)=i\}}\right) d_s L_s^0(B_i) \\ &- \frac{1}{2}\sum_{i=1}^{n}\left\{\int_0^t 1_{\{u_s(n)=i\}} d_s L_s^0\left(B_i^+ - R\right) + \int_0^t 1_{\{v_s(n)=i\}} d_s L_s^0\left(T - B_i^*\right)\right\}. \end{aligned} \tag{3.4.12}$$

In order to price options with respect to $S(t)$ one should ensure that $S(t)$ does not admit arbitrage possibilities and the natural question which arises at this point is the following: Can we find an equivalent probability measure Q such that S is a Q-sigma martingale (see [116] for definitions)? Since our process S is continuous we can reformulate the question as: Can we find an equivalent probability measure Q such that S is a Q local martingale[1]? We first give the following useful remark which is a part of Theorem 1 in [117].

Remark 3.4.4 *Let $X(t) = X_0 + M(t) + V(t)$ be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$. Let $C_t = [X, X]_t = [M, M]_t$, $0 \le t \le T$. A necessary condition for the existence of an equivalent martingale measure is that $dV << dC$.*

Consequence 3.4.5 *Since local time is singular, we observe that the total variation of the bounded variation part in Equation (3.4.12) cannot be absolutely continuous with respect to the quadratic variation of the martingale. It follows that the set of equivalent martingale measures is empty and thus such a market contains arbitrage opportunities.*

3.4.2 (In)complete market with hidden arbitrage

We consider in this Section a model where $\{S(t)\}_{t\ge 0}$ denotes a stochastic process modeling the price of a risky asset, and $\{R(t)\}_{t\ge 0}$ denotes the value of a risk free money market

[1] In fact since S is continuous and since all continuous sigma martingales are in fact local martingales, we only need to concern ourselves with local martingales

account. We assume a given filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the "usual hypothesis". In such a market, a trading strategy (a, b) is self-financing if a is predictable, b is optional, and

$$a(t)S(t) + b(t)R(t) = a(0)S(0) + b(0)R(0) + \int_0^t a(s)dS(s) + \int_0^t b(s)dR(s) \tag{3.4.13}$$

for all $0 \leq t \leq T$. For convenience, we let $S_0 = 0$ and $R(t) \equiv 1$ (thus the interest rate $r = 0$), so that $dR(t) = 0$, and Equation (3.4.13) becomes

$$a(t)S(t) + b(t)R(t) = b(0) + \int_0^t a(s)dS(s).$$

Definition 3.4.6 *(See [68].)*

1. *We call a random variable $H \in \mathcal{F}_T$ a contingent claim. Further, a contingent claim H is said to be Q-redundant if for a probability measure Q there exists a self-financing strategy (a, b) such that*

$$V^Q(t) = E_Q\left[H | \mathcal{F}_t\right] = b(0) + \int_0^t a(s)dS(s), \tag{3.4.14}$$

where $\{V(t)\}_{t\geq 0}$ is the value of the portfolio.

2. *A market $(S(t), R(t)) = (S(t), 1)$ is Q-complete if every $H \in L^1(\mathcal{F}_T, Q)$ is Q-redundant.*

Define the process $\left(M^S(t)\right)_{t>0}$ as follows

$$M^S(t) = \frac{1}{2}\sum_{i=1}^{n}\int_0^t \left(1_{\{u_s(n)=i\}}1_{\{B_i(s)>0\}} - 1_{\{v_s(n)=i\}}1_{\{B_i(s)\leq 0\}}\right) dB_i(s). \tag{3.4.15}$$

Then the following theorem is immediate from Theorem 3.2 in [68].

Theorem 3.4.7 *Suppose there exists a unique probability measure P^* equivalent to P such that $M^S(t)$ is a P^*-local martingale. Then the market $(S(t), 1)$ is P^*-complete.*

Proof. Omitted. ■

Proposition 3.4.8 *Suppose that $n \geq 2$. Then, there exists no unique martingale measure P^* such that $M^S(t)$ is a P^*-local martingale.*

Proof. Because of Equation (3.4.15), we observe that $M^S(t)$ is a P-martingale. Let us construct another equivalent martingale measure P^*. For this purpose assume *wlog* that $u_s(n)$ and $v_s(n)$ are given by

$$u_s(n) = \min\left\{i \in \{1, ..., n\} : B_i^+(t) = R(t)\right\}$$

and

$$v_s(n) = \min\left\{i \in \{1, ..., n\} : B_i^*(t) = T(t)\right\}.$$

Now define the process h as

$$h(t) = 1_{\{A(t)\}},$$

where

$$A(t) = \{\omega \in \Omega : \beta(t, \omega) = 0\},$$

with

$$\beta(s) = \sum_{i=1}^{n}\left(1_{\{u_s(n)=i\}}1_{\{B_i(s)>0\}} - 1_{\{v_s(n)=i\}}1_{\{B_i(s)\leq 0\}}\right). \tag{3.4.16}$$

One finds that $Pr[A(t)] > 0$ for all t. Let us define the equivalent measure P^* with respect to a density process Z_t given by

$$Z_t = \mathcal{E}[N]_t.$$

Here $\mathcal{E}(N)$ denotes the Doléans-Dade exponential of the martingale N_t defined by

$$N_t = \sum_{i=1}^{n}\int_0^t h(s)\, dB_i(s).$$

Then it follows from the Girsanov-Meyer theorem (see [116]) that $M^S(t)$ has a P^*-semimartingale decomposition with a bounded variation part given by

$$\int_0^t 2h(s)\, d\left\langle M^S, M^S\right\rangle_s.$$

We have that

$$\int_0^t 2h(s)d\left\langle M^S, M^S\right\rangle_s = \frac{1}{2}\int_0^t h(s)\beta(s)ds.$$

Since $h\beta = 0$, it follows that

$$\int_0^t h(s)d\left\langle M^S, M^S\right\rangle_s = 0.$$

Thus $M^S(t)$ is a P^*-martingale. Since P is also a martingale measure with $P \neq P^*$ the proof follows. ■

Remark 3.4.9 *In the case $n = 1$ (a single Bid/Ask), the market becomes complete since the process $\beta(t)$, defined by Equation (3.4.16) in the proof is equal to* $\mathrm{sgn}(B(t))$*. Therefore the unique martingale measure is P.*

We can then deduce the following theorem on our process $S(t)$.

Theorem 3.4.10 *Suppose that $S = \{S(t)\}_{t\geq 0}$ is given by Equation (3.4.12), and $\left\{M^S(t)\right\}_{t\geq 0}$ is given by Equation (3.4.15). Then*

1. *For $n = 1$ (a single Bid/Ask), the market $(S(t), 1)$ is P-complete and admits the arbitrage opportunity of Equation (3.4.17).*

2. *For $n \geq 2$ (more than a single Bid/Ask), the market $(S(t), 1)$ is incomplete and arbitrage exists.*

Proof. From Theorem 3.4.8, we know that the market is P-complete for $n = 1$ and incomplete for $n > 1$. Let P such that $M^S(t)$ is a P-local martingale.

For $n = 1$, let us construct an arbitrage strategy. Let

$$a_s = 1_{\{\mathrm{supp}(d[M^S, M^S])\}^c}(s), \tag{3.4.17}$$

where $\mathrm{supp}\left(d\left[M^S, M^S\right]\right)$ denotes the ω by ω support of the (random) measure $d\left[M^S, M^S\right]_s(\omega)$; that is, for fixed ω it is the smallest closed set in $\mathbb{R}_+$ such that $d\left[M^S, M^S\right]_s$ does not charge its complement. Compare the proof of Proposition 3.4.8.

Let

$$H = H(T) = \frac{1}{2}\sum_{i=1}^{n}\int_0^T \left(1_{\{u_s(n)=i\}} + 1_{\{v_s(n)=i\}}\right) d_s L_s^0(B_i)$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left\{\int_0^T 1_{\{u_s(n)=i\}} d_s L_s^0\left(B_i^+ - R\right) + \int_0^T 1_{\{v_s(n)=i\}} d_s L_s^0\left(T - B_i^*\right)\right\}.$$

Assume *wlog* that $H \in L^1(P)$. Then by Theorem 3.4.7, there exists a self financing strategy (j_t, b) such that

$$H = H(T) = E\left[H(T)\right] + \int_0^T j(s)dS(s).$$

However, by Equation 3.4.17, we also have

$$H_T = 0 + \int_0^T a(s)dH(s).$$

Moreover, we have $\int_0^t a(s)dM^S(s) = 0,\ \ 0 \le t \le T$, by construction of the process a. Hence,

$$H = H(T) = 0 + \int_0^T a(s)dS(s),$$

which is an arbitrage opportunity. ■

3.5 Pricing and insider trading with respect to $S(t)$

In this Section we discuss a framework introduced in [27], which enables us pricing of contingent claims with respect to the price process $S(t)$ of the previous sections. We even consider the case of insider trading, that is the case of an investor, who has access to insider information. To this end we need some notions.

We consider a market driven by the stock price process $S(t)$ on a filtered probability space $(\Omega, \mathcal{H}, \{\mathcal{H}_t\}_{t\ge 0}, \mathbb{P})$. We assume that, the decisions of the trader are based on market information given by the filtration $\{\mathcal{G}_t\}_{0\le t\le T}$ with $\mathcal{H}_t \subset \mathcal{G}_t$ for all $t \in [0,T]$, $T > 0$ being a fixed terminal time. In this context an insider strategy is represented by an $\mathcal{G}_t$-adapted process $\varphi(t)$ and we interpret all anticipating integrals as the forward integral defined in [95] and

[121].

In such a market, a natural tool to describe the self-financing portfolio is the forward integral of an integrand process Y with respect to an integrator S, denoted by $\int_0^t Y\, d^- S$. See Chapter 1 or [121]. The following definitions and concepts are consistent with those given in [27].

Definition 3.5.1 *A self-financing portfolio is a pair (V_0, a) where V_0 is the initial value of the portfolio and a is a $\mathcal{G}_t$-adapted and S-forward integrable process specifying the number of shares of S held in the portfolio. The market value process V of such a portfolio at time $t \in [0, T]$, is given by*

$$V(t) = V_0 + \int_0^t a(s)\, d^- S(s), \tag{3.5.1}$$

while $b(t) = V(t) - S(t)a(t)$ constitutes the number of shares of the less risky asset held.

3.5.1 $\mathcal{A}$-martingales

Now, we briefly review the definition of $\mathcal{A}$-martingales which generalizes the concept of a martingale. We refer to [27] for more information about this notion. Throughout this Section, $\mathcal{A}$ will be a real linear space of measurable processes indexed by $[0, 1)$ with paths which are bounded on each compact interval of $[0, 1)$.

Definition 3.5.2 *A process $X = \{X(t)\}_{0 \le t \le T}$ is said to be a $\mathcal{A}$-martingale if every θ in $\mathcal{A}$ is X-improperly forward integrable (see Chapter 1) and*

$$E\left[\int_0^t \theta(s) d^- X(s)\right] = 0 \text{ for every } 0 \le t \le T \tag{3.5.2}$$

Definition 3.5.3 *A process $X = (X(t), 0 \le t \le T)$ is said to be $\mathcal{A}$-semimartingale if it can be written as the sum of an $\mathcal{A}$-martingale M and a bounded variation process V, with $V(0) = 0$.*

Remark 3.5.4

1. *Let X be a continuous $\mathcal{A}$-martingale with X belonging to $\mathcal{A}$, then, the quadratic variation of X exists improperly. In fact, if $\int_0^\cdot X(t)d^-X(t)$ exists improperly, then one can show that $[X,X]$ exists improperly and $[X,X] = X^2 - X^2(0) - 2\int_0^\cdot X(s)d^-X(s)$. See [27] for details.*

2. *Let X a continuous square integrable martingale with respect to some filtration $\mathcal{F}$. Suppose that every process in $\mathcal{A}$ is the restriction to $[0,T)$ of a process $(\theta(t),\ 0 \le t \le T)$ which is $\mathcal{F}$-adapted. Moreover, suppose that its paths are left continuous with right limits and $E\left[\int_0^T \theta^2(t)d[X]_t\right] < \infty$. Then X is an $\mathcal{A}$-martingale.*

3.5.2 Completeness and arbitrage: $\mathcal{A}$-martingale measures

We first recall some definitions and notions introduced in [27].

Definition 3.5.5 *Let h be a self-financing portfolio in $\mathcal{A}$, which is S-improperly forward integrable and X its wealth process. Then h is an $\mathcal{A}$-*arbitrage *if $X(T) = \lim_{t\to T} X(t)$ exists almost surely, $Pr[X(T) \ge 0] = 1$ and $Pr[X(T) > 0] > 0$.*

Definition 3.5.6 *If there is no $\mathcal{A}$-arbitrage, the market is said to be $\mathcal{A}$-*arbitrage free*.*

Definition 3.5.7 *A probability measure $Q \sim P$ is called a $\mathcal{A}$-*martingale measure *if with respect to Q the process S is an $\mathcal{A}$-martingale according to Definition 3.5.2.*

We need the following assumption. See [27].

Assumption 3.5.8 *Suppose that for all h in $\mathcal{A}$ the following condition holds.*
h is S-improperly forward integrable and

$$\int_0^\cdot d^- \int_0^t h(s)d^-S(s) = \int_0^\cdot h(t)d^-S(t) = \int_0^\cdot h(t)d^- \int_0^t d^-S(s) \tag{3.5.3}$$

The proof of the following proposition can be found in [27].

Proposition 3.5.9 *Under Assumption 3.5.8, if there exists an $\mathcal{A}$-martingale measure Q, the market is $\mathcal{A}$-arbitrage free.*

Definition 3.5.10 *A contingent claim is an $\mathcal{F}$-measurable random variable. Let $\mathcal{L}$ be the set of all contingent claims the investor is interested in.*

Definition 3.5.11

1. *A contingent claim C is called $\mathcal{A}$-*attainable* if there exists a self-financing trading portfolio $(X(0), h)$ with h in $\mathcal{A}$, which is S-improperly forward integrable, and whose terminal portfolio value coincides with C, i.e.,*

$$\lim_{t \to T} X(t) = C \quad P\text{-a.s.}$$

Such a portfolio strategy h is called a replicating *or* hedging portfolio *for C, and $X(0)$ is the* replication price *for C.*

2. *A $\mathcal{A}$-arbitrage free market is called $(\mathcal{A}, \mathcal{L})$-*complete* if every contingent claim in L is attainable.*

Assumption 3.5.12 *For every $\mathcal{G}_0$-measurable random variable η, and h in $\mathcal{A}$ the process $u = h\eta$, belongs to $\mathcal{A}$.*

Proposition 3.5.13 *Suppose that the market is $\mathcal{A}$-arbitrage free, and that Assumption 3.5.8 is realized. Then the replication price of an attainable contingent claim is unique*

Proof. Let Q be a given measure equivalent to P. For such a Q, let $\mathcal{A}$ be a set of all strategies ($\mathcal{G}_t$-adapted) such that Equation (3.5.2) in definition 3.5.2 is satisfied. Then, it follows from Proposition 3.5.9 that our market $(S(t), 1)$ in Section 3.4.2 is $\mathcal{A}$-arbitrage free.

■

In the final section, we shall discuss attainability of claims in connection with a concrete set $\mathcal{A}$ of trading strategies.

3.5.3 Hedging with respect to $S(t)$

In this Section, we want to determine hedging strategies for a certain class of European options with respect to the price process $S(t)$ of Section 3.4.2.

Let us now assume that $n = 1$ (a single Bid/Ask). Then, the price process S is the sum of a Wiener process and a continuous process with zero quadratic variation; moreover, we have that $d[S]_t = \frac{1}{4}\beta^2(t) = \frac{1}{4}$, where $\beta(t)$ is given by Equation (3.4.16). We can derive the following proposition which is similar to Proposition 5.29 in [27].

Proposition 3.5.14 *Let ψ be a function in $C^0(\mathbb{R})$ of polynomial growth. Suppose that there exist $(v(t,x),\ \ 0 \le t \le T,\ \ x \in \mathbb{R})$ of class $C^{1,2}([0,T) \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ which is a solution of the following Cauchy problem*

$$\begin{cases} \partial_t v(t,x) + \frac{1}{8}\partial_{yy} v(t,y) & = & 0 \ \text{ on } [0,T) \times \mathbb{R} \\ v(T,y) & = & \psi(y) \end{cases} \tag{3.5.4}$$

Set

$$h(t) = \partial_y v(t, S(t)),\ \ 0 \le t \le T,\ \ X(0) = v(0, S(0)).$$

Then $(X(0); h)$ is a self-financing portfolio replicating the contingent claim $(\psi S(T))$.

In particular, $(S(t), 1)$ is $\mathcal{A}, \mathcal{L}$-complete, where $\mathcal{A}$ is given by

$$\mathcal{A} = \{(\phi(t, S(t)),\ \ 0 \le t \le T) \,:\, \phi : [0,T] \times \mathbb{R} \to \mathbb{R} \ \text{Borel}$$
$$\text{measurable, of polynomial growth and lower bounded}\},$$

and $\mathcal{L}$ by all claims as stated in this Proposition.

Proof. The proof is a direct consequence of Itô's Lemma for forward integrals. See Proposition 5.29 in [27]. ∎

Part II

Malliavin calculus applied to optimal control under asymmetry of information

Chapter 4

Malliavin calculus applied to optimal control of stochastic partial differential equations with jumps

4.1 Introduction

In this Chapter we aim at using Malliavin calculus to prove a general stochastic maximum principle for stochastic partial differential equations (SPDE's) with jumps under partial information. More precisely, the controlled process is given by a quasilinear stochastic heat equation driven by a Wiener process and a Poisson random measure. Further the control processes are assumed to be adapted to a subfiltration of the filtration generated by the driving noise of the controlled process. Our Chapter is inspired by ideas developed in Meyer-Brandis, Øksendal & Zhou [88], where the authors establish a general stochastic maximum principle for SDE's based on Malliavin calculus. The results obtained in this Chapter can be considered a generalization of [88] to the setting of SPDE's.

There is already a vast literature on the stochastic maximum principle. The reader is *e.g.,* referred to [10, 11, 7, 52, 128, 102, 136] and the references therein. Let us mention that the authors in [10, 128], resort to stochastic maximum principles to study partially observed optimal control problems for diffusions, that is the controls under consideration are based on noisy observations described by the state process. Our Chapter covers the partial observation case in [10, 11, 128], since we deal with controls being adapted to a general subfiltration of the underlying reference filtration. Further, our Malliavin calculus approach to stochastic control of SPDE's allows for optimization of very general performance functionals. Thus our method is useful to examine control problems of non-Markovian type, which cannot be solved by stochastic dynamic programming. Another important advantage of our technique is that we may relax the assumptions on our Hamiltonian, considerably. For example, we do not need to impose concavity on the Hamiltonian. See *e.g.,* [102, 7]. We remark that the authors in [7] prove a sufficient and necessary maximum principle for partial information control of jump diffusions. However, their method relies on an adjoint equation which often turns out to be unsolvable.

We shall give an outline of our Chapter: In Section 4.2, we introduce a framework for our partial information control problem. Then in Section 4.3, we prove a general maximum principle for SPDE's by invoking Malliavin calculus. See Theorem 4.3.3. In Section 4.4, we use the results of the previous section to solve a partial information optimal harvesting problem (see Theorem 4.4.1). Further we inquire into a portfolio optimization problem under partial observation. The latter problem boils down to a partial observation problem of jump diffusions, which cannot be captured by the framework of [88].

4.2 Framework

In the following, let $\{B_s\}_{0\leq s\leq T}$ be a Brownian motion and $\widetilde{N}(dz, ds) = N(dz, ds) - ds\nu(dz)$ a compensated Poisson random measure associated with a Lévy process with Lévy measure ν on the (complete) filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$. In the sequel, we assume that the Lévy measure ν fulfills

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty,$$

where $\mathbb{R}_0 := \mathbb{R} \backslash \{0\}$.

Consider the controlled stochastic reaction-diffusion equation of the form

$$\begin{aligned} d\Gamma(t,x) =& \Big[L\Gamma(t,x) + b(t,x,\Gamma(t,x),\nabla_x \Gamma(t,x),u(t,x),\omega) \Big] dt \\ &+ \sigma(t,x,\Gamma(t,x),\nabla_x \Gamma(t,x),u(t,x),\omega) dB(t) \\ &+ \int_{\mathbb{R}} \theta(t,x,\Gamma(t,x),\nabla_x \Gamma(t,x),u(t,x),z,\omega) \widetilde{N}(dz,dt), \qquad (4.2.1) \\ (t,x) \in& \; [0,T] \times G \end{aligned}$$

with boundary condition

$$\begin{aligned} \Gamma(0,x) =& \xi(x), \;\; x \in \overline{G}, \\ \Gamma(t,x) =& \eta(t,x), \;\; (t,x) \in (0,T) \times \partial G. \end{aligned}$$

Here L is a partial differential operator of order m and ∇_x the gradient acting on the space variable $x \in \mathbb{R}^n$ and $G \subset \mathbb{R}^n$ is an open set. Further

$$\begin{aligned} b(t,x,\gamma,\gamma',u,\omega) &: \;\; [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \Omega \longrightarrow \mathbb{R} \\ \sigma(t,x,\gamma,\gamma',u,\omega) &: \;\; [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \Omega \longrightarrow \mathbb{R} \\ \theta(t,x,\gamma,\gamma',u,z,\omega) &: \;\; [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \mathbb{R}_0 \times \Omega \longrightarrow \mathbb{R} \\ \xi(x) &: \;\; \overline{G} \longrightarrow \mathbb{R} \\ \eta(t,x) &: \;\; (0,T) \times \partial G \longrightarrow \mathbb{R} \end{aligned}$$

are Borel measurable functions, where $U \subset \mathbb{R}$ is a closed convex set. The process

$$u : [0,T] \times G \times \Omega \longrightarrow U$$

is called an *admissible control* if the system (4.2.1) has a unique (strong) solution $\Gamma = \Gamma^{(u)}$ such that $u(t,x)$ is adapted with respect to a subfiltration

$$\mathcal{E}_t \subset \mathcal{F}_t, \;\; 0 \leq t \leq T, \qquad (4.2.2)$$

and such that

$$E\left[\int_0^T \int_G |f(t,x,\Gamma(t,x),u(t,x),\omega)|\,dx\,dt + \int_G |g(x,\Gamma(T,x),\omega)|\,dx\right] < \infty$$

for some given C^1 functions that define the performance functional (see Equation (4.2.3) below)

$$\begin{aligned} f &: [0,T]\times G\times\mathbb{R}\times U\times\Omega \longrightarrow \mathbb{R}\,, \\ g &: G\times\mathbb{R}\times\Omega \longrightarrow \mathbb{R}. \end{aligned}$$

A sufficient set of conditions, which ensures the existence of a unique strong solution of (4.2.1), is *e.g.,* given by the requirement that the coefficients b,σ,θ satisfy a certain linear growth and Lipschitz condition and that the operator L is bounded and coercive with respect to some Gelfand triple. For more general information on the theory of SPDE's the reader may consult *e.g.,* [28], [70].

Note that one possible subfiltration $\mathcal{E}_t$ of the type (4.2.2) is the δ-delayed information given by

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+};\;\; t \geq 0$$

where $\delta \geq 0$ is a given constant delay.

The σ-algebra $\mathcal{E}_t$ can be interpreted as the entirety of information at time t the controller has access to. We shall denote by $\mathcal{A} = \mathcal{A}_{\mathcal{E}}$ the class of all such admissible controls.

For admissible controls $u \in \mathcal{A}$ define the *performance functional*

$$J(u) = \mathbb{E}\left[\int_0^T \int_G f(t,x,\Gamma(t,x),u(t,x),\omega)\,dx\,dt + \int_G g(x,\Gamma(T,x),\omega)\,dx\right]. \tag{4.2.3}$$

The optimal control problem is to find the maximum and the maximizer of the performance, i.e. determine the *value* $J^* \in \mathbb{R}$ and the *optimal control* $u^* \in \mathcal{A}$ such that

$$J^* = \sup_{u\in\mathcal{A}} J(u) = J(u^*) \tag{4.2.4}$$

4.3 A generalized maximum principle for stochastic partial differential equations with jumps

In this Section we want to derive a general stochastic maximum principle by means of Malliavin calculus. To this end, let us briefly review some basic concepts of this theory. As for definitions and further information on Malliavin calculus, we refer to [94] or [31].

4.3.1 Some elementary concepts of Malliavin calculus for Lévy processes

In the sequel consider a Brownian motion $B(t)$ on the filtered probability space

$$(\Omega^{(1)}, \mathcal{F}^{(1)}, \{\mathcal{F}_t^{(1)}\}_{0\leq t\leq T}, P^{(1)}),$$

where $\{\mathcal{F}_t^{(1)}\}_{0\leq t\leq T}$ is the $P^{(1)}-$augmented filtration generated by B_t with $\mathcal{F}^{(1)} = \mathcal{F}_T^{(1)}$. Further we assume that a Poisson random measure $N(dt, dz)$ associated with a Lévy process is defined on the stochastic basis

$$(\Omega^{(2)}, \mathcal{F}^{(2)}, \{\mathcal{F}_t^{(2)}\}_{0\leq t\leq T}, P^{(2)}).$$

See [13, 127] for more information about Lévy processes.

The starting point of Malliavin calculus is the following observation which goes back to K. Itô [66]: Square integrable functionals of $B(t)$ and $\widetilde{N}(dt, dz)$ enjoy the chaos representation property, that is

(i) If $F \in L^2(\mathcal{F}^{(1)}, P^{(1)})$ then

$$F = \sum_{n\geq 0} I_n^{(1)}(f_n) \tag{4.3.1}$$

for a unique sequence of symmetric $f_n \in L^2(\lambda^n)$, where λ is the Lebesgue measure and

$$I_n^{(1)}(f_n) := n! \int_0^T \int_0^{t_n} \cdots \left(\int_0^{t_2} f_n(t_1, \cdots, t_n) dB(t_1) \right) dB(t_2) \cdots dB(t_n), \;\; n \in \mathbb{N}$$

the n-fold iterated stochastic integral with respect B_t. Here $I_n^{(1)}(f_0) := f_0$ for constants f_0.

(ii) Similarly, if $G \in L^2(\mathcal{F}^{(2)}, P^{(2)})$, then

$$G = \sum_{n \geq 0} I_n^{(2)}(g_n), \tag{4.3.2}$$

for a unique sequence of kernels g_n in $L^2((\lambda \times \nu)^n)$, which are symmetric with respect to $(t_1, z_1), \cdots, (t_n, z_n)$. Here $I_n^{(2)}(g_n)$ is given by

$$I_n^{(2)}(g_n) := n! \int_0^T \int_{\mathbb{R}_0} \int_0^{t_n} \int_{\mathbb{R}_0} \cdots \left(\int_0^{t_2} \int_{\mathbb{R}_0} g_n(t_1, z_1, \cdots, t_n, z_n) \right) \widetilde{N}(dt_1, dz_1) \cdots \widetilde{N}(dt_n, dz_n),$$

$n \in \mathbb{N}$.

It follows from the Itô isometry that

$$\|F\|^2_{L^2(P^{(1)})} = \sum_{n \geq 0} n! \, \|f_n\|^2_{L^2(\lambda^n)}$$

and

$$\|G\|^2_{L^2(P^{(2)})} = \sum_{n \geq 0} n! \, \|g_n\|^2_{L^2((\lambda \times \nu)^n)} .$$

Definition 4.3.1 (Malliavin derivatives D_t and $D_{t,z}$)

(i) *Denote by $\mathbb{D}^{(1)}_{1,2}$ the stochastic Sobolev space of all $F \in L^2(\mathcal{F}^{(1)}, P^{(1)})$ with chaos expansion (4.3.1) such that*

$$\|F\|^2_{\mathbb{D}^{(1)}_{1,2}} := \sum_{n \geq 0} n n! \, \|f_n\|^2_{L^2(\lambda^n)} < \infty.$$

Then the Malliavin derivative D_t of $F \in \mathbb{D}^{(1)}_{1,2}$ in the direction of the Brownian motion B is defined as

$$D_t F = \sum_{n \geq 1} n I^{(1)}_{n-1}(\widetilde{f}_{n-1}),$$

where $\widetilde{f}_{n-1}(t_1, \cdots, t_{n-1}) := f_n(t_1, \cdots, t_{n-1}, t)$.

(ii) *Similarly, let $\mathbb{D}^{(2)}_{1,2}$ be the space of all $G \in L^2(\mathcal{F}^{(2)}, P^{(2)})$ with chaos representation (4.3.2) satisfying*

$$\|G\|^2_{\mathbb{D}^{(2)}_{1,2}} := \sum_{n \geq 0} n n! \, \|g_n\|^2_{L^2((\lambda \times \nu)^n)} < \infty.$$

Then the Malliavin derivative $D_{t,z}$ of $G \in \mathbb{D}_{1,2}^{(2)}$ in the direction of the pure jump Lévy process $\eta_t := \int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(dt, dz)$ is defined as

$$D_{t,z}G := \sum_{n \geq 1} nI_{n-1}^{(2)}(\widetilde{g}_{n-1}),$$

where $\widetilde{g}_{n-1}(t_1, z_1, \cdots, t_{n-1}, z_{n-1}) := g_n(t_1, z_1, \cdots, t_{n-1}, z_{n-1}, t, z)$.

A crucial argument in the proof of our general maximum principle (Theorem 4.3.3) rests on duality formulas for the Malliavin derivatives D_t and $D_{t,z}$. (See [94] and [32].)

Lemma 4.3.2 (Duality formula for D_t and $D_{t,z}$)

(i) *Require that $\varphi(t)$ is $\mathcal{F}_t^{(1)}-$adapted with $E_{P^{(1)}}\left[\int_0^T \varphi^2(t)\, dt\right] < \infty$ and $F \in \mathbb{D}_{1,2}^{(1)}$. Then*

$$E_{P^{(1)}}\left[F \int_0^T \varphi(t)\, dB(t)\right] = E_{P^{(1)}}\left[\int_0^T \varphi(t) D_t F\, dt\right].$$

(ii) *Assume that $\psi(t,z)$ is $\mathcal{F}_t^{(2)}-$adapted with $E_{P^{(2)}}\left[\int_0^T \int_{\mathbb{R}_0} \psi^2(t,z)\, \nu(dz)\, dt\right] < \infty$ and $G \in \mathbb{D}_{1,2}^{(2)}$. Then*

$$E_{P^{(2)}}\left[G \int_0^T \int_{\mathbb{R}_0} \psi(t,z) \widetilde{N}(dt, dz)\right] = E_{P^{(2)}}\left[\int_0^T \int_{\mathbb{R}_0} \psi(t,z) D_{t,z} G\, \nu(dz)\, dt\right].$$

From now on, our stochastic basis will be

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P),$$

where $\Omega = \Omega^{(1)} \times \Omega^{(2)}$, $\mathcal{F} = \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}$, $\mathcal{F}_t = \mathcal{F}_t^{(1)} \times \mathcal{F}_t^{(2)}$, $P = P^{(1)} \times P^{(2)}$.

We remark that we may state the duality relations in Lemma 4.3.2 in terms of P.

4.3.2 Assumptions

In view of the optimization problem (4.2.4) we require the following conditions 1–5:

1. The functions $b,\ \sigma,\ \theta,\ f,\ g$ are contained in C^1 with respect to the arguments $\Gamma \in \mathbb{R}$ and $u \in U$.

2. For all $0 < t \leq r < T$ and all bounded $\mathcal{E}_t \otimes \mathcal{B}(\mathbb{R})-$measurable random variables α, the control

$$\beta_\alpha(s,x) := \alpha \cdot \chi_{[t,r]}(s),\ 0 \leq s \leq T\,, \tag{4.3.3}$$

where $\chi_{[t,T]}$ denotes the indicator function on $[t,T]$, is an admissible control.

3. For all $u, \beta \in \mathcal{A}_\mathcal{E}$ with β bounded there exists a $\delta > 0$ such that

$$u + y\beta \in \mathcal{A}_\mathcal{E} \tag{4.3.4}$$

for all $y \in (-\delta, \delta)$, and such that the family

$$\Big\{ \frac{\partial}{\partial\gamma} f(t,x,\Gamma^{u+y\beta}(t,x), u(t,x) + y\beta(t,x),\omega) \frac{d}{dy}\Gamma^{u+y\beta}(t,x)$$
$$+ \frac{\partial}{\partial u} f(t,x,\Gamma^{u+y\beta}(t,x), u(t,x) + y\beta(t,x),\omega)\beta(t,x) \Big\}_{y\in(-\delta,\delta)}$$

is $\lambda \times P \times \mu-$uniformly integrable;

$$\Big\{ \frac{\partial}{\partial\gamma} g(T,x,\Gamma^{u+y\beta}(T,x),\omega) \frac{d}{dy}\Gamma^{u+y\beta}(T,x) \Big\}_{y\in(-\delta,\delta)}$$

is $P \times \mu-$uniformly integrable.

4. For all $u, \beta \in \mathcal{A}_\mathcal{E}$ with β bounded the process

$$Y(t,x) = Y^\beta(t,x) = \frac{d}{dy}\Gamma^{(u+y\beta)}(t,x)\Big|_{y=0}$$

exists and

$$LY(t,x) = \frac{d}{dy} L\Gamma^{(u+y\beta)}(t,x)\Big|_{y=0}$$
$$\nabla_x Y(t,x) = \frac{d}{dy} \nabla_x\Gamma^{(u+y\beta)}(t,x)\Big|_{y=0}$$

Further suppose that $Y(t,x)$ follows the SPDE

$$
\begin{aligned}
Y(t,x) = &\int_0^t \Big[LY(s,x) + Y(s,x)\frac{\partial}{\partial\gamma} b(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x),\omega) \\
&+ \nabla_x Y(s,x)\nabla_{\gamma'} b(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x),\omega)\Big]\, ds \\
&+ \int_0^t \Big[Y(s,x)\frac{\partial}{\partial\gamma}\sigma(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),\omega) \\
&+ \nabla_x Y(s,x)\nabla_{\gamma'}\sigma(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),\omega)\Big]\, dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \Big[Y(s^-,x)\frac{\partial}{\partial\gamma}\theta(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),z,\omega) \\
&+ \nabla_x Y(s^-,x)\nabla_{\gamma'}\theta(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),z,\omega)\Big]\, \widetilde{N}(dz,ds) \\
&+ \int_0^t \Big[\beta(s,x)\frac{\partial}{\partial u} b(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x),\omega)\Big]\, ds \\
&+ \int_0^t \beta(s,x)\frac{\partial}{\partial u}\sigma(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),\omega)\, dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \beta(s^-,x)\frac{\partial}{\partial u}\theta\,(s,x,\Gamma(s,x),\nabla_x\Gamma(t,x),u(s,x),z,\omega)\; \widetilde{N}(dz,ds)\,, \\
(t,x) \in &[0,T]\times G\,, \qquad (4.3.5)
\end{aligned}
$$

with

$$
\begin{aligned}
Y(0,x) =&0,\ x\in\overline{G}\,, \\
Y(t,x) =&0,\ (t,x)\in(0,T)\times\partial G\,.
\end{aligned}
$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)$, $\nabla_{\gamma'} = \left(\frac{\partial}{\partial\gamma'_1},\cdots,\frac{\partial}{\partial\gamma'_n}\right)$ and $\gamma' = \left(\frac{\partial\Gamma}{\partial x_1},\cdots,\frac{\partial\Gamma}{\partial x_n}\right) = (\gamma'_1,\cdots,\gamma'_n)$

The proof of our maximum principle (Theorem 4.3.3) necessitates a certain probabilistic representation of solutions of the SPDE (4.3.5). Compare [79] in the Gaussian case. To this end, we need some notations and conditions.

In what follows we need some notation:

Let $m\in\mathbb{N}$, $0<\delta\le 1$. Denote by $C^{m,\delta}$ the space of all m-times continuously differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\|f\|_{m+\delta;K} := \|f\|_{m;K} + \sum_{|\alpha|=m} \sup_{x,y\in K, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\|x-y\|^\delta} < \infty$$

for all compact sets $K \subset \mathbb{R}^n$, where

$$\|f\|_{m;K} := \sup_{x\in K} \frac{|f(x)|}{(1+\|x\|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x\in K} |D^\alpha f(x)| .$$

For the multi-index of non-negative integers $\alpha = (\alpha_1, \cdots, \alpha_d)$ the operator D^α is defined as

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^d)^{\alpha_d}},$$

where $|\alpha| := \sum_{i=1}^d \alpha_i$.

Further introduce for sets $K \subset \mathbb{R}^n$ the norm

$$\|g\|^{\sim}_{m+\delta;K} := \|g\|^{\sim}_{m;K} + \sum_{|\alpha|=m} \left\|D_x^\alpha D_y^\alpha g\right\|^{\sim}_{\delta;K} ,$$

where

$$\|g\|^{\sim}_{\delta;K} := \sup_{\substack{x,y,x',y'\in K \\ x\neq y, x'\neq y'}} \frac{\left|g(x,y) - g(x',y) - g(x,y') + g(x',y')\right|}{\|x-x'\|^\delta \, \|y-y'\|^\delta}$$

and

$$\|g\|^{\sim}_{m;K} := \sup_{x,y\in K} \frac{|g(x,y)|}{(1+\|x\|)(1+\|y\|)} + \sum_{1\leq|\alpha|\leq m} \sup_{x,y\in K} \left|D_x^\alpha D_y^\alpha g(x,y)\right| .$$

We shall simply write $\|g\|^{\sim}_{m+\delta}$ for $\|g\|^{\sim}_{m+\delta;\mathbb{R}^n}$.

Define

$$\widetilde{b}_i(t,x) = \frac{\partial}{\partial \gamma_i'} b(t,x,\Gamma(t,x), \nabla_x\Gamma(t,x), u(t,x), \omega), i = 1, \cdots, n \tag{4.3.6}$$

$$\widetilde{\sigma}_i(t,x) = \frac{\partial}{\partial \gamma_i'} \sigma(t,x,\Gamma(t,x), \nabla_x\Gamma(t,x), u(t,x), \omega), i = 1, \cdots, n \tag{4.3.7}$$

$$\widetilde{\theta}_i(t,x) = \frac{\partial}{\partial \gamma_i'} \theta(t,x,\Gamma(t,x), \nabla_x\Gamma(t,x), u(t,x), z, \omega), i = 1, \cdots, n \tag{4.3.8}$$

$$b^*(t,x) = \frac{\partial}{\partial \gamma} b(t,x,\Gamma(t,x), \nabla_x\Gamma(t,x), u(t,x), \omega) \tag{4.3.9}$$

$$\sigma^*(t,x) = \frac{\partial}{\partial \gamma}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{4.3.10}$$

$$\theta^*(t,x,z) = \frac{\partial}{\partial \gamma}\theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega) \tag{4.3.11}$$

$$b_u(t,x) := \beta(s,x)\frac{\partial}{\partial u}b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{4.3.12}$$

$$\sigma_u(t,x) := \beta(s,x)\frac{\partial}{\partial u}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \tag{4.3.13}$$

Set

$$\begin{aligned}
F_i(x,dt) &:= \widetilde{b}_i(t,x)\,dt + \widetilde{\sigma}_i(t,x)\,dB(t), \quad i = 1,\cdots,n \\
F_{n+1}(x,dt) &:= b^*(t,x)\,dt + \sigma^*(t,x)\,dB(t) + \int_{\mathbb{R}_0}\theta^*(t,x,z)\widetilde{N}(dt,dz) \\
F_{n+2}(x,t) &:= \int_0^t b_u(s,x)\,ds + \int_0^t \sigma_u(s,x)\,dB(s)
\end{aligned}$$

Define the symmetric matrix function $(A^{ij}(x,y,s)_{1\le i,j\le n+2}$ given by

$$\begin{aligned}
A^{ij}(x,y,s) &= \widetilde{\sigma}_i(s,x)\cdot\widetilde{\sigma}_j(s,y), \quad i,j = 1,\cdots,n, \\
A^{i,n+1}(x,y,s) &= \widetilde{\sigma}_i(s,x)\sigma^*(s,y), \quad i = 1,\cdots,n \\
A^{i,n+2}(x,y,s) &= \widetilde{\sigma}_i(s,x)\sigma_u(s,y), \quad i = 1,\cdots,n
\end{aligned}$$

and

$$\begin{aligned}
A^{n+1,n+1}(x,y,s) &= \sigma^*(s,x)\cdot\sigma^*(s,y) \\
A^{n+1,n+2}(x,y,s) &= \sigma^*(s,x)\cdot\sigma_u(s,y) \\
A^{n+2,n+2}(x,y,s) &= \sigma_u(s,x)\cdot\sigma_u(s,y)
\end{aligned}$$

We make the following assumptions:

D1 $\frac{\partial}{\partial u}\theta(t,x,\Gamma(t,x), \ \nabla_x\Gamma(t,x),u(t,x),z,\omega) \equiv 0, \ \widetilde{\theta}_i(t,x) \equiv 0, \ i = 1,\cdots,n.$

D2 $\sigma^*(t,x),\theta^*(t,x,z), \ \widetilde{\sigma}_i(t,x), \ i = 1,\cdots,n$ are measurable deterministic functions.

D3 $\sum_{i,j=1}^{n+2}\int_0^T \left\|A^{ij}(\cdot,\cdot,s)\right\|_{m+\delta}^{\sim} ds < \infty$ and
$\int_0^T \left\{ (\sum_{i=1}^n \left\|\widehat{b}_i(s,\cdot)\right\|_{m+\delta}) + \|b^*(s,\cdot)\|_{m+\delta} + \|b_u(s,\cdot)\|_{m+\delta} \right\} ds < \infty$ a.e.
for some $m \ge 3$ and $\delta > 0$.

D4 There exists a measurable function $(z \longmapsto \beta(r,z))$ such that

$$\left| D_x^\alpha \theta^*(t,x,z) - D_x^\alpha \theta^*(t,x',z) \right| \leq \beta(r,z) \left\| x - x' \right\|^\delta$$

and

$$\int_{\mathbb{R}_0} |\beta(r,z)|^p \,\nu(dz) < \infty$$

for all $p \geq 2, |\alpha| \leq 2, 0 \leq t \leq T$ and x, x' with $\|x\| \leq r, \left\|x'\right\| \leq r$.

D5 There exist measurable functions $\alpha(z) \leq 0 \leq \beta(z)$ such that

$$-1 < \alpha(z) \leq \theta^*(t,x,z) \leq \beta(z) \text{ for all } t,x,z$$

and

$$\int_{\mathbb{R}_0} |\beta(z)|^p \,\nu(dz) + \int_{\mathbb{R}_0} (\alpha(z) - \log(1+\alpha(z)))^{p/2} \,\nu(dz) < \infty \text{ for all } p \geq 2.$$

In the following we assume that the differential operator L in Equation (4.3.5) is of the form

$$L_s u = L_s^{(1)} u + L_s^{(2)} u,$$

where

$$L_s^{(1)} u := \frac{1}{2} \sum_{i,j=1}^n a^{ij}(x,s) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n b_i(x,s) \frac{\partial u}{\partial x^i} + d(x,s) u$$

and

$$\begin{aligned} L_s^{(2)} u \quad := \quad & \frac{1}{2} \sum_{i,j=1}^n A^{ij}(x,x,s) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n \left(A^{i,n+1}(x,x,s) + \frac{1}{2} C_i(x,s) \right) \frac{\partial u}{\partial x^i} \\ & + \frac{1}{2} \left(D(x,s) + A^{n+1,n+1}(x,x,s) \right) u \end{aligned}$$

with

$$C_j(x,s) := \sum_{j=1}^n \frac{\partial A^{ij}}{\partial y^i}(x,y,s) \Big|_{y=x}, \quad i = 1, \cdots, n$$

and

$$D(x,s) := \sum_{j=1}^n \frac{\partial A^{i,n+1}}{\partial y^i}(x,y,s) \Big|_{y=x}.$$

We require the following conditions:

D6 $L_t^{(1)}$ is an elliptic differential operator.

D7 There exists a non-negative symmetric continuous matrix function $(a^{ij}(x,y,s))_{1\le i,j\le n}$ such that $a^{ij}(x,x,s) = a^{ij}(x,s)$. Further it is assumed that

$$\sum_{i,j=1}^{n} \left\| a^{ij}(\cdot,s) \right\|_{m+1+\delta} \le K \text{ for all } s$$

for a constant K and some $m \ge 3$, $\delta > 0$.

D8 The functions $b_i(x,s), i = 1,\cdots,n$ are continuous in (x,s) and satisfy

$$\sum_{i=1}^{n} \left\| b_i(\cdot,s) \right\|_{m+\delta} \le C \text{ for all } s$$

for a constant C and some $m \ge 3$, $\delta > 0$.

D9 The function $d(x,s)$ is continuous in (x,s) and belongs to $C^{m,\delta}$ for some $m \ge 3$, $\delta > 0$. In addition a^{ij} is bounded and $d/(1 + \|x\|)$ is bounded from the above.

D10 The functions b^*, σ^* and d^* are uniformly bounded.

Now let $X(x,t) = (X_1(x,t),\cdots,X_n(x,t))$ be a $C^{k,\gamma}$−valued Brownian motion , that is a continuous process $X(t,\cdot) \in C^{k,\gamma}$ with independent increments (see [79]) on another probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$. Assume that this process has local characteristic $a^{ij}(x,y,t)$ and $m(x,t) = b(x,t) - c(x,t)$, where the correction term $c(x,t)$ is given by

$$c_i(x,t) = \frac{1}{2}\int_0^t \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x^j}(x,y,s)\bigg|_{y=x} ds, \quad i = 1,\cdots,n.$$

Then, let us consider on the product space $(\Omega\times\widehat{\Omega},\mathcal{F}\times\widehat{\mathcal{F}},P\times\widehat{P})$ the first order SPDE

$$\begin{aligned} v(x,t) = &\sum_{i=1}^{n}\int_0^t (X_i(x,\circ ds) + F_i(x,\circ ds))\frac{\partial v}{\partial x^i} \\ &+ \int_0^t (d(x,s) + F_{n+1}(x,\circ ds))v + F_{n+2}(x,t), \end{aligned} \tag{4.3.14}$$

where $\circ dt$ stands for non-linear integration in the sense of Stratonovich (see [79]).

Using the definition of $X(x,t)$ the equation (4.3.14) can be recast as

$$\begin{aligned}v(x,t) = &\int_0^t L_s v(x,s)ds + \sum_{i=1}^n Y_i^*(x,ds)\frac{\partial v}{\partial x^i} \\ &+ \sum_{i=1}^n \int_0^t F_i(x,ds)\frac{\partial v}{\partial x^i} + \int_0^t F_{n+1}(x,ds)v \\ &+ F_{n+2}(x,t), \qquad (4.3.15)\end{aligned}$$

where $Y^*(x,t) = (Y_1^*(x,t),...,Y_n^*(x,t))$ is the martingale part of $X(t,x)$. So applying the expectation $E_{\widehat{P}}$ to both sides of the latter equation gives the following representation for the solution to system 4.3.5:

$$Y(t,x) = E_{\widehat{P}}\left[v(x,t)\right]$$

See also the proof of Theorem 6.2.5 in [79]. Now let $\varphi_{s,t}$ be the solution of the Stratonovich SDE

$$\varphi_{s,t}(x) = x - \int_s^t G(\varphi_{s,r}(x), \circ dr),$$

where $G(x,t) := (X_1(x,t)+F_1(x,t),\cdots,X_n(x,t)+F_n(x,t))$. Then by employing the proof of Theorem 6.1.8 and Theorem 6.1.9 in [79] with respect to a generalized Itô formula in [20] one obtains the following explicit representation of $v(t,x)$:

$$\begin{aligned}&v(x,t) \\ &= \int_0^t \exp\left\{\frac{1}{2}\int_s^t \sigma^*(r,\varphi_{t,r}(x))^2 dr + \int_s^t b^*(r,\varphi_{t,r}(x))\,dr + \sigma^*(r,\varphi_{t,r}(x))\,\widehat{d}B(r)\right. \\ &\quad + \int_s^t\int_{\mathbb{R}_0} \left(\log(1+\theta^*(r,\varphi_{t,r}(x),z)) - \theta^*(r,\varphi_{t,r}(x),z)\right)dr \\ &\quad \left.+ \int_s^t\int_{\mathbb{R}_0} \log(1+\theta^*(r,\varphi_{t,r}(x),z)\,\widetilde{N}(\widehat{d}r,dz)\right\}\times \\ &\left(\beta(s,x)\frac{\partial}{\partial u}b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\,ds\right. \\ &\quad \left.+\beta(s,x)\frac{\partial}{\partial u}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\circ\widehat{d}B(s)\right), \qquad (4.3.16)\end{aligned}$$

where $\widehat{d}$ denotes backward integration and where the inverse flow $\varphi_{t,s} = \varphi_{s,t}^{-1}$ solves the backward Stratonovich SDE

$$\varphi_{t,s}^{(i)}(x) = x_i + \int_s^t \widetilde{b}_i(r,\varphi_{t,r}(x))\,dr + \int_s^t \widetilde{\sigma}_i(r,\varphi_{t,r}(x))\circ\widehat{d}B(r),\;\; i=1,...,n.$$

For later use, let us consider the general case, when

$$\begin{aligned} Y(0,x) =& f(x),\ x \in \overline{G}\,, \\ Y(t,x) =& 0,\ \ (t,x) \in (0,T) \times \partial G\,, \end{aligned}$$

holds, where $f \in C^{m,\delta}$.

Then, $v(x,t)$ is described by

$$\begin{aligned} v(x,t) =& f(x) + \int_0^t L_s v(x,s)ds + \sum_{i=1}^n Y_i^*(x,ds)\frac{\partial v}{\partial x^i} \\ & + \sum_{i=1}^n \int_0^t F_i(x,ds)\frac{\partial v}{\partial x^i} + \int_0^t F_{n+1}(x,ds)v \\ & + F_{n+2}(x,t). \end{aligned}$$

Using the same reasoning, we obtain:

$$\begin{aligned} & v(x,t) \\ =& \exp\left\{\frac{1}{2}\int_s^t \sigma^*(r,\varphi_{t,r}(x))^2 dr + \int_s^t b^*(r,\varphi_{t,r}(x))\,dr + \sigma^*(r,\varphi_{t,r}(x))\,\widehat{d}B(r)\right. \\ & + \int_s^t \int_{\mathbb{R}_0} (\log(1+\theta^*(r,\varphi_{t,r}(x),z)) - \theta^*(r,\varphi_{t,r}(x),z))\,dr \\ & \left. + \int_s^t \int_{\mathbb{R}_0} \log(1+\theta^*(r,\varphi_{t,r}(x),z)\,\widetilde{N}(\widehat{d}r,dz)\right\} \times f(\varphi_{t,0}(x)) \\ & + \int_0^t \exp\left\{\frac{1}{2}\int_s^t \sigma^*(r,\varphi_{t,r}(x))^2 dr + \int_s^t b^*(r,\varphi_{t,r}(x))\,dr + \sigma^*(r,\varphi_{t,r}(x))\,\widehat{d}B(r)\right. \\ & + \int_s^t \int_{\mathbb{R}_0} (\log(1+\theta^*(r,\varphi_{t,r}(x),z)) - \theta^*(r,\varphi_{t,r}(x),z))\,dr \\ & \left. + \int_s^t \int_{\mathbb{R}_0} \log(1+\theta^*(r,\varphi_{t,r}(x),z)\,\widetilde{N}(\widehat{d}r,dz)\right\} \times \\ & \left(\beta(s,x)\frac{\partial}{\partial u} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\,ds\right. \\ & \left. + \beta(s,x)\frac{\partial}{\partial u}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega) \circ \widehat{d}B(s)\right), \end{aligned} \tag{4.3.17}$$

Finally, we require the following conditions:

5. Suppose that for all $u \in \mathcal{A}_{\mathcal{E}}$ the processes

$$
\begin{aligned}
K(t,x) :=& \frac{\partial}{\partial \gamma} g(x, \Gamma(T,x)) + \int_t^T \frac{\partial}{\partial \gamma} f\left(s, x, \Gamma(s,x), u(s,x)\right) ds \\
D_t K(t,x) :=& D_t \frac{\partial}{\partial \gamma} g(x, \Gamma(T,x)) + \int_t^T D_t \left(\frac{\partial}{\partial \gamma} f(s, x, \Gamma(s,x), u(s,x)) \right) ds \\
D_{t,z} K(t,x) :=& D_{t,z} \frac{\partial}{\partial \gamma} g(x, \Gamma(T,x)) + \int_t^T D_{t,z} \left(\frac{\partial}{\partial \gamma} f(s, x, \Gamma(s,x), u(s,x)) \right) ds \\
H_0(s,x,\gamma,\gamma',u) :=& K(s,x) b(s,x,\gamma,\gamma',u,\omega) + D_s K(s,x) \sigma(s,x,\gamma,\gamma',u,\omega) \qquad (4.3.18) \\
&+ \int_{\mathbb{R}} D_{s,z} K(s,x) \theta(s,x,\gamma,\gamma',u,z,\omega)\, \nu(dz) \\
Z(t,s,x) :=& \exp\left\{ \int_s^t F_{n+1}\left(x, \circ \widehat{dr}\right) \right\}, \qquad (4.3.19) \\
p(t,x) :=& K(t,x) + \int_t^T \left\{ \frac{\partial}{\partial \gamma} H_0(s,x,\Gamma(s,x), \nabla_x \Gamma(s,x), u(s,x)) + L^* K(s,x) \right. \\
&\left. + \nabla_x^* \left(\nabla_{\gamma'} H_0(s,x,\Gamma(s,x), \nabla_x \Gamma(s,x), u(s,x)) \right) \right\} Z(t,s,\varphi_{s,t}(x))\, ds \\
q(t,x) :=& D_t p(t,x) \\
r(t,x,z) :=& D_{t,z} p(t,x); \;\; t \in [0,T], \;\; z \in \mathbb{R}_0, \;\; x \in G\,.
\end{aligned}
$$

are well-defined and where $\varphi_{s,t}$ and $\varphi_{t,s}^{(i)}$ are defined as before.

Assume also that

$$
\begin{aligned}
& E\Bigg[\int_0^T \int_G \Bigg\{ |K(t,x)| \Bigg(|LY(t,x)| + \left| Y(t,x)\frac{\partial}{\partial\gamma} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\right| \\
& + \left|\beta(t,x)\frac{\partial}{\partial u} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\right| \\
& + \left|\nabla_x Y(t,x)\nabla_{\gamma'} b(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\right|\Bigg) \\
& + |D_t K(t,x)| \Bigg(\left| Y(t,x)\frac{\partial}{\partial\gamma}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\right| \\
& + \left|\nabla_x Y(t,x)\nabla_{\gamma'}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\right| \\
& + \left|\beta(t,x)\frac{\partial}{\partial u}\sigma(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),\omega)\right|\Bigg) \\
& + \int_{\mathbb{R}} |D_{t,z}K(t,x)| \Bigg(\left| Y(t,x)\frac{\partial}{\partial\gamma}\theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega)\right| \\
& + \left|\nabla_x Y(t,x)\nabla_{\gamma'}\theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega)\right| \\
& + \left|\beta(t,x)\frac{\partial}{\partial u}\theta(t,x,\Gamma(t,x),\nabla_x\Gamma(t,x),u(t,x),z,\omega)\right|\Bigg)\nu(dz) \\
& + \left|\beta(t,x)\frac{\partial}{\partial u} f(t,x,\Gamma(t,x),u(t,x))\right|\Bigg\}\,dt\,dx\Bigg] \\
& < \infty.
\end{aligned}
$$

Here L^* is the dual operator of L. Further, the densely defined operator ∇_x^* stands for the adjoint of ∇_x, that is

$$
(g, \nabla_x f)_{L^2(G;\mathbb{R}^n)} = (\nabla_x^* g, f)_{L^2(G;\mathbb{R})} \tag{4.3.20}
$$

for all $f \in Dom\,(\nabla_x)\,, g \in Dom\,(\nabla_x^*)$. For example, if $g = (g_1, ..., g_n) \in C_0^\infty(G;\mathbb{R}^n)$, then $\nabla_x^* g = \sum\limits_{j=1}^{n} \frac{\partial g_j}{\partial x_j}$.

Let us comment that $D_t K(t,x)$ and $D_{t,z}K(t,x)$ in condition 5 exist, if *e.g.,* coefficients b, σ, θ fulfill a global Lipschitz condition, f is independent of u in condition 1 and the operator L is the generator of a strongly continuous semigroup. See *e.g.,* [94, 126] and [21, Section 5]. Now let us introduce the *general Hamiltonian*

$$
H : [0,T] \times G \times \mathbb{R} \times \mathbb{R}^n \times U \times \Omega \longrightarrow \mathbb{R}
$$

by

$$
\begin{aligned}
H(t,x,\gamma,\gamma',u,\omega) \quad &:= \quad f(t,x,\gamma,u,\omega)+p(t,x)b(t,x,\gamma,\gamma',u,\omega)+D_t p(t,x)\sigma(t,x,\gamma,\gamma',u,\omega) \\
&\quad + \int_{\mathbb{R}} D_{t,z}p(t,x)\theta(t,x,\gamma,\gamma',u,z,\omega)\,\nu(dz). \qquad (4.3.21)
\end{aligned}
$$

4.3.3 A general stochastic maximum principle for a partial information control problem

We can now state a general stochastic maximum principle for our partial information control problem (4.2.4):

Theorem 4.3.3 *Retain the conditions 1-5. Assume that* $\widehat{u} \in \mathcal{A}_{\mathcal{E}}$ *is a* critical point *of the performance functional* $J(u)$ *given by (4.2.4), that is*

$$
\frac{d}{dy}J(\widehat{u}+y\beta)\bigg|_{y=0} = 0 \qquad (4.3.22)
$$

for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$*. Then*

$$
E\left[E_Q\left[\int_G \frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x))dx\right]\bigg|\,\mathcal{E}_t\right] = 0 \quad a.e.\ in \quad (t,x,\omega), \qquad (4.3.23)
$$

where

$$
\begin{aligned}
\widehat{\Gamma}(t,x) \quad &= \quad \Gamma^{(\widehat{u})}(t,x), \\
\widehat{H}(t,x,\gamma,\gamma',u,\omega) \quad &= \quad f(t,x,\gamma,u,\omega)+\widehat{p}(t,x)b(t,x,\gamma,\gamma',u,\omega)+D_t\widehat{p}(t,x)\sigma(t,x,\gamma,\gamma',u,\omega) \\
&\quad + \int_{\mathbb{R}} D_{t,z}\widehat{p}(t,x)\theta(t,x,\gamma,\gamma',u,z,\omega)\nu(dz)\,,
\end{aligned}
$$

with

$$
\begin{aligned}
\widehat{p}(t,x) \quad &= \quad \widehat{K}(t,x)+\int_t^T \left\{\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x,\widehat{\Gamma},\widehat{\Gamma}',\widehat{u},\omega)+L^*\widehat{K}(s,x)\right. \\
&\quad \left. +\nabla_x^*\left(\nabla_{\gamma'}H_0(s,x,\widehat{\Gamma},\widehat{\Gamma}',\widehat{u},\omega)\right)\right\}\widehat{Z}(t,s,\widehat{\varphi}_{s,t}(x))ds\,,
\end{aligned}
$$

$$
\widehat{K}(t,x) \quad = \quad \frac{\partial}{\partial\gamma}g(x,\widehat{\Gamma}(T,x),\omega)+\int_t^T \frac{\partial}{\partial\gamma}f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega)ds,
$$

where $\widehat{\varphi}_{s,t}$ is the solution of the of the Stratonovich SDE

$$\varphi_{s,t}(x) = x - \int_s^t \widehat{G}(\widehat{\varphi}_{s,r}(x), \circ dr),$$

with $\widehat{G}(x,t) := (X_1(x,t) + \widehat{F}_1(x,t), \cdots, X_n(x,t) + \widehat{F}_n(x,t))$,

$$\begin{aligned}
\widehat{\varphi}_{s,t}(x) &= x - \int_s^t \widehat{G}(\varphi_{s,r}(x), \circ dr), \\
\widehat{F}_i(x,dt) &:= \widehat{\widehat{b}}_i(t,x)\,dt + \widehat{\widehat{\sigma}}_i(t,x)\,dB(t), \quad i = 1, \cdots, n \\
\widehat{F}_{n+1}(x,dt) &:= \widehat{b}^*(t,x)\,dt + \widehat{\sigma}^*(t,x)\,dB(t) + \int_{\mathbb{R}_0} \widehat{\theta}^*(t,x,z)\widetilde{N}(dt,dz) \\
\widehat{F}_{n+2}(x,t) &:= \int_0^t \widehat{b}_u(s,x)\,ds + \int_0^t \widehat{\sigma}_u(s,x)\,dB(s) \\
\widehat{\widehat{b}}_i(t,x) &= \frac{\partial}{\partial \gamma_i'} b(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), \omega), \quad i = 1, \cdots, n \\
\widehat{\widehat{\sigma}}_i(t,x) &= \frac{\partial}{\partial \gamma_i'} \sigma(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), \omega), \quad i = 1, \cdots, n \\
\widehat{\widehat{\theta}}_i(t,x) &= \frac{\partial}{\partial \gamma_i'} \theta(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), z, \omega), \quad i = 1, \cdots, n \\
\widehat{b}^*(t,x) &= \frac{\partial}{\partial \gamma} b(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), \omega) \\
\widehat{\sigma}^*(t,x) &= \frac{\partial}{\partial \gamma} \sigma(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), \omega) \\
\widehat{\theta}^*(t,x,z) &= \frac{\partial}{\partial \gamma} \theta(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), z, \omega) \\
\widehat{b}_u(t,x) &:= \beta(s,x) \frac{\partial}{\partial u} b(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), \omega) \\
\widehat{\sigma}_u(t,x) &:= \beta(s,x) \frac{\partial}{\partial u} \sigma(t,x,\widehat{\Gamma}(t,x), \nabla_x \widehat{\Gamma}(t,x), \widehat{u}(t,x), \omega),
\end{aligned}$$

and

$$\widehat{Z}(t,s) := \exp\left\{ \int_s^t F_{n+1}\left(\widehat{\varphi}_{s,r}(x), \circ \widehat{dr}\right) \right\},$$

Remark 4.3.4 *We remark that in Theorem 4.3.3 the partial derivatives of H and H_0 with respect to u, γ, and γ' only refer to differentiation at places where the arguments appear in the coefficients of the definitions (4.3.18) and (4.3.21).*

Proof. See Appendix A, Section A.2. ■

4.4 Applications

In this Section we take aim at two applications of Theorem 4.3.3 : The first one pertains to partial information optimal harvesting, whereas the other one refers to portfolio optimization under partial observation.

4.4.1 Partial information optimal harvesting

Assume that $\Gamma(t,x)$ describes the density of a population (e.g. fish) at time $t \in (0,T)$ and at the location $x \in G \subset \mathbb{R}^d$. Further suppose that $\Gamma(t,x)$ is modeled by the stochastic-reaction diffusion equation

$$\begin{aligned} d\Gamma(t,x) &= \left[\frac{1}{2}\Delta\Gamma(t,x) + b(t)\Gamma(t,x) - c(t)\right] dt + \sigma(t)\Gamma(t,x)\, dB(t) \\ &\quad + \int_{\mathbb{R}} \theta(t,z)\Gamma(t,x)\widetilde{N}(dz,dt),\ (t,x) \in [0,T] \times G, \qquad (4.4.1) \\ &\quad \text{where } \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial X_i^2} \text{is the Laplacian,} \end{aligned}$$

with boundary condition

$$\begin{aligned} \Gamma(0,x) &= \xi(x),\ x \in \overline{G} \\ \Gamma(t,x) &= \eta(t,x),\ (t,x) \in (0,T) \times \partial G. \end{aligned}$$

where b, σ, θ, c are given processes such that D1–D10 in Section 4.3.2 are fulfilled.

The process $c(t) \geq 0$ is our harvesting rate, which is assumed to be a $\mathcal{E}_t-$predictable admissible control.

We aim to maximize both expected cumulative utility of consumption and the terminal size of the population subject to the performance functional

$$J(c) = E\left[\int_G \int_0^T \zeta(s)U(c(s))\, ds\, dx + \int_G \xi\Gamma^{(c)}(T,x)\, dx\right], \qquad (4.4.2)$$

where $U : [0,+\infty) \longrightarrow \mathbb{R}$ is a C^1 utility function, $\zeta(s) = \zeta(s,x,\omega)$ is an $\mathcal{F}_t-$predictable process and $\xi = \xi(\omega)$ is an $\mathcal{F}_T-$measurable random variable such that

$$E\left[\int_G |\zeta(t,x)|\,dx\right] < \infty \text{ and } E\left[\xi^2\right] < \infty.$$

We want to find an admissible control $\hat{c} \in A_{\mathcal{E}}$ such that

$$\sup_{c\in A_{\mathcal{E}}} J(c) = J(\hat{c}). \tag{4.4.3}$$

Note that condition 1 of Section 4.3.2 is fulfilled. Using the same arguments in [10] it can be verified that the linear SPDE (4.4.1) also satisfies conditions 2–4. Using the previous notation, we note that in this case, with $u = c$,

$$f(t,x,\Gamma(t,x),c(t),\omega) = \zeta(s,\omega)U(c(t)); \;\; g(x,\Gamma(t,x),\omega) = \xi(\omega)\Gamma^{(c)}(t,x).$$

Hence

$$\begin{aligned}
K(t,x) =&\frac{\partial}{\partial\gamma}g(x,\Gamma(T,x),\omega) + \int_t^T \frac{\partial}{\partial\gamma}f(s,x,\Gamma(s,x),u(s,x),\omega)\,ds = \xi(\omega),\\
H_0(t,x,\gamma,c) =&\xi(\omega)\left(b(t,x)\gamma - c\right) + D_t\xi(\omega)\sigma(t)\gamma + \int_{\mathbb{R}} D_{t,z}\xi(\omega)\theta(t,z)\gamma\,\nu(dz)\,dt,\\
I(t,s,x) =& \left(b(t,x)\xi(\omega) + D_t\xi(\omega)\sigma(t) + \int_{\mathbb{R}} D_{t,z}\xi(\omega)\theta(t,z)\,\nu(dz)\right) \times Z(t,s,\widehat{\varphi}_{s,t}(x)),\\
I_1(t,s,x) =&I_2(t,s,x) = 0,\\
Z(s,t,x) =&\exp\left\{\int_t^s F_{n+1}(x,\circ\widehat{dr})\right\},\\
F_{n+1}(x,dt) =&b(t)\,dt + \sigma(t)\,dB(t) + \int_{\mathbb{R}_0}\theta(t,z)\widetilde{N}(dt,dz).
\end{aligned}$$

In this case we have $\varphi_{s,t}(x) = x$ since $K(s,x) = \xi(\omega)$ if follows that $L^*K(s,x) = 0$, in addition, H_0 does not depend on γ' and then $\nabla_x^*\left(\nabla_{\gamma'}H_0(s,x,\Gamma(s,x),\nabla_x\Gamma(s,x),u(s,x))\right) = 0$. Therefore

$$p(t,x) = \xi(\omega) + \int_t^T \left(b(t,x)\xi(\omega) + D_t\xi(\omega)\sigma(t) + \int_{\mathbb{R}} D_{t,z}\xi(\omega)\theta(t,z)\nu(dz)\right) Z(t,r,\widehat{\varphi}_{s,t}(x))\,dr, \tag{4.4.4}$$

and the Hamiltonian becomes

$$\begin{aligned}
H(t,x,\gamma,c) \;\;=\;\; & \zeta(t)U(c) + p(t,x)\left(b(t,x)\Gamma(t,x) - c(t)\right) + D_tp(t,x)\sigma(t)\\
&+ \int_{\mathbb{R}_0} D_{t,z}p(t,x)\theta(t,z)\nu(dz).
\end{aligned} \tag{4.4.5}$$

Then, $\hat{c} \in \mathcal{A}_{\mathcal{E}}$ is an optimal control for the problem (4.4.3) if we have:

$$\begin{aligned} 0 &= E\left[E_Q\left[\int_G \frac{\partial}{\partial c} H(t,x,\hat{\Gamma}(t,x),\hat{c}(t))\,dx\right]\bigg|\,\mathcal{E}_t\right] \\ &= E\left[E_Q\left[\int_G \{\zeta(t)U'(\hat{c}(t)) - p(t,x)\}\,dx\right]\bigg|\,\mathcal{E}_t\right] \\ &= U'(\hat{c}(t))E\left[E_Q\left[\int_G \zeta(t,x)\,dx\right]\bigg|\,\mathcal{E}_t\right] - E\left[E_Q\left[\int_G p(t,x)\,dx\right]\bigg|\,\mathcal{E}_t\right] \end{aligned}$$

We have proved a theorem similar to Theorem 4.2 in [88]:

Theorem 4.4.1 *If there exists an optimal harvesting rate $\hat{c}(t)$ of problem (4.4.3), then it satisfies the equation*

$$U'(\hat{c}(t))E\left[E_Q\left[\int_G \zeta(t,x)\,dx\right]\bigg|\,\mathcal{E}_t\right] = E\left[E_Q\left[\int_G p(t,x)\,dx\right]\bigg|\,\mathcal{E}_t\right]. \tag{4.4.6}$$

4.4.2 Application to optimal stochastic control of jump diffusion with partial observation

In this Subsection we want to apply ideas of non-linear filtering theory in connection with Theorem 4.3.3 to solve a portfolio optimization problem, where the trader has limited access to market information (Example 4.4.3). As for general background information on non-linear filtering theory the reader may *e.g.,* consult [10]. For the concrete setting that follows below see also [86] and [90].

Suppose that the state process $X(t) = X^{(u)}(t)$ and the observation process $Z(t)$ are described by the following system of SDE's:

$$\begin{aligned} dX(t) =& \alpha(X(t),u(t))\,dt + \beta(X(t),u(t))\,dB^X(t), \\ dZ(t) =& h(t,X(t))\,dt + dB^Z(t) + \int_{\mathbb{R}_0} \xi\, N_\lambda(dt,d\xi), \end{aligned} \tag{4.4.7}$$

where $(B^X(t); B^Z(t)) \in \mathbb{R}^2$ is a Wiener process independent of the initial value $X(0)$, and N_λ is an integer valued random measure with predictable compensator

$$\mu(dt,d\xi,\omega) = \lambda(t,X_t,\xi)\,dt\,\nu(d\xi),$$

for a Lévy measure ν and an intensity rate function $\lambda(t,x,\xi)$, such that the increments of N_λ are conditionally independent with respect to the filtration generated by B_t^X. Further $u(t)$ is a control process which takes values in a closed convex set $U \subset \mathbb{R}$ and which is adapted to the filtration $\mathcal{G}_t$ generated by the observation process $Z(t)$. The coefficients $\alpha : \mathbb{R} \times U \longrightarrow \mathbb{R}$, $\beta : \mathbb{R} \times U \longrightarrow \mathbb{R}$, $\lambda : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_0 \longrightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are twice continuously differentiable.

In what follows we shall assume that a strong solution $X(t) = X^{(u)}(t)$ of System (4.4.7), if it exists, takes values in a given Borel set $G \subseteq \mathbb{R}$. Let us introduce the performance functional

$$J(u) := E\left[\int_0^T f(X(t), Z(t), u(t))\, dt + g(X(T), Z(T))\right],$$

where $f : G \times \mathbb{R} \times U \longrightarrow \mathbb{R}$, $g : G \times \mathbb{R} \longrightarrow \mathbb{R}$ are (lower) bounded C^1 functions. We want to find the maximizer u^* of J, that is

$$J^* = \sup_{u \in \mathcal{A}} J(u) = J(u^*), \tag{4.4.8}$$

where $\mathcal{A}$ is the set of admissible controls consisting of $\mathcal{G}_t-$predictable controls u such that System (4.4.7) admits a unique strong solution.

We shall now briefly outline how the optimal control problem (4.4.8) for SDE's with partial observation can be transformed into one for SPDE's with complete information. See *e.g.,* [10] and [86] for details. In the sequel we assume that $\lambda(t,x,\xi) > 0$ for all t, x, ξ and that the exponential process

$$\begin{aligned} M_t \quad := \quad & \exp\left\{\int_0^t h(X(s))\, dB^Z(s) - \frac{1}{2}\int_0^t h^2(X(s))\, ds \right. \\ & \left. + \int_0^t \int_{\mathbb{R}_0} \log \lambda(s, X(s), \xi) N_\lambda(ds, d\xi) + \int_0^t \int_{\mathbb{R}_0} [1 - \lambda(s, X(s), \xi)]\, ds\, \nu(d\xi)\right\}; \quad t \geq 0 \end{aligned}$$

is well defined and a martingale. Define the change of measure

$$dQ' = M_T dP$$

and set

$$N_T = M_T^{-1}$$

Using the Girsanov theorem for random measures and the uniqueness of semimartingale characteristics (see *e.g.,* [67]), one sees that the processes in System (4.4.7) get decoupled under the measure Q' in the sense the system is transformed to

$$\begin{aligned} dX(t) &= \alpha(X(t), u(t))\, dt + \beta(X(t), u(t))\, dB^X(t), \\ dZ(t) &= dB(t) + dL(t), \end{aligned}$$

where $Z(t)$ is a Levy process independent of Brownian motion $B^X(t)$, and consequently independent of $X(t)$, under Q'. Here

$$B(t) = B^Z(t) - \int_0^t h(X(s))\, ds$$

is the Brownian motion part and

$$L(t) = \int_0^t \int_{\mathbb{R}_0} \xi N(dt, d\xi)$$

is the pure jump component associated to the Poisson random measure $N(dt, d\xi) = N_\lambda(dt, d\xi)$ with compensator given by $ds\nu(d\xi)$. Define the differential operator $A = A_{z,u}$ by

$$A\phi(x) = A_u\phi(x) = \alpha(x, u)\frac{d\phi}{dx}(x) + \frac{1}{2}\beta^2(x, u)\frac{d^2\phi}{dx^2}(x)$$

for $\phi \in C_0^2(\mathbb{R})$. Hence A_u is the generator of $X(t)$, if u is constant. Set

$$a(x, u) = \frac{1}{2}\beta^2(x, u). \tag{4.4.9}$$

Then the adjoint operator A^* of A is given by

$$A^*\phi = \frac{\partial}{\partial x}\left(a(x, u)\frac{d\phi}{dx}(x)\right) + \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial x}(x, u)\phi(x)\right) - \frac{\partial}{\partial x}\left(a(x, u)\phi(x)\right). \tag{4.4.10}$$

Let us assume that the initial condition $X(0)$ has a density p_0 and that there exists a unique strong solution $\Phi(t, x)$ of the following SPDE (Zakai equation)

$$d\Phi(t, x) = A^*\Phi(t, x)\, dt + h(x)\Phi(t, x)\, dB(t) + \int_{\mathbb{R}_0} [\lambda(t, x, \xi) - 1]\Phi(t, x)\, \widetilde{N}(dt, d\xi), \tag{4.4.11}$$

with

$$\Phi(0,x) = p_0(x)$$

Then $\Phi(t,x)$ is the *unnormalized conditional density* of $X(t)$ given $\mathcal{G}_t$ and satisfies:

$$E_{Q'}[\phi(X(t))N_t|\mathcal{G}_t] = \int_{\mathbb{R}} \phi(x)\Phi(t,x)dx \tag{4.4.12}$$

for all $\phi \in C_b(\mathbb{R})$.

Using Relations (4.4.12) and (4.4.11) under the change of measure Q' and the definition of the performance functional we obtain that

$$\begin{aligned} J(u) &= E\left[\int_0^T f(X(t),Z(t),u(t))dt + g(X(T),Z(T))\right] \\ &= E_{Q'}\left[\left\{\int_0^T f(X(t),Z(t),u(t))dt + g(X(T),Z(T))\right\} N_T\right] \\ &= E_{Q'}\left[\int_0^T f(X(t),Z(t),u(t))N_t dt + g(X(T),Z(T))N_T\right] \\ &= E_{Q'}\left[\int_0^T E_Q\left[f(X(t),Z(t),u(t))N_t \mid \mathcal{G}_t\right]dt + E_Q\left[g(X(T),Z(T))N_T \mid \mathcal{G}_t\right]\right] \\ &= E_{Q'}\left[\int_0^T \int_G f(x,Z(t),u(t))\Phi(t,x)dxdt + \int_G g(x,Z(T))\Phi(T,x)dx\right] \end{aligned}$$

The observation process $Z(t)$ is a Q'-Lévy process. Hence the partial observation control problem (4.4.8) reduces to a SPDE control problem under complete information. More precisely, our control problem is equivalent to the maximization problem

$$\sup_u E_{Q'}\left[\int_0^T \int_G f(x,Z(t),u(t))\Phi(t,x)\,dx\,dt + \int_G g(x,Z(T))\Phi(T,x)\,dx\right] \tag{4.4.13}$$

where Φ solves the SPDE (4.4.11). So the latter problem can be tackled by means of the maximum principle of Section 4.3.3.

For convenience, let us impose that a in Equation (4.4.9) is independent of the control, *i.e.*,

$$a(x,u) = a(x)\,.$$

Denote by $\mathcal{A}_1$ the set $u \in \mathcal{A}$ for which Equation (4.4.11) has a unique solution. Consider the general stochastic Hamiltonian (if existent) of the control problem (4.4.13) given by

$$H(t,x,\phi,\phi',u,\omega) = f(t,x,Z(t),u)\phi + p(t,x)b(t,x,\phi,\phi',u) + D_t p(t,x)h(x)\phi \\ + \int_{\mathbb{R}_0} D_{t,z}p(t,x)[\lambda(t,x,\xi)-1]\phi\,\nu(dz), \tag{4.4.14}$$

where

$$b(t,x,\phi,\phi',u) = \left(\frac{d^2a}{dx^2}(x) - \alpha(x,u)\right)\phi + \left(\frac{da}{dx}(x) - \alpha(x,u)\right)\phi'$$

and where $p(t,x)$ is defined as in Equation (4.3.21) with

$$g(x,\phi,\omega) = g(x,Z(T))\phi$$

and

$$L\psi(x) = a(x)\frac{d^2\psi}{dx^2}(x),\ \ \psi \in C_0^2(\mathbb{R}).$$

Assume that the conditions 1–5 in Section 4.3.2 are satisfied with respect to Problem (4.4.13) for controls $u \in \mathcal{A}_1$. Then by the general stochastic maximum principle (Theorem 4.3.3) applied to the partial information control problem (4.4.8) we find that

$$E\left[E_Q\left[\int_G \frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Phi},\widehat{\Phi}',\hat{u},\omega)\,dx\middle|\mathcal{G}_t\right]\right] = 0, \tag{4.4.15}$$

if $\hat{u} \in \mathcal{A}_1$ is an optimal control.

4.4.3 Optimal consumption with partial observation

Let us illustrate the maximum principle by inquiring into the following portfolio optimization problem with partial observation: Assume the wealth $X(t)$ at time t of an investor is modeled by

$$dX(t) = [\mu X(t) - u(t)]\ dt + \sigma X(t)\,dB^X(t), 0 \le t \le T,$$

where $m \in \mathbb{R}, \sigma \neq 0$ are constants, $B^X(t)$ a Brownian motion and $u(t) \ge 0$ the *consumption rate*. Suppose that the initial value $X(0)$ has the density $p_0(x)$ and that $u(t)$ is adapted to the filtration $\mathcal{G}_t$ generated by the observation process

$$dZ(t) = mX(t)\,dt + dB^Z(t) + \int_{\mathbb{R}_0} \xi\,N_\lambda(dt,d\xi),\ \ Z(0) = 0\,,$$

where m is a constant. As before we require that $(B^X(t), B^Z(t))$ is a Brownian motion independent of the initial value $X(0)$, and that N_λ is an integer valued random measure as described in System (4.4.7). Further, let us restrict the wealth process $X(t)$ to be bounded from below by a threshold $\zeta > 0$ for $0 \le t \le T$. The investor intends to maximize the expected utility of his consumption and terminal wealth according to the performance criterion

$$J(u) = E\left[\int_0^T \frac{u^r(t)}{r}\,dt + \theta X^r(T)\right],\ r \in (0,1),\ \ \theta > 0. \tag{4.4.16}$$

So we are dealing with a partial observation control problem of the type (4.4.8) (for $G = [\zeta, \infty)$). Here, the operator A in (4.4.11) has the form

$$A\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) + [\mu x - u]\phi'(x), \tag{4.4.17}$$

(where $'$ denotes the differentiation with respect to x) and hence

$$A^*\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) - [\mu x - u]\phi'(x) - \mu\phi(x)\,. \tag{4.4.18}$$

Therefore the Zakai equation becomes

$$\begin{aligned} d\Phi(t,x) &= \left[\frac{1}{2}\sigma^2 x^2 \Phi''(t,x) - [\mu x - u]\,\Phi'(t,x) - \mu\Phi(t,x)\right]dt + x\Phi(t,x)\,dB(t) \\ &\quad + \int_{\mathbb{R}_0} [\lambda(t,x,\xi) - 1]\Phi(t,x)\,\widetilde{N}(dt,d\xi), \\ \Phi(0,x) &= p_0(x)\,,\ \ x > \zeta, \\ \Phi(t,0) &= 0\,,\ \ t \in (0,T), \end{aligned} \tag{4.4.19}$$

where $\widetilde{N}(dt,d\xi)$ is a compensated Poisson random measure under the corresponding measure Q'. Since $L\psi = \frac{1}{2}\sigma^2 x^2 \frac{d^2\psi}{dx^2}(x)$ is uniformly elliptic for $x > \zeta$ there exists a unique strong solution of SPDE (4.4.19). Further one verifies that condition 4 of Section 4.3.2 is fulfilled. See [10]. So our problem amounts to finding an admissible $\hat{u} \in \mathcal{A}_1$ such that

$$J_1(\hat{u}) = \sup_{u \in \mathcal{A}_1} J_1(u), \tag{4.4.20}$$

where

$$J_1(u) = E_{Q'}\left[\int_0^T \int_G \frac{u^r(t)}{r}\Phi(t,x)\,dx\,dt + \int_V \theta x^r \Phi(T,x)\,dx\right].$$

Our assumptions imply that condition 1 of Section 4.3.2 holds. Further, by exploiting the linearity of the SPDE (4.4.19) one shows as in [11] that also the conditions 2–4 in Section 4.3.2 are fulfilled. Using the notation of (4.4.14) we see that

$$
\begin{aligned}
f(x, Z(t), u(t)) =& \frac{u^r(t)}{r}, \\
g(x, Z(T)) =& \theta x^r, \\
L^*\Phi(t,x) =& \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\Phi(t,x), \\
K(t,x) =& \theta x^r(T) + \int_t^T \frac{u^r(s)}{r} ds, \\
H_0(t,x,\phi,\phi',u) =& \left[(-\mu x - u)\phi'(t,x) - \mu\phi(t,x)\right] K(t,x) + D_t K(t,x) x\phi \\
&+ \int_{\mathbb{R}_0} D_{t,z}K(t,x)[\lambda(t,x,\xi) - 1]\phi\,\nu(d\xi) \\
I(t,s,x) =& \left(-\mu K(s,x) + D_s K(s,x)x + \int_{\mathbb{R}_0} D_{s,z}K_1(s,x)[\lambda(s,x,\xi)-1]\nu(d\xi)\right) \times \\
& Z(t,s,\widehat{\varphi}_{s,t}(x)), \\
I_1(t,s,x) =& \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} K(s,x) \times Z(t,s,\widehat{\varphi}_{s,t}(x)), \\
I_2(t,s,x) =& \frac{\partial}{\partial x}\left[(-\mu x - u)K(s,x)\right] Z(t,s,\widehat{\varphi}_{s,t}(x)), \\
Z(t,s,x) =& \exp\left\{\int_t^s F_{n+1}\left(x, \circ\widehat{dr}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
F_{n+1}(x,dt) =& \mu dt + x dB(t) + \int_{\mathbb{R}_0} [\lambda(t,x,\xi) - 1]\,\widetilde{N}(dt,d\xi), \\
F_i(x,dt) = F(x,dt) =& -[\mu x - u]\,dt, \quad i = 1, \cdots, n.
\end{aligned}
$$

In this case we have $\varphi_{s,t}(x) = x + \int_t^s G(\varphi_{s,r}(x), \circ dr)$, where $G(x,t) = X(x,t) + F(t,x)$. Then

$$
p(t,x) = K(t,x) + \int_t^T \Big(I_1(r,s,x) + I_2(r,s,x) + I_3(r,s,x)\Big) dr. \tag{4.4.21}
$$

So the Hamiltonian (if it exists) becomes

$$
\begin{aligned}
H(t,x,\phi,\phi',u) \quad = \quad & \frac{u^r(t)}{r}\phi + \left[(-\mu x - u)\phi'(t,x) - \mu\phi(t,x)\right] p(t,x) \\
& + D_t p(t,x) x\phi + \int_{\mathbb{R}_0} D_{t,z}p(t,x)[\lambda(t,x,\xi) - 1]\phi\,\nu(d\xi).
\end{aligned}
$$

Hence, if $\hat{u}$ is an optimal control of the problem (4.4.8) such that the Hamiltonian is well-defined, then it follows from Relations (4.4.15) and (4.4.12) that

$$\begin{aligned} 0 &= E_{Q'}\left[E_Q\left[\int_G \frac{\partial}{\partial u} H(t,x,\widehat{\Phi},\widehat{\Phi}',\widehat{u})\,dx\right]\Big|\,\mathcal{G}_t\right] \\ &= E_{Q'}\left[E_Q\left[\int_G \left\{u^{r-1}(t)\widehat{\Phi}(t,x)+\widehat{\Phi}'(t,x)\widehat{p}(t,x)\right\}dx\right]\Big|\,\mathcal{G}_t\right]. \end{aligned}$$

Thus we get

$$u^{r-1}(t) = -\frac{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}'(t,x)\widehat{p}(t,x)\,dx\right]\Big|\,\mathcal{G}_t\right]}{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}(t,x)dx\right]\Big|\,\mathcal{G}_t\right]}.$$

Using integration by parts and (4.4.12) implies that

$$\begin{aligned} u^*(t) &= \left(-\frac{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}'(t,x)\widehat{p}(t,x)\,dx\right]\Big|\,\mathcal{G}_t\right]}{E_{Q'}\left[E_Q\left[\int_G \widehat{\Phi}(t,x)\,dx\right]\Big|\,\mathcal{G}_t\right]}\right)^{\frac{1}{r-1}} \\ &= \left(\frac{E_Q\left[E_{Q'}\left[\int_G \widehat{\Phi}(t,x)\widehat{p}'(t,x)\,dx\Big|\,\mathcal{G}_t\right]\right]}{E_Q\left[E_{Q'}\left[\int_G \widehat{\Phi}(t,x)dx\Big|\,\mathcal{G}_t\right]\right]}\right)^{\frac{1}{r-1}} \\ &= \left(E_Q\left[\frac{E_{Q'}\left[\int_G \widehat{\Phi}(t,x)\widehat{p}'(t,x)\,dx\Big|\,\mathcal{G}_t\right]}{E_{Q'}\left[\int_G \widehat{\Phi}(t,x)\,dx\Big|\,\mathcal{G}_t\right]}\right]\right)^{\frac{1}{r-1}} \\ &= \left(E_Q\left[\frac{E_Q\left[\widehat{p}'(t,X(t))N_t\,|\,\mathcal{G}_t\right]}{E_Q\left[N_t\,|\,\mathcal{G}_t\right]}\right]\right)^{\frac{1}{r-1}} \\ &= E_Q\left[E\left[\widehat{p}'(t,X(t))\,|\,\mathcal{G}_t\right]\right]^{\frac{1}{r-1}}. \end{aligned}$$

So if $u^*(t)$ maximizes (4.4.16) then $u^*(t)$ necessarily satisfies

$$\begin{aligned} u^*(t) &= E_Q\left[E\left[\widehat{p}'(t,X(t))\,|\,\mathcal{G}_t\right]\right]^{\frac{1}{r-1}} \\ &= E\left[E_Q\left[\widehat{p}'(t,X(t))\right]\,|\,\mathcal{G}_t\right]^{\frac{1}{r-1}}. \end{aligned} \tag{4.4.22}$$

Theorem 4.4.2 *Suppose that $\widehat{u} \in \mathcal{A}_{\mathcal{G}_t}$ is an optimal portfolio for the partial observation control problem*

$$\sup_{u\in\mathcal{A}_{\mathcal{G}_t}} E\left[\int_0^T \frac{u^r(t)}{r}dt + \theta X^r(T)\right],\ r\in(0,1),\ \ \theta>0,$$

with the wealth and the observation processes $X(t)$ and $Z(t)$ at time t given by

$$dX(t) = [\mu X(t) - u(t)]\, dt + \sigma X(t) dB^X(t), 0 \leq t \leq T,$$
$$dZ(t) = mX(t)dt + dB^Z(t) + \int_{\mathbb{R}_0} \xi N_\lambda(dt, d\xi).$$

Then

$$u^*(t) \quad = \quad E\left[E_Q\left[\widehat{p}'(t, X(t))\right] \big| \, \mathcal{G}_t\right]^{\frac{1}{r-1}}. \tag{4.4.23}$$

Remark 4.4.3 *Note that the last example cannot be treated within the framework of [88], since the random measure $N_\lambda(dt, d\xi)$ is not necessarily a functional of a Lévy process. Let us also mention that the SPDE maximum principle studied in [102] does not apply to Example 4.4.3. This is due to the fact the corresponding Hamiltonian in [102] fails to be concave.*

Chapter 5

Uniqueness of decompositions of skorohod-semimartingales

5.1 Introduction

Let $X(t) = X(t, \omega);\ \ t \in [0, T],\ \ \omega \in \Omega$ be a stochastic process of the form

$$X(t) = \zeta + \int_0^t \alpha(s)\, ds + \int_0^t \beta(s)\, \delta B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\, \widetilde{N}(dz, \delta s), \tag{5.1.1}$$

where ζ is a random variable, α is an integrable measurable process, $\beta(s)$ and $\gamma(s, z)$ are measurable processes such that $\beta\chi_{[0,t]}(\cdot)$ and $\gamma\chi_{[0,t]}(\cdot)$ are Skorohod integrable with respect to B_s and $\widetilde{N}(dz, ds)$ respectively, and the stochastic integrals are interpreted as Skorohod integrals. Here $B(s) = B(s, \omega)$ and $\widetilde{N}(dz, ds) = \widetilde{N}(dz, ds, \omega)$ is a Brownian motion and and independent Poisson random measure, respectively. Such processes are called *Skorohod-semimartingales*. The purpose of this chapter is to prove that the decomposition (5.1.1) is *unique*, in the sense that if $X(t) = 0$ for all $t \in [0, T]$ then

$$\zeta = \alpha(\cdot) = \beta(\cdot) = \gamma(\cdot, \cdot) = 0$$

(see Theorem 5.3.5).

This is an extension of a result by Nualart and Pardoux [95], who proved the uniqueness of such a decomposition in the Brownian case (i.e., $\widetilde{N} = 0$) and with additional assumption on β.

We obtain Theorem 5.3.5 as a special case of a more general decomposition uniqueness theorem for an extended class of Skorohod integral processes with values in in the space of generalized random variables $\mathcal{G}^*$. See Theorem 5.3.3. Our proof uses white noise theory of Lévy processes. In Section 5.2 we give a brief review of this theory and in Section 5.3 we prove our main theorem.

Our decomposition uniqueness is motivated by applications in anticipative stochastic control theory, including insider trading in finance. See Chapters 6 and 7.

5.2 A concise review of Malliavin calculus and white noise analysis

This Section provides the mathematical framework of our chapter which will be used in Section 5.3. Here we want to briefly recall some basic facts from both Malliavin calculus and white noise theory. See [31, 84] and [94] for more information on Malliavin calculus. As for white noise theory we refer the reader to [30, 64, 65, 77, 81, 97] and [101].

In the sequel denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space on $\mathbb{R}$ and by $\mathcal{S}'(\mathbb{R})$ its topological dual. Then in virtue of the celebrated Bochner-Minlos theorem there exists a unique probability measure μ on the Borel sets of the conuclear space $\mathcal{S}'(\mathbb{R})$ (i.e. $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$)such that

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle\omega,\phi\rangle}\mu(d\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}} \tag{5.2.1}$$

holds for all $\phi \in \mathcal{S}(\mathbb{R})$, where $\langle\omega, \phi\rangle$ is the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $\phi \in \mathcal{S}(\mathbb{R})$. The measure μ is called the *Gaussian white noise measure* and the triple

$$\big(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu\big) \tag{5.2.2}$$

is referred to as (Gaussian) *white noise probability space*.

Consider the Doleans-Dade exponential

$$\widetilde{e}(\phi,\omega)=e^{\langle\omega,\phi\rangle-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}, \tag{5.2.3}$$

which is holomorphic in ϕ around zero. Hence there exist *generalized Hermite polynomials* $H_n(\omega)\in\left((\mathcal{S}(\mathbb{R}))^{\widehat{\otimes}n}\right)^{'}$ (i.e. dual of $n-$th completed symmetric tensor product of $\mathcal{S}(\mathbb{R})$) such that

$$\widetilde{e}(\phi,\omega)=\sum_{n\geq 0}\frac{1}{n!}\left\langle H_n(\omega),\phi^{\otimes n}\right\rangle \tag{5.2.4}$$

for all ϕ in a neighborhood of zero in $\mathcal{S}(\mathbb{R})$. One verifies that the orthogonality relation

$$\int_{\mathcal{S}^{'}(\mathbb{R})}\left\langle H_n(\omega),\phi^{(n)}\right\rangle\left\langle H_n(\omega),\psi^{(n)}\right\rangle\mu(d\omega)=\begin{cases} n!\left(\phi^{(n)},\psi^{(n)}\right)_{L^2(\mathbb{R}^n)}, & m=n\\ 0 & m\neq n\end{cases} \tag{5.2.5}$$

is fulfilled for all $\phi^{(n)}\in(\mathcal{S}(\mathbb{R}))^{\widehat{\otimes}n}$, $\psi^{(m)}\in(\mathcal{S}(\mathbb{R}))^{\widehat{\otimes}m}$. From this relation we obtain that the mappings $(\phi^{(n)}\longmapsto\langle H_n(\omega),\phi^{(n)}\rangle)$ from $(\mathcal{S}(\mathbb{R}))^{\widehat{\otimes}n}$ to $L^2(\mu)$ have unique continuous extensions

$$I_n:\widehat{L}^2(\mathbb{R}^n)\longrightarrow L^2(\mu),$$

where $\widehat{L}^2(\mathbb{R}^n)$ is the space of square integrable symmetric functions. It turns out that $L^2(\mu)$ admits the orthogonal decomposition

$$L^2(\mu)=\sum_{n\geq 0}\oplus I_n(\widehat{L}^2(\mathbb{R}^n)). \tag{5.2.6}$$

Note that that $I_n(\phi^{(n)})$ can be considered an $n-$fold iterated Itô integral $\phi^{(n)}\in\widehat{L}^2(\mathbb{R}^n)$ with respect to a Brownian motion $B(t)$ on our white noise probability space. In particular

$$I_1(\varphi\chi_{[0,T]})=\left\langle H_1(\omega),\varphi\chi_{[0,T]}\right\rangle=\int_0^T\varphi(t)\,dB(t),\ \varphi\in L^2(\mathbb{R}). \tag{5.2.7}$$

Let $F\in L^2(\mu)$. It follows from (5.2.6) that

$$F=\sum_{n\geq 0}\left\langle H_n(\cdot),\phi^{(n)}\right\rangle \tag{5.2.8}$$

for unique $\phi^{(n)}\in\widehat{L}^2(\mathbb{R}^n)$. Further require that

$$\sum_{n\geq 1}nn!\left\|\phi^{(n)}\right\|^2_{\widehat{L}^2(\mathbb{R}^n)}<\infty. \tag{5.2.9}$$

Then the *Malliavin derivative* D_t of F in the direction $B(t)$ is defined by

$$D_t F = \sum_{n\geq 1} n \left\langle H_{n-1}(\cdot), \phi^{(n)}(\cdot, t) \right\rangle .$$

Denote by $\mathbb{D}_{1,2}$ the stochastic Sobolev space which consists of all $F \in L^2(\mu)$ such that (5.2.9) is satisfied. The Malliavin derivative $D.$ is a linear operator from $\mathbb{D}_{1,2}$ to $L^2(\lambda\times\mu)$ (λ Lebesgue measure). The adjoint operator δ of $D.$ as a mapping from $Dom(\delta) \subset L^2(\lambda\times\mu)$ to $L^2(\mu)$ is called *Skorohod integral*. The Skorohod integral can be regarded as a generalization of the Itô integral and one also uses the notation

$$\delta(u\chi_{[0,T]}) = \int_0^T u(t)\, \delta B(t) \tag{5.2.10}$$

for Skorohod integrable (not necessarily adapted) processes $u \in L^2(\lambda\times\mu)$ (i.e. $u \in Dom(\delta)$).

In view of Section 5.3 we give the construction of the dual pair of spaces $((\mathcal{S}), (\mathcal{S})^*)$, which was first introduced by Hida [63] in white noise analysis: Consider the self-adjoint operator

$$A = 1 + t^2 - \frac{d^2}{dt^2}$$

on $\mathcal{S}(\mathbb{R}) \subset L^2(\mu)$. Then the *Hida test function space* $(\mathcal{S})$ is the space of all square integrable functionals f with chaos expansion

$$f = \sum_{n\geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle$$

such that

$$\|f\|_{0,p}^2 := \sum_{n\geq 0} n! \left\| (A^{\otimes n})^p \phi^{(n)} \right\|_{L^2(\mathbb{R}^n)}^2 < \infty \tag{5.2.11}$$

for all $p \geq 0$. We mention that $(\mathcal{S})$ is a nuclear Fréchet algebra, that is a countably Hilbertian nuclear space with respect to the seminorms $\|\cdot\|_{0,p}\,, p \geq 0$ and an algebra with respect to ordinary multiplication of functions. The topological dual $(\mathcal{S})^*$ of $(\mathcal{S})$ is the *Hida distribution space*.

Another useful dual pairing which was studied in [115] is $(\mathcal{G}, \mathcal{G}^*)$. Denote by N the Ornstein-Uhlenbeck operator (or number operator). The space of smooth random variables $\mathcal{G}$ is the space of all square integrable functionals f such that

$$\|f\|_q^2 := \left\|e^{qN} f\right\|_{L^2(\mu)}^2 < \infty \tag{5.2.12}$$

for all $q \geq 0$. The dual of $\mathcal{G}$ denoted by $\mathcal{G}^*$ is called space of generalized random variables.

We have the following interrelations of the above spaces in the sense of inclusions:

$$(\mathcal{S}) \hookrightarrow \mathcal{G} \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathcal{G}^* \hookrightarrow (\mathcal{S})^*. \tag{5.2.13}$$

In what follows we define the *white noise differential operator*

$$\partial_t = D_t|_{(\mathcal{S})} \tag{5.2.14}$$

as the restriction of the Malliavin derivative to the Hida test function space. It can be shown that ∂_t maps $(\mathcal{S})$ into itself, continuously. We denote by $\partial_t^* : (\mathcal{S})^* \longrightarrow (\mathcal{S})^*$ the adjoint operator of ∂_t. We mention the following crucial link between ∂_t^* and δ:

$$\int_0^T u(t)\, \delta B(t) = \int_0^T \partial_t^* u(t)\, dt, \tag{5.2.15}$$

where the integral on the right hand side is defined on $(\mathcal{S})^*$ in the sense of Bochner. In fact, the operator ∂_t^* can be represented as Wick multiplication with Brownian white noise $\dot{B}(t) = \frac{dB(t)}{dt}$, *i.e.*,

$$\partial_t^* u = u \diamond \dot{B}(t), \tag{5.2.16}$$

where $\diamond$ represents the Wick or Wick-Grassmann product. See [65].

We now shortly elaborate a white noise framework for pure jump Lévy processes: Let A be a positive self-adjoint operator on $L^2(X, \pi)$, where $X = \mathbb{R} \times \mathbb{R}_0$ ($\mathbb{R}_0 := \mathbb{R} \backslash \{0\}$)and $\pi = \lambda \times v$. Here ν is the Lévy measure of a (square integrable) Lévy process η_t. Assume that A^{-p} is of Hilbert-Schmidt type for some $p > 0$. Then denote by $\mathcal{S}(X)$ the standard countably Hilbert space constructed from A. See *e.g.*, [97] or [64]. Let $\mathcal{S}'(X)$ be the dual of $\mathcal{S}(X)$. In what follows we impose the following conditions on $\mathcal{S}(X)$:

(i) Each $f \in \mathcal{S}(X)$ has a ($\pi-$a.e.) continuous version.

(ii) The evaluation functional $\delta_t : \mathcal{S}(X) \longrightarrow \mathbb{R}$; $f \longmapsto f(t)$ belongs to $\mathcal{S}'(X)$ for all t.

(iii) The mapping $(t \longmapsto \delta_t)$ from X to $\mathcal{S}'(X)$ is continuous.

Then just as in the Gaussian case we obtain by the Bochner-Minlos theorem the *(pure jump) Lévy noise measure* τ on $\mathcal{B}(\mathcal{S}'(X))$ which satisfies

$$\int_{\mathcal{S}'(X)} e^{i\langle \omega,\phi\rangle}\, \tau(d\omega) = \exp(\int_X (e^{i\phi} - 1)\, \pi(dx)) \tag{5.2.17}$$

for all $\phi \in \mathcal{S}(X)$.

We remark that analogously to the Gaussian case each $F \in L^2(\tau)$ has the unique chaos decomposition

$$F = \sum_{n\geq 0} \left\langle C_n(\cdot), \phi^{(n)} \right\rangle \tag{5.2.18}$$

for $\phi^{(n)} \in \widehat{L}^2(X,\pi)$ (space of square integrable symmetric functions on X). Here $C_n(\omega) \in \left((\mathcal{S}(X))^{\widehat{\otimes} n}\right)'$ are *generalized Charlier polynomials*. Note that $\left\langle C_n(\cdot), \phi^{(n)} \right\rangle$ can be viewed as the $n-$fold iterated Itô integral of $\phi^{(n)}$ with respect to the compensated Poisson random measure $\widetilde{N}(dz,dt) := N(dz,dt) - v(dz)dt$ associated with the pure jump Lévy process

$$\eta_t = \left\langle C_1(\cdot), z\chi_{[0,t]} \right\rangle = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(dz,ds). \tag{5.2.19}$$

Similarly to the Gaussian case we define the *(pure jump) Lévy-Hida test function space* $(\mathcal{S})_\tau$ as the space of all $f = \sum_{n\geq 0} \left\langle C_n(\cdot), \phi^{(n)} \right\rangle \in L^2(\tau)$ such that

$$\|f\|^2_{0,\pi,p} := \sum_{n\geq 0} n! \left\| (A^{\otimes n})^p \phi^{(n)} \right\|^2_{L^2(X^n,\pi^n)} < \infty \tag{5.2.20}$$

for $p \geq 0$.

Suppressing the notational dependence on τ we mention that the spaces $(S)^*$, $\mathcal{G}$, $\mathcal{G}^*$ and the operators $D_{t,z}$, $\partial_{t,z}$, $\partial^*_{t,z}$ can be introduced in the same way as in the Gaussian case. For example Equation (5.2.15) takes the form

$$\int_0^T \int_{\mathbb{R}_0} u(t,z)\, \widetilde{N}(dz,\delta t) = \int_0^T \int_{\mathbb{R}_0} \partial^*_{t,z} u(t,z)\, \nu(dz)\, dt, \tag{5.2.21}$$

where the left hand side denotes the Skorohod integral of $u(\cdot,\cdot)$ with respect to $\widetilde{N}(\cdot,\cdot)$, for Skorohod integrable processes $u \in L^2(\tau \times \pi)$. See *e.g.,* [81] or [66]. Similar to the Brownian motion case, (see (5.2.16)), one can prove the representation

$$\partial^*_{t,z} u = u \diamond \dot{\widetilde{N}}(z,t), \tag{5.2.22}$$

where $\dot{\widetilde{N}}(z,t) = \frac{\widetilde{N}(dz,dt)}{\nu(dz)\times dt}$ is the white noise of $\widetilde{N}$. See [65] and [101].

In the sequel we choose the white noise probability space

$$(\Omega, \mathcal{F}, P) = \big(\mathcal{S}'(\mathbb{R}) \times \mathcal{S}'(X), \mathcal{B}(\mathcal{S}'(\mathbb{R})) \otimes \mathcal{B}(\mathcal{S}'(X))), \mu \times \tau\big) \tag{5.2.23}$$

and we suppose that the above concepts are defined with respect to this stochastic basis.

5.3 Main results

In this Section we aim at establishing a uniqueness result for decompositions of Skorohod-semimartingales. Let us clarify the latter notion in the following:

Definition 5.3.1 (Skorohod-semimartingale) *Assume that a process $X_t, 0 \le t \le T$ on the probability space (5.2.23) has the representation*

$$X_t = \zeta + \int_0^t \alpha(s)\,ds + \int_0^t \beta(s)\,\delta B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\,\widetilde{N}(dz,\delta s) \tag{5.3.1}$$

for all t. Here we require that $\beta\chi_{[0,t]}(\cdot)$ resp. $\gamma\chi_{[0,t]}(\cdot)$ are Skorohod integrable with respect to B_t respectively $\widetilde{N}(dz,dt)$ for all $0 \le t \le T$. Further ζ is a random variable and α a process such that

$$\int_0^T |\alpha(s)|\,ds < \infty \text{ } P\text{-a.e.}$$

Then X_t is called a Skorohod-semimartingale.

Obviously, the Skorohod-semimartingale is a generalization of semimartingales of the type

$$X_t = \zeta + \int_0^t \alpha(s)\,ds + \int_0^t \beta(s)\,dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\,\widetilde{N}(dz,ds),$$

where β, γ are predictable Itô integrable processes w.r.t. to some filtration $\mathcal{F}_t$ and where ζ is $\mathcal{F}_0$-measurable. The Skorohod-semimartingale also extends the concepts of the Skorohod integral processes

$$\int_0^t \beta(s)\,\delta B(s) \text{ and } \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\,\widetilde{N}(dz,\delta s),\;\; 0 \le t \le T.$$

Further it is worth mentioning that the increments of the Skorohod integral process $Y(t) := \int_0^t \beta(s)\,\delta B(s)$ satisfy the following orthogonality relation:

$$E\left[Y(t) - Y(s)\,\middle|\mathcal{F}_{[s,t]^c}\right] = 0,\;\; s < t,$$

where $\mathcal{F}_{[s,t]^c}$ is the $\sigma-$algebra generated by the increments of the Brownian motion in the complement of the interval $[s,t]$. See [94] or [108]. We point out that Skorohod integral processes may exhibit very rough path properties. For example consider the Skorohod SDE

$$Y(t) = \eta + \int_0^t Y(s)\,\delta B(s),\;\; \eta = sign(B(1)),\;\; 0 \le t \le 1.$$

It turns out that the Skorohod integral process $X(t) = Y(t) - \eta$ possesses discontinuities of the second kind. See [19]. Another surprising example is the existence of continuous Skorohod integral processes $\int_0^t \beta(s)\,\delta B(s)$ with a quadratic variation, which is essentially bigger than the expected process $\int_0^t \beta^2(s)\,ds$. See [9].

In order to prove the uniqueness of Skorohod-semimartingale decompositions we need the following result which is of independent interest:

Theorem 5.3.2 *Let ∂_t^* and $\partial_{t,z}^*$ be the white noise operators of Section 5.2. Then*
(i) ∂_t^* maps $\mathcal{G}^*\backslash\{0\}$ into $(S)^*\backslash\mathcal{G}^*$.
(ii) The operator

$$(u \longmapsto \int_{\mathbb{R}_0} \partial_{t,z}^* u(t,z)\,\nu(dz))$$

maps $\mathcal{G}^*\backslash\{0\}$ into $(S)^*\backslash\mathcal{G}^*$.
(iii)

$$\partial_t^* + \int_{\mathbb{R}_0} \partial_{t,z}^*(\cdot)\,\nu(dz) : \mathcal{G}^*\backslash\{0\} \times \mathcal{G}^*\backslash\{0\} \longrightarrow (S)^*\backslash\mathcal{G}^*.$$

Proof. Without loss of generality it suffices to show that

$$\partial_t^* \text{ maps } \mathcal{G}^*\backslash\{0\} \text{ into } (S)^*\backslash\mathcal{G}^*.$$

For this purpose consider a $F \in \mathcal{G}^*\backslash\{0\}$ with formal chaos expansion

$$F = \sum_{n\geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle.$$

where $\phi^{(n)} \in \widehat{L}^2(\mathbb{R}^n)$. One checks that $\langle H_n(\cdot), \phi^{(n)} \rangle$ can be written as

$$\left\langle H_n(\cdot), \phi^{(n)} \right\rangle = \sum_{|\alpha|=n} c_\alpha \left\langle H_n(\cdot), \xi^{\widehat{\otimes}\alpha} \right\rangle$$

where

$$c_\alpha = \left(\phi^{(n)}, \xi^{\widehat{\otimes}\alpha}\right)_{L^2(\mathbb{R}^n)} \tag{5.3.2}$$

with

$$\xi^{\widehat{\otimes}\alpha} = \xi_1^{\widehat{\otimes}\alpha_1} \widehat{\otimes} ... \widehat{\otimes} \xi_k^{\widehat{\otimes}\alpha_k}$$

for Hermite functions $\xi_k, k \geq 1$ and multiindices $\alpha = (\alpha_1, ..., \alpha_k), \alpha_i \in \mathbb{N}_0$. Here $|\alpha| := \sum_{i=1}^k \alpha_i$. By Equation (5.2.5) we know that

$$\infty > \left\| \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \right\|_{L^2(\mu)}^2 = \sum_{|\alpha|=n} \alpha! c_\alpha^2.$$

Assume that

$$\partial_t^* F \in \mathcal{G}^*. \tag{5.3.3}$$

Then $\partial_t^* F$ has a formal chaos expansion

$$\partial_t^* F = \sum_{n\geq 0} \left\langle H_n(\cdot), \psi^{(n)} \right\rangle.$$

Thus it follows from of the definition of ∂_t^* (see Section 5.2) that

$$\infty > \left\| \left\langle H_n(\cdot), \psi^{(n)} \right\rangle \right\|_{L^2(\mu)}^2 = \sum_{|\gamma|=n} \gamma! \left(\sum_{\alpha+\varepsilon^{(m)}=\gamma} c_\alpha \cdot \xi_m(t) \right)^2, \tag{5.3.4}$$

where the multiindex $\varepsilon^{(m)}$ is defined as

$$\varepsilon^{(m)}(i) = \begin{cases} 1, & i = m \\ 0 & \text{else} \end{cases}.$$

On the one hand we observe that

$$\sum_{|\gamma|=n}\gamma!\left(\sum_{\alpha+\varepsilon^{(m)}=\gamma}c_\alpha\cdot\xi_m(t)\right)^2$$
$$=\sum_{k=1}^{n}\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k\\ a_1+\dots+a_k=n}}a_1!\cdot\dots\cdot a_k!\sum_{i_1>i_2>\dots>i_k}\left(\sum_{m\geq 1}c_{a_1\varepsilon^{(i_1)}+\dots+a_k\varepsilon^{(i_k)}-\varepsilon^{(m)}}\cdot\xi_m(t)\right)^2,$$

where coefficients are set equal to zero, if not defined. So we get that

$$\left\|\left\langle H_n(\cdot),\psi^{(n)}\right\rangle\right\|^2_{L^2(\mu)}$$
$$=\sum_{k=1}^{n}\sum_{\substack{(a_1,\dots,a_k)\in\mathbb{N}^k\\ a_1+\dots+a_k=n}}a_1!\cdot\dots\cdot a_k!a_1!\cdot\dots\cdot a_k!\sum_{i_1>i_2>\dots>i_k}\left(\sum_{j=1}^{k}c_{a_1\varepsilon^{(i_1)}+\dots+a_k\varepsilon^{(i_k)}-\varepsilon^{(i_j)}}\cdot\xi_{i_j}(t)\right)^2.$$
(5.3.5)

By our assumption there exist $n^*\in\mathbb{N}_0$, $a_2^*,\dots,a_{k_0}^*\in\mathbb{N}$, pairwise unequal $i_2^*,\dots,i_{k_0}^*$, $k_0\leq n^*-1$ such that

$$a_2^*+\dots+a_{k_0}^*=n^*-1$$

and

$$c_{a_2^*\varepsilon^{(i_2^*)}+\dots+a_{k_0}^*\varepsilon^{(i_{k_0}^*)}}\neq 0. \tag{5.3.6}$$

On the other hand it follows from Equation (5.3.5) for $n=n^*$ that

$$\left\|\left\langle H_n(\cdot),\psi^{(n)}\right\rangle\right\|^2_{L^2(\mu)}$$
$$\geq a_2^*!\cdots a_{k_0}^*!\sum_{i_1^*>\max(i_2^*,\cdots,i_{k_0}^*)}\left(\sum_{j=1}^{k_0}c_{\varepsilon^{(i_1^*)}+a_2^*\varepsilon^{(i_2^*)}+\dots+a_{k_0}^*\varepsilon^{(i_{k_0}^*)}-\varepsilon^{(i_j^*)}}\cdot\xi_{i_j^*}(t)\right)^2$$
$$=a_2^*!\cdots a_{k_0}^*!\sum_{i_1^*>\max(i_2^*,\cdots,i_{k_0}^*)}\sum_{j_1,j_2=1}^{k_0}\left(c_{\varepsilon^{(i_1^*)}+a_2^*\varepsilon^{(i_2^*)}+\dots+a_{k_0}^*\varepsilon^{(i_{k_0}^*)}-\varepsilon^{(i_{j_1}^*)}}\cdot\xi_{i_{j_1}^*}(t)\right.$$
$$\left.\cdot c_{\varepsilon^{(i_1^*)}+a_2^*\varepsilon^{(i_2^*)}+\dots+a_{k_0}^*\varepsilon^{(i_{k_0}^*)}-\varepsilon^{(i_{j_2}^*)}}\cdot\xi_{i_{j_2}^*}(t)\right)$$
$$=:A_1+A_2+A_3, \tag{5.3.7}$$

where

$$A_1 = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} (c_{a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}})^2 \cdot (\xi_{i_1^*}(t))^2,$$

$$A_2 = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} \sum_{j=2}^{k_0} (c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_j^*}(t))^2,$$

$$A_3 = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2 = 1}}^{K_0} \Big(c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t)$$
$$\cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \Big).$$

The first term A_1 in (5.3.7) diverges to ∞ because of (5.3.6). The second term is positive. The last term A_3 can be written as

$$\begin{aligned}
& A_3 \\
&= a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} 2 \sum_{j=2}^{k_0} \Big(c_{a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}} \cdot \xi_{i_1^*}(t) \\
&\quad \cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_j^*}(t) \Big) \\
&\quad + a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2 = 1}}^{K_0} \Big(c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t) \\
&\quad \cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \Big) \\
&=: A_{3,1} + A_{3,2},
\end{aligned} \tag{5.3.8}$$

where

$$A_{3,1} = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} 2 \sum_{j=2}^{k_0} \Big(c_{a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}} \cdot \xi_{i_j^*}(t)$$
$$\cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_1^*}(t) \Big),$$

$$A_{3,2} = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \cdots, i_{k_0}^*)} \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2 = 1}}^{K_0} \Big(c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t)$$
$$\cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \cdots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \Big).$$

By means of relation (5.3.2) and the properties of basis elements one can show that the term $A_{3,1}$ in (5.3.8) converges $t-$a.e. The other term $A_{3,2}$ with Hermite functions which do not depend on the summation index converges by assumption, too.

We conclude that

$$\left\| \left\langle H_{n^*}(\cdot), \psi^{(n^*)} \right\rangle \right\|^2_{L^2(\mu)} = \infty,$$

which contradicts (5.3.4) and it contradicts (5.3.3), too.

It follows that

$$\partial_t^* \text{ maps } \mathcal{G}^* \backslash \{0\} \text{ into } (S)^* \backslash \mathcal{G}^*.$$

The proofs of (ii) and (iii) are similar. ■

We are now ready to prove the main result of this chapter:

Theorem 5.3.3 ***[Decomposition uniqueness for general Skorohod processes]*** *Consider a stochastic process X_t of the form*

$$X(t) = \zeta + \int_0^t \alpha(s)\, ds + \int_0^t \beta(s)\, \delta B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\, \widetilde{N}(dz, \delta s),$$

where $\beta\chi_{[0,t]}$, $\gamma\chi_{[0,t]}$ are Skorohod integrable for all t. Further require that $\alpha(t)$ is in $\mathcal{G}^*$ a.e. and that α is Bochner-integrable w.r.t. $\mathcal{G}^*$ on the interval $[0,T]$. Suppose that

$$X(t) = 0 \textit{ for all } 0 \leq t \leq T.$$

Then

$$\zeta = 0,\ \alpha = 0,\ \beta = 0,\ \gamma = 0 \ \textit{a.e.}$$

Proof. Because of Equations (5.2.15) and (5.2.21) it follows that

$$\begin{aligned} X(t) =& \zeta + \int_0^t \alpha(s)\, ds + \int_0^t \partial_s^* \beta(s)\, ds + \int_0^t \int_{\mathbb{R}_0} \partial_{s,z}^* \gamma(s,z)\, \nu(dz)\, ds \\ =& 0, \quad 0 \leq t \leq T. \end{aligned}$$

Thus

$$\alpha(t) + \partial_t^* \beta(t) + \int_{\mathbb{R}_0} \partial_{t,z}^* \gamma(t,z)\, \nu(dz) = 0 \text{ a.e.}$$

Therefore

$$\partial_t^* \beta(t) + \int_{\mathbb{R}_0} \partial_{t,z}^* \gamma(t,z)\, \nu(dz) \in \mathcal{G}^* \text{ a.e.}$$

Then Theorem 5.3.2 implies

$$\beta = 0,\ \gamma = 0 \text{ a.e.}$$

■

Remark 5.3.4 *We mention that Theorem 5.3.3 is a generalization of a result in [95] in the Gaussian case, when $\beta \in \mathbb{L}^{1,2}$, that is*

$$\|\beta\|_{1,2}^2 := \|\beta\|_{L^2(\lambda\times\mu)}^2 + \|D.\beta\|_{L^2(\lambda\times\lambda\times\mu)}^2 < \infty.$$

As a special case of Theorem 5.3.3, we get the following:

Theorem 5.3.5 ***[Decomposition uniqueness for Skorohod-semimartingales]***
Let X_t be a Skorohod-semimartingale of the form

$$X(t) = \zeta + \int_0^t \alpha(s)\, ds + \int_0^t \beta(s)\, \delta B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\, \widetilde{N}(dz, \delta s),$$

where $\alpha(t) \in L^2(P)$ for all t. Then if

$$X(t) = 0 \text{ for all } \ 0 \le t \le T.$$

we have

$$\zeta = 0, \alpha = 0,\ \beta = 0,\ \gamma = 0 \text{ a.e.}$$

Example 5.3.6 *Assume in Theorem 5.3.3 that $\gamma \equiv 0$. Further require $\alpha(t) \in L^p(\mu)$ $0 \le t \le T$ for some $p > 1$. Since $L^p(\mu) \subset \mathcal{G}^*$ for all $p > 1$ (see [115]) it follows from Theorem 5.3.3 that if $X(t) = 0,\ 0 \le t \le T$ then $\zeta = 0,\ \alpha = 0,\ \beta = 0$ a.e.*

Example 5.3.7 *Denote by $L_t(x)$ the local time of the Brownian motion. Consider the Donsker delta function $\delta_x(B(t))$ of $B(t)$, which is a mapping from $[0,T]$ into $\mathcal{G}^*$. The*

Donsker delta function can be regarded as a time-derivative of the local time $L_t(x)$, that is

$$L_t(x) = \int_0^t \delta_x(B(s))\, ds$$

for all x a.e. See e.g. [64]. So we see from Theorem 5.3.3 that the random field

$$X(t) = \zeta + L_t(x) + \int_0^t \beta(s)\, \delta B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\, \widetilde{N}(dz, \delta s)$$

has a unique decomposition. We remark that we obtain the same result if we generalize $L_t(x)$ to be a local time of a diffusion process (as constructed in [114]) or the local time of a Lévy process (as constructed in [85]). Finally, we note that the unique decomposition property carries over to the case when X_t has the form

$$X(t) = \zeta + A(t) + \int_0^t \beta(s)\, \delta B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\, \widetilde{N}(dz, \delta s),$$

where $A(t)$ is a positive continuous additive functional with the representation

$$A(t) = \int_{\mathbb{R}} L_t(x)\, m(dx),$$

where m is a finite measure. See [13] or [53].

Chapter 6

A general stochastic maximum principle for insider control

6.1 Introduction

In the classical Black-Scholes model, and in most problems of stochastic analysis applied to finance, one of the fundamental hypotheses is the homogeneity of information that market participants have. This homogeneity does not reflect reality. In fact, there exist many types of agents in the market, who have different levels of information. In this Chapter, we are focusing on agents who have additional information (insider), and show that, it is important to understand how an optimal control is affected by particular pieces of such information.

In the following, let $\{B_s\}_{0\leq s\leq T}$ be a Brownian motion and $\widetilde{N}(dz,ds) = N(dz,ds) - ds\nu(dz)$ be a compensated Poisson random measure associated with a Lévy process with Lévy measure ν on the (complete) filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$. In the sequel, we assume that the Lévy measure ν fulfills

$$\int_{\mathbb{R}_0} z^2\,\nu(dz) < \infty,$$

where $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$.

Suppose that the state process $X(t) = X^{(u)}(t,\omega);\ t \geq 0,\ \omega \in \Omega$ is a controlled Itô-Lévy process in $\mathbb{R}$ of the form:

$$\begin{cases} d^-(X)(t) &= b(t,X(t),u(t))\,dt + \sigma(t,X(t),u(t))\,d^-B(t) \\ &\quad + \int_{\mathbb{R}_0} \theta(t,X(t),u(t),z)\,\widetilde{N}(dz,d^-t); \\ X(0) &= x \in \mathbb{R} \end{cases} \tag{6.1.1}$$

Here we have supposed that we are given a filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$ such that

$$\mathcal{F}_t \subset \mathcal{G}_t,\ \ t \in [0,T], \tag{6.1.2}$$

representing the information available to the controller at time t.

Since $B(t)$ and $\widetilde{N}(dz,dt)$ need not be a semimartingale with respect to $\{\mathcal{G}_t\}_{t\geq 0}$, the two last integrals in (6.1.1) are *anticipating* stochastic integral that we interpret as *forward* integrals.

The control process

$$u : [0,T] \times \Omega \longrightarrow U,$$

is called an *admissible control* if (6.1.1) has a unique (strong) solution $X(\cdot) = X^{(u)}(\cdot)$ such that $u(\cdot)$ is adapted with respect to the sup-filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$.

The choice of forward integration is motivated by the possible applications to optimal portfolio problems for insiders as in Section 6.6. (See for *e.g.,* [14, 33, 32].) Moreover, the applications are not restricted to this area and include all situations of optimization problems in anticipating environment. (See *e.g.,* [104].)

More significantly, the problem we are dealing with is the following. Suppose that we are given a performance functional of the form

$$J(u) := E\left[\int_0^T f(t,X(t),u(t))\,dt + g(X(T))\right],\ u \in \mathcal{A}_{\mathcal{G}}, \tag{6.1.3}$$

where $\mathcal{A}_{\mathcal{G}}$ is a family of admissible controls $u(\cdot)$ contained in the set of $\mathcal{G}_t$-adapted controls. Consider

$$\begin{aligned} f &: [0,T] \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}\,, \\ g &: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}, \end{aligned}$$

where f is an $\mathbb{F}$-adapted process for each $x \in \mathbb{R}$, $u \in U$ and g is an $\mathcal{F}_T$-measurable random variable for each $x \in \mathbb{R}$ satisfying

$$E\left[\int_0^T |f(t, X(t), u(t))|\, dt \,+\, |g(X(T))|\right] < \infty \text{ for all, } u \in \mathcal{A}_{\mathcal{G}},$$

The goal is to find the optimal control u^* of the following insider control problem

$$\Phi_{\mathcal{G}} = \sup_{u \in \mathcal{A}_{\mathcal{G}}} J(u) = J(u^*)\,. \tag{6.1.4}$$

We use Malliavin calculus to prove a general stochastic maximum principle for stochastic differential equations (SDE's) with jumps under additional information. The main result here is difficult to apply because of the appearance of some terms, which all depend on the control. We then consider the special case when the coefficients of the controlled process $X(\cdot)$ do not depend on X; we call such processes *controlled Itô-Lévy processes*. In this case, we give a necessary condition for the existence of optimal control. Using the uniqueness of decomposition of a Skorohod-semimartingale (see [34]), we derive more precise results when our enlarged filtration is first chaos generated (the class of such filtrations contains the class of initially enlarged filtrations and also advanced information filtrations). We apply our results to maximize the expected utility of terminal wealth for the insider. We show that there does not exist an optimal portfolio for the insider. For the advanced information case, this conclusion is in accordance with the results in [14] and [33], since the Brownian motion is not a semimartingale with respect to the advanced information filtration. It follows that the stock price is not a semimartingale with respect to that filtration either. Hence, we can deduce that the market has an arbitrage for the insider in this case, by Theorem 7.2 in [29].

In the initial enlargement of filtration case, knowing the terminal value of the stock price, we also prove that there does not exist an optimal portfolio for the insider. This result is a generalization of a result in [72], where the same conclusion is obtained in the special case when the utility function is the logarithm function and there are no jumps in the stock price. The other application pertains to optimal insider consumption. We show that there exists an optimal insider consumption, and in some special cases the optimal consumption can be expressed explicitly.

The Chapter is structured as follows: In Section 6.2, we briefly recall some basic concepts of Malliavin calculus and its connection to the theory of forward integration. In Section 6.3, we use Malliavin calculus to obtain a maximum principle (*i.e.,* necessary and sufficient conditions) for this general non-Markovian insider information stochastic control problem. Section 6.4 considers the special of Itô-Lévy processes. In Section 6.5, we apply our results to some special cases of filtrations. Section 6.6 and 6.7 are respectively application to optimal insider portfolio, and optimal insider consumption.

6.2 Framework

In this Section we briefly recall some basic concepts of Malliavin calculus and its connection to the theory of forward integration. We refer to Section 4.3 of Chapter 4 for more informations about Malliavin calculus. As for the theory of forward integration the reader may consult [95, 121, 124] and [32].

6.2.1 Malliavin calculus and forward integral

In this Section we briefly recall some basic concepts of Malliavin calculus and forward integrations related to this Chapter. We refer to [95, 121, 124] and [32] for more information about these integrals.

A crucial argument in the proof of our general maximum principle rests on duality formulas for the Malliavin derivatives D_t and $D_{t,z}$. (See [94] or [31].)

Lemma 6.2.1 *We recall that the Skorohod integral with respect to B resp. $\widetilde{N}(dt,dz)$ is defined as the adjoint operator of $D_\cdot : \mathbb{D}_{1,2}^{(1)} \longrightarrow L^2(\lambda\times P^{(1)})$ resp. $D_{\cdot,\cdot} : \mathbb{D}_{1,2}^{(2)} \longrightarrow L^2(\lambda\times\nu\times P^{(2)})$. Thus if we denote by*

$$\int_0^T (\cdot)\,\delta B_t \text{ and } \int_0^T\int_{\mathbb{R}_0} (\cdot)\,\widetilde{N}(dt,dz)$$

the corresponding adjoint operators the following duality relations are satisfied:

(i)

$$E_{P^{(1)}}\left[F\int_0^T \varphi(t)\delta B_t\right] = E_{P^{(1)}}\left[\int_0^T \varphi(t)D_tF\,dt\right] \tag{6.2.1}$$

for all $F\in\mathbb{D}_{1,2}^{(1)}$ and all Skorohod integrable $\varphi\in L^2(\lambda\times P^{(1)})$ (i.e. φ in the domain of the adjoint operator).

(ii)

$$E_{P^{(2)}}\left[G\int_0^T\int_{\mathbb{R}_0} \psi(t,z)\widetilde{N}(\delta t,dz)\right] = E_{P^{(2)}}\left[\int_0^T\int_{\mathbb{R}_0} \psi(t,z)D_{t,z}G\nu(dz)dt\right] \tag{6.2.2}$$

for all $G\in\mathbb{D}_{1,2}^{(2)}$ and all Skorohod integrable $\psi\in L^2(\lambda\times\nu\times P^{(2)})$.

Forward integral and Malliavin calculus for $B(\cdot)$

This section constitutes a brief review of the forward integral with respect to the Brownian motion. Let $B(t)$ be a Brownian motion on a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},P)$, and $T>0$ a fixed horizon.

Definition 6.2.2 *Let $\phi:[0,T]\times\Omega\to\mathbb{R}$ be a measurable process. The forward integral of ϕ with respect to $B(\cdot)$ is defined by*

$$\int_0^T \phi(t,\omega)\,d^-B(t) = \lim_{\epsilon\to 0}\int_0^T \phi(t,\omega)\frac{B(t+\epsilon)-B(t)}{\epsilon}\,dt, \tag{6.2.3}$$

if the limit exist in probability, in which case ϕ is called forward integrable.

Note that if ϕ is càdlàg and forward integrable, then

$$\int_0^T \phi(t,\omega)\,d^-B(t) = \lim_{\Delta t\to 0}\sum_j \phi(t_j)\Delta B(t_j). \tag{6.2.4}$$

where the sum is taken over the points of a finite partition of $[0,T]$.

Definition 6.2.3 *Let $\mathcal{M}^B$ denote the set of stochastic functions $\phi : [0,T] \times \Omega \to \mathbb{R}$ such that:*

1. $\phi \in L^2([0,T] \times \Omega)$, *$u(t) \in \mathbb{D}^B_{1,2}$ for almost all t and satisfies*
$$E\left(\int_0^T |\phi(t)|^2 \, dt \, + \int_0^T \int_0^T |D_u \phi(t)|^2 \, dudt\right) < \infty \, .$$
We will denoted by $L^{1,2}[0,T]$ the class of such processes.

2. $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{u-\epsilon}^{u} \phi(t) dt = \phi(u)$ *for a.a $u \in [0,T]$ in $L^{1,2}[0,T]$,*

3. $D_{t+}\phi(t) := \lim_{s \to t+} D_s \phi(t)$ *exists in $L^1((0,T) \otimes \Omega)$ uniformly in $t \in [0,T]$.*

We let $\mathbb{M}^B_{1,2}$ be the closure of the linear span of $\mathcal{M}^B$ with respect to the norm given by
$$\|\phi\|_{\mathbb{M}^B_{1,2}} := \|\phi\|_{\mathbb{L}^{1,2}[0,T]} \, + \|D_{t+}\phi(t)\|_{L^1((0,T)\otimes\Omega)}$$

Then we have the relation between the forward integral and the Skorohod integral (see [76, 31]):

Lemma 6.2.4 *If $\phi \in \mathbb{M}^B_{1,2}$ then it is forward integrable and*
$$\int_0^T \phi(t) d^- B(t) = \int_0^T \phi(t) \delta B(t) \, + \int_0^T D_{t+}\phi(t) dt \, . \tag{6.2.5}$$
Moreover
$$E\left[\int_0^T \phi(t) d^- B(t)\right] = E\left[\int_0^T D_{t+}\phi(t) dt\right] \, . \tag{6.2.6}$$

Using (6.2.5) and the duality formula for the Skorohod integral see *e.g.*, [31], we deduce the following result.

Corollary 6.2.5 *Suppose $\phi \in \mathbb{M}^B_{1,2}$ and $F \in \mathbb{D}^B_{1,2}$ then*
$$\begin{aligned} E\left[F \int_0^T \phi(t) d^- B(t)\right] &= E\left[F \int_0^T \phi(t) \delta B(t) \, + F \int_0^T D_{t+}\phi(t) dt\right] \\ &= E\left[\int_0^T \phi(t) D_t F \, dt \, + \int_0^T F \, D_{t+}\phi(t) dt\right] \end{aligned} \tag{6.2.7}$$

Proposition 6.2.6 *Let $\mathcal{H}$ be a given fixed σ-algebra and $\varphi : [0,T] \times \Omega \to \mathbb{R}$ be a $\mathcal{H}$-measurable process. Set $X(t) = E[B(t)|\mathcal{H}]$. Then*

$$E\left[\int_0^T \varphi(t)d^-B(t)\middle|\mathcal{H}\right] = E\left[\int_0^T \varphi(t)d^-X(t)\right] \tag{6.2.8}$$

Proof. Using uniform convergence on compacts in $L^1(P)$ and the definition of forward integration in the sense of Russo-Vallois (see [121]) we observe that

$$\begin{aligned}
E\,[\int_0^T \varphi(t)d^-B(t)\,|\mathcal{H}] =& E\,[\lim_{\epsilon\to 0^+}\int_0^T \varphi(t)\frac{B(t+\epsilon)-B(t)}{\epsilon}dt\,|\mathcal{H}] \\
=& L^1(P) - \lim_{\epsilon\to 0^+} E\,[\int_0^T \varphi(t)\frac{B(t+\epsilon)-B(t)}{\epsilon}dt\,|\mathcal{H}] \\
=& \lim_{\epsilon\to 0^+}\int_0^T \varphi(t)E\,[\,\frac{B(t+\epsilon)-B(t)}{\epsilon}\,|\mathcal{H}]\,dt \\
=& \lim_{\epsilon\to 0^+}\int_0^T \varphi(t)\frac{X(t+\epsilon)-X(t)}{\epsilon}dt \\
=& \int_0^T \varphi(t)d^-X(t), \quad \text{in the ucp sense}
\end{aligned}$$

and the result follows. ■

Definition 6.2.7 *Let $(\mathcal{H}_t)_{t\geq 0}$ be a given filtration and $\varphi : [0,T] \times \Omega \to \mathbb{R}$ be a $\mathcal{H}$-adapted process. The conditional forward integral of ϕ with respect to $B(\cdot)$ is defined by*

$$\int_0^T \varphi(t)\,E\,[\,d^-B(t)\,|\mathcal{H}_{t^-}] = \lim_{\epsilon\to 0}\int_0^T \varphi(t)\frac{E\,[\,B(t+\epsilon)-B(t)\,|\mathcal{H}_{t^-}]}{\epsilon}\,dt, \tag{6.2.9}$$

if the limit exist ucp sense.

Remark 6.2.8 *Note that Definition 6.2.7 is different from Proposition 6.2.6 except if $\mathcal{H}_t = \mathcal{H}$ for all t*

Forward integral and Malliavin calculus for $\widetilde{N}(\cdot,\cdot)$

In this section, we review the forward integral with respect to the Poisson random measure $\widetilde{N}$.

Definition 6.2.9 *The forward integral*

$$J(\phi) := \int_0^T \int_{\mathbb{R}_0} \phi(t,z)\widetilde{N}(dz,d^-t)\,,$$

with respect to the Poisson random measure $\widetilde{N}$*, of a càdlàg stochastic function* $\phi(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}$*, with* $\phi(t,z) = \phi(\omega,t,z)$, $\omega \in \Omega$*,is defined as*

$$J(\phi) = \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}} \phi(t,z) 1_{U_m} \widetilde{N}(dz,dt)\,,$$

if the limit exist in $L^2(\mathbb{P})$*. Here* $U_m, m = 1,2,\cdots$*, is an increasing sequence of compact sets* $U_m \subseteq \mathbb{R}\backslash\{0\}$ *with* $\nu(U_m) < \infty$ *such that* $\lim_{m\to\infty} U_m = \mathbb{R}\backslash\{0\}$*.*

Definition 6.2.10 *Let* $\mathcal{M}^{\widetilde{N}}$ *denote the set of stochastic functions* $\phi : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ *such that:*

1. $\phi(t,z,\omega) = \phi_1(t,\omega)\phi_2(t,z,\omega)$ *where* $\phi_1(\omega,t) \in \mathbb{D}^{\widetilde{N}}_{1,2}$ *is càdlàg and* $\phi_2(\omega,t,z)$ *is adapted such that*
$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \phi_2(t,z)\nu(dz)dt\right] < \infty\,,$$
2. $D_{t+,z}\phi := \lim_{s\to t+} D_{s,z}\phi$ *exists in* $L^2(\mathbb{P}\times\lambda\times\nu)$*,*
3. $\phi(t,z) + D_{t+,z}\phi(t,z)$ *is Skorohod integrable.*

We let $\mathbb{M}^{\widetilde{N}}_{1,2}$ be the closure of the linear span of $\mathcal{M}^B$ with respect to the norm given by

$$\|\phi\|_{\mathbb{M}^{\widetilde{N}}_{1,2}} := \|\phi\|_{L^2(\mathbb{P}\times\lambda\times\nu)} + \|D_{t+,z}\phi(t,z)\|_{L^2(\mathbb{P}\times\lambda\times\nu)}$$

Then we have the relation between the forward integral and the Skorohod integral (see [32, 31]):

Lemma 6.2.11 *If* $\phi \in \mathbb{M}^{\widetilde{N}}_{1,2}$ *then it is forward integrable and*

$$\int_0^T \int_{\mathbb{R}} \phi(t,z)\widetilde{N}(dz,d^-t) = \int_0^T \int_{\mathbb{R}} D_{t+,z}\phi(t,z)\nu(dz)dt + \int_0^T \int_{\mathbb{R}} (\phi(t,z)+D_{t+,z}\phi(t,z))\widetilde{N}(dz,\delta t)\,. \tag{6.2.10}$$

Moreover

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \phi(t,z)\widetilde{N}(dz,d^-t)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} D_{t+,z}\phi(t,z)\nu(dz)dt\right]. \tag{6.2.11}$$

Then by (6.2.10) and duality formula for Skorohod integral for Poisson process see [31], we have

Corollary 6.2.12 *Suppose* $\phi \in \mathbb{M}_{1,2}^{\widetilde{N}}$ *and* $F \in \mathbb{D}_{1,2}^{\widetilde{N}}$*, then*

$$\begin{aligned}
\mathbb{E}\left[F\int_0^T\int_{\mathbb{R}}\phi(t,z)\widetilde{N}(dz,d^-t)\right] =&\mathbb{E}\left[F\int_0^T\int_{\mathbb{R}}D_{t+,z}\phi(t,z)\nu(dz)dt\right]\\
&+\mathbb{E}\left[F\int_0^T\int_{\mathbb{R}}(\phi(t,z)\,+\,D_{t+,z}\phi(t,z))\widetilde{N}(dz,\delta t)\right]\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\phi(t,z)D_{t,z}F\nu(dz)dt\right]\\
&+\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}(F\,+\,D_{t,z}F)D_{t+,z}\phi(t,z)\nu(dz)dt\right]. \quad (6.2.12)
\end{aligned}$$

6.3 A Stochastic Maximum Principle for insider

In view of the optimization problem (6.1.4) we require the following conditions 1–5:

1. The functions $b:[0,T]\times\mathbb{R}\times U\times\Omega\to\mathbb{R}$, $\sigma:[0,T]\times\mathbb{R}\times U\times\Omega\to\mathbb{R}$, $\theta:[0,T]\times\mathbb{R}\times U\times\mathbb{R}_0\times\Omega\to\mathbb{R}$, $f:[0,T]\times\mathbb{R}\times U\times\Omega\to\mathbb{R}$ and $g:\mathbb{R}\times\Omega\to\mathbb{R}$ are contained in C^1 with respect to the arguments $x\in\mathbb{R}$ and $u\in U$ for each $t\in\mathbb{R}$ and a.a $\omega\in\Omega$.

2. For all $r,t\in(0,T)$, $t\le r$ and all bounded $\mathcal{G}_t-$measurable random variables $\alpha=\alpha(\omega)$, $\omega\in\Omega$, the control

$$\beta_\alpha(s):=\alpha(\omega)\chi_{[t,r]}(s),\ 0\le s\le T\,, \quad (6.3.1)$$

is an admissible control i.e., belongs to $\mathcal{A}_{\mathcal{G}}$ (here $\chi_{[t,T]}$ denotes the indicator function on $[t,T]$).

3. For all $u,\beta\in\mathcal{A}_{\mathcal{G}}$ with β bounded, there exists a $\delta>0$ such that

$$u+y\beta\in\mathcal{A}_{\mathcal{G}}\,,\ \text{for all } y\in(-\delta,\delta) \quad (6.3.2)$$

and such that the family

$$\left\{\frac{\partial}{\partial x}f(t,X^{u+y\beta}(t),u(t)+y\beta(t))\frac{d}{dy}X^{u+y\beta}(t)\right.$$
$$\left.+\frac{\partial}{\partial u}f(t,X^{u+y\beta}(t),u(t)+y\beta(t))\beta(t)\right\}_{y\in(-\delta,\delta)}$$

is $\lambda \times \mathbb{P}-$uniformly integrable and

$$\left\{ g'(X^{u+y\beta}(T)) \frac{d}{dy} X^{u+y\beta}(T) \right\}_{y \in (-\delta,\delta)}$$

is $\mathbb{P}-$uniformly integrable.

4. For all $u, \beta \in \mathcal{A}_{\mathcal{G}}$ with β bounded the process

$$Y(t) = Y_\beta(t) = \left. \frac{d}{dy} X^{(u+y\beta)}(t) \right|_{y=0}$$

exists and follows the SDE

$$\begin{aligned} dY^u_\beta(t) = & Y_\beta(t^-) \left[\frac{\partial}{\partial x} b(t, X^u(t), u(t))\, dt + \frac{\partial}{\partial x} \sigma(t, X^u(t), u(t))\, d^- B(t) \right. \\ & \left. + \int_{\mathbb{R}_0} \frac{\partial}{\partial x} \theta(t, X^u(t), u(t), z)\, \widetilde{N}(dz, d^- t) \right] \\ & + \beta(t) \left[\frac{\partial}{\partial u} b(t, X^u(t), u(t))\, dt + \frac{\partial}{\partial u} \sigma(t, X^u(t), u(t))\, d^- B(t) \right. \\ & \left. + \int_{\mathbb{R}_0} \frac{\partial}{\partial u} \theta(t, X^u(t), u(t), z)\, \widetilde{N}(dz, d^- t) \right] \qquad (6.3.3) \\ Y(0) = & 0 \end{aligned}$$

5. Suppose that for all $u \in \mathcal{A}_{\mathcal{G}}$ the processes

$$
\begin{aligned}
K(t) &:= g'(X(T)) + \int_t^T \frac{\partial}{\partial x} f(s, X(s), u(s))\, ds && (6.3.4)\\
D_t K(t) &:= D_t g'(X(T)) + \int_t^T D_t \frac{\partial}{\partial x} f(s, X(s), u(s))\, ds \\
D_{t,z} K(t) &:= D_{t,z} g'(X(T)) + \int_t^T D_{t,z} \frac{\partial}{\partial x} f(s, X(s), u(s))\, ds \\
H_0(s,x,u) &:= K(s) \left(b(s,x,u) + D_{s+}\sigma(s,x,u) + \int_{\mathbb{R}_0} D_{s+,z}\theta(s,x,u,z)\, \nu(dz) \right) \\
&\quad + D_s K(s) \sigma(s,x,u) && (6.3.5)\\
&\quad + \int_{\mathbb{R}_0} D_{s,z} K(s) \Big\{ \theta(s,x,u,z) + D_{s+,z}\theta(s,x,u,z) \Big\}\, \nu(dz) \\
G(t,s) &:= \exp \Bigg(\int_t^s \left\{ \frac{\partial b}{\partial x}(r, X(r), u(r)) - \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2 (r, X(r), u(r)) \right\} dr \\
&\quad + \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), u(r))\, dB^-(r) \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln \left(1 + \frac{\partial \theta}{\partial x}(r, X(r), u(r), z) \right) - \frac{\partial \theta}{\partial x}(r, X(r), u(r), z) \right\} \nu(dz)\, dr \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln \left(1 + \frac{\partial \theta}{\partial x}\left(r, X(r^-), u(r^-), z\right) \right) \right\} \widetilde{N}(dz, d^- r) \Bigg) && (6.3.6)
\end{aligned}
$$

$$
\begin{aligned}
p(t) &:= K(t) + \int_t^T \frac{\partial}{\partial x} H_0(s, X(s), u(s)) G(t,s)\, ds && (6.3.7)\\
q(t) &:= D_t p(t) && (6.3.8)\\
r(t,z) &:= D_{t,z} p(t); \quad t \in [0,T],\ z \in \mathbb{R}_0\,. && (6.3.9)
\end{aligned}
$$

are well-defined.

Now let us introduce the *general Hamiltonian of an insider*

$$H : [0,T] \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}$$

by

$$
\begin{aligned}
H(t,x,u,\omega) := p(t) &\left(b(t,x,u,\omega) + D_{t+}\sigma(t,x,u,\omega) + \int_{\mathbb{R}_0} D_{t+,z}\theta(t,x,u,\omega)\, \nu(dz) \right) \\
&+ f(t,x,u,\omega) + q(t)\sigma(t,x,u,\omega) \\
&+ \int_{\mathbb{R}_0} r(t,z) \Big\{ \theta(t,x,u,z,\omega) + D_{t+,z}\theta(t,x,u,z,\omega) \Big\} \nu(dz) \qquad (6.3.10)
\end{aligned}
$$

We can now state a general stochastic maximum principle for our control problem (6.1.4):

Theorem 6.3.1 *1. Retain the conditions 1–5. Assume that $\widehat{u} \in \mathcal{A}_{\mathcal{G}}$ is a* critical point *of the performance functional $J(u)$ in (6.1.4), that is*

$$\frac{d}{dy} J(\widehat{u} + y\beta)\Big|_{y=0} = 0 \tag{6.3.11}$$

for all bounded $\beta \in \mathcal{A}_{\mathcal{G}}$. Then

$$E\left[\frac{\partial}{\partial u}\widehat{H}(t, \widehat{X}(t), \widehat{u}(t))\Big|\mathcal{G}_t\right] + E[A] = 0 \quad a.e. \ in \ \ (t,\omega), \tag{6.3.12}$$

where A is given by Equation (A.3.12)

$$\begin{aligned}
\widehat{X}(t) =& X^{(\widehat{u})}(t),\\
\widehat{H}(t, \widehat{X}(t), u) =& p(t)\Big(b(t, \widehat{X}, u) + D_{t+}\sigma(t, \widehat{X}, u) + \int_{\mathbb{R}_0} D_{t+,z}\theta(t, \widehat{X}, u)\, \nu(dz)\Big)\\
&+ f(t, \widehat{X}, u) + q(t)\sigma(t, \widehat{X}, u)\\
&+ \int_{\mathbb{R}_0} r(t,z)\Big\{\theta(t, \widehat{X}, u, z) + D_{t+,z}\theta(t, \widehat{X}, u, z)\Big\}\nu(dz)
\end{aligned} \tag{6.3.13}$$

with

$$\begin{aligned}
\widehat{p}(t) =& \widehat{K}(t) + \int_t^T \frac{\partial}{\partial x}\widehat{H}_0(s, \widehat{X}(s), \widehat{u}(s))\widehat{G}(t,s)\, ds\,, \qquad (6.3.14)\\
\widehat{K}(t) :=& g'(\widehat{X}(T)) + \int_t^T \frac{\partial}{\partial x} f(s, \widehat{X}(s), \widehat{u}(s)) ds,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{G}(t,s) :=& \exp\Bigg(\int_t^s \left\{\frac{\partial b}{\partial x}\Big(r, \widehat{X}(r), u(r)\Big) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2\Big(r, \widehat{X}(r), u(r)\Big)\right\} dr\\
&+ \int_t^s \frac{\partial \sigma}{\partial x}\Big(r, \widehat{X}(r), u(r)\Big)\, dB^-(r)\\
&+ \int_t^s \int_{\mathbb{R}_0}\left\{\ln\Big(1 + \frac{\partial \theta}{\partial x}\Big(r, \widehat{X}(r), u(r), z\Big)\Big) - \frac{\partial \theta}{\partial x}\Big(r, \widehat{X}(r), u(r), z\Big)\right\}\nu(dz)dt\\
&+ \int_t^s \int_{\mathbb{R}_0}\left\{\ln\Big(1 + \frac{\partial \theta}{\partial x}\Big(r, \widehat{X}(r^-), u(r^-), z\Big)\Big)\right\}\widetilde{N}(dz, d^-r)\Bigg)\\
\widehat{H}(t,x,u) =& \widehat{K}(t)\Big(b(t,x,u) + D_{t+}\sigma(t,x,u) + \int_{\mathbb{R}_0} D_{t+,z}\theta(t,x,u)\,\nu(dz)\Big)\\
&+ D_t\widehat{K}(t)\sigma(t,x,u) + f(t,x,u)\\
&+ \int_{\mathbb{R}_0} D_{t,z}\widehat{K}(t)\Big\{\theta(t,x,u,z) + D_{t+,z}\theta(t,x,u,z)\Big\}\nu(dz)
\end{aligned}$$

2. *Conversely, suppose there exists $\widehat{u} \in \mathcal{A}_{\mathcal{G}}$ such that (6.3.12) holds. Then $\widehat{u}$ satisfies (6.3.11).*

Proof. See Appendix A, Section A.3. ■

6.4 Controlled Itô-Lévy processes

The main result of the previous section (Theorem 6.3.1) is difficult to apply because of the appearance of the terms $Y(t)$, $D_{t+}Y(t)$ and $D_{t+,z}Y(t)$, which all depend on the control u. However, consider the special case when the coefficients do not depend on X, i.e., when

$$\begin{aligned} b(t,x,u,\omega) = b(t,u,\omega), \quad \sigma(t,x,u,\omega) = \sigma(t,u,\omega) \\ \text{and } \theta(t,x,u,z,\omega) = \theta(t,u,z,\omega). \end{aligned} \tag{6.4.1}$$

Then the equation (6.1.1) gets the form

$$\begin{cases} d^-(X)(t) &= b(t,u(t),\omega)dt + \sigma(t,u(t),\omega)d^-B_t \\ & \quad + \int_{\mathbb{R}_0} \theta(t,u(t),z,\omega)\widetilde{N}(dz,d^-t); \\ X(0) &= x \in \mathbb{R} \end{cases} \tag{6.4.2}$$

We call such processes *controlled Itô-Lévy processes*.

In this case, Theorem 6.3.1 simplifies to the following

Theorem 6.4.1 *Let $X(t)$ be a controlled Itô-Lévy process as given in Equation (6.4.2). Retain the conditions (1)-(5) as in Theorem 6.3.1.*

Then the following are equivalent:

1. *$\widehat{u} \in \mathcal{A}_{\mathcal{G}}$ is a critical point of $J(u)$,*

2. $$E\left[L(t)\alpha + M(t)D_{t+}\alpha + \int_{\mathbb{R}_0} R(t,z)D_{t+,z}\alpha\,\nu(dz)\right] = 0$$

for all $\mathcal{G}_t$-measurable $\alpha \in \mathbb{D}_{1,2}$ and all $t \in [0,T]$, where

$$\begin{aligned} L(t) =& K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + \frac{\partial f(t)}{\partial u} \\ &+ \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\Big)\nu(dz) + D_tK(t)\frac{\partial \sigma(t)}{\partial u}, \end{aligned} \tag{6.4.3}$$

$$M(t) = K(t)\frac{\partial \sigma(t)}{\partial u} \tag{6.4.4}$$

and

$$R(t,z) = \{K(t) + D_{t,z}K(t)\}\left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right). \tag{6.4.5}$$

Proof.

1. It is easy to see that in this case, $p(t) = K(t),\ \ q(t) = D_tK(t),\ \ r(t,z) = D_{t,z}K(t)$ and the general Hamiltonian H given by (6.3.10) is reduced to H_1 given as follows

$$\begin{aligned} H_1(s,x,u,\omega) :=& K(s)\left(b(s,u,\omega) + D_{s+}\sigma(s,u,\omega) + \int_{\mathbb{R}_0} D_{s+,z}\theta(s,u,\omega)\nu(dz)\right) \\ &+ D_sK(s)\sigma(s,u,\omega) + f(s,x,u,\omega) \\ &+ \int_{\mathbb{R}_0} D_{s,z}K(s)\Big\{\theta(s,u,z,\omega) + D_{s+,z}\theta(s,u,z,\omega)\Big\}\nu(dz). \end{aligned}$$

Then, performing the same calculus lead to

$$\begin{aligned} A_1 =& A_3 = A_5 = 0, \\ A_2 =& E\left[\int_t^{t+h}\left\{K(t)\left(\frac{\partial b(s)}{\partial u} + D_{s+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \gamma(s)}{\partial u}\nu(dz)\right) + \frac{\partial f(s)}{\partial u}\right.\right. \\ &\left.\left. + \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \gamma(s)}{\partial u}\Big)\nu(dz) + D_sK(s)\frac{\partial \sigma(s)}{\partial u}\right\}\alpha ds\right], \\ A_4 =& E\left[\int_t^{t+h} K(s)\frac{\partial \sigma(s)}{\partial u}D_{s+}\alpha ds\right], \\ A_6 =& E\left[\int_t^{t+h}\int_{\mathbb{R}_0}\{K(s) + D_{s,z}K(s)\}\left(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \gamma(s)}{\partial u}\right)\nu(dz)D_{s+,z}\alpha ds\right]. \end{aligned}$$

It follows that

$$\begin{aligned}
\frac{d}{dh}A_2\Big|_{h=0} &= E\Bigg[\Bigg\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + \frac{\partial f(t)}{\partial u} \\
&\quad + \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz) + D_tK(t)\frac{\partial \sigma(t)}{\partial u}\Bigg\}\,\alpha\Bigg], \\
\frac{d}{dh}A_4\Big|_{h=0} &= E\left[K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\alpha\right], \\
\frac{d}{dh}A_6\Big|_{h=0} &= E\left[\int_{\mathbb{R}_0}\{K(t) + D_{t,z}K(t)\}\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz)D_{t+,z}\alpha\right].
\end{aligned}$$

This means that

$$\begin{aligned}
0 = E\Bigg[&\Bigg\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + \frac{\partial f(t)}{\partial u} \\
&\int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz) + D_tK(t)\frac{\partial \sigma(t)}{\partial u}\Bigg\}\,\alpha \\
&+ K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\alpha + \left\{\int_{\mathbb{R}_0}\{K(t) + D_{t,z}K(t)\}\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz)\right\}D_{t+,z}\alpha\Bigg],
\end{aligned}$$

and the first implication follows.

2. The converse part follows from the arguments used in the proof of Theorem 6.3.1.

■

6.5 Application to special cases of filtrations

The results obtained so far are for given general sup-filtrations. To provide some concrete examples, let us confine ourselves to particular cases of filtrations. We consider the case of an insider who has an additional information compared to the standard normally informed investor.

- It can be the case of an insider who always has advanced information compared to the honest trader. This means that if $\mathcal{G}_t$ and $\mathcal{F}_t$ represent respectively the flows of informations of the insider and the honest investor then we can write that $\mathcal{G}_t \supset \mathcal{F}_{t+\delta(t)}$ where $\delta(t) > 0$;

- it can also be the case of a trader who has at the initial date particular information about the future (initial enlargement of filtration). This means that if $\mathcal{G}_t$ and $\mathcal{F}_t$ represent respectively the flows of informations of the insider and the honest investor then we can write that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ where L is a random variable.

6.5.1 Filtrations satisfying conditions (Co1) and (Co2)

In the following we need the notion of *D-commutativity* of a σ-algebra.

Definition 6.5.1 *A sub-σ-algebra $\mathcal{A} \subseteq \mathcal{F}$ is D-commutable if for all $F \in \mathbb{D}_{2,1}$ the conditional expectation $E[F \,|\mathcal{A}]$ belongs to $\mathbb{D}_{2,1}$ and*

$$D_t E[F \,|\mathcal{A}] = E[D_t F \,|\mathcal{A}], \tag{Co1}$$

$$D_{t,z} E[F \,|\mathcal{A}] = E[D_{t,z} F \,|\mathcal{A}] \tag{Co2}$$

Theorem 6.5.2 *Suppose that $\widehat{u} \in \mathcal{A}_{\mathcal{G}}$ is a critical point for $J(u)$. Assume that $\mathcal{G}_t$ is D-commutable for all t. Further require that the set of $\mathcal{G}_t-$measurable elements in the space of smooth random variables $\mathcal{G}$ (see [115]) are dense in $L^2(\mathcal{G}_t)$ for all t and that $E[M(t) \,|\mathcal{G}_t]$ and $E[R(t,z) \,|\mathcal{G}_t]$ are Skorohod integrable. Then*

$$\begin{aligned} 0 = & \int_0^s E[L(t) \,|\mathcal{G}_{t_0}]\, h(t)\, dt + \int_0^s E[M(t) \,|\mathcal{G}_{t_0}]\, h(t)\, \delta B_t \\ & + \int_0^s \int_{\mathbb{R}_0} E[R(t,z)|\, \mathcal{G}_{t_0}]\, h(t)\, \widetilde{N}(\delta t, dz). \end{aligned} \tag{6.5.1}$$

for all $h \in L^2([0,T])$ with $\operatorname{supp} .h \subseteq [t_0, T]$.

Proof. Without lost of generality, we give the proof for the Brownian motion case, and the ones of the pure jump case and mixed case follow.

Let fix a $t_0 \in [0,T)$. Then, by assumption it follows that for all $\mathcal{G}_{t_0}-$measurable $\alpha \in \mathcal{G}$ and $h \in L^2([0,T])$ with

$$\operatorname{supp} .h \subseteq [t_0, T], \;\; t_0 \leq t \leq T,$$

$$0 = \left\langle \int_0^T E[L(t) \,|\mathcal{G}_{t_0}]\, h(t) dt, \alpha \right\rangle + \left\langle E[\int_0^T M(t) h(t) \delta B_t \,|\mathcal{G}_{t_0}], \alpha \right\rangle.$$

On the other hand the duality relation (6.2.1) implies

$$\begin{aligned}\left\langle E\left[\int_0^T M(t)h(t)\delta B_t \middle| \mathcal{G}_{t_0}\right], \alpha\right\rangle &= E\left[\int_0^T M(t)h(t)\delta B_t E\left[\alpha|\,\mathcal{G}_{t_0}\right]\right] \\ &= E\left[\int_0^T M(t)h(t)D_t E\left[\alpha|\,\mathcal{G}_{t_0}\right]dt\right] \\ &= E\left[\int_0^T M(t)h(t)E\left[D_t\alpha|\,\mathcal{G}_{t_0}\right]dt\right] \\ &= E\left[\int_0^T E\left[M(t)h(t)|\,\mathcal{G}_{t_0}\right]D_t\alpha dt\right] \\ &= \left\langle \int_0^T E\left[M(t)\mid\mathcal{G}_{t_0}\right]h(t)\delta B_t, \alpha\right\rangle\end{aligned}$$

for all $\alpha \in \mathcal{G}$. So

$$E\left[\int_0^T M(t)h(t)\delta B_t \middle| \mathcal{G}_{t_0}\right] = \int_0^T E\left[M(t)\mid\mathcal{G}_{t_0}\right]h(t)\delta B_t.$$

Hence, by denseness, we obtain that

$$0 = \int_0^T E\left[L(t)\,|\mathcal{G}_{t_0}\right]h(t)dt + \int_0^T E\left[M(t)\,|\mathcal{G}_{t_0}\right]h(t)\delta B_t.$$

■

To provide some concrete examples let us confine ourselves to the following type of filtrations $\mathcal{H}$. Given an increasing family of $\mathbb{G} = \{G_t\}_{t\in[0,T]}$ Borel sets $G_t \supset [0,t]$. Define

$$\mathcal{H} = \mathcal{F}_{\mathcal{G}} = \{\mathcal{G}_t\}_{t\geq 0} \text{ where } \mathcal{F}_{\mathcal{G}} = \sigma\left\{\int_0^T \chi_U(s)dB(s);\, U \subset G_t\right\} \vee \mathcal{N} \tag{6.5.2}$$

where $\mathcal{N}$ is the collection of $P-$null sets. Then Conditions (Co1) and (Co2) hold (see Proposition 3.12 in [31]). Examples of filtrations of type (6.5.2) are

$$\begin{aligned}\mathcal{H}_1 &= \mathcal{F}_{t+\delta(t)}, \\ \mathcal{H}_2 &= \mathcal{F}_{[0,t]\cup O},\end{aligned}$$

where O is an open set contained in $[0,T]$.

It is easily seen that filtrations of type (6.5.2) satisfy conditions of Theorem 6.5.2 as well. Hence, we have

Corollary 6.5.3 *Suppose that $\{\mathcal{G}_t\}_{0\leq t\leq T}$ is given by (6.5.2). Then, Equation (6.5.1) holds.*

Using Theorem 5.3.3 in Chapter 5, it follows that

Theorem 6.5.4 *Suppose that $\mathcal{G}_t$ is of type (6.5.2). Then there exist a critical point $\widehat{u}$ for the performance functional $J(u)$ in (6.1.3) if and only if the following three conditions hold:*

$$(i)\ E\left[L(t)|\,\mathcal{G}_t\right] = 0,$$

$$(ii)\ E\left[M(t)|\,\mathcal{G}_t\right] = 0,$$

$$(iii)\ E\left[R(t,z)|\,\mathcal{G}_t\right] = 0.$$

where L, M and R are respectively given by Equations (6.4.3), (6.4.4) and (6.4.5).

Proof. It follows from the uniqueness of decomposition of Skorohod-semimartingale processes of type (6.5.1) (See Theorem 5.3.3.) ■

We show in Appendix A, Section A.4, that, using a technique based on chaos expansion, we can obtain similar results, when $\mathcal{G}_t$ is of type (6.5.2).

Remark 6.5.5 *Not all filtrations satisfy conditions (Co1) and (Co2). An important example is the following: Choose the σ-field $\mathcal{H}$ to be $\sigma(B(T))$, where $\{B(s)\}_{s\geq 0}$, is the Wiener process (Brownian motion) starting at 0 and $T>0$ is fixed. Then, $\mathcal{H}$ is not D-commutable. In fact, let $F = B(t)$ for some $t < T$ and choose s such that $t < s < T$. Then*

$$D_s E\left[B(t)|\mathcal{H}\right] = D_s\left(\frac{t}{T}B(T)\right) = \frac{t}{T},$$

while

$$E\left[D_s B(t)|\mathcal{H}\right] = E\left[0|\mathcal{H}\right] = 0.$$

It follows from the preceding Remark that the technique using in the preceding Section cannot be apply to the σ-algebra of the type $\mathcal{F}_t \vee \sigma(B_T)$, and hence we need a different approach to discuss such cases.

6.5.2 A different approach

In this Section, we consider σ-algebras which do not necessarily satisfy conditions (Co1) and (Co2).

Theorem 6.5.6 *Let $t_0 \in [0,T]$, put $\mathcal{H} = \mathcal{G}_{t_0}$. Further, require that there exists a set $\mathcal{A} = \mathcal{A}_{t_0} \subseteq \mathbb{D}_{1,2} \cap L^2(\mathcal{H})$ and a measurable set $\mathcal{M} \subset [t_0,T]$ such that $E\left[L(t)|\,\mathcal{H}\right]\cdot\chi_{[0,T]\cap\mathcal{M}}$, $E\left[M(t)|\,\mathcal{H}\right]\cdot \chi_{[0,T]\cap\mathcal{M}}$ and $E\left[R(t,z)|\,\mathcal{H}\right]\cdot\chi_{[0,T]\cap\mathcal{M}}$ are Skorohod integrable and*

(i) *$D_t\alpha$ and $D_{t,z}\alpha$ are $\mathcal{H}$-measurable, for all $\alpha \in \mathcal{A}$, $t \in \mathcal{M}$.*

(ii) *$D_{t+}\alpha = D_t\alpha$ and $D_{t+,z}\alpha = D_{t,z}\alpha$ for all $\alpha \in \mathcal{A}$ and a.a. t, z, $t \in \mathcal{M}$.*

(iii) *Span$\mathcal{A}$ is dense in $L^2(\mathcal{H})$.*

(iv)

$$E\left[L(t)\alpha + M(t)D_{t+}\alpha + \int_{\mathbb{R}_0} R(t,z)D_{t+,z}\alpha\,\nu(dz)\right] = 0$$

Then for all $h = \chi_{[t_0,s)}(t)\chi_{\mathcal{M}}(t)$

$$\begin{aligned}0 = E\Bigg[&\int_0^T E\left[L(t)\,|\mathcal{H}\right]h(t)dt + \int_0^T E\left[M(t)\,|\mathcal{H}\right]h(t)\delta B_t \\ &+ \int_0^T\int_{\mathbb{R}_0} E\left[R(t,z)|\,\mathcal{H}\right]h(t)\,\widetilde{N}(\delta t,dz)\Bigg|\,\mathcal{H}\Bigg]. \qquad (6.5.3)\end{aligned}$$

Proof. Let $\alpha = E\left[F|\,\mathcal{H}\right]$ for all $F \in \mathcal{A}$. Further, choose a $h \in L^2([0,T])$ with $h = \chi_{[t_0,s)}(t)\chi_{\mathcal{M}}(t)$. By assumption, we see that

$$\begin{aligned}0 = &\left\langle \int_0^T E\left[L(t)\,|\mathcal{H}\right]h(t)dt, \alpha\right\rangle + \left\langle E\left[\int_0^T M(t)h(t)\delta B_t\Big|\,\mathcal{H}\right], \alpha\right\rangle \\ &+ \left\langle E\left[\int_0^T\int_{\mathbb{R}_0} R(t,z)h(t)\,\widetilde{N}(\delta t,dz)\Big|\,\mathcal{H}\right], \alpha\right\rangle.\end{aligned}$$

On the other hand, the duality relation (6.2.1) and (ii) imply that

$$\begin{aligned}\left\langle E\left[\int_0^T M(t)h(t)\delta B_t \middle| \mathcal{H}\right], \alpha\right\rangle &= E\left[\int_0^T M(t)h(t)\delta B_t E\left[F\middle|\mathcal{H}\right]\right] \\ &= E\left[\int_0^T M(t)h(t)D_t E\left[F\middle|\mathcal{H}\right]dt\right] \\ &= E\left[\int_0^T E\left[M(t)\middle|\mathcal{H}\right]h(t)D_t E\left[F\middle|\mathcal{H}\right]dt\right] \\ &= E\left[\int_0^T E\left[M(t)\,\middle|\mathcal{H}\right]h(t)\delta B_t \cdot E\left[F\middle|\mathcal{H}\right]\right] \\ &= \left\langle \int_0^T E\left[M(t)\,\middle|\mathcal{H}\right]h(t)\delta B_t, \alpha\right\rangle.\end{aligned}$$

In the same way, we show that

$$\left\langle E\left[\int_0^T\int_{\mathbb{R}_0} R(t,z)h(t)\,\widetilde{N}(\delta t,dz)\middle|\mathcal{H}\right],\alpha\right\rangle = \left\langle \int_0^T\int_{\mathbb{R}_0} E\left[R(t,z)\,\middle|\mathcal{H}\right]h(t)\,\widetilde{N}(\delta t,dz),\alpha\right\rangle.$$

Then it follows from (iv) that

$$0 = E\left[\int_0^T E\left[L(t)\,|\mathcal{H}\right]h(t)dt + \int_0^T E\left[M(t)\,|\mathcal{H}\right]h(t)\delta B_t + \int_0^T\int_{\mathbb{R}_0} E\left[R(t,z)|\,\mathcal{H}\right]h(t)\,\widetilde{N}(\delta t,dz)\middle|\mathcal{H}\right].$$

for all $h \in L^2\left([0,T]\right)$ with with supp $.h \subseteq (t_0,T]$, $\;t_0 \le t \le T,$.

■

Theorem 6.5.7 *[Brownian motion case] Assume that the conditions in Theorem 6.5.6 are in force and* $\theta = 0$. *In addition, we require that* $E\left[M(t)|\,\mathcal{G}_{t-}\right] \in \mathbb{M}^B_{1,2}$ *and is forward integrable with respect to* $E\left[\,d^-B(t)\,|\mathcal{G}_{t-}\right]$. *Then*

$$\begin{aligned}0 = &\int_0^T E\left[L(t)\,|\mathcal{G}_{t-}\right]h_0(t)dt + \int_0^T E\left[M(t)\,|\mathcal{G}_{t-}\right]h_0(t)E\left[\,d^-B\,|\mathcal{G}_{t-}\right] \\ &- \int_0^T D_{t^+}E\left[M(t)|\,\mathcal{G}_{t-}\right]h_0(t)dt \qquad (6.5.4)\end{aligned}$$

Proof. We apply the preceding result to $h(t) = h_0(t)\chi_{[t_i,t_{i+1}]}(t)$, where $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} = T$ is a partition of $[0,T]$. From Equation (6.5.3), we have

$$\begin{aligned}0 = &\int_{t_i}^{t_{i+1}} E\left[L(t)\,|\mathcal{G}_{t_i}\right]h(t)dt + E\left[\int_{t_i}^{t_{i+1}} E\left[M(t)\,|\mathcal{G}_{t_i}\right]h(t)\delta B_t\middle|\mathcal{G}_{t_i}\right] \\ &+ E\left[\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}_0} E\left[R(t,z)|\,\mathcal{G}_{t_i}\right]h(t)\,\widetilde{N}(\delta t,dz)\middle|\mathcal{G}_{t_i}\right]. \qquad (6.5.5)\end{aligned}$$

By Lemma 6.2.4 and by assumption, we know that

$$\int_{t_i}^{t_{i+1}} E\left[M(t)\,|\mathcal{G}_{t_i}\right] h_0(t)\delta B_t = \int_{t_i}^{t_{i+1}} E\left[M(t)\,|\mathcal{G}_{t_i}\right] h_0(t) d^- B(t) - \int_{t_i}^{t_{i+1}} D_{t^+} E\left[M(t)|\,\mathcal{G}_{t_i}\right] h_0(t) dt. \quad (6.5.6)$$

Substituting (6.5.6) into (6.5.5) and summing over all i and taking the limit as $\Delta t_i \to 0$, we get

$$0 = \lim_{\substack{\Delta t_i \to 0 \\ n\to\infty}} \left\{ \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} E\left[L(t)\,|\mathcal{G}_{t_i}\right] h_0(t) dt + \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} E\left[M(t)\,|\mathcal{G}_{t_i}\right] h_0(t) \frac{E\left[B(t_{i+1}) - B(t_i)|\,\mathcal{G}_{t_i}\right]}{\Delta t_i} \Delta t_i - \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} D_{t^+} E\left[M(t)\,|\mathcal{G}_{t_i}\right] h_0(t) dt \right\},$$

in the topology of uniform convergence in probability.

Hence, by Definition 6.2.7, we get the result. ■

Important examples of a σ-algebras $\mathcal{H}$ satisfying condition of Theorem 6.5.6 are σ-algebras of the following type which are first chaos generated (see [96]), that is

$$\mathcal{H} = \sigma(I_1(h_i), i \in \mathbb{N}, h_i \in L^2([0,T])) \vee \mathcal{N}, \quad (6.5.7)$$

where $\mathcal{N}$ is the collection of $P-$null sets. Concrete examples of these σ-algebras are

$$\mathcal{H}_3 = \mathcal{F}_t \vee \sigma(B(T)),$$

or

$$\mathcal{H}_4 = \mathcal{F}_t \vee \sigma\left(B(t + \Delta t_n)\right); \;\; n = 1, 2, ...$$

Lemma 6.5.8 *Suppose that* $\mathcal{H} = \mathcal{H}_2 = \mathcal{F}_t \vee \sigma(B(T))$. *Then*

$$E\left[B(t)\mid \mathcal{H}_{t_0}\right] = \frac{T-t}{T-t_0} B(t_0) + \frac{t-t_0}{T-t_0} B(T) \text{ for all } t > t_0.$$

In particular

$$E\left[B(t+\varepsilon)\mid \mathcal{H}_t\right] = B(t) + \frac{\varepsilon}{T-t}\left(B(T) - B(t)\right)$$

Proof. We have that

$$E\left[B(t)\mid\mathcal{H}_{t_0}\right]=\int_0^{t_0}\varphi(t,s)dB(s)+C(t)B(T).$$

On one hand, we have

$$\begin{aligned}t=E\left[E\left[B(t)\mid\mathcal{H}_{t_0}\right]B(T)\right]=&E\left[\left(\int_0^{t_0}\varphi(t,s)dB(s)\right)B(T)\right]+C(t)T\\=&\int_0^{t_0}\varphi(t,s)ds+C(t)T.\end{aligned}\tag{6.5.8}$$

On the other hand

$$\begin{aligned}u=E\left[E\left[B(t)\mid\mathcal{H}_{t_0}\right]B(u)\right]=&E\left[\left(\int_0^{t_0}\varphi(t,s)dB(s)\right)B(u)\right]+C(t)u\\=&\int_0^{u}\varphi(t,s)ds+C(t)u,\ \text{for all } u<t.\end{aligned}\tag{6.5.9}$$

Differentiating Equation (6.5.9) with respect to u, it follows that

$$\varphi(t,u)+C(t)=1.$$

Substituting φ by its value in Equation(6.5.8), we obtain $C(t)=\frac{t-t_0}{T-t_0}$ and then $\varphi(t,s)=\frac{T-t_0}{T-t_0}$. Therefore, the result follows. ■

Corollary 6.5.9 *Suppose that $\mathcal{H}=\mathcal{H}_2=\mathcal{F}_t\vee\sigma(B(T))$. Then*

$$E\left[\,d^-B\,|\mathcal{H}_{t-}\right]=\frac{B(T)-B(t)}{T-t}dt.$$

We now consider a generalization of the previous example:

For each $t\in[0,T)$, let $\{\delta_n\}_{n=0}^{\infty}=\{\delta_n(t)\}_{n=0}^{\infty}$ be a given decreasing sequence of numbers $\delta_n(t)\geq 0$ such that

$$t+\delta_n(t)\in[t,T]\text{ for all } n.$$

Define

$$\mathcal{H}=\mathcal{H}_4=\mathcal{F}_t\vee\sigma\left(B(t+\delta_n(t))\right);\ \ n=1,2,\cdots\tag{6.5.10}$$

Then, at each time t, the σ-algebra $\mathcal{H}_4(t)$ contains full information about the values of the Brownian motion at the future times $t + \delta_n(t);\ n = 1, 2, \cdots$ The amount of information that this represents, depends on the density of the sequence $\delta_n(t)$ near 0. Define

$$\rho_k(t) = \frac{1}{\delta_{k+1}^2} (\delta_k - \delta_{k+1}) \ln \left(\ln \left(\frac{1}{\delta_k - \delta_{k+1}} \right) \right); \quad k = 1, 2, \cdots \tag{6.5.11}$$

We may regard $\rho_k(t)$ as a measure of how small $\delta_k - \delta_{k+1}$ is compared to δ_{k+1}. If $\rho_k(t) \to 0$, then $\delta_k \to 0$ slowly, which means that the controller has at time t many immediate future values of $B(t+\delta_k(t));\ \ k = 1, 2, \cdots$, at her disposal when making her control value decision. For example, if

$$\delta_k(t) = \left(\frac{1}{k}\right)^p \text{ for some } p > 0,$$

then we see that

$$\lim_{k\to\infty} \rho_k(t) = \begin{cases} 0 & \text{if} \quad p < 1 \\ 1 & \text{if} \quad p = 1 \\ \infty & \text{if} \quad p > 1 \end{cases} \tag{6.5.12}$$

Lemma 6.5.10 *Suppose that $\mathcal{H} = \mathcal{H}_4$ as in (6.5.10) and that*

$$\lim_{k\to\infty} \rho_k(t) = 0 \quad \textit{in probability, uniformly in } t \in [0, T). \tag{6.5.13}$$

Then

$$E\left[d^- B(t) \,|\, \mathcal{H}_{t^-}\right] = d^- B(t);\ t \in [0, T)$$

Proof. For each $\varepsilon > 0$, choose $\delta_k = \delta_k^{(\varepsilon)}$ such that

$$\delta_{k+1} < \varepsilon \le \delta_k.$$

Then

$$
\begin{aligned}
&\frac{1}{\varepsilon}E\left[B(t+\varepsilon)-B(t)\,|\,\mathcal{H}_{t-}\right]\\
=&\frac{1}{\varepsilon}E\left[B(t+\varepsilon)-B(t)\,|\,\mathcal{F}_{t+\delta_{k+1}(t)}\vee\sigma\left(B(t+\delta_k(t))\right)\right]\\
=&\frac{1}{\varepsilon}\left[\frac{\delta_k-\varepsilon}{\delta_k-\delta_{k+1}}B(t+\delta_{k+1})+\frac{\varepsilon-\delta_{k+1}}{\delta_k-\delta_{k+1}}B(t+\delta_k)-B(t)\right]\\
=&\frac{1}{\varepsilon}\left[B(t+\delta_{k+1})-B(t)+\frac{\varepsilon-\delta_{k+1}}{\delta_k-\delta_{k+1}}\left\{B(t+\delta_k)-B(t+\delta_{k+1})\right\}\right]\\
=&\frac{\delta_{k+1}}{\varepsilon}\cdot\frac{1}{\delta_{k+1}}\left[B(t+\delta_{k+1})-B(t)\right]+\frac{\varepsilon-\delta_{k+1}}{\varepsilon(\delta_k-\delta_{k+1})}\left[B(t+\delta_k)-B(t+\delta_{k+1})\right]
\end{aligned}
$$

Note that

$$\frac{\varepsilon-\delta_{k+1}}{\varepsilon(\delta_k-\delta_{k+1})}\le\frac{1}{\delta_{k+1}}$$

and, by the law of iterated logarithm for Brownian motion (See e.g [119], p. 56),

$$
\begin{aligned}
&\overline{\lim_{k\to\infty}}\frac{1}{\delta_{k+1}}\left|B(t+\delta_k)-B(t+\delta_{k+1})\right|\\
&=\overline{\lim_{k\to\infty}}\frac{1}{\delta_{k+1}}\left[(\delta_k-\delta_{k+1})\ln\left(\ln\left(\frac{1}{\delta_k-\delta_{k+1}}\right)\right)\right]^{\frac{1}{2}}=0\text{ a.s.,}
\end{aligned}
$$

uniformly in t, by assumption (6.5.13).

Therefore, since

$$\frac{\delta_{k+1}}{\delta_k}\le\frac{\delta_{k+1}}{\varepsilon}\le 1,\ \text{ for all } k$$

and

$$\frac{\delta_{k+1}}{\delta_k}\to 1\text{ a.s., } k\to\infty,\ \text{ again by (6.5.13),}$$

we conclude that, using Definition 6.2.7,

$$
\begin{aligned}
\int_0^T\varphi(t)E\left[d^-B(t)\,|\,\mathcal{H}_{t-}\right]=&\lim_{\varepsilon\to 0}\int_0^T\varphi(t)\frac{E\left[B(t+\varepsilon)-B(t)\,|\,\mathcal{H}_{t-}\right]}{\varepsilon}\,dt\\
=&\lim_{k\to\infty}\int_0^T\varphi(t)\frac{B(t+\delta_{k+1})-B(t)}{\delta_{k+1}}\,dt=\int_0^T\varphi(t)\,d^-B(t)
\end{aligned}
$$

in probability, for all bounded forward-integrable $\mathcal{H}$-adapted processes φ. This proves the lemma. ■

6.6 Application to optimal insider portfolio

Consider a financial market with two investments possibilities:

1. A risk free asset, where the unit price $S_0(t)$ at time t is given by

$$dS_0(t) = r(t)S_0(t)\,dt, \quad S_0(0) = 1. \tag{6.6.1}$$

2. A risky asset, where the unit price $S_1(t)$ at time t is given by the stochastic differential equation

$$dS_1(t) = S_1(t^-)\left[\mu(t)dt + \sigma_0(t)dB^-(t) + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^-t,dz)\right], \quad S_1(0) > 0. \tag{6.6.2}$$

Here $r(t) \geq 0$, $\mu(t), \sigma_0(t)$, and $\gamma(t,z) \geq -1+\epsilon$ (for some constant $\epsilon > 0$) are given $\mathcal{G}_t$-predictable, forward integrable processes, where $\{\mathcal{G}_t\}_{t\in[0,T]}$ is a given filtration such that

$$\mathcal{F}_t \subset \mathcal{G}_t \text{ for all } t \in [0,T] \tag{6.6.3}$$

Suppose a trader in this market is an insider, in the sense that she has access to the information represented by $\mathcal{G}_t$ at time t. This means that if she chooses a portfolio $u(t)$, representing the *amount* she invests in the risky asset at time t, then this portfolio is a $\mathcal{G}_t$-predictable stochastic process.

The corresponding wealth process $X(t) = X^{(u)}(t)$ will then satisfies the (forward) SDE

$$\begin{aligned} d^-X(t) =& \frac{X(t)-u(t)}{S_0(t)}dS_0(t) + \frac{u(t)}{S_1(t)}d^-S_1(t) \\ =& X(t)r(t)dt + u(t)\Big[(\mu(t)-r(t))\,dt + \sigma_0(t)dB^-(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^-t,dz)\Big], \quad t \in [0,T], \qquad (6.6.4) \\ X(0) =& x > 0. \qquad (6.6.5) \end{aligned}$$

By choosing $S_0(t)$ as a numeraire, we can, without loss of generality, assume that

$$r(t) = 0 \tag{6.6.6}$$

from now on. Then Equations (6.6.4) and (6.6.5) simplify to

$$\begin{cases} d^- X(t) &= u(t)\left[\mu(t)dt + \sigma_0(t)dB^-(t) + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^-t, dz)\right], \\ X(0) &= x > 0. \end{cases} \tag{6.6.7}$$

This is a controlled Itô-Lévy process of the type discussed in section 6.4 and we can apply the results of that section to the problem of the insider to maximize the expected utility of the terminal wealth, *i.e.,* to find $\Phi(x)$ and $u^* \in \mathcal{A}_{\mathcal{G}}$ such that

$$\Phi(x) = \sup_{u\in\mathcal{A}_{\mathcal{G}}} E\left[U\left(X^{(u)}(T)\right)\right] = E\left[U\left(X^{(u^*)}(T)\right)\right], \tag{6.6.8}$$

where $U : \mathbb{R}_+ \to \mathbb{R}$ is a given utility function, assumed to be concave, strictly increasing and C^1. In this case the processes $K(t)$, $L(t)$, $M(t)$ and $R(t,z)$, given respectively by Equations (6.3.4), (6.4.3), (6.4.4) and (6.4.5), take the form

$$K(t) = U'(X(T)), \tag{6.6.9}$$

$$\begin{aligned} L(t) =& U'(X(T))\left[\mu(t) + D_{t+}\sigma_0(t) + \int_{\mathbb{R}_0} D_{t+,z}\gamma(t,z)\,\nu(dz)\right] \\ &+ \int_{\mathbb{R}_0} D_{t,z}U'(X(T))\left[\gamma(t,z) + D_{t+,z}\gamma(t,z)\right]\nu(dz) + D_tU'(X(T))\,\sigma_0(t), \end{aligned} \tag{6.6.10}$$

$$M(t) = U'(X(T))\,\sigma_0(t), \tag{6.6.11}$$

$$R(t,z) = \left\{U'(X(T)) + D_{t,z}U'(X(T))\right\}\left\{\gamma(t,z) + D_{t+,z}\gamma(t,z)\right\}. \tag{6.6.12}$$

6.6.1 Case $\mathcal{G}_t = \mathcal{F}_{G_t}$, $G_t \supset [0,t]$

In this case, $\mathcal{G}_t$ satisfies Equation (6.5.2) and hence conditions (Co1) and (Co2). Therefore, Theorem 6.5.4 of Section 6.4 gives the following:

Theorem 6.6.1 *Suppose that* $P\left(\lambda\left\{t \in [0,T];\ \sigma_0(t) \neq 0\right\} > 0\right) > 0$ *where* λ *denotes the Lebesgue measure on* $\mathbb{R}$ *and that* $\mathcal{G}_t$ *is given by (6.5.2). Then, there does not exist an optimal portfolio* $u^* \in \mathcal{A}_{\mathcal{G}}$ *of the insider's portfolio problem (6.6.8).*

Proof. Suppose an optimal portfolio exists. Then we have seen that in either of the cases, the conclusion is that

$$E\left[L(t)|\,\mathcal{G}_t\right] = E\left[M(t)|\,\mathcal{G}_t\right] = E\left[R(t,z)|\,\mathcal{G}_t\right] = 0$$

for a.a. $t \in [0,T]$, $z \in \mathbb{R}_0$. In particular,

$$E\left[M(t)|\,\mathcal{G}_t\right] = E\left[U'\left(X(T)\right)\middle|\,\mathcal{G}_t\right]\sigma_0(t) = 0, \text{ for a.a } t \in [0,T].$$

Since $U' > 0$, this contradicts our assumption about U. Hence an optimal portfolio cannot exist. ■

Remark 6.6.2 *In the case that $\mathcal{G}_t = \mathcal{H}_i$, $i = 1$ or $i = 3$ it is known that $B(\cdot)$ is not a semimartingale with respect to $\{\mathcal{G}_t\}$ and hence an optimal portfolio cannot exists, by Theorem 3.8 in [14] and Theorem 15 in [33]. It follows that $S_1(t)$ is not a $\mathcal{G}_t$-semimartingale either and hence we can even deduce that the market has an arbitrage for the insider in this case, by Theorem 7.2 in [29]*

6.6.2 Case $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T))$

In this case, $\mathcal{G}_t$ is not D-commutable, therefore, we apply results from Section 6.5.2. We have seen that

$$E\left[\, d^- B\,|\mathcal{G}_{t-}\right] = \frac{B(T) - B(t)}{T - t} dt$$

(Corollary 6.5.9). It follows that

Theorem 6.6.3 *[Brownian motion case] Assume that $\mu(t) = \mu_0$, $\sigma_0(t) = \sigma_0$, $\gamma(t,z) = 0$ and conditions in Theorem 6.5.6 hold. In addition, require that*

1. $E\left[M(t)|\,\mathcal{G}_{t-}\right] \in \mathbb{M}_{1,2}^B$
2. $\underline{\lim}_{t\uparrow T} E\left[\left|D_{t^+}E\left[M(t)|\,\mathcal{G}_{t-}\right]\right|\right] < \infty.$
3. $\underline{\lim}_{t\uparrow T} E\left[|L(t)|\right] < \infty.$

Then, there does not exist a critical point of the performance functional $J(u)$ in (6.1.3).

Proof. Assume that there is a critical point of the performance functional $J(u)$ in (6.1.3). It follows from Theorems 6.4.1, 6.5.6 and 6.5.7 that Equation 6.5.4 holds. Replacing

$K(t)$, $L(t)$, and $M(t)$ by their given expressions in Equations (6.6.9), (6.6.10) and (6.6.11), Equation (6.5.4) becomes

$$0 = E\left[\mu_0 U'(X(T)) + \sigma_0 D_t U'(X(T))\ |\mathcal{G}_{t-}\right] + E\left[U'(X(T))\sigma_0\ |\mathcal{G}_{t-}\right] \frac{B(T)-B(t)}{T-t}$$
$$- D_{t+}E\left[\sigma_0 U'(X(T))\middle|\, \mathcal{G}_{t-}\right],\ \text{a.e } t \tag{6.6.13}$$

Taking the limit as $t \uparrow T$, the second term in Equation (6.6.13) goes to ∞. Therefore, there is no critical point for the performance functional $J(u)$ in (6.1.3). ■

Remark 6.6.4 *This result is a generalization of a result in [72], where the same conclusion was obtained in the special case when*

$$U(x) = \ln(x)$$

6.6.3 Case $\mathcal{G}_t = \mathcal{H} = \mathcal{H}_4$

In this case, we have seen that under the condition of Lemma 6.5.10

$$E\left[d^- B(t)\ |\, \mathcal{H}_{t-}\right] = d^- B(t).$$

Therefore, we get

Theorem 6.6.5 *Suppose that, with $\mathcal{G}_t$ as above, the conditions of Theorem 6.5.7 are satisfied. Then u is a critical point for $J(u) = E\left[U(X^u(T))\right]$ if and only if*

$$E\left[L(t)|\, \mathcal{G}_{t-}\right] - D_{t+}E\left[M(t)|\, \mathcal{G}_{t-}\right] = 0, \tag{6.6.14}$$

and

$$E\left[M(t)|\, \mathcal{G}_{t-}\right] = 0,\ \textit{for a.a } t \in [0,T]. \tag{6.6.15}$$

Proof. It follows from Equation (6.5.4) and the uniqueness of decomposition of forward processes. ■

Corollary 6.6.6 *Suppose $\mathcal{G}_t$ is as in Theorem 6.6.5 and that $P\left(\lambda\left\{t \in [0,T];\ \sigma_0(t) \neq 0\right\} > 0\right) > 0$ where λ denotes the Lebesgue measure on $\mathbb{R}$. Then, there does not exist an optimal portfolio $u^* \in \mathcal{A}_{\mathcal{G}}$ for the performance $J(u) = E\left[U(X^u(T))\right]$.*

Proof. It follows from Equation (6.6.15) and the property of the utility function U. ■

6.7 Application to optimal insider consumption

Suppose we have a cash flow $X(t) = X^{(u)}(t)$ given by

$$\begin{cases} dX(t) &= (\mu(t) - u(t))\,dt + \sigma(t)dB(t) + \displaystyle\int_{\mathbb{R}_0} \theta(t,z)\widetilde{N}(dt,dz), \\ X(0) &= x \in \mathbb{R}. \end{cases} \tag{6.7.1}$$

Here $\mu(t),\ \sigma(t)$ and $\theta(t,z)$ are given $\mathcal{G}_t$-predictable processes and $u(t) \geq 0$ is our *consumption* rate, assumed to be adapted to a given insider filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$ where

$$\mathcal{F}_t \subset \mathcal{G}_t \text{ for all } t.$$

Let $f(t,u,\omega);\ t \in [0,T],\ u \in \mathbb{R},\ \omega \in \Omega$ be a given $\mathcal{F}_T$-measurable utility process. Assume that $u \to f(t,u,\omega)$ is strictly increasing, concave and C^1 for a.a (t,ω).

Let $g(x,\omega);\ x \in \mathbb{R},\ \omega \in \Omega$ be a given $\mathcal{F}_T$-measurable random variable for each x. Assume that $x \to g(x,\omega)$ is concave for a.a ω. Define the performance functional J by

$$J(u) = E\left[\int_0^T f(t,u(t),\omega)\,dt + g\left(X^{(u)}(T),\omega\right)\right];\ u \in \mathcal{A}_{\mathcal{G}},\ u \geq 0. \tag{6.7.2}$$

Note that $u \to J(u)$ is concave, so $u = \widehat{u}$ maximizes $J(u)$ if and only if $\widehat{u}$ is a critical point of $J(u)$.

Theorem 6.7.1 *(Optimal insider consumption I)*

$\widehat{u}$ is an optimal insider consumption rate for the performance functional J in Equation (6.7.2) if and only if

$$E\left[\frac{\partial}{\partial u}f(t,\widehat{u}(t),\omega)\bigg|\mathcal{G}_t\right] = E\left[g'\left(X^{(\widehat{u})}(T),\omega\right)\bigg|\mathcal{G}_t\right]. \tag{6.7.3}$$

Proof. In this case we have

$$\begin{aligned} K(t) &= g'\left(X^{(u)}(T)\right) \\ L(t) &= -\,g'\left(X^{(u)}(T)\right) + \frac{\partial}{\partial u}f(t,\widehat{u}(t)) \\ M(t) &= R(t,z) = 0 \end{aligned}$$

Therefore Theorem 6.4.1 gives $\widehat{u}$ is a critical point for $J(u)$ if and only if

$$0 = E\left[L(t)|\,\mathcal{G}_t\right] = E\left[\left.\frac{\partial}{\partial u} f(t, \widehat{u}(t))\right|\mathcal{G}_t\right] + E\left[-g'\left(X^{(\widehat{u})}(T)\right)\Big|\,\mathcal{G}_t\right].$$

∎

Since $X^{(\widehat{u})}(T)$ depends on $\widehat{u}$, Equation (6.7.3) does not give the value of $\widehat{u}(t)$ directly. However, in some special cases $\widehat{u}$ can be found explicitly:

Corollary 6.7.2 *(Optimal insider consumption II)*

Assume that

$$g(x, \omega) = \lambda(\omega)x \tag{6.7.4}$$

for some $\mathcal{G}_T$-measurable random variable $\lambda > 0$.

Then the optimal consumption rate $\widehat{u}(t)$ is given by

$$E\left[\left.\frac{\partial}{\partial u} f(t, \widehat{u}, \omega)\right|\mathcal{G}_t\right]_{u=\widehat{u}(t)} = E\left[\lambda|\,\mathcal{G}_t\right]. \tag{6.7.5}$$

Thus we see that an optimal consumption rate exists, for *any* given insider information filtration $\{\mathcal{G}_t\}_{t\geq 0}$. It is not necessary to be in a semimartingale setting.

Chapter 7

Stochastic Differential Games in Insider markets via Malliavin Calculus

7.1 Introduction

In real world, market agents have access to different levels of information and it is important to understand what value particular pieces of information have. This chapter is devoted to the study of a class of two-player stochastic differential games in which the players have different information on the payoff. The different agents invest different amounts of capital in order to optimize their utility. We derive necessary and sufficient conditions for the existence of Nash-equilibria for this game and characterize these for various levels of information asymmetry. The framework is the one of stochastic differential games with anticipative strategy sets.

In the following, let $\{B_s\}_{0\le s\le T}$ be a Brownian motion and $\widetilde{N}(dz, ds) = N(dz, ds) - ds\nu(dz)$ be a compensated Poisson random measure associated with a Lévy process with Lévy measure ν on the (complete) filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$. In the sequel, we assume that the Lévy measure ν fulfills

$$\int_{\mathbb{R}_0} z^2 \, \nu(dz) < \infty,$$

where $\mathbb{R}_0 := \mathbb{R} \backslash \{0\}$.

Suppose that the state process $X(t) = X^{(u)}(t, \omega)$; $t \geq 0$, $\omega \in \Omega$ is a controlled Itô-Lévy process in $\mathbb{R}$ of the form:

$$\begin{cases} d^- X(t) &= b(t, X(t), u_0(t), \omega) \, dt + \sigma(t, X(t), u_0(t), \omega) \, d^- B(t) \\ & \quad + \int_{\mathbb{R}_0} \gamma(t, X(t), u_0(t), u_1(t,z), z, \omega) \, \widetilde{N}(dz, d^- t); \\ X(0) &= x \in \mathbb{R} \end{cases} \tag{7.1.1}$$

Where the coefficients $b : [0,T] \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}$, and $\gamma : [0,T] \times \mathbb{R} \times U \times K \times \mathbb{R}_0 \times \Omega \longrightarrow \mathbb{R}$ are measurable functions, where $U \subset \mathbb{R}^2$, $K \subset \mathbb{R} \times \mathbb{R}_0$ are given open convex sets. Here we consider filtrations $\left\{\mathcal{G}_t^i\right\}_{t \in [0,T]}$, $i = 1, 2$ such that

$$\mathcal{F}_t \subset \mathcal{G}_t^i \subset \mathcal{F}_T, \quad t \in [0,T], \quad i = 1, 2, \tag{7.1.2}$$

representing the information available to the controller at time t.

Since $B(t)$ and $\widetilde{N}(dz, dt)$ need not be a semimartingale with respect to $\{\mathcal{G}_t^i\}_{t \geq 0}$, $i = 1, 2$, the two last integrals in (7.1.1) are *anticipating* stochastic integrals that we interpret as *forward* integrals.

The control processes $u_0(t)$ and $u_1(t,z)$ with values in given open convex sets U and K respectively for a.a $t \in [0,T]$, $z \in \mathbb{R}_0$ are called *admissible controls* if (7.1.1) has a unique (strong) solution $X = X^{(u_0, u_1)}$ such that the components of $u_0(\cdot)$ and $u_1(\cdot, \cdot)$ are adapted to the considered filtrations $\left\{\mathcal{G}_t^1\right\}_{t \in [0,T]}$ and $\left\{\mathcal{G}_t^2\right\}_{t \in [0,T]}$ respectively.

Let $f : [0,T] \times \mathbb{R} \times U \times K \times \Omega \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ be given measurable functions and the given performance functionals for players are as follows:

$$J_i(u_0, u_1) := E^x \left[\int_0^T f_i(t, X(t), u_0(t), u_1(t,z), \omega) \, \mu(dz) dt + g_i(X(T), \omega) \right], \; i = 1, 2, \tag{7.1.3}$$

where μ is a measure on the given measurable space $(\Omega, \mathcal{F}_T)$ and $E^x = E$ denotes the expectation with respect to P given that $X(0) = x$. Suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form

$$u_0(t) = (\pi_0(t), \theta_0(t))\,;\ t \in [0,T]\,, \tag{7.1.4}$$

$$u_1(t,z) = (\pi_1(t,z), \theta_1(t))\,;\ t \in [0,T] \times \mathbb{R}_0. \tag{7.1.5}$$

Let $\mathcal{A}_\Pi$ (respectively $\mathcal{A}_\Theta$) denote the given family of controls $\pi = (\pi_0, \pi_1)$ (respectively $\theta = (\theta_0, \theta_1)$) such that they are contained in the set of $\mathcal{G}_t^1$-adapted controls (respectively $\mathcal{G}_t^2$-adapted controls), (7.1.1) has a unique strong solution up to time T and

$$E^x\left[\int_0^T |f_i(t, X(t), u_0(t), u_1(t,z), \omega)|\ \mu(dz)dt\ +\ |g_i(X(T), \omega)|\right] < \infty,\ i = 1, 2.$$

The insider information non-zero-sum stochastic differential games problem we analyze is the following:

Problem 7.1.1 *Find* $(\pi^*, \theta^*) \in \mathcal{A}_\Pi \times \mathcal{A}_\Theta$ *(if it exists) such that*

1. $J_1(\pi, \theta^*) \leq J_1(\pi^*, \theta^*)$ *for all* $\pi \in \mathcal{A}_\Pi$

2. $J_2(\pi^*, \theta) \leq J_2(\pi^*, \theta^*)$ *for all* $\theta \in \mathcal{A}_\Theta$

The pair (π^*, θ^*) is called a Nash Equilibrium (if it exists). The intuitive idea is that there are two players, Player I and Player II. While Player I controls π, Player II controls θ. Each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy (i.e., by changing unilaterally). Player I and Player II are in Nash Equilibrium if each player is making the best decision she can, taking into account the other players decision. Note that since we allow b, σ, γ, f and g to be stochastic processes and since our controls are also $\mathcal{G}_t^1$-adapted (respectively $\mathcal{G}_t^2$-adapted), this problem is not of Markovian type and hence cannot be embedded into the framework of dynamic programming.

This chapter is inspired by ideas developed both in Chapter 6 where we use Malliavin calculus to derive a general maximum principle for anticipative stochastic control, and, in An et al [4], where the authors derived a general maximum principle for stochastic differential games with partial information. This Chapter covers the insider case in [44], since we deal with controls being adapted to general supfiltrations of the underlying reference filtration. Moreover, our Malliavin calculus approach to stochastic differential games with insider information for *Itô-Lévy processes* allows for optimization of very general performance functionals. We apply our results to a worst case scenario portfolio problem in finance under additional information. We show that there does not exist a Nash-equilibrium for the insider. We prove that there exists a Nash-equilibrium insider consumption, and in some special cases the optimal solution can be expressed explicitly.

The framework in this Chapter is the same as in Chapter 6. The reader is then referred to Section 6.2 of that chapter for a brief recall on some basic concepts of Malliavin calculus and its connection to the theory forward integration, and also some notions and results from white noise analysis.

7.2 A stochastic maximum principle for the stochastic differential games for insider

We now return to Problem 7.1.1 given in the introduction. We make the following assumptions:

1. The functions $b : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\gamma : [0,T] \times \mathbb{R} \times U \times K \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \Omega \to \mathbb{R}$ are contained in C^1 with respect to the arguments $x \in \mathbb{R}$, $u_0 \in U$ and $u_1 \in K$ for each $t \in [0,T]$ and a.a. $\omega \in \Omega$.

2. For all $s, r, t \in (0,T)$, $t \leq r$ and all bounded $\mathcal{G}_t^2$-measurable (respectively $\mathcal{G}_t^1$-measurable) random variables $\alpha = \alpha(\omega)$ (respectively $\xi = \xi(\omega)$), $\omega \in \Omega$, the controls $\beta_\alpha(s) :=$

$\left(0, \beta_\alpha^i(s)\right)$ and $\eta_\xi(s) := \left(0, \eta_\xi^i(s)\right)$ for $i = 1, 2$ with

$$\beta_\alpha^i(s) := \alpha^i(\omega)\chi_{[t,r]}(s), \qquad 0 \le s \le T, \tag{7.2.1}$$

respectively

$$\eta_\xi^i(s) := \xi^i(\omega)\chi_{[t,r]}(s), \qquad 0 \le s \le T, \tag{7.2.2}$$

belong to $\mathcal{A}_\Pi$ (respectively $\mathcal{A}_\Theta$). Also, we will denote the transposes of the vectors β and η by β^*, η^* respectively.

3. For all $\pi, \beta \in \mathcal{A}_\Pi$ with β bounded, there exists a $\delta_1 > 0$ such that

$$\pi + y\beta \in \mathcal{A}_\Pi\,, \text{ for all } y \in (-\delta_1, \delta_1), \tag{7.2.3}$$

and such that the family

$$\Big\{ \frac{\partial}{\partial x} f_1(t, X^{(\pi+y\beta,\theta)}(t), \pi + y\beta, \theta, z) \frac{d}{dy} X^{(\pi+y\beta,\theta)}(t) \\ + \nabla_\pi f_1(t, X^{(\pi+y\beta,\theta)}(t), \pi + y\beta, \theta, z)\beta^*(t) \Big\}_{y\in(-\delta_1,\delta_1)}$$

is $\lambda \times \nu \times \mathbb{P}-$uniformly integrable and

$$\left\{ g'(X^{(\pi+y\beta,\theta)}(T)) \frac{d}{dy} X^{(\pi+y\beta,\theta)}(T) \right\}_{y\in(-\delta_1,\delta_1)}$$

is $\mathbb{P}-$uniformly integrable. Similarly, for all $\theta, \eta \in \mathcal{A}_\Theta$ with η bounded, there exists a $\delta_2 > 0$ such that

$$\theta + v\eta \in \mathcal{A}_\Theta\,, \text{ for all } v \in (-\delta_2, \delta_2), \tag{7.2.4}$$

and such that the family

$$\Big\{ \frac{\partial}{\partial x} f_2(t, X^{(\pi,\theta+v\eta)}(t), \pi, \theta + v\eta, z) \frac{d}{dy} X^{(\pi,\theta+v\eta)}(t) \\ + \nabla_\theta f_2(t, X^{(\pi,\theta+v\eta)}(t), \pi, \theta + v\eta, z)\eta^*(t) \Big\}_{v\in(-\delta_2,\delta_2)}$$

is $\lambda \times \nu \times \mathbb{P}-$uniformly integrable and

$$\left\{ g'(X^{(\pi,\theta+v\eta)}(T)) \frac{d}{dy} X^{(\pi,\theta+v\eta)}(T) \right\}_{v\in(-\delta_2,\delta_2)}$$

is $\mathbb{P}-$uniformly integrable.

4. For all $\pi, \beta \in \mathcal{A}_\Pi$ and $\theta, \eta \in \mathcal{A}_\Theta$ with β, η bounded the processes

$$Y(t) = Y_\beta(t) = \frac{d}{dy} X^{(\pi+y\beta,\theta)}(t)\Big|_{y=0}, \quad V(t) = V_\eta(t) = \frac{d}{dv} X^{(\pi,\theta+v\eta)}(t)\Big|_{v=0}$$

exist and follow the SDE, respectively:

$$\begin{aligned} dY_\beta^\pi(t) = & Y_\beta(t^-)\Big[\frac{\partial}{\partial x} b(t, X(t), \pi_0(t), \theta_0(t))\, dt + \frac{\partial}{\partial x}\sigma(t, X(t), \pi_0(t), \theta_0(t))\, d^- B(t) \\ & + \int_{\mathbb{R}_0} \frac{\partial}{\partial x}\gamma\left(t, X(t^-), \pi_0(t), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z\right) \widetilde{N}(dz, d^- t)\Big] \\ & + \beta^*(t)\left[\nabla_\pi b(t, X(t), \pi_0(t), \theta_0(t))\, dt + \nabla_\pi \sigma(t, X(t), \pi_0(t), \theta_0(t))\, d^- B(t)\right. \\ & \left. + \int_{\mathbb{R}_0} \nabla_\pi \gamma\left(t, X(t^-), \pi_0(t), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z\right) \widetilde{N}(dz, d^- t)\right] \end{aligned} \tag{7.2.5}$$

$$Y(0) = 0$$

and

$$\begin{aligned} dV_\eta^\theta(t) = & V_\eta(t^-)\Big[\frac{\partial}{\partial x} b(t, X(t), \pi_0(t), \theta_0(t))\, dt + \frac{\partial}{\partial x}\sigma(t, X(t), \pi_0(t), \theta_0(t))\, d^- B(t) \\ & + \int_{\mathbb{R}_0} \frac{\partial}{\partial x}\gamma\left(t, X(t^-), \pi_0(t), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z\right) \widetilde{N}(dz, d^- t)\Big] \\ & + \eta^*(t)\left[\nabla_\theta b(t, X(t), \pi_0(t), \theta_0(t))\, dt + \nabla_\theta \sigma(t, X(t), \pi_0(t), \theta_0(t))\, d^- B(t)\right. \\ & \left. + \int_{\mathbb{R}_0} \nabla_\theta \gamma\left(t, X(t^-), \pi_0(t), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z\right) \widetilde{N}(dz, d^- t)\right] \end{aligned} \tag{7.2.6}$$

$$V(0) = 0$$

5. Suppose that for all $\pi \in \mathcal{A}_\Pi$ and $\theta \in \mathcal{A}_\Theta$ the following processes

$$K_i(t) := g_i'(X(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial}{\partial x} f_i(s, X(s), \pi, \theta, z_1)\, \mu(dz_1) ds, \tag{7.2.7}$$

$$D_t K_i(t) := D_t g_i'(X(T)) + \int_t^T D_t \frac{\partial}{\partial x} f_i(s, X(s), \pi, \theta, z_1)\, \mu(dz_1) ds,$$

$$D_{t,z} K_i(t) := D_{t,z} g_i'(X(T)) + \int_t^T \int_{\mathbb{R}_0} D_{t,z} \frac{\partial}{\partial x} f_i(s, X(s), \pi, \theta, z_1)\, \mu(dz_1) ds,$$

$$\begin{aligned} H_i^0(s, x, \pi, \theta) &:= K_i(s)\Big(b(s, x, \pi_0, \theta_0) + D_{s+}\sigma(s, x, \pi_0, \theta_0) \\ &\quad + \int_{\mathbb{R}_0} D_{s+,z}\gamma(s, x, \pi, \theta, z)\, \nu(dz)\Big) + D_s K(s) \sigma(s, x, \pi_0, \theta_0) \\ &\quad + \int_{\mathbb{R}_0} D_{s,z} K(s) \Big\{ \gamma(s, x, \pi, \theta, z) + D_{s+,z}\gamma(s, x, \pi, \theta, z) \Big\}\, \nu(dz), \end{aligned} \tag{7.2.8}$$

$$\begin{aligned} G(t, s) := \exp \Bigg[&\int_t^s \left\{ \frac{\partial b}{\partial x}(r, X(r), \pi_0(r), \theta_0(r)) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2 (r, X(r), \pi_0(r), \theta_0(r)) \right\} dr \\ &+ \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), \pi_0(r), \theta_0(r))\, d^- B(r) \\ &+ \int_t^s \int_{\mathbb{R}_0} \left\{ \ln\left(1 + \frac{\partial \gamma}{\partial x}(r, X(r), \pi, \theta, z)\right) - \frac{\partial \gamma}{\partial x}(r, X(r), \pi, \theta, z) \right\} \nu(dz)\, dt \\ &+ \int_t^s \int_{\mathbb{R}_0} \left\{ \ln\left(1 + \frac{\partial \gamma}{\partial x}(r, X(r^-), \pi(r^-, z), \theta(r^-, z), z)\right) \right\} \widetilde{N}(dz, d^- r) \Bigg], \end{aligned} \tag{7.2.9}$$

$$p_i(t) := K_i(t) + \int_t^T \frac{\partial}{\partial x} H_i^0(s, X(s), \pi_0(s), \pi_1(s, z), \theta_0(s), \theta_1(s, z)) G(t, s)\, ds, \tag{7.2.10}$$

$$q_i(t) := D_t p_i(t), \tag{7.2.11}$$

$$r_i(t, z) := D_{t,z} p_i(t), \tag{7.2.12}$$

all exist for $i = 1, 2,\ 0 \le t \le s \le T,\ z_1, z \in \mathbb{R}_0$.

Now let us introduce the *general Hamiltonians of insiders*.

Definition 7.2.1 *The general stochastic Hamiltonians for the stochastic differential games for insiders in Problem 7.1.1 are the functions*

$$H_i(t, x, \pi, \theta, \omega) : [0, T] \times \mathbb{R} \times U \times K \times \Omega \longrightarrow \mathbb{R},\ i = 1, 2$$

defined by

$$H_i\,(t,x,\pi,\theta,\omega) \;:=\; \int_{\mathbb{R}_0} f_i(t,x,\pi,\theta,z,\omega)\,\mu(dz) + p_i(t)\Big(b(t,x,\pi_0,\theta_0,\omega)\;+D_{t+}\sigma(t,x,\pi_0,\theta_0,\omega)$$
$$+\int_{\mathbb{R}_0} D_{t+,z}\gamma(t,x,\pi,\theta,z,\omega)\,\nu(dz)\Big) + q_i(t)\sigma(t,x,\pi_0,\theta_0,\omega)$$
$$+\int_{\mathbb{R}_0} r_i(t,z)\Big\{\gamma(t,x,\pi,\theta,z,\omega)+D_{t+,z}\gamma(t,x,\pi,\theta,z,\omega)\Big\}\nu(dz), \qquad (7.2.13)$$

where $\pi=(\pi_0,\pi_1)$ *and* $\theta=(\theta_0,\theta_1)$

We can now state a general stochastic maximum principle of insider for zero-sum games:

Theorem 7.2.2 ***[Maximum principle for insider non zero-sum games]***

Retain conditions 1-5.

(i) *Suppose* $(\widehat{\pi},\widehat{\theta})\in\mathcal{A}_\Pi\times\mathcal{A}_\Theta$ *is a Nash equilibrium, i.e.*

1. $J_1(\pi,\widehat{\theta})\le J_1(\widehat{\pi},\widehat{\theta})$ *for all* $\pi\in\mathcal{A}_\Pi$

2. $J_2(\widehat{\pi},\theta)\le J_2(\widehat{\pi},\widehat{\theta})$ *for all* $\theta\in\mathcal{A}_\Theta$

Then

$$E\Big[\,\nabla_\pi \widehat{H}_1(t,X^{\pi,\widehat{\theta}}(t),\pi,\widehat{\theta},\omega)\Big|_{\pi=\widehat{\pi}}\Big|\,\mathcal{G}_t^2\Big]\;+\;E[A]=0\quad a.e.\ \ in\ \ (t,\omega), \qquad (7.2.14)$$

and

$$E\Big[\,\nabla_\theta \widehat{H}_2(t,X^{\widehat{\pi},\theta}(t),\widehat{\pi},\theta,\omega)\Big|_{\theta=\widehat{\theta}}\Big|\,\mathcal{G}_t^1\Big]\;+\;E[B]=0\quad a.e.\ \ in\ \ (t,\omega), \qquad (7.2.15)$$

where A *is given by Equation (A.5.21) and* B *is defined in a similar way.*

$$\widehat{X}(t)=X^{(\widehat{\pi},\widehat{\theta})}(t),$$
$$\widehat{H}_i\Big(t,\widehat{X}(t),\pi,\theta,\omega\Big) := \int_{\mathbb{R}_0} f_i(t,\widehat{X}(t),\pi,\theta,z,\omega)\,\mu(dz) \qquad (7.2.16)$$
$$+\,\widehat{p}_i(t)\Big(b(t,\widehat{X}(t),\pi_0,\theta_0,\omega)\;+D_{t+}\sigma(t,\widehat{X}(t),\pi_0,\theta_0,\omega)$$
$$+\int_{\mathbb{R}_0} D_{t+,z}\gamma(t,\widehat{X}(t),\pi,\theta,z,\omega)\,\nu(dz)\Big)$$
$$+\,\widehat{q}_i(t)\sigma(t,\widehat{X}(t),\pi_0,\theta_0,\omega)$$
$$+\int_{\mathbb{R}_0}\widehat{r}_i(t,z)\Big\{\gamma(t,\widehat{X}(t),\pi,\theta,z,\omega)+D_{t+,z}\gamma(t,\widehat{X}(t),\pi,\theta,z,\omega)\Big\}\nu(dz),$$

with

$$\widehat{p}_i(t) := \widehat{K}_i(t) + \int_t^T \frac{\partial}{\partial x} \widehat{H}_i^0(s, \widehat{X}(s), \widehat{\pi}(s), \widehat{\theta}(s)) \widehat{G}(t,s)\, ds, \tag{7.2.17}$$

$$\widehat{K}_i(t) := g_i'(\widehat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial}{\partial x} f_i(s, \widehat{X}(s), \widehat{\pi}(s,z), \widehat{\theta}(s,z), z)\, \mu(dz) ds, \tag{7.2.18}$$

$$\begin{aligned} \widehat{H}_i^0(s, \widehat{X}, \widehat{\pi}, \widehat{\theta}) := \widehat{K}_i(s) \Big(& b(s, \widehat{X}, \widehat{\pi}_0, \widehat{\theta}_0) + D_{s+}\sigma(s, \widehat{X}, \widehat{\pi}_0, \widehat{\theta}_0) \\ & + \int_{\mathbb{R}_0} D_{s+,z}\gamma(s, \widehat{X}, \widehat{\pi}, \widehat{\theta}, z)\, \nu(dz) \Big) + D_s K_i(s) \sigma(s, \widehat{X}, \widehat{\pi}_0, \widehat{\theta}_0) \\ & + \int_{\mathbb{R}_0} D_{s,z} K_i(s) \Big\{ \gamma(s, \widehat{X}, \widehat{\pi}, \widehat{\theta}, z) + D_{s+,z}\gamma(s, \widehat{X}, \widehat{\pi}, \widehat{\theta}, z) \Big\}\, \nu(dz), \end{aligned} \tag{7.2.19}$$

$$\begin{aligned} \widehat{G}(t,s) := \exp \Bigg[& \int_t^s \left\{ \frac{\partial b}{\partial x} \left(r, \widehat{X}(r), \widehat{\pi}_0(r), \widehat{\theta}_0(r) \right) - \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2 \left(r, \widehat{X}(r), \widehat{\pi}_0(r), \widehat{\theta}_0(r) \right) \right\} dr \\ & + \int_t^s \frac{\partial \sigma}{\partial x} \left(r, \widehat{X}(r), \widehat{\pi}_0(r), \widehat{\theta}_0(r) \right) d^- B(r) \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln \left(1 + \frac{\partial \gamma}{\partial x} \left(r, \widehat{X}(r), \widehat{\pi}, \widehat{\theta}, z \right) \right) - \frac{\partial \gamma}{\partial x} \left(r, \widehat{X}(r), \widehat{\pi}, \widehat{\theta}, z \right) \right\} \nu(dz)\, dt \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln \left(1 + \frac{\partial \gamma}{\partial x} \left(r, \widehat{X}(r^-), \widehat{\pi}(r^-, z), \widehat{\theta}(r^-, z), z \right) \right) \right\} \widetilde{N}(dz, d^- r) \Bigg]. \end{aligned} \tag{7.2.20}$$

(ii) *Conversely, suppose* $(\widehat{\pi}, \widehat{\theta}) \in \mathcal{A}_\Pi \times \mathcal{A}_\Theta$ *such Equations (7.2.14) and (7.2.15) hold. Then*

$$\left. \frac{\partial J_1}{\partial y} (\widehat{\pi} + y\beta, \widehat{\theta}) \right|_{y=0} = 0 \text{ for all } \beta, \tag{7.2.21}$$

$$\left. \frac{\partial J_2}{\partial v} (\widehat{\pi}, \widehat{\theta} + v\eta) \right|_{v=0} = 0 \text{ for all } \eta, \tag{7.2.22}$$

In particular, if

$$\pi \to J_1(\pi, \widehat{\theta}),$$

and

$$\theta \to J_2(\widehat{\pi}, \theta)$$

are concave, then $\left(\widehat{\pi}, \widehat{\theta} \right)$ *is a Nash equilibrium.*

Proof. See Appendix A, Section A.5. ■

7.2.1 Zero-sum games

Here, we suppose that the given performance functional for Player I is the negative of that for Player II, i.e.,

$$J_1(u_0, u_1) := E\left[\int_0^T f(t, X(t), u_0(t), u_1(t,z), \omega)\, \mu(dz)dt \,+\, g(X(T), \omega)\right] = -J_2(u_0, u_1) \tag{7.2.23}$$

where $E = E_P^x$ denotes the expectation with respect to P given that $X(0) = x$. Suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form (7.1.4) and (7.1.5). Let $\mathcal{A}_\Pi$ (respectively $\mathcal{A}_\Theta$) denote the given family of controls $\pi = (\pi_0, \pi_1)$ (respectively $\theta = (\theta_0, \theta_1)$) such that they are contained in the set of $\mathcal{G}_t^1$-adapted controls (respectively $\mathcal{G}_t^2$-adapted controls), (7.1.1) has a unique strong solution up to time T and

$$E\left[\int_0^T |f(t, X(t), u_0(t), u_1(t,z), \omega)|\; \mu(dz)dt \,+\, |g(X(T), \omega)|\right] < \infty. \tag{7.2.24}$$

Then the insider information zero-sum stochastic differential games problem is the following:

Problem 7.2.3 *Find $\pi^* \in \mathcal{A}_\Pi^{\mathcal{G}^2}$ and $\theta^* \in \mathcal{A}_\Theta^{\mathcal{G}^1}$ and $\Phi \in \mathbb{R}$ (if it exists) such that*

$$\Phi = \inf_{\theta \in \mathcal{A}_\Theta^{\mathcal{G}^1}} \Big(\sup_{\pi \in \mathcal{A}_\Pi^{\mathcal{G}^2}} J(\pi, \theta)\Big) = J(\pi^*, \theta^*) = \sup_{\pi \in \mathcal{A}_\Pi^{\mathcal{G}^2}} \Big(\inf_{\theta \in \mathcal{A}_\Theta^{\mathcal{G}^1}} J(\pi, \theta)\Big) \tag{7.2.25}$$

Such a control (π^*, θ^*) is called an *optimal control* (if it exists). The intuitive idea is that while Player I controls π, Player II controls θ. The actions of the players are antagonistic, which means that between player I and II there is a payoff $J(\pi, \theta)$ and it is a reward for Player I and cost for Player II. Note that since we allow b, σ, γ, f and g to be stochastic processes and also because our controls are $\mathcal{G}_t^1$-adapted, and $\mathcal{G}_t^2$-adapted respectively, this problem is not of Markovian type and can not be solved by dynamic programming.

Theorem 7.2.4 ***[Maximum principle for insider zero-sum games]***

Retain conditions 1-5.

(i) *Suppose* $(\widehat{\pi}, \widehat{\theta}) \in \mathcal{A}_\Pi \times \mathcal{A}_\Theta$ *is a directional critical point for* $J(\pi, \theta)$*, in the sense that for all bounded* $\beta \in \mathcal{A}_\Pi$ *and* $\eta \in \mathcal{A}_\Theta$*, there exists* $\delta > 0$ *such that* $\widehat{\pi} + y\beta \in \mathcal{A}_\Pi$, $\widehat{\theta} + v\eta \in \mathcal{A}_\Theta$ *for all* $y, v \in (-\delta, \delta)$ *and*

$$c(y, v) := J(\widehat{\pi} + y\beta, \widehat{\theta} + v\eta), \quad y, v \in (-\delta, \delta)$$

has a critical point at zero, i.e.,

$$\frac{\partial c}{\partial y}(0,0) = \frac{\partial c}{\partial v}(0,0) = 0. \tag{7.2.26}$$

Then

$$E\left[\nabla_\pi \widehat{H}(t, X^{\pi,\widehat{\theta}}(t), \pi, \widehat{\theta}, \omega)\Big|_{\pi=\widehat{\pi}} \Big| \mathcal{G}_t^2 \right] + E[A] = 0 \quad a.e. \text{ in } \ (t, \omega), \tag{7.2.27}$$

and

$$E\left[\nabla_\theta \widehat{H}(t, X^{\widehat{\pi},\theta}(t), \widehat{\pi}, \theta, \omega)\Big|_{\theta=\widehat{\theta}} \Big| \mathcal{G}_t^1 \right] + E[B] = 0 \quad a.e. \text{ in } \ (t, \omega), \tag{7.2.28}$$

where A *and* B *are given as in the previous theorem.*

$$\begin{aligned}
\widehat{X}(t) &= X^{(\widehat{\pi},\widehat{\theta})}(t), \\
\widehat{H}\left(t, \widehat{X}(t), \pi, \theta, \omega\right) &:= \int_{\mathbb{R}_0} f(t, \widehat{X}(t), \pi, \theta, z, \omega)\, \mu(dz) \\
&\quad + \widehat{p}(t)\Big(b(t, \widehat{X}(t), \pi_0, \theta_0, \omega) + D_{t+}\sigma(t, \widehat{X}(t), \pi_0, \theta_0, \omega) \\
&\quad + \int_{\mathbb{R}_0} D_{t+,z}\gamma(t, \widehat{X}(t), \pi, \theta, z, \omega)\, \nu(dz)\Big) \\
&\quad + \widehat{q}(t)\sigma(t, \widehat{X}(t), \pi_0, \theta_0, \omega) \\
&\quad + \int_{\mathbb{R}_0} \widehat{r}(t,z)\Big\{\gamma(t, \widehat{X}(t), \pi, \theta, z, \omega) + D_{t+,z}\gamma(t, \widehat{X}(t), \pi, \theta, z, \omega)\Big\}\nu(dz),
\end{aligned} \tag{7.2.29}$$

with

$$\widehat{p}(t) := \widehat{K}(t) + \int_t^T \frac{\partial}{\partial x}\widehat{H}^0(s,\widehat{X}(s),\widehat{\pi}(s),\widehat{\theta}(s))\widehat{G}(t,s)\,ds \tag{7.2.30}$$

$$\widehat{K}(t) := g'(\widehat{X}(T)) \;+\; \int_t^T \int_{\mathbb{R}_0} \frac{\partial}{\partial x} f(s,\widehat{X}(s),\widehat{\pi}(s,z),\widehat{\theta}(s,z),z)\,\mu(dz)ds \tag{7.2.31}$$

$$\begin{aligned}\widehat{H}^0(s,\widehat{X},\widehat{\pi},\widehat{\theta}) := \widehat{K}(s)\Big(&b(s,\widehat{X},\widehat{\pi}_0,\widehat{\theta}_0) \;+ D_{s+}\sigma(s,\widehat{X},\widehat{\pi}_0,\widehat{\theta}_0)\\ &+\int_{\mathbb{R}_0} D_{s+,z}\gamma(s,\widehat{X},\widehat{\pi},\widehat{\theta},z)\,\nu(dz)\Big) + D_sK(s)\sigma(s,\widehat{X},\widehat{\pi}_0,\widehat{\theta}_0)\\ &+\int_{\mathbb{R}_0} D_{s,z}K(s)\Big\{\gamma(s,\widehat{X},\widehat{\pi},\widehat{\theta},z) + D_{s+,z}\gamma(s,\widehat{X},\widehat{\pi},\widehat{\theta},z)\Big\}\,\nu(dz)\end{aligned} \tag{7.2.32}$$

$$\begin{aligned}\widehat{G}(t,s) := \exp\Bigg[&\int_t^s \left\{\frac{\partial b}{\partial x}\left(r,\widehat{X}(r),\widehat{\pi}_0(r),\widehat{\theta}_0(r)\right) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2\left(r,\widehat{X}(r),\widehat{\pi}_0(r),\widehat{\theta}_0(r)\right)\right\} dr\\ &+\int_t^s \frac{\partial \sigma}{\partial x}\left(r,\widehat{X}(r),\widehat{\pi}_0(r),\widehat{\theta}_0(r)\right)\,dB^-(r)\\ &+\int_t^s\int_{\mathbb{R}_0}\left\{\ln\left(1+\frac{\partial\gamma}{\partial x}\left(r,\widehat{X}(r),\widehat{\pi},\widehat{\theta},z\right)\right) - \frac{\partial\gamma}{\partial x}\left(r,\widehat{X}(r),\widehat{\pi},\widehat{\theta},z\right)\right\}\nu(dz)\,dt\\ &+\int_t^s\int_{\mathbb{R}_0}\left\{\ln\left(1+\frac{\partial\gamma}{\partial x}\left(r,\widehat{X}(r^-),\widehat{\pi}(r^-,z),\widehat{\theta}(r^-,z),z\right)\right)\right\}\widetilde{N}(dz,d^-r)\Bigg]\end{aligned} \tag{7.2.33}$$

(ii) *Conversely, suppose that there exists a* $(\widehat{\pi},\widehat{\theta}) \in \mathcal{A}_\Pi \times \mathcal{A}_\Theta$ *such that Equations (7.2.27) and (7.2.28) hold. Then* $(\widehat{\pi},\widehat{\theta})$ *satisfies 7.2.26.*

7.3 Controlled Itô-Lévy processes

The main result of the previous section (Theorem 7.2.2) is difficult to apply because of the appearance of the terms $Y(t)$, $D_{t+}Y(t)$ and $D_{t+,z}Y(t)$, which all depend on the control u. However, consider the special case when the coefficients do not depend on X, i.e., when

$$b(t,x,u,\omega) = b(t,u,\omega), \quad \sigma(t,x,u,\omega) = \sigma(t,u,\omega)$$
$$\text{and } \theta(t,x,u,z,\omega) = \theta(t,u,z,\omega). \tag{7.3.1}$$

Then equation (7.1.1) takes the form

$$\begin{cases} d^-X(t) &= b(t,u(t),\omega)dt + \sigma(t,u(t),\omega)d^-B(t) \\ & \quad + \int_{\mathbb{R}_0} \theta(t,u(t),z,\omega)\widetilde{N}(dz,d^-t); \\ X(0) &= x \in \mathbb{R} \end{cases} \tag{7.3.2}$$

We call such processes *controlled Itô-Lévy processes.*

In this case, Theorem 7.2.2 simplifies to the following

Theorem 7.3.1 *Let $X(t)$ be a controlled Itô-Lévy process as given in Equation (7.3.2). Assume that the conditions 1-5 as in Theorem 7.2.2 are in force.*

Then the following statements are equivalent:

(i) *$(\widehat{\pi},\widehat{\theta})$ is a directional critical point for $J_i(\pi,\theta)$ for $i=1,2$ in the sense that for all bounded $\beta \in \mathcal{A}_\Pi$ and $\eta \in \mathcal{A}_\Theta$, there exists $\delta > 0$ such that $\widehat{\pi}+y\beta \in \mathcal{A}_\Pi$, $\widehat{\theta}+v\eta \in \mathcal{A}_\Theta$ for all $y,v \in (-\delta,\delta)$.*

(ii)

$$E\Big[L_\pi(t)\alpha + M_\pi(t)D_{t+}\alpha + \int_{\mathbb{R}_0} R_\pi(t,z)D_{t+,z}\alpha\,\nu(dz)\Big] = 0$$

and

$$E\Big[L_\theta(t)\xi + M_\theta(t)D_{t+}\xi + \int_{\mathbb{R}_0} R_\theta(t,z)D_{t+,z}\xi\,\nu(dz)\Big] = 0$$

for all α Malliavin differentiable and all $t \in [0,T]$, where

$$\begin{aligned} L_\pi(t) = {} & \widehat{K}_1(t)\Big(\nabla_\pi b(t) + D_{t+}\nabla_\pi\sigma(t) + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi\gamma(t,z)\nu(dz)\Big) \\ & + \nabla_\pi f_1(t) + D_t\widehat{K}_1(t)\nabla_\pi\sigma(t) \\ & + \int_{\mathbb{R}_0} D_{t,z}\widehat{K}_1(t)\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz), \end{aligned} \tag{7.3.3}$$

$$M_\pi(t) = \widehat{K}_1(t)\nabla_\pi\sigma(t), \tag{7.3.4}$$

$$R_\pi(t,z) = \Big\{\widehat{K}_1(t) + D_{t,z}\widehat{K}_1(t)\Big\}\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big), \tag{7.3.5}$$

$$
\begin{aligned}
L_\theta(t) = &\widehat{K}_2(t)\left(\nabla_\theta b(t) \,+\, D_{t+}\nabla_\theta\sigma(t) \,+\, \int_{\mathbb{R}_0} D_{t+,z}\nabla_\theta\gamma(t,z)\nu(dz)\right) \\
&+ \nabla_\theta f_2(t) \,+\, D_t\widehat{K}_2(t)\nabla_\theta\sigma(t) \\
&+ \int_{\mathbb{R}_0} D_{t,z}\widehat{K}_2(t)\Big(\nabla_\theta\gamma(t,z) + D_{t+,z}\nabla_\theta\gamma(t,z)\Big)\nu(dz), \qquad (7.3.6)
\end{aligned}
$$

$$
M_\theta(t) = \widehat{K}_2(t)\nabla_\theta\sigma(t) \qquad (7.3.7)
$$

and

$$
R_\theta(t,z) = \left\{\widehat{K}_2(t) + D_{t,z}\widehat{K}_2(t)\right\}\Big(\nabla_\theta\gamma(t,z) + D_{t+,z}\nabla_\theta\gamma(t,z)\Big). \qquad (7.3.8)
$$

In particular, if

$$
\pi \longrightarrow J_1(\pi,\widehat{\theta})
$$

and

$$
\theta \longrightarrow J_2(\widehat{\pi},\theta),
$$

are concave, then $\left(\widehat{\pi},\widehat{\theta}\right)$ *is a Nash equilibrium.*

Proof. It is easy to see that in this case, $p(t) = K(t),\ q(t) = D_tK(t),\ r(t,z) = D_{t,z}K(t)$ and the general Hamiltonian $H_i,\ i = 1,2$ given by Equation (7.2.13) is reduced to H_i given as follows

$$
\begin{aligned}
H_i\,(t,x,\pi,\theta,\omega) := &\int_{\mathbb{R}_0} f_i(t,\pi,\theta,z,\omega)\,\mu(dz) + p_i(t)\Big(b(t,\pi_0,\theta_0,\omega) \,+ D_{t+}\sigma(t,\pi_0,\theta_0,\omega) \\
&+ \int_{\mathbb{R}_0} D_{t+,z}\gamma(t,\pi,\theta,z,\omega)\,\nu(dz)\Big) + q_i(t)\sigma(t,\pi_0,\theta_0,\omega) \\
&+ \int_{\mathbb{R}_0} r_i(t,z)\Big\{\gamma(t,\pi,\theta,z,\omega) + D_{t+,z}\gamma(t,\pi,\theta,z,\omega)\Big\}\nu(dz),
\end{aligned}
$$

(i) Performing the same calculation leads to

$$
\begin{aligned}
A_1 &= A_3 = A_5 = 0, \\
A_2 &= E\left[\int_t^{t+h} \left\{ \widehat{K}_1(t)\left(\nabla_\pi b(s) + D_{s+}\nabla_\pi\sigma(s) + \int_{\mathbb{R}_0} D_{ts,z}\nabla_\pi\gamma(t,z)\nu(dz)\right)\right.\right. \\
&\quad + D_t\widehat{K}_1(t)\nabla_\pi\sigma(t) \; + \int_{\mathbb{R}_0} \nabla_\pi f_1(s,z)\mu(dz) \\
&\quad \left.\left. + \int_{\mathbb{R}_0} D_{s,z}\widehat{K}_1(t)\Big(\nabla_\pi\gamma(s,z) + D_{s,z}\nabla_\pi\gamma(s,z)\Big)\nu(dz)\right\}\alpha\, ds\right], \\
A_4 &= E\left[\int_t^{t+h} \widehat{K}_1(t)\nabla_\pi\sigma(s)D_{s+}\alpha\, ds\right], \\
A_6 &= E\left[\int_t^{t+h}\int_{\mathbb{R}_0}\Big(\widehat{K}_1(t) + D_{s,z}\widehat{K}_1(t)\Big)\Big\{\nabla_\pi\gamma(s,z) \; + \; D_{s+,z}\nabla_\pi\gamma(s,z)\Big\}\nu(dz)D_{s+,z}\alpha ds\right],
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{d}{dh}A_2\bigg|_{h=0} =& E\left[\left\{\widehat{K}_1(t)\left(\nabla_\pi b(t) \; + \; D_{t+}\nabla_\pi\sigma(t) \; + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi\gamma(t,z)\nu(dz)\right)\right.\right. \\
&\quad + \nabla_\pi f_1(t) \; + \; D_t\widehat{K}_1(t)\nabla_\pi\sigma(t) \\
&\quad \left.\left. + \int_{\mathbb{R}_0} D_{t,z}\widehat{K}_1(t)\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz)\right\}\alpha\right], \\
\frac{d}{dh}A_4\bigg|_{h=0} =& E\left[\widehat{K}_1(t)\nabla_\pi\sigma(t)D_{t+}\alpha\right], \\
\frac{d}{dh}A_6\bigg|_{h=0} =& E\left[\int_{\mathbb{R}_0}\Big\{\widehat{K}_1(t) + D_{t,z}\widehat{K}_1(t)\Big\}\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz)D_{t+,z}\alpha\right].
\end{aligned}
$$

This means that

$$
\begin{aligned}
0 = E&\left[\left\{\widehat{K}_1(t)\left(\nabla_\pi b(t) \; + \; D_{t+}\nabla_\pi\sigma(t) \; + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi\gamma(t,z)\nu(dz)\right)\right.\right. \\
&+ \nabla_\pi f_1(t) \; + \; D_t\widehat{K}_1(t)\nabla_\pi\sigma(t) \\
&\left. + \int_{\mathbb{R}_0} D_{t,z}\widehat{K}_1(t)\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz)\right\}\alpha \\
&+ \widehat{K}_1(t)\nabla_\pi\sigma(t)D_{t+}\alpha \\
&\left. + \int_{\mathbb{R}_0}\Big\{\widehat{K}_1(t) + D_{t,z}\widehat{K}_1(t)\Big\}\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz)D_{t+,z}\alpha\right].
\end{aligned}
$$

Performing the same computation for H_2, the result follows. This completes the proof for (i).

(ii) The converse part follows from the arguments used in the proof of Theorem 7.2.2.

■

7.3.1 Zero-sum games

Under the same hypothesis as given in Section 7.2.1, if we assume that the controlled process is of Itô-Lévy type, Theorem 7.2.4 becomes

Theorem 7.3.2 *Let $X(t)$ be a controlled Itô-Lévy process as given in Equation (7.3.2). Retain the conditions 1-5 as in Theorem 7.2.2.*

Then the following statements are equivalent:

(i) *$(\widehat{\pi}, \widehat{\theta})$ is a directional critical point for $J(\pi, \theta)$ in the sense that for all bounded $\beta \in \mathcal{A}_\Pi$ and $\eta \in \mathcal{A}_\Theta$, there exists $\delta > 0$ such that $\widehat{\pi} + y\beta \in \mathcal{A}_\Pi$, $\widehat{\theta} + v\eta \in \mathcal{A}_\Theta$ for all $y, v \in (-\delta, \delta)$ and*

$$c(y, v) := J(\widehat{\pi} + y\beta, \widehat{\theta} + v\eta), \quad y, v \in (-\delta, \delta)$$

has a critical point at 0, i.e.,

$$\frac{\partial c}{\partial y}(0,0) = \frac{\partial c}{\partial v}(0,0) = 0. \tag{7.3.9}$$

(ii)

$$E\left[L_\pi(t)\alpha \;+\; M_\pi(t)D_{t+}\alpha \;+\; \int_{\mathbb{R}_0} R_\pi(t,z)D_{t+,z}\alpha\,\nu(dz)\right] = 0$$

and

$$E\left[L_\theta(t)\xi \;+\; M_\theta(t)D_{t+}\xi \;+\; \int_{\mathbb{R}_0} R_\theta(t,z)D_{t+,z}\xi\,\nu(dz)\right] = 0$$

for all α Malliavin differentiable and all $t \in [0,T]$, where

$$L_\pi(t) = \widehat{K}(t)\left(\nabla_\pi b(t) + D_{t+}\nabla_\pi\sigma(t) + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi\gamma(t,z)\nu(dz)\right)$$
$$+ \nabla_\pi f(t) + D_t\widehat{K}(t)\nabla_\pi\sigma(t)$$
$$+ \int_{\mathbb{R}_0} D_{t,z}\widehat{K}(t)\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz), \tag{7.3.10}$$

$$M_\pi(t) = \widehat{K}(t)\nabla_\pi\sigma(t), \tag{7.3.11}$$

$$R_\pi(t,z) = \left\{\widehat{K}(t) + D_{t,z}\widehat{K}(t)\right\}\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big), \tag{7.3.12}$$

$$L_\theta(t) = \widehat{K}(t)\left(\nabla_\theta b(t) + D_{t+}\nabla_\theta\sigma(t) + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\theta\gamma(t,z)\nu(dz)\right)$$
$$+ \nabla_\theta f(t) + D_t\widehat{K}(t)\nabla_\theta\sigma(t)$$
$$+ \int_{\mathbb{R}_0} D_{t,z}\widehat{K}(t)\Big(\nabla_\theta\gamma(t,z) + D_{t+,z}\nabla_\theta\gamma(t,z)\Big)\nu(dz), \tag{7.3.13}$$

$$M_\theta(t) = \widehat{K}(t)\nabla_\theta\sigma(t) \tag{7.3.14}$$

and

$$R_\theta(t,z) = \left\{\widehat{K}(t) + D_{t,z}\widehat{K}(t)\right\}\Big(\nabla_\theta\gamma(t,z) + D_{t+,z}\nabla_\theta\gamma(t,z)\Big). \tag{7.3.15}$$

7.3.2 Some special cases revisited

In this Section, we consider the same sup-filtration given in Section 6.5 of Chapter 6.

Let $\mathcal{H}$ be one of the following sup-filtrations,

$$\mathcal{H}_1 = \mathcal{F}_{t+\delta(t)},$$

$$\mathcal{H}_2 = \mathcal{F}_{[0,t]\cup O},$$

$$\mathcal{H}_3 = ,\mathcal{F}_t \vee \sigma(B_T)$$

$$\mathcal{H}_4 = \mathcal{F}_t \vee \sigma\left(B(t+\Delta t_n)\right);\ \ n = 1, 2, .. - .$$

where O is an open set contained in $[0,T]$.

From now on we assume that the following conditions are fulfilled:

Fix a $t_0 \in [0,T]$. Then

(C1) There exist a $\mathcal{A}^i = \mathcal{A}^i_{t_0} \subseteq \mathbb{D}_{1,2} \cap L^2(\mathcal{G}^i_{t_0}),\ i = 1,2$ and a measurable $\mathcal{M}^i \subset [t_0, T],\ i = 1,2$ such that $D_t\alpha$ and $D_{t,z}\alpha$ are $\mathcal{G}^i_{t_0}$-measurable, for all $\alpha \in \mathcal{A}^i,\ \ t \in \mathcal{M}^i,\ i = 1,2,$

(C2) $D_{t+}\alpha = D_t\alpha$ and $D_{t+,z}\alpha = D_{t,z}\alpha$ for all $\alpha \in \mathcal{A}^i$ and a.a. $t, z,\ t \in \mathcal{M}^i,\ i = 1,2.$

(C3) $\mathcal{A}^i$ is total in $L^2(\mathcal{G}^i_{t_0}),\ \ i = 1,2,$

(C4) $E[M_\theta(t)|\mathcal{G}^1_{t_0}]\cdot\chi_{[0,t]\cap\mathcal{M}^1}$, $E[R_\theta(t,z)|\mathcal{G}^1_{t_0}]\cdot\chi_{[0,t]\cap\mathcal{M}^1}$, $E[M_\pi(t)|\mathcal{G}^2_{t_0}]\cdot\chi_{[0,t]\cap\mathcal{M}^2}$ and $E[R_\pi(t,z))|\mathcal{G}^2_{t_0}]\cdot\chi_{[0,t]\cap\mathcal{M}^2}$ are Skorohod integrable for all t,

(C5) $\int_0^T \{|E[L_\theta(t)|\mathcal{G}^1_{t_0}]| + |E[L_\pi(t)|\mathcal{G}^1_{t_0}]|\}dt < \infty$ a.e.,

where L_π, M_π, L_θ, M_θ, R_π and R_θ are defined as in (7.3.3), (7.3.4), (7.3.6), (7.3.7), (7.3.5) and (7.3.8).

We can then deduce the following results from the arguments in Theorems 6.5.2 and 6.5.4 in Section 6.5 of Chapter 6.

Theorem 7.3.3 *Suppose that $\mathcal{G}^i_t,\ i = 1,2$ is of type (6.5.2) and that b, σ and γ do not depend on the controlled process $X(\cdot)$. Further require that **(C3)–(C5)** hold. Then the following are equivalent:*

(i) *$(\widehat{\pi}, \widehat{\theta})$ is a directional critical point for $J_i(\pi, \theta)$ for $i = 1,2$ in the sense that for all bounded $\beta \in \mathcal{A}_\Pi$ and $\eta \in \mathcal{A}_\Theta$, there exists a $\delta > 0$ such that $\widehat{\pi} + y\beta \in \mathcal{A}_\Pi$, $\widehat{\theta} + v\eta \in \mathcal{A}_\Theta$ for all $y, v \in (-\delta, \delta)$.*

(ii)

$$(1)\, E\left[L_\pi(t)|\, \mathcal{G}^2_t\right] = E\left[M_\pi(t)|\, \mathcal{G}^2_t\right] = E\left[R_\pi(t,Z)|\, \mathcal{G}^2_t\right] = 0,$$

$$(2)\, E\left[L_\theta(t)|\, \mathcal{G}^1_t\right] = E\left[M_\theta(t)|\, \mathcal{G}^1_t\right] = E\left[R_\theta(t,z)|\, \mathcal{G}^1_t\right] = 0,$$

where L_π, M_π, L_θ, and M_θ are given by (7.3.3), (7.3.4), (7.3.6) and (7.3.7) respectively. In particular, if

$$\pi \to J_1(\pi, \widehat{\theta})$$

and

$$\theta \to J_2(\widehat{\pi}, \theta),$$

are concave, then $\left(\widehat{\pi}, \widehat{\theta}\right)$ *is a Nash-equilibrium.*

Proof. It follows from the proof of Theorems 6.5.2 and 6.5.4 of Chapter 6. ■

If $\mathcal{G}_t^i$, $i = 1, 2$ is of type $\mathcal{H}_4$, then from Theorem 6.6.5, we have

Theorem 7.3.4 *[Brownian motion case] Suppose that b and σ do not depend on X and that*

$$\mathcal{G}_t^i = \mathcal{H}_4, \quad i = 1, 2$$

*Assume that **(C1)–(C5)** are valid for $\mathcal{M} \in (t_0, T]$. In addition, we require that $E\left[M_\pi(t)|\,\mathcal{G}_{t-}^2\right]$, $E\left[M_\theta(t)|\,\mathcal{G}_{t-}^1\right] \in \mathbb{M}_{1,2}^B$ and are forward integrable with respect to $E\left[\,d^-B(t)\,\middle|\mathcal{G}_{t-}^2\right]$ and $E\left[\,d^-B(t)\,\middle|\mathcal{G}_{t-}^1\right]$ respectively. Then the following statements are equivalent:*

(i) $(\widehat{\pi}, \widehat{\theta})$ *is a directional critical point for $J_i(\pi, \theta)$ for $i = 1, 2$ in the sense that for all bounded $\beta \in \mathcal{A}_\Pi$ and $\eta \in \mathcal{A}_\Theta$, there exists a $\delta > 0$ such that $\widehat{\pi} + y\beta \in \mathcal{A}_\Pi$, $\widehat{\theta} + v\eta \in \mathcal{A}_\Theta$ for all $y, v \in (-\delta, \delta)$.*

(ii)

$$\begin{aligned}
&E\left[L_\pi(t)|\,\mathcal{G}_{t-}^2\right] - D_{t^+} E\left[M_\pi(t)|\,\mathcal{G}_{t-}^2\right] = 0,\\
&E\left[M_\pi(t)|\,\mathcal{G}_{t-}^2\right] = 0,\\
&E\left[L_\theta(t)|\,\mathcal{G}_{t-}^1\right] - D_{t^+} E\left[M_\theta(t)|\,\mathcal{G}_{t-}^1\right] = 0,\\
&E\left[M_\theta(t)|\,\mathcal{G}_{t-}^1\right] = 0, \ \textit{for a.a } t \in [0, T].
\end{aligned}$$

where L_π, M_π, R_π, L_θ, M_θ and R_θ are given by (7.3.3), (7.3.4), (7.3.5), (7.3.6), (7.3.7) and (7.3.8) respectively. In particular, if

$$\pi \to J_1(\pi, \widehat{\theta})$$

and

$$\theta \to J_2(\widehat{\pi}, \theta),$$

are concave, then $\left(\widehat{\pi}, \widehat{\theta}\right)$ *is a Nash-equilibrium.*

Proof. See proof of Theorem6.6.5. ■

In the next section, we apply our results to model a competition of two heterogeneously informed agents in the market. We particularly focus on a game between the market and the trader. We assume that the mean relative growth rate $\theta(t)$ of the risky asset is not known to the trader, but subject to uncertainty.

7.4 Application to optimal and competing-insider trading

Consider a financial market with two investments possibilities:

1. A risk free asset, where the unit price $S_0(t)$ at time t is given by

$$dS_0(t) = r(t)S_0(t)\,dt, \quad S_0(0) = 1. \tag{7.4.1}$$

2. A risky asset, where the unit price $S_1(t)$ at time t is given by the stochastic differential equation

$$dS_1(t) = S_1(t^-)\left[\theta(t)dt + \sigma_0(t)d^-B(t) + \int_{\mathbb{R}_0}\gamma(t,z)\widetilde{N}(d^-t,dz)\right], \quad S_1(0) > 0. \tag{7.4.2}$$

Here $r(t) \geq 0$, $\theta(t), \sigma_0(t)$, and $\gamma(t,z) \geq -1+\epsilon$ (for some constant $\epsilon > 0$) are given $\mathcal{G}_t^1$-predictable, forward integrable processes, where $\{\mathcal{G}_t^1\}_{t\in[0,T]}$ is a given filtration such that

$$\mathcal{F}_t \subset \mathcal{G}_t^1 \text{ for all } t \in [0,T] \tag{7.4.3}$$

Suppose a trader in this market is an insider, in the sense that she has access to information represented by $\mathcal{G}_t^2$ at time t (with $\mathcal{F}_t \subset \mathcal{G}_t^2$ for all $t \in [0,T]$). Assume that $\mathcal{G}_t^1 \subset \mathcal{G}_t^2$ (e.g. $\mathcal{G}_t^1 = \mathcal{F}_t$). Let $\pi(t) = \pi(t,\omega)$ be a portfolio representing the *amount* invested by her in the risky asset at time t. Then this portfolio is a $\mathcal{G}_t^2$-predictable stochastic process and hence the corresponding wealth process $X(t) = X^{(\pi,\theta)}(t)$ will then satisfy the (forward) SDE

$$
\begin{aligned}
d^-X(t) &= \frac{X(t)-\pi(t)}{S_0(t)} dS_0(t) + \frac{\pi(t)}{S_1(t)} d^-S_1(t) \\
&= X(t)r(t)dt + \pi(t)\Big[(\theta(t) - r(t))\, dt + \sigma_0(t) d^-B(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^-t, dz)\Big], \quad t \in [0,T], \qquad (7.4.4) \\
X(0) &= x > 0. \qquad (7.4.5)
\end{aligned}
$$

By choosing $S_0(t)$ as a numeraire, we can, without loss of generality, assume that

$$
r(t) = 0 \tag{7.4.6}
$$

from now on. Then Equations (7.4.4) and (7.4.5) simplify to

$$
\begin{cases}
d^-X(t) = \pi(t)\left[\theta(t)dt + \sigma_0(t)d^-B(t) + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^-t,dz)\right], \\
X(0) = x > 0.
\end{cases} \tag{7.4.7}
$$

This is a controlled Itô-Lévy process of the type discussed in Section 7.3. Let us assume that the mean relative growth rate $\theta(t)$ of the risky asset is not known to the trader, but subject to uncertainty. We may regard θ as a market scenario or a stochastic control of the market, which is playing against the trader. Let $\mathcal{A}_{\Pi}^{\mathcal{G}^2}$ and $\mathcal{A}_{\Theta}^{\mathcal{G}^1}$ denote the set of admissible controls π, θ, respectively. The *worst case insider information scenario optimal problem* for the trader is to find $\pi^* \in \mathcal{A}_{\Pi}^{\mathcal{G}^2}$ and $\theta^* \in \mathcal{A}_{\Theta}^{\mathcal{G}^1}$ and $\Phi \in \mathbb{R}$ such that

$$
\begin{aligned}
\Phi &= \inf_{\theta \in \mathcal{A}_{\Theta}^{\mathcal{G}^1}} \Big(\sup_{\pi \in \mathcal{A}_{\Pi}^{\mathcal{G}^2}} E[U(X^{\theta,\pi})(T)]\Big) \\
&= E\left[U(X^{\theta^*,\pi^*})(T)\right] \qquad (7.4.8)
\end{aligned}
$$

where $U : \mathbb{R}_+ \to \mathbb{R}$ is a given utility function, assumed to be concave, strictly increasing and C^1. We want to study this problem by using results of Section 7.3. In this case, the processes $K(t)$,$L(t)$, $M(t)$ and $R(t,z)$ which are given respectively by equations (7.2.7), (7.3.3), (7.3.4), (7.3.5), (7.3.6), (7.3.7) and (7.3.8) become

$$K_1(t) = K_2(t) = U'(X(T)), \tag{7.4.9}$$

$$L_\pi(t) = U'(X(T))\left[\theta(t) + D_{t+}\sigma_0(t) + \int_{\mathbb{R}_0} D_{t+,z}\gamma(t,z)\,\nu(dz)\right] \tag{7.4.10}$$
$$+ \int_{\mathbb{R}_0} D_{t,z}U'(X(T))\left[\gamma(t,z) + D_{t+,z}\gamma(t,z)\right]\nu(dz) + D_tU'(X(T))\,\sigma_0(t),$$

$$M_\pi(t) = U'(X(T))\,\sigma_0(t), \tag{7.4.11}$$

$$R_\pi(t,z) = \left\{U'(X(T)) + D_{t,z}U'(X(T))\right\}\left\{\gamma(t,z) + D_{t+,z}\gamma(t,z)\right\}. \tag{7.4.12}$$

$$L_\theta(t) = U'(X(T))\,\pi \tag{7.4.13}$$

$$M_\theta(t) = R_\theta(t,z) = 0 \tag{7.4.14}$$

7.4.1 Case $\mathcal{G}_t = \mathcal{F}_{G_t},\ G_t \supset [0,t]$

In this case, it follows from Theorem 7.3.3 and Theorem 7.3.4 of Section 7.3 that:

Theorem 7.4.1 *Suppose that $P(\lambda\{t \in [0,T];\ \sigma_0(t) \neq 0\} > 0) > 0$ where λ denotes the Lebesgue measure on $\mathbb{R}$ and that $\mathcal{G}_t^i,\ i = 1,2$ is given by (6.5.2).*
Then there does not exist an optimal solution $(\pi^*, \theta^*) \in \mathcal{A}_\Pi^{\mathcal{G}^2} \times \mathcal{A}_\Theta^{\mathcal{G}^1}$ of the stochastic differential game (7.4.8).

Proof. Suppose an optimal portfolio exists. Then we have seen that

$$E\left[L_\pi(t)|\,\mathcal{G}_t^2\right] = E\left[M_\pi(t)|\,\mathcal{G}_t^2\right] = E\left[R_\pi(t,z)|\,\mathcal{G}_t^2\right] = E\left[L_\theta(t)|\,\mathcal{G}_t^1\right] = 0$$

for a.a. $t \in [0,T],\ z \in \mathbb{R}_0$. In particular,

$$E\left[M_\pi(t)|\,\mathcal{G}_t^2\right] = E\left[U'(X(T))\middle|\,\mathcal{G}_t^2\right]\sigma_0(t) = 0,$$

or,

$$E\left[L_\theta(t)|\,\mathcal{G}_t^1\right] = E\left[U'(X(T))\middle|\,\mathcal{G}_t^1\right]\pi(t) = 0.$$

Since $U' > 0$, this contradicts our assumption about U. Hence an optimal solution cannot exist. ∎

7.4.2 Case $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T))$

In this case, using Corollary 6.5.9 and Theorem 6.6.3 in Section 6.6.2 of Chapter 6, we have

Theorem 7.4.2 *(Knowing the terminal value of the risky asset) Suppose that* $\sigma_0(t) \neq 0$ $\mathcal{G}_t^1 = \mathcal{F}_t$ *and* $\mathcal{G}_t^2 = \mathcal{F}_t \vee \sigma(S_1(T))$, $t \in [0,T]$ *and the coefficients* $\theta(t), \sigma_0(t) = \sigma_0 \neq 0$ *and* $\gamma(t,z) \equiv 0$ *are deterministic. Further, require that the conditions* ***(C1)–(C5)*** *hold for* $\mathcal{M} \in (t_0, T]$ *and that*

1. $E\left[M_\theta(t)|\,\mathcal{G}_{t_0}^1\right],\ E\left[M_\pi(t)|\,\mathcal{G}_{t_0}^2\right] \in \mathbb{M}_{1,2}^B$
2. $\underline{\lim}_{t\uparrow 0} E\left[\left|D_{t+}E\left[M_{\phi_i}(t)|\,\mathcal{G}_{t_0}^2\right]\right|\right] < \infty$ *for* $\phi_1 = \theta$ *and* $\phi_2 = \pi$.
3. $\underline{\lim}_{t\uparrow 0} E\left[|L_{\phi_i}(t)|\right] < \infty$ *for* $\phi_1 = \theta$ *and* $\phi_2 = \pi$.

Then, there does not exist an optimal solution for the insider stochastic differential game.

Proof. Since $S_1(t)$ can be written as

$$S_1(t) = S_1(0)\, \exp\left(\int_0^T \left\{\theta(t) - \frac{1}{2}\sigma_0^2(t)\right\} dt + \int_0^T \sigma_0(t)\, dB(t)\right), \tag{7.4.15}$$

One finds that $\mathcal{G}_t^2 = \mathcal{H}_t^2$. Hence the result follows from Theorem 6.6.3 in Chapter 6. ∎

Remark 7.4.3 *This result is a generalization of a result in [44], where the same conclusion was obtained in the special case when*

$$U(x) = \ln(x)$$

Remark 7.4.4 *It can be shown (see Chapter 6) that Theorem 7.4.2 also applies e.g., to cases, when the terminal value* $S_1(T)$ *is given by* $\max_{o\leq t\leq T} B(t)$ *or* $\eta(T)$*, where* η *is a Lévy process.*

7.4.3 Case $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(B(t+\Delta t_n)\right);\ n = 1, 2, \cdots$ with $\frac{\Delta t_{n+1}}{\Delta t_n} \to 1$ as $n \to \infty$

It follows from Theorem 7.3.4 that:

Theorem 7.4.5 *Suppose $\mathcal{G}_t^i$ is as in Theorem 7.3.4 and that $P\left(\lambda\left\{t \in [0,T];\ \sigma_0(t) \neq 0\right\} > 0\right) > 0$ where λ denotes the Lebesgue measure on $\mathbb{R}$. Then, there does not exist an optimal solution $(\theta^*, \pi^*) \in \mathcal{A}_{\Pi}^{\mathcal{G}^2} \times \mathcal{A}_{\Theta}^{\mathcal{G}^1}$ for the performance functional $J(\theta,\pi) = E\left[U(X^{\theta,\pi}(T))\right]$.*

Proof. See proof of Corollary 6.6.6. ■

7.5 Application to optimal insider consumption

Suppose we have a cash flow $X(t) = X^{(\pi,\theta)}(t)$ given by

$$\begin{cases} dX(t) &= (\theta(t) - u(t))\,dt + \sigma(t)dB(t) + \displaystyle\int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(dt,dz), \\ X(0) &= x \in \mathbb{R}. \end{cases} \tag{7.5.1}$$

Here $\theta(t),\ \sigma(t)$ and $\theta(t,z)$ are given $\mathcal{F}_T$-measurable processes and $\pi(t) \geq 0$ is the *consumption* rate, assumed to be adapted to a given insider filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$ where

$$\mathcal{F}_t \subset \mathcal{G}_t \text{ for all } t$$

Let $f(t,\pi,\theta,\omega);\ t \in [0,T],\ \pi,\ \theta \in \mathbb{R},\ \omega \in \Omega$ be a given $\mathcal{F}_T$-measurable utility process. Assume that $u \to f(t,\pi,\theta,\omega)$ is strictly increasing, concave and C^1 for a.a (t,ω).
Let $g(x,\omega);\ x \in \mathbb{R},\ \omega \in \Omega$ be a given $\mathcal{F}_T$-measurable random variable for each x. Assume that $u \to g(x,\omega)$ is concave for a.a ω. Define the performance functional J by

$$J(\pi,\theta) = E\left[\int_0^T f(t,\pi(t),\theta(t),\omega)\,dt + g\left(X^{(u)}(T),\omega\right)\right];\ u \in \mathcal{A}_{\mathcal{G}},\ u \geq 0 \tag{7.5.2}$$

Note that $\pi \to J(\pi,\widehat{\theta})$ and $\theta \to J(\widehat{\pi},\theta)$ are concave, so $(\widehat{\pi},\widehat{\theta})$ is a Nash-equilibrium if and only if $(\widehat{\pi},\widehat{\theta})$ is a critical point of $J(\pi,\theta)$.

Theorem 7.5.1 *[Optimal insider consumption stochastic differential game consumption I] $(\widehat{\pi},\widehat{\theta})$ is a Nash-equilibrium of insider consumption rate for the performance functional J in Equation (7.5.2) if and only if*

$$-E\left[\frac{\partial}{\partial\theta} f(t,\widehat{\pi}(t),\widehat{\theta}(t),\omega)\middle|\,\mathcal{G}_t\right] = E\left[\frac{\partial}{\partial\pi} f(t,\widehat{\pi}(t),\widehat{\theta}(t),\omega)\middle|\,\mathcal{G}_t\right] = E\left[g'\left(X^{(\widehat{\pi},\widehat{\theta})}(T),\omega\right)\middle|\,\mathcal{G}_t\right] \tag{7.5.3}$$

Proof. In this case we have

$$\begin{aligned}
K_1(t) &= K_2(t) = g\left(X^{(\pi,\theta)}(T)\right) \\
L_\pi(t) &= -\,g\left(X^{(\pi,\theta)}(T)\right) + \frac{\partial}{\partial \pi} f(t,\widehat{\pi}(t),\widehat{\theta}(t)) \\
L_\theta(t) &= g\left(X^{(\pi,\theta)}(T)\right) + \frac{\partial}{\partial \theta} f(t,\widehat{\pi}(t),\widehat{\theta}(t)) \\
M_\pi(t) &= R_\pi(t,z) = \; M_\theta = R_\theta = 0
\end{aligned}$$

Therefore $(\widehat{\pi},\widehat{\theta})$ is a critical point for $J(\pi,\theta)$ if and only if

$$\begin{aligned}
0 &= E\left[L_\pi(t)\middle|\, \mathcal{G}_t\right] = E\left[L_\theta(t)\middle|\, \mathcal{G}_t\right] \\
&= E\left[\frac{\partial}{\partial \pi} f(t,\widehat{\pi}(t),\widehat{\theta}(t))\middle|\, \mathcal{G}_t\right] + E\left[-g'\left(X^{(\widehat{\pi},\widehat{\theta}(t))}(T)\right)\middle|\, \mathcal{G}_t\right] \\
&= E\left[\frac{\partial}{\partial \theta} f(t,\widehat{\pi}(t),\widehat{\theta}(t))\middle|\, \mathcal{G}_t\right] + E\left[g'\left(X^{(\widehat{\pi},\widehat{\theta}(t))}(T)\right)\middle|\, \mathcal{G}_t\right]
\end{aligned}$$

■

Since $X^{(\widehat{\pi},\widehat{\theta})}(T)$ depends on $(\widehat{\pi},\widehat{\theta})$, Equation (7.5.3) does not give the value of $\widehat{\pi}(t)$ (respectively $\widehat{\theta}(t)$) directly.

However, in some special cases $\widehat{\pi}$ and $\widehat{\theta}(t)$ can be found explicitly:

Corollary 7.5.2 *(Optimal insider stochastic differential game consumption II)*

Assume that

$$g(x,\omega) = \lambda(\omega)x \tag{7.5.4}$$

for some $\mathcal{F}_T$-measurable random variable $\lambda \geq 0$.

Then the Nash-equilibrium $(\widehat{\pi}(t),\widehat{\theta}(t))$ of the stochastic differential game (7.5.2) is given by

$$E\left[\frac{\partial}{\partial \pi} f(t,\widehat{\pi},\widehat{\theta},\omega)\middle|\, \mathcal{G}_t\right]_{\pi=\widehat{\pi}(t)} = E\left[\lambda\middle|\, \mathcal{G}_t\right], \tag{7.5.5}$$

$$E\left[\frac{\partial}{\partial \theta} f(t,\widehat{\pi},\widehat{\theta},\omega)\middle|\, \mathcal{G}_t\right]_{\theta=\widehat{\theta}(t)} = -\,E\left[\lambda\middle|\, \mathcal{G}_t\right] \tag{7.5.6}$$

Thus we see that the Nash-equilibrium exists, for *any* given insider information filtration $\{\mathcal{G}_t\}_{t\geq 0}$.

Appendix A

Proofs of Some Theorems

A.1 Proof of Proposition 1.2.7

Here we give a proof of Proposition 1.2.7.

Proof. of Proposition 1.2.7.

Motivated by [122], we set

$$R_\epsilon(F,X,s)=F(s+\epsilon,X_{s+\epsilon})-F(s,X_s)-\frac{\partial F}{\partial s}(s,X_{s+\epsilon})\epsilon-\frac{\partial F}{\partial x}(s,X)(X_{s+\epsilon}-X_s).\text{(A.1.1)}$$

Put $C_\epsilon\left(F(\cdot,X),G(\cdot,Y)\right)=C_\epsilon^{(1)}+I_\epsilon$, where

$$\begin{aligned}
C_\epsilon^{(1)}=&\frac{1}{\epsilon}\int_0^{\cdot}\frac{\partial F}{\partial x}(s,X)\frac{\partial G}{\partial y}(s,Y)(X_{s+\epsilon}-X_s)(Y_{s+\epsilon}-Y_s)ds,\\
I_\epsilon=&\sum_{i=1}^{7}I_\epsilon^{(i)},\ \text{with}\\
I_\epsilon^{(1)}=&\frac{1}{\epsilon}\int_0^{\cdot}\frac{\partial G}{\partial x}(s,Y_s)R_\epsilon(F,X,s)(Y_{s+\epsilon}-Y_s)ds,\\
I_\epsilon^{(2)}=&\frac{1}{\epsilon}\int_0^{\cdot}\frac{\partial F}{\partial x}(s,X_s)R_\epsilon(G,Y,s)(X_{s+\epsilon}-X_s)ds,\\
I_\epsilon^{(3)}=&\frac{1}{\epsilon}\int_0^{\cdot}R_\epsilon(G,Y,s)R_\epsilon(F,X,s)ds,\\
I_\epsilon^{(4)}=&\int_0^{\cdot}\frac{\partial G}{\partial s}(s,Y_{s+\epsilon})R_\epsilon(F,X,s)ds,
\end{aligned}$$

$$I_\epsilon^{(5)} = \int_0^{\cdot} \frac{\partial F}{\partial s}(s, X_{s+\epsilon}) R_\epsilon(G, Y, s) ds,$$
$$I_\epsilon^{(6)} = \int_0^{\cdot} \frac{\partial F}{\partial s}(s, X_{s+\epsilon}) \frac{\partial G}{\partial x}(s, Y_s)(Y_{s+\epsilon} - Y_s) ds,$$

$$I_\epsilon^{(7)} = \int_0^{\cdot} \frac{\partial G}{\partial s}(s, Y_{s+\epsilon}) \frac{\partial F}{\partial x}(s, X_s)(X_{s+\epsilon} - X_s) ds.$$

By proposition 1.2 in [122], $C_\epsilon^{(1)}$ converges *ucp* to $\int_0^{\cdot} \frac{\partial F}{\partial x}(s, X) \frac{\partial G}{\partial y}(s, Y) d[X, Y](s)$. We will check that $I_\epsilon^{(i)}$ converges *ucp* to 0, $1 \leq i \leq 7$. Since F is a $C^{1,1}$ function, one can verify that

$$\begin{aligned} R_\epsilon(F, X, s) \quad &= \quad \left(\int_0^1 \left[\frac{\partial F}{\partial x}(s, \alpha X_s + (1-\alpha) X_{s+\epsilon}) - \frac{\partial F}{\partial x}(s, X_s) d\alpha \right] \right)(X_{s+\epsilon} - X_s) \\ &+ \left(\int_0^1 \left[\frac{\partial F}{\partial u}(\alpha s + (1-\alpha)(s+\epsilon), X_{s+\epsilon}) - \frac{\partial F}{\partial u}(s, X_{s+\epsilon}) d\alpha \right] \right) \epsilon. \end{aligned}$$

Let $T > 0$ fixed. Then X is uniformly continuous on $[0, T+1]$ and $X([0, T+1]) \subset [-K, K]$. We set ρ_1 (respectively ρ_2 and ρ_3) to be the modulus of continuity $\frac{\partial F}{\partial x}$ on $[-K, K]$ (respectively $\frac{\partial F}{\partial u}$, X on $[0, T+1]$). The functions ρ_1, ρ_2 and ρ_3 are positive, increasing and converge to 0, at 0. Moreover K, ρ_1, ρ_2 and ρ_3 depend on ω. We have

$$|R_\epsilon(F, X, s)| \leq \rho_1(\rho_3(\epsilon)) |X_{s+\epsilon} - X_s| + \rho_3(\epsilon)\epsilon \;\; s \in [0, T], 0 < \epsilon < 1. \tag{A.1.2}$$

Now, $\left(\frac{\partial F}{\partial x}(s, X_s); 0 \leq s \leq T+1 \right)$ is bounded by M_1 (M_1 is random); therefore,

$$\sup_{0 \leq t \leq T} I_\epsilon^{(1)} \leq M_1 \left(\rho_1(\rho_3(\epsilon)) \frac{1}{\epsilon} \int_0^T |X_{s+\epsilon} - X_s| |Y_{s+\epsilon} - Y_s| ds + \rho_3(\epsilon) \int_0^T |Y_{s+\epsilon} - Y_s| ds \right).$$

By the Cauchy-Schwartz inequality, we have

$$\sup_{0 \leq t \leq T} I_\epsilon^{(1)} \leq M_1 \left(\rho_1(\rho_3(\epsilon)) \{ C_\epsilon(X, X)(T) C_\epsilon(Y, Y)(T) \}^{\frac{1}{2}} + \epsilon \rho_3(\epsilon) \{ T C_\epsilon(Y, Y)(T) \}^{\frac{1}{2}} \right).$$

Since $(C_\epsilon(X, X)(T),\ 0 < \epsilon < 1)$ and $(C_\epsilon(Y, Y)(T),\ 0 < \epsilon < 1)$ are bounded, it follows that $I_\epsilon^{(1)}$ converges *ucp* to 0. By symmetry, $I_\epsilon^{(2)}$ converges *ucp* to 0. Substituting X for Y, F for G, Equation (A.1.2) becomes

$$|R_\epsilon(G, Y, s)| \leq \rho_4(\epsilon) |Y_{s+\epsilon} - Y_s| + \rho_5(\epsilon)\epsilon \;\; s \in [0, T],\ \ 0 < \epsilon < 1.$$

where $\lim_{\epsilon\to 0}\rho_4(\epsilon) = 0 = \lim_{\epsilon\to 0}\rho_5(\epsilon)$. We have

$$\begin{aligned}\sup_{0\le t\le T} I_\epsilon^{(3)} &\le \rho_4(\epsilon)\rho_1(\rho_3(\epsilon))\frac{1}{\epsilon}\int_0^T |X_{s+\epsilon}-X_s|\,|Y_{s+\epsilon}-Y_s|\,ds + \rho_5(\epsilon)\rho_1(\rho_3(\epsilon))\int_0^T |X_{s+\epsilon}-X_s|\,ds \\ &\quad +\rho_3(\epsilon)\rho_4(\epsilon)\int_0^T |Y_{s+\epsilon}-Y_s|\,ds + \epsilon\rho_3(\epsilon)\rho_5(\epsilon).\end{aligned}$$

As before, the previous inequality implies that $I_\epsilon^{(3)}$ converges *ucp* to 0. On the other hand, $\left(\frac{\partial F}{\partial s}(s, X_s);\, 0\le s\le T+1\right)$ is bounded by M_2; therefore,

$$\sup_{0\le t\le T} I_\epsilon^{(5)} \le M_2\left(\rho_4(\epsilon)\int_0^T |Y_{s+\epsilon}-Y_s|\,ds + T\rho_5(\epsilon)\epsilon\right),$$

and the convergence follows. By symmetry $I_\epsilon^{(4)}$ converges *ucp* to 0. Using the same arguments, it is easy to show that $I_\epsilon^{(6)}$ and $I_\epsilon^{(7)}$ converges *ucp* to 0. ■

A.2 Proof of Theorem 4.3.3

Since $\widehat{u} \in \mathcal{A}_{\mathcal{E}}$ is a critical point, there exists for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ a $\delta > 0$ as in (4.3.4). We conclude that

$$\begin{aligned} 0 &= \frac{d}{dy} J(\widehat{u} + y\beta)\Big|_{y=0} = E\Big[\int_0^T \int_G \Big(\frac{\partial}{\partial\gamma} f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega)\widehat{Y}^\beta(s,x) \\ &\quad + \frac{\partial}{\partial u} f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega)\beta(s,x)\Big)\, dx\, ds + \int_G \frac{\partial}{\partial\gamma} g(x,\widehat{\Gamma}(T,x),\omega)\widehat{Y}^\beta(T,x)\, dx\Big], \end{aligned}$$

where $\widehat{Y}^\beta$ is defined as in condition 4 with $u = \widehat{u}$ and fulfills

$$\begin{aligned} \widehat{Y}^\beta(t,x) = \int_0^t \Big[& L\widehat{Y}^\beta(s,x) + \widehat{Y}^\beta(s,x)\frac{\partial}{\partial\gamma} b(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(s,x),\widehat{u}(s,x)) \\ & + \nabla_x \widehat{Y}^\beta(s,x)\nabla_{\gamma'} b(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(s,x),\widehat{u}(s,x))\Big]\, ds \\ & + \int_0^t \Big[\widehat{Y}^\beta(s,x)\frac{\partial}{\partial\gamma}\sigma(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(s,x)) \\ & + \nabla_x \widehat{Y}^\beta(s,x)\nabla_{\gamma'}\sigma(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(s,x))\Big]\, dB(s) \\ & + \int_0^t \int_{\mathbb{R}} \Big[\widehat{Y}^\beta(s^-,x)\frac{\partial}{\partial\gamma}\theta(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(s,x),z) \\ & + \nabla_x \widehat{Y}^\beta(s^-,x)\nabla_{\gamma'}\theta(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(s,x),z)\Big]\, \widetilde{N}(dz,ds) \\ & + \int_0^t \Big[\beta(s,x)\frac{\partial}{\partial u} b(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(s,x),\widehat{u}(s,x))\Big]\, ds \\ & + \int_0^t \beta(s,x)\frac{\partial}{\partial u}\sigma(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(s,x))\, dB(s) \\ & + \int_0^t \int_{\mathbb{R}} \beta(s^-,x)\frac{\partial}{\partial u}\theta(s,x,\widehat{\Gamma}(s,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(s,x),z)\, \widetilde{N}(dz,ds) \qquad \text{(A.2.1)} \\ (t,x) \in [0,T]\times G & \end{aligned}$$

with

$$\begin{aligned} \widehat{Y}^\beta(0,x) &= 0,\ x \in \overline{G} \\ \widehat{Y}^\beta(t,x) &= 0,\ (t,x) \in (0,T)\times\partial G. \end{aligned}$$

Using the shorthand notation $\frac{\partial}{\partial\gamma} f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega) = \frac{\partial}{\partial\gamma} f(s,x)$, $\frac{\partial}{\partial u} f(s,x,\widehat{\Gamma}(s,x),\widehat{u}(s,x),\omega) = \frac{\partial}{\partial u} f(s,x)$ and similarly for $\frac{\partial g}{\partial\gamma}$, $\frac{\partial b}{\partial\gamma}$, $\frac{\partial b}{\partial u}$, $\frac{\partial\sigma}{\partial\gamma}$, $\frac{\partial\sigma}{\partial u}$, $\frac{\partial\theta}{\partial\gamma}$ and $\frac{\partial\theta}{\partial u}$, we can write

$$
\begin{aligned}
& E\left[\int_G \frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x)) \widehat{Y}^\beta(T,x) dx\right] \\
= & \int_G E\left[\frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x)) \widehat{Y}^\beta(T,x)\right] dx \\
= & \int_G E\left[\frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x)) \left(\int_0^T \left[L\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial \gamma} b(t,x) \widehat{Y}^\beta(t,x)\right.\right.\right. \\
& \left. + \nabla_x \widehat{Y}^\beta(t,x) \frac{\partial}{\partial \gamma'} b(t,x) + \beta(t,x) \frac{\partial}{\partial u} b(t,x)\right] dt \\
& + \int_0^T \left[\frac{\partial}{\partial \gamma} \sigma(t,x,) \widehat{Y}^\beta(t,x) + \nabla_x \widehat{Y}^\beta(s,x) \frac{\partial}{\partial \gamma'} \sigma(t,x) + \frac{\partial}{\partial u} \sigma(t,x) \beta(t,x)\right] dB(t) \\
& \left.\left. + \int_0^T \int_{\mathbb{R}_0} \left[\frac{\partial}{\partial \gamma} \theta(t,x,z) + \nabla_x \widehat{Y}^\beta(s,x) \frac{\partial}{\partial \gamma'} \theta(t,x,z) + \frac{\partial}{\partial u} \theta(t,x,z) \beta(t^-,x)\right] \widetilde{N}(dt,dz)\right)\right] dx
\end{aligned}
$$

Then by the duality formulas (Lemma 4.3.2) we get that

$$
\begin{aligned}
& E\left[\int_G \frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x)) \widehat{Y}^\beta(T,x)\, dx\right] \\
= & \int_G E\left[\int_0^T \left(\frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x)) \left[L\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial \gamma} b(t,x) \widehat{Y}^\beta(t,x)\right.\right.\right. \\
& \left. + \nabla_{\gamma'} b(t,x) \nabla_x Y(t,x) + \frac{\partial}{\partial u} b(t,x) \beta(t,x)\right] \\
& + D_t \left(\frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x))\right) \left[\frac{\partial}{\partial \gamma} \sigma(t,x) \widehat{Y}^\beta(t,x) + \nabla_{\gamma'} \sigma(t,x) \nabla_x \widehat{Y}^\beta(t,x)\right. \\
& \left. + \frac{\partial}{\partial u} \sigma(t,x) \beta(t,x)\right] + \int_{\mathbb{R}_0} \left\{D_{t,z} \left(\frac{\partial}{\partial \gamma} g(x, \widehat{\Gamma}(T,x))\right) \left[\frac{\partial}{\partial \gamma} \theta(t,x,z) \widehat{Y}^\beta(t^-,x)\right.\right. \\
& \left.\left.\left.\left. + \nabla_{\gamma'} \theta(t,x,z) \nabla_x \widehat{Y}^\beta(t^-,x) + \frac{\partial}{\partial u} \theta(t,x,z) \beta(t^-,x)\right]\right\} \nu(dz)\right) dt\right] dx. \qquad \text{(A.2.2)}
\end{aligned}
$$

Further we similarly obtain by duality and Fubini's theorem that

$$
\begin{aligned}
& E\left[\int_0^T \int_G \frac{\partial}{\partial \gamma} f(t,x) \widehat{Y}^\beta(t,x)\, dx\, dt\right] \\
= \;& E\left[\int_0^T \int_G \frac{\partial}{\partial \gamma} f(t,x) \left(\int_0^t \left\{ L\widehat{Y}^\beta(s,x) + \frac{\partial}{\partial \gamma} b(s,x) \widehat{Y}^\beta(s,x) \right.\right.\right. \\
& \left. + \frac{\partial}{\partial u} b(s,x)\beta(s,x) + \nabla_{\gamma'} b(s,x,)\nabla_x Y(s,x) \right\} ds \\
& + \int_0^t \left\{ \frac{\partial}{\partial \gamma}\sigma(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'}\sigma(s,x)\nabla_x \widehat{Y}^\beta(s,x) + \frac{\partial}{\partial u}\sigma(s,x)\beta(s,x)\right\} dB(s) \\
& \left.\left. + \int_0^t \left\{ \frac{\partial}{\partial \gamma}\theta(s,x,z)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'}\theta(s,x,z)\nabla_x \widehat{Y}^\beta(t,x) + \frac{\partial}{\partial u}\theta(s,x,z)\beta(t,x)\right\} \widetilde{N}(dz,ds)\right) dxdt\right] \\
= \;& \int_G E\left[\int_0^T \left(\int_0^t \left\{ \frac{\partial}{\partial \gamma} f(t,x) \left[L\widehat{Y}^\beta(s,x) + \frac{\partial}{\partial \gamma} b(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'} b(s,x,)\nabla_x Y(s,x) \right.\right.\right.\right. \\
& \left. + \frac{\partial}{\partial u} b(s,x)\beta(s,x)\right] + D_s\left(\frac{\partial}{\partial \gamma} f(t,x)\right)\left[\frac{\partial}{\partial \gamma}\sigma(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'}\sigma(s,x)\nabla_x\widehat{Y}^\beta(s,x)\right. \\
& \left. + \frac{\partial}{\partial u}\sigma(s,x)\beta(s,x)\right] + + \int_{\mathbb{R}_0} D_{s,z}\left(\frac{\partial}{\partial \gamma} f(t,x)\right)\left[\frac{\partial}{\partial \gamma}\theta(s,x,z)\widehat{Y}^\beta(s,x)\right. \\
& \left.\left.\left.\left. + \nabla_{\gamma'}\theta(s,x,z)\nabla_x\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial u}\theta(s,x,z)\beta(t,x)\right] \nu(dz)\right\} ds\right) dt\right] dx \\
= \;& \int_G E\left[\int_0^T \left\{\left(\int_s^T \frac{\partial}{\partial \gamma} f(t,x) dt\right) \left[L\widehat{Y}^\beta(s,x) + \frac{\partial}{\partial \gamma} b(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'} b(s,x,)\nabla_x Y(s,x) \right.\right.\right. \\
& \left. + \frac{\partial}{\partial u} b(s,x)\beta(s,x)\right] + \left(\int_s^T D_s \frac{\partial}{\partial \gamma} f(t,x)dt\right)\left[\frac{\partial}{\partial \gamma}\sigma(s,x)\widehat{Y}^\beta(s,x) + \nabla_{\gamma'}\sigma(s,x)\nabla_x\widehat{Y}^\beta(s,x)\right. \\
& \left. + \frac{\partial}{\partial u}\sigma(s,x)\beta(s,x)\right] + \int_{\mathbb{R}_0} \left(\int_s^T D_{s,z}\frac{\partial}{\partial \gamma} f(t,x)dt\right)\left[\frac{\partial}{\partial \gamma}\theta(s,x,z)\widehat{Y}^\beta(s,x)\right. \\
& \left.\left.\left. + \nabla_{\gamma'}\theta(s,x,z)\nabla_x\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial u}\theta(s,x,z)\beta(t,x)\right] \nu(dz)\right\} ds\right] dx
\end{aligned}
$$

Changing the notation $s \to t$, this becomes

$$
\begin{aligned}
= \;& \int_G E\left[\int_0^T \left\{\left(\int_t^T \frac{\partial}{\partial \gamma} f(s,x) ds\right) \left[L\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial \gamma} b(t,x)\widehat{Y}^\beta(t,x) \right.\right.\right. \\
& \left. + \frac{\partial}{\partial u} b(t,x)\beta(t,x) + \nabla_{\gamma'} b(t,x,)\nabla_x Y(t,x)\right] \\
& + \left(\int_t^T D_t \frac{\partial}{\partial \gamma} f(s,x)ds\right)\left[\frac{\partial}{\partial \gamma}\sigma(t,x)\widehat{Y}^\beta(t,x) + \nabla_{\gamma'}\sigma(t,x)\nabla_x\widehat{Y}^\beta(t,x)\right. \\
& \left. + \frac{\partial}{\partial u}\sigma(t,x)\beta(t,x)\right] + \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z}\frac{\partial}{\partial \gamma} f(s,x)ds\right)\left[\frac{\partial}{\partial \gamma}\theta(t,x,z)\widehat{Y}^\beta(t,x)\right. \\
& \left.\left.\left. + \nabla_{\gamma'}\theta(t,x,z)\nabla_x\widehat{Y}^\beta(t,x) + \frac{\partial}{\partial u}\theta(t,x,z)\beta(t,x)\right] \nu(dz)\right\} dt\right] dx \qquad \text{(A.2.3)}
\end{aligned}
$$

Thus by the definition of $\widehat{K}(t,x)$ and combining with Equations (4.3.22)-(A.2.3) it follows that

$$\begin{aligned}
&E\Big[\int_G\int_0^T\Big\{\widehat{K}(t,x)\Big[L\widehat{Y}^\beta(t,x)+\frac{\partial}{\partial\gamma}b(t,x)\widehat{Y}^\beta(t,x)+\nabla_{\gamma'}b(t,x)\nabla_x\widehat{Y}^\beta(t,x)\\
&+\frac{\partial}{\partial u}b(t,x)\beta(t,x)\Big]+D_t\widehat{K}(t,x)\Big[\frac{\partial}{\partial\gamma}\sigma(t,x)\widehat{Y}^\beta(t,x)+\nabla_{\gamma'}\sigma(t,x)\widehat{Y}^\beta(t,x)\\
&+\frac{\partial}{\partial u}\sigma(t,x)\beta(t,x)\Big]+\int_{\mathbb{R}_0}\Big\{D_{t,z}\widehat{K}(t,x)\Big[\frac{\partial}{\partial\gamma}\theta(t,x,z)\widehat{Y}^\beta(r,x)\\
&+\nabla_{\gamma'}\theta(t,x,z)\nabla_x\widehat{Y}^\beta(r,x)+\frac{\partial}{\partial u}\theta(t,x,z)\beta(t,x)\Big]\Big\}\nu(dz)\\
&+\frac{\partial}{\partial u}f(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x),\omega)\beta(t,x)\Big\}\,dtdx\Big]=0 \qquad\qquad \text{(A.2.4)}
\end{aligned}$$

We observe that for all $\beta=\beta_\alpha\in\mathcal{A}_\mathcal{E}$ of the form $\beta_\alpha(s,x)=\alpha\chi_{[t,t+h]}(s)$ for some $t,h\in(0,T)$, $t+h\le T$ as defined in Equation (4.3.3)

$$\widehat{Y}^{\beta_\alpha}(s,x)=0,\ 0\le s\le t, x\in G\,.$$

Then by inspecting Equation (A.2.4) we have that

$$A_1+A_2+A_3+A_4=0 \qquad\qquad \text{(A.2.5)}$$

where

$$\begin{aligned}
A_1 &= E\Big[\int_G\int_t^T\Big\{\widehat{K}(s,x)\frac{\partial}{\partial\gamma}b(s,x)+D_s\widehat{K}(s,x)\frac{\partial}{\partial\gamma}\sigma(s,x)\\
&\quad+\int_{\mathbb{R}}D_{s,z}\widehat{K}(s,x)\frac{\partial}{\partial\gamma}\theta(s,x,z)\nu(dz)\Big\}\widehat{Y}^{\beta_\alpha}(s,x)\,ds\,dx\Big]\\
A_2 &= E\Big[\int_G\int_t^{t+h}\Big\{\widehat{K}(s,x)\frac{\partial}{\partial u}b(s,x)+D_s\widehat{K}(s,x)\frac{\partial}{\partial u}\sigma(s,x)\\
&\quad+\int_{\mathbb{R}}D_{s,z}\widehat{K}(s,x)\frac{\partial}{\partial u}\theta(s,x,z)\nu(dz)+\frac{\partial}{\partial u}f(s,x)\Big\}\alpha\,ds\,dx\Big]\\
A_3 &= E\Big[\int_G\int_t^T\widehat{K}(s,x)L\widehat{Y}^{\beta_\alpha}(s,x)\,dx\,dt\Big]\\
A_4 &= E\Big[\int_G\int_t^T\Big\{\widehat{K}(s,x)\frac{\partial}{\partial\gamma'}b(s,x)+D_s\widehat{K}(s,x)\nabla_{\gamma'}\sigma(s,x)\\
&\quad+\int_{\mathbb{R}}D_{s,z}\widehat{K}(s,x)\nabla_{\gamma'}\theta(s,x,z)\,\nu(dz)\Big\}\nabla_x\widehat{Y}^{\beta_\alpha}(s,x)\,ds\,dx\Big]
\end{aligned}$$

Note by the definition of $\widehat{Y}^{\beta_\alpha}$ with $\widehat{Y}^{\beta_\alpha}(s,x) = Y(s,x)$ and $s \geq t+h$ the process $Y(s,x)$ follows the following SPDE

$$\begin{aligned} d\widehat{Y}(s,x) =& \left\{ L\widehat{Y}^{\beta_\alpha}(s,x) + \widehat{Y}^{\beta_\alpha}(s^-,x)\frac{\partial}{\partial\gamma}b(s,x) + \nabla_x\widehat{Y}^{\beta_\alpha}(s^-,x)\nabla_{\gamma'}b(s,x)\right\} ds \\ &+ \left\{ \widehat{Y}^{\beta_\alpha}(s^-,x)\frac{\partial}{\partial\gamma}\sigma(s,x) + \nabla_x\widehat{Y}^{\beta_\alpha}(s^-,x)\nabla_{\gamma'}\sigma(s,x)\right\} dB(s) \\ &+ \int_{\mathbb{R}_0} \left\{ \widehat{Y}^{\beta_\alpha}(s^-,x)\frac{\partial}{\partial\gamma}\theta(s,x,z) + \nabla_x\widehat{Y}^{\beta_\alpha}(s^-,x)\nabla_{\gamma'}\theta(s,x,z)\right\} \widetilde{N}(dz,dr) \end{aligned}$$

Using notation (4.3.6)-(4.3.13) and assumption D1 we have

$$\begin{aligned} d\widehat{Y}(s,x) =& L\widehat{Y}^{\beta_\alpha}(s,x) + \widehat{Y}^{\beta_\alpha}(s^-,x)\left\{ b^*(s,x)\,ds + \sigma^*(s,x)\,dB(s) + \int_{\mathbb{R}_0}\theta^*(s,x,z)\,\widetilde{N}(dz,dr)\right\} \\ &+ \sum_{i=1}^n \frac{\partial}{\partial x_i}\widehat{Y}^{\beta_\alpha}(s^-,x)\left\{ \widetilde{b}_i(s,x)\,ds + \widetilde{\sigma}_i(s,x)\,dB(s)\right\}, \end{aligned} \tag{A.2.6}$$

for $s \geq t+h$ with initial condition $Y(t+h,x) \neq 0$ at time $t+h$. Equation (A.2.6) can be solve explicitly using the stochastic flow theory of the preceding section.

Let us consider the equation (see p. 297/298 in [79])

$$\eta_s(y) = \int_0^s \eta_r(y)F_{n+1}(\varphi_{0,r}(x), \circ dr) + \int_0^s F_{n+2}(\varphi_{0,r}(x), \circ dr).$$

Then

$$\begin{aligned} \eta_s(y) =& \int_0^{t+h} \eta_r(y)F_{n+1}(\varphi_{0,r}(x), \circ dr) + \int_0^{t+h} F_{n+2}(\varphi_{0,r}(x), \circ dr) \\ &+ \int_{t+h}^s \eta_r(y)F_{n+1}(\varphi_{0,r}(x), \circ dr) + \int_{t+h}^s F_{n+2}(\varphi_{0,r}(x), \circ dr) \\ =& \eta_{t+h}(y) + \int_{t+h}^s \eta_r(y)F_{n+1}(\varphi_{0,r}(x), \circ dr) + 0 \\ & \eta_{t+h}(y) + \int_{t+h}^s \eta_r(y)F_{n+1}(\varphi_{0,r}(x), \circ dr). \end{aligned}$$

So it follows that

$$\eta_s(y) = \eta_{t+h}(y)\exp\left\{\int_{t+h}^s F_{n+1}(\varphi_{0,r}(x), \circ dr)\right\}.$$

Thus

$$\begin{aligned} v(x,s) &= \eta_s(y)|_{y=\varphi_{s,0}(x)} = \eta_{t+h}(y)|_{y=\varphi_{s,0}(x)}\exp\left\{\int_{t+h}^s F_{n+1}(\varphi_{s,r}(x), \circ\widehat{dr})\right\} \\ &= v(\varphi_{s,t+h}(x), t+h)\exp\left\{\int_{t+h}^s F_{n+1}(\varphi_{s,r}(x), \circ\widehat{dr})\right\}. \end{aligned}$$

Therefore we obtain that

$$
\begin{aligned}
Y(s,x) =& E_Q\left[v(\varphi_{s,t+h}(x), t+h)\exp\left\{\int_{t+h}^{s} F_{n+1}(\varphi_{s,r}(x), \circ\widehat{dr})\right\}\right] \\
=& E_{\widehat{P}}\Big[v(\varphi_{s,t+h}(x), t+h) Z(t+h, s, \varphi_{s,r}(x))\Big].
\end{aligned}
\tag{A.2.7}
$$

where $Z(t,s,x),\ s \geq t$ is given by (4.3.19). For notational convenience, we set

$$
Q = \widehat{P}.
$$

Recall that $v(x,s)$ satisfies the SPDE (4.3.15).

In addition recall that

$$
Y(t,x) = E_{\widehat{P}}[v(t,x)]. \tag{A.2.8}
$$

Put

$$
\begin{aligned}
\widehat{H}_0(s,x,\gamma,\gamma',u) &= \widehat{K}(s,x)b(s,x,\gamma,\gamma',u) + D_s\widehat{K}(s,x)\sigma(s,x,\gamma,\gamma',u) \\
&\quad + \int_{\mathbb{R}} D_{s,z}\widehat{K}(s,x)\theta(s,x,\gamma,\gamma',z,u)\,\nu(dz)
\end{aligned}
\tag{A.2.9}
$$

$$
A_1 = E\left[\int_G \int_t^T \frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Y}(s,x)\,ds\,dx\right]
$$

Differentiating with respect to h at $h=0$ we get

$$
\begin{aligned}
\frac{d}{dh}A_1\Big|_{h=0} &= \frac{d}{dh}E\left[\int_G \int_t^{t+h} \frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0} \\
&\quad + \frac{d}{dh}E\left[\int_G \int_{t+h}^{T} \frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0}
\end{aligned}
\tag{A.2.10}
$$

Since $Y(t,x) = 0$ we see that

$$
\frac{d}{dh}E\left[\int_G \int_t^{t+h} \frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0} = 0. \tag{A.2.11}
$$

Therefore by Equation (A.2.7), we get

$$
\begin{aligned}
\frac{d}{dh}A_1\Big|_{h=0} &= \frac{d}{dh}E\left[\int_G \int_{t+h}^{T} \frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[v\left(t+h, \widehat{\varphi}_{s,t+h}(x)\right)\widehat{Z}\left(t+h,s,\widehat{\varphi}_{s,r}(x)\right)\right]ds\,dx\right]_{h=0} \\
&= \int_G \int_t^T \frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[v\left(t+h, \widehat{\varphi}_{s,t+h}(x)\right)\widehat{Z}\left(t+h,s,\widehat{\varphi}_{s,r}(x)\right)\right]\right]_{h=0} ds\,dx \\
&= \int_G \int_t^T \frac{d}{dh}E\left[\frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)E_Q\left[v\left(t+h, \widehat{\varphi}_{s,t+h}(x)\right)\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right)\right]\right]_{h=0} ds\,dx.
\end{aligned}
\tag{A.2.12}
$$

By Equation (4.3.15)

$$v(t+h,x) = \int_t^{t+h} L_s v(x,s)ds + \sum_{i=1}^n \int_t^{t+h} Y_i^*(x,ds)\frac{\partial v}{\partial x^i} + \sum_{i=1}^n \int_t^{t+h} F_i(x,ds)\frac{\partial v}{\partial x^i} + \int_0^t F_{n+1}(x,ds)v + \alpha \int_t^{t+h} \left\{\frac{\partial}{\partial u} b(r,x)\,dr + \frac{\partial}{\partial u}\sigma(r,x)\,dB(r)\right\}, \tag{A.2.13}$$

Then, using Equations (A.2.12) and (A.2.13), we get

$$\frac{d}{dh}A_1\Big|_{h=0} = A_{1,1} + A_{1,2} + A_{1,3}, \tag{A.2.14}$$

where

$$A_{1,1} = \int_G \int_t^T \frac{d}{dh} E\left[\frac{\partial}{\partial \gamma}\widehat{H}_0(s,x)E_Q\left[\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right) \times \left\{\int_t^{t+h} L_s\widehat{v}(x,r)\,dr + \int_0^t F_{n+1}(x,dr)\widehat{v}(x,r)\right\}\right]\right]_{h=0} ds\,dx \tag{A.2.15}$$

$$A_{1,2} = \int_G \int_t^T \frac{d}{dh} E\left[\frac{\partial}{\partial \gamma}\widehat{H}_0(s,x)E_Q\left[\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right) \times \alpha \int_t^{t+h}\left\{\frac{\partial}{\partial u} b(r,\varphi_{t+h,r}(x))\,dr + \frac{\partial}{\partial u}\sigma(r,\varphi_{t+h,r}(x))dB(r)\right\}\right]\right]_{h=0} ds\,dx, \tag{A.2.16}$$

$$A_{1,3} = \int_G \int_t^T \frac{d}{dh} E\left[\frac{\partial}{\partial \gamma}\widehat{H}_0(s,x)E_Q\left[\widehat{Z}\left(t,s,\widehat{\varphi}_{s,r}(x)\right) \times \left\{\sum_{i=1}^n \int_t^{t+h} Y_i^*(x,dr)\frac{\partial \widehat{v}}{\partial x^i}(x,r) + \sum_{i=1}^n \int_t^{t+h} F_i(x,dr)\frac{\partial \widehat{v}}{\partial x^i}(x,r)\right\}\right]\right]_{h=0} ds\,dx. \tag{A.2.17}$$

Since $\widehat{Y}(t,x) = 0$ we have that $v(t,x) = 0$ and then

$$A_{1,1} = A_{1,3} = 0$$

By the duality formula and applying Fubini's theorem repeatedly, $A_{1,2}$ becomes

$$\begin{aligned} A_{1,2} &= \int_G \int_t^T \frac{d}{dh} E\left[E_Q\left[\alpha \int_t^{t+h}\left\{\frac{\partial}{\partial u} b(r,\varphi_{t+h,r}(x))I(t,s,x) + \frac{\partial}{\partial u}\sigma(r,\varphi_{t+h,r}(x))D_r I(t,s,x)\right\}\right]\right]_{h=0} ds\,dx, \\ &= \int_G \int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u} b(t,\varphi_{t+h,t}(x))I(t,s,x) + \frac{\partial}{\partial u}\sigma(t,\varphi_{t+h,t}(x))D_t I(t,s,x)\right\}\right]\right] ds\,dx, \end{aligned} \tag{A.2.18}$$

where $I(t,s,x) = \frac{\partial}{\partial\gamma}\widehat{H}_0(s,x)\widehat{Z}(t,s,\widehat{\varphi}_{s,t}(x))$.

This implies that

$$\begin{aligned}
\frac{d}{dh}A_1\Big|_{h=0} =& A_{1,2} \\
=& \int_G\int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,\varphi_{t,t}(x))I(t,s,x)\right.\right.\right. \\
&\left.\left.\left.+\frac{\partial}{\partial u}\sigma(t,\varphi_{t,t}(x))D_tI(t,s,x)\right\}\right]\right]\,ds\,dx. \\
=& \int_G\int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,x)I(t,s,x)+\frac{\partial}{\partial u}\sigma(t,x)D_tI(t,s,x)\right\}\right]\right]\,ds\,dx,
\end{aligned} \tag{A.2.19}$$

where the last equality follows from the fact that $\widehat{\varphi}_{t,t}(x) = x$. Moreover, we see that

$$\frac{d}{dh}A_2\Big|_{h=0} = \int_G E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,x)\widehat{K}(t,x)+\frac{\partial}{\partial u}\sigma(t,x)D_t\widehat{K}(t,x)+\frac{\partial}{\partial u}f(t,x)\right\}\right]\,ds\,dx \tag{A.2.20}$$

Then, using the adjoint operators L^* and ∇_x^* (see (4.3.20)) we get

$$\begin{aligned}
A_3 &= E\left[\int_G\int_t^T \widehat{K}(s,x)L\widehat{Y}^{\beta_\alpha}(s,x)dx\,dt\right] \\
&= E\left[\int_G\int_t^T L^*\widehat{K}(s,x)\widehat{Y}^{\beta_\alpha}(s,x)\,dx\,dt\right] \\
A_4 &= E\left[\int_G\int_t^T \left\{\widehat{K}(s,x)\nabla_{\gamma'}b(s,x)+D_s\widehat{K}(s,x)\nabla_{\gamma'}\sigma(s,x)\right.\right. \\
&\quad \left.\left.+\int_{\mathbb{R}}D_{s,z}\widehat{K}(s,x)\nabla_{\gamma'}\theta(s,x,z)\nu(dz)\right\}\nabla_x\widehat{Y}^{\beta_\alpha}(s,x)\,ds\,dx\right] \\
&= E\left[\int_G\int_t^T \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Y}^{\beta_\alpha}(s,x)\,ds\,dx\right]
\end{aligned}$$

Differentiating with respect to h at $h=0$ gives

$$\begin{aligned}
\frac{d}{dh}A_3\Big|_{h=0} &= \frac{d}{dh}E\left[\int_G\int_t^{t+h} L^*\widehat{K}(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0} \\
&\quad +\frac{d}{dh}E\left[\int_G\int_{t+h}^T L^*\widehat{K}(s,x)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0},
\end{aligned} \tag{A.2.21}$$

$$\begin{aligned}
\frac{d}{dh}A_4\Big|_{h=0} &= \frac{d}{dh}E\left[\int_G\int_t^{t+h} \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0} \\
&\quad +\frac{d}{dh}E\left[\int_G\int_{t+h}^T \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Y}(s,x)\,ds\,dx\right]_{h=0}.
\end{aligned} \tag{A.2.22}$$

Using the same arguments as before, it can be shown that

$$\frac{d}{dh}A_3\bigg|_{h=0} = \int_G \int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,x)I_1(t,s,x) + \frac{\partial}{\partial u}\sigma(t,x)D_tI_1(t,s,x)\right\}\right]\right] ds\,dx, \tag{A.2.23}$$

$$\frac{d}{dh}A_4\bigg|_{h=0} = \int_G \int_t^T E_Q\left[E\left[\alpha\left\{\frac{\partial}{\partial u}b(t,x)I_2(t,s,x) + \frac{\partial}{\partial u}\sigma(t,x)D_tI_2(t,s,x)\right\}\right]\right] ds\,dx, \tag{A.2.24}$$

where $I_1(t,s,x) = L^*\widehat{K}(s,x)\widehat{Z}(t,s,\varphi_{s,t}(x))$ and $I_2(t,s,x) = \nabla_x^*\left(\nabla_{\gamma'}\widehat{H}_0(s,x)\right)\widehat{Z}(t,s,\varphi_{s,t}(x))$.
Therefore, differentiating Equation (A.2.5) with respect to h at $h=0$ yields

$$\begin{aligned} E_Q\Bigg[E\Bigg[\alpha\int_G\Bigg\{\frac{\partial}{\partial u}f(t,x) + \left(\widehat{K}(t,x) + \int_t^T\Big(I(t,s,x) + I_1(t,s,x) + I_2(t,s,x)\Big)ds\right)\frac{\partial}{\partial u}b(t,x) \\ +D_t\left(\widehat{K}(t,x) + \int_t^T\Big(I(t,s,x) + I_1(t,s,x) + I_2(t,s,x)\Big)ds\right)\frac{\partial}{\partial u}\sigma(t,x)\Bigg\}\,dx\Bigg]\Bigg] = 0 \end{aligned} \tag{A.2.25}$$

By the definition of $\widehat{p}(t,x)$, we have

$$\widehat{p}(t,x) = \widehat{K}(t,x) + \int_t^T\Big(I(t,s,x) + I_1(t,s,x) + I_2(t,s,x)\Big)ds.$$

We can then write (A.2.25), as

$$\begin{aligned} & E_Q\Bigg[E\Bigg[\int_G\frac{\partial}{\partial u}\Big\{f(t,x,\Gamma,\widehat{u},\omega) + p(t,x)b(t,x,\Gamma,\Gamma',\widehat{u},\omega) + D_tp(t,x)\sigma(t,x,\Gamma,\Gamma',\widehat{u},\omega) \\ & + \int_{\mathbb{R}} D_{t,z}p(t,x)\theta(t,x,\Gamma,\Gamma',\widehat{u},z,\omega)\nu(dz)\Big\}\,\alpha\,dx\Bigg]\Bigg] \\ = & \ 0 \end{aligned}$$

Since this holds for all bounded $\mathcal{E}_t-$measurable random variables α, we conclude that

$$E_Q\left[E\left[\int_G\frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x))\,dx\,\bigg|\,\mathcal{E}_t\right]\right] = 0 \text{ a.e. in } (t,x,\omega),$$

which means

$$E\left[E_Q\left[\int_G\frac{\partial}{\partial u}\widehat{H}(t,x,\widehat{\Gamma}(t,x),\nabla_x\widehat{\Gamma}(t,x),\widehat{u}(t,x))\,dx\right]\bigg|\,\mathcal{E}_t\right] = 0 \text{ a.e. in } (t,x,\omega),$$

which completes the proof.

A.3 Proof of Theorem 6.3.1

1. Since $\widehat{u} \in \mathcal{A}_{\mathcal{G}}$ is a critical point for $J(u)$, there exists a $\delta > 0$ as in Equaation (6.3.2) for all bounded $\beta \in \mathcal{A}_{\mathcal{G}}$. Thus

$$
\begin{aligned}
0 &= \frac{d}{dy} J(\widehat{u} + y\beta)\Big|_{y=0} \qquad \text{(A.3.1)} \\
&= E\left[\int_0^T \left\{\frac{\partial}{\partial x} f(t, X(t), u(t))\widehat{Y}(t) + \frac{\partial}{\partial u} f(t, X(t), u(t))\beta(t)\right\} ds + g'(X(T))\widehat{Y}(T)\right],
\end{aligned}
$$

where $\widehat{Y} = Y_\beta^{\widehat{u}}$ is as defined in Eqution (6.3.3).

We study the two summands separately.

$$
\begin{aligned}
& E\left[g'(X(T))Y(T)\right] \\
=& E\left[g'(X(T))\left(\int_0^T \left\{\frac{\partial b(t)}{\partial x} Y(t) + \frac{\partial b(t)}{\partial u}\beta(t)\right\} dt\right.\right. \\
&+ \int_0^T \left\{\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right\} d^- B(t) \\
&\left.\left.+ \int_0^T \int_{\mathbb{R}_0} \left\{\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right\} \widetilde{N}(dz, d^- t)\right)\right] \\
=& E\left[\int_0^T g'(X(T))\left\{\frac{\partial b(t)}{\partial x} Y(t) + \frac{\partial b(t)}{\partial u}\beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T D_t g'(X(T))\left\{\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T g'(X(T)) D_{t+}\left(\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right) dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\left\{\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right\} \nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z} g'(X(T))\right\} D_{t+,z}\left(\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right) \nu(dz)dt\right]
\end{aligned}
$$

$$
\begin{aligned}
= & E\left[\int_0^T \left\{g'(X(T))\frac{\partial b(t)}{\partial x} + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\frac{\partial \theta(t)}{\partial x}\nu(dz)\right\} Y(t)dt\right] \\
& + E\left[\int_0^T \left\{g'(X(T))\frac{\partial b(t)}{\partial u} + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\frac{\partial \theta(t)}{\partial u}\nu(dz)\right\} \beta(t)dt\right] \\
& + E\left[\int_0^T g'(X(T)) D_{t+}\frac{\partial \sigma(t)}{\partial x} Y(t)dt\right] + E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial x} D_{t+}Y(t)dt\right] \\
& + E\left[\int_0^T g'(X(T)) D_{t+}\frac{\partial \sigma(t)}{\partial u} \beta(t)dt\right] + E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial u} D_{t+}\beta(t)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} D_{t+,z}\frac{\partial \theta(t)}{\partial x} Y(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} D_{t+,z}\frac{\partial \theta(t)}{\partial u} \beta(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right] \\
= & E\left[\int_0^T \left\{g'(X(T))\left(\frac{\partial b(t)}{\partial x} + D_{t+}\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial x}\nu(dz)\right) + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial x}\right.\right. \\
& \left.\left. + \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\left(\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right)\nu(dz)\right\} Y(t)dt\right] \\
& + E\left[\int_0^T \left\{g'(X(T))\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial u}\right.\right. \\
& \left.\left. + \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right)\nu(dz)\right\} \beta(t)dt\right] \\
& + E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial x} D_{t+}Y(t)dt\right] \\
& + E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial u} D_{t+}\beta(t)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right]
\end{aligned}
$$

Similarly, we have using both Fubini and duality theorems,

$$
\begin{aligned}
& E\left[\int_0^T \frac{\partial}{\partial x} f(t) Y(t) dt\right] \\
= & E\Bigg[\int_0^T \frac{\partial}{\partial x} f(t) \Bigg(\int_0^t \left\{\frac{\partial b(s)}{\partial x} Y(s) + \frac{\partial b(s)}{\partial u} \beta(s)\right\} ds \\
& + \int_0^t \left\{\frac{\partial \sigma(s)}{\partial x} Y(s) + \frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} d^- B(s) \\
& + \int_0^t \int_{\mathbb{R}_0} \left\{\frac{\partial \theta(s)}{\partial x} Y(s) + \frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \widetilde{N}(dz, d^- s)\Bigg) dt\Bigg] \\
= & E\left[\int_0^T \left(\int_0^t \frac{\partial f(t)}{\partial x} \left\{\frac{\partial b(s)}{\partial x} Y(s) + \frac{\partial b(s)}{\partial u} \beta(s)\right\} ds\right) dt\right] \\
& + E\left[\int_0^T \left(\int_0^t D_s \frac{\partial f(t)}{\partial x} \left\{\frac{\partial \sigma(s)}{\partial x} Y(s) + \frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} ds\right) dt\right] \\
& + E\left[\int_0^T \left(\int_0^t \frac{\partial f(t)}{\partial x} D_{s+} \left\{\frac{\partial \sigma(s)}{\partial x} Y(s) + \frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} ds\right) dt\right] \\
& + E\left[\int_0^T \left(\int_0^t \int_{\mathbb{R}_0} D_{s,z} \frac{\partial f(t)}{\partial x} \left\{\frac{\partial \theta(s)}{\partial x} Y(s) + \frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \nu(dz) ds\right) dt\right] \\
& + E\Bigg[\int_0^T \Bigg(\int_0^t \int_{\mathbb{R}_0} \left\{\frac{\partial f(t)}{\partial x} + D_{s,z} \frac{\partial f(t)}{\partial x}\right\} \times \\
& \quad D_{s+,z} \left(\frac{\partial \theta(s)}{\partial x} Y(s) + \frac{\partial \theta(s)}{\partial u} \beta(s)\right) \nu(dz) ds\Bigg) dt\Bigg] \\
= & E\left[\int_0^T \left(\int_s^T \frac{\partial f(t)}{\partial x} dt\right) \left\{\frac{\partial b(s)}{\partial x} Y(s) + \frac{\partial b(s)}{\partial u} \beta(s)\right\} ds\right] \\
& + E\left[\int_0^T \left(\int_s^T D_s \frac{\partial f(t)}{\partial x} dt\right) \left\{\frac{\partial \sigma(s)}{\partial x} Y(s) + \frac{\partial \sigma(s)}{\partial u} \beta(s)\right\}\right] \\
& + E\left[\int_0^T \left(\int_s^T \frac{\partial f(t)}{\partial x} dt\right) D_{s+} \left\{\frac{\partial \sigma(s)}{\partial x} Y(s) + \frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} ds\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_s^T D_{s,z} \frac{\partial f(t)}{\partial x} dt\right) \left\{\frac{\partial \theta(s)}{\partial x} Y(s) + \frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \nu(dz) ds\right] \\
& + E\Bigg[\int_0^T \int_{\mathbb{R}_0} \left(\int_s^T \left\{\frac{\partial f(t)}{\partial x} + D_{s,z} \frac{\partial f(t)}{\partial x}\right\} dt\right) \times \\
& \quad D_{s+,z} \left\{\frac{\partial \theta(s)}{\partial x} Y(s) + \frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \nu(dz) ds\Bigg]
\end{aligned}
$$

Changing the notation $s \to t$, this becomes

$$
\begin{aligned}
&= E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \left\{\frac{\partial b(t)}{\partial x} Y(t) + \frac{\partial b(t)}{\partial u} \beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T \left(\int_t^T D_t \frac{\partial f(s)}{\partial x} ds\right) \left\{\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u} \beta(t)\right\}\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z} \frac{\partial f(s)}{\partial x} ds\right) \left\{\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u} \beta(t)\right\} \nu(dz) dt\right] \\
&+ E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) D_{t+} \left\{\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u} \beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_t^T \left\{\frac{\partial f(s)}{\partial x} + D_{t,z} \frac{\partial f(s)}{\partial x}\right\} ds\right)\right. \\
&\quad \left.\left(D_{t+,z} \left\{\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u} \beta(t)\right\}\right) \nu(dz) dt\right] \qquad \text{(A.3.2)} \\
&= E\left[\int_0^T \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial b(t)}{\partial x} + D_{t+} \frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t+,z} \frac{\partial \theta(t)}{\partial x} \nu(dz)\right)\right.\right. \\
&\quad + \left(\int_t^T D_t \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial x} \\
&\quad \left.\left. + \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z} \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial \theta(t)}{\partial x} + D_{t+,z} \frac{\partial \theta(t)}{\partial x}\right) \nu(dz)\right\} Y(t) dt\right] \\
&\quad + E\left[\int_0^T \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial b(t)}{\partial u} + D_{t+} \frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z} \frac{\partial \theta(t)}{\partial u} \nu(dz)\right)\right.\right. \\
&\quad + \left(\int_t^T D_t \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial u} \\
&\quad \left.\left. + \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z} \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z} \frac{\partial \theta(t)}{\partial u}\right) \nu(dz)\right\} \beta(t) dt\right] \\
&\quad + E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial x} D_{t+} Y(t) dt\right] \\
&\quad + E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial u} D_{t+} \beta(t) dt\right] \\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} + D_{t,z} \frac{\partial f(s)}{\partial x} ds\right)\right\} \left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z} \frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z} Y(t) \nu(dz) dt\right] \\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} + D_{t,z} \frac{\partial f(s)}{\partial x} ds\right)\right\} \left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z} \frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z} \beta(t) \nu(dz) dt\right]
\end{aligned}
$$

Recall that

$$K(t) \quad := \quad g'(X(T)) + \int_t^T \frac{\partial}{\partial x} f(s, X(s), u(s)) ds$$

and combining Equations (6.3.11)-(6.3.14) and (A.3.2), it follows that

$$
\begin{aligned}
0 = E\Bigg[\int_0^T \Bigg\{ & K(t)\left(\frac{\partial b(t)}{\partial x} + D_{t+}\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial x}\nu(dz)\right) + D_t K(t)\frac{\partial \sigma(t)}{\partial x} \\
& + \int_{\mathbb{R}_0} D_{t,z}K(t)\left(\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right)\nu(dz)\Bigg\} Y(t)dt\Bigg] \\
& + E\Bigg[\int_0^T \Bigg\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + D_t K(t)\frac{\partial \sigma(t)}{\partial u} \\
& + \int_{\mathbb{R}_0} D_{t,z}K(t)\left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right)\nu(dz) + \frac{\partial f(t)}{\partial u}\Bigg\}\beta(t)dt\Bigg] \\
& + E\left[\int_0^T K(t)\frac{\partial \sigma(t)}{\partial x}D_{t+}Y(t)dt\right] \\
& + E\left[\int_0^T K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\beta(t)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \{K(t) + D_{t,z}K(t)\}\left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0} \{K(t) + D_{t,z}K(t)\}\left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right]
\end{aligned}
\tag{A.3.3}
$$

We observe that for all $\beta_\alpha \in \mathcal{A}_\mathcal{G}$ given as $\beta_\alpha(s) := \alpha\chi_{[t,t+h]}(s)$, for some $t,\, h \in (0,T),\ \ t+h \leq T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_t-$measurable. Then $Y^{(\beta_\alpha)}(s) = 0$ for $0 \leq s \leq t$ and hence Equation (A.3.3) becomes

$$
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0 \tag{A.3.4}
$$

Where

$$
\begin{aligned}
A_1 = E\Bigg[\int_t^T \Bigg\{ & K(t)\left(\frac{\partial b(s)}{\partial x} + D_{s+}\frac{\partial \sigma(s)}{\partial x} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial x}\nu(dz)\right) \\
& + \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz) + D_s K(s)\frac{\partial \sigma(s)}{\partial x}\Bigg\} Y^{(\beta_\alpha)}(s)ds\Bigg] \\
A_2 = E\Bigg[\int_t^{t+h} \Bigg\{ & K(t)\left(\frac{\partial b(s)}{\partial u} + D_{s+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial u}\nu(dz)\right) + \frac{\partial f(s)}{\partial u} \\
& + \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz) + D_s K(s)\frac{\partial \sigma(s)}{\partial u}\Bigg\} \alpha ds\Bigg] \\
A_3 = E\Bigg[\int_t^T & K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y^{(\beta_\alpha)}(s)ds\Bigg] \\
A_4 = E\Bigg[\int_t^{t+h} & K(s)\frac{\partial \sigma(s)}{\partial u}D_{s+}\alpha ds\Bigg]
\end{aligned}
$$

$$A_5 = E\left[\int_t^T \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\} \Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz) D_{s+,z} Y^{(\beta_\alpha)}(s) ds\right]$$
$$A_6 = E\left[\int_t^{t+h} \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\} \Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz) D_{s+,z} \alpha ds\right]$$

Note by the definition of Y, with $Y(s) = Y^{(\beta_\alpha)}(s)$ and $s \geq t+h$, the process $Y(s)$ follows the dynamics

$$dY(s) = Y(s^-)\left[\frac{\partial b}{\partial x}(s)ds + \frac{\partial \sigma}{\partial x}(s)d^-B(s) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(s,z)\widetilde{N}(dz, d^-s)\right], \quad \text{(A.3.5)}$$

for $s \geq t+h$ with initial condition $Y(t+h)$ in time $t+h$. By Itô formula for forward integral, this equation can be solved explicitly and we get

$$Y(s) = Y(t+h)G(t+h,s), \quad s \geq t+h \quad \text{(A.3.6)}$$

where, in general, for $s \geq t$,

$$\begin{aligned} G(t,s) \quad := \quad & \exp\Bigg(\int_t^s \left\{\frac{\partial b}{\partial x}(r,X(r),u(r),\omega) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2 (r,X(r),u(r),\omega)\right\} dr \\ & + \int_t^s \frac{\partial \sigma}{\partial x}(r,X(r),u(r),\omega)\, dB^-(r) \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial \theta}{\partial x}(r,X(r),u(r),\omega)\right) - \frac{\partial \theta}{\partial x}(r,X(r),u(r),\omega)\right\}\nu(dz)dt \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial \theta}{\partial x}(r,X(r^-),u(r^-),\omega)\right)\right\}\widetilde{N}(dz, d^-r)\Bigg). \end{aligned}$$

Note that $G(t,s)$ does not depend on h, but $Y(s)$ does. Defining H_0 as in Equation (6.3.5), it follows that

$$A_1 = E\left[\int_t^T \frac{\partial H_0}{\partial x}(s)Y(s)ds\right].$$

Differentiating with respect to h at $h=0$, we get

$$\frac{d}{dh}A_1\Big|_{h=0} = \frac{d}{dh}E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)Y(s)ds\right]_{h=0} + \frac{d}{dh}E\left[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)Y(s)ds\right]_{h=0}.$$

Since $Y(t)=0$, we see that

$$\frac{d}{dh}E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)Y(s)ds\right]_{h=0} = 0$$

Therefore, by Equation (A.3.6),

$$\begin{aligned}\frac{d}{dh}A_1\Big|_{h=0} &= \frac{d}{dh}E\left[\int_{t+h}^{T}\frac{\partial H_0}{\partial x}(s)Y(t+h)G(t+h,s)ds\right]_{h=0}\\ &= \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)Y(t+h)G(t+h,s)\right]_{h=0}ds\\ &= \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)G(t,s)Y(t+h)\right]_{h=0}ds,\end{aligned}$$

where, $Y(t+h)$ is given by

$$\begin{aligned}Y(t+h) = &\int_t^{t+h} Y(r^-)\left[\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)d^-B(r) + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(r,z)\widetilde{N}(dz,d^-r)\right]\\ &+ \alpha\int_t^{t+h}\left[\frac{\partial b}{\partial u}(r)dr + \frac{\partial \sigma}{\partial u}(r)d^-B(r) + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial u}(r,z)\widetilde{N}(dz,d^-r)\right].\end{aligned}$$

Therefore, by the two preceding equalities,

$$\frac{d}{dh}A_1\Big|_{h=0} = A_{1,1}+A_{1,2},$$

where

$$\begin{aligned}A_{1,1} = &\int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)G(t,s)\alpha\int_t^{t+h}\left\{\frac{\partial b}{\partial u}(r)dr + \frac{\partial \sigma}{\partial u}(r)d^-B(r)\right.\right.\\ &\left.\left. + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial u}(r,z)\widetilde{N}(dz,d^-r)\right\}\right]_{h=0}ds,\end{aligned}$$

and

$$\begin{aligned}A_{1,2} = &\int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)G(t,s)\int_t^{t+h}Y(r^-)\left\{\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)d^-B(r)\right.\right.\\ &\left.\left. + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(r,z)\widetilde{N}(dz,d^-r)\right\}\right]_{h=0}ds.\end{aligned}$$

Applying again the duality formula, we have

$$\begin{aligned}A_{1,1} = &\int_t^T \frac{d}{dh}E\left[\alpha\int_t^{t+h}\left\{\frac{\partial b}{\partial u}(r)F(t,s) + \frac{\partial \sigma}{\partial u}(r)D_rF(t,s) + F(t,s)D_{r^+}\frac{\partial \sigma}{\partial u}(r)\right.\right.\\ &+ \int_{\mathbb{R}_0}\left\{\left(\frac{\partial \theta}{\partial u}(r,z) + D_{r^+,z}\frac{\partial \theta}{\partial u}(r,z)\right)D_{r,z}F(t,s)\right.\\ &\left.\left.\left. + D_{r^+,z}\frac{\partial \theta}{\partial u}(r,z)F(t,s)\right\}\nu(dz)\right\}dr\right]_{h=0}ds\\ = &\int_t^T E\left[\alpha\left\{\left(\frac{\partial b}{\partial u}(t) + D_{t^+}\frac{\partial \sigma}{\partial u}(t) + \int_{\mathbb{R}_0}D_{t^+,z}\frac{\partial \theta}{\partial u}(t,z)\nu(dz)\right)F(t,s)\right.\right.\\ &\left.\left. + \frac{\partial \sigma}{\partial u}(t)D_tF(t,s) + \int_{\mathbb{R}_0}\left(\frac{\partial \theta}{\partial u}(t,z) + D_{t^+,z}\frac{\partial \theta}{\partial u}(t,z)\right)D_{t,z}F(t,s)\nu(dz)\right\}\right]ds,\end{aligned}$$

where we have put

$$F(t,s) = \frac{\partial H_0}{\partial x}(s)G(t,s)$$

Since $Y(t) = 0$ we see that

$$A_{1,2} = 0.$$

We conclude that

$$\begin{aligned}
\frac{d}{dh}A_1\Big|_{h=0} &= A_{1,1} \qquad\qquad \text{(A.3.7)}\\
&= \int_t^T \frac{d}{dh}E\Big[\alpha\int_t^{t+h}\Big\{\frac{\partial b}{\partial u}(r)F(t,s) + \frac{\partial \sigma}{\partial u}(r)D_rF(t,s)\\
&\quad + F(t,s)D_{r+}\frac{\partial \sigma}{\partial u}(r) + \int_{\mathbb{R}_0}\Big\{\Big(\frac{\partial \theta}{\partial u}(r,z) + D_{r^+,z}\frac{\partial \theta}{\partial u}(r,z)\Big)D_{r,z}F(t,s)\\
&\quad + \ D_{r^+,z}\frac{\partial \theta}{\partial u}(r,z)F(t,s)\Big\}\nu(dz)\Big\}dr\Big]_{h=0} ds\\
&= \int_t^T E\Big[\alpha\Big\{\Big(\frac{\partial b}{\partial u}(t) + D_{t^+}\frac{\partial \sigma}{\partial u}(t) + \int_{\mathbb{R}_0} D_{t^+,z}\frac{\partial \theta}{\partial u}(t,z)\nu(dz)\Big)F(t,s)\\
&\quad + \frac{\partial \sigma}{\partial u}(t)D_tF(t,s)\\
&\quad + \int_{\mathbb{R}_0}\Big(\frac{\partial \theta}{\partial u}(t,z)D_{t^+,z}\frac{\partial \theta}{\partial u}(t,z)\Big)D_{t,z}F(t,s)\nu(dz)\Big\}\Big]ds,
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
\frac{d}{dh}A_2\Big|_{h=0} =& E\Big[\Big\{K(t)\Big(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t,z)}{\partial u}\nu(dz)\Big)\\
&+ \frac{\partial f(t)}{\partial u} + D_tK(t)\frac{\partial \sigma(t,z)}{\partial u}\\
&+ \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t,z)}{\partial u} + D_{t+,z}\frac{\partial \theta(t,z)}{\partial u}\Big)\nu(dz)\Big\}\alpha\Big], \qquad \text{(A.3.8)}\\
\frac{d}{dh}A_4\Big|_{h=0} =& E\Big[K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\alpha\Big], \qquad \text{(A.3.9)}\\
\frac{d}{dh}A_6\Big|_{h=0} =& E\Big[\int_{\mathbb{R}_0}\{K(t) + D_{t,z}K(t)\}\Big(\frac{\partial \theta(t,z)}{\partial u} + D_{t+,z}\frac{\partial \theta(t,z)}{\partial u}\Big)\nu(dz)D_{t+,z}\alpha\Big].
\end{aligned}$$

(A.3.10)

On the other hand, differentiating A_3 with respect to h at $h = 0$, we get

$$\begin{aligned}
\frac{d}{dh}A_3\Big|_{h=0} &= \frac{d}{dh}E\Big[\int_t^{t+h} K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y(s)ds\Big]_{h=0}\\
&\quad + \frac{d}{dh}E\Big[\int_{t+h}^T K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y(s)ds\Big]_{h=0}.
\end{aligned}$$

Since $Y(t) = 0$, we see that

$$\begin{aligned}
\frac{d}{dh}A_3\Big|_{h=0} &= \frac{d}{dh}E\left[\int_{t+h}^{T} K(s)\frac{\partial\sigma(s)}{\partial x}D_{s+}\Big(Y(t+h)G(t+h,s)\Big)ds\right]_{h=0} \\
&= \int_t^T \frac{d}{dh}E\left[K(s)\frac{\partial\sigma(s)}{\partial x}D_{s+}\Big(Y(t+h)G(t+h,s)\Big)\right]_{h=0} ds \\
&= \int_t^T \frac{d}{dh}E\left[K(s)\frac{\partial\sigma(s)}{\partial x}\Big(D_{s+}G(t+h,s)\cdot Y(t+h)\right. \\
&\quad \left.+ D_{s+}Y(t+h)\cdot G(t+h,s)\Big)\right]_{h=0} ds \\
&= \int_t^T \frac{d}{dh}E\left[K(s)\frac{\partial\sigma(s)}{\partial x}\cdot D_{s+}Y(t+h)G(t,s)\right]_{h=0} ds.
\end{aligned}$$

For

$$\begin{aligned}
Y(t+h) &= \int_t^{t+h} Y(r^-)\left[\frac{\partial b}{\partial x}(r)dr + \frac{\partial\sigma}{\partial x}(r)d^-B(r) + \int_{\mathbb{R}_0}\frac{\partial\theta}{\partial x}b(r,z)\widetilde{N}(dz,d^-r)\right] \\
&\quad + \alpha\int_t^{t+h}\left[\frac{\partial b}{\partial u}(r)dr + \frac{\partial\sigma}{\partial u}(r)d^-B(r) + \int_{\mathbb{R}_0}\frac{\partial\theta}{\partial u}(r,z)\widetilde{N}(dz,d^-r)\right],
\end{aligned}$$

Using the definition of $\widehat{p}$ and $\widehat{H}$ given respectively by Equations (6.3.14) and (6.3.13) in the Theorem, it follows from Equation (A.3.4) that

$$E\left[\frac{\partial}{\partial u}\widehat{H}(t,\widehat{X}(t),\widehat{u}(t))\Big|\,\mathcal{G}_t\right] + E[A] = 0 \text{ a.e. in } (t,\omega), \tag{A.3.11}$$

where

$$A = \frac{d}{dh}A_3\Big|_{h=0} + \frac{d}{dh}A_4\Big|_{h=0} + \frac{d}{dh}A_5\Big|_{h=0} + \frac{d}{dh}A_6\Big|_{h=0} \tag{A.3.12}$$

2. Conversely, suppose there exists $\widehat{u} \in \mathcal{A}_{\mathcal{G}}$ such that Equation (6.3.12) holds. Then by reversing the previous arguments, we obtain that Equation (A.3.4) holds for all

$\beta_\alpha(s) := \alpha\chi_{[t,t+h]}(s) \in \mathcal{A}_\mathcal{G}$, where

$$\begin{aligned}
A_1 &= E\Bigg[\int_t^T \Bigg\{K(t)\left(\frac{\partial b(s)}{\partial x} + D_{s+}\frac{\partial \sigma(s)}{\partial x} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial x}\nu(dz)\right) \\
&\quad + \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz) + D_sK(s)\frac{\partial \sigma(s)}{\partial x}\Bigg\} Y^{(\beta_\alpha)}(s)ds\Bigg] \\
A_2 &= E\Bigg[\int_t^{t+h} \Bigg\{K(t)\left(\frac{\partial b(s)}{\partial u} + D_{s+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial u}\nu(dz)\right) + \frac{\partial f(s)}{\partial u} \\
&\quad + \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz) + D_sK(s)\frac{\partial \sigma(s)}{\partial u}\Bigg\} \alpha ds\Bigg] \\
A_3 &= E\left[\int_t^T K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y^{(\beta_\alpha)}(s)ds\right] \\
A_4 &= E\left[\int_t^{t+h} K(s)\frac{\partial \sigma(s)}{\partial u}D_{s+}\alpha ds\right]
\end{aligned}$$

$$\begin{aligned}
A_5 &= E\left[\int_t^T \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\}\Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz)D_{s+,z}Y^{(\beta_\alpha)}(s)ds\right] \\
A_6 &= E\left[\int_t^{t+h} \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\}\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz)D_{s+,z}\alpha ds\right]
\end{aligned}$$

for some $t, h \in (0,T)$, $t+h \le T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_t-$measurable. Hence, these equalities hold for all linear combinations of β_α. Since all bounded $\beta \in \mathcal{A}_\mathcal{G}$ can be approximated pointwise boundedly in (t,ω) by such linear combinations, it follows that Equation (A.3.4) holds for all bounded $\beta \in \mathcal{A}_\mathcal{G}$. Hence, by reversing the remaining part of the previous proof, we conclude that

$$\frac{d}{dy}J_1(\widehat{u} + y\beta)\Big|_{y=0} = 0, \text{ for all } \beta,$$

and then $\widehat{u}$ satisfies Relation (6.3.11).

A.4 A chaos expansion approach when $\mathcal{G}_t$ is of type (6.5.2)

Theorem A.4.1 *Suppose that* b, σ *and* θ *do not depend on* X *and that* $\mathcal{G}_t$ *is of type (6.5.2). Then there exist an optimal control* $\widehat{u}$ *for the performance functional* $J(u)$ *in (6.1.3) if and only if the following three conditions hold:*

$$(i)\ E\left[L(t)|\,\mathcal{G}_t\right]=0,$$

$$(ii)\ E\left[M(t)|\,\mathcal{G}_t\right]=0,$$

$$(iii)\ E\left[\int_{\mathbb{R}_0} R(t,z)\nu(dz)|\,\mathcal{G}_t\right]=0,$$

where $L, M,$ *and* R *are given by (6.4.3), (6.4.4) and (6.4.5).*

Proof. In order to prove the theorem, we consider the Brownian motion case and the Poisson random measure case separately.

Brownian motion case: Choose $\psi_1,\cdots,\psi_n\in C\,[0,T]$ and

$$\alpha=\ I_n\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)$$

where

$$\varphi_i(t_i)=\psi_i(t_i)\chi_{A_i}(t_i);\ \ 1\leq i\leq n. \tag{A.4.1}$$

Then

$$D_{t+}\alpha=\ n\,I_{n-1}\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n(\cdot,\cdots,t^+)\right)$$

where

$$\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)(t_1,\cdots,t_{n-1},t)=\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\bigotimes}_{j\in\{1,\cdots,n\}\setminus\{i\}}\varphi_j\right)\cdot\varphi_i(t).$$

Since $\chi_{A_t}(t^+)=1$, this implies that

$$D_{t+}\alpha=\sum_{i=1}^{n}I_{n-1}\left(\widehat{\bigotimes}_{j\in\{1,\cdots,n\}\setminus\{i\}}\varphi_j\right)\cdot\psi_i(t).$$

We have

$$
\begin{aligned}
0 &= E\left[L(t)\,\alpha + M(t)\,D_{t+}\alpha\right] \\
&= E\left[L(t)\,I_n\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)\right] + \sum_{i=1}^{n} E\left[M(t)\,I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n\}\setminus\{i\}}}\varphi_j\right)\right]\cdot\psi_i(t) \\
&= E\left[L(t)\,I_n\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)\right] + \sum_{i=1}^{n-1} E\left[M(t)\,I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n\}\setminus\{i\}}}\varphi_j\right)\right]\cdot\psi_i(t) \\
&+ E\left[M(t)\,I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n-1\}}}\varphi_j\right)\right]\cdot\psi_n(t)
\end{aligned}
\tag{A.4.2}
$$

Let $\varepsilon > 0$ and choose ψ_n such that $|\psi_n| \le 1$ and

$$
\psi_n(s) = \begin{cases} 1 & \text{if } |s-t| < \varepsilon \\ 0 & \text{if } |s-t| \ge 2\varepsilon. \end{cases}
$$

Then, applying both the Cauchy-Schwartz inequality and the Itô isometry to the first term, we have

$$
\begin{aligned}
\left|E\left[L(t)\,\alpha\right]\right| &= \left|E\left[L(t)\,I_n\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)\right]\right| \\
&\le E\left[L_t^2\right]^{\frac{1}{2}}\cdot E\left[I_n^2\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)\right]^{\frac{1}{2}} \\
&= E\left[L_t^2\right]^{\frac{1}{2}}\cdot E\left[\int_0^{t+\delta}\cdots\int_0^{t+\delta}\left(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_{n-1}\widehat{\otimes}\varphi_n\right)^2 ds_1\cdots ds_n\right]^{\frac{1}{2}}
\end{aligned}
$$

The choice of ψ_n leads to the convergence of the second factor of the last equality to zero. It follows that $E\left[L(t)\,\alpha\right]$ goes to zero as $\varepsilon \to 0$.

In the same way, we prove that the second term in equality (A.4.2) goes to zero as $\varepsilon \to 0$.

Then, we can conclude that

$$
E\left[M(t)\,I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n-1\}}}\varphi_j\right)\right] = 0,
$$

and then

$$
E\left[M(t)|\,\mathcal{G}_t\right] = 0.
$$

The preceding equality implies that $E\left[M(t)\, D_{t+}\alpha\right]=0$. In fact, we have

$$\begin{aligned} E\left[M(t) D_{t+}\alpha\right] &= E\left[E\left[M(t) D_{t+}\alpha\right] \mid \mathcal{G}_t\right] \\ &= E\left[E\left[M(t) \mid \mathcal{G}_t\right] \cdot D_{t+}\alpha\right] \\ &= 0 \end{aligned}$$

It then follows that $E\left[L(t) \mid \mathcal{G}_t\right] = 0$.

Jump case: Choose

$$\begin{aligned} \alpha &= I_n\left(g_1 \widehat{\otimes} \cdots \widehat{\otimes} g_{n-1} \widehat{\otimes} g_n\right) \\ D_{t+}\alpha &= n\, I_{n-1}\left(g_1 \widehat{\otimes} \cdots \widehat{\otimes} g_{n-1} \widehat{\otimes} g_n\left((\cdot,\cdot),\cdots,(\cdot,\cdot),(t,z)\right)\right) \end{aligned}$$

$$\left(g_1 \widehat{\otimes} \cdots \widehat{\otimes} g_{n-1} \widehat{\otimes} g_n\right)\left((t_1,z_1),\cdots,(t_{n-1},z_{n-1}),(t,z)\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n\}\setminus\{i\}}} g_j\right)\cdot g_i(t,z).$$

This implies that

$$D_{t+}\alpha = \sum_{i=1}^{n} I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n\}\setminus\{i\}}} g_j\right)\cdot g_i(t,z).$$

We have

$$\begin{aligned} 0 &= E\left[L(t)\alpha + \int_{\mathbb{R}_0} R(t,z)\, D_{t+,z}\alpha\, \nu(dz)\right] \\ &= E\left[L(t)\, I_n\left(g_1 \widehat{\otimes} \cdots \widehat{\otimes} g_{n-1} \widehat{\otimes} g_n\right)\right] + \sum_{i=1}^{n} E\left[\int_{\mathbb{R}_0} R(t,z)\, I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n\}\setminus\{i\}}} g_j\right)\cdot g_i(t,z)\,\nu(dz)\right] \\ &= E\left[L(t)\, I_n\left(g_1 \widehat{\otimes} \cdots \widehat{\otimes} g_{n-1} \widehat{\otimes} g_n\right)\right] + \sum_{i=1}^{n-1} \int_{\mathbb{R}_0} E\left[R(t,z)\, I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n\}\setminus\{i\}}} g_j\right)\right] g_i(t,z)\,\nu(dz) \\ &+ \int_{\mathbb{R}_0} E\left[R(t,z)\, I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n-1\}}} g_j\right)\right] g_n(t,z)\,\nu(dz). \end{aligned} \tag{A.4.3}$$

Choose $g_i(t,z) = \varphi_i(t) f_i(z),\ \ i = 1,\cdots,n$ and define φ_i as in Equation (A.4.1). Choosing $\psi_i,\ i = 1,\cdots,n$ as before, we have by applying again both the Cauchy-Schwartz inequality and the Itô isometry that the first two terms go to 0 as ε converges to 0.

The last term gives

$$E\left[\int_{\mathbb{R}_0} R(t,z)\, I_{n-1}\left(\widehat{\bigotimes_{j\in\{1,\cdots,n-1\}}} g_j\right) f_n(z)\,\nu(dz)\right] = 0. \tag{A.4.4}$$

Since $I_{n-1}\left(\widehat{\bigotimes}_{j\in\{1,\cdots,n-1\}} g_j\right)$ does not depend on z, Equation (A.4.4) becomes

$$E\left[\int_{\mathbb{R}_0} R(t,z) f_n(z)\,\nu(dz)\, I_{n-1}\left(\widehat{\bigotimes}_{j\in\{1,\cdots,n-1\}} g_j\right)\right] = 0. \tag{A.4.5}$$

Equation (A.4.5) holds for all f_n, we can then choose $f_n = 1$ and it follows that

$$E\left[\int_{\mathbb{R}_0} R(t,z)\,\nu(dz)\middle|\,\mathcal{G}_t\right] = 0.$$

The same arguments as in the Brownian case lead to $E\left[\,L(t)|\,\mathcal{G}_t\right] = 0$.

In order to have the whole filtration generated by the Itô-Lévy process, we can define α as

$$\alpha = I_n^B(\omega_1)\cdot I_n^{\widetilde{N}}(\omega_2)$$

and perform the same computations. The result follows. ■

A.5 Proof of Theorem 7.2.2

The proof relies on a combination of arguments in [4] and in Chapter 6

(i) Suppose $(\widehat{\pi}, \widehat{\theta}) \in \mathcal{A}_\Pi \times \mathcal{A}_\Theta$ is a Nash equilibrium. Since 1 and 2 hold for all π and θ, $(\widehat{\pi}, \widehat{\theta})$ is a directional critical point for $J_i(\pi, \theta)$ for $i = 1, 2$ in the sense that for all bounded $\beta \in \mathcal{A}_\Pi$ and $\eta \in \mathcal{A}_\Theta$, there exists $\delta > 0$ such that $\widehat{\pi} + y\beta \in \mathcal{A}_\Pi$, $\widehat{\theta} + v\eta \in \mathcal{A}_\Theta$ for all $y, v \in (-\delta, \delta)$. Then we have

$$
\begin{aligned}
0 = & \frac{\partial}{\partial y} J_1(\widehat{\pi} + y\beta, \widehat{\theta})\Big|_{y=0} \qquad \text{(A.5.1)}\\
= & E^x \Bigg[\int_0^T \int_{\mathbb{R}_0} \bigg\{ \frac{\partial}{\partial x} f_1(t, \widehat{X}(t), \widehat{\pi}_0(t), \widehat{\pi}_1(t,z), \widehat{\theta}_0(t), \widehat{\theta}_1(t,z), z) \, \frac{d}{dy} X^{(\widehat{\pi}+y\beta, \theta)}(t)\Big|_{y=0} \\
& + \nabla_\pi f_1(t, X^{(\pi,\widehat{\theta})}(t), \pi_0(t), \pi_1(t,z), \widehat{\theta}_0(t), \widehat{\theta}_1(t,z), z)\Big|_{\widehat{\pi}=\pi} \beta^*(t) \bigg\} \mu(dz)dt \\
& + g'(X(T)) \frac{d}{dy} X^{(\widehat{\pi}+y\beta, \theta)}(t)\Big|_{y=0} \Bigg] \\
= & E^x \Bigg[\int_0^T \int_{\mathbb{R}_0} \bigg\{ \frac{\partial}{\partial x} f_1(t, \widehat{X}(t), \widehat{\pi}_0(t), \widehat{\pi}_1(t,z), \widehat{\theta}_0(t), \widehat{\theta}_1(t,z), z) \widehat{Y}(t) \\
& + \nabla_\pi f_1(t, X^{(\pi,\widehat{\theta})}(t), \pi_0(t), \pi_1(t,z), \widehat{\theta}_0(t), \widehat{\theta}_1(t,z), z)\Big|_{\widehat{\pi}=\pi} \beta^*(t) \bigg\} \mu(dz)dt \\
& + g'(\widehat{X}(T)) \widehat{Y}(t) \Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{Y}(t) = \widehat{Y}_\beta(t) = & \frac{d}{dy} X^{(\widehat{\pi}+y\beta, \widehat{\theta})}(t)\Big|_{y=0} \qquad \text{(A.5.2)}\\
= & \int_0^t \bigg\{ \frac{\partial}{\partial x} b(s, \widehat{X}(s), \widehat{\pi}_0(s), \widehat{\theta}_0(s)) Y(s) \\
& + \nabla_\pi b(s, X^{\pi,\widehat{\theta}}(s), \pi_0(s), \widehat{\theta}_0(s))\Big|_{\pi=\widehat{\pi}} \beta^*(s) \bigg\} ds \\
& + \int_0^t \bigg\{ \frac{\partial}{\partial x} \sigma(s, \widehat{X}(s), \widehat{\pi}_0(s), \widehat{\theta}_0(s)) Y(s) \\
& + \nabla_\pi \sigma(s, X^{\pi,\widehat{\theta}}(s), \pi_0(s), \widehat{\theta}_0(s))\Big|_{\pi=\widehat{\pi}} \beta^*(s) \bigg\} dB^-(s) \\
& + \int_0^t \int_{\mathbb{R}_0} \bigg\{ \frac{\partial}{\partial x} \gamma(s, \widehat{X}(s^-), \widehat{\pi}_0(s^-), \widehat{\theta}_0(s^-), z) Y(s) \\
& + \nabla_\pi \gamma(s, X^{\pi,\widehat{\theta}}(s^-), \pi_0(s^-), \widehat{\theta}_0(s^-), z)\Big|_{\pi=\widehat{\pi}} \beta^*(s) \bigg\} \widetilde{N}(dz, d^-s)
\end{aligned}
$$

We study the three summands separately. Using the short notation $\frac{\partial}{\partial x} f_1(t, \widehat{X}(t), \widehat{\pi}, \widehat{\theta}, z) = \frac{\partial}{\partial x} f_1(t, z)$, $\nabla_\pi f_1(t, X^{(\pi,\widehat{\theta})}(t), \pi, \widehat{\theta}, z)\Big|_{\widehat{\pi}=\pi}$ and similarly for $\frac{\partial}{\partial x} b$, $\nabla_\pi b$, $\frac{\partial}{\partial x}\sigma$, $\nabla_\pi \sigma$, $\frac{\partial}{\partial x}\gamma$ and $\nabla_\pi \gamma$.

By the duality formulas (6.2.7) and (6.2.12) and the Fubini theorem, we get

$$
\begin{aligned}
& E\left[g_1'(X(T))Y(T)\right] \\
&= E\Bigg[g_1'(X(T))\Bigg(\int_0^T \left\{\frac{\partial b}{\partial x}(t)Y(t) \,+\, \nabla_\pi b(t)\,\beta^*(t)\right\} dt \\
&\quad + \int_0^T \left\{\frac{\partial \sigma}{\partial x}(t)Y(t) \,+\, \nabla_\pi \sigma(s)\,\beta^*(t)\right\} d^- B(t) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \left\{\frac{\partial \gamma}{\partial x}(t, z_1)Y(t) \,+\, \nabla_\pi \gamma(s, z_1)\,\beta^*(t)\right\} \widetilde{N}(dz_1, d^- t)\Bigg)\Bigg] \\
&= E\left[\int_0^T g_1'(X(T))\left\{\frac{\partial b}{\partial x}(t)Y(t) \,+\, \nabla_\pi b(t)\,\beta^*(t)\right\} dt\right] \\
&\quad + E\left[\int_0^T D_t g_1'(X(T))\left\{\frac{\partial \sigma}{\partial x}(t, z_1)Y(t) \,+\, \nabla_\pi \sigma(t, z_1)\,\beta^*(t)\right\} dt\right] \\
&\quad + E\left[\int_0^T g_1'(X(T)) D_{t+}\left(\frac{\partial \sigma}{\partial x}(t)Y(t) \,+\, \nabla_\pi \sigma(t)\,\beta^*(t)\right) dt\right] \\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} D_{t,z_1} g_1'(X(T))\left\{\frac{\partial \gamma}{\partial x}(t, z_1)Y(t) \,+\, \nabla_\pi \gamma(t, z_1)\,\beta^*(t)\right\} \nu(dz_1)dt\right] \\
&\quad + E\Big[\int_0^T \int_{\mathbb{R}_0} \left\{g_1'(X(T)) + D_{t,z_1} g_1'(X(T))\right\} D_{t+,z_1}\Big(\frac{\partial \gamma}{\partial x}(t, z_1)Y(t) \\
&\quad + \nabla_\pi \gamma(t, z_1)\,\beta^*(t)\Big)\nu(dz_1)dt\Big]
\end{aligned}
$$

Changing notation $z_1 \to z$ and using the multidimensional product rule for Malliavin derivatives, this becomes

$$
\begin{aligned}
&= E\left[\int_0^T \left\{ g_1'(X(T))\left(\frac{\partial b}{\partial x}(t) + D_{t+}\frac{\partial \sigma}{\partial x}(t) + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z)\nu(dz)\right)\right.\right.\\
&\quad \left.\left. + D_t g_1'(X(T))\frac{\partial \sigma}{\partial x}(t) + \int_{\mathbb{R}_0} D_{t,z} g_1'(X(T))\left(\frac{\partial \gamma}{\partial x}(t,z) + D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z)\right)\nu(dz)\right\} Y(t)dt\right]\\
&\quad + E\left[\int_0^T \left\{ g_1'(X(T))\left(\nabla_\pi b(t) + D_{t+}\nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi \gamma(t,z)\nu(dz)\right)\right.\right.\\
&\quad \left.\left. + D_t g_1'(X(T))\nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t,z} g_1'(X(T))\left(\nabla_\pi \gamma(t,z) + D_{t+,z}\nabla_\pi \gamma(t,z)\right)\nu(dz)\right\} \beta^*(t)dt\right]\\
&\quad + E\left[\int_0^T g_1'(X(T))\frac{\partial \sigma}{\partial x}(t) D_{t+}Y(t)dt\right] + E\left[\int_0^T g_1'(X(T))\nabla_\pi \sigma(t)\, D_{t+}\beta^*(t)dt\right]\\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g_1'(X(T)) + D_{t,z}g_1'(X(T))\right\}\left\{\frac{\partial \gamma}{\partial x}(t,z) + D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z)\right\} D_{t+,z}Y(t)\nu(dz)dt\right]\\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g_1'(X(T)) + D_{t,z}g_1'(X(T))\right\}\left\{\nabla_\pi \gamma(t,z) + D_{t+,z}\nabla_\pi \gamma(t,z)\right\} D_{t+,z}\beta^*(t)\nu(dz)dt\right].
\end{aligned}
\tag{A.5.3}
$$

Similarly, we have using both Fubini and duality formulas (6.2.7) and (6.2.12), we get

$$
\begin{aligned}
& E^x\left[\int_0^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)Y(t)\mu(dz)dt\right]\\
&= E^x\left[\int_0^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)\left(\int_0^t \left\{\frac{\partial b}{\partial x}(s)Y(s) + \nabla_\pi b(s)\beta^*(s)\right\}ds\right.\right.\\
&\quad + \int_0^t \left\{\frac{\partial \sigma}{\partial x}(s)Y(s) + \nabla_\pi \sigma(s)\beta^*(s)\right\}d^-B(s)\\
&\quad \left.\left. + \int_0^t \int_{\mathbb{R}_0} \left\{\frac{\partial \gamma}{\partial x}(s,z_1)Y(s) + \nabla_\pi \gamma(s,z_1)\right\}\widetilde{N}(dz_1,d^-s)\right)dt\right]\\
&= E^x\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_0^t \frac{\partial f_1}{\partial x}(t,z)\left\{\frac{\partial b}{\partial x}(s)Y(s) + \nabla_\pi b(s)\beta^*(s)\right\}ds\right)\mu(dz)dt\right]\\
&\quad + E^x\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_0^t D_s\frac{\partial f_1}{\partial x}(t,z)\left\{\frac{\partial \sigma}{\partial x}(s)Y(s) + \nabla_\pi \sigma(s)\beta^*(s)\right\}ds\right)\mu(dz)dt\right]\\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_0^t \frac{\partial f_1}{\partial x}(t,z)D_{s+}\left\{\frac{\partial \sigma(s)}{\partial x}Y(s) + \nabla_\pi \sigma(s)\,\beta^*(s)\right\}ds\right)\mu(dz)dt\right]\\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_0^t \int_{\mathbb{R}_0} D_{s,z_1}\frac{\partial f_1}{\partial x}(t,z)\left\{\frac{\partial \gamma}{\partial x}(s,z_1)Y(s) + \nabla_\pi \gamma(s,z_1)\beta^*(s)\right\}\nu(dz_1)ds\right)\mu(dz)dt\right]\\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_0^t \int_{\mathbb{R}_0} \left\{\frac{\partial f_1}{\partial x}(t,z) + D_{s,z_1}\frac{\partial f_1}{\partial x}(t,z)\right\}\times\right.\right.\\
&\quad \left.\left. D_{s+,z_1}\left(\frac{\partial \gamma}{\partial x}(s,z_1)Y(s) + \nabla_\pi \gamma(s,z_1)\beta(s)\right)\nu(dz_1)ds\right)\mu(dz)dt\right].
\end{aligned}
$$

Changing notation $t_1 \to t$ and $z_1 \to z$ this becomes

$$
\begin{aligned}
&= E^x \Bigg[\int_0^T \Bigg\{ \Bigg(\int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \Bigg(\frac{\partial b}{\partial x}(t) + D_{t+}\frac{\partial \sigma}{\partial x}(t) + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z)\nu(dz) \Bigg) \\
&\quad + \Bigg(\int_t^T \int_{\mathbb{R}_0} D_t \frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \frac{\partial \sigma}{\partial x}(t) \\
&\quad + \int_{\mathbb{R}_0} \Bigg(\int_t^T \int_{\mathbb{R}_0} D_{t,z}\frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \Bigg(\frac{\partial \gamma}{\partial x}(t,z) + D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z) \Bigg) \nu(dz) \Bigg\} Y(t)dt \Bigg] \\
&\quad + E^x \Bigg[\int_0^T \Bigg\{ \Bigg(\int_t^T \int_{\mathbb{R}_0} \frac{\partial f}{\partial x}(s,z)\mu(dz)ds \Bigg) \Bigg(\nabla_\pi b(t) + D_{t+}\nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi \gamma(t,z)\nu(dz) \Bigg) \\
&\quad + \Bigg(\int_t^T \int_{\mathbb{R}_0} D_t \frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \nabla_\pi \sigma(t) \\
&\quad + \Bigg(\int_t^T \int_{\mathbb{R}_0} D_{t,z}\frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \Big(\nabla_\pi \gamma(t,z) + D_{t+,z}\nabla_\pi \gamma(t,z) \Big) \nu(dz) \Bigg\} \beta^*(t)dt \Bigg] \\
&\quad + E^x \Bigg[\int_0^T \Bigg(\int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \frac{\partial \sigma(t)}{\partial x} D_{t+}Y(t)dt \Bigg] \\
&\quad + E^x \Bigg[\int_0^T \Bigg(\int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z)\mu(dz)ds \Bigg) \nabla_\pi \sigma(t) D_{t+}\beta^*(t)dt \Bigg] \\
&\quad + E^x \Bigg[\int_0^T \int_{\mathbb{R}_0} \Bigg\{ \int_t^T \int_{\mathbb{R}_0} \Bigg(\frac{\partial f}{\partial x}(s,z) + D_{t,z}\frac{\partial f}{\partial x}(s,z) \Bigg) \mu(dz)ds \Bigg\} \times \\
&\quad \Bigg\{ \frac{\partial \gamma}{\partial x}(t) + D_{t+,z}\frac{\partial \gamma}{\partial x}(t) \Bigg\} D_{t+,z}Y(t)\nu(dz)dt \Bigg] \\
&\quad + E^x \Bigg[\int_0^T \int_{\mathbb{R}_0} \Bigg\{ \int_t^T \int_{\mathbb{R}_0} \Bigg(\frac{\partial f_1}{\partial x}(s,z) + D_{t,z}\frac{\partial f_1}{\partial x}(s,z) \Bigg) \mu(dz)ds \Bigg\} \times \\
&\quad \Big\{ \nabla_\pi \gamma(t,z) + D_{t+,z}\nabla_\pi \gamma(t,z) \Big\} D_{t+,z}\beta^*(t)\nu(dz)dt \Bigg].
\end{aligned} \tag{A.5.4}
$$

Recall that

$$
K(t) \quad := \quad g_1'(X(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)ds,
$$

so

$$
\widehat{K}_1(t) \quad := \quad g_1'(\widehat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)ds. \tag{A.5.5}
$$

By combining (A.5.3)-(A.5.4), we get

$$
\begin{aligned}
0 = E\Bigg[\int_0^T \Bigg\{ & \widehat{K}_1 \left(\frac{\partial b}{\partial x}(t) + D_{t+}\frac{\partial \sigma}{\partial x}(t) + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z)\nu(dz)\right) \\
& + D_t\widehat{K}_1\frac{\partial \sigma}{\partial x}(t) + \int_{\mathbb{R}_0} D_{t,z}\widehat{K}_1\left(\frac{\partial \gamma}{\partial x}(t,z) + D_{t+,z}\frac{\partial \gamma}{\partial x}(t,z)\right)\nu(dz)\Bigg\} Y(t)dt\Bigg] \\
& + E\Bigg[\int_0^T \Bigg\{\widehat{K}_1\left(\nabla_\pi b(t) + D_{t+}\nabla_\pi\sigma(t) + \int_{\mathbb{R}_0} D_{t+,z}\nabla_\pi\gamma(t,z)\nu(dz)\right) \\
& + D_t\widehat{K}_1\nabla_\pi\sigma(t) + \int_{\mathbb{R}_0} D_{t,z}\widehat{K}_1\Big(\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big)\nu(dz)\Bigg\}\beta^*(t)dt\Bigg] \\
& + E\left[\int_0^T \widehat{K}_1\frac{\partial\sigma(t)}{\partial x}D_{t+}Y(t)dt\right] + E\left[\int_0^T \widehat{K}_1\nabla_\pi\sigma(t)D_{t+}\beta^*(t)dt\right] \\
& + E\left[\int_0^T\int_{\mathbb{R}_0}\left(\widehat{K}_1 + D_{t,z}\widehat{K}_1\right)\left\{\frac{\partial\gamma}{\partial x}(t) + D_{t+,z}\frac{\partial\gamma}{\partial x}(t)\right\}D_{t+,z}Y(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T\int_{\mathbb{R}_0}\left(\widehat{K}_1 + D_{t,z}\widehat{K}_1\right)\Big\{\nabla_\pi\gamma(t,z) + D_{t+,z}\nabla_\pi\gamma(t,z)\Big\}D_{t+,z}\beta^*(t)\nu(dz)dt\right] \\
& + E\left[\int_0^T\int_{\mathbb{R}_0}\nabla_\pi f_1(t,z)\beta^*(t)\mu(dz)dt\right].
\end{aligned} \tag{A.5.6}
$$

Now apply this to $\beta = \beta_\alpha \in \mathcal{A}_\Pi$ given as $\beta_\alpha(s) := \alpha\chi_{[t,t+h]}(s)$, for some $t, h \in (0,T)$, $t+h \le T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_t^2-$measurable. Then $Y^{(\beta_\alpha)}(s) = 0$ for $0 \le s \le t$ and hence Equation (A.5.6) becomes

$$
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0, \tag{A.5.7}
$$

where

$$
\begin{aligned}
A_1 = E^x\Bigg[\int_t^T\Bigg\{ & \widehat{K}_1(t)\left(\frac{\partial b}{\partial x}(s) + D_{s+}\frac{\partial\sigma}{\partial x}(s) + \int_{\mathbb{R}_0}D_{s+,z}\frac{\partial\gamma}{\partial x}(s,z)\nu(dz)\right) + D_t\widehat{K}_1(t)\frac{\partial\sigma}{\partial x}(t) \\
& + \int_{\mathbb{R}_0}D_{s,z}\widehat{K}_1(t)\left(\frac{\partial\gamma}{\partial x}(s,z) + D_{s+,z}\frac{\partial\gamma}{\partial x}(s,z)\right)\nu(dz)\Bigg\}Y^{(\beta_\alpha)}(s)\,ds\Bigg],
\end{aligned} \tag{A.5.8}
$$

$$
\begin{aligned}
A_2 = E^x\Bigg[\int_t^{t+h}\Bigg\{ & \widehat{K}_1(t)\left(\nabla_\pi b(s) + D_{s+}\nabla_\pi\sigma(s) + \int_{\mathbb{R}_0}D_{ts,z}\nabla_\pi\gamma(t,z)\nu(dz)\right) + D_t\widehat{K}_1(t)\nabla_\pi\sigma(t) \\
& + \int_{\mathbb{R}_0}D_{s,z}\widehat{K}_1(t)\Big(\nabla_\pi\gamma(s,z) + D_{s,z}\nabla_\pi\gamma(s,z)\Big)\nu(dz) + \int_{\mathbb{R}_0}\nabla_\pi f_1(s,z)\mu(dz)\Bigg\}\alpha\,ds\Bigg],
\end{aligned} \tag{A.5.9}
$$

$$A_3 = E^x \left[\int_t^T \widehat{K}_1(t) \frac{\partial \sigma(s)}{\partial x} D_{s+} Y^{(\beta_\alpha)}(s)\, ds \right], \tag{A.5.10}$$

$$A_4 = + \ E \left[\int_t^{t+h} \widehat{K}_1(t) \nabla_\pi \sigma(s) D_{s+} \alpha \, ds \right], \tag{A.5.11}$$

$$A_5 = E^x \left[\int_t^T \int_{\mathbb{R}_0} \left(\widehat{K}_1(t) + D_{s,z} \widehat{K}_1(t) \right) \left\{ \frac{\partial \gamma}{\partial x}(s) \ + \ D_{s+,z} \frac{\partial \gamma}{\partial x}(t) \right\} \nu(dz) D_{s+,z} Y^{(\beta_\alpha)}(s)\, ds \right], \tag{A.5.12}$$

$$A_6 = E^x \left[\int_t^{t+h} \int_{\mathbb{R}_0} \left(\widehat{K}_1(t) + D_{s,z} \widehat{K}_1(t) \right) \left\{ \nabla_\pi \gamma(s,z) \ + \ D_{s+,z} \nabla_\pi \gamma(s,z) \right\} \nu(dz) D_{s+,z} \alpha ds \right]. \tag{A.5.13}$$

Note by the definition of Y, with $Y(s) = Y^{(\beta_\alpha)}(s)$ and $s \geq t+h$, the process $Y(s)$ follows the dynamics

$$dY(s) = Y(s^-) \left[\frac{\partial b}{\partial x}(s) ds \ + \ \frac{\partial \sigma}{\partial x}(s) d^- B(s) \ + \ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(s^-, z) \widetilde{N}(dz, d^- s) \right], \tag{A.5.14}$$

for $s, \geq t+h$ with initial condition $Y(t+h)$ in time $t+h$. By the Itô formula for forward integrals, this equation can be solved explicitly and we get

$$Y(s) = Y(t+h) G(t+h, s), \ \ s \geq t+h, \tag{A.5.15}$$

where, in general, for $s \geq t$,

$$\begin{aligned} G(t,s) := \exp \Bigg(& \int_t^s \left\{ \frac{\partial b}{\partial x}(r) - \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2 (r) \right\} dr + \int_t^s \frac{\partial \sigma}{\partial x}(r) dB^-(r) \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln \left(1 + \frac{\partial \gamma}{\partial x}(r,z) \right) - \frac{\partial \gamma}{\partial x}(r,z) \right\} \nu(dz) dt \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln \left(1 + \frac{\partial \gamma}{\partial x}(r^-, z) \right) \right\} \widetilde{N}(dz, d^- r) \Bigg). \end{aligned}$$

Note that $G(t,s)$ does not depend on h, but $Y(s)$ does. Defining H_0^1 as in Equation (7.2.8), it follows that

$$A_1 = E^x \left[\int_t^T \frac{\partial \widehat{H}_0^1}{\partial x}(s) Y(s) ds \right].$$

Where $\widehat{H}_0^1(s) = H_0^1(s, \widehat{X}(s), \widehat{\pi}, \widehat{\theta})$.

Differentiating with respect to h at $h=0$, we get

$$\frac{d}{dh} A_1 \Big|_{h=0} = \frac{d}{dh} E^x \left[\int_t^{t+h} \frac{\partial \widehat{H}_0^1}{\partial x}(s) Y(s) ds \right]_{h=0} + \frac{d}{dh} E^x \left[\int_{t+h}^T \frac{\partial \widehat{H}_0^1}{\partial x}(s) Y(s) ds \right]_{h=0}.$$

Since $Y(t) = 0$, we see that

$$\frac{d}{dh} E^x \left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s) Y(s) ds \right]_{h=0} = 0.$$

Therefore, by (A.5.15), we get

$$\begin{aligned}
\frac{d}{dh} A_1 \Big|_{h=0} =& \frac{d}{dh} E^x \left[\int_{t+h}^T \frac{\partial \widehat{H}_0^1}{\partial x}(s) Y(t+h) G(t+h,s) ds \right]_{h=0} \\
=& \int_t^T \frac{d}{dh} E^x \left[\frac{\partial \widehat{H}_0^1}{\partial x}(s) Y(t+h) G(t+h,s) \right]_{h=0} ds \\
=& \int_t^T \frac{d}{dh} E^x \left[\frac{\partial \widehat{H}_0^1}{\partial x}(s) G(t,s) Y(t+h) \right]_{h=0} ds,
\end{aligned}$$

where, $Y(t+h)$ is given by

$$\begin{aligned}
Y(t+h) = & \int_t^{t+h} Y(r^-) \left[\frac{\partial b}{\partial x}(r) dr \, + \, \frac{\partial \sigma}{\partial x}(r) d^- B(r) \, + \, \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(r^-, z) \widetilde{N}(dz, d^- r) \right] \\
& + \alpha \int_t^{t+h} \left[\nabla_\pi b(r) dr \, + \, \nabla_\pi \sigma(r) d^- B(r) \, + \, \int_{\mathbb{R}_0} \nabla_\pi \gamma(r^-, z) \widetilde{N}(dz, d^- r) \right].
\end{aligned}$$

Therefore, by the two preceding equalities,

$$\frac{d}{dh} A_1 \Big|_{h=0} = A_{1,1} + A_{1,2},$$

where

$$\begin{aligned}
A_{1,1} = & \int_t^T \frac{d}{dh} E^x \left[\frac{\partial H_0}{\partial x}(s) G(t,s) \alpha \int_t^{t+h} \left\{ \nabla_\pi b(r) dr \, + \, \nabla_\pi \sigma(r) d^- B(r) \right.\right. \\
& \left.\left. + \int_{\mathbb{R}_0} \nabla_\pi \gamma(r^-, z) \widetilde{N}(dz, d^- r) \right\} \right]_{h=0} ds,
\end{aligned}$$

and

$$\begin{aligned}
A_{1,2} = & \int_t^T \frac{d}{dh} E^x \left[\frac{\partial H_0}{\partial x}(s) G(t,s) \int_t^{t+h} Y(r^-) \left\{ \frac{\partial b}{\partial x}(r) dr \, + \, \frac{\partial \sigma}{\partial x}(r) d^- B(r) \right.\right. \\
& \left.\left. + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(r, z) \widetilde{N}(dz, d^- r) \right\} \right]_{h=0} ds.
\end{aligned}$$

Applying again the duality formula, we have

$$
\begin{aligned}
A_{1,1} = & \int_t^T \frac{d}{dh} E^x \Big[\alpha \int_t^{t+h} \{ \nabla_\pi b(r) F_1(t,s) + \nabla_\pi \sigma(r) D_r F_1(t,s) \\
& + F_1(t,s) D_{r^+} \nabla_\pi \sigma(r) + \int_{\mathbb{R}_0} \big\{ \big(\nabla_\pi \gamma(r,z) + D_{r^+,z} \nabla_\pi \gamma(r,z) \big) D_{r,z} F_1(t,s) \\
& + D_{r^+,z} \nabla_\pi \gamma(r,z) F_1(t,s) \big\} \nu(dz) \big\} dr \Big]_{h=0} ds \\
= & \int_t^T E^x \Big[\alpha \Big\{ \Big(\nabla_\pi b(t) + D_{t^+} \nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t^+,z} \nabla_\pi \gamma(t,z) \nu(dz) \Big) F_1(t,s) \\
& \nabla_\pi \sigma(t) D_t F_1(t,s) + \int_{\mathbb{R}_0} \big(\nabla_\pi \gamma(t,z) + D_{t^+,z} \nabla_\pi \gamma(t,z) \big) D_{t,z} F_1(t,s) \nu(dz) \Big\} \Big] ds,
\end{aligned}
$$

where we have put

$$F_1(t,s) = \frac{\partial \widehat{H}_0^1}{\partial x}(s) G(t,s).$$

Since $Y(t) = 0$ we see that

$$A_{1,2} = 0.$$

We conclude that

$$
\begin{aligned}
\frac{d}{dh} A_1 \Big|_{h=0} = & A_{1,1} && \text{(A.5.16)} \\
= & \int_t^T E \Big[\alpha \Big\{ \Big(\nabla_\pi b(t) + D_{t^+} \nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t^+,z} \nabla_\pi \gamma(t,z) \nu(dz) \Big) F_1(t,s) \\
& + \nabla_\pi \sigma(t) D_t F_1(t,s) + \int_{\mathbb{R}_0} \big(\nabla_\pi \gamma(t,z) + D_{t^+,z} \nabla_\pi \gamma(t,z) \big) D_{t,z} F_1(t,s) \nu(dz) \Big\} \Big] ds,
\end{aligned}
$$

Moreover, we see that

$$
\begin{aligned}
\frac{d}{dh} A_2 \Big|_{h=0} = & E \Big[\Big\{ \widehat{K}_1(t) \Big(\nabla_\pi b(t) + D_{t+} \nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t+,z} \nabla_\pi \gamma(t,z) \nu(dz) \Big) \\
& + \nabla_\pi f_1(t) + D_t \widehat{K}_1(t) \nabla_\pi \sigma(t) \\
& + \int_{\mathbb{R}_0} D_{t,z} \widehat{K}_1(t) \Big(\nabla_\pi \gamma(t,z) + D_{t+,z} \nabla_\pi \gamma(t,z) \Big) \nu(dz) \Big\} \alpha \Big], && \text{(A.5.17)} \\
\frac{d}{dh} A_4 \Big|_{h=0} = & E \Big[\widehat{K}_1(t) \nabla_\pi \sigma(t) D_{t+} \alpha \Big], && \text{(A.5.18)} \\
\frac{d}{dh} A_6 \Big|_{h=0} = & E \Big[\int_{\mathbb{R}_0} \Big\{ \widehat{K}_1(t) + D_{t,z} \widehat{K}_1(t) \Big\} \Big(\nabla_\pi \gamma(t,z) + D_{t+,z} \nabla_\pi \gamma(t,z) \Big) \nu(dz) D_{t+,z} \alpha \Big]. \\
& && \text{(A.5.19)}
\end{aligned}
$$

On the other hand, differentiating A_3 with respect to h at $h = 0$, we get

$$\begin{aligned}\frac{d}{dh}A_3\Big|_{h=0} =& \frac{d}{dh}E\left[\int_t^{t+h}\widehat{K}_1(s)\frac{\partial\sigma(s)}{\partial x}D_{s+}Y(s)ds\right]_{h=0}\\ &+\frac{d}{dh}E\left[\int_{t+h}^{T}\widehat{K}_1(s)\frac{\partial\sigma(s)}{\partial x}D_{s+}Y(s)ds\right]_{h=0}.\end{aligned}$$

Since $Y(t) = 0$, we see that

$$\begin{aligned}\frac{d}{dh}A_3\Big|_{h=0} =& \frac{d}{dh}E\left[\int_{t+h}^{T}\widehat{K}_1(s)\frac{\partial\sigma(s)}{\partial x}D_{s+}\Big(Y(t+h)G(t+h,s)\Big)ds\right]_{h=0}\\ =& \int_t^T\frac{d}{dh}E\left[\widehat{K}_1(s)\frac{\partial\sigma(s)}{\partial x}D_{s+}\Big(Y(t+h)G(t+h,s)\Big)\right]_{h=0}ds\\ =& \int_t^T\frac{d}{dh}E\Bigg[\widehat{K}_1(s)\frac{\partial\sigma(s)}{\partial x}\Big(D_{s+}G(t+h,s)\cdot Y(t+h)\\ &+D_{s+}Y(t+h)\cdot G(t+h,s)\Big)\Bigg]_{h=0}ds\\ =& \int_t^T\frac{d}{dh}E\left[\widehat{K}_1(s)\frac{\partial\sigma(s)}{\partial x}\cdot D_{s+}Y(t+h)G(t,s)\right]_{h=0}ds.\end{aligned}$$

Using the definition of $\widehat{p}$ and $\widehat{H}_1$ given respectively by Equations (7.2.17) and (7.2.16) in the theorem, it follows by (A.5.7) that

$$E\left[\nabla_\pi\widehat{H}_1(t,\widehat{X}(t),\widehat{u}(t))\Big|\,\mathcal{G}_t^2\right] + E[A] = 0 \quad \text{a.e. in} \quad (t,\omega), \tag{A.5.20}$$

where

$$A = \frac{d}{dh}A_3\Big|_{h=0} + \frac{d}{dh}A_4\Big|_{h=0} + \frac{d}{dh}A_5\Big|_{h=0} + \frac{d}{dh}A_6\Big|_{h=0}. \tag{A.5.21}$$

Similarly, we have

$$\begin{aligned}0 =& \frac{\partial}{\partial v}J_2(\widehat{\pi},\widehat{\theta}+v\eta)\Big|_{v=0} \qquad\qquad (A.5.22)\\ =& E^x\Bigg[\int_0^T\int_{\mathbb{R}_0}\Bigg\{\frac{\partial}{\partial x}f_2(t,\widehat{X}(t),\widehat{\pi}_0(t),\widehat{\pi}_1(t,z),\widehat{\theta}_0(t),\widehat{\theta}_1(t,z),z)\widehat{V}(t)\\ &+\nabla_\pi f_1(t,X^{(\widehat{\pi},\theta)}(t),\widehat{\pi}_0(t),\widehat{\pi}_1(t,z),\theta_0(t),\theta_1(t,z),z)\Big|_{\widehat{\theta}=\theta}\eta^*(t)\Bigg\}\mu(dz)dt\\ &+g'(\widehat{X}(T))\widehat{V}(t)\Bigg],\end{aligned}$$

where

$$
\begin{aligned}
\widehat{V}(t) = \widehat{V}_\eta(t) &= \frac{d}{dv} X^{(\widehat{\pi},\widehat{\theta}+v\eta)}(t)\Big|_{v=0} \qquad \text{(A.5.23)}\\
&= \int_0^t \left\{ \frac{\partial}{\partial x} b(s,\widehat{X}(s),\widehat{\pi}_0(s),\widehat{\theta}_0(s))V(s) + \nabla_\pi b(s,X^{\widehat{\pi},\theta}(s),\widehat{\pi}_0(s),\theta_0(s))\Big|_{\theta=\widehat{\theta}} \eta^*(s) \right\} ds\\
&+ \int_0^t \left\{ \frac{\partial}{\partial x} \sigma(s,\widehat{X}(s),\widehat{\pi}_0(s),\widehat{\theta}_0(s))V(s) + \nabla_\pi \sigma(s,X^{\widehat{\pi},\theta}(s),\widehat{\pi}_0(s),\theta_0(s))\Big|_{\theta=\widehat{\theta}} \eta^*(s) \right\} dB^-(s)\\
&+ \int_0^t \int_{\mathbb{R}_0} \left\{ \frac{\partial}{\partial x} \gamma(s,\widehat{X}(s^-),\widehat{\pi}_0(s^-),\widehat{\theta}_0(s^-),z)V(s) \right.\\
&\left. + \nabla_\pi \gamma(s,X^{\widehat{\pi},\theta}(s^-),\widehat{\pi}_0(s^-),\theta_0(s^-),z)\Big|_{\theta=\widehat{\theta}} \eta^*(s) \right\} \widetilde{N}(dz,d^-s).
\end{aligned}
$$

Define

$$D(s) = D(t+h)G(t+h,s); \;\; s \geq t+h,$$

where $G(t,s)$ is defined as in Equation (7.2.20). Using similar arguments, we get

$$E\left[\nabla_\pi \widehat{H}_2(t,\widehat{X}(t),\widehat{u}(t)) \Big| \mathcal{G}_t^1 \right] + E[B] = 0 \;\; \text{a.e. in} \;\; (t,\omega),$$

where B is given in the same way as A. This completes the proof of (i).

(ii) Conversely, suppose that there exist $\widehat{\pi} \in \mathcal{A}_\Pi$ such that Equation (7.2.14) holds. Then by reversing the previous arguments, we obtain that Equation (A.5.7) holds for all $\beta_\alpha(s) := \alpha\chi_{[t,t+h]}(s \in \mathcal{A}_\Pi)$, where $A_1, ..., A_6$ are given by Equation (A.5.10),..., Equation (A.5.13) respectively, for some $t,h \in (0,T)$, $t+h \leq T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_t^2-$measurable. Hence, these equalities hold for all linear combinations of β_α. Since all bounded $\beta \in \mathcal{A}_\Pi$ can be approximated pointwise boundedly in (t,ω) by such linear combinations, it follows that Equation (A.5.7) holds for all bounded $\beta \in \mathcal{A}_\Pi$. Hence, by reversing the remaining part of the previous proof, we conclude that

$$\frac{\partial J_1}{\partial y}(\widehat{\pi}+y\beta,\widehat{\theta})\Big|_{y=0} = 0, \;\text{for all}\; \beta.$$

Similarly, suppose that there exist $\widehat{\theta} \in \mathcal{A}_\Theta$ such that Equation (7.2.15) holds. Then, the above argument leads us to conclude that

$$\frac{\partial J_2}{\partial v}(\widehat{\pi},\widehat{\theta}+v\eta)\Big|_{v=0} = 0, \;\text{for all}\; \eta.$$

This complete the proof.

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