Group Invariant Solutions for Turbulent Boundary Layer Flows described by Eddy Viscosity

Trishna Mistry

School of Computational and Applied Mathematics

University of the Witwatersrand

A dissertation submitted for the degree of

Master of Science

2009

Declaration

I declare that this dissertation is my own unaided work unless otherwise acknowledged. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination to any other instituition.

(Signature)

(Date)

Acknowledgements

I thank my supervisor, Professor DP Mason, for his guidance and continual support. The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the NRF.

Abstract

The study of turbulence is important in a wide variety of applications. Some examples are: the flow of fluid once a sluice gate is opened; the flow of air over the wing of an aircraft; the flow of air from a ventilation system. A very important concept to consider when dealing with turbulence is the boundary layer that develops around an object as air or fluid moves past the object. The boundary layer is a thin layer in which the velocity gradients increase rapidly from zero at the contact surface, to match the mainstream velocity further away from the surface. The velocity is zero at the contact surface due to the viscous nature of the fluid or air molecules moving past the surface. Within the boundary layer either laminar or turbulent flow, or both types of flow, occur. Laminar boundary layers have been analysed rigorously in the past. In this dissertation we analyse turbulent boundary layers and turbulent wall jets. The turbulence is represented by introducing the concept of eddy viscosity. The Reynolds averaged boundary layer equations are used in this dissertation. The group invariant solution of the two–dimensional turbulent boundary layer equations are derived by finding the Lie point symmetries of the equation. The mainstream velocity is allowed to be an arbitrary function. The Lie point symmetries of the equation modelling the wall jet are used to solve for a group invariant solution. The conserved quantity for the wall jet is found which enables us to solve the problem.

Contents

Chapter 1

Introduction

1.1 Introduction

The study of turbulent flow is very important in order to predict flow rates, drag, flow separation and heat transfer. Furthermore most flows which occur in practical applications are turbulent [30]. An effective way in which turbulent flow can be represented is by eddy viscosity. The problems which are considered in this dissertation use the concept of eddy viscosity.

A boundary layer is the layer around an object which is formed due to the viscous property of fluids. As the mainstream moves past an object or along a surface the fluid tends to stick to the contact surface. In our analysis we use the cartesian co-ordinate system with the x–axis parallel to the contact surface and the y–axis perpendicular to the contact surface.

Some assumption is usually made about the eddy viscosity. The simplest assumption is the hypothesis made by Prandtl(1942) that the eddy viscosity is constant across the boundary layer and is proportional to the product of the maximum mean velocity and the width of the boundary layer [11].

In this dissertation we will not impose Prandtl's Hypothesis. We investigate group invariant solutions of the two–dimensional boundary layer equations with eddy viscosity where the eddy viscosity is allowed to vary across the boundary layer as well as in the direction of flow. Hence we allow for a variation in y as well. Furthermore we allow the mainstream velocity to be an arbitrary function of x which will be less restrictive than the usual assumption of a power law in x. The group invariant solutions for the boundary layer equations are derived using the Lie point symmetries of the equations.

The equations for the two–dimensional turbulent free jet have already been analysed where eddy viscosity is present [13]. The wall jet is intermediate between a boundary layer and a free jet. Examples of wall jets are shown in Figures (1.1) and (1.2). For the next part of the dissertation we consider a two–dimensional turbulent wall jet with eddy viscosity. We once again consider the Lie point symmetries of the

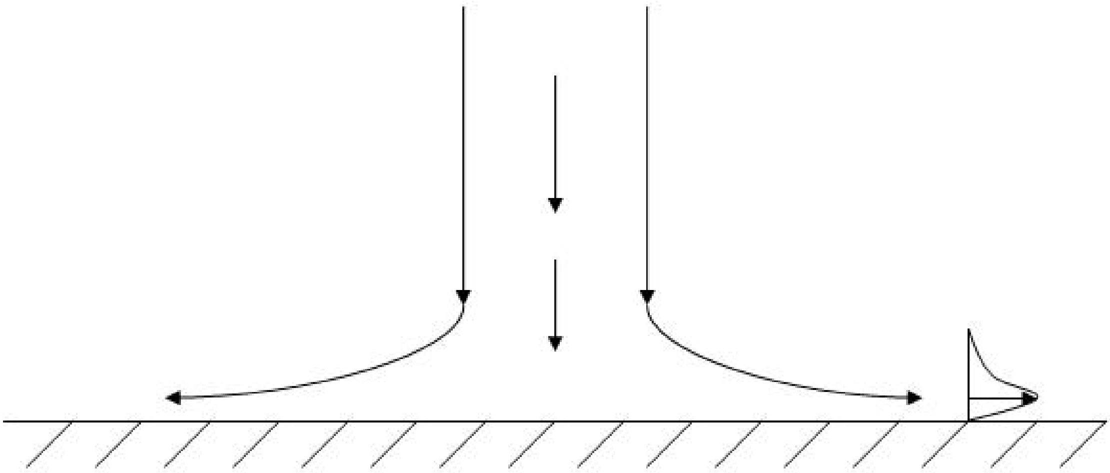

Figure 1.1: Two-dimensional wall jet

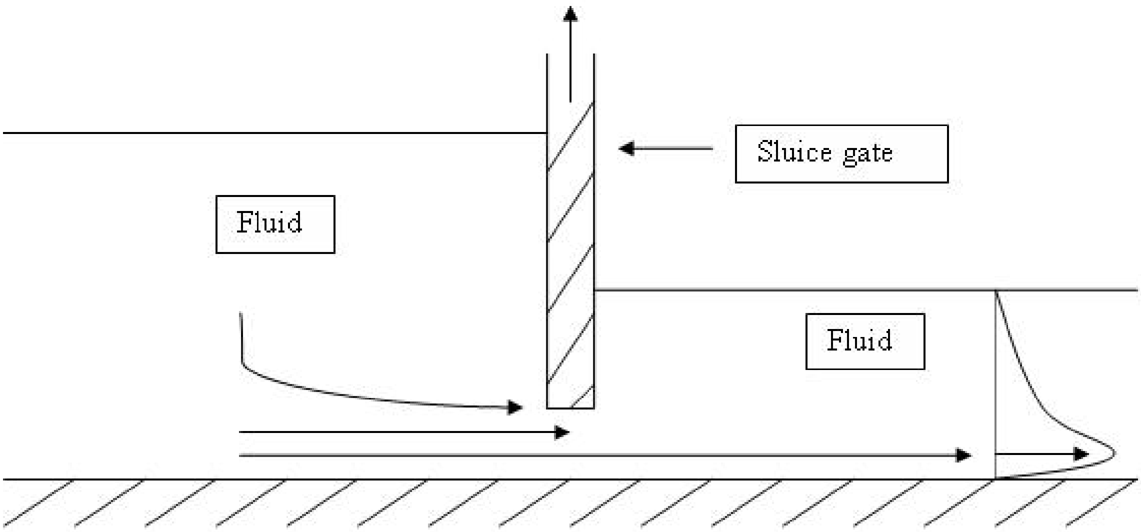

Figure 1.2: A two-dimensional wall jet which is created when a sluice gate is opened

resulting equations. In order to derive the solution a conserved quantity is required. We derive this conserved quantity by an *ad hoc* method.

1.2 Background to eddy viscosity and the Reynolds stresses

The concept of eddy viscosity was first introduced by Joseph Boussinesq in 1877 [32]. Eddy viscosity can be described as a quantity that characterizes the transport and dissipation of energy in the motion of a fluid. In turbulent flow it is common practice to ignore the eddies or vortices in the small scale motion and to rather calculate the eddy viscosity in the large scale motion of the fluid [33]. The eddy viscosity is a property of the flow and not a property of the fluid.

The Reynolds stress can be thought of as an effective force which represents the net transport of momentum by fluctuating velocities [33]. Thus in effect turbulence transports momentum. The turbulence term contributes more to the transportation of momentum than the viscous term when we consider simple shear flow. The Reynolds stress can be expressed as a stress tensor. The Reynolds stress can be thought of as the apparent or virtual stress of turbulent flow.

1.3 Turbulent boundary layers described by eddy viscosity

Turbulent motion can be described as motion in which irregular fluctuations occur. Irregular fluctuations can be thought of as mixing or eddying motion. These fluctuations are so complex and varied that they seem almost impossible to model exactly in a mathematical sense. However the effect that this mixing motion has on the course of the flow is very important and hence cannot be neglected. Furthermore we need some form of representation of this mixing motion in order to attain equilibrium of forces. The effects would be the same if we increased the viscosity by factors of one hundred, ten thousand, or even more [30].

When we deal with turbulent motion we deal with large Reynolds numbers. In [30] it is shown in detail that there is a large movement of energy from the mainstream flow to the large eddies but the energy is dissipated mainly through the small eddies. It is also shown that this process occurs in a narrow strip inside the boundary layer close to the wall.

The most striking feature of turbulent flow is that the velocity and pressure at a fixed point will always fluctuate in very irregular ways. The volume elements of fluid which perform these fluctuations are small but not single molecules and hence can be visualised. Their size will continuously change and determines the scale of turbulence.

We need to be able to formulate a substantial theory for turbulent boundary layers in order to predict flow rates, drag, flow separation and heat transfer. Practical prediction of turbulent flow starts with Reynolds' averaging [30]. In this averaging we divide each flow variable into a mean added to a fluctuation. The resulting turbulent boundary layer equations have an extra term which represents the Reynolds shear stress [30]. These shear and normal stresses are not known beforehand and hence need to be modelled.

There are various models which are used. There is the K–ε model which is based on the coupled transport equations for the turbulent energy density K and the turbulent dissipation rate ε [33]. Statistical concepts were also developed in order to deal with the unknown stress [30].

In this dissertation we use the idea that the presence of the fluctuations manifests itself in an increase in apparent viscosity of the mainstream flow. In analogy to the coefficient of viscosity in Stokes' law for laminar flow, Boussinesq introduced a mixing coefficient for the Reynolds stress [30]. This turbulent mixing coefficient corresponds to the viscosity, μ, in laminar flow and is thus aptly called eddy viscosity. The main consequence of using this idea is that eddy viscosity is a property of the flow and not a property of the fluid.

1.4 Structure of the boundary layer

The concept of the boundary layer was introduced by Ludwig Prandtl(1874-1953) in 1904. This was a significant accomplishment as prior to this, solutions to the general fluid equations did not explain the observed flow effects. Prandtl realized that there was a significant difference between the relative magnitude of the inertial and viscous forces close to the surface when compared to a distance away from the surface.

If we consider a viscous fluid moving past an obstacle or along a surface then the molecules closest to the surface or obstacle will stick to it and hence become stationary. This is due to the viscous nature of most real fluids and the roughness of the surface on a molecular scale. The next layer of molecules will collide into these stationary molecules and hence slow down. A similar process will continue with the collisions becoming less frequent as we move further away from the contact surface. Thus we have a tangential flow alongside the object which increases rapidly from being zero at the surface to matching the mainstream at some distance away from the surface. The region in which this rapid increase is occurring is referred to as the boundary layer around an object. An example of a boundary layer is shown in Figure (1.3).

The mainstream flow will react to the boundary layer in the same way as it would to the physical surface of the object. Hence the boundary layer determines the effective shape of the object which is usually different from its physical shape. To further complicate matters the boundary layer may separate from the object itself and hence the effective shape of the object will be greatly altered in comparison to the physical shape. This separation occurs when the flow in the boundary layer has very low energy compared to that of the outer mainstream. This separation is not favourable when we consider flow along the wing of an aircraft as it effectively increases the pressure drag. In this case the flow is deliberately altered to become turbulent so as to decrease the overall drag [10].

The thickness of the boundary layer is dependant on the Reynolds number if the flow is laminar and as we move along the object or some distance past the obstacle, the thickness will usually increase. The Reynolds number (Re), is dependent on

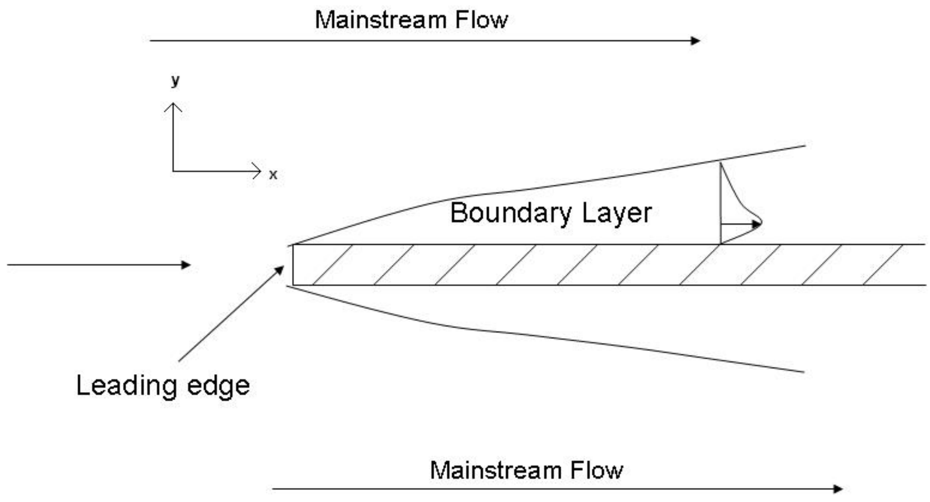

Figure 1.3: Two-dimensional boundary layer

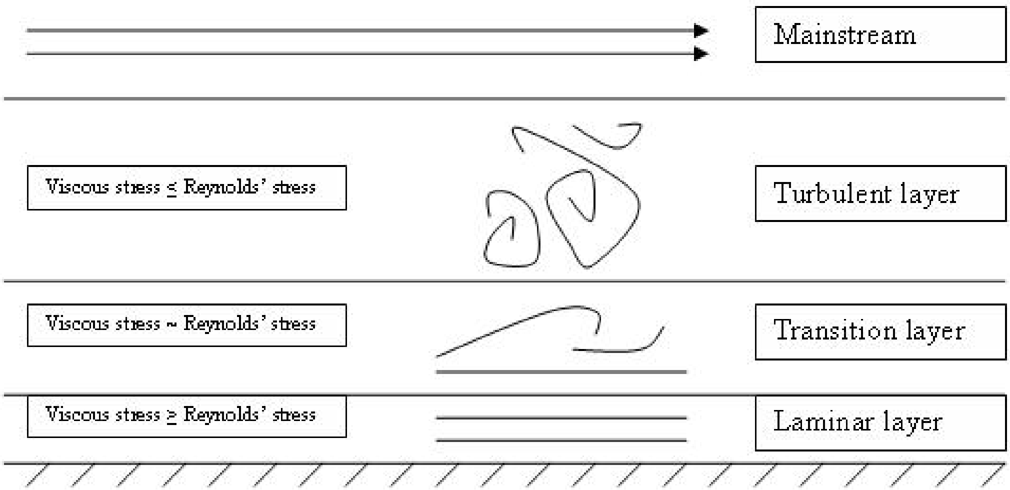

Figure 1.4: Schematic diagram of the structure of a turbulent boundary layer

the characteristic velocity of the mainstream (U); the characteristic length from the beginning of the disturbance in the flow (L); and the kinematic viscosity of the fluid (ν) :

$$Re = \frac{UL}{\nu}. \tag{1.1}$$

The Reynolds number can also be thought of as the ratio of inertial force to viscous force.

A boundary layer may be turbulent or laminar. In laminar flow the fluid moves in smooth layers which slide past each other and hence the shear stresses are low. The thickness of the laminar boundary layer increases with distance from the start of the boundary layer and increases as the Reynolds number decreases. Furthermore a laminar boundary layer will only develop when the Reynolds number is of order

10^3 to 10^5.

Turbulent boundary layers have swirling or disordered flow and they occur for high Reynolds numbers. The Reynolds number for turbulent flow is defined as :

$$Re = \frac{UL}{\nu + \epsilon_0}, \tag{1.2}$$

where U, L and ν are defined as before and ϵ_0 is the mean value for the kinematic eddy viscosity. Turbulent boundary layers develop for Reynolds numbers of order greater than about 10^5.

It is in turbulent flow that we experience eddies at many scales which dissipate as we move further away from the obstacle. Turbulent flow can occur because of rough surfaces and pockets of fluid can be seen to move across layers and hence we see mixing of layers and hence large shear stresses are observed. In the calculation of turbulent flow we rely on eddy viscosity.

Under certain conditions where the shear stresses increase as the fluid moves further downstream, a point may be reached where the boundary layer becomes unsteady and hence turbulent. In this case there may exist a transition layer in which the flow is neither laminar nor turbulent. This is a typical phenomena which occurs along the surface of an airfoil [10].

When we consider boundary conditions for turbulent flow then we still need to consider the no slip condition. The reason for this is because inside the turbulent boundary layer we have three different layers as shown in Figure (1.4). Very close to the contact surface, an extremely thin laminar layer in which viscous forces are greater then inertial forces is found. This layer satisfies the no slip boundary condition at the contact surface. Above this we have a transition layer in which there are large velocity fluctuations and hence the turbulent shearing stresses become comparable to the viscous stresses. Lastly, the turbulent layer is found above this in which the turbulent stresses outweigh the viscous stresses some distance away from the wall [30].

1.5 Previous work in this field

There have been tremendous advances in the study of laminar boundary layer theory in which the eddy viscosity does not play a role [27]. The group invariant solution has been derived using the Lie point symmetries of the boundary layer equations [6].

The group invariant solution of the two–dimensional free laminar jet [20] and the two–dimensional laminar wall jet [28] have been derived using the Lie point symmetries of the boundary layer equations.

Previous work in turbulent boundary layers has considered the scaling symmetries of the boundary layer equations where the eddy viscosity was allowed to be a power law in x [11]. The mainstream velocity was also assumed to be a power law in x. Avramenko et al [3] considered symmetry of turbulent boundary layer flows with three different models of the eddy viscosity. In this dissertation the approach is different. We do not assume models for the eddy viscosity. The form of the eddy viscosity is not prescribed a priori. It is obtained from the condition that the Lie point symmetries of the Navier-Stokes equation with eddy viscosity exist and therefore that the group invariant solution exists.

The group invariant solution for a two–dimensional free turbulent jet described by eddy viscosity has been derived using Lie point symmetries [13].

Previous work on wall jets with eddy viscosity considered the eddy viscosity to be a power law in x only [11].

Several papers have been published on the application of symmetry methods to turbulent flow. Bruzon et al [4] and Gandarias et al [9] have investigated symmetry reductions of a model for turbulence. Cantwell [5] derived a set of similarity rules that define the scaling properties of a wide range of geometrically simple turbulent flows. Oberlack [24] and Oberlack and Khujadze [25] applied symmetry methods to turbulent shear flow and turbulent boundary layers while $\ddot{U}$nal [31] considered the application of equivalence transformations to turbulence. Holmes, Lumley and Berkooz [14] used coherent structures and dynamical systems in order to model turbulence including turbulence in boundary layers.

Grebenev, Oberlack and Grishkov [12] used approximate Lie symmetries of the Navier-Stokes equations for applications to scaling phenomenon arising in turbulence. We will not consider approximate symmatries in this dissertation.

New scaling laws for turbulent boundary layers which were derived using Lie group symmetry methods, have been tested against experimental data for zero-pressure-gradient turbulent boundary layers by Lindgren, Osterlund and Johansson [19]. The results were found to fit very well with the experimental data over most of the boundary layer except the outermost part of the boundary layer.

1.6 Derivation of equations

In this section the Navier-Stokes equation in cartesian coordinates is considered and it is shown how turbulence can easily be modelled into this classical equation. The Reynolds averaged equations are then derived. Finally the concept of eddy viscosity is introduced in order to close the system of equations.

1.6.1 Background to the derivation of turbulent boundary layer equations

We have already seen that turbulent motion can be thought of as mixing or eddying motion in which irregular fluctuations occur. This turbulence is responsible for large resistance in pipes, drag on ships and aeroplanes and can have the disastrous effects of losing turbines and turbocompressors. However turbulence also increases the pressure along the wings of an aeroplane and hence stops the flow along the wing from separating. To understand turbulence is of vital importance from both view points.

If we consider a fixed point in space, then we notice that when the motion is turbulent, the velocity and pressure at this point will not remain constant with time. Both the velocity and pressure will perform very irregular fluctuations of high frequency. The lumps of fluid which perform these fluctuations are not single molecules but rather macroscopic fluid balls of varying size. Even if the velocity fluctuation is not very large it will still have a decisive effect on the whole course of the motion. The size of the fluid lumps continually agglomerate and disintegrate and they determine the scale of the turbulence [30]. The size of these fluid lumps is determined by external conditions associated with the flow. For example, by the mesh or honeycomb through which the stream had passed [30].

By taking this into consideration we can see that fluid lumps have their own motion which is superimposed on the mean flow. Hence we can conveniently describe turbulent flow mathematically by splitting it into its mean motion added to its fluctuation.

Let $v_x(x, y, z, t)$ denote the velocity in the x–direction, $v_y(x, y, z, t)$ the velocity in the y–direction and $v_z(x, y, z, t)$ denote the velocity in the z–direction. Let $p(x, y, z, t)$ denote the pressure. Let $\bar{v}_x(x, y, z)$, $\bar{v}_y(x, y, z)$ and $\bar{v}_z(x, y, z)$ denote the time averages of the velocities in the x, y and z directions, respectively. Let $\bar{p}(x, y, z)$ denote the time average of the pressure. Let the fluctuations be represented by $v'_x(x, y, z, t)$, $v'_y(x, y, z, t)$, $v'_z(x, y, z, t)$ and $p'(x, y, z, t)$. We can thus write the following relations for the velocity components and the pressure :

$$v_x = \bar{v}_x + v'_x; \qquad v_y = \bar{v}_y + v'_y; \qquad v_z = \bar{v}_z + v'_z; \qquad p = \bar{p} + p'. \tag{1.3}$$

As we will only be dealing with incompressible fluids, the density is constant. If the fluid were compressible then we would also need to split the density variable into its mean added to its fluctuation. We define the time average at a fixed point in space. For example

$$\bar{v}_x(x, y, z) = \frac{1}{t_1} \int_{t_0}^{t_0+t_1} v_x(x, y, z, t)\, dt. \tag{1.4}$$

We note that t_1 is taken to be sufficiently large for the mean values to be completely independent of time. Thus, as the fluctuations occur in a very short period of time we see that the time-averages of all the quantities describing the fluctuations are equal to zero :

$$\overline{v'_x} = 0; \qquad \overline{v'_y} = 0; \qquad \overline{v'_z} = 0; \qquad \overline{p'} = 0. \tag{1.5}$$

A very important concept that we must now understand is that the fluctuations v'_x, v'_y, v'_z influence the mean motion $\bar{v}_x$, $\bar{v}_y$, $\bar{v}_z$ such that the mean flow shows an apparent increase in resistance to deformation. This means that the fluctuations cause an apparent increase in the viscosity of the mean flow. The increased apparent viscosity of the mean flow is the central concept of all theories of turbulent motion. We will see later that this apparent increase is accounted for by introducing an additional viscosity termed the eddy viscosity.

1.6.2 Derivation of the Reynolds' averaged equations

We consider an incompressible fluid in cartesian coordinates in three dimensions. The body force per unit mass $\underline{F}$, is neglected. The continuity equation for the fluid is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{1.6}$$

and the components of the Navier-Stokes equation are

$$\rho \left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \nabla^2 v_x, \tag{1.7}$$

$$\rho \left[\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right] = -\frac{\partial p}{\partial y} + \mu \nabla^2 v_y, \tag{1.8}$$

$$\rho \left[\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \nabla^2 v_z, \tag{1.9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.10}$$

and μ is the viscosity of the fluid and ρ is the density of the fluid.

We now substitute equations (1.3) into the continuity equation (1.6):

$$\frac{\partial(\bar{v}_x + v'_x)}{\partial x} + \frac{\partial(\bar{v}_y + v'_y)}{\partial y} + \frac{\partial(\bar{v}_z + v'_z)}{\partial z} = 0. \tag{1.11}$$

Now we take time averages and use (1.5). Thus equation (1.11) reduces to

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} + \frac{\partial \bar{v}_z}{\partial z} = 0. \tag{1.12}$$

In the derivation of (1.12) we used the rule that if f is a dependent variable then,

$$\overline{\overline{f}} = \overline{f}. \tag{1.13}$$

We now substitute equation (1.12) into equation (1.11) and we see that the fluctuations also satisfy the continuity equation :

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} + \frac{\partial v'_z}{\partial z} = 0. \tag{1.14}$$

Using the continuity equation (1.6) we can rewrite equations (1.7), (1.8) and (1.9) as follows :

$$\rho \left[\frac{\partial v_x}{\partial t} + \frac{\partial (v_x^2)}{\partial x} + \frac{\partial (v_x\, v_y)}{\partial y} + \frac{\partial (v_x\, v_z)}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \nabla^2 v_x, \tag{1.15}$$

$$\rho \left[\frac{\partial v_y}{\partial t} + \frac{\partial (v_y\, v_x)}{\partial x} + \frac{\partial (v_y^2)}{\partial y} + \frac{\partial (v_y\, v_z)}{\partial z} \right] = -\frac{\partial p}{\partial y} + \mu \nabla^2 v_y, \tag{1.16}$$

$$\rho \left[\frac{\partial v_z}{\partial t} + \frac{\partial (v_z\, v_x)}{\partial x} + \frac{\partial (v_z\, v_y)}{\partial y} + \frac{\partial (v_z^2)}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \nabla^2 v_z. \tag{1.17}$$

We now substitute (1.3) into equations (1.15), (1.16) and (1.17). We will consider the calculations for the $x-$component in detail. The calculations for the y and z components are exactly the same. The $x-$component thus becomes :

$$\begin{aligned}\rho\Big[\frac{\partial v'_x}{\partial t} &+ \frac{\partial}{\partial x}(\bar{v}_x\bar{v}_x + 2\bar{v}_x v'_x + v'_x v'_x) + \frac{\partial}{\partial y}(\bar{v}_x\bar{v}_y + \bar{v}_x v'_y + \bar{v}_y v'_x + v'_x v'_y) \\ &+\frac{\partial}{\partial z}(\bar{v}_x\bar{v}_z + \bar{v}_x v'_z + \bar{v}_z v'_x + v'_x v'_z)\Big] = -\frac{\partial(\bar{p}+p')}{\partial x} + \mu\nabla^2(\bar{v}_x + v'_x).\end{aligned} \tag{1.18}$$

We now take the time average defined by (1.4). The time average of the product of fluctuations and the square of fluctuations do not vanish. Equation (1.18) becomes :

$$\rho\left[\frac{\partial(\bar{v}_x\bar{v}_x)}{\partial x} + \frac{\partial(\bar{v}_x\bar{v}_y)}{\partial y} + \frac{\partial(\bar{v}_x\bar{v}_z)}{\partial z}\right] = -\frac{\partial\bar{p}}{\partial x} + \mu\nabla^2\bar{v}_x + \frac{\partial(-\rho\overline{v'_x v'_x})}{\partial x} + \frac{\partial(-\rho\overline{v'_x v'_y})}{\partial y} + \frac{\partial(-\rho\overline{v'_x v'_z})}{\partial z}. \tag{1.19}$$

We now use equation (1.12) to rewrite the left hand side of equation (1.19) and we expand the right hand side of equation (1.19) using again the continuity equation (1.12) :

$$\begin{aligned}\rho\left[\bar{v}_x\frac{\partial\bar{v}_x}{\partial x} + \bar{v}_y\frac{\partial\bar{v}_x}{\partial y} + \bar{v}_z\frac{\partial\bar{v}_x}{\partial z}\right] &= \frac{\partial}{\partial x}\left[-\bar{p} + \mu\left(\frac{\partial\bar{v}_x}{\partial x} + \frac{\partial\bar{v}_x}{\partial x}\right) - \rho\overline{v'_x v'_x}\right] \\ +\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\bar{v}_x}{\partial y} + \frac{\partial\bar{v}_y}{\partial x}\right) - \rho\overline{v'_x v'_y}\right] &+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\bar{v}_x}{\partial z} + \frac{\partial\bar{v}_z}{\partial x}\right) - \rho\overline{v'_x v'_z}\right].\end{aligned} \tag{1.20}$$

Equation (1.20) can be written as

$$\rho\left[\bar{v}_x\frac{\partial\bar{v}_x}{\partial x} + \bar{v}_y\frac{\partial\bar{v}_x}{\partial y} + \bar{v}_z\frac{\partial\bar{v}_x}{\partial z}\right] = \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}, \tag{1.21}$$

where

$$\tau_{ik} = -\bar{p}\delta_{ik} + \mu\left(\frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_i}\right) - \rho\overline{v_i'v_k'}, \tag{1.22}$$

and

$$\delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

Equation (1.21) can also be expressed as

$$\rho\frac{D\bar{v}_x}{Dt} = \tau_{kx,k}, \tag{1.23}$$

where the repeated index k is summed over x, y and z and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{v}_x\frac{\partial}{\partial x} + \bar{v}_y\frac{\partial}{\partial y} + \bar{v}_z\frac{\partial}{\partial z}. \tag{1.24}$$

Equation (1.23) has the same form as the $x-$component of Cauchy's first law of motion,

$$\rho\frac{Dv_i}{Dt} = \tau_{ki,k} + \rho F_i, \tag{1.25}$$

where F_i is the body force per unit mass which is neglected in (1.23). We can therefore interpret (1.22) as a generalisation to turbulent flow of the Navier-Poisson law relating the stress tensor to the rate-of-strain tensor in an incompressible viscous fluid in laminar flow.

In (1.22) the term $\mu\left(\frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_i}\right)$ is the shear stress tensor or viscous stress tensor that arises in laminar flow. However, the last term, $-\rho\overline{v_i'v_k'}$ is a new term which is not seen in laminar flow. This last term which has a total of six components satisfies the properties of a second order cartesian tensor. The tensor $-\rho\overline{v_i'v_k'}$ is called the Reynolds stress tensor. The Reynolds stresses arise due to turbulent fluctuations. The Reynolds stresses outweigh the viscous stresses in turbulent flow.

The main challenge in studying turbulent motion is how to model the Reynolds' stresses. In order to close the system of equations we need to model the Reynolds stresses in an appropriate way. Many models have been suggested with various degrees of success. We will use a model which was introduced by Boussinesq in (1877). Boussinesq chose to model the Reynolds stresses as an apparent increase in the viscosity of the fluid. He suggested that a function of proportionality which relates the Reynolds stresses to the spatial gradient of the mean velocity be introduced, in the same way that the viscosity relates the shear stresses to the spatial gradient of the fluid velocity in laminar flow. This function of proportionality is termed the eddy viscosity :

$$\tau_{ik}' = -\rho\overline{v_i'v_k'} = A\left(\frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_i}\right), \tag{1.26}$$

where A is the eddy viscosity. Define

$$\epsilon = \frac{A}{\rho}. \tag{1.27}$$

Then ϵ is the kinematic eddy viscosity. We will refer to the kinematic eddy viscosity simply as the eddy viscosity. The eddy viscosity is a property of the flow and it is not a property of the fluid. Hence it can depend on position and velocity. Later in our analysis of the wall jet, we will consider the eddy viscosity a function of position and velocity. In our analysis of the turbulent boundary layer we will consider the eddy viscosity to depend only on position and not on velocity :

$$\epsilon = \epsilon(x, y, z). \tag{1.28}$$

Substituting equation (1.26) into equation (1.20) we obtain :

$$\begin{aligned}\bar{v}_x\frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y\frac{\partial \bar{v}_x}{\partial y} + \bar{v}_z\frac{\partial \bar{v}_x}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_x}{\partial x}\right)\right] \\ &+\frac{\partial}{\partial y}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial y} + \frac{\partial \bar{v}_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial z} + \frac{\partial \bar{v}_z}{\partial x}\right)\right].\end{aligned} \tag{1.29}$$

where ν is the kinematic viscosity of the fluid. Equation (1.29) is the x–component of the Reynolds equations for turbulent flow of an incompressible fluid. The $y-$ and $z-$components are derived in the same way and are :

$$\begin{aligned}\bar{v}_x\frac{\partial \bar{v}_y}{\partial x} + \bar{v}_y\frac{\partial \bar{v}_y}{\partial y} + \bar{v}_z\frac{\partial \bar{v}_y}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial x}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial y} + \frac{\partial \bar{v}_y}{\partial x}\right)\right] \\ &+\frac{\partial}{\partial y}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_y}{\partial y} + \frac{\partial \bar{v}_y}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_y}{\partial z} + \frac{\partial \bar{v}_z}{\partial y}\right)\right],\end{aligned} \tag{1.30}$$

$$\begin{aligned}\bar{v}_x\frac{\partial \bar{v}_z}{\partial x} + \bar{v}_y\frac{\partial \bar{v}_z}{\partial y} + \bar{v}_z\frac{\partial \bar{v}_z}{\partial z} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \frac{\partial}{\partial x}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial z} + \frac{\partial \bar{v}_z}{\partial x}\right)\right] \\ &+\frac{\partial}{\partial y}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_z}{\partial y} + \frac{\partial \bar{v}_y}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_z}{\partial z} + \frac{\partial \bar{v}_z}{\partial z}\right)\right].\end{aligned} \tag{1.31}$$

The sum

$$E(x, y, z) = \nu + \epsilon(x, y, z, \bar{v}_x, \bar{v}_y, \bar{v}_z), \tag{1.32}$$

is referred to as the effective viscosity [26]. Equation (1.22) for the stress tensor becomes

$$\tau_{ik} = -\bar{p}\delta_{ik} + \rho E\left(\frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_i}\right). \tag{1.33}$$

1.6.3 Derivation of two-dimensional turbulent boundary layer equations

Prandtl's equations for laminar boundary layer theory are derived by non- dimensionalising the Navier Stokes equations and then letting the Reynolds number tend to infinity. We will use the same approach to derive the turbulent boundary layer equations by starting with the Reynolds equations for turbulent flow which we have derived in Section 1.6.2. In this dissertation we deal with the two-dimensional turbulent boundary layer and hence the $z-$component vanishes identically. The Reynolds equations for turbulent flow in two dimensions are :

$$\bar{v}_x\frac{\partial \bar{v}_x}{\partial x}+\bar{v}_y\frac{\partial \bar{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{\partial}{\partial x}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial x}+\frac{\partial \bar{v}_x}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_x}{\partial y}+\frac{\partial \bar{v}_y}{\partial x}\right)\right], \tag{1.34}$$

$$\bar{v}_x\frac{\partial \bar{v}_y}{\partial x}+\bar{v}_y\frac{\partial \bar{v}_y}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}+\frac{\partial}{\partial x}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_y}{\partial x}+\frac{\partial \bar{v}_x}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[(\nu+\epsilon)\left(\frac{\partial \bar{v}_y}{\partial y}+\frac{\partial \bar{v}_y}{\partial y}\right)\right] \tag{1.35}$$

and the continuity equation is

$$\frac{\partial \bar{v}_x}{\partial x}+\frac{\partial \bar{v}_y}{\partial y}=0. \tag{1.36}$$

The $x-$coordinate is along the boundary layer and the $y-$coordinate is perpendicular to the boundary layer. The following dimensionless variables are used to non-dimensionalise (1.34) and (1.35) :

$$x=Lx^*;\;\; y=\frac{L}{\sqrt{Re}}y^*;\;\; \bar{v}_x=U\bar{v}_x^*;\;\; \bar{v}_y=\frac{U}{\sqrt{Re}}\bar{v}_y^*;\;\; \bar{p}=\rho U^2\bar{p}^*;\;\; E^*=\frac{\nu+\epsilon}{\nu+\epsilon_0}. \tag{1.37}$$

The Reynolds number Re is defined by (1.2) and the star denotes a non-dimensional variable. Using (1.37) and (1.2) for Re in equations (1.34) and (1.35) we arrive at the following two equations :

$$\bar{v}_x^*\frac{\partial \bar{v}_x^*}{\partial x^*}+\bar{v}_y^*\frac{\partial \bar{v}_x^*}{\partial y^*} = -\frac{\partial \bar{p}^*}{\partial x^*}+\frac{1}{Re}\frac{\partial}{\partial x^*}\left[E^*\left(\frac{\partial \bar{v}_x^*}{\partial x^*}+\frac{\partial \bar{v}_x^*}{\partial x^*}\right)\right]+\frac{\partial}{\partial y^*}\left[E^*\left(\frac{\partial \bar{v}_x^*}{\partial y^*}+\frac{1}{Re}\frac{\partial \bar{v}_y^*}{\partial x^*}\right)\right], \tag{1.38}$$

$$\frac{1}{Re}\bar{v}_x^*\frac{\partial \bar{v}_y^*}{\partial x^*}+\frac{1}{Re}\bar{v}_y^*\frac{\partial \bar{v}_y^*}{\partial y^*} = -\frac{\partial \bar{p}^*}{\partial y^*}+\frac{\partial}{\partial x^*}\left[E^*\left(\frac{1}{Re^2}\frac{\partial \bar{v}_y^*}{\partial x^*}+\frac{1}{Re}\frac{\partial \bar{v}_x^*}{\partial y^*}\right)\right]+\frac{2}{Re}\frac{\partial}{\partial y^*}\left[E^*\frac{\partial \bar{v}_y^*}{\partial x^*}\right]. \tag{1.39}$$

We now let $Re\to\infty$ and hence $\frac{1}{Re}\to 0$. Equations (1.38) and (1.39) reduce to :

$$\bar{v}_x^*\frac{\partial \bar{v}_x^*}{\partial x^*}+\bar{v}_y^*\frac{\partial \bar{v}_x^*}{\partial y^*} = -\frac{\partial \bar{p}^*}{\partial x^*}+\frac{\partial}{\partial y^*}\left[E^*\frac{\partial \bar{v}_x^*}{\partial y^*}\right], \tag{1.40}$$

$$0=-\frac{\partial \bar{p}^*}{\partial y^*}. \tag{1.41}$$

Equations (1.40) and (1.41) are in non-dimensional form. We re-express (1.40) and

(1.41) in dimensional form using (1.37). This gives the Reynolds equations for a turbulent boundary layer in dimensional variables :

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left[(\nu + \epsilon)\frac{\partial \bar{v}_x}{\partial y}\right], \tag{1.42}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}. \tag{1.43}$$

Equations (1.42), (1.43) and the continuity equation (1.36) are three equations for the three unknowns $\bar{v}_x(x, y)$, $\bar{v}_y(x, y)$ and $\bar{p}(x, y)$. They are the equations we study in our analysis of turbulent boundary layers using eddy viscosity.

1.6.4 Lie point symmetries

In our investigation of turbulent boundary layers we make use of the Lie point symmetry approach. A brief outline of how we do this will be given in this section. We will later introduce a stream function, ψ, into (1.42) which will leave us with a third order partial differential equation. Hence we consider one dependent variable ψ and two independent variables x and y.

Consider the third order partial differential equation

$$F = 0, \tag{1.44}$$

where F is a function of x, y, ψ and the partial derivatives of ψ up to third order. The Lie point symmetry generators,

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}, \tag{1.45}$$

of the partial differential equation (1.44) are obtained by solving the determining equation [15],

$$X^{[3]}F|_{F=0} = 0, \tag{1.46}$$

where

$$X^{[3]} = X + \zeta_1\frac{\partial}{\partial \psi_x} + \zeta_2\frac{\partial}{\partial \psi_y} + \zeta_{11}\frac{\partial}{\partial \psi_{xx}} + \zeta_{12}\frac{\partial}{\partial \psi_{xy}} + \zeta_{22}\frac{\partial}{\partial \psi_{yy}}$$
$$+\zeta_{111}\frac{\partial}{\partial \psi_{xxx}} + \zeta_{112}\frac{\partial}{\partial \psi_{xxy}} + \zeta_{122}\frac{\partial}{\partial \psi_{xyy}} + \zeta_{222}\frac{\partial}{\partial \psi_{yyy}}, \tag{1.47}$$

and

$$\zeta_i = D_i(\eta) - \psi_s\, D_i(\xi^s), \tag{1.48}$$

$$\zeta_{ij} = D_j(\zeta_i) - \psi_{is}\, D_j(\xi^s), \tag{1.49}$$

$$\zeta_{ijk} = D_k(\zeta_{ij}) - \psi_{ijs}\, D_k(\xi^s), \tag{1.50}$$

with summation over repeated indices [15]. The operators D_1 and D_2 of total differentiation with respect to x and y respectively are defined by :

$$D_1 = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \cdots, \tag{1.51}$$

$$D_2 = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \cdots. \tag{1.52}$$

One of the partial derivatives of ψ is replaced in the determining equation (1.46) using (1.44). We then solve for ξ^1, ξ^2 and η by equating the coefficients of the partial derivatives of ψ in the determining equation (1.46). This gives a system of linear partial differential equations from which we obtain ξ^1, ξ^2 and η.

1.6.5 Group invariant solution

Once we have found ξ^1, ξ^2 and η, we substitute these results into the generator (1.45). We then assume that $\psi = \Phi(x, y)$ is a solution to the partial differential equation (1.44). Hence $\psi - \Phi(x, y)$ must be invariant when operated on by the generator X :

$$X(\psi - \Phi(x, y))|_{\psi=\Phi(x,y)} = 0. \tag{1.53}$$

Equation (1.53) is a first order quasi-linear partial differential equation for $\Phi(x, y)$. It is solved using the method of characteristics, to reduce it to a system of ordinary differential equations. This will lead to a group invariant solution for $\Phi(x, y)$ and hence for $\psi(x, y)$ in terms of a similarity variable.

Substituting this group invariant solution into the partial differential equation (1.44) reduces the partial differential equation to an ordinary differential equation. At this point we impose the boundary conditions and solve the equation either analytically or numerically. We may find the Lie point symmetries of the ordinary differential equation in order to reduce the order of the ordinary differential equation.

Once we have solved the ordinary differential equation we will substitute the result into the group invariant solution for $\psi(x, y)$.

1.7 Boundary conditions

There are two main boundary conditions that we impose when analysing the two-dimensional turbulent boundary layer. The $x-$axis is taken to be parallel to the contact surface and the $y-$axis is taken to be perpendicular to the contact surface. We choose the origin to be at the leading edge as shown in Figure (1.3).

The first condition we impose is that there is no slip at the contact surface. Even though the flow is turbulent, there is a thin laminar sublayer within a turbulent boundary layer and hence we still need to impose the no-slip condition. This is illustrated in Figure (1.4). This thin laminar sublayer will always occur. Even if the fluid is not very viscous, on a molecular scale there will be no slipping at the surface and hence this condition will still hold. Thus :

$$\bar{v}_x(x,0) = 0. \tag{1.54}$$

The second condition that we impose is that as we move away from the outer layer of the turbulent boundary layer we need the mean fluid velocity to match that of the mainstream. This is commonly known as mainstream matching. Hence

$$\bar{v}_x(x,\infty) = \bar{U}(x), \tag{1.55}$$

where $\bar{U}(x)$ is the velocity of the mainstream.

1.8 Structure of dissertation

In Chapter 1 we gained a better understanding of turbulent boundary layers. We looked at the concepts of eddy viscosity and Reynolds stresses. The structure of the turbulent boundary layer was described and the boundary conditions were outlined. The derivation of the turbulent boundary layer equations described by eddy viscosity was presented. The theory for finding the Lie point symmetries and group invariant solutions was outlined.

In Chapter 2 we look at the two-dimensional turbulent boundary layer equations. We first consider the scaling tranformation on these equations where the mainstream velocity is taken to be a power law solution. Various special cases are also considered. We next find the Lie point symmetries of the turbulent boundary layer equations and using these point symmetries we find the group invariant solution to the partial differential equation which describes turbulent flow. In order to simplify calculations the Lie point symmetry generator and group invariant solution were scaled. We then consider various forms of the eddy viscosity and derive the corresponding ordinary differential equations. We first consider the eddy viscosity to be a power law in x. Next we consider the eddy viscosity to be of the form $m(x)y^{\frac{6}{7}}$.

In Chapter 3 we consider the two-dimensional turbulent wall jet. We derive the conserved quantity for different forms of the eddy viscosity. We consider four forms of the eddy viscosity. Firstly we consider when the eddy viscosity is a function of x only. Secondly we consider the eddy viscosity of the form $M(x)\bar{v}_x^n(x,y)$ for $n>0$. Thirdly we consider eddy viscosity to be a general function of the mainstream mean velocity in the x-direction, that is $M(x)F(\bar{v}_x(x,y))$. Lastly we consider the eddy viscosity to be a general function of x and y. We then consider the group invariant solution for $E(x) = E_0x^n$ and go on to find the analytical solution. Using the analytical solution we plot a range of different graphs and investigate the effect the different constants have on the shape of the wall jet. In the next section we find the Lie point symmetries of the turbulent wall jet equation when the eddy viscosity is of the form $M(x)F(\bar{v}_x(x,y))$. Using the Lie point symmetries we find the group invariant solution and we also find the solution of the ordinary differential equation in parametric form. We plot our parametric solution and compare the graphs which arise from varying the constants.

In Chapter 4 we briefly summarise our results and provide some conclusions.

In Appendix A we outline the derivation of the Lie point symmetry of the turbulent boundary layer equations which were used in Chapter 2. In Appendix B the derivation of the Lie point symmetry of the turbulent wall jet equations which were used in Chapter 3 is given.

Chapter 2

Two-dimensional turbulent boundary layer

2.1 Introduction

In this chapter we will investigate the two-dimensional turbulent boundary layer described by eddy viscosity. We first perform a scaling symmetry analysis. This applies only when the free stream velocity and the effective viscosity $E = \nu + \epsilon$ satisfy power laws in x where the $x-$axis is along the boundary layer. We then perform a Lie point symmetry analysis of the turbulent boundary layer equations. The Lie point symmetries are used to reduce the partial differential equation for the stream function to an ordinary differential equation.

We will derive the laminar boundary layer solutions by taking the eddy viscosity to be zero. The turbulent boundary layer solutions are derived next by considering various forms for the eddy viscosity.

2.2 Turbulent boundary layer equations

We will reformulate the two-dimensional turbulent boundary layer equations as one equation expressed in terms of the stream function.

The two-dimensional turbulent boundary layer equations are (1.42), (1.43) and the continuity equation (1.36) :

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left[(\nu + \epsilon)\frac{\partial \bar{v}_x}{\partial y}\right], \tag{2.1}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}, \tag{2.2}$$

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} = 0. \tag{2.3}$$

From equation (2.2) we can deduce that $\bar{p}$ is a function of x only, that is $\bar{p} = \bar{p}(x)$.

We also note that we can write,

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \frac{\partial \bar{U}}{\partial t} + \bar{U}\frac{\partial \bar{U}}{\partial x}, \tag{2.4}$$

where $\bar{U}(x)$ is the mean value of the mainstream velocity which is in the x–direction. Equation (2.4) is the x–component of the time averaged Euler equation for inviscid flow in the x–direction just outside the boundary layer. Since we are dealing with steady state flows, $\frac{\partial \bar{U}}{\partial t} = 0$ and hence

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \bar{U}(x)\frac{d\bar{U}(x)}{dx}. \tag{2.5}$$

We now substitute (2.5) into (2.1) :

$$\bar{v}_x\frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y\frac{\partial \bar{v}_x}{\partial y} = \bar{U}(x)\frac{d\bar{U}(x)}{dx} + \frac{\partial}{\partial y}\left[(\nu + \epsilon)\frac{\partial \bar{v}_x}{\partial y}\right]. \tag{2.6}$$

Since (2.3) is satisfied we can introduce a stream function $\psi(x, y)$ defined by

$$\bar{v}_x = \frac{\partial \psi}{\partial y}, \qquad \bar{v}_y = -\frac{\partial \psi}{\partial x}. \tag{2.7}$$

Equation (2.3) is identically satisfied by (2.7). Equation (2.6) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \bar{U}(x)\frac{d\bar{U}(x)}{dx} + \frac{\partial}{\partial y}\left((\nu + \epsilon)\frac{\partial^2 \psi}{\partial y^2}\right). \tag{2.8}$$

We write (2.8) in terms of the effective viscosity

$$E(x, y) = \nu + \epsilon(x, y). \tag{2.9}$$

The equation that we will be investigating is

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \bar{U}(x)\frac{d\bar{U}(x)}{dx} + \frac{\partial}{\partial y}\left(E(x, y)\frac{\partial^2 \psi}{\partial y^2}\right). \tag{2.10}$$

2.3 Scaling transformation

In this section we apply a scaling transformation to equation (2.10) as described by Dresner [8]. This will not be possible for arbitrary forms of $\bar{U}(x)$ and $E(x, y)$. It is possible if $\bar{U}(x)$ and $E(x, y)$ are power laws in x :

$$\bar{U}(x) = \bar{U}_0 x^n; \qquad E(x) = \nu + \epsilon(x) = E_0 x^{\beta}. \tag{2.11}$$

where $\bar{U}_0$ and E_0 are constants. In the power law for $E(x)$, ν is neglected when compared with $\epsilon(x)$. This is a good approximation because the kinematic eddy viscosity $\epsilon(x)$ is 10^2 or 10^3 times larger than the kinematic viscosity ν.

2.3.1 Power law for mainstream mean velocity with $n \neq 0$ and $\bar{U}_0 \neq 0$

We first consider the case $n \neq 0$ and $\bar{U}_0 \neq 0$. We will see that the case $n = 0$ and/or $\bar{U}_0 = 0$ has to be treated separately and this will be done in the next subsection.

Using (2.11), equation (2.10) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = n\bar{U}_0^2 x^{2n-1} + E_0 x^\beta \frac{\partial^3 \psi}{\partial y^3}. \tag{2.12}$$

Consider the scaling transformation :

$$\bar{x} = \lambda^a x, \qquad \bar{y} = \lambda^b y, \qquad \bar{\psi} = \lambda^c \psi. \tag{2.13}$$

Equation (2.12) becomes

$$\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = n\bar{U}_0^2 \lambda^{-2na-2b+2c}\, \bar{x}^{2n-1} + E_0 \lambda^{-(1+\beta)a+b+c}\, \bar{x}^\beta\, \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3}. \tag{2.14}$$

Now (2.12) is invariant under the scaling transformation (2.13) if

$$\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = n\bar{U}_0^2 \bar{x}^{2n-1} + E_0 \bar{x}^\beta \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3}. \tag{2.15}$$

We are assuming that $n \neq 0$ and $\bar{U}_0 \neq 0$. Thus (2.12) is invariant under the scaling transformation (2.13) provided

$$-2na - 2b + 2c = 0, \tag{2.16}$$

$$-(1+\beta)a + b + c = 0. \tag{2.17}$$

Solving (2.16) and (2.17) for b and c in terms of a, gives

$$b = \frac{1}{2}(1+\beta-n)a, \qquad c = \frac{1}{2}(1+\beta+n)a. \tag{2.18}$$

Thus (2.12) is invariant under the transformation

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^{\frac{1}{2}(1+\beta-n)a} y, \quad \bar{\psi} = \lambda^{\frac{1}{2}(1+\beta+n)a} \psi, \tag{2.19}$$

where a is arbitrary.

Suppose that the solution of (2.12) is

$$\psi = f(x, y). \tag{2.20}$$

Then the solution of (2.15) is

$$\bar{\psi} = f(\bar{x}, \bar{y}), \tag{2.21}$$

where there is no overhead bar on f in (2.21) because (2.12) and (2.15) have the same form. Substitute (2.19) into (2.21) and use (2.20) for ψ. This gives

$$\lambda^{\frac{1}{2}(1+\beta+n)a} f(x, y) = f(\lambda^a x, \lambda^{\frac{1}{2}(1+\beta-n)a} y). \tag{2.22}$$

To obtain the functional form for f differentiate (2.22) with respect to λ and then set $\lambda = 1$. The following first order linear partial differential equation for $f(x, y)$ is obtained :

$$x\frac{\partial f}{\partial x} + \frac{1}{2}(1+\beta-n)y\frac{\partial f}{\partial y} = \frac{1}{2}(1+\beta+n)f. \tag{2.23}$$

Equation (2.23) is independent of the constant a. The differential equations of the characteristic curves of (2.23) are

$$\frac{dx}{x} = \frac{dy}{\frac{1}{2}(1+\beta-n)y} = \frac{df}{\frac{1}{2}(1+\beta+n)f}. \tag{2.24}$$

The first and second terms in (2.24) give

$$\frac{y}{x^{\frac{1}{2}(1+\beta-n)}} = c_1, \tag{2.25}$$

where c_1 is a constant. The first and last terms in (2.24) give

$$\frac{f}{x^{\frac{1}{2}(1+\beta+n)}} = c_2, \tag{2.26}$$

where c_2 is a constant. The general solution of (2.23) is

$$c_2 = F(c_1) \tag{2.27}$$

or using (2.25) and (2.26),

$$f(x, y) = x^{\frac{1}{2}(1+\beta+n)} F\left(\frac{y}{x^{\frac{1}{2}(1+\beta-n)}}\right) \tag{2.28}$$

where F is an arbitrary function. But $\psi = f(x, y)$ and so the functional form for ψ is

$$\psi = x^{\frac{1}{2}(1+\beta+n)} F(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}(1+\beta-n)}}. \tag{2.29}$$

We now substitute ψ given by (2.29) into the partial differential equation (2.12). The following ordinary differential equation for $F(\xi)$ is obtained :

$$2E_0\frac{d^3F}{d\xi^3} + (n+\beta+1)F\frac{d^2F}{d\xi^2} + 2n\left[{U_0}^2 - \left(\frac{dF}{d\xi}\right)^2\right] = 0. \tag{2.30}$$

Consider now the mainstream matching boundary condition which is

$$y \to \infty : \qquad \bar{v}_x(x,y) = \bar{U}(x) = \bar{U}_0 x^n. \tag{2.31}$$

But

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = x^n \frac{dF}{d\xi}, \tag{2.32}$$

and $\xi \to \infty$ as $y \to \infty$. Thus (2.31) becomes

$$\xi \to \infty : \qquad x^n \frac{dF}{d\xi}(\infty) = \bar{U}_0 x^n \tag{2.33}$$

and therefore

$$\frac{dF}{d\xi}(\infty) = \bar{U}_0. \tag{2.34}$$

In order to simplify the ordinary differential equation (2.30) and the boundary condition (2.34), consider the transformation

$$\xi = A\eta \quad \textit{and} \quad F(\xi) = BG(\xi). \tag{2.35}$$

where A and B are constants still to be determined. Suppose also that $n \neq -(1+\beta)$. We find that equation (2.30) and the boundary condition (2.34) transform to

$$\frac{2E_0}{AB(n+\beta+1)} \frac{d^3G}{d\eta^3} + G\frac{d^2G}{d\eta^2} + \frac{2n}{(n+\beta+1)}\left[\frac{A^2\bar{U}_0^2}{B^2} - \left(\frac{dG}{d\eta}\right)^2\right] = 0, \tag{2.36}$$

$$\frac{B}{A}\frac{dG}{d\eta}(\infty) = \bar{U}_0. \tag{2.37}$$

We choose

$$\frac{2E_0}{AB(n+\beta+1)} = 1 \quad \textit{and} \quad \frac{B}{A} = \bar{U}_0. \tag{2.38}$$

Solving (2.38) for A and B we find

$$A = \sqrt{\frac{2E_0}{\bar{U}_0(n+\beta+1)}}, \qquad B = \sqrt{\frac{2E_0\bar{U}_0}{n+\beta+1}}. \tag{2.39}$$

Substituting (2.39) into the differential equation (2.36) and the boundary condition (2.37), we obtain

$$\frac{d^3G}{d\eta^3} + G\frac{d^2G}{d\eta^2} + \frac{2n}{(n+\beta+1)}\left[1 - \left(\frac{dG}{d\eta}\right)^2\right] = 0, \tag{2.40}$$

$$\frac{dG}{d\eta}(\infty) = 1. \tag{2.41}$$

When $\beta = 0$, (2.40) reduces to the Falkner-Skan equation.

Finally, using (2.35) and (2.39), the stream function (2.29) becomes

$$\psi(x,y) = \left(\frac{2E_0\bar{U}_0}{n+\beta+1}\right)^{\frac{1}{2}} x^{\frac{1}{2}(n+\beta+1)}G(\eta), \tag{2.42}$$

where

$$\eta = \left(\frac{(n+\beta+1)\bar{U}_0}{2E_0}\right)^{\frac{1}{2}} \frac{y}{x^{\frac{1}{2}(1+\beta-n)}}. \tag{2.43}$$

We will not make the Prandtl hypothesis but it can be used to compare the results. Prandtl's hypothesis applied to a two-dimensional boundary layer states that the eddy viscosity is constant across the boundary layer and is proportional to the product of the maximum mean velocity and the width of the boundary layer [11]. Since $\epsilon(x) > \nu$ we will neglect ν compared with $\epsilon(x)$ and apply Prandtl's hypothesis to the effective viscosity E. The maximum mean velocity is the mainstream velocity $\bar{U}_0 x^n$ and from (2.43), the width of the boundary layer is proportional to $x^{\frac{1}{2}(1+\beta-n)}$. Thus, from Prandtl's hypothesis,

$$E_0 x^{\beta} \propto \bar{U}_0 x^n x^{\frac{1}{2}(1+\beta-n)} \tag{2.44}$$

and therefore when Prandtl's hypothesis applies, $n = \beta - 1$.

2.3.2 Mainstream mean velocity constant or zero

We now consider the mainstream velocity $\bar{U}(x)$ to be constant or zero. The effective viscosity $E(x,y)$ is still a power law in x.

$$\bar{U}(x) = \bar{U}_0 \qquad \textit{or} \qquad \bar{U}(x) = 0, \qquad\qquad E(x) = E_0 x^{\beta}, \tag{2.45}$$

where $\bar{U}_0$ is a constant. The results of Subsection 2.3.1 do not apply when $n = 0$ or $\bar{U}_0 = 0$. The case in which $\bar{U}_0$ is a non-zero constant corresponds to flow past a flat plate with constant mean velocity. The case $\bar{U}_0 = 0$ corresponds to the boundary layer induced by a stretching sheet [7], [29] or to a free jet emerging through an orifice [13].

Equation (2.10) becomes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = E_0 x^{\beta}\frac{\partial^3\psi}{\partial y^3}. \tag{2.46}$$

We consider again the following scaling transformation :

$$\bar{x} = \lambda^a x, \qquad \bar{y} = \lambda^b y, \qquad \bar{\psi} = \lambda^c \psi. \tag{2.47}$$

Equation (2.46) becomes

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} = E_0\lambda^{-(1+\beta)a+b+c}\ \bar{x}^{\beta}\ \frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}. \tag{2.48}$$

The transformation (2.47) will leave equation (2.46) invariant if

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} = E_0\bar{x}^{\beta}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}. \tag{2.49}$$

Thus (2.46) is invariant under the scaling transformation (2.47) provided

$$c = (1+\beta)a - b. \tag{2.50}$$

Hence (2.46) is invariant under the scaling transformation

$$\bar{x} = \lambda^a x, \qquad \bar{y} = \lambda^b y, \qquad \bar{\psi} = \lambda^{(1+\beta)a-b}\psi. \tag{2.51}$$

In order to obtain a functional form for ψ we suppose that

$$\psi = f(x, y). \tag{2.52}$$

This means that the solution of (2.49) is

$$\bar{\psi} = f(\bar{x}, \bar{y}). \tag{2.53}$$

We substitute the transformation (2.51) in (2.53) and use (2.52) for ψ. We obtain

$$\lambda^{(1+\beta)a-b} f = f(\lambda^a x, \lambda^b y). \tag{2.54}$$

Differentiating (2.54) with respect to λ and then setting $\lambda = 1$, we obtain the following first order linear partial differential equation for f :

$$ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} = [(1+\beta)a - b]f. \tag{2.55}$$

This partial differential equation can readily be solved by considering the differential equations of the characteristic curves,

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{df}{((1+\beta)a - b)f}. \tag{2.56}$$

From the first and second terms in (2.56),

$$\frac{y}{x^{\alpha}} = c_1, \qquad\qquad \alpha = \frac{b}{a} \tag{2.57}$$

and from the first and third terms

$$\frac{f}{x^{1+\beta-\alpha}} = c_2, \tag{2.58}$$

where c_1 and c_2 are constants. The general solution is

$$c_2 = F(c_1), \tag{2.59}$$

that is,

$$f(x,y) = x^{1+\beta-\alpha} F(\frac{y}{x^\alpha}), \tag{2.60}$$

where F is an arbitrary function. But $\psi = f(x,y)$ and so the functional form for ψ is

$$\psi = x^{1+\beta-\alpha} F(\xi), \qquad \xi = \frac{y}{x^\alpha}. \tag{2.61}$$

The solution depends on the ratio α of b to a and not on b and a separately.

We now substitute ψ into the partial differential equation (2.46). It is found that $F(\xi)$ satisfies the ordinary differential equation

$$E_0 \frac{d^3F}{d\xi^3} + (1+\beta-\alpha) F \frac{d^2F}{d\xi^2} - (1+\beta-2\alpha) \left(\frac{dF}{d\xi}\right)^2 = 0. \tag{2.62}$$

The parameter α is not yet determined. It is found from the boundary conditions and from conserved quantities.

i) $\mathbf{\bar{U}_0 \neq 0}$

Consider first flow past a flat plate for which $\bar{U}(x) = \bar{U}_0$ where $\bar{U}_0 \neq 0$. Then from mainstream matching

$$y \to \infty : \qquad \bar{v}_x(x,y) = \bar{U}_0. \tag{2.63}$$

But

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = x^{1+\beta-2\alpha} \frac{dF}{d\xi} \tag{2.64}$$

and

$$\xi = \frac{y}{x^\alpha} \to \infty \quad as \quad y \to \infty. \tag{2.65}$$

Thus (2.63) becomes

$$\xi \to \infty : \qquad x^{1+\beta-2\alpha} \frac{dF}{d\xi}(\infty) = \bar{U}_0 \tag{2.66}$$

Since the right hand side of (2.66) is independent of x, the exponent of x on the left hand side must vanish. Hence

$$\alpha = \frac{1}{2}(1+\beta) \tag{2.67}$$

The stream function (2.61) becomes

$$\psi = x^{\frac{1}{2}(1+\beta)} F(\xi), \qquad \xi = \frac{y}{x^{\frac{1}{2}(1+\beta)}}, \tag{2.68}$$

where from (2.62), $F(\xi)$ satisfies the ordinary differential equation

$$E_0 \frac{d^3F}{d\xi^3} + \frac{1}{2}(1+\beta) F \frac{d^2F}{d\xi^2} = 0 \tag{2.69}$$

subject to the boundary condition (2.66),

$$\frac{dF}{d\xi}(\infty) = \bar{U}_0 \tag{2.70}$$

To remove E_0 and β from the differential equation (2.69) and $\bar{U}_0$ from the boundary condition (2.70) we again make the transformation

$$\xi = A\eta \quad \textit{and} \quad F(\xi) = BG(\xi). \tag{2.71}$$

Equations (2.69) and (2.70) become

$$\frac{d^3G}{d\eta^3} + \frac{1}{2}(1+\beta)\frac{AB}{E_0} G \frac{d^2G}{d\eta^2} = 0, \tag{2.72}$$

$$\frac{B}{A}\frac{dG}{d\eta}(\infty) = \bar{U}_0. \tag{2.73}$$

Choose

$$\frac{1}{2}(1+\beta)\frac{AB}{E_0} = 1, \qquad \frac{B}{A} = \bar{U}_0 \tag{2.74}$$

and therefore

$$A = \sqrt{\frac{2E_0}{\bar{U}_0(1+\beta)}}, \qquad B = \sqrt{\frac{2E_0\bar{U}_0}{1+\beta}}. \tag{2.75}$$

Equations (2.72) and (2.73) become

$$\frac{d^3G}{d\eta^3} + G\frac{d^2G}{d\eta^2} = 0, \tag{2.76}$$

$$\frac{dG}{d\eta}(\infty) = 1. \tag{2.77}$$

The stream function (2.68) is

$$\psi = \sqrt{\frac{2E_0\bar{U}_0}{1+\beta}}\, x^{\frac{1}{2}(1+\beta)}\, G(\eta), \tag{2.78}$$

where

$$\eta = \sqrt{\frac{(1+\beta)\bar{U}_0}{2E_0}}\,\frac{y}{x^{\frac{1}{2}(1+\beta)}}. \tag{2.79}$$

Equation (2.76) is the Blasius equation. Equations (2.76) to (2.79) agree with (2.40) to (2.43) with $n = 0$.

ii) $\bar{\mathbf{U}}_{\mathbf{0}} = \mathbf{0}$

Mainstream matching gives (2.66) with $\bar{U}_0 = 0$:

$$x^{1+\beta-2\alpha}\frac{dF}{d\xi}(\infty) = 0 \tag{2.80}$$

Unlike (2.66), α cannot be determined from (2.80). Other conditions have to be used to determine α.

For a boundary layer induced by a stretching sheet, in laminar flow α is determined from the boundary condition at $y = 0$ [7],[29].

For a free jet α is determined from conservation of mean momentum flux in the $x-$direction [13]. We will not consider the free jet in this dissertation. For a wall jet α is determined from a conserved quantity. This will be discussed in Chapter 3.

We have seen that the scaling transformation reduces the partial differential equation to an ordinary differential equation when

$$\bar{U}(x) = \bar{U}_0 x^n, \qquad E(x) = E_0 x^\beta. \tag{2.81}$$

We now perform a Lie group analysis for general transformations to investigate if more general results can be derived.

The ordinary differential equations derived in this section will be studied after the Lie group analysis has been performed.

2.4 Lie point symmetries

We now find the Lie point symmetries of the turbulent boundary layer equation with stream function ψ. We rearrange equation (2.10) as

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \bar{U}(x)\frac{d\bar{U}(x)}{dx} - \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2\psi}{\partial y^2}\right) = 0. \tag{2.82}$$

In order to simplify our calculations we let,

$$W(x) = \bar{U}(x)\frac{d\bar{U}(x)}{dx}. \tag{2.83}$$

We can later easily solve for $\bar{U}(x)$ if $W(x)$ is known. Thus equation (2.82) becomes,

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - W(x) - \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2 \psi}{\partial y^2}\right) = 0. \qquad (2.84)$$

From the notation of Section 1.6.4

$$F(\psi_x, \psi_y, \psi_{xy}, \psi_{yy}, \psi_{yyy}) = \psi_y \psi_{xy} - \psi_x \psi_{yy} - E_y \psi_{yy} - E\psi_{yyy} - W(x) = 0, \quad (2.85)$$

where variable subscripts denote partial differentiation with respect to the variable that is in the subscript.

In order to find the Lie point symmetries we need to solve the determining equation (1.46),

$$X^{[3]}F|_{F=0} = 0, \qquad (2.86)$$

where we will not need all the terms of generator (1.47) defined in Section 1.6.4. We will only need

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial \psi_x} + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{12} \frac{\partial}{\partial \psi_{xy}} + \zeta_{22} \frac{\partial}{\partial \psi_{yy}} + \zeta_{222} \frac{\partial}{\partial \psi_{yyy}}, \qquad (2.87)$$

as the other partial derivatives of ψ do not occur in equation (2.85). We need to find ζ_1, ζ_2, ζ_{12}, ζ_{22} and ζ_{222} using the formulae (1.48), (1.49) and (1.50) provided in Section 1.6.4. Details of these calculations are presented in Appendix A.1

Using (2.87) and (2.85), the determining equation (2.86) to find the Lie point symmetries of (2.84) becomes

$$\begin{aligned} -E_{xy}\psi_{yy}\xi^1 - E_x \psi_{yyy}\xi^1 - W_x \xi^1 - E_{yy}\psi_{yy}\xi^2 - E_y \psi_{yyy}\xi^2 \\ -\psi_{yy}\zeta_1 + \psi_{xy}\zeta_2 + \psi_y \zeta_{12} - \psi_x \zeta_{22} - E_y \zeta_{22} - E\zeta_{222}|_{F=0} = 0. \end{aligned} \qquad (2.88)$$

Now ψ_{yyy} is replaced using the partial differential equation (2.85) :

$$\psi_{yyy} = \frac{1}{E}\psi_y \psi_{xy} - \frac{1}{E}\psi_x \psi_{yy} - \frac{1}{E}E_y \psi_{yy} - \frac{W}{E}. \qquad (2.89)$$

Substituting the prolongation formulae (A.10), (A.11), (A.12), (A.13) and (A.14) which were calculated in Appendix A.1 and the expression for ψ_{yyy} from equation (2.89) into (2.88), gives us the determining equation in expanded form. From this expanded equation which is shown in full in Appendix A.2, the coefficients of the partial derivatives of ψ can be equated and hence expressions for ξ^1, ξ^2 and η can be obtained. Details of these calculations are given in Appendix A.2.

The following expressions for ξ^1, ξ^2 and η were found in Appendix A.2

$$\xi^1 = a(x), \qquad (2.90)$$

$$\xi^2 = C_2 y + e(x), \tag{2.91}$$

$$\eta = (C_1 + C_2)\psi + C_3, \tag{2.92}$$

where $a(x)$ and $e(x)$ are arbitrary functions of x and C_1, C_2 and C_3 are arbitrary constants.

The following first order linear partial differential equation for $E(x, y)$ must be satisfied,

$$a(x)\frac{\partial E}{\partial x} + [C_2 y + e(x)]\frac{\partial E}{\partial y} = [2C_2 + C_1 - a'(x)]E, \tag{2.93}$$

and the following ordinary differential equation for $W(x)$ must also be satisfied,

$$a(x)\frac{dW}{dx} + (a'(x) - 2C_1)W(x) = 0, \tag{2.94}$$

in order for the Lie point symmetries to exist.

2.5 Group invariant solution : general case, $a(x) \neq 0$, $C_1 + C_2 \neq 0$

In this dissertation we will consider only the general case in which $a(x) \neq 0$ and $C_1 + C_2 \neq 0$. The special cases may be treated in the same way.

In this section the group invariant solution of the partial differential equation (2.84) is found using the Lie point symmetry generator (1.45) :

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}. \tag{2.95}$$

Using (2.90), (2.91) and (2.92) for $\xi^1(x, y, \psi)$, $\xi^2(x, y, \psi)$ and $\eta(x, y, \psi)$ the generator (2.95) becomes

$$X = a(x)\frac{\partial}{\partial x} + (C_2 y + e(x))\frac{\partial}{\partial y} + ((C_1 + C_2)\psi + C_3)\frac{\partial}{\partial \psi}. \tag{2.96}$$

We assume $\psi = \Phi(x, y)$ is a group invariant solution of the partial differential equation (2.84) as outlined in Section 1.6.5 . This means that,

$$X\Big(\psi - \Phi(x, y)\Big)|_{\psi=\Phi(x,y)} = 0, \tag{2.97}$$

where X is defined by (2.96). Thus,

$$\left(a(x)\frac{\partial}{\partial x} + (C_2 y + e(x))\frac{\partial}{\partial y} + ((C_1 + C_2)\psi + C_3)\frac{\partial}{\partial \psi}\right)(\psi - \Phi(x, y))\,|_{\psi=\Phi(x,y)} = 0. \tag{2.98}$$

Equation (2.98) can be written as

$$-a(x)\frac{\partial\Phi}{\partial x} - (C_2 y + e(x))\frac{\partial\Phi}{\partial y} + ((C_1 + C_2)\psi + C_3)|_{\psi=\Phi(x,y)} = 0. \tag{2.99}$$

Imposing $\psi = \Phi(x, y)$ in (2.99) we arrive at the following first order linear partial differential equation

$$a(x)\frac{\partial\Phi}{\partial x} + (C_2 y + e(x))\frac{\partial\Phi}{\partial y} = (C_1 + C_2)\Phi + C_3. \tag{2.100}$$

The differential equations of the characteristic curves of (2.100) are

$$\frac{dx}{a(x)} = \frac{dy}{C_2 y + e(x)} = \frac{d\Phi}{(C_1 + C_2)\Phi + C_3}. \tag{2.101}$$

From the first and second terms in (2.101)

$$R_1 = e^{-C_2 \int \frac{1}{a(x)} dx} y - \int^x \frac{e(x)}{a(x)}\, e^{-C_2 \int \frac{1}{a(x)} dx}\, dx, \tag{2.102}$$

where R_1 is a constant. We define

$$B(x) = \int \frac{1}{a(x)} dx \quad \text{and} \quad D(x) = \int^x \frac{e(x)}{a(x)}\, e^{-C_2 B(x)}\, dx. \tag{2.103}$$

Thus (2.102) becomes

$$R_1 = e^{-C_2 B(x)} y - D(x). \tag{2.104}$$

Consider the case $C_1 + C_2 \neq 0$. Using the first and third terms of (2.101) we obtain

$$R_2 = \left(\Phi + \frac{C_3}{C_1 + C_2}\right) e^{-(C_1+C_2)B(x)}, \tag{2.105}$$

where R_2 is a constant. The general solution is

$$R_2 = F(R_1), \tag{2.106}$$

that is

$$\Phi(x, y) = e^{(C_1+C_2)B(x)} F(e^{-C_2 B(x)} y - D(x)) - \frac{C_3}{C_1 + C_2}, \tag{2.107}$$

where F is an arbitrary function. But $\psi = \Phi(x, y)$ and so the functional form for ψ is

$$\psi = e^{(C_1+C_2)B(x)} F(\eta) - \frac{C_3}{C_1 + C_2}, \qquad \eta = e^{-C_2 B(x)} y - D(x). \tag{2.108}$$

We now solve for the functional form of the eddy viscosity $E(x, y)$ from the linear partial differential equation (2.93) :

$$a(x)\frac{\partial E}{\partial x} + [C_2 y + e(x)]\frac{\partial E}{\partial y} = [2C_2 + C_1 - a'(x)]E, \tag{2.109}$$

which must be satisfied in order for the Lie point symmetries of (2.84) to exist. The differential equations of the characteristic curves of (2.109) are

$$\frac{dx}{a(x)} = \frac{dy}{C_2 y + e(x)} = \frac{dE}{[2C_2 + C_1 - a'(x)]E}. \tag{2.110}$$

The first and second terms of (2.110) are the same as in (2.101) and therefore we have again

$$R_1 = ye^{-C_2 B(x)} - D(x). \tag{2.111}$$

From the first and third terms of (2.110)

$$R_3 = Ea(x)e^{-(2C_2+C_1)B(x)}, \tag{2.112}$$

where R_3 is a constant. The general solution is

$$R_3 = G(R_1), \tag{2.113}$$

that is

$$Ea(x)e^{-(2C_2+C_1)B(x)} = G(e^{-C_2 B(x)}y - D(x)), \tag{2.114}$$

where G is an arbitrary function. Hence the general functional form for $E(x, y)$ is

$$E(x,y) = \frac{1}{a(x)}\, e^{(2C_2+C_1)\, B(x)}\, G(\eta), \qquad \eta = e^{-C_2 B(x)}y - D(x). \tag{2.115}$$

In order for the Lie point symmetries to exist, $W(x)$ must satisfy the first order ordinary differential equation (2.94) :

$$\frac{dW}{dx} + \frac{(a'(x) - 2C_1)}{a(x)}W(x) = 0, \tag{2.116}$$

which is a linear ordinary differential equation which can readily be solved. The solution to (2.116) is

$$W(x) = \frac{C_7}{a(x)}e^{2C_1 B(x)}, \tag{2.117}$$

where C_7 is a constant. From (2.83) we have that

$$W(x) = \bar{U}(x)\frac{d\bar{U}(x)}{dx} = \frac{1}{2}\frac{d}{dx}\Big(\bar{U}^2(x)\Big). \tag{2.118}$$

Thus from (2.117)

$$\frac{1}{2}\frac{d}{dx}\Big(\bar{U}^2(x)\Big) = \frac{C_7}{a(x)}e^{2C_1B(x)}. \tag{2.119}$$

Integrating both sides of (2.119) with respect to x gives

$$\bar{U}^2(x) = 2C_7\int^x \frac{1}{a(x)}\, e^{2C_1B(x)}dx + C_8, \tag{2.120}$$

where C_8 is a constant and therefore

$$\bar{U}(x) = \pm\sqrt{2C_7\int^x \frac{1}{a(x)}\, e^{2C_1B(x)}dx + C_8}, \tag{2.121}$$

which using (2.103) simplifies to

$$\bar{U}(x) = \pm\sqrt{\frac{C_7}{C_1}\, e^{2C_1B(x)} + C_8}, \qquad C_1 \neq 0 \tag{2.122}$$

$$\bar{U}(x) = \pm\sqrt{2C_7B(x) + C_8}, \qquad C_1 = 0 \tag{2.123}$$

In summary, we have the following expressions for the stream function $\psi(x, y)$, the effective viscosity $E(x, y)$, $W(x)$ and the mainstream mean velocity $\bar{U}(x)$ provided $C_1 + C_2 \neq 0$ and $a(x) \neq 0$:

$$\psi(x, y) = e^{(C_1+C_2)B(x)}F(\eta) - \frac{C_3}{C_1 + C_2}, \tag{2.124}$$

$$E(x, y) = \frac{1}{a(x)}\, e^{(2C_2+C_1)\, B(x)}\, G(\eta), \tag{2.125}$$

$$W(x) = \frac{C_7}{a(x)}e^{2C_1B(x)}, \tag{2.126}$$

$$\bar{U}(x) = \pm\sqrt{\frac{C_7}{C_1}\, e^{2C_1B(x)} + C_8}, \qquad C_1 \neq 0 \tag{2.127}$$

$$\bar{U}(x) = \pm\sqrt{2C_7B(x) + C_8}, \qquad C_1 = 0 \tag{2.128}$$

where

$$\eta = e^{-C_2B(x)}y - D(x), \tag{2.129}$$

and where $B(x)$ and $D(x)$ are defined by (2.103). Considering (2.129) we note that in order for $\eta = 0$ when $y = 0$, we require $D(x) = 0$. This can be achieved by setting $e(x) = 0$ since $e(x)$ can be arbitrarily chosen. For comparison $a(x)$ for a laminar jet can be obtained from (2.109) by setting E a constant :

$$a(x) = (C_1 + 2C_2)x + k, \tag{2.130}$$

where k is a constant.

There are three special cases that are apparent at this stage. The first is $a(x) = 0$. The second arises when $C_1 + C_2 = 0$ and the third is if $C_1 = 0$.

We now substitute (2.124), (2.125) and (2.126) into the partial differential equation (2.84)

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - W(x) - \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2 \psi}{\partial y^2}\right) = 0. \tag{2.131}$$

Details of the calculation are presented in Appendix $A.3$ assuming $C_1 + C_2 \neq 0$ and $a(x) \neq 0$. The following third order ordinary differential equation in $F(\eta)$ is found,

$$\frac{d}{d\eta}\left(G\frac{d^2F}{d\eta^2}\right) + (C_1 + C_2)F\frac{d^2F}{d\eta^2} - C_1\left(\frac{dF}{d\eta}\right)^2 + C_7 = 0. \tag{2.132}$$

2.6 Scaling of the Lie point symmetry generator and group invariant solutions : general case $a(x) \neq 0$, $C_1 + C_2 \neq 0$

In this section the Lie point symmetry generator is scaled in order to simplify calculations that arise in the derivation of the turbulent boundary layer solutions. This leads to the scaling of $\psi(x,y)$, $E(x,y)$, $W(x)$ and η which were defined in Section 2.5.

We consider the Lie point symmetry generator (2.96):

$$X = a(x)\frac{\partial}{\partial x} + (C_2 y + e(x))\frac{\partial}{\partial y} + ((C_1 + C_2)\psi + C_3)\frac{\partial}{\partial \psi}. \tag{2.133}$$

Assume that $C_2 \neq 0$. Scaling the generator by dividing through by C_2 gives

$$\frac{X}{C_2} = \frac{a(x)}{C_2}\frac{\partial}{\partial x} + \left(y + \frac{e(x)}{C_2}\right)\frac{\partial}{\partial y} + \left(\left(\frac{C_1}{C_2} + 1\right)\psi + \frac{C_3}{C_2}\right)\frac{\partial}{\partial \psi}. \tag{2.134}$$

Define

$$\begin{gathered}\bar{X} = \frac{X}{C_2}, \quad \bar{a}(x) = \frac{a(x)}{C_2}, \quad \bar{e}(x) = \frac{e(x)}{C_2}, \quad \bar{G}(\eta) = \frac{G(\eta)}{C_2}, \\ \alpha_1 = \frac{C_1}{C_2}, \quad \alpha_2 = \frac{C_3}{C_2}, \quad \bar{C}_7 = \frac{C_7}{C_2}, \quad \bar{C}_8 = C_8\end{gathered} \tag{2.135}$$

which implies that

$$B(x) = \int^x \frac{1}{a(x)}dx = \frac{1}{C_2}\int^x \frac{1}{\bar{a}(x)}dx = \frac{\bar{B}(x)}{C_2}, \tag{2.136}$$

and

$$D(x) = \int^x \frac{e(x)}{a(x)}\, e^{-C_2 B(x)}\, dx = \int^x \frac{\bar{e}(x)}{\bar{a}(x)}\, e^{-\bar{B}(x)}\, dx = \bar{D}(x). \tag{2.137}$$

Substitute (2.135), (2.136) and (2.137) into $\psi(x,y)$, $E(x,y)$, $W(x)$ and η which are given by (2.124) to (2.126) and (2.129). Then for $\alpha_1 \neq -1$ and $\bar{a}(x) \neq 0$,

$$\psi(x,y) = e^{(\alpha_1+1)\bar{B}(x)}F(\eta) - \frac{\alpha_2}{\alpha_1+1}, \tag{2.138}$$

$$E(x,y) = \frac{1}{\bar{a}(x)}\, e^{(2+\alpha_1)\,\bar{B}(x)}\, \bar{G}(\eta), \tag{2.139}$$

$$W(x) = \frac{\bar{C}_7}{\bar{a}(x)} e^{2\alpha_1\bar{B}(x)}, \tag{2.140}$$

$$\bar{U}(x) = \pm\sqrt{\frac{\bar{C}_7}{\alpha_1}\, e^{2\alpha_1\bar{B}(x)} + \bar{C}_8}, \qquad \alpha_1 \neq 0 \tag{2.141}$$

$$\bar{U}(x) = \pm\sqrt{2\bar{C}_7\bar{B}(x) + \bar{C}_8}, \qquad \alpha_1 = 0 \tag{2.142}$$

$$\eta = e^{-\bar{B}(x)}y - \bar{D}(x). \tag{2.143}$$

We note that in (2.143) $\bar{D}(x)$ can be set equal to 0 so that $\eta = 0$ when $y = 0$.

Substituting the transformations (2.135) into the ordinary differential equation (2.132) leads us to :

$$\frac{d}{d\eta}\left(\bar{G}\frac{d^2F}{d\eta^2}\right) + (\alpha_1+1)F\frac{d^2F}{d\eta^2} - \alpha_1\left(\frac{dF}{d\eta}\right)^2 + \bar{C}_7 = 0. \tag{2.144}$$

Transforming the partial differential equation for $E(x,y)$ yields

$$\bar{a}(x)\frac{\partial E}{\partial x} + (y + \bar{e}(x))\frac{\partial E}{\partial y} = (2 + \alpha_1 - \bar{a}'(x))E. \tag{2.145}$$

The Lie point symmetry which generates the solution is

$$\bar{X} = \bar{a}(x)\frac{\partial}{\partial x} + (y + \bar{e}(x))\frac{\partial}{\partial y} + ((\alpha_1+1)\psi + \alpha_2)\frac{\partial}{\partial \psi}. \tag{2.146}$$

2.7 Turbulent boundary layer solutions for given $E(x,y)$: general case $a(x) \neq 0$, $\alpha_1 \neq -1$

In this section we will investigate various forms for the effective viscosity, $E(x,y)$ and we will derive the functional form for the mainstream velocity, $\bar{U}(x)$, using the matching boundary condition with the mainstream flow. The corresponding ordinary differential equation for $F(\eta)$ will be derived. We will also see that the limiting cases lead to the laminar boundary layer solutions.

2.7.1 Power law in x for effective viscosity

In this section we assume that the effective viscosity is a power law in x, that is

$$E(x) = E_0 x^n. \tag{2.147}$$

We know that the effective viscosity must satisfy the first order linear partial differential equation (2.145) :

$$\bar{a}(x)\frac{\partial E}{\partial x} + (y + \bar{e}(x))\frac{\partial E}{\partial y} = (2 + \alpha_1 - \bar{a}'(x))E. \tag{2.148}$$

Substituting (2.147) into (2.148) yields the following ordinary differential equation :

$$\frac{d\bar{a}(x)}{dx} + \frac{n}{x}\bar{a}(x) = 2 + \alpha_1. \tag{2.149}$$

Equation (2.149) is linear and has integrating factor x^n. Thus (2.149) can be written as,

$$\frac{d}{dx}\left(\bar{a}(x)x^n\right) = (2 + \alpha_1)x^n. \tag{2.150}$$

We now integrate both sides with respect to x. This leads us to two different cases.

Case (i) $\mathbf{n \neq -1}$

Integrating equation (2.150) gives us a functional form for $\bar{a}(x)$:

$$\bar{a}(x) = (2 + \alpha_1)\frac{x}{n+1} + \alpha_3 x^{-n}, \tag{2.151}$$

where α_3 is a constant. Thus

$$\bar{B}(x) = \int^x \frac{1}{\bar{a}(x)}dx \tag{2.152}$$

$$= \int \frac{(n+1)x^n}{(2+\alpha_1)x^{n+1} + (n+1)\alpha_3}dx \tag{2.153}$$

$$= \frac{1}{(2+\alpha_1)} \ln\Big((2+\alpha_1)x^{n+1} + (n+1)\alpha_3\Big) + constant, \quad \alpha_1 \neq -2. \tag{2.154}$$

When we take the exponential this additive constant becomes a multiplicative constant which can be incorporated in $F(\eta)$. We therefore take the constant to be zero. We substitute $\bar{a}(x)$ and $\bar{B}(x)$ into (2.139) :

$$E(x, y) = (n+1)x^n\bar{G}(\eta). \tag{2.155}$$

Equating the right hand side of (2.147) and (2.155) gives

$$\bar{G}(\eta) = \frac{E_0}{n+1}. \tag{2.156}$$

Substituting (2.151) and (2.154) into (2.140) for $W(x)$ we obtain

$$W(x) = \bar{C}_7(n+1)(2+\alpha_1)^{\frac{\alpha_1-2}{\alpha_1+2}} x^n \left[x^{n+1} + \frac{(n+1)}{(2+\alpha_1)}\alpha_3\right]^{\frac{\alpha_1-2}{\alpha_1+2}}. \tag{2.157}$$

Define

$$W_0 = \bar{C}_7(n+1)(2+\alpha_1)^{\frac{\alpha_1-2}{\alpha_1+2}}, \quad p = \frac{\alpha_1-2}{\alpha_1+2}. \tag{2.158}$$

Then

$$W(x) = W_0 x^n \left[x^{n+1} + \frac{(n+1)}{(2+\alpha_1)}\alpha_3\right]^p. \tag{2.159}$$

From equation (2.118) we know that

$$W(x) = \bar{U}(x)\frac{d\bar{U}(x)}{dx}. \tag{2.160}$$

Substituting (2.159) into (2.160) and integrating both sides of the equation with respect to x we obatin

$$\bar{U}^2(x) = 2\frac{W_0}{n+1}\int k^p dk + 2C_6, \tag{2.161}$$

where C_6 is an arbitrary constant and

$$k(x) = x^{n+1} + \frac{(n+1)}{(\alpha_1+2)}\alpha_3. \tag{2.162}$$

There are two cases depending on whether $p = -1$ or $p \neq -1$.

Subcase(i a) $\mathbf{n \neq -1,\ p = -1}$:

This case corresponds to $\alpha_1 = 0$ and hence $C_1 = 0$. Equation (2.161) gives

$$\bar{U}(x) = \pm\sqrt{2\frac{W_0}{n+1}\ln\left(x^{n+1} + \frac{(n+1)}{2}\alpha_3\right) + C_8}, \quad C_8 = 2C_6 \tag{2.163}$$

where C_8 is an arbitrary constant. We now impose the mainstream matching condition

$$y = \infty: \qquad \bar{v}_x(x,y) = \bar{U}(x) = \pm\sqrt{2\frac{W_0}{n+1}\ln\left(x^{n+1} + \frac{(n+1)}{2}\alpha_3\right) + C_8} \tag{2.164}$$

But using (2.138) and (2.143) for ψ and η with $\alpha_1 = 0$ we obtain

$$\bar{v}_x(x, y) = \frac{\partial \psi}{\partial y} = \frac{dF}{d\eta}. \tag{2.165}$$

Also $\eta \to \infty$ as $y \to \infty$. Thus (2.164) becomes

$$\eta = \infty : \qquad \frac{dF}{d\eta}(\infty) = \pm\sqrt{2\frac{W_0}{n+1}\ln\left(x^{n+1} + \frac{(n+1)}{2}\alpha_3\right) + C_8}. \tag{2.166}$$

In equation (2.166) the left hand side is constant and the right hand side is a function of x. In order to match the left hand side and right hand side of (2.166) it is necessary that $W_0 = 0$. Thus

$$\frac{dF}{d\eta}(\infty) = \pm\bar{U}_0, \tag{2.167}$$

where $\bar{U}_0 = \sqrt{C_8}$ is an arbitrary constant. This also implies that $\bar{U}(x)$ is constant,

$$\bar{U}(x) = \bar{U}_0. \tag{2.168}$$

Thus by (2.160)

$$W(x) = 0. \tag{2.169}$$

and therefore from (2.157)

$$\bar{C}_7 = 0. \tag{2.170}$$

The results (2.138), (2.139), (2.143) and (2.144) still hold. Substituting $\alpha_1 = 0$, (2.156) and (2.170) into (2.144) we arrive at the following ordinary differential equation :

$$\frac{E_0}{(n+1)}\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0. \tag{2.171}$$

In order to simplify the ordinary differential equation (2.171) and boundary condition (2.167) we make the transformation

$$\xi = A\eta, \qquad F(\eta) = BH(\xi). \tag{2.172}$$

where A and B are constants still to be specified. Equations (2.167) and (2.171) become

$$AB\frac{dH}{d\xi}(\infty) = \pm\bar{U}_0, \tag{2.173}$$

$$\frac{d^3H}{d\xi^3} + \frac{(n+1)}{E_0}\frac{B}{A}H\frac{d^2H}{d\xi^2} = 0. \tag{2.174}$$

We will take $\bar{U}_0 > 0$ and consider only the + sign in (2.173). The case of the negative sign can be treated similarly. Choose

$$AB = \bar{U}_0, \qquad \frac{(n+1)}{E_0}\frac{B}{A} = 1. \tag{2.175}$$

Hence

$$A = \left((n+1)\frac{\bar{U}_0}{E_0} \right)^{\frac{1}{2}}, \qquad B = \left(\frac{E_0 \bar{U}_0}{n+1} \right)^{\frac{1}{2}} \tag{2.176}$$

and (2.173) and (2.174) reduce to

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.177}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} = 0. \tag{2.178}$$

Equation (2.178) is the Blasius equation. The similarity variable ξ is defined by

$$\xi = \left((n+1)\frac{\bar{U}_0}{E_0} \right)^{\frac{1}{2}} \left[\frac{y}{[2x^{n+1} + (n+1)\alpha_3]^{\frac{1}{2}}} - \bar{D}(x) \right] \tag{2.179}$$

and the stream function by

$$\psi(x,y) = \left(\frac{E_0 \bar{U}_0}{n+1} \right)^{\frac{1}{2}} \left[2x^{n+1} + (n+1)\alpha_3 \right]^{\frac{1}{2}} H(\xi) - \alpha_2. \tag{2.180}$$

The solution to the ordinary differential equation will be investigated later in this Section. We now consider equation (2.161) with $p \neq -1$.

Subcase(i b) $\mathbf{n \neq -1,\ p \neq -1}$:

This case corresponds to $\alpha_1 \neq 0$ and hence $C_1 \neq 0$. Integrating (2.161) gives

$$\bar{U}(x) = \pm \sqrt{ \frac{2W_0}{(n+1)(p+1)} \left(x^{n+1} + \frac{(n+1)}{\alpha_1 + 2}\alpha_3 \right)^{p+1} + C_8 }, \tag{2.181}$$

where C_8 is an arbitrary constant. We now impose the mainstream matching condition

$$y = \infty : \qquad \bar{v}_x(x,\infty) = \bar{U}(x) = \pm \sqrt{ \frac{2W_0}{(n+1)(p+1)} \left(x^{n+1} + \frac{(n+1)}{\alpha_1 + 2}\alpha_3 \right)^{p+1} + C_8 } \tag{2.182}$$

But using (2.138), (2.143) and (2.154) it follows that

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = \left((2+\alpha_1)x^{n+1} + (n+1)\alpha_3 \right)^{\frac{(p+1)}{2}} \frac{dF}{d\eta}, \tag{2.183}$$

and $\eta \to \infty$ as $y \to \infty$. Thus as $\eta \to \infty$, (2.182) becomes

$$\left((2+\alpha_1)x^{n+1} + (n+1)\alpha_3\right)^{\frac{p+1}{2}} \frac{dF}{d\eta}(\infty) = \pm\sqrt{\frac{2W_0}{(n+1)(p+1)}\left(x^{n+1} + \frac{(n+1)}{\alpha_1+2}\alpha_3\right)^{p+1} + C_8}. \tag{2.184}$$

In order for the left hand side and right hand side to match we deduce that $C_8 = 0$ and

$$\frac{dF}{d\eta}(\infty) = \pm\sqrt{\frac{2W_0(1-p)^{p+1}}{(n+1)(p+1)4^{p+1}}}, \tag{2.185}$$

Now from (2.181)

$$\bar{U}(x) = \pm\bar{U}_0\left[x^{n+1} + \frac{n+1}{\alpha_1+2}\alpha_3\right]^{\frac{p+1}{2}}. \tag{2.186}$$

where

$$\bar{U}_0 = \sqrt{\frac{2W_0}{(p+1)(n+1)}}. \tag{2.187}$$

Thus

$$W_0 = \frac{1}{2}(p+1)(n+1)\bar{U}_0^2, \tag{2.188}$$

and hence

$$\frac{dF}{d\eta}(\infty) = \pm\bar{U}_0\left(\frac{1-p}{4}\right)^{\frac{p+1}{2}} \tag{2.189}$$

Using equations (2.158) and (2.188) we can solve for $\bar{C}_7$ in terms of $\bar{U}_0$ to obtain

$$\bar{C}_7 = \frac{(p+1)(1-p)^p\bar{U}_0^2}{2^{2p+1}}. \tag{2.190}$$

Using (2.156), (2.158) and (2.190) the differential equation (2.144) becomes

$$\frac{d}{d\eta}\left(\frac{E_0}{n+1}\frac{d^2F}{d\eta^2}\right) + \left(\frac{3+p}{1-p}\right)\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right) - \left(\frac{5+3p}{1-p}\right)\left(\frac{dF}{d\eta}\right)^2 + \frac{(p+1)(1-p)^p\bar{U}_0^2}{2^{2p+1}} = 0. \tag{2.191}$$

We use the following transformations to simplify the ordinary differential equation (2.191) and boundary condition (2.189).

$$\xi = A\eta, \qquad F(\eta) = BH(\xi), \tag{2.192}$$

where A and B are constants not yet specified. Substituting transformations (2.192) into the mainstream matching boundary condition (2.189) we obtain

$$AB\frac{dH}{d\xi}(\infty) = \pm\bar{U}_0\left(\frac{1-p}{4}\right)^{\frac{p+1}{2}}. \tag{2.193}$$

We take $\bar{U}_0 > 0$ and consider only the $+$ sign. The $-$ sign can be treated similarly. Choosing

$$AB = \bar{U}_0 \left(\frac{1-p}{4}\right)^{\frac{p+1}{2}}, \tag{2.194}$$

leads to the transformed boundary condition

$$\frac{dH}{d\xi}(\infty) = 1. \tag{2.195}$$

Transform the equation (2.191) using (2.192). This gives

$$\begin{aligned} \frac{d^3H}{d\xi^3} + \frac{(p+3)(n+1)}{(1-p)E_0}\frac{B}{A}H\frac{d^2H}{d\xi^2} - \frac{2(p+1)(n+1)}{(1-p)E_0}\frac{B}{A}\left(\frac{dH}{d\xi}\right)^2 \\ + \frac{(n+1)(p+1)(1-p)^p\bar{U}_0^2}{2^{2p+1}E_0}\frac{1}{A^3B} = 0, \end{aligned} \tag{2.196}$$

and choose the coefficient of $H\frac{d^2H}{d\xi^2}$ in (2.196) to be 1, that is

$$\frac{(p+3)(n+1)}{(1-p)E_0}\frac{B}{A} = 1. \tag{2.197}$$

Solving for A and B using (2.194) and (2.197) we obtain

$$A = \sqrt{\frac{(n+1)(p+3)\bar{U}_0}{2^{(p+1)}E_0}(1-p)^{\frac{p-1}{2}}}, \quad B = \sqrt{\frac{\bar{U}_0 E_0 (1-p)^{\frac{3+p}{2}}}{(n+1)(p+3)2^{p+1}}} \tag{2.198}$$

where

$$p \neq -3, \qquad \alpha_1 \neq -1, \qquad C_1 + C_2 \neq 0. \tag{2.199}$$

Substituting A and B into (2.196) leads us to the following third order ordinary differential equation

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2(p+1)}{(p+3)}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0. \tag{2.200}$$

The similarity variable ξ is given by

$$\xi = \sqrt{\frac{(n+1)(p+3)\bar{U}_0}{2^{(p+1)}E_0}(1-p)^{\frac{p-1}{2}}}\left[\frac{y}{\left[\frac{4}{1-p}x^{n+1} + (n+1)\alpha_3\right]^{\frac{1-p}{4}}} - \bar{D}(x)\right] \tag{2.201}$$

and the stream function $\psi(x,y)$ is

$$\psi(x,y) = \left[\frac{4}{1-p}x^{n+1} + (n+1)\alpha_3\right]^{\frac{3+p}{4}} \sqrt{\frac{\bar{U}_0 E_0 (1-p)^{\frac{3+p}{2}}}{(n+1)(p+3)2^{p+1}}}H(\xi) - \frac{\alpha_2}{\alpha_1 + 1}. \tag{2.202}$$

We note that when we used the scaling transformation we obtained the ordinary differential equation (2.40)

$$\frac{d^3G}{d\eta^3} + G\frac{d^2G}{d\eta^2} + \frac{2n}{(n+\beta+1)}\left[1 - \left(\frac{dG}{d\eta}\right)^2\right] = 0, \tag{2.203}$$

where

$$E(x,y) = E_0 x^{\beta}, \qquad \bar{U}(x) = \bar{U}_0 x^n. \tag{2.204}$$

In the solution using the Lie point symmetries we obtained (2.200)

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2(p+1)}{(p+3)}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0, \tag{2.205}$$

where

$$E(x,y) = E_0 x^n, \qquad \bar{U}(x) = \bar{U}_0 x^{\frac{1}{2}(n+1)(p+1)}, \tag{2.206}$$

if $\alpha_3 = 0$. Consider the coefficient $\frac{2n}{(n+\beta+1)}$ in (2.203) and replace n by $\frac{1}{2}(n+1)(p+1)$ and β by n. Then

$$\frac{2n}{(n+\beta+1)} = \frac{2(p+1)}{(p+3)} \tag{2.207}$$

Equations (2.203) and (2.205) are therefore consistent.

Case (ii) $\mathbf{n = -1}$

We now go back to equation (2.150) and integrate with respect to x when $n = -1$. Thus

$$\bar{a}(x) = (\alpha_1 + 2)x\ln x + \alpha_5 x, \tag{2.208}$$

where α_5 is an arbitrary constant and from definition (2.136) for $\bar{B}(x)$,

$$\bar{B}(x) = \frac{1}{\alpha_1 + 2}\ \ln\left(\alpha_5 + (\alpha_1 + 2)\ln x\right). \tag{2.209}$$

We substitute (2.208) and (2.209) into (2.139) to obtain

$$E(x,y) = \frac{\bar{G}(\eta)}{x}. \tag{2.210}$$

Equating the right hand side of (2.147) and (2.210) gives

$$\bar{G}(\eta) = E_0, \qquad n = -1. \tag{2.211}$$

Substituting (2.208) and (2.209) into (2.140) for $W(x)$ we obtain

$$W(x) = \frac{\bar{C}_7}{x}(2+\alpha_1)^{\frac{\alpha_1-2}{\alpha_1+2}}\left[\ln x + \frac{\alpha_5}{(2+\alpha_1)}\right]^{\frac{\alpha_1-2}{\alpha_1+2}}. \tag{2.212}$$

Define

$$W_0 = (2+\alpha_1)^{\frac{\alpha_1-2}{\alpha_1+2}} \left(\frac{4}{1-p}\right)^p \bar{C}_7. \tag{2.213}$$

where p is defined in (2.158). Then (2.212) becomes

$$W(x) = \frac{W_0}{x}\left[\ln x + \frac{\alpha_5}{(2+\alpha_1)}\right]^p. \tag{2.214}$$

From equation (2.118) we know that

$$W(x) = \bar{U}(x)\frac{d\bar{U}(x)}{dx}. \tag{2.215}$$

Substituting (2.214) into (2.215) and integrating both sides of the equation with respect to x we see that there are two cases which arise once again.

$$\bar{U}^2(x) = 2W_0\int k^p dk + C_6, \tag{2.216}$$

where C_6 is an arbitrary constant and

$$k(x) = \ln x + \frac{\alpha_5}{(2+\alpha_1)}. \tag{2.217}$$

There are two cases which are apparent at this stage. These cases depend on whether $p=-1$ or $p\neq -1$.

Subcase(ii a) $\mathbf{n=-1,\ p=-1}$:

This case corresponds to $\alpha_1 = 0$ and hence $C_1 = 0$. Equation (2.216) gives

$$\bar{U}(x) = \pm\sqrt{2W_0\ln\left(\ln x + \frac{\alpha_5}{2}\right) + C_6}, \tag{2.218}$$

We now impose the mainstream matching condition

$$y=\infty: \qquad \bar{v}_x(x,\infty) = \bar{U}(x) = \pm\sqrt{2W_0\ln\left(\ln x + \frac{\alpha_5}{2}\right) + C_6}. \tag{2.219}$$

But

$$\bar{v}_x(x,y) = \frac{\partial\psi}{\partial y} = \frac{dF}{d\eta}, \tag{2.220}$$

since $\alpha_1 = 0$ and $\eta\to\infty$ as $y\to\infty$. Thus (2.219) becomes

$$\eta=\infty: \qquad \frac{dF}{d\eta}(\infty) = \pm\sqrt{2W_0\ln\left(\ln x + \frac{\alpha_5}{2}\right) + 2C_9}. \tag{2.221}$$

In equation (2.221) the left hand side is constant and the right hand side is a function of x. In order to match the left hand side and right hand side of (2.221), it is necessary that $W_0 = 0$. Thus

$$\frac{dF}{d\eta}(\infty) = \pm\bar{U}_0, \tag{2.222}$$

where $\bar{U}_0 = \sqrt{C_6}$ is an arbitrary constant. This implies that $\bar{U}(x)$ is constant :

$$\bar{U}(x) = \pm\bar{U}_0. \tag{2.223}$$

and by (2.215)

$$W(x) = 0. \tag{2.224}$$

In order for (2.224) to hold, we see from (2.212) that

$$\bar{C}_7 = 0. \tag{2.225}$$

The results (2.138), (2.139), (2.143) and (2.144) still hold. Substituting $\alpha_1 = 0$, (2.211) and (2.225) into (2.144) we arrive at the following ordinary differential equation :

$$E_0\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0. \tag{2.226}$$

In order to simplify (2.222) and (2.226) we make the transformation

$$\xi = A\eta, \qquad F(\eta) = BH(\xi) \tag{2.227}$$

where A and B are constants still to be determined. Equations (2.222) and (2.226) become

$$AB\frac{dH}{d\xi}(\infty) = \pm\bar{U}_0, \tag{2.228}$$

$$\frac{d^3H}{d\xi^3} + \frac{1}{E_0}H\frac{d^2H}{d\xi^2} = 0. \tag{2.229}$$

We will take $\bar{U}_0 > 0$ and consider only the + sign in (2.228). Choose

$$AB = \bar{U}_0, \qquad \frac{1}{E_0}\frac{B}{A} = 1 \tag{2.230}$$

and therefore

$$A = \left(\frac{\bar{U}_0}{E_0}\right)^{\frac{1}{2}}, \qquad B = (E_0\bar{U}_0)^{\frac{1}{2}} \tag{2.231}$$

Equations (2.228) and (2.229) become

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.232}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} = 0. \tag{2.233}$$

Equation (2.233) is the Blasius equation. The similarity variable is

$$\xi = \left(\frac{\bar{U}_0}{E_0}\right)^{\frac{1}{2}} \left[\frac{y}{(\alpha_5 + 2\ln x)^{\frac{1}{2}}} - \bar{D}(x)\right], \tag{2.234}$$

and the stream function is

$$\psi(x,y) = \Big(E_0 \bar{U}_0(\alpha_5 + 2\ln x)\Big)^{\frac{1}{2}} H(\xi) - \alpha_2. \tag{2.235}$$

Subcase(ii b) $\mathbf{n = -1,\ p \neq -1}$:

This case corresponds to the case when $\alpha_1 \neq 0$ and hence $C_1 \neq 0$. Integrating both sides of (2.216) with respect to x and solving for $\bar{U}(x)$ gives

$$\bar{U}(x) = \pm\sqrt{\frac{2W_0}{(p+1)}\left(\ln x + \frac{\alpha_5}{(2+\alpha_1)}\right)^{p+1} + C_6}, \tag{2.236}$$

We now impose the mainstream matching condition

$$y = \infty : \qquad \bar{v}_x(x,y) = \bar{U}(x) = \pm\sqrt{\frac{2W_0}{(p+1)}\left(\ln x + \frac{\alpha_5}{(2+\alpha_1)}\right)^{p+1} + C_6} \tag{2.237}$$

But using (2.138), (2.143) and (2.209)

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = ((2+\alpha_1)\ln x + \alpha_5)^{\frac{(p+1)}{2}} \frac{dF}{d\eta}, \tag{2.238}$$

and $\eta \to \infty$ as $y \to \infty$. Thus as $\eta \to \infty$, (2.237) becomes

$$((2+\alpha_1)\ln x + \alpha_5)^{\frac{(p+1)}{2}} \frac{dF}{d\eta}(\infty) = \pm\sqrt{\frac{2W_0}{(p+1)}\left(\ln x + \frac{\alpha_5}{(2+\alpha_1)}\right)^{p+1} + C_6}. \tag{2.239}$$

In order for the left hand side and right hand side to match we deduce that $C_6 = 0$ and that

$$\frac{dF}{d\eta}(\infty) = \pm\sqrt{\frac{2W_0(1-p)^{p+1}}{(p+1)4^{p+1}}}, \tag{2.240}$$

Now from (2.236)

$$\bar{U}(x) = \pm\bar{U}_0\left(\ln x + \frac{\alpha_5(1-p)}{4}\right)^{\frac{p+1}{2}}, \tag{2.241}$$

where

$$\bar{U}_0 = \sqrt{\frac{2W_0}{(p+1)}}. \tag{2.242}$$

and therefore

$$\frac{d\bar{U}}{dx} = \pm\bar{U}_0 \frac{p+1}{2x}\left(\ln x + \frac{\alpha_5(1-p)}{4}\right)^{\frac{p-1}{2}}. \tag{2.243}$$

Using (2.214) and (2.215) we find that

$$W_0 = \frac{1}{2}(p+1)\bar{U}_0^2, \tag{2.244}$$

and hence

$$\frac{dF}{d\eta}(\infty) = \pm\bar{U}_0\left(\frac{1-p}{4}\right)^{\frac{p+1}{2}}. \tag{2.245}$$

Using equations (2.213) and (2.244) we can solve for $\bar{C}_7$ in terms of $\bar{U}_0$. This gives

$$\bar{C}_7 = \frac{(p+1)(1-p)^p\bar{U}_0^2}{2^{2p+1}}. \tag{2.246}$$

Using (2.211), (2.213) and (2.246) the differential equation (2.144) becomes

$$E_0\frac{d}{d\eta}\left(\frac{d^2F}{d\eta^2}\right) + \left(\frac{3+p}{1-p}\right)\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right) - \left(\frac{5+3p}{1-p}\right)\left(\frac{dF}{d\eta}\right)^2 + \frac{(p+1)(1-p)^p\bar{U}_0^2}{2^{2p+1}} = 0. \tag{2.247}$$

Using the following transformations we simplify the boundary condition (2.245) and the ordinary differential equation (2.247) :

$$\xi = A\eta, \qquad F(\eta) = BH(\xi). \tag{2.248}$$

Substituting transformations (2.248) into the mainstream matching boundary condition (2.245) gives

$$AB\frac{dH}{d\xi}(\infty) = \pm\bar{U}_0\left(\frac{1-p}{4}\right)^{\frac{p+1}{2}}, \tag{2.249}$$

We take $\bar{U}_0 > 0$ and consider only the + sign. The − sign can be treated similarly. Choosing

$$AB = \bar{U}_0\left(\frac{1-p}{4}\right)^{\frac{p+1}{2}}, \tag{2.250}$$

leads to the transformed boundary condition

$$\frac{dH}{d\xi}(\infty) = 1. \tag{2.251}$$

Transform the equation using (2.248). We obtain

$$\frac{d^3H}{d\xi^3} + \frac{(p+3)}{(1-p)E_0}\frac{B}{A}H\frac{d^2H}{d\xi^2} - \frac{2(p+1)}{(1-p)E_0}\frac{B}{A}\left(\frac{dH}{d\xi}\right)^2 + \frac{(p+1)(1-p)^p\bar{U}_0^2}{2^{2p+1}E_0}\frac{1}{A^3B} = 0 \tag{2.252}$$

and choose the coefficient of $H\frac{d^2H}{d\xi^2}$ in (2.252) to be 1, that is

$$\frac{(p+3)}{(1-p)E_0}\frac{B}{A} = 1. \tag{2.253}$$

Solving for A and B using (2.250) and (2.253) gives

$$A = \sqrt{\frac{\bar{U}_0(p+3)}{E_0 2^{p+1}}(1-p)^{\frac{p-1}{2}}}, \quad B = \sqrt{\frac{\bar{U}_0 E_0}{(p+3)2^{p+1}}(1-p)^{\frac{p+3}{2}}} \tag{2.254}$$

Substituting A and B into (2.252) leads to the following third order ordinary differential equation

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2(p+1)}{(p+3)}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0. \tag{2.255}$$

We note that this is the same differential equation that we obtained for the case $n \neq -1$ and $p \neq -1$. The only difference is the form of the mainstream velocity, $\bar{U}(x)$. Using (2.138), (2.143) and (2.209) the similarity variable is

$$\xi = \left[\frac{\bar{U}_0(p+3)}{E_0 2^{p+1}}(1-p)^{\frac{p-1}{2}}\right]^{\frac{1}{2}} \left[\frac{y}{\left[\alpha_5 + \frac{4\ln x}{1-p}\right]^{\frac{1-p}{4}}} - \bar{D}(x)\right] \tag{2.256}$$

and using (2.138) and (2.209) the stream function is

$$\psi(x,y) = \left[\frac{\bar{U}_0 E_0}{(p+3)2^{p+1}}(1-p)^{\frac{p+3}{2}}\right]^{\frac{1}{2}} \left[\left[\alpha_5 + \frac{4}{(1-p)}\ln x\right]^{\frac{3+p}{4}} H(\xi) - \frac{\alpha_2}{\alpha_1+1}\right] \tag{2.257}$$

Since $\bar{U}(x)$ is given by (2.241), define

$$m = \frac{1}{2}(p+1). \tag{2.258}$$

Then (2.241) becomes (taking the + sign)

$$\bar{U}(x) = \bar{U}_0\left[\ln x + \frac{\alpha_5(1-p)}{4}\right]^m \tag{2.259}$$

and (2.255) reduces to the Falkner-Skan equation

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2m}{(1+m)}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0. \tag{2.260}$$

The mainstream velocity $\bar{U}(x)$ is not a power law and therefore cannot be compared to the scaling transformation solution.

We now make a final transformation of the parameters to write the results in a more convenient form. For simplicity we take $\bar{D}(x) = 0$ and $\alpha_2 = 0$ because they play no part in the solution. Choosing $\bar{D}(x) = 0$ means that $\xi = 0$ when $y = 0$.

In summary when the effective viscosity, $E(x)$, is a power law in x we have the following results :

<u>n = −1</u> :

$$E(x) = \frac{E_0}{x}, \tag{2.261}$$

p = −1 ($\alpha_1 = 0$) :

$$\bar{U}(x) = \bar{U}_0, \tag{2.262}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.263}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} = 0. \tag{2.264}$$

$$\xi = \left(\frac{\bar{U}_0}{2E_0}\right)^{\frac{1}{2}} \frac{y}{\left(\ln\frac{x}{x_0}\right)^{\frac{1}{2}}}, \tag{2.265}$$

$$\psi(x, y) = (2E_0\bar{U}_0)^{\frac{1}{2}} \left[\ln\frac{x}{x_0}\right]^{\frac{1}{2}} H(\xi). \tag{2.266}$$

where

$$\ln x_0 = -\frac{\alpha_5}{2}. \tag{2.267}$$

p ≠ −1 ($\alpha_1 \neq 0$) :

$$\bar{U}(x) = \bar{U}_0\left[\ln\left(\frac{x}{x_0}\right)\right]^m, \tag{2.268}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.269}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2m}{m+1}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0. \tag{2.270}$$

$$\xi = \left[\frac{\bar{U}_0(m+1)}{2E_0}\right]^{\frac{1}{2}} \frac{y}{\left[\ln\frac{x}{x_0}\right]^{\frac{1}{2}(1-m)}} \tag{2.271}$$

$$\psi(x, y) = \left[\frac{2\bar{U}_0E_0}{m+1}\right]^{\frac{1}{2}} \left[\ln\left(\frac{x}{x_0}\right)\right]^{\frac{1}{2}(m+1)} H(\xi) \tag{2.272}$$

where

$$\ln x_0 = -\frac{1}{4}(1-p)\alpha_5, \qquad m = \frac{p+1}{2}. \tag{2.273}$$

<u>n ≠ −1</u> :

$$E(x) = E_0x^n, \tag{2.274}$$

$\mathbf{p = -1}$ $(\alpha_1 = \mathbf{0})$:

$$\bar{U}(x) = \bar{U}_0, \tag{2.275}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.276}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} = 0. \tag{2.277}$$

$$\xi = \left(\frac{\bar{U}_0(n+1)}{2E_0}\right)^{\frac{1}{2}} \frac{y}{[x^{n+1} + x_0^{n+1}]^{\frac{1}{2}}} \tag{2.278}$$

$$\psi(x,y) = \left(\frac{2E_0\bar{U}_0}{n+1}\right)^{\frac{1}{2}} \left[x^{n+1} + x_0^{n+1}\right]^{\frac{1}{2}} H(\xi) \tag{2.279}$$

where

$$x_0^{n+1} = -\frac{1}{2}(n+1)\alpha_3. \tag{2.280}$$

$\mathbf{p \neq -1}$ $(\alpha_1 \neq \mathbf{0})$:

$$\bar{U}(x) = \bar{U}_0 \left[x^{n+1} + x_0^{n+1}\right]^{\frac{m}{n+1}}, \tag{2.281}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.282}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2m}{m+n+1}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0. \tag{2.283}$$

$$\xi = \left[\frac{(m+n+1)\bar{U}_0}{2E_0}\right]^{\frac{1}{2}} \frac{y}{\left[x^{n+1} - x_0^{n+1}\right]^{\frac{n+1-m}{2(n+1)}}} \tag{2.284}$$

$$\psi(x,y) = \left[\frac{2\bar{U}_0E_0}{(n+m+1)}\right]^{\frac{1}{2}} \left[x^{n+1} - x_0^{n+1}\right]^{\frac{m+n+1}{2(n+1)}} H(\xi). \tag{2.285}$$

where

$$x_0^{n+1} = -\frac{1}{4}(n+1)(1-p)\alpha_3, \qquad \frac{p+1}{2} = \frac{m}{n+1} \tag{2.286}$$

2.7.2 Numerical solution for $n \neq -1$ and $p \neq -1$

We will now consider the numerical solution for flow past a flat plate for the case $n \neq -1$ where $p = -1$. Choose $\bar{D}(x) = 0$ so that $\xi = 0$ when $y = 0$. Choose $\alpha_3 = 0$ so that $x = 0$ is the leading edge and take $\alpha_2 = 0$. Thus

$$E(x) = E_0x^n, \tag{2.287}$$

$$\bar{U}(x) = \bar{U}_0, \tag{2.288}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.289}$$

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} = 0, \tag{2.290}$$

$$\xi = \left[\frac{\bar{U}_0(n+1)}{2E_0}\right]^{\frac{1}{2}} \frac{y}{x^{\frac{1}{2}(n+1)}}, \tag{2.291}$$

$$\psi(x,y) = \left[\frac{2E_0\bar{U}_0}{n+1}\right]^{\frac{1}{2}} x^{\frac{1}{2}(n+1)}H(\xi). \tag{2.292}$$

Now

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = \bar{U}_0\frac{dH}{d\xi} \tag{2.293}$$

and

$$\bar{v}_y(x,y) = -\frac{\partial \psi}{\partial x} = \left(\frac{E_0\bar{U}_0(n+1)}{2}\right)^{\frac{1}{2}} x^{\frac{1}{2}(n-1)}\left[-H(\xi) + \xi\frac{dH}{d\xi}\right]. \tag{2.294}$$

The flat plate is at $y = 0$, $x \geq 0$. Consider the boundary conditions. First we impose the no slip boundary condition at $y = 0$:

$$\bar{v}_x(x,0) = 0. \tag{2.295}$$

When $y = 0$, then $\xi = 0$ and (2.293) gives

$$\frac{dH}{d\xi}(0) = 0. \tag{2.296}$$

We also impose the no suction or blowing condition at the boundary :

$$\bar{v}_y(x,0) = 0. \tag{2.297}$$

Using (2.294) we see that when $y = 0$, that is $\xi = 0$, then

$$H(0) = 0, \tag{2.298}$$

as $n \neq -1$, $E_0 \neq 0$ and $\bar{U}_0 \neq 0$. The mainstream matching condition, $\bar{v}_x(x,\infty) = \bar{U}_0$, has already been imposed. It yields the boundary condition (2.289) :

$$\frac{dH}{d\xi}(\infty) = 1. \tag{2.299}$$

Thus we must solve the Blasius equation, (2.290) using the three boundary conditions (2.296), (2.298) and (2.299). The analytical solution for the Blasius equation is not known. We will solve the equation numerically by converting the boundary value problem into two initial value problems ([17], [18], [22], [23]).

Consider the Blasius equation (2.290),

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} = 0. \tag{2.300}$$

Using the scaling transformation

$$\bar{\xi} = \lambda^a \xi, \qquad \bar{H} = \lambda^b H \tag{2.301}$$

equation (2.300) becomes

$$\frac{d^3\bar{H}}{d\bar{\xi}^3} + \lambda^{-a-b}\bar{H}\frac{d^2\bar{H}}{d\bar{\xi}^2} = 0. \tag{2.302}$$

But (2.300) is invariant under the scaling transformation (2.301) if

$$\frac{d^3\bar{H}}{d\bar{\xi}^3} + \bar{H}\frac{d^2\bar{H}}{d\bar{\xi}^2} = 0. \tag{2.303}$$

Thus (2.300) is invariant under the scaling transformation (2.301) if

$$b = -a, \tag{2.304}$$

Thus (2.300) is invariant under the transformation

$$\bar{\xi} = \lambda^a \xi, \qquad \bar{H} = \lambda^{-a} H, \tag{2.305}$$

and therefore

$$H(\xi) = \lambda^a \bar{H}(\lambda^a \xi). \tag{2.306}$$

We can either write $\lambda^a = \mu$ or let $a = 1$. Let $a = 1$:

$$H(\xi) = \lambda \bar{H}(\lambda \xi). \tag{2.307}$$

Thus

$$\frac{dH}{d\xi} = \lambda^2 \frac{d\bar{H}}{d\bar{\xi}}, \qquad \frac{d^2H}{d\xi^2} = \lambda^3 \frac{d^2\bar{H}}{d\bar{\xi}^2}. \tag{2.308}$$

Consider now the boundary conditions (2.296), (2.298) and (2.299). Using (2.307) and (2.308) we have

$$\bar{H}(0) = 0, \tag{2.309}$$

$$\frac{d\bar{H}}{d\bar{\xi}}(0) = 0, \tag{2.310}$$

$$\frac{d\bar{H}}{d\bar{\xi}}(\infty) = \frac{1}{\lambda^2}, \tag{2.311}$$

Choose

$$\frac{d^2\bar{H}}{d\bar{\xi}^2}(0) = 1, \tag{2.312}$$

Then

$$\frac{d^2 H}{d\xi^2}(0) = \lambda^3, \tag{2.313}$$

The boundary value problem is replaced by the following two initial value problems.

<u>Initial value problem 1</u>

$$\frac{d^3\bar{H}}{d\bar{\xi}^3} + \bar{H}\frac{d^2\bar{H}}{d\bar{\xi}^2} = 0. \tag{2.314}$$

$$\bar{H}(0) = 0, \tag{2.315}$$

$$\frac{d\bar{H}}{d\bar{\xi}}(0) = 0, \tag{2.316}$$

$$\frac{d^2\bar{H}}{d\bar{\xi}^2}(0) = 1. \tag{2.317}$$

The solution for $\bar{H}(\bar{\xi})$ is then used to calculate

$$\lambda = \frac{1}{\left(\frac{d\bar{H}}{d\bar{\xi}}(\infty)\right)^{\frac{1}{2}}}. \tag{2.318}$$

<u>Initial value problem 2</u>

$$\frac{d^3 H}{d\xi^3} + H\frac{d^2 H}{d\xi^2} = 0. \tag{2.319}$$

$$H(0) = 0, \tag{2.320}$$

$$\frac{dH}{d\xi}(0) = 0, \tag{2.321}$$

$$\frac{d^2 H}{d\xi^2}(0) = \lambda^3, \tag{2.322}$$

where λ is calculated from (2.318) in Problem 1. Problem 1 gives only λ. Problem 2 gives the solution $H(\xi)$ of the Blasius equation. Mathematica was used to solve Problem 1 and obtain λ. It was found that $\lambda = 0.747839$. Problem 2 was then solved using Mathematica with λ given by Problem 1. The velocity profile was obtained by plotting $\frac{dH}{d\xi}$ against ξ.

We see in Figure 2.1 that as $\xi \to \infty$, $\frac{dH}{d\xi}(\infty) \to 1$ in agreement with the main-stream matching boundary condition (2.299).

We now estimate the turbulent boundary layer thickness. From Figure 2.1, the graph of the numerical solution of $\frac{\bar{v}_x}{U_0}$ against ξ shows that $\frac{\bar{v}_x}{U_0} \approx 1$ when $\xi = O(1)$, that is using (2.291), when

$$y = O\left(\left(\frac{2E_0 x^{n+1}}{\bar{U}_0(n+1)}\right)^{\frac{1}{2}}\right). \tag{2.323}$$

Let

$$\delta(x) = \textit{ thickness of boundary layer at } x. \tag{2.324}$$

Then

$$\delta(x) = O\left(\left(\frac{2E_0 x^{n+1}}{\bar{U}_0(n+1)}\right)^{\frac{1}{2}}\right). \tag{2.325}$$

A choice sometimes made is

$$\delta(x) = \left(\frac{2E_0 x^{n+1}}{\bar{U}_0(n+1)}\right)^{\frac{1}{2}}. \tag{2.326}$$

The width of the boundary layer is proportional to $E_0^{\frac{1}{2}}$. The increase in the effective viscosity causes an increase in diffusion effects in the mean flow which leads to an increase in the boundary layer thickness. The thickness of the boundary layer is proportional to $x^{\frac{1}{2}(n+1)}$ where x is the distance from the leading edge. For laminar flow, $n = 0$, and the boundary layer thickness is parabolic since $\delta^2(x) \propto x$.

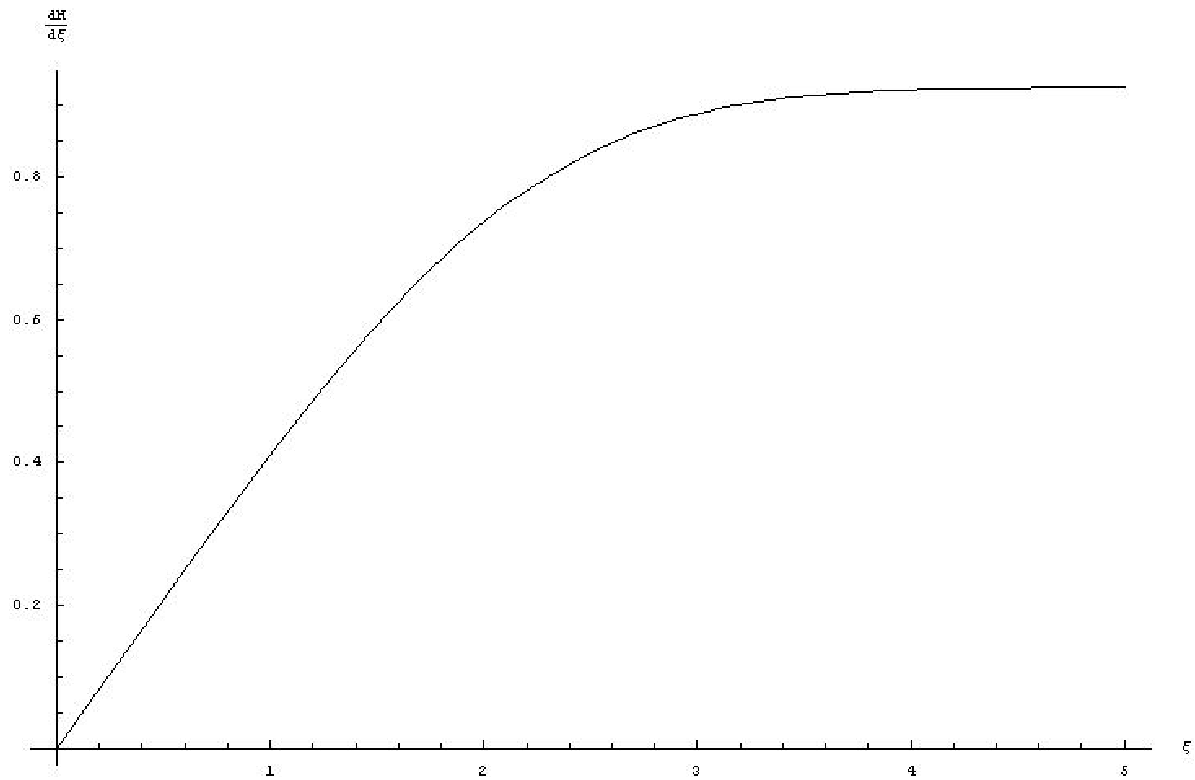

Figure 2.1: Two-dimensional turbulent boundary layer on a flat plate

2.7.3 Eddy viscosity of the form $E(x,y) = m(x)y^{\frac{6}{7}}$

In this section we assume that the eddy viscosity has the following form :

$$E(x,y) = m(x)y^{\frac{6}{7}}. \tag{2.327}$$

where $m(x)$ has still to be specified. We will assume that $m(x) \neq 0$ since the fluid is viscous and turbulent.

The motivation for this form for $E(x,y)$ is as follows [11]. Blasius (1913) proposed the empirical formula, based on an investigation of turbulent pipe flow,

$$\tau_{yx}(x,0) = 0.0225\rho U^2 \left(\frac{\nu}{Ua}\right)^{\frac{1}{4}}, \tag{2.328}$$

where U is the maximum mean velocity and a is the radius of the pipe. Since $\tau_{yx}(x,0)$ is governed by conditions near the wall it has been argued that U and a may be replaced by $\bar{v}_x(x,y)$ and y. Thus near the boundary

$$\tau_{yx}(x,y) = 0.0225\rho\bar{v}_x^2(x,y)\left(\frac{\nu}{\bar{v}_x(x,y)y}\right)^{\frac{1}{4}}. \tag{2.329}$$

Since the left hand side of (2.329) tends to a finite non-zero limit as $y \to 0$ it follows that near the boundary for fixed x,

$$\bar{v}_x(x,y) \propto y^{\frac{1}{7}}. \tag{2.330}$$

Now from (1.22) and (1.26)

$$\tau_{xy}(x,y) = \rho E(x,y)\left(\frac{\partial \bar{v}_x}{\partial y} + \frac{\partial \bar{v}_y}{\partial x}\right) \tag{2.331}$$

To investigate which term in (2.331) can be neglected in the boundary layer, make (2.331) dimensionless using (1.37). Equation (2.331) becomes

$$\tau^*_{xy} = \frac{E^*}{\sqrt{Re}}\left(\frac{\partial \bar{v}^*_x}{\partial y^*} + \frac{1}{Re}\frac{\partial \bar{v}^*_y}{\partial x^*}\right) \tag{2.332}$$

In the boundary layer, $Re >> 1$ and therefore (2.332) approximates to

$$\tau^* = \frac{E^*}{\sqrt{Re}}\frac{\partial \bar{v}^*_x}{\partial y^*}. \tag{2.333}$$

Hence in the boundary layer, approximation (2.331) reduces to

$$\tau_{xy}(x,y) = \rho E(x,y)\frac{\partial \bar{v}_x}{\partial y}(x,y) \tag{2.334}$$

Using again the observation that $\tau_{xy}(x,y)$ tends to a finite non-zero value as $y \to 0$ for fixed x and also equation (2.330) it follows from (2.334) that

$$E(x,y) \propto y^{\frac{6}{7}} \tag{2.335}$$

Thus we choose

$$E(x,y) = m(x)y^{\frac{6}{7}} \tag{2.336}$$

where m(x) is still to be specified. We will assume that (2.336) holds throughout the whole width of the boundary layer even although $E(x,y) \to \infty$ as $y \to \infty$.

We know that the effective viscosity must satisfy the first order quasi-linear partial differential equation (2.145) :

$$\bar{a}(x)\frac{\partial E}{\partial x} + (y + \bar{e}(x))\frac{\partial E}{\partial y} = (2 + \alpha_1 - \bar{a}'(x))E. \tag{2.337}$$

Substituting (2.327) into (2.337) and equating coefficients of y^0 and y^1 yields the following results :

$\mathbf{y^0}$:

$$\bar{e}(x) = 0, \tag{2.338}$$

$\mathbf{y^1}$:

$$\frac{d\bar{a}}{dx} + \frac{m'(x)}{m(x)}\bar{a}(x) = \alpha_1 + \frac{8}{7}. \tag{2.339}$$

Solving (2.339) with general $m(x)$ gives

$$\bar{a}(x) = \frac{(\alpha_1 + \frac{8}{7})}{m(x)}(M(x) + C), \tag{2.340}$$

where C is a constant and

$$M(x) = \int^x m(x)dx, \qquad \frac{dM}{dx} = m(x). \tag{2.341}$$

From (2.136),

$$\bar{B}(x) = \int^x \frac{dx}{\bar{a}(x)} = \frac{1}{(\alpha_1 + \frac{8}{7})} \ln{(M(x) + C)} + constant, \tag{2.342}$$

provided $\alpha_1 + \frac{8}{7} \neq 0$. Again we take the *constant* to be zero because, if it is non-zero, the resulting multiplicative constant can be incorporated into $F(\eta)$. Consider first $\bar{G}(\eta)$. Substituting $\bar{a}(x)$ and $\bar{B}(x)$ into (2.139),

$$E(x,y) = \frac{m(x)}{\alpha_1 + \frac{8}{7}}[M(x) + C]^{\frac{\frac{6}{7}}{\alpha_1 + \frac{8}{7}}}\bar{G}(\eta). \tag{2.343}$$

Equating the right hand sides of (2.327) and (2.343) we obtain

$$\bar{G}(\eta) = \left(\alpha_1 + \frac{8}{7}\right) y^{\frac{6}{7}} [M(x) + C]^{\frac{-6}{7\alpha_1+8}}. \qquad (2.344)$$

But using (2.143) for η we have

$$\eta = \frac{y}{[M(x) + C]^{\frac{1}{\alpha_1+\frac{8}{7}}}} - \bar{D}(x) \qquad (2.345)$$

and therefore

$$\bar{G}(\eta) = \left(\alpha_1 + \frac{8}{7}\right) \eta^{\frac{6}{7}}, \qquad \bar{D}(x) = 0. \qquad (2.346)$$

Consider next $\bar{U}(x)$. Substituting (2.340) and (2.342) into (2.140) for $W(x)$ we obtain

$$W(x) = \frac{\bar{C}_7}{\alpha_1 + \frac{8}{7}} m(x) \left(1 - \frac{7\alpha_1 - 8}{7\alpha_1 + 8}\right) [M(x) + C]^{\frac{\alpha_1 - \frac{8}{7}}{\alpha_1 + \frac{8}{7}}}. \qquad (2.347)$$

Define

$$p = \frac{7\alpha_1 - 8}{7\alpha_1 + 8}, \qquad W_0 = \frac{7}{16}(1-p)\bar{C}_7 \qquad (2.348)$$

The definition of p in this section is different from p in Section 2.7.1. Then

$$W(x) = W_0 m(x) [M(x) + C]^p. \qquad (2.349)$$

But from (2.118),

$$W(x) = \frac{d}{dx}\left(\frac{1}{2}\bar{U}^2\right) \qquad (2.350)$$

and substituting (2.349) into (2.350) we obtain

$$\bar{U}^2(x) = 2W_0 \int^x k^p dk + C_6, \qquad (2.351)$$

where

$$k(x) = M(x) + C. \qquad (2.352)$$

There are two cases which depend on whether $p = -1$ or $p \neq -1$. The case $p = 1$ is not possible.

Case (i) $\mathbf{p = -1}$

This case corresponds to $\alpha_1 = 0$ and hence $C_1 = 0$. Equation (2.351) gives

$$\bar{U}(x) = \pm\sqrt{2W_0 \ln(M(x) + C) + C_6}, \qquad (2.353)$$

We now impose the mainstream matching condition

$$y = \infty : \qquad \bar{v}_x(x, \infty) = \bar{U}(x) = \pm\sqrt{2W_0 \ln(M(x) + C) + C_6}. \tag{2.354}$$

But from (2.138) and (2.143) and since $\alpha_1 = 0$,

$$\bar{v}_x(x, y) = \frac{\partial \psi}{\partial y} = \frac{dF}{d\eta}, \tag{2.355}$$

and $\eta \to \infty$ as $y \to \infty$. Thus (2.354) becomes

$$\eta = \infty : \qquad \frac{dF}{d\eta}(\infty) = \pm\sqrt{2W_0 \ln(M(x) + C) + C_6}. \tag{2.356}$$

The left hand side of (2.356) is constant and if $W_0 \neq 0$ the right hand side is a function of x. In order to match the left hand side and right hand side of (2.356), we set $W_0 = 0$. Thus

$$\frac{dF}{d\eta}(\infty) = \pm\bar{U}_0, \tag{2.357}$$

where $\bar{U}_0 = \sqrt{C_6}$ is an arbitrary constant. This implies that $\bar{U}(x)$ is constant :

$$\bar{U}(x) = \pm\bar{U}_0. \tag{2.358}$$

Since $W_0 = 0$ we have that

$$W(x) = 0. \tag{2.359}$$

and

$$\bar{C}_7 = 0. \tag{2.360}$$

Substituting $\alpha_1 = 0$, (2.346) and (2.360) into (2.144) we arrive at the following ordinary differential equation for $F(\eta)$:

$$\frac{d}{d\eta}\left(\eta^{\frac{6}{7}} \frac{d^2F}{d\eta^2}\right) + \frac{7}{8} F \frac{d^2F}{d\eta^2} = 0. \tag{2.361}$$

Using the transformations

$$\xi = A\eta, \qquad F(\eta) = BH(\xi) \tag{2.362}$$

we simplify the boundary condition (2.357) and the ordinary differential equation (2.361). Using (2.362), (2.357) and (2.361) become

$$AB\frac{dH}{d\xi}(\infty) = \pm\bar{U}_0, \tag{2.363}$$

$$\frac{d}{d\xi}\left(\xi^{\frac{6}{7}} \frac{d^2H}{d\xi^2}\right) + \frac{7}{8} H \frac{d^2H}{d\xi^2} = 0. \tag{2.364}$$

We will take $\bar{U}_0 > 0$ and consider only the $+$ sign in (2.363). The negative sign can be treated similarly. Choose

$$AB = \bar{U}_0, \qquad \frac{7}{8} B A^{-\frac{1}{7}} = 1. \tag{2.365}$$

Thus

$$A = \left(\frac{7}{8}\bar{U}_0\right)^{\frac{7}{8}}, \qquad B = \left(\frac{8}{7}\right)^{\frac{7}{8}} \bar{U}_0^{\frac{1}{8}} \tag{2.366}$$

and (2.363) and (2.364) become

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.367}$$

$$\frac{d}{d\xi}\left(\xi^{\frac{6}{7}} \frac{d^2 H}{d\xi^2}\right) + H \frac{d^2 H}{d\xi^2} = 0. \tag{2.368}$$

The similarity variable ξ is

$$\xi = \left(\frac{7\bar{U}_0}{8(M(x)+C)}\right)^{\frac{7}{8}} y, \tag{2.369}$$

and the stream function is

$$\psi(x,y) = \left(\frac{8}{7}\bar{U}_0^{\frac{1}{7}}(M(x)+C)\right)^{\frac{7}{8}} H(\xi) - \alpha_2. \tag{2.370}$$

Case (ii) p $\neq -1$

This case corresponds to $\alpha_1 \neq 0$ and hence $C_1 \neq 0$. Integrating both sides of (2.351) and solving for $\bar{U}(x)$ gives

$$\bar{U}(x) = \pm\sqrt{\frac{2W_0}{(p+1)}(M(x)+C)^{p+1} + C_6}, \tag{2.371}$$

We now impose the mainstream matching condition

$$y = \infty: \qquad \bar{v}_x(x,\infty) = \bar{U}(x) = \pm\sqrt{\frac{2W_0}{(p+1)}(M(x)+C)^{p+1} + C_6}. \tag{2.372}$$

But from (2.138) and (2.143),

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = [M(x)+C]^{\frac{(p+1)}{2}} \frac{dF}{d\eta}, \tag{2.373}$$

and $\eta \to \infty$ as $y \to \infty$. Thus (2.372) becomes

$$[M(x)+C]^{\frac{(p+1)}{2}} \frac{dF}{d\eta}(\infty) = \pm\sqrt{\frac{2W_0}{(p+1)}(M(x)+C)^{p+1} + C_6}. \tag{2.374}$$

In order for the left hand side and right hand side to match we deduce that $C_6 = 0$ and

$$\frac{dF}{d\eta}(\infty) = \pm\sqrt{\frac{2W_0}{(p+1)}}. \tag{2.375}$$

Now from (2.371),

$$\bar{U}(x) = \pm\bar{U}_0[M(x)+C]^{\frac{p+1}{2}}. \tag{2.376}$$

where

$$\bar{U}_0 = \sqrt{\frac{2W_0}{(p+1)}}. \tag{2.377}$$

Thus, from (2.375)

$$\frac{dF}{d\eta}(\infty) = \pm\bar{U}_0. \tag{2.378}$$

and solving (2.377) for W_0 gives

$$W_0 = \frac{1}{2}(1+p)\bar{U}_0^2 \tag{2.379}$$

Using (2.348) and (2.379) we can solve for $\bar{C}_7$ in terms of $\bar{U}_0$:

$$\bar{C}_7 = \frac{8\bar{U}_0^2(p+1)}{7(1-p)}. \tag{2.380}$$

Using (2.346), (2.348) and (2.380) the differential equation (2.144) becomes

$$\frac{d}{d\eta}\left(\eta^{\frac{6}{7}}\frac{d^2F}{d\eta^2}\right) + \frac{(15+p)}{16}\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right) - \frac{(23+9p)}{16}\left(\frac{dF}{d\eta}\right)^2 + \frac{1}{2}(1+p)\bar{U}_0^2 = 0. \tag{2.381}$$

Using the following transformations we simplify the boundary condition (2.378) and the ordinary differential equation (2.381) :

$$\xi = A\eta, \qquad F(\eta) = BH(\xi). \tag{2.382}$$

Substituting transformations (2.382) into the mainstream matching boundary condition (2.378) gives

$$AB\frac{dH}{d\xi}(\infty) = \pm\bar{U}_0. \tag{2.383}$$

We will take $\bar{U}_0 > 0$ and consider only the + sign in (2.383). Choosing

$$AB = \bar{U}_0, \tag{2.384}$$

leads to the transformed boundary condition

$$\frac{dH}{d\xi}(\infty) = 1. \tag{2.385}$$

The transformation of the differential equation (2.381) is

$$\frac{d}{d\xi}\left(\xi^{\frac{6}{7}}\frac{d^2H}{d\xi^2}\right) + \frac{1}{16}(p+15)\frac{B}{A^{\frac{1}{7}}}H\frac{d^2H}{d\xi^2} - \frac{1}{2}(p+1)\frac{B}{A^{\frac{1}{7}}}\left(\frac{dH}{d\xi}\right)^2 + \frac{1}{2}(p+1)\bar{U}_0^2\frac{1}{A^{\frac{15}{7}}B} = 0, \tag{2.386}$$

and choose the coefficient of $H\frac{d^2H}{d\xi^2}$ in (2.386) to be unity, that is

$$\frac{1}{16}(p+15)\frac{B}{A^{\frac{1}{7}}} = 1. \tag{2.387}$$

Solving for A and B using (2.384) and (2.387) gives

$$A = \left[\frac{\bar{U}_0(p+15)}{16}\right]^{\frac{7}{8}}, \quad B = \left[\bar{U}_0\left(\frac{16}{p+15}\right)^7\right]^{\frac{1}{8}}. \tag{2.388}$$

Substituting A and B into (2.386) leads us to the following third order ordinary differential equation for $H(\xi)$:

$$\frac{d}{d\xi}\left(\xi^{\frac{6}{7}}\frac{d^2H}{d\xi^2}\right) + H\frac{d^2H}{d\xi^2} + \frac{8(p+1)}{(p+15)}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0. \tag{2.389}$$

The similarity variable is

$$\xi = \left[\frac{(15+p)}{16}\bar{U}_0\right]^{\frac{7}{8}} \frac{y}{[M(x)+C]^{\frac{7}{16}(1-p)}} \tag{2.390}$$

and the stream function is

$$\psi(x,y) = \left[\bar{U}_0\left(\frac{16}{p+15}\right)^7\right]^{\frac{1}{8}} (M(x)+C)^{\frac{1}{16}(15+p)}H(\xi) - \frac{7(1-p)}{(15+p)}\alpha_2 \tag{2.391}$$

We now make a final transformation of the parameters to write the results in a more convenient form. For simplicity we take $\alpha_2 = 0$ because it plays no part in the solution.

In summary the results of this section are as follows :

$$E(x,y) = m(x)y^{\frac{6}{7}}. \tag{2.392}$$

$\mathbf{s \neq 0}$ $(\mathbf{p \neq -1},\ \boldsymbol{\alpha_1} \neq \mathbf{0})$:

$$\bar{U}(x) = \bar{U}_0[M(x)+C]^s, \tag{2.393}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.394}$$

$$\frac{d}{d\xi}\left(\xi^{\frac{6}{7}}\frac{d^2H}{d\xi^2}\right) + H\frac{d^2H}{d\xi^2} + \frac{8s}{(s+7)}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0, \tag{2.395}$$

$$\xi = \left[\frac{(s+7)}{8}\bar{U}_0\right]^{\frac{7}{8}} \frac{y}{[M(x)+C]^{\frac{7}{8}(1-s)}} \tag{2.396}$$

$$\psi(x,y) = \left[\frac{8}{s+7}\bar{U}_0^{\frac{1}{7}}\right]^{\frac{7}{8}} [M(x)+C]^{\frac{7+s}{8}} H(\xi), \tag{2.397}$$

where

$$s = \frac{p+1}{2}. \tag{2.398}$$

$\mathbf{s = 0\ (p = -1,\ \alpha_1 = 0)}$:

$$\bar{U}(x) = \bar{U}_0. \tag{2.399}$$

$$\frac{dH}{d\xi}(\infty) = 1, \tag{2.400}$$

$$\frac{d}{d\xi}\left(\xi^{\frac{6}{7}}\frac{d^2H}{d\xi^2}\right) + H\frac{d^2H}{d\xi^2} = 0, \tag{2.401}$$

$$\xi = \left[\frac{(7)}{8}\bar{U}_0\right]^{\frac{7}{8}} \frac{y}{[M(x)+C]^{\frac{7}{8}}}, \tag{2.402}$$

$$\psi(x,y) = \left[\frac{8}{7}\bar{U}_0^{\frac{1}{7}}\right]^{\frac{7}{8}} (M(x)+C)^{\frac{7}{8}} H(\xi), \tag{2.403}$$

Equations (2.399) to (2.403) are obtained by setting $s = 0$ in (2.393) to (2.397).

2.8 Concluding remarks

The mainstream matching boundary condition was used to determine some of the parameters in the fluid flow. Boundary conditions also need to be imposed at $y = 0$ to complete the mathematical formulation. Unlike the mainstream matching condition the boundary conditions at $y = 0$ do not in general further determine the parameters in the flow. The no slip and no suction or blowing boundary conditions could be used as in the solution of the Blasius equation in Section 2.7.1. More general boundary conditions could be considered such as slip or suction or blowing at the boundary. The group invariant solution would then determine the form of the slip velocity or the suction or blowing velocity. For example, if the mainstream velocity is a power law then for a group invariant solution the slip and suction or blowing velocities may also have to be power laws with the powers determined by the group invariant solution.

For power law effective viscosity the general form of the differential equation was

$$\frac{d^3H}{d\xi^3} + H\frac{d^2H}{d\xi^2} + \frac{2m}{m+n+1}\left[1 - \left(\frac{dH}{d\xi}\right)^2\right] = 0, \tag{2.404}$$

which reduces to the Blasius equation when $m = 0$ and the Falkner-Skan equation when $n = 0$. Existence and uniqueness of solutions have been investigated for the Falkner-Skan equation with no slip and no suction or blowing boundary conditions [1]. If $m \geq 0$ the solution is unique and there is no reversed flow. If $-0.0904 \leq m < 0$ then there are two solutions. One solution has no reversed flow and the other has a region of reversed flow near the boundary. If $m < -0.0904$ there are no solutions without reversed flow. It would be of interest to investigate the effect of eddy viscosity, $n \neq 0$, on these results.

We considered only the general case $C_1 + C_2 \neq 0$. For laminar flow when $C_1 + C_2 = 0$ the mainstream velocity is

$$\bar{U}(x) = \frac{\bar{U}_0}{x}, \tag{2.405}$$

which describes flow in a converging or diverging channel [1], [27]. The special case $C_1 + C_2 = 0$ would therefore describe the effect of eddy viscosity on this flow.

Chapter 3

Two-dimensional turbulent wall jet

3.1 Introduction

In this chapter we will investigate the two-dimensional turbulent wall jet. We will first derive the equations and conserved quantities. We will then consider a special case where the effective viscosity is of the form $E(x, \bar{v}_x(x, y))$. We will derive the Lie point symmetries for this special case and the group invariant solution will be found. We will see that the results are consistent with those found in Chapter 2. We will then find the conserved quantity for this special case.

The analysis of the turbulent wall jet differs in two ways from that of the turbulent boundary layer :

i) The mainstream mean velocity, $\bar{U}(x)$, in the turbulent wall jet is zero.

ii) We derive a conserved quantity which must be satisfied in order for the solution to exist.

3.2 Derivation of equations for the two-dimensional turbulent wall jet

We consider the two-dimensional boundary layer equation

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = \bar{U}(x) \frac{d\bar{U}(x)}{dx} + \frac{\partial}{\partial y}\left[E(x, y) \frac{\partial \bar{v}_x}{\partial y} \right] \tag{3.1}$$

and the continuity equation

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} = 0, \tag{3.2}$$

which were derived in Chapter 2.

When considering a turbulent wall jet the mainstream velocity vanishes, that is

$$\bar{U}(x) = 0. \tag{3.3}$$

Hence equation (3.1) becomes

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = \frac{\partial}{\partial y}\left[E(x,y)\frac{\partial \bar{v}_x}{\partial y}\right]. \tag{3.4}$$

Equations (3.4) and (3.2) are used to describe the two-dimesional turbulent wall jet.

3.3 Derivation of conserved quantities

In this section we will consider the derivation of the conserved quantity for different forms of the effective viscosity. We will consider the different conditions which will need to be satisfied for each form of the effective viscosity.

The four forms of the effective viscosity we consider are:

i) $E = E(x)$
ii) $E = M(x)\,\bar{v}_x^n(x,y), \qquad n > 0$
iii) $E = M(x)\,f(\bar{v}_x(x,y))$
iv) $E = E(x,y)$

The initial steps in the derivation of the conserved quantity are the same. We integrate equation (3.4) with respect to y from y to ∞:

$$\int_y^\infty \bar{v}_x(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial x}dy^* + \int_y^\infty \bar{v}_y(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}dy^* \\ = \int_y^\infty \frac{\partial}{\partial y^*}\left[E(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}\right]dy^*. \tag{3.5}$$

Integrate the second term in (3.5) by parts:

$$\int_y^\infty \bar{v}_y(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}dy^* = \bar{v}_x(x,\infty)\bar{v}_y(x,\infty) \\ -\bar{v}_x(x,y)\bar{v}_y(x,y) - \int_y^\infty \bar{v}_x(x,y^*)\frac{\partial \bar{v}_y(x,y^*)}{\partial y^*}dy^*. \tag{3.6}$$

We use the mainstream matching boundary condition which since $\bar{U}(x) = 0$, is

$$\bar{v}_x(x,\infty) = 0, \tag{3.7}$$

to deduce that the first term on the right hand side of equation (3.6) vanishes. We note that $\bar{v}_y(x,\infty)$ is finite and hence this is possible. Using the continuity equation

(3.2), (3.6) can be written as

$$\int_y^\infty \bar{v}_y(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}dy^* = -\bar{v}_x(x,y)\bar{v}_y(x,y) + \int_y^\infty \bar{v}_x(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial x}dy^*. \quad (3.8)$$

Now the term on the right hand side of (3.5) can be integrated to give

$$\int_y^\infty \frac{\partial}{\partial y^*}\left[E(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}\right]dy^* = E(x,\infty)\frac{\partial \bar{v}_x(x,\infty)}{\partial y} - E(x,y)\frac{\partial \bar{v}_x(x,y)}{\partial y}. \quad (3.9)$$

We make the assumption that

$$E(x,\infty)\frac{\partial \bar{v}_x(x,\infty)}{\partial y} = 0. \quad (3.10)$$

We will check that for the results derived, condition (3.10) is satisfied. Thus (3.9) becomes

$$\int_y^\infty \frac{\partial}{\partial y^*}\left[E(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}\right]dy^* = -E(x,y)\frac{\partial \bar{v}_x(x,y)}{\partial y}. \quad (3.11)$$

Subsituting (3.8) and (3.11) into (3.5) gives

$$2\int_y^\infty \bar{v}_x(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}dy^* - \bar{v}_x(x,y)\bar{v}_y(x,y) = -E(x,y)\frac{\partial \bar{v}_x(x,y)}{\partial y}. \quad (3.12)$$

We can rewrite the first term in (3.12) as

$$2\int_y^\infty \bar{v}_x(x,y^*)\frac{\partial \bar{v}_x(x,y^*)}{\partial y^*}dy^* = \frac{\partial}{\partial x}\int_y^\infty \bar{v}_x^2(x,y^*)dy^*. \quad (3.13)$$

Substituting (3.13) into (3.12) gives

$$\frac{\partial}{\partial x}\int_y^\infty \bar{v}_x^2(x,y^*)dy^* - \bar{v}_x(x,y)\bar{v}_y(x,y) = -E(x,y)\frac{\partial \bar{v}_x(x,y)}{\partial y}. \quad (3.14)$$

Multiply (3.14) by $\bar{v}_x(x,y)$ and integrate with respect to y from 0 to ∞ :

$$\int_0^\infty \bar{v}_x(x,y)\frac{\partial}{\partial x}\left[\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right]dy - \int_0^\infty \bar{v}_x^2(x,y)\bar{v}_y(x,y)dy$$
$$= -\int_0^\infty E(x,y)\bar{v}_x(x,y)\frac{\partial \bar{v}_x(x,y)}{\partial y}dy. \quad (3.15)$$

Now the first term of (3.15) can be written as

$$\int_0^\infty \bar{v}_x(x,y)\frac{\partial}{\partial x}\left[\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right]dy = \frac{d}{dx}\int_0^\infty \left[\bar{v}_x(x,y)\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)\right]dy$$
$$-\int_0^\infty \frac{\partial \bar{v}_x(x,y)}{\partial x}\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy. \quad (3.16)$$

Using the continuity equation we can write the last term in equation (3.16) as

$$-\int_0^\infty \frac{\partial \bar{v}_x(x,y)}{\partial x}\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy = \int_0^\infty \frac{\partial \bar{v}_y(x,y)}{\partial y}\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy \tag{3.17}$$

Integrating the right hand side of (3.17) by parts gives

$$-\int_0^\infty \frac{\partial \bar{v}_x(x,y)}{\partial x}\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy = -\bar{v}_y(x,0)\int_0^\infty \bar{v}_x^2(x,y^*)dy^* + \int_0^\infty \bar{v}_y(x,y)\bar{v}_x^2(x,y)dy \tag{3.18}$$

because

$$\bar{v}_y(x,\infty)\int_\infty^\infty \bar{v}_x^2(x,y^*)dy^* = 0. \tag{3.19}$$

Substituting (3.18) into (3.16) we have

$$\begin{aligned}\int_0^\infty \bar{v}_x(x,y)\frac{\partial}{\partial x}\left[\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right]dy &= \frac{d}{dx}\left[\int_0^\infty \bar{v}_x(x,y)\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy\right] \\ &\quad -\bar{v}_y(x,0)\int_0^\infty \bar{v}_x^2(x,y^*)dy^* + \int_0^\infty \bar{v}_y(x,y)\bar{v}_x^2(x,y)dy.\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.15) and simplifying we obtain

$$\begin{aligned}\frac{d}{dx}\left[\int_0^\infty \bar{v}_x(x,y)\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy\right] &= \bar{v}_n(x)\int_0^\infty \bar{v}_x^2(x,y)dy \\ &\quad -\frac{1}{2}\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy,\end{aligned} \tag{3.21}$$

where

$$\bar{v}_n(x) = \bar{v}_y(x,0). \tag{3.22}$$

Define

$$J(x) = \int_0^\infty \bar{v}_x(x,y)\left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^*\right)dy \tag{3.23}$$

Then (3.21) becomes

$$\frac{dJ}{dx} = \bar{v}_n(x)\int_0^\infty \bar{v}_x^2(x,y)dy - \frac{1}{2}\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy. \tag{3.24}$$

3.3.1 Effective viscosity $E = E(x)$

When $E = E(x)$,

$$\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy = E(x)\int_0^\infty \frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy = -E(x)\bar{v}_s^2(x) \tag{3.25}$$

where the velocity of slip $\bar{v}_s(x)$ is

$$\bar{v}_s(x) = \bar{v}_x(x,0). \tag{3.26}$$

Equation (3.24) becomes

$$\frac{dJ}{dx} = \bar{v}_n(x) \int_0^\infty \bar{v}_x^2(x,y)dy + \frac{1}{2}E(x)\bar{v}_s^2(x). \tag{3.27}$$

Thus if

$$\bar{v}_n(x) \int_0^\infty \bar{v}_x^2(x,y)dy + \frac{1}{2}E(x)\bar{v}_s^2(x) = 0, \tag{3.28}$$

then

$$\frac{dJ}{dx} = 0 \tag{3.29}$$

and therefore

$$J = \int_0^\infty \bar{v}_x(x,y) \left(\int_y^\infty \bar{v}_x^2(x,y^*)dy^* \right) dy = \textit{ constant independent of } x. \tag{3.30}$$

If there is no suction or blowing at the boundary and no slip then $\bar{v}_n(x) = 0$ and $\bar{v}_s(x) = 0$ and (3.28) is satisfied.

If there is blowing at the boundary then $\bar{v}_n(x) > 0$ and (3.28) cannot be satisfied even if there is no slip.

If there is suction at the boundary then $\bar{v}_n(x) < 0$ and (3.28) can be satisfied provided there is slip at the boundary.

3.3.2 Effective viscosity $E = M(x)\bar{v}_x^n(x,y)$, $n > 0$

Consider an effective viscosity of the form

$$E = M(x)\bar{v}_x^n(x,y), \qquad n > 0 \tag{3.31}$$

where $M(x)$ is an arbitrary function of x. Since $\bar{v}_x(x,\infty) = 0$, $E \to 0$ as $y \to \infty$ provided $n > 0$. We therefore assume that $n > 0$. Using (3.31),

$$\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy = \frac{2}{(n+2)}M(x)\int_0^\infty \frac{\partial}{\partial y}\left(\bar{v}_x^{n+2}(x,y)\right)dy = -\frac{2}{(n+2)}M(x)\bar{v}_s^{n+2}(x) \tag{3.32}$$

Thus (3.24) becomes

$$\frac{dJ}{dx} = \bar{v}_n(x)\int_0^\infty \bar{v}_x^2(x,y)dy + \frac{1}{(n+2)}M(x)\bar{v}_s^{n+2}(x). \tag{3.33}$$

Thus if

$$\bar{v}_n(x)\int_0^\infty \bar{v}_x^2(x,y)dy + \frac{1}{(n+2)}M(x)\bar{v}_s^{n+2}(x) = 0, \tag{3.34}$$

then (3.29) is satisfied and J is a conserved quantity.

If there is no suction or blowing at the boundary then $\bar{v}_n(x) = 0$ and if there is no slip then $\bar{v}_s(x) = 0$. Condition (3.34) is then satisfied.

In general, $\bar{v}_s(x) > 0$ and $M(x) > 0$ for $E > 0$. Thus (3.34) has a solution only if $\bar{v}_n(x) < 0$ which describes suction at the boundary.

Equation (3.31) gives an effective viscosity which depends on y and is bounded as $y \to \infty$.

We saw in Section 2.7.3, based on the empirical formula of Blasius, that as $y \to 0$,

$$\bar{v}_x(x,y) \propto y^{\frac{1}{7}} \tag{3.35}$$

and

$$E(x,y) = m(x)y^{\frac{6}{7}}. \tag{3.36}$$

Thus as $y \to 0$,

$$E(x,y) = M(x)\bar{v}_x^6(x,y). \tag{3.37}$$

Glauert [11] considered the special case $n = 6$ and assumed that (3.37) is valid throughout the whole width of the jet although it only really applies in a limited region near the wall. We will consider general $n > 0$ but compare our results with the special case $n = 6$ since it is based on experiment.

3.3.3 Effective viscosity $E = M(x)f(\bar{v}_x(x,y))$

Consider an effective viscosity of the form

$$E = M(x)f(\bar{v}_x(x,y)) \tag{3.38}$$

where $M(x)$ is an arbitrary function of x and $f(\bar{v}_x)$ is bounded as $y \to \infty$. Now

$$\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy = 2M(x)\int_0^\infty \bar{v}_x(x,y)f(\bar{v}_x(x,y))\frac{\partial \bar{v}_x(x,y)}{\partial y}dy \tag{3.39}$$

Define $G(\bar{v}_x(x,y))$ by

$$\bar{v}_x(x,y)f(\bar{v}_x(x,y)) = \frac{dG}{d\bar{v}_x}. \tag{3.40}$$

Then (3.39) becomes

$$\begin{aligned}\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy &= 2M(x)\int_0^\infty \frac{dG}{d\bar{v}_x}\frac{\partial \bar{v}_x}{\partial y}dy \\ &= 2M(x)\int_0^\infty \frac{\partial G}{\partial y}dy \\ &= 2M(x)[G(\bar{v}_x(x,\infty)) - G(\bar{v}_x(x,0))]. \end{aligned} \tag{3.41}$$

Since $\bar{v}_x(x,\infty) = 0$ and $\bar{v}_x(x,0) = \bar{v}_s(x)$, (3.41) becomes

$$\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy = 2M(x)[G(0) - G(\bar{v}_s(x))]. \tag{3.42}$$

Substituition of (3.42) into (3.24) gives

$$\frac{dJ}{dx} = \bar{v}_n(x) \int_0^\infty \bar{v}_x^2(x,y)dy + M(x)[G(\bar{v}_s(x)) - G(0)]. \tag{3.43}$$

Thus if

$$\bar{v}_n(x) \int_0^\infty \bar{v}_x^2(x,y)dy + M(x)[G(\bar{v}_s(x)) - G(0)] = 0 \tag{3.44}$$

then (3.29) is valid and J is again a conserved quantity.

If there is no slip then $\bar{v}_s(x) = 0$ and (3.44) reduces to the condition

$$\bar{v}_n(x) = 0. \tag{3.45}$$

Hence if there is no suction or blowing and no slip then (3.44) is satisfied and J is a constant independent of x.

3.3.4 Effective viscosity $E = E(x,y)$

From (3.24), if

$$\bar{v}_n(x) \int_0^\infty \bar{v}_x^2(x,y)dy - \frac{1}{2}\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2(x,y)\right)dy = 0 \tag{3.46}$$

then (3.29) is satisfied and J is a conserved quantity. Equation (3.46) is the most general condition on $E(x,y)$ for J to be a conserved quantity.

The results of the section are summarised in Table 3.1.

Effective viscosity	**Condition**	**Conserved quantity**
$E = E(x)$	$\bar{v}_n \int_0^\infty \bar{v}_x^2(x,y)dy + \frac{1}{2}E(x)\bar{v}_s^2 = 0$	J
$E = M(x)\bar{v}_x^n$ $n > 0$	$\bar{v}_n \int_0^\infty \bar{v}_x^2(x,y)dy + \frac{1}{(n+2)}M(x)\bar{v}_s^{n+2} = 0$	J
$E = M(x)f(\bar{v}_x)$	$\bar{v}_n \int_0^\infty \bar{v}_x^2(x,y)dy + M(x)[G(\bar{v}_s) - G(0)] = 0$ $\bar{v}_x F(\bar{v}_x) = \frac{dG}{d\bar{v}_x}$	J
$E = E(x,y)$	$\bar{v}_n \int_0^\infty \bar{v}_x^2(x,y)dy - \frac{1}{2}\int_0^\infty E(x,y)\frac{\partial}{\partial y}\left(\bar{v}_x^2\right)dy = 0$	J

Table 3.1: Effective viscosity and condition for J, given by (3.23), to be a conserved quantity.

3.4 Effective viscosity of the form $E = E(x)$

In this section we will first look at the group invariant solution when the effective viscosity is a general function of x, that is $E = E(x)$. We will then consider the special case when the effective viscosity is in the form of a power law, that is $E(x) = E_0 x^n$. We will show that the group invariant solution and the scaling transformation leads to the same results for the stream function and ordinary differential equations. We then use the conserved quantity derived in Section 3.3.1 to solve for the unknown constant. Finally we solve the ordinary differential equation analytically and plot the solution.

3.4.1 Group invariant solution for $E = E(x)$

We use the results from Chapter 2 and we set the mainstream mean velocity, $\bar{U}(x)$ to be zero. Consider the equations for the two-dimensional turbulent wall jet (3.2) and (3.4) :

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} = 0, \tag{3.47}$$

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = \frac{\partial}{\partial y}\left[E(x,y)\frac{\partial \bar{v}_x}{\partial y}\right]. \tag{3.48}$$

We introduce the stream function defined by (2.7),

$$\bar{v}_x = \frac{\partial \psi}{\partial y}, \qquad \bar{v}_y = -\frac{\partial \psi}{\partial x}. \tag{3.49}$$

Equation (3.47) is identically satisfied by (3.49).Equation (3.48) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2 \psi}{\partial y^2}\right). \tag{3.50}$$

We observe that equation (2.82) is identical to (3.50) if we set $\bar{U}(x) = 0$ in (2.82). But we have already derived the group invariant solution for (2.82) and hence we can use these results for the wall jet by setting the mainstream mean velocity to be zero, that is $\bar{U}(x) = 0$.

For this section we only require the group invariant solution when $E = E(x)$. The group invariant solution for the wall jet when $E = E(x)$ can be derived from equations (2.138), (2.139) and (2.143). Thus we have

$$\psi(x,y) = e^{(\alpha_1+1)\bar{B}(x)} F(\eta), \tag{3.51}$$

$$E(x) = \frac{G_0}{\bar{a}(x)}\, e^{(2+\alpha_1)\,\bar{B}(x)}, \qquad \frac{1}{\bar{a}(x)} = \frac{d\bar{B}}{dx} \tag{3.52}$$

$$\eta = e^{-\bar{B}(x)} y, \tag{3.53}$$

where G_0 is a constant. In (2.139) we have chosen $\bar{G}(\eta) = G_0$ because $E = E(x)$. We set $\alpha_2 = 0$ in (2.138) because it plays no part in the solution and $\bar{D}(x) = 0$ so that $\eta = 0$ when $y = 0$.

The ordinary differential equation (2.144) becomes :

$$G_0\frac{d^3F}{d\eta^3}+(\alpha_1+1)F\frac{d^2F}{d\eta^2}-\alpha_1\left(\frac{dF}{d\eta}\right)^2=0. \tag{3.54}$$

since $\bar{C}_7=0$ because $\bar{U}(x)=0$.

3.4.2 Group invariant solution for $E(x)=E_0x^n$

In this section we consider a special case of the effective viscosity, that is

$$E(x)=E_0x^n, \tag{3.55}$$

From Section 2.7.1 we know that when the effective viscosity is of this form then we can solve for $\bar{a}(x)$ and if we consider the case when $n\neq -1$ then we have from (2.151) that

$$\bar{a}(x)=(2+\alpha_1)\frac{x}{n+1}+\alpha_3x^{-n}, \tag{3.56}$$

where α_3 is a constant. We also have (2.154) which gives us $\bar{B}(x)$

$$\bar{B}(x)=\frac{1}{(2+\alpha_1)}\ln\Big((2+\alpha_1)\mathrm{x}^{\mathrm{n}+1}+(\mathrm{n}+1)\alpha_3\Big)+\text{constant},\ \ \alpha_1\neq-2. \tag{3.57}$$

As with the boundary layer we take the additive constant in (3.57) to be zero. We now show that the equations can be transformed to the form (2.61) and (2.62) obtained by the scaling transformation. Substituting (3.56) and (3.57) into (3.51) to (3.53) and setting $\alpha_3=0$ we have

$$\psi(x,y)=(\alpha_1+2)^{1-\frac{1}{(\alpha_1+2)}}x^{\frac{(\alpha_1+1)(n+1)}{\alpha_1+2}}F(\eta), \tag{3.58}$$

$$E(x)=E_0x^n, \tag{3.59}$$

$$\eta=(\alpha_1+2)^{-\frac{1}{(\alpha_1+2)}}x^{-\frac{(n+1)}{\alpha_1+2}}y, \tag{3.60}$$

where

$$E_0=G_0(n+1). \tag{3.61}$$

The ordinary differential equation (3.54) becomes

$$E_0\frac{d^3F}{d\eta^3}+(\alpha_1+1)(n+1)F\frac{d^2F}{d\eta^2}-\alpha_1(n+1)\left(\frac{dF}{d\eta}\right)^2=0, \tag{3.62}$$

In order to simplify the expression for $\psi(x,y)$ and η we make the following transformation

$$\xi=B\eta,\qquad K(\xi)=CF(\eta), \tag{3.63}$$

where

$$B = (\alpha_1 + 2)^{\frac{1}{(\alpha_1+2)}}, \qquad C = (\alpha_1 + 2)^{1-\frac{1}{(\alpha_1+2)}}. \tag{3.64}$$

Equations (3.58) and (3.60) become

$$\psi(x, y) = x^{\frac{(\alpha_1+1)(n+1)}{(\alpha_1+2)}} K(\xi), \tag{3.65}$$

$$\xi = x^{-\frac{n+1}{\alpha_1+2}} y, \tag{3.66}$$

The ordinary differential equation (3.62) transforms to

$$E_0 \frac{d^3K}{d\xi^3} + \frac{(\alpha_1 + 1)(n + 1)}{(\alpha_1 + 2)} K \frac{d^2K}{d\xi^2} - \frac{\alpha_1(n + 1)}{(\alpha_1 + 2)} \left(\frac{dK}{d\xi}\right)^2 = 0, \tag{3.67}$$

Let

$$\alpha = \frac{n + 1}{\alpha_1 + 2}, \tag{3.68}$$

Using (3.68) we see that (3.65), (3.66) and (3.67) become

$$\psi(x, y) = x^{1+n-\alpha} K(\xi), \tag{3.69}$$

$$\xi = \frac{y}{x^\alpha}, \tag{3.70}$$

$$E_0 \frac{d^3K}{d\xi^3} + (1 + n - \alpha) K \frac{d^2K}{d\xi^2} - (1 + n - 2\alpha) \left(\frac{dK}{d\xi}\right)^2 = 0, \tag{3.71}$$

We note that these results correspond to (2.61) and (2.62) which we found using the scaling transformation in Section 2.3.2 with n replaced by β.

3.4.3 Analytical solution for $E(x) = E_0 x^n$

Consider (3.69) and (3.70) where $K(\xi)$ satisfies the ordinary differential equation (3.71). The conserved quantity (3.30),

$$J = \int_0^\infty \bar{v}_x(x, y) \left(\int_y^\infty \bar{v}_x^2(x, y^*) dy^* \right) dy = \textit{ constant independent of } x, \tag{3.72}$$

is used to determine α. In order for (3.72) to hold we require that the condition (3.28),

$$\bar{v}_n(x) \int_0^\infty \bar{v}_x^2(x, y) dy + \frac{1}{2} E(x) \bar{v}_s^2(x) = 0, \tag{3.73}$$

is satisfied.

We will now consider boundary conditions which we will use in order to solve for $K(\xi)$. We will assume that there is no slipping at the boundary, that is

$$\bar{v}_x(x, 0) = \bar{v}_s(x) = 0. \tag{3.74}$$

We will also assume that there is no suction or blowing at the boundary and hence

$$\bar{v}_y(x,0) = \bar{v}_n(x) = 0. \tag{3.75}$$

Since we are dealing with a wall jet that emerges into a fluid that is at rest, we will have that the mainstream mean velocity in the $x-$direction will be at rest at a large distance from the wall, that is

$$\bar{v}_x(x,\infty) = 0 \tag{3.76}$$

and

$$\frac{\partial \bar{v}_x(x,\infty)}{\partial y} = 0. \tag{3.77}$$

Thus we see using (3.74) and (3.75) that condition (3.73) is identically satisfied and hence we can use the conserved quantity (3.72) to determine α.

We first express J in (3.72) in terms of the stream function, (3.49) :

$$J = \int_0^\infty \frac{\partial \psi(x,y)}{\partial y} \left(\int_y^\infty \left[\frac{\partial \psi(x,y^*)}{\partial y^*} \right]^2 dy^* \right) dy. \tag{3.78}$$

We now transform from y to ξ at a fixed point x. From (3.70)

$$dy = x^\alpha \, d\xi, \quad dy^* = x^\alpha \, d\xi^*, \quad \frac{\partial \xi}{\partial y} = \frac{1}{x^\alpha}, \tag{3.79}$$

and from (3.69)

$$\bar{v}_x(x,y) = \frac{\partial \psi}{\partial y} = x^{1+n-2\alpha} \frac{dK}{d\xi}. \tag{3.80}$$

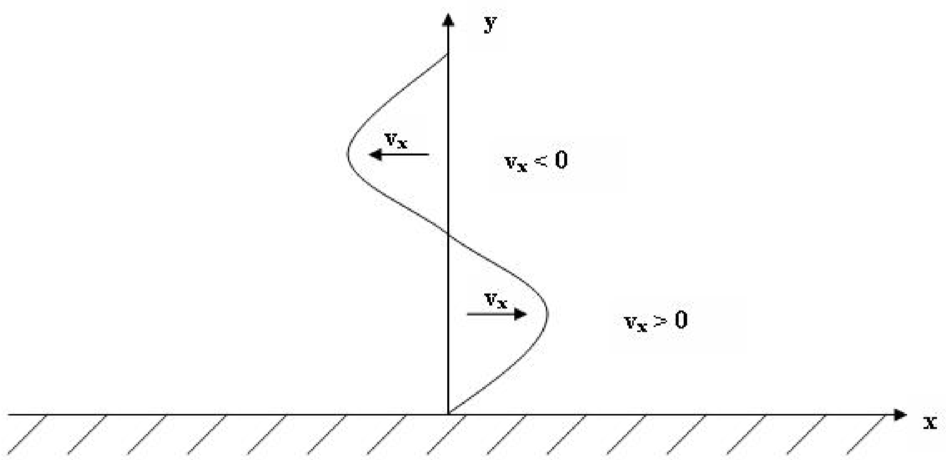

Figure 3.1: Backflow in a two-dimensional wall jet

Thus using (3.79) and (3.80) in (3.78) we have

$$J = x^{3(1+n)-4\alpha} \int_0^\infty \frac{dK}{d\xi} \left(\int_\xi^\infty \left[\frac{dK}{d\xi^*} \right]^2 d\xi^* \right) d\xi. \tag{3.81}$$

Now J could vanish if $\frac{dK}{d\xi} < 0$ for part of the range $0 \leq \xi \leq \infty$. From (3.80) a negative value, $\frac{dK}{d\xi} < 0$, would describe backflow as shown in Figure 3.1.

We assume that

$$J \neq 0. \tag{3.82}$$

The special case of $J = 0$, is of interest and must be treated separately. The constant α is not determined from the method that follows when $J = 0$. Since

$$\int_0^\infty \frac{dK}{d\xi} \left(\int_\xi^\infty \left[\frac{dK}{d\xi^*} \right]^2 d\xi^* \right) d\xi \neq 0, \tag{3.83}$$

from (3.81) it follows that J is independent of x provided

$$\alpha = \frac{3}{4}(1+n), \tag{3.84}$$

which implies from (3.68) that $\alpha_1 = -\frac{2}{3}$. The result $\alpha_1 = -\frac{2}{3}$ is independent of n and therefore also applies for laminar flow. Substituting (3.84) into (3.69), (3.70) and (3.71) we have

$$\psi(x, y) = x^{\frac{1}{4}(1+n)} K(\xi), \tag{3.85}$$

$$\xi = \frac{y}{x^{\frac{3}{4}(1+n)}}, \tag{3.86}$$

$$\frac{4E_0}{1+n} \frac{d^3K}{d\xi^3} + K \frac{d^2K}{d\xi^2} + 2\left(\frac{dK}{d\xi} \right)^2 = 0, \tag{3.87}$$

Using (3.49), (3.85) and (3.86), $\bar{v}_x$ and $\bar{v}_y$ in terms of $K(\xi)$ are

$$\bar{v}_x(x, y) = \frac{1}{x^{\frac{1}{2}(1+n)}} \frac{dK}{d\xi} \tag{3.88}$$

$$\bar{v}_y(x, y) = \frac{1}{4} x^{-\frac{3}{4}+\frac{1}{4}n} (1+n) \left[-K(\xi) + 3\xi \frac{dK}{d\xi} \right] \tag{3.89}$$

We see from (3.88) that for $n > 1$, $\bar{v}_x(x, 0) \rightarrow \infty$ as $x \rightarrow 0$. The orifice in the wall in Figure 1.2 is at $x = 0$. The long narrow orifice is assumed to be infinitely thin. In order to have a finite volume of flow and a finite momentum it is necessary to have an infinite fluid velocity at the orifice.

The boundary conditions (3.74) to (3.77) can be written in terms of $K(\xi)$ as, since $n \neq -1$,

$$\bar{v}_x(x, 0) = 0 : \quad \frac{dK(0)}{d\xi} = 0, \tag{3.90}$$

$$\bar{v}_y(x,0) = 0: \quad K(0) = 0, \tag{3.91}$$

$$\bar{v}_x(x,\infty) = 0: \quad \tfrac{dK(\infty)}{d\xi} = 0, \tag{3.92}$$

$$\frac{\partial \bar{v}_x(x,\infty)}{\partial y} = 0: \quad \tfrac{d^2K(\infty)}{d\xi^2} = 0. \tag{3.93}$$

In summary we have two cases for the two-dimensional turbulent wall jet :

<u>$\mathbf{J \neq 0}$</u> :

$$\psi(x,y) = x^{\frac{1}{4}(1+n)} K(\xi), \tag{3.94}$$

$$\xi = \frac{y}{x^{\frac{3}{4}(1+n)}}, \tag{3.95}$$

$$\frac{4E_0}{1+n}\frac{d^3K}{d\xi^3} + K\frac{d^2K}{d\xi^2} + 2\left(\frac{dK}{d\xi}\right)^2 = 0, \tag{3.96}$$

$$K(0) = 0, \qquad \frac{dK(0)}{d\xi} = 0, \qquad \frac{dK(\infty)}{d\xi} = 0, \qquad \frac{d^2K(\infty)}{d\xi^2} = 0. \tag{3.97}$$

<u>$\mathbf{J = 0}$</u> :

$$\alpha \ \texttt{not determined}, \tag{3.98}$$

$$\int_0^\infty \frac{dK}{d\xi}\left(\int_\xi^\infty \left[\frac{dK}{d\xi^*}\right]^2 d\xi^*\right) d\xi = 0, \tag{3.99}$$

$$\psi(x,y) = x^{1+n-\alpha} K(\xi), \tag{3.100}$$

$$\xi = \frac{y}{x^\alpha}, \tag{3.101}$$

$$E_0\frac{d^3K}{d\xi^3} + (1+n-\alpha)K\frac{d^2K}{d\xi^2} - (1+n-2\alpha)\left(\frac{dK}{d\xi}\right)^2 = 0. \tag{3.102}$$

We will consider only the case $J \neq 0$.

In order to remove E_0 from equation (3.96) we make the transformation

$$\vartheta = A_1\xi, \qquad K = B_1 G, \tag{3.103}$$

where A_1 and B_1 are constants. Equation (3.96) transforms to

$$4\frac{E_0}{(1+n)}\frac{A_1}{B_1}\frac{d^3G}{d\vartheta^3} + G\frac{d^2G}{d\vartheta^2} + 2\left(\frac{dG}{d\vartheta}\right)^2 = 0. \tag{3.104}$$

Choose

$$4\frac{E_0}{(1+n)}\frac{A_1}{B_1} = 1. \tag{3.105}$$

Since only the ratio $\frac{A_1}{B_1}$ is determined, choose

$$A_1 = 1 \tag{3.106}$$

and therefore

$$B_1 = \frac{4E_0}{(1+n)}. \tag{3.107}$$

Thus we can rewrite the summary for $J \neq 0$ in terms of the transformation as follows :

<u>$\mathbf{J \neq 0}$</u> :

$$\psi(x,y) = \frac{4E_0}{(1+n)} x^{\frac{1}{4}(1+n)} G(\xi), \tag{3.108}$$

$$\xi = \frac{y}{x^{\frac{3}{4}(1+n)}}, \tag{3.109}$$

$$\frac{d^3G}{d\xi^3} + G\frac{d^2G}{d\xi^2} + 2\left(\frac{dG}{d\xi}\right)^2 = 0, \tag{3.110}$$

$$G(0) = 0, \qquad \frac{dG(0)}{d\xi} = 0, \qquad \frac{dG(\infty)}{d\xi} = 0, \qquad \frac{d^2G(\infty)}{d\xi^2} = 0, \tag{3.111}$$

$$\bar{v}_x(x,y) = \frac{4E_0}{(1+n)} \frac{1}{x^{\frac{1}{2}(1+n)}} \frac{dG}{d\xi}, \tag{3.112}$$

$$\bar{v}_y(x,y) = E_0 \frac{1}{x^{\frac{1}{4}(3-n)}} \left[3\xi\frac{dG}{d\xi} - G\right]. \tag{3.113}$$

We will now outline the solution given by Glauert [11] of the ordinary differential equation (3.110) subject to the boundary conditions (3.111). [11] Multiply (3.110) by G.

$$G\frac{d^3G}{d\xi^3} + G^2\frac{d^2G}{d\xi^2} + 2G\left(\frac{dG}{d\xi}\right)^2 = 0. \tag{3.114}$$

But

$$G^2\frac{d^2G}{d\xi^2} + 2G\left(\frac{dG}{d\xi}\right)^2 = \frac{d}{d\xi}\left(G^2\frac{dG}{d\xi}\right), \tag{3.115}$$

and

$$G\frac{d^3G}{d\xi^3} = \frac{d}{d\xi}\left(G\frac{d^2G}{d\xi^2}\right) - \frac{1}{2}\frac{d}{d\xi}\left(\left(\frac{dG}{d\xi}\right)^2\right). \tag{3.116}$$

Using (3.115) and (3.116), (3.114) becomes

$$\frac{d}{d\xi}\left(G\frac{d^2G}{d\xi^2}\right) - \frac{1}{2}\frac{d}{d\xi}\left(\left(\frac{dG}{d\xi}\right)^2\right) + \frac{d}{d\xi}\left(G^2\frac{dG}{d\xi}\right) = 0. \tag{3.117}$$

Integrate (3.117) with respect to ξ :

$$G\frac{d^2G}{d\xi^2} - \frac{1}{2}\left(\frac{dG}{d\xi}\right)^2 + G^2\frac{dG}{d\xi} = D_1, \qquad (3.118)$$

where D_1 is a constant. Consider the boundary conditions (3.111) at $\xi = 0$. Now $\frac{d^2G(0)}{d\xi^2}$ is finite because the stress at the wall, $\xi = 0$, is finite. In the boundary layer approximation

$$\tau_{yx}(x,0) = \rho E_0 x^n \frac{\partial \bar{v}_x}{\partial y} = \rho E_0 x^n \frac{\partial \bar{v}_x(x,0)}{\partial y} = \frac{4\rho E_0^2}{(1+n)} x^{-\frac{1}{4}(5+n)} \frac{d^2G(0)}{d\xi^2}. \qquad (3.119)$$

Since $\tau_{yx}(x,0)$ is finite it follows that $\frac{d^2G(0)}{d\xi^2}$ is finite. Substituting (3.111) into (3.118) gives

$$D_1 = 0. \qquad (3.120)$$

Thus (3.118) becomes

$$G\frac{d^2G}{d\xi^2} - \frac{1}{2}\left(\frac{dG}{d\xi}\right)^2 + G^2\frac{dG}{d\xi} = 0, \qquad (3.121)$$

There are two ways of integrating (3.121).
Method 1 :

Multiply (3.121) by G^γ where the constant γ is chosen later :

$$G^{1+\gamma}\frac{d^2G}{d\xi^2} - \frac{1}{2}G^\gamma\left(\frac{dG}{d\xi}\right)^2 + G^{2+\gamma}\frac{dG}{d\xi} = 0. \qquad (3.122)$$

Now

$$G^{1+\gamma}\frac{d^2G}{d\xi^2} = \frac{d}{d\xi}\left(G^{1+\gamma}\frac{dG}{d\xi}\right) - (1+\gamma)G^\gamma\left(\frac{dG}{d\xi}\right)^2 \qquad (3.123)$$

and

$$G^{2+\gamma}\frac{dG}{d\xi} = \frac{1}{3+\gamma}\frac{d}{d\xi}\left(G^{3+\gamma}\right). \qquad (3.124)$$

Using (3.123) and (3.124), (3.122) becomes

$$\frac{d}{d\xi}\left(G^{1+\gamma}\frac{dG}{d\xi}\right) - \frac{1}{2}(3+2\gamma)G^\gamma\left(\frac{dG}{d\xi}\right)^2 + \frac{1}{(3+\gamma)}\frac{d}{d\xi}\left(G^{3+\gamma}\right) = 0. \qquad (3.125)$$

Choosing

$$\gamma = -\frac{3}{2}, \qquad (3.126)$$

(3.125) becomes

$$\frac{d}{d\xi}\left(G^{-\frac{1}{2}}\frac{dG}{d\xi}\right) + \frac{2}{3}\frac{d}{d\xi}\left(G^{\frac{3}{2}}\right) = 0. \qquad (3.127)$$

Integrating (3.127) with respect to ξ gives

$$\frac{1}{G^{\frac{1}{2}}}\frac{dG}{d\xi} + \frac{2}{3}G^{\frac{3}{2}} = D_2, \tag{3.128}$$

where D_2 is an arbitrary constant.

<u>Method 2</u> :

Let

$$\frac{dG}{d\xi} = H. \tag{3.129}$$

Then

$$\frac{d^2G}{d\xi^2} = \frac{dH}{d\xi} = \frac{dH}{dG}\frac{dG}{d\xi} = H\frac{dH}{dG}. \tag{3.130}$$

Using (3.129) and (3.130), (3.121) becomes

$$\frac{dH}{dG} - \frac{1}{2G}H = -G. \tag{3.131}$$

Equation (3.131) is a first order ordinary differential equation for H with G as independent variable. The integrating factor is $\frac{1}{G^{\frac{1}{2}}}$. Thus (3.131) can be written as

$$\frac{d}{dG}\left(\frac{H}{G^{\frac{1}{2}}}\right) = -G^{\frac{1}{2}}. \tag{3.132}$$

Integrating (3.132) with respect to G gives

$$\frac{1}{G^{\frac{1}{2}}}\frac{dG}{d\xi} + \frac{2}{3}G^{\frac{3}{2}} = D_2, \tag{3.133}$$

where D_2 is an arbitrary constant. Equation (3.133) agrees with (3.128) obtained by Method 1.

We cannot impose the boundary condition at $\xi = 0$ on (3.133) because

$$\frac{1}{G^{\frac{1}{2}}(0)}\frac{dG(0)}{d\xi} \tag{3.134}$$

is not determined. We therefore impose the boundary condition at $\xi = \infty$. We do not know $G(\infty)$. Let

$$G(\infty) = G_\infty. \tag{3.135}$$

The constant G_∞ will be determined later. From (3.133),

$$D_2 = \frac{2}{3}G_\infty^{\frac{3}{2}}. \tag{3.136}$$

Thus (3.133) becomes

$$\frac{dG}{d\xi} = \frac{2}{3}G^{\frac{1}{2}}(G_\infty^{\frac{3}{2}} - G^{\frac{3}{2}}). \tag{3.137}$$

Equation (3.137) is a variable separable first order ordinary differential equation, that is

$$\frac{dG}{G^{\frac{1}{2}}(G_{\infty}^{\frac{3}{2}} - G^{\frac{3}{2}})} = \frac{2}{3}d\xi. \tag{3.138}$$

Let

$$\frac{G}{G_{\infty}} = g^2. \tag{3.139}$$

Thus equation (3.138) becomes

$$\frac{dg}{(1-g)(1+g+g^2)} = \frac{G_{\infty}}{3}d\xi. \tag{3.140}$$

Splitting the left hand side using partial fractions and integrating both sides of (3.140) leads to

$$\int \frac{dg}{1-g} + \int \frac{(g+2)dg}{g^2+g+1} = G_{\infty}\xi + k, \tag{3.141}$$

where k is a constant. Now

$$\int \frac{dg}{1-g} = -\ln(1-\mathrm{g}) \tag{3.142}$$

and

$$\int \frac{g+2}{(g^2+g+1)}dg = \frac{1}{2}\int \frac{(2g+1)dg}{(g^2+g+1)} + \frac{3}{2}\int \frac{dg}{g^2+g+1}. \tag{3.143}$$

But

$$\frac{1}{2}\int \frac{(2g+1)dg}{(g^2+g+1)} = \ln(\mathrm{g}^2+\mathrm{g}+1)^{\frac{1}{2}} \tag{3.144}$$

and

$$\frac{3}{2}\int \frac{dg}{g^2+g+1} = \frac{3}{2}\int \frac{dg}{(g+\frac{1}{2})^2+\frac{3}{4}} = \sqrt{3}\arctan\left(\frac{2\mathrm{g}+1}{\sqrt{3}}\right). \tag{3.145}$$

Thus (3.141) becomes

$$\ln\left[\frac{(\mathrm{g}^2+\mathrm{g}+1)^{\frac{1}{2}}}{1-\mathrm{g}}\right] + \sqrt{3}\arctan\left(\frac{2\mathrm{g}+1}{\sqrt{3}}\right) = \mathrm{G}_{\infty}\xi + \mathrm{k}. \tag{3.146}$$

Using (3.139) and (3.111) we see that

$$g(0) = \left(\frac{G(0)}{G_{\infty}}\right)^{\frac{1}{2}} = 0. \tag{3.147}$$

Imposing the boundary condition at $\xi = 0$ on (3.146) gives

$$k = \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) \tag{3.148}$$

and (3.146) becomes

$$\ln\left[\frac{(\mathrm{g}^2+\mathrm{g}+1)^{\frac{1}{2}}}{1-\mathrm{g}}\right]+\sqrt{3}\left[\arctan\left(\frac{2\mathrm{g}+1}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\right]=\mathrm{G}_\infty\xi. \qquad (3.149)$$

But [2]

$$\arctan\theta-\arctan\phi=\arctan\left(\frac{\theta-\phi}{1+\theta\phi}\right) \qquad (3.150)$$

and therefore

$$\arctan\left(\frac{2\mathrm{g}+1}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)=\arctan\left(\frac{\sqrt{3}\mathrm{g}}{2+\mathrm{g}}\right). \qquad (3.151)$$

Thus (3.149) becomes

$$\ln\left[\frac{(\mathrm{g}^2+\mathrm{g}+1)^{\frac{1}{2}}}{1-\mathrm{g}}\right]+\sqrt{3}\arctan\left(\frac{\sqrt{3}\mathrm{g}}{2+\mathrm{g}}\right)=\mathrm{G}_\infty\xi. \qquad (3.152)$$

In order to calculate G_∞ we use the conserved quantity J given by (3.81),

$$J=\int_0^\infty\frac{dK}{d\xi}\left(\int_\xi^\infty\left[\frac{dK}{d\xi^*}\right]^2 d\xi^*\right)d\xi. \qquad (3.153)$$

Now from (3.103) and (3.107) we have that

$$K(\xi)=\frac{4E_0}{(1+n)}G(\xi). \qquad (3.154)$$

Substituting (3.154) into (3.153) we have

$$J=\frac{64E_0^3}{(1+n)^3}\int_0^\infty\frac{dG}{d\xi}\left(\int_\xi^\infty\left[\frac{dG}{d\xi^*}\right]^2 d\xi^*\right)d\xi. \qquad (3.155)$$

which may be rewritten as

$$J=\frac{64E_0^3}{(1+n)^3}\int_0^\infty\left(\int_G^\infty\left(\frac{dG}{d\xi}\right)^* dG^*\right)dG. \qquad (3.156)$$

Substituting (3.137) into (3.156), we have

$$J=\frac{128E_0^3}{3(1+n)^3}\int_0^\infty\left[\int_G^\infty\left(G_\infty^{\frac{3}{2}}G^{*\frac{1}{2}}-G^{*2}\right)dG^*\right]dG. \qquad (3.157)$$

Integrating (3.157) and using $G(\infty)=G_\infty$ and $G(0)=0$ gives

$$J=\frac{32E_0^3G_\infty^4}{5(1+n)^3} \qquad (3.158)$$

and therefore

$$G_\infty = \left[\frac{5J(1+n)^3}{32E_0^3}\right]^{\frac{1}{4}}. \tag{3.159}$$

Thus, in summary, we have from (3.137), (3.139), (3.152) and (3.159)

$$G = G_\infty g^2, \tag{3.160}$$

$$\frac{dG}{d\xi} = \frac{2}{3}G_\infty^2 g(1-g^3), \tag{3.161}$$

$$\xi = \frac{1}{G_\infty}\ln\left[\frac{(\mathrm{g}^2+\mathrm{g}+1)^{\frac{1}{2}}}{1-\mathrm{g}}\right] + \frac{\sqrt{3}}{\mathrm{G}_\infty}\arctan\left(\frac{\sqrt{3}\mathrm{g}}{2+\mathrm{g}}\right), \tag{3.162}$$

$$\xi = \frac{y}{x^{\frac{3}{4}(1+n)}}, \tag{3.163}$$

$$G_\infty = \left[\frac{5J(1+n)^3}{32E_0^3}\right]^{\frac{1}{4}}, \tag{3.164}$$

$$\psi(x,y) = \frac{4E_0}{(1+n)}x^{\frac{1}{4}(1+n)}G(\xi), \tag{3.165}$$

$$\bar{v}_x(x,y) = \frac{4E_0}{(1+n)}\frac{1}{x^{\frac{1}{2}(1+n)}}\frac{dG}{d\xi}, \tag{3.166}$$

$$\bar{v}_y(x,y) = E_0\frac{1}{x^{\frac{1}{4}(3-n)}}\left[3\xi\frac{dG}{d\xi} - G\right]. \tag{3.167}$$

Equations (3.160) and (3.162) are parametric equations for G in terms of ξ. The parameter is g where $0 \le g \le 1$.

3.4.4 Discussion of the solution for $E(x) = E_0 x^n$

Consider now properties of the solution.

(i) Shear stress at the wall

In the boundary layer approximation the shear stress at the wall is

$$\tau_{yx}(x,0) = \frac{4\rho E_0^2}{(1+n)}x^{-\frac{1}{4}(5+n)}\frac{d^2G(0)}{d\xi^2}. \tag{3.168}$$

In order to evaluate (3.168), differentiate (3.137) with respect to ξ. This gives

$$\frac{d^2G}{d\xi^2} = \frac{1}{3G^{\frac{1}{2}}}[G_\infty^{\frac{3}{2}} - 4G^{\frac{3}{2}}]\frac{dG}{d\xi} \tag{3.169}$$

and substituting (3.137) for $\frac{dG}{d\xi}$ into (3.169) we obtain

$$\frac{d^2G}{d\xi^2} = \frac{2}{9}[G_\infty^{\frac{3}{2}} - 4G^{\frac{3}{2}}][G_\infty^{\frac{3}{2}} - G^{\frac{3}{2}}]. \tag{3.170}$$

Thus since $G(0) = 0$,

$$\frac{d^2G}{d\xi^2}(0) = \frac{2}{9}G_\infty^3 \tag{3.171}$$

and using (3.164) for G_∞, (3.168) becomes

$$\tau_{yx}(x,0) = \frac{\rho}{9}\left[\frac{125J^3(1+n)^5}{8E_0}\right]^{\frac{1}{4}} \frac{1}{x^{\frac{1}{4}(5+n)}}. \tag{3.172}$$

The stress $\tau_{yx}(x,0)$ is positive as required. The result due to Glauert for a laminar two-dimensional jet is obtained from (3.172) by setting $n = 0$ [11]. In Blasius flow past a flat plate $G''(0)$ is calculated numerically. It cannot be derived analytically, as for the wall jet. Since E_0 for a turbulent wall jet is 100 or even 1000 times greater than E_0 for a laminar jet, the shear stress at the wall in a turbulent jet is less than that for a laminar jet. The drag is decreased in the turbulent flow because diffusion is increased and therefore the velocity gradients normal to the wall are decreased. Since $1000^{\frac{1}{4}}$ is only 5.62 the decrease in the drag is not as great as may have been expected from the large increase in E_0.

(ii) Maximum value of $\bar{\mathbf{v}}_\mathbf{x}(\mathbf{x},\mathbf{y})$

To find the maximum value of the velocity $\bar{v}_x(x,y)$ as y varies at a given point x, differentiate (3.166) with respect to y keeping x constant :

$$\frac{\partial \bar{v}_x}{\partial y} = \frac{4E_0}{(1+n)} \frac{1}{x^{\frac{5}{4}(1+n)}} \frac{d^2G}{d\xi^2}. \tag{3.173}$$

From (3.170),

$$\frac{d^2G}{d\xi^2} = 0 \ \ when \ \ G = G_\infty \ or \ G = \left(\frac{1}{4}\right)^{\frac{2}{3}} G_\infty = 0.396G_\infty. \tag{3.174}$$

Now $G = G_\infty$ when $y = \infty$. When $G(\xi) = 0.396G_\infty$, from (3.161),

$$\frac{dG}{d\xi} = 0.315G_\infty^2. \tag{3.175}$$

Thus from (3.166) and using (3.164) for G_∞,

$$\bar{v}_{x\ max}(x) = 0.315\left[\frac{5}{2}\frac{J(1+n)}{E_0}\right]^{\frac{1}{2}} \frac{1}{x^{\frac{1}{2}(1+n)}}. \tag{3.176}$$

At position x, the maximum velocity of the wall jet is decreased by the turbulence since E_0 for a turbulent jet is much larger than E_0 for a laminar jet.

(iii) Width of the wall jet

From (3.162) and (3.163) for ξ, the width, W, of the wall jet is proportional to

$$W \propto \frac{1}{G_\infty} x^{\frac{3}{4}(1+n)} = \left[\frac{32E_0^3}{5J(1+n)^3}\right]^{\frac{1}{4}} x^{\frac{3}{4}(1+n)} \tag{3.177}$$

Equation (3.177) shows that the width increases like $E_0^{\frac{3}{4}}$ as the effective viscosity, E_0, increases due to the increase in diffusion and decreases like $J^{-\frac{1}{4}}$ as the strength of the jet, J, is increased. The dependence on n is more complicated and depends on whether $0 < x < 1$ or $x > 1$.

(iv) Prandtl′s hypothesis

In order to obtain an estimate for the exponent n in the power law $E(x) = E_0 x^n$ consider again Prandtl's hypothesis which states that the eddy viscosity is constant across a boundary layer and is proportional to the product of the maximum mean velocity and the width of the boundary layer [11]. The eddy viscosity $E(x) = E_0 x^n$ is constant across the wall jet because it is independent of y. Now the maximum mean velocity, from (3.176), is proportional to $x^{-\frac{1}{2}(1+n)}$ and from (3.177) the width of the wall jet is proportional to $x^{\frac{3}{4}(1+n)}$. Thus by Prandtl's hypothesis,

$$E_0 x^n \propto x^{-\frac{1}{2}(1+n)} x^{\frac{3}{4}(1+n)} \tag{3.178}$$

and therefore $n = \frac{1}{3}$.

A wall jet is between a free jet and a boundary layer. We will not adopt the Prandtl hypothesis but the calculation gives a value of n for comparison.

(v) Velocity profile

The $x-$component of the fluid velocity is given by (3.166) and using (3.161) it becomes

$$\bar{v}_x(x, y) = \left[\frac{10J(1+n)}{9E_0}\right]^{\frac{1}{2}} \frac{1}{x^{\frac{1}{2}(1+n)}} g(1 - g^3). \tag{3.179}$$

In the following pages we plot $\bar{v}_x(x, y)$ against y given in parametric form by (3.179) and (3.162) for $0 \leq g \leq 1$ at a fixed value of x in order to investigate the dependence of $\bar{v}_x(x, y)$ on J, n and E_0.

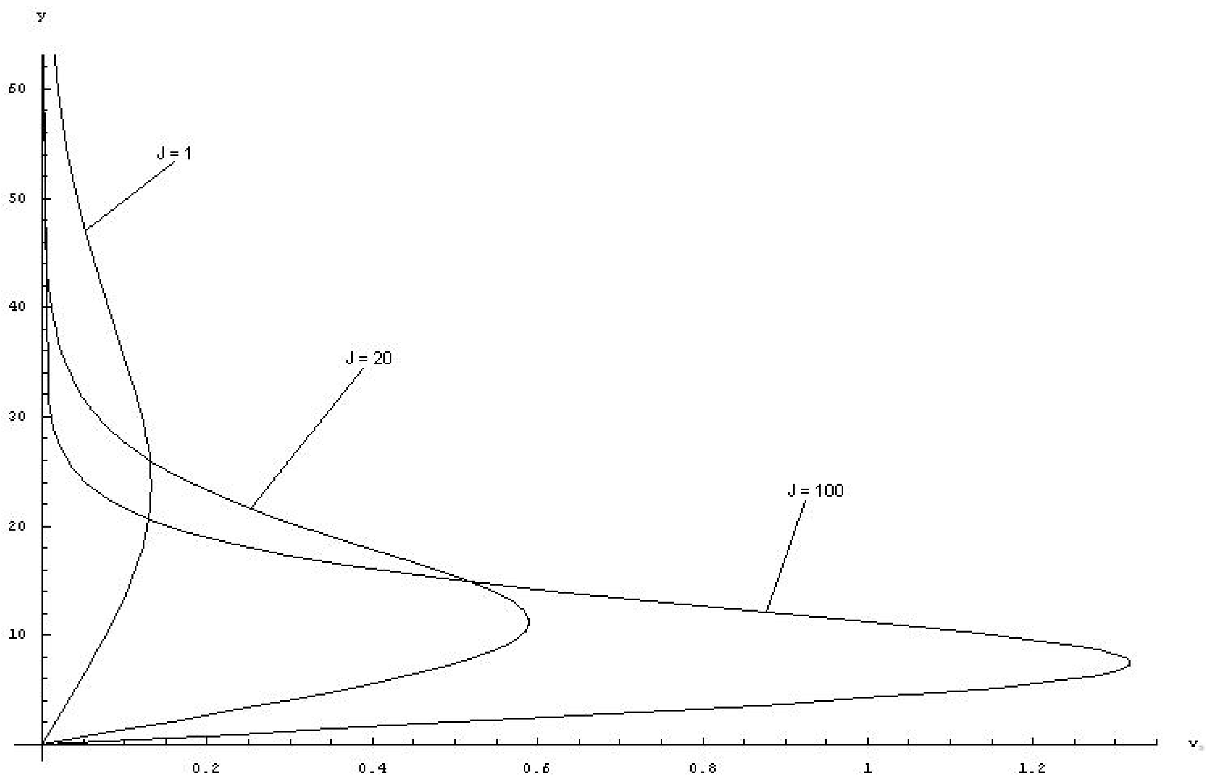

Figure 3.2: Velocity profile of two-dimensional wall jet varying J with $n = 6$, $E_0 = 100$, $x = 1$

In Figure 3.2, $\bar{v}_x(x, y)$ is plotted against y with n and E_0 kept fixed at $n = 6$ and $E_0 = 100$ while J takes the values $J = 1$, $J = 20$ and $J = 100$. As the strength of the jet, J, increases the maximum velocity, which by (3.176) is proportional to $J^{\frac{1}{2}}$, increases. The width of the jet, which by (3.177) is proportional to $J^{-\frac{1}{4}}$, decreases as J increases. Thus as the strength of the wall jet increases it becomes longer and narrower.

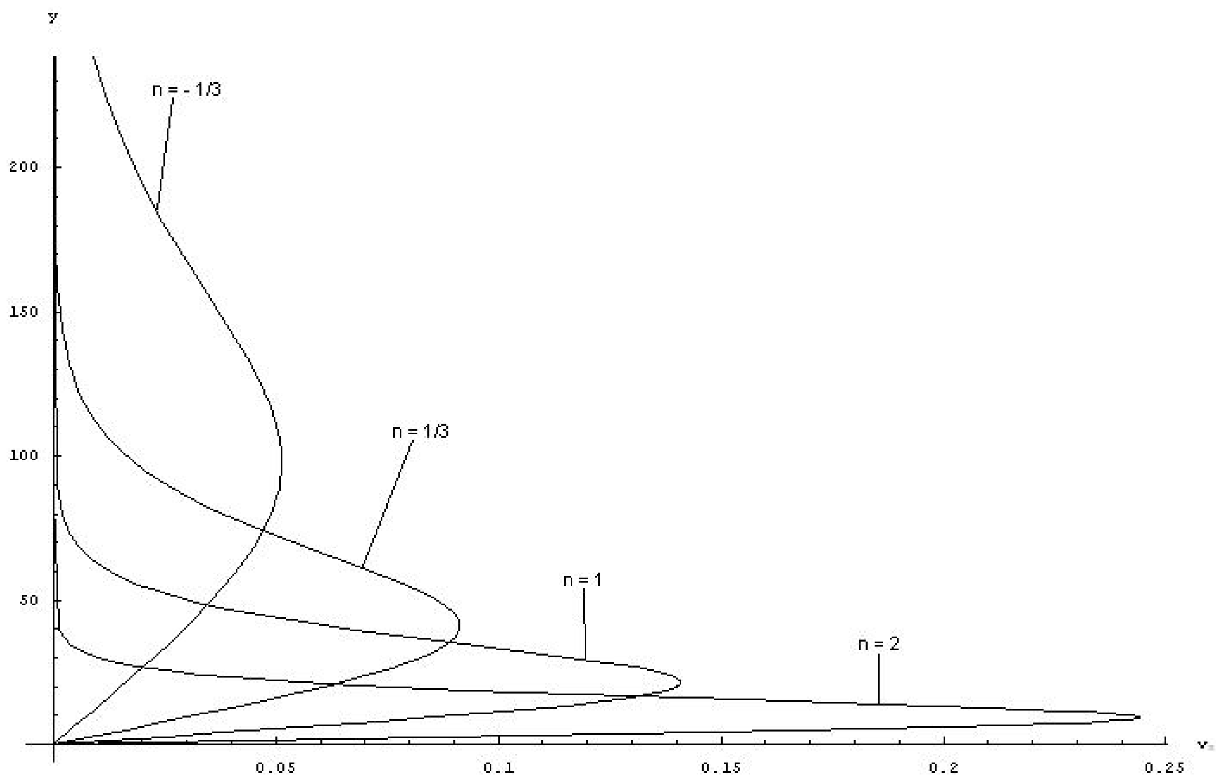

Figure 3.3: Velocity profile of two-dimensional wall jet varying n with $J = 1$, $E_0 = 100$, $x = 0.5$

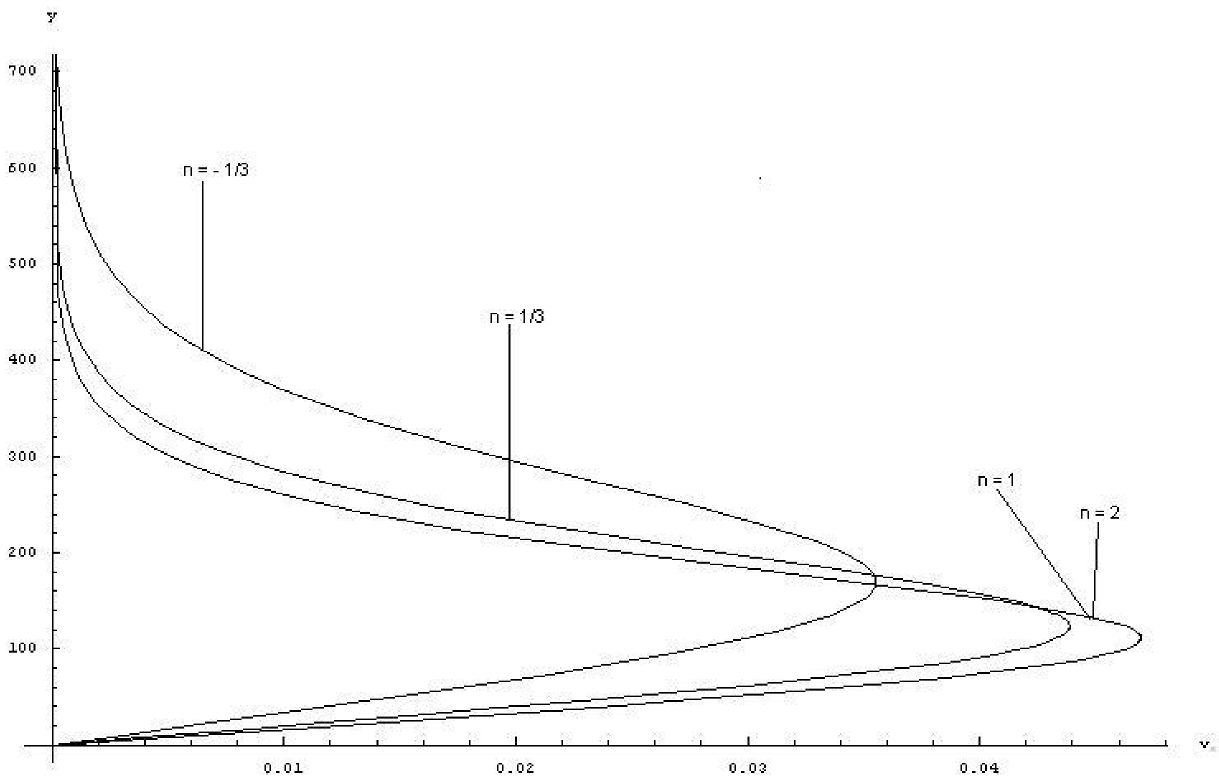

Figure 3.4: Velocity profile of two-dimensional wall jet varying n with $J = 1$, $E_0 = 100$, $x = 1.5$

In Figures 3.3 and 3.4, $\bar{v}_x(x, y)$ is plotted against y with J and E_0 kept fixed at $J = 1$ and $E_0 = 100$ while n takes the values $n = -\frac{1}{3}$, $n = \frac{1}{3}$, which is the value of n if Prandtl's hypothesis is satisfied, $n = 1$ and $n = 2$. In Figure 3.3, $x = 0.5$ and in Figure 3.4, $x = 1.5$. As n increases the wall jet becomes longer and narrower. Increasing x shortens and broadens the wall jet because there is more time for diffusion to occur across the jet.

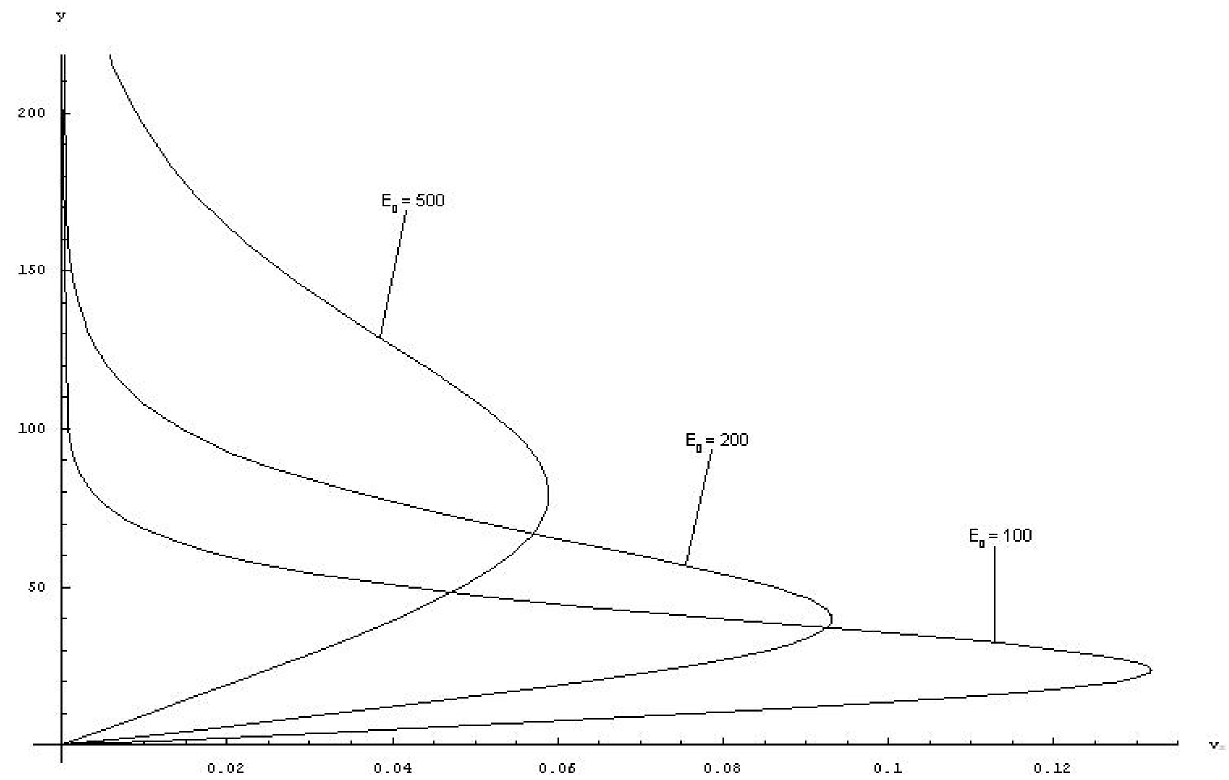

Figure 3.5: Velocity profile of two-dimensional wall jet varying E_0 with $n = 6$, $J = 1$, $x = 1$

In Figure 3.5, $\bar{v}_x(x, y)$ is plotted against y with J and n kept fixed at $J = 1$ and $n = 6$ while E_0 takes the values $E_0 = 100$, $E_0 = 200$ and $E_0 = 500$. As E_0 increases, the maximum velocity, which by (3.176) is proportional to $E_0^{-\frac{1}{2}}$, decreases. However the width, which from (3.177) is proportional to $E_0^{\frac{3}{4}}$, increases as E_0 increases. Thus as the effective viscosity increases, the wall jet becomes shorter and broader.

3.5 Effective viscosity of the form $E(x, v_x) = M(x)F(v_x)$

In this section we will consider the Lie point symmetries of the two-dimensional turbulent boundary layer equation where the effective viscosity is a function of x and of $\bar{v}_x$, the mean velocity in the x-direction :

$$E(x, \bar{v}_x) = M(x) f(\bar{v}_x), \tag{3.180}$$

where $M(x)$ is an arbitrary function of x. We saw in Section 3.3.3 that if there is no suction or blowing and no slip at the boundary then J given by (3.22) is a conserved quantity when the effective viscosity is given by (3.180). This conserved quantity is required in the solution.

The two-dimensional turbulent boundary layer equation in terms of the stream function is given by (2.84). For a wall jet, $W(x) = 0$ and (2.84) reduces to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial}{\partial y}\left(E(x, \bar{v}_x)\frac{\partial^2 \psi}{\partial y^2}\right) = 0, \tag{3.181}$$

where $E(x, \bar{v}_x)$ is given by (3.180). Substituting (3.180) into (3.181), we have

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - M(x)\frac{df}{d\psi_y}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 - M(x)f(\psi_y)\frac{\partial^3 \psi}{\partial y^3} = 0. \tag{3.182}$$

3.5.1 Derivation of Lie point symmetries

From the notation of Section 1.6.4

$$F(x, \psi_x, \psi_y, \psi_{xy}, \psi_{yy}, \psi_{yyy}) = \psi_y \psi_{xy} - \psi_x \psi_{yy} - M(x)\frac{df}{d\psi_y}\psi_{yy}^2 - M(x)f(\psi_y)\psi_{yyy} = 0, \tag{3.183}$$

where variable subscripts denote partial differentiation with respect to the variable that is in the subscript.

In order to find the Lie point symmetries we need to solve the determining equation (1.46),

$$X^{[3]}F|_{F=0} = 0, \tag{3.184}$$

where we will not need all the terms of generator (1.47). We will only need

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial \psi_x} + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{12}\frac{\partial}{\partial \psi_{xy}} + \zeta_{22}\frac{\partial}{\partial \psi_{yy}} + \zeta_{222}\frac{\partial}{\partial \psi_{yyy}}, \tag{3.185}$$

as the other partial derivatives of ψ do not occur in equation (3.183). We need to derive ζ_1, ζ_2, ζ_{12}, ζ_{22} and ζ_{222} using the formulae (1.48), (1.49) and (1.50). Details of these calculations are presented in Appendix A.1

Using (3.185) and (3.183), the determining equation (3.184) to find the Lie point symmetries of (3.182) becomes

$$\begin{aligned} &-\frac{dM}{dx}\frac{df}{d\psi_y}\psi_{yy}^2 \xi^1 - \frac{dM}{dx} f(\psi_y)\psi_{yyy}\xi^1 - \psi_{yy}\zeta_1 + \psi_{xy}\zeta_2 \\ &\quad -M(x)\frac{d^2 f}{d\psi_y^2}\psi_{yy}^2\zeta_2 - M(x)\frac{df}{d\psi_y}\psi_{yyy}\zeta_2 + \psi_y\zeta_{12} - \psi_x\zeta_{22} \\ &\qquad -2M(x)\frac{df}{d\psi_y}\psi_{yy}\zeta_{22} - M(x)f(\psi_y)\zeta_{222}|_{F=0} = 0. \end{aligned} \tag{3.186}$$

Now $M(x)f(\psi_y)\psi_{yyy}$ is replaced using the partial differential equation (3.183) :

$$M(x)f(\psi_y)\psi_{yyy} = \psi_y\psi_{xy} - \psi_x\psi_{yy} - M(x)\frac{df}{d\psi_y}\psi_{yy}^2. \tag{3.187}$$

Substituting the prolongation formulae (A.10), (A.11), (A.12), (A.13) and (A.14) which were calculated in Appendix A.1 and the expression for $M(x)f(\psi_y)\psi_{yyy}$ from equation (3.187) into (3.186), gives us the determining equation in expanded form. From this expanded equation which is presented in full in Appendix B, the coefficients of the partial derivatives of ψ can be equated and hence expressions for ξ^1, ξ^2 and η can be obtained. Details of these calculations are given in Appendix B.

In order to simplify the calculations we specified $f(\psi_y)$ as

$$f(\psi_y) = f_0 \psi_y^n, \qquad n > 0, \tag{3.188}$$

where f_0 is a constant. The condition $n > 0$ is to keep $f(\psi_y)$ finite as $\psi_y \to 0$ which occurs as $y \to \infty$ and $y \to 0$. We incorporate f_0 into $M(x)$ in (3.180) which is equivalent to taking $f_0 = 1$.

The following expressions for ξ^1, ξ^2 and η were found in Appendix B :

$$\xi^1 = A(x), \tag{3.189}$$

$$\xi^2 = c_3 y + K(x), \tag{3.190}$$

$$\eta = c_1 \psi + c_2, \tag{3.191}$$

where $A(x)$ and $K(x)$ are arbitrary functions of x and c_1, c_2 and c_3 are arbitrary constants provided

$$n \neq -1, \ \textit{and}\ n \neq -\frac{3}{2}. \tag{3.192}$$

The conditions (3.192) are satisfied since we are assuming that $n > 0$.

The following ordinary differential equation for $M(x)$ must be satisfied

$$\frac{dM}{dx} + \left[\frac{1}{A(x)} \frac{dA}{dx} + ((n-1)c_1 - (n+1)c_3) \frac{1}{A(x)} \right] M = 0, \tag{3.193}$$

in order for the Lie point symmetry to exist. At this point we note that the Lie point symmetry generator is the same as (2.96) derived in Chapter 2 for the boundary layer

$$X^{BL} = a(x)\frac{\partial}{\partial x} + (C_2 y + e(x))\frac{\partial}{\partial y} + ((C_1 + C_2)\psi + C_3)\frac{\partial}{\partial \psi}, \tag{3.194}$$

where the superscript ^{BL} describes the boundary layer. The Lie point symmetry generator given by (3.189) to (3.191) is

$$X^{WJ} = A(x)\frac{\partial}{\partial x} + (c_3 y + K(x))\frac{\partial}{\partial y} + (c_1 \psi + c_2)\frac{\partial}{\partial \psi}, \tag{3.195}$$

where the superscript ^{WJ} describes the wall jet. Since C_1, C_2, C_3, c_1, c_2 and c_3 are all arbitrary constants the results are in effect the same. The transformation from (3.194) to (3.195) is

$$C_2 = c_3, \quad C_1 + C_2 = c_1, \quad C_3 = c_2, \quad a(x) = A(x), \quad e(x) = K(x). \tag{3.196}$$

3.5.2 Derivation of group invariant solution

In this section we derive the group invariant solution using (3.195). We drop the superscript of ^{WJ} for simplicity.The derivation of a group invariant solution was outlined in Section 1.6.5.

Let $\psi = \Phi(x, y)$ be a group invariant solution of the partial differential equation (3.182). Then

$$X\,(\psi \;-\; \Phi(x, y))\,|_{\psi=\Phi(x,y)} = 0, \tag{3.197}$$

where X is defined by (3.195). Thus,

$$\left(A(x)\frac{\partial}{\partial x} + (c_3 y + K(x))\frac{\partial}{\partial y} + (c_1\psi + c_2)\frac{\partial}{\partial \psi}\right)(\psi \;-\; \Phi(x, y))\,|_{\psi=\Phi(x,y)} = 0. \tag{3.198}$$

Equation (3.198) can be written as

$$-A(x)\frac{\partial \Phi}{\partial x} - (c_3 y + K(x))\frac{\partial \Phi}{\partial y} + (c_1\psi + c_2)|_{\psi=\Phi(x,y)} = 0, \tag{3.199}$$

and imposing $\psi = \Phi(x, y)$ in (3.199) we obtain the first order quasi-linear partial differential equation

$$A(x)\frac{\partial \Phi}{\partial x} + (c_3 y + K(x))\frac{\partial \Phi}{\partial y} = c_1\Phi + c_2. \tag{3.200}$$

The differential equations of the characteristic curves of (3.200) are

$$\frac{dx}{A(x)} = \frac{dy}{c_3 y + K(x)} = \frac{d\Phi}{c_1\Phi + c_2}. \tag{3.201}$$

From the first and second terms in (3.201)

$$e^{-c_3\int \frac{1}{A(x)}dx}y - \int^x \frac{K(x)}{A(x)}\, e^{-c_3\int \frac{1}{A(x)}dx}\, dx = R_1, \tag{3.202}$$

where R_1 is a constant. We define

$$L(x) = \int^x \frac{1}{A(x)}dx \quad \textit{and} \quad H(x) = \int^x \frac{K(x)}{A(x)}\, e^{-c_3 L(x)}\, dx. \tag{3.203}$$

Thus (3.202) becomes

$$e^{-c_3 L(x)}y - H(x) = R_1. \tag{3.204}$$

Consider the case $c_1 \neq 0$. Using the first and third terms of (3.201) we obtain

$$(\Phi + \frac{c_2}{c_1})e^{-c_1 L(x)} = R_2, \tag{3.205}$$

where R_2 is a constant. The general solution is

$$R_2 = F(R_1), \tag{3.206}$$

that is,

$$\Phi(x,y) = e^{c_1 L(x)} F(e^{-c_3 L(x)} y - H(x)) - \frac{c_2}{c_1}, \tag{3.207}$$

where F is an arbitrary function. But $\psi = \Phi(x,y)$ and so the functional form for ψ is

$$\psi = e^{c_1 L(x)} F(\eta) - \frac{c_2}{c_1}, \qquad \eta = e^{-c_3 L(x)} y - H(x). \tag{3.208}$$

We now obtain the general solution for $M(x)$ which satisfies the first order linear ordinary differential equation (3.193) :

$$\frac{dM}{dx} + \left[\frac{1}{A(x)}\frac{dA}{dx} + \left((n-1)c_1 - (n+1)c_3\right)\frac{1}{A(x)}\right] M = 0. \tag{3.209}$$

The solution to (3.209) is

$$M(x) = \frac{M_0}{A(x)} exp[\left((n+1)c_3 - (n-1)c_1\right) L(x)], \tag{3.210}$$

where M_0 is an arbitrary constant.

In summary, we have the following expressions for the stream function $\psi(x,y)$, the effective viscosity $E(x,y)$ and $M(x)$:

$$\psi(x,y) = e^{c_1 L(x)} F(\eta) - \frac{c_2}{c_1}, \tag{3.211}$$

$$E(x,y) = M(x) f(\psi_y), \tag{3.212}$$

$$M(x) = \frac{M_0}{A(x)} exp[\left((n+1)c_3 - (n-1)c_1\right) L(x)], \tag{3.213}$$

where

$$\eta = e^{-c_3 L(x)} y - H(x), \qquad f(\psi_y) = \psi_y^n, \qquad n \neq -1, n \neq -\frac{3}{2}, \tag{3.214}$$

and $L(x)$ and $H(x)$ are defined by (3.203).

We now substitute (3.211), (3.212), (3.213) and (3.214) into the partial differential equation (3.182)

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - M(x)\frac{df}{d\psi_y}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 - M(x) f(\psi_y)\frac{\partial^3 \psi}{\partial y^3} = 0. \tag{3.215}$$

Details of the calculation are presented in Appendix $B.3$ assuming $c_1 \neq 0$ and $A(x) \neq 0$. The following third order ordinary differential equation for $F(\eta)$ is found

:

$$\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2}\right] + \frac{c_1}{M_0}\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right) - \frac{(2c_1 - c_3)}{M_0}\left(\frac{dF}{d\eta}\right)^2 = 0. \tag{3.216}$$

The differential equation (3.216) does not depend on the ratio $\frac{c_2}{c_1}$ or on $H(x)$. The mean velocity components $\bar{v}_x$ and $\bar{v}_y$ are defined in terms of the derivatives of $\psi(x, y)$ and therefore do not depend on $\frac{c_2}{c_1}$. Thus c_2 and $H(x)$ can be chosen suitably. We choose

$$c_2 = 0 \tag{3.217}$$

and in order that $\eta = 0$ when $y = 0$ we choose $H(x) = 0$. To give $H(x) = 0$ we choose, from (3.203),

$$K(x) = 0. \tag{3.218}$$

Thus (3.211) to (3.214) become provided $c_1 \neq 0$,

$$\psi(x, y) = e^{c_1 L(x)} F(\eta), \tag{3.219}$$

$$E(x, y) = M(x)\psi_y^n, \quad n > 0 \tag{3.220}$$

$$M(x) = \frac{M_0}{A(x)} exp[((n+1)c_3 - (n-1)c_1)\, L(x)], \tag{3.221}$$

$$\eta = e^{-c_3 L(x)} y, \tag{3.222}$$

where from (3.203),

$$L(x) = \int^x \frac{dx}{A(x)}, \qquad A(x) = \frac{dL}{dx} \tag{3.223}$$

and from (3.189) to (3.191),

$$X = A(x)\frac{\partial}{\partial x} + c_3 y\frac{\partial}{\partial y} + c_1\psi\frac{\partial}{\partial \psi}. \tag{3.224}$$

3.5.3 Conserved quantity

We saw in Section 3.3.2 that when the effective viscosity is of the form

$$E(x, y, v_x) = M(x)\bar{v}_x(x, y)^n \tag{3.225}$$

then

$$J = \int_0^\infty \bar{v}_x(x, y)\left(\int_y^\infty \bar{v}_x^2(x, y^*)dy^*\right) dy = \; constant\ independent\ of\ x, \tag{3.226}$$

provided (3.34) holds, that is provided

$$\bar{v}_n(x)\int_0^\infty \bar{v}_x^2(x, y)dy + \frac{1}{(n+2)} M(x)\bar{v}_s^{n+2}(x) = 0. \tag{3.227}$$

Glauert [11] considered the case $n = 6$ based on the empirical law of Blasius at the boundary. We will consider general $n > 0$.

We assume that (3.227) is satisfied so that J is a conserved quantity. Substitute the group invariant solution (3.219) where η is given by (3.222) into (3.226). Change the variable of integration from y to η at any given fixed point x. Then

$$dy = e^{c_3 L(x)} d\eta \tag{3.228}$$

and

$$\bar{v}_x(x, y) = \frac{\partial \psi}{\partial y} = e^{(C_1 - C_3)L(x)} \frac{dF}{d\eta}. \tag{3.229}$$

Using (3.228) and (3.229), equation (3.226) becomes

$$J = e^{(3c_1 - c_3)L(x)} \int_0^\infty \frac{dF}{d\eta} \left(\int_\eta^\infty \left(\frac{dF}{d\eta^*} \right)^2 d\eta^* \right) d\eta. \tag{3.230}$$

Thus J is a constant independent of x provided

$$c_3 = 3c_1. \tag{3.231}$$

Substitute (3.231) and (3.229) into (3.216), (3.219), (3.220), (3.230), (3.222) and (3.224):

$$\frac{d}{d\eta} \left[\left(\frac{dF}{d\eta} \right)^n \frac{d^2 F}{d\eta^2} \right] + \frac{c_1}{M_0} \frac{d}{d\eta} \left(F \frac{dF}{d\eta} \right) + \frac{c_1}{M_0} \left(\frac{dF}{d\eta} \right)^2 = 0. \tag{3.232}$$

$$\psi(x, y) = e^{c_1 L(x)} F(\eta), \tag{3.233}$$

$$E(x, y) = \frac{M_0}{A(x)} exp(4c_1 L(x)) \left(\frac{dF}{d\eta} \right)^n, \quad n > 0, \tag{3.234}$$

$$\eta = e^{-3c_1 L(x)} y, \tag{3.235}$$

$$J = \int_0^\infty \frac{dF}{d\eta} \left(\int_\eta^\infty \left(\frac{dF}{d\eta^*} \right)^2 d\eta^* \right) d\eta, \tag{3.236}$$

$$X = A(x) \frac{\partial}{\partial x} + 3c_1 y \frac{\partial}{\partial y} + c_1 \psi \frac{\partial}{\partial \psi}, \tag{3.237}$$

where

$$L(x) = \int^x \frac{dx}{A(x)}. \tag{3.238}$$

To simplify the notation

$$H(x) = exp(c_1 L(x)). \tag{3.239}$$

Then

$$\frac{c_1}{A(x)} = \frac{H'(x)}{H(x)} \tag{3.240}$$

and equations (3.233) to (3.235) and (3.237) become

$$\psi(x,y) = H(x)F(\eta), \tag{3.241}$$

$$E(x,y) = \frac{M_0}{c_1} H'(x) H(x)^3 \left(\frac{dF}{d\eta}\right)^n, \quad n > 0, \tag{3.242}$$

$$\eta = \frac{y}{H(x)^3}, \tag{3.243}$$

$$X = \frac{H(x)}{H'(x)} \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial \psi}. \tag{3.244}$$

Finally, in order to remove $\frac{c_1}{M_0}$ from the ordinary differential equation let

$$\bar{\eta} = P\eta \tag{3.245}$$

where P is a constant still to be determined. Then (3.232) becomes

$$\frac{M_0}{c_1} P^{n+1} \frac{d}{d\bar{\eta}} \left[\left(\frac{dF}{d\bar{\eta}}\right)^n \frac{d^2F}{d\bar{\eta}^2} \right] + \frac{d}{d\bar{\eta}} \left(F \frac{dF}{d\bar{\eta}} \right) + \left(\frac{dF}{d\bar{\eta}}\right)^2 = 0. \tag{3.246}$$

Choose

$$P = \left(\frac{c_1}{M_0}\right)^{\frac{1}{n+1}} \tag{3.247}$$

and define

$$E_0 = \frac{M_0}{c_1}. \tag{3.248}$$

Equations (3.246), (3.236) and (3.241) to (3.244) become

$$\frac{d}{d\bar{\eta}} \left[\left(\frac{dF}{d\bar{\eta}}\right)^n \frac{d^2F}{d\bar{\eta}^2} \right] + \frac{d}{d\bar{\eta}} \left(F \frac{dF}{d\bar{\eta}} \right) + \left(\frac{dF}{d\bar{\eta}}\right)^2 = 0, \tag{3.249}$$

$$\psi(x,y) = H(x)F(\bar{\eta}), \tag{3.250}$$

$$E(x,y) = E_0^{\frac{1}{1+n}} H'(x) H^3(x) \left(\frac{dF}{d\bar{\eta}}\right)^n, \quad n > 0 \tag{3.251}$$

$$\bar{\eta} = \frac{y}{E_0^{\frac{1}{1+n}} H(x)^3}, \tag{3.252}$$

$$J = E_0^{-\frac{1}{1+n}} \int_0^\infty \frac{dF}{d\bar{\eta}} \left(\int_{\bar{\eta}}^\infty \left(\frac{dF}{d\bar{\eta}^*} \right)^2 d\bar{\eta}^* \right) d\bar{\eta}, \tag{3.253}$$

$$X = \frac{H(x)}{H'(x)} \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial \psi}. \tag{3.254}$$

A boundary condition on $H(x)$ is required. The long narrow orifice from which the jet emerges is assumed to be infinitely thin. Since the volume of the mean flow and the mean momentum are finite we must assume that the mean fluid velocity at the orifice is infinite. Now

$$\bar{v_{\hat{x}}}(\hat{x}, 0) = \frac{\partial \psi}{\partial y} = \frac{1}{E_0^{\frac{1}{1+n}}} \frac{1}{H(x)^2} \frac{dF}{d\bar{\eta}}(0) \tag{3.255}$$

and for $\bar{v_{\hat{x}}}(\hat{x}, 0) \to \infty$ as $x \to 0$ it is necessary that

$$H(0) = 0. \tag{3.256}$$

Consider now the boundary conditions. We will assume that there is no suction or blowing at the boundary, $y = 0$, and that there is no slip at the boundary. Then $\bar{v}_n = 0$ and $\bar{v}_s = 0$ and condition (3.227) is satisfied. The solution for a laminar jet with $\bar{v}_n \neq 0$ and $\bar{v}_s \neq 0$ has been considered by Merkin and Needham(1986)[21]. We also assume that $\bar{v}_x(x, y) \to 0$ as $y \to \infty$. Now

$$\bar{v_x}(x, y) = \frac{\partial \psi}{\partial y} = \frac{1}{E_0^{\frac{1}{1+n}}} \frac{1}{H(x)^2} \frac{dF}{d\bar{\bar{\eta}}}(\bar{\eta}), \tag{3.257}$$

$$\bar{v_y}(x, y) = -\frac{\partial \psi}{\partial x} = H'(x) \left(3\bar{\eta} \frac{dF}{d\bar{\eta}} - F(\bar{\eta}) \right), \tag{3.258}$$

and therefore

$$\bar{v}_x(x, 0) = 0 : \qquad \frac{dF}{d\bar{\eta}}(0) = 0, \tag{3.259}$$

$$\bar{v}_y(x, 0) = 0 : \qquad F(0) = 0, \tag{3.260}$$

provided $H'(x) \neq 0$ and

$$\bar{v}_x(x, \infty) = 0 : \qquad \frac{dF}{d\bar{\eta}}(\infty) = 0. \tag{3.261}$$

The problem is summarised as follows. The overhead bar on η is suppressed to simplify the notation.

$$\frac{d}{d\eta} \left[\left(\frac{dF}{d\eta} \right)^n \frac{d^2 F}{d\eta^2} \right] + \frac{d}{d\eta} \left(F \frac{dF}{d\eta} \right) + \left(\frac{dF}{d\eta} \right)^2 = 0, \tag{3.262}$$

$$F(0) = 0, \qquad \frac{dF}{d\eta}(0) = 0, \qquad \frac{dF}{d\eta}(\infty) = 0, \tag{3.263}$$

$$J = E_0^{-\frac{1}{1+n}} \int_0^\infty \frac{dF}{d\eta} \left(\int_\eta^\infty \left(\frac{dF}{d\eta^*} \right)^2 d\eta^* \right) d\eta. \tag{3.264}$$

$$\psi(x, y) = H(x) F(\eta), \tag{3.265}$$

$$E(x, y) = E_0^{\frac{1}{1+n}} H'(x) H^3(x) \left(\frac{dF}{d\eta} \right)^n, \quad n > 0, \tag{3.266}$$

$$\eta = \frac{y}{E_0^{\frac{1}{1+n}} H(x)^3}, \tag{3.267}$$

$$H(0) = 0, \tag{3.268}$$

$$X = \frac{H(x)}{H'(x)} \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial \psi}, \tag{3.269}$$

where E_0 and J are given and $H(x)$ is either given or determined from a given condition on the effective viscosity.

Consider first the limit of a laminar jet. Then $n = 0$ and (3.266) becomes

$$E = E_0 H'(x) H(x)^3. \tag{3.270}$$

Since E is constant for a laminar jet,

$$H(x)^3 \frac{dH}{dx} = \alpha \tag{3.271}$$

where α is a constant. Integrating (3.271) gives

$$H(x) = (4\alpha x + \beta)^{\frac{1}{4}} \tag{3.272}$$

where β is a constant. But $H(0) = 0$ and therefore $\beta = 0$. Substituting (3.272) into (3.270) gives

$$E = E_0 \alpha. \tag{3.273}$$

We interpret E_0 as the kinematic viscosity of the jet. Thus $\alpha = 1$ and

$$H(x) = (4x)^{\frac{1}{4}}. \tag{3.274}$$

Equations (3.262) to (3.269) become

$$\frac{d^3 F}{d\eta^3} + \frac{d}{d\eta} \left(F \frac{dF}{d\eta} \right) + \left(\frac{dF}{d\eta} \right)^2 = 0, \tag{3.275}$$

$$F(0) = 0, \qquad \frac{dF}{d\eta}(0) = 0, \qquad \frac{dF}{d\eta}(\infty) = 0, \tag{3.276}$$

$$J = \frac{1}{E_0} \int_0^\infty \frac{dF}{d\eta} \left(\int_\eta^\infty \left(\frac{dF}{d\eta^*} \right)^2 d\eta^* \right) d\eta, \tag{3.277}$$

$$\psi(x,y) = \sqrt{2}x^{\frac{1}{4}}F(\eta), \tag{3.278}$$

$$E = E_0, \tag{3.279}$$

$$\eta = \frac{1}{E_0}\frac{y}{(4x)^{\frac{3}{4}}}, \tag{3.280}$$

$$X = 4x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}. \tag{3.281}$$

The solution of this problem was obtained in parametric form by Glauert in [11] and is given in Section 3.4.3 with $n = 0$.

3.5.4 Solution for $n > 0$

Consider now the solution of the differential equation (3.262) for $n > 0$ subject to the boundary conditions (3.263). Multiply (3.262) by F.

$$F\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2}\right] + F^2\frac{d^2F}{d\eta^2} + 2F\left(\frac{dF}{d\eta}\right)^2 = 0. \tag{3.282}$$

Now

$$F\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2}\right] = \frac{d}{d\eta}\left[F\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2}\right] - \frac{1}{(n+2)}\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^{n+2}\right] \tag{3.283}$$

and

$$F^2\frac{d^2F}{d\eta^2} + 2F\left(\frac{dF}{d\eta}\right)^2 = \frac{d}{d\eta}\left[F^2\frac{dF}{d\eta}\right]. \tag{3.284}$$

Using (3.283) and (3.284), equation (3.282) becomes

$$\frac{d}{d\eta}\left[F\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2}\right] - \frac{1}{(n+2)}\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^{n+2}\right] + \frac{d}{d\eta}\left[F^2\frac{dF}{d\eta}\right] = 0. \tag{3.285}$$

Integrating with respect to η, we obtain

$$F\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2} - \frac{1}{(n+2)}\left(\frac{dF}{d\eta}\right)^{n+2} + F^2\frac{dF}{d\eta} = C \tag{3.286}$$

where C is a constant.

To obtain C we consider the boundary condition (3.263) at $\eta = 0$. We need the behaviour of $F''(\eta)$ as $\eta \to 0$. Since $F'(0) = 0$ it follows that

$$F'(\eta) = O(\eta^\alpha), \qquad F''(\eta) = O(\eta^{\alpha-1}), \qquad F(\eta) = O(\eta^{\alpha+1}), \tag{3.287}$$

as $\eta \to 0$ where $\alpha > 0$. Thus

$$F\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2} = O(\eta^{(n+2)\alpha}), \tag{3.288}$$

as $\eta \to 0$. Thus imposing the boundary condition (3.263) at $\eta = 0$ gives $C = 0$ and therefore

$$F\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2} - \frac{1}{(n+2)}\left(\frac{dF}{d\eta}\right)^{n+2} + F^2\frac{dF}{d\eta} = 0. \tag{3.289}$$

Multiply (3.289) by F^β where the constant β has still to be chosen :

$$F^{1+\beta}\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2} - \frac{1}{(n+2)}F^\beta\left(\frac{dF}{d\eta}\right)^{n+2} + F^{2+\beta}\frac{dF}{d\eta} = 0. \tag{3.290}$$

Now

$$F^{1+\beta}\left(\frac{dF}{d\eta}\right)^n \frac{d^2F}{d\eta^2} = \frac{1}{1+n}\frac{d}{d\eta}\left[F^{1+\beta}\left(\frac{dF}{d\eta}\right)^{n+1}\right] - \frac{1+\beta}{1+n}F^\beta\left(\frac{dF}{d\eta}\right)^{n+2} \tag{3.291}$$

and

$$F^{2+\beta}\frac{dF}{d\eta} = \frac{1}{3+\beta}\frac{d}{d\eta}(F^{3+\beta}). \tag{3.292}$$

Using (3.291) and (3.292), equation (3.290) becomes

$$\frac{d}{d\eta}\left[F^{1+\beta}\left(\frac{dF}{d\eta}\right)^{n+1}\right] - \left(\beta + \frac{2n+3}{n+2}\right)F^\beta\left(\frac{dF}{d\eta}\right)^{n+2} + \frac{n+1}{3+\beta}\frac{d}{d\eta}(F^{3+\beta}) = 0 \tag{3.293}$$

In order to remove the second term from (3.293), choose

$$\beta = -\left(\frac{2n+3}{n+2}\right). \tag{3.294}$$

Equation (3.293) reduces to

$$\frac{d}{d\eta}\left[F^{-\left(\frac{n+1}{n+2}\right)}\left(\frac{dF}{d\eta}\right)^{n+1}\right] + \frac{(n+1)(n+2)}{(n+3)}\frac{d}{d\eta}(F^{\frac{n+3}{n+2}}) = 0. \tag{3.295}$$

Integrating with respect to η gives

$$F^{-\left(\frac{n+1}{n+2}\right)}\left(\frac{dF}{d\eta}\right)^{n+1} + \frac{(n+1)(n+2)}{(n+3)}F^{\frac{n+3}{n+2}} = D, \tag{3.296}$$

where D is constant.

To investigate if the constant D can be obtained from the boundary conditions (3.263) at $\eta = 0$ consider the behaviour of the first term in (3.296) for small η. From

(3.287),

$$F^{-\left(\frac{n+1}{n+2}\right)}\left(\frac{dF}{d\eta}\right)^{n+1} = O(\eta^{\left(\frac{n+1}{n+2}\right)(\alpha(n+1)-1)}) \tag{3.297}$$

and therefore

$$F^{-\left(\frac{n+1}{n+2}\right)}\left(\frac{dF}{d\eta}\right)^{n+1} \to 0 \quad as \quad \eta \to 0 \tag{3.298}$$

provided

$$\alpha > \frac{1}{(n+1)} \tag{3.299}$$

Condition (3.299) is not necessarily satisfied. We therefore impose the boundary condition at large values of η. Suppose that

$$\frac{dF}{d\eta} = 0 \quad at \quad \eta = \eta_0 \tag{3.300}$$

and let

$$F_0 = F(\eta_0) \tag{3.301}$$

For a laminar jet, $n = 0$, and $\eta_0 = \infty$ consistent with the boundary condition (3.276)(Glauert 1956). We will see that for a turbulent jet with $n > 0$, η_0 is finite. From (3.296) and (3.300),

$$D = \frac{(n+1)(n+2)}{(n+3)} F_0^{\frac{n+3}{n+2}} \tag{3.302}$$

and equation (3.296) becomes

$$\left[\frac{d}{d\eta}\left(\frac{F}{F_0}\right)\right]^{n+1} = \frac{(n+1)(n+2)}{(n+3)} F_0^{1-n}\left(\frac{F}{F_0}\right)^{\frac{n+1}{n+2}}\left[1 - \left(\frac{F}{F_0}\right)^{\frac{n+3}{n+2}}\right]. \tag{3.303}$$

Define

$$g = \left(\frac{F}{F_0}\right)^{\frac{n+1}{n+2}} \tag{3.304}$$

Equation (3.303) becomes

$$\frac{dg}{d\eta} = \left(\frac{n+1}{n+2}\right)\left(\frac{(n+1)(n+2)}{(n+3)}\right)^{\frac{1}{n+1}} F_0^{\frac{1-n}{1+n}}\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}} \tag{3.305}$$

and therefore

$$\int^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}} = \left(\frac{n+1}{n+2}\right)\left(\frac{(n+1)(n+2)}{(n+3)}\right)^{\frac{1}{n+1}} F_0^{\frac{1-n}{1+n}}\eta + K, \tag{3.306}$$

where K is a constant. But since $F(0) = 0$,

$$\eta = 0: \qquad g(0) = 0 \tag{3.307}$$

and therefore

$$K = \int^{0} \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}}. \tag{3.308}$$

Thus (3.306) becomes

$$\eta = \left(\frac{n+2}{n+1}\right)\left(\frac{(n+3)}{(n+1)(n+2)}\right)^{\frac{1}{n+1}} F_0^{\frac{n-1}{n+1}} \int_0^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}}. \tag{3.309}$$

For the solution (3.309) to extend to $\eta = \infty$ the integral in (3.309) must diverge as $g \to 1$. To investigate the behaviour of the integral as $g \to 1$, let

$$h(\eta) = 1 - g(\eta) \tag{3.310}$$

and consider the behaviour of the integral as $h(\eta) \to 0$. Now

$$\int_0^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}} = \int_h^1 \frac{dh}{\left[1 - (1-h)^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}} \tag{3.311}$$

and expanding for small h,

$$\int_h \frac{dh}{\left[1 - (1-h)^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}} = \left(\frac{n+1}{n+3}\right)^{\frac{1}{n+1}} \int_h \frac{1}{h^{\frac{1}{n+1}}}(1 + O(h))dh, \tag{3.312}$$

as $h \to 0$. But

$$\int_h \frac{dh}{h^{\frac{1}{n+1}}} = \begin{cases} - \\ \text{lnh}, & n = 0 \\ -\frac{(n+1)}{n}\, h^{\frac{n}{n+1}}, & n > 0. \end{cases}$$

Thus for $n = 0$, the integral is divergent as $h \to 0$ and the solution extends to infinity. Hence $\eta_0 = \infty$. For $n > 0$ the integral is convergent as $h \to 0$, the solution does not extend to infinity and η_0 is finite.

In summary we have

$$g = \left(\frac{F}{F_0}\right)^{\frac{n+1}{n+2}}, \qquad 0 \leq g \leq 1, \tag{3.313}$$

$$\frac{dg}{d\eta} = \left(\frac{n+1}{n+2}\right)\left(\frac{(n+1)(n+2)}{(n+3)}\right)^{\frac{1}{n+1}} F_0^{\frac{1-n}{1+n}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}, \tag{3.314}$$

$$\int_0^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}} = \left(\frac{n+1}{n+2}\right)\left(\frac{(n+1)(n+2)}{(n+3)}\right)^{\frac{1}{n+1}} F_0^{\frac{1-n}{1+n}} \eta. \tag{3.315}$$

These results reduce to the results of Glauert [11] for a laminar wall jet when $n = 0$ and for a turbulent wall jet when $n = 6$.

It remains to calculate F_0 which is obtained from the conserved quantity (3.264),

$$JE_0^{\frac{1}{1+n}} = \int_0^{\eta_0} \frac{dF}{d\eta} \left[\int_{\eta}^{\eta_0} \left(\frac{dF}{d\eta^*} \right)^2 d\eta^* \right] d\eta, \tag{3.316}$$

where $F_0 = F(\eta_0)$ and we have seen that η_0 is finite for $n > 0$. Now

$$\frac{dF}{d\eta} d\eta = dF, \qquad \left(\frac{dF}{d\eta} \right)^* d\eta^* = dF^* \tag{3.317}$$

and thus (3.316) becomes

$$JE_0^{\frac{1}{1+n}} = \int_0^{F_0} \left[\int_F^{F_0} \left(\frac{dF}{d\eta} \right)^* dF^* \right] dF. \tag{3.318}$$

Let

$$\hat{F} = \frac{F}{F_0}. \tag{3.319}$$

Substituting (3.319) into (3.318) gives

$$JE_0^{\frac{1}{1+n}} = F_0^3 \int_0^1 \left[\int_{\hat{F}}^1 \left(\frac{d\hat{F}}{d\eta} \right)^* d\hat{F}^* \right] d\hat{F}. \tag{3.320}$$

But from equation (3.303)

$$\frac{d\hat{F}}{d\eta} = \left[\frac{(n+1)(n+2)}{(n+3)} \right]^{\frac{1}{n+1}} F_0^{\frac{1-n}{1+n}} \hat{F}^{\frac{1}{n+2}} \left[1 - \hat{F}^{\frac{n+3}{n+2}} \right]^{\frac{1}{n+1}} \tag{3.321}$$

and (3.320) becomes

$$JE_0^{\frac{1}{1+n}} = F_0^{2(\frac{2+n}{1+n})} \left[\frac{(n+1)(n+2)}{(n+3)} \right]^{\frac{1}{n+1}} \int_0^1 \left[\int_{\hat{F}}^1 \hat{F}^{\frac{1}{n+2}} \left[1 - \hat{F}^{\frac{n+3}{n+2}} \right]^{\frac{1}{n+1}} d\hat{F} \right] d\hat{F}. \tag{3.322}$$

Let

$$\hat{F}^{\frac{n+3}{n+2}} = u, \qquad \left(\frac{n+3}{n+2} \right) \hat{F}^{\frac{1}{n+2}} d\hat{F} = du. \tag{3.323}$$

Thus (3.322) transforms to

$$JE_0^{\frac{1}{1+n}} = F_0^{2(\frac{2+n}{1+n})} \left(\frac{n+2}{n+3} \right) \left[\frac{(n+1)(n+2)}{(n+3)} \right]^{\frac{1}{n+1}} \int_0^1 \left[\int_u^1 (1-u)^{\frac{1}{n+1}} du \right] d\hat{F}. \tag{3.324}$$

But

$$\int_0^1 \left[\int_u^1 (1-u)^{\frac{1}{n+1}} du \right] d\hat{F} = \frac{n+1}{n+2} \int_0^1 \left(1 - \hat{F}^{\frac{n+3}{n+2}} \right)^{\frac{n+2}{n+1}} d\hat{F} \tag{3.325}$$

and (3.324) becomes

$$JE_0^{\frac{1}{1+n}} = F_0^{2(\frac{2+n}{1+n})} \left(\frac{n+1}{n+3}\right) \left[\frac{(n+1)(n+2)}{(n+3)}\right]^{\frac{1}{n+1}} I(n). \tag{3.326}$$

where

$$I(n) = \int_0^1 \left(1 - w^{\frac{n+3}{n+2}}\right)^{\frac{n+2}{n+1}} dw. \tag{3.327}$$

Solving (3.326) for F_0 gives

$$F_0 = \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{(1+n)}{2(2+n)}} \left[\frac{n+3}{(n+1)(n+2)}.\right]^{\frac{1}{2}}. \tag{3.328}$$

We can now eliminate F_0 from the results for $F(\eta)$, η and $\frac{dF}{d\eta}$. From (3.313) we have

$$F(\eta) = \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{(1+n)}{2(2+n)}} \left[\frac{n+3}{(n+1)(n+2)}.\right]^{\frac{1}{2}} g^{\frac{n+2}{n+1}} \tag{3.329}$$

and from (3.315)

$$\eta = \left(\frac{n+2}{n+1}\right) \left[\frac{(n+3)}{(n+1)(n+2)}\right]^{\frac{1}{2}} \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{n-1}{2(n+2)}} \int_0^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}}. \tag{3.330}$$

Now

$$\frac{dF}{d\eta} = \left(\frac{n+2}{n+1}\right) F_0 g^{\frac{1}{n+1}} \frac{dg}{d\eta} \tag{3.331}$$

and using (3.314) for $\frac{dg}{d\eta}$ and (3.328) for F_0 we obtain

$$\frac{dF}{d\eta} = \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{1}{2+n}} g^{\frac{1}{n+1}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}. \tag{3.332}$$

The following is a summary of the results for $n \geq 0$:

$$I(n) = \int_0^1 \left(1 - w^{\frac{n+3}{n+2}}\right)^{\frac{n+2}{n+1}} dw, \tag{3.333}$$

$$\eta = \left(\frac{n+2}{n+1}\right) \left[\frac{(n+3)}{(n+1)(n+2)}\right]^{\frac{1}{2}} \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{n-1}{2(n+2)}} \int_0^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}}, \tag{3.334}$$

$$F(\eta) = \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{(1+n)}{2(2+n)}} \left[\frac{n+3}{(n+1)(n+2)}.\right]^{\frac{1}{2}} g^{\frac{n+2}{n+1}}, \tag{3.335}$$

$$\frac{dF}{d\eta} = \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{1}{2+n}} g^{\frac{1}{n+1}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}, \tag{3.336}$$

$$\frac{dg}{d\eta} = \left(\frac{n+1}{n+2}\right)\left(\frac{(n+1)(n+2)}{(n+3)}\right)^{\frac{1}{2}}\left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{1-n}{2(2+n)}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}, \tag{3.337}$$

where $0 \leq g \leq 1$,

$$E(x, y) = E_0^{\frac{1}{1+n}} H'(x)H^3(x)\left(\frac{dF}{d\eta}\right)^n, \tag{3.338}$$

$$\psi(x, y) = H(x)F(\eta), \tag{3.339}$$

$$\eta = \frac{y}{E_0^{\frac{1}{n+1}} H^3(x)}, \tag{3.340}$$

$$\bar{v}_x(x, y) = \frac{1}{E_0^{\frac{1}{n+1}} H^2(x)} \frac{dF}{d\eta}, \tag{3.341}$$

$$\bar{v}_y(x, y) = H'(x)\left(3\eta\frac{dF}{d\eta} - F(\eta)\right), \tag{3.342}$$

$$X = \frac{H(x)}{H'(x)}\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}, \tag{3.343}$$

where $H(x)$ is determined from the effective viscosity $E(x, y)$, subject to the condition

$$H(0) = 0 \tag{3.344}$$

and E_0 and J, the strength of the wall jet, are given constants.

3.5.5 Discussion of the solution

(i) Expression for H(x)

The function $H(x)$ is determined from the given form for the effective viscosity. Expressing the effective viscosity in terms of the mean velocity $\bar{v}_x(x, y)$ using (3.338) and (3.341) gives

$$E(x, y) = E_0 H'(x)H^{2n+3}(x)(\bar{v}_x(x, y))^n. \tag{3.345}$$

Consider

$$H^{2n+3}\frac{dH}{dx} = 1, \qquad H(0) = 0, \tag{3.346}$$

so that (3.345) reduces to

$$E(x, y) = E_0(\bar{v}_x(x, y))^n. \tag{3.347}$$

The solution of the differential equation (3.346) is

$$H(x) = [2(n+2)x]^{\frac{1}{2(n+2)}}. \tag{3.348}$$

Equation (3.348) reduces to (3.274) for the laminar wall jet when $n = 0$. The effective viscosity (3.347) is the special case in which $M(x) = E_0 = constant$.

(ii) Shear stress at the wall

In the boundary layer approximation, the shear stress at the wall is

$$\tau_{yx}(x,0) = \rho E \frac{\partial \bar{v}_x}{\partial y}(x,0). \tag{3.349}$$

Now from (3.340) and (3.341)

$$\frac{\partial \bar{v}_x}{\partial y} = \frac{1}{E_0^{\frac{2}{1+n}} H^5(x)} \frac{d^2F}{d\eta^2} \tag{3.350}$$

and using (3.338) for E, (3.349) becomes

$$\tau_{yx}(x,0) = \frac{\rho}{E_0^{\frac{1}{n+1}}} \frac{H'(x)}{H^2(x)} \frac{1}{n+1} \frac{d}{d\eta} \left[\left(\frac{dF}{d\eta} \right)^{n+1} \right] |_{\eta=0}. \tag{3.351}$$

Now using (3.336) and (3.337),

$$\frac{1}{n+1}\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^{n+1}\right] = \left[\frac{(n+1)}{(n+2)(n+3)}\right]^{\frac{1}{2}} \left[\frac{(n+2)JE_0^{\frac{1}{n+1}}}{I(n)}\right]^{\frac{1}{2}\left(\frac{3+n}{2+n}\right)} \left[1 - 2\left(\frac{n+2}{n+1}\right) g^{\frac{n+3}{n+1}}\right] \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}} \tag{3.352}$$

and since $g(0) = 0$,

$$\tau_{yx}(x,0) = \rho \frac{H'(x)}{H^2(x)} \left[\frac{(n+1)}{(n+2)(n+3)}\right]^{\frac{1}{2}} \left[\frac{(n+2)J}{I(n)}\right]^{\frac{1}{2}\left(\frac{3+n}{2+n}\right)} E_0^{-\frac{1}{2(2+n)}}. \tag{3.353}$$

If we take (3.348) for $H(x)$ then

$$\frac{H'(x)}{H^2(x)} = [2(n+2)x]^{-\left(\frac{2n+5}{2(n+2)}\right)} \tag{3.354}$$

and (3.353) becomes

$$\tau_{yx}(x,0) = \frac{\rho}{(n+2)} (2x)^{-\left(\frac{2n+5}{2(n+2)}\right)} \left(\frac{n+1}{n+3}\right)^{\frac{1}{2}} \left(\frac{J}{I(n)}\right)^{\frac{1}{2}\left(\frac{3+n}{2+n}\right)} E_0^{-\frac{1}{2(2+n)}}. \tag{3.355}$$

The stress at the wall increases as the strength of the jet, J, increases. It decreases as the effective viscosity E_0 increases. An increase in effective viscosity causes an increase in diffusion which leads to a decrease in the velocity gradient normal to the wall. The stress, like the mean velocity $\bar{v}_x(x,0)$, is infinite at the orifice $x = 0$.

(iii) Maximum value of the mean velocity

The turning point of $\bar{v}_x(x,y)$ as y increases from the wall is obtained from (3.350). Now, from (3.336) and (3.337),

$$\frac{d^2F}{d\eta^2} = \left[\frac{n+1}{(n+2)(n+3)}\right]^{\frac{1}{2}} \left[\frac{(n+2)JE_0^{\frac{1}{1+n}}}{I(n)}\right]^{\frac{3-n}{2(2+n)}} g^{-\frac{n}{n+1}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1-n}{1+n}} \left[1 - 2\left(\frac{n+2}{n+1}\right) g^{\frac{n+3}{n+1}}\right] \tag{3.356}$$

and hence $\frac{\partial \bar{v}_x}{\partial y} = 0$ when

$$g = 1 \qquad if \qquad n > 1, \tag{3.357}$$

$$g = g_{max} = \left[\frac{n+1}{2(n+2)}\right]^{\frac{n+1}{n+3}}, \qquad n \geq 0. \tag{3.358}$$

But $g = 1$ corresponds to $\eta = \eta_0$ which is not a turning point. The turning point is given by (3.358). Since $n > 0$, $0 < g_{max} < 1$. Thus using (3.358)

$$\bar{v}_{x\,max}(x) = \frac{1}{H^2(x)} \left[\frac{(n+2)J}{I(n)E_0}\right]^{\frac{1}{2+n}} \left[\frac{n+1}{2(n+2)}\right]^{\frac{1}{n+3}} \left[\frac{n+3}{2(n+2)}\right]^{\frac{1}{n+1}}. \tag{3.359}$$

But using (3.348) for $H(x)$,

$$\frac{1}{H^2(x)} = [2(n+2)x]^{-\frac{1}{n+2}} \tag{3.360}$$

and therefore

$$\bar{v}_{x\,max}(x) = \left[\frac{n+1}{2(n+2)}\right]^{\frac{1}{n+3}} \left[\frac{n+3}{2(n+2)}\right]^{\frac{1}{n+1}} \left[\frac{J}{2I(n)E_0 x}\right]^{\frac{1}{2+n}}. \tag{3.361}$$

The maximum velocity increases as the strength, J, of the wall jet increases and decreases as the effective viscosity increases. It tends to infinity at the orifice $x = 0$.

(iv) Velocity profile

Consider now the graphs of $\bar{v}_x(x,y)$ against y at fixed values of x. From (3.336) and (3.341),

$$\bar{v}_x(x,y) = \frac{1}{H^2(x)} \left[\frac{(n+2)J}{E_0 I(n)}\right]^{\frac{1}{2+n}} g^{\frac{1}{1+n}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{1+n}}. \tag{3.362}$$

and using (3.348) for $H(x)$, (3.362) becomes

$$\bar{v}_x(x,y) = \left[\frac{J}{2E_0 I(n) x}\right]^{\frac{1}{2+n}} g^{\frac{1}{1+n}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{1+n}}. \tag{3.363}$$

Also, from (3.334) and (3.340) and with $H(x)$ given by (3.348),

$$y = [2E_0 x]^{\frac{3}{2(n+2)}} \left(\frac{n+2}{n+1}\right) \left[\frac{n+3}{n+1}\right]^{\frac{1}{2}} \left[\frac{J}{I(n)}\right]^{\frac{n-1}{2(n+2)}} \int_0^g \frac{dg}{\left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1}{n+1}}}. \tag{3.364}$$

It also follows from (3.350) and (3.356) that

$$\frac{\partial \bar{v}_x}{\partial y} = \frac{1}{[2(n+2)E_0 x]^{\frac{5}{2(n+2)}}} \left[\frac{n+1}{(n+2)(n+3)}\right]^{\frac{1}{2}} \left[\frac{(n+2)J}{I(n)}\right]^{\frac{3-n}{2(2+n)}}$$
$$g^{-\frac{n}{n+1}} \left[1 - g^{\frac{n+3}{n+1}}\right]^{\frac{1-n}{1+n}} \left[1 - 2\frac{n+2}{n+1} g^{\frac{n+3}{n+1}}\right]. \tag{3.365}$$

Thus

$$0 \le n < 1 \quad : \quad \frac{\partial \bar{v}_x}{\partial y} \to 0 \ as\ g \to 1$$
$$n = 1 \quad : \quad \frac{\partial \bar{v}_x}{\partial y} \to \ finite\ negative\ value\ as\ g \to 1$$
$$n > 1 \quad : \quad \frac{\partial \bar{v}_x}{\partial y} \to -\infty \ as\ g \to 1$$

In the following pages we consider the graphs of $\bar{v}_x(x,y)$ against y, given in parametric form by (3.363) and (3.364) where g is the parameter, $0 \le g \le 1$. The dependence on the parameters J, n and E_0 is investigated by keeping two constant and varying the third parameter.

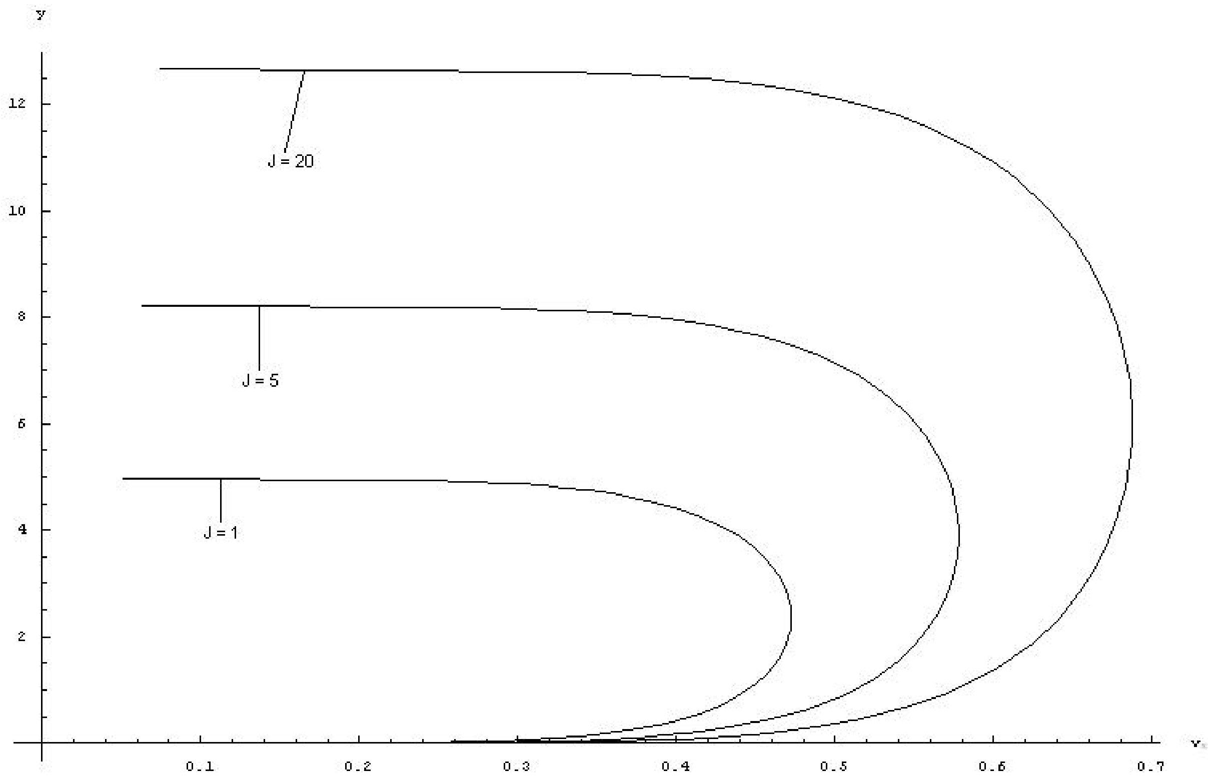

Figure 3.6: Velocity profile of two-dimensional wall jet varying J with $n = 6$, $E_0 = 100$, $x = 1$

In Figure 3.6, $\bar{v}_x(x, y)$ is plotted against y at $x = 1$ with n and E_0 kept fixed at $n = 6$ and $E_0 = 100$ while J takes the values $J = 1$, $J = 5$ and $J = 20$. We see that as the strength of the jet, J, increases the maximum velocity increases which agrees with the result that the maximum velocity is proportional to $J^{\frac{1}{2+n}}$. Also the value of η_0 increases and the jet becomes wider.

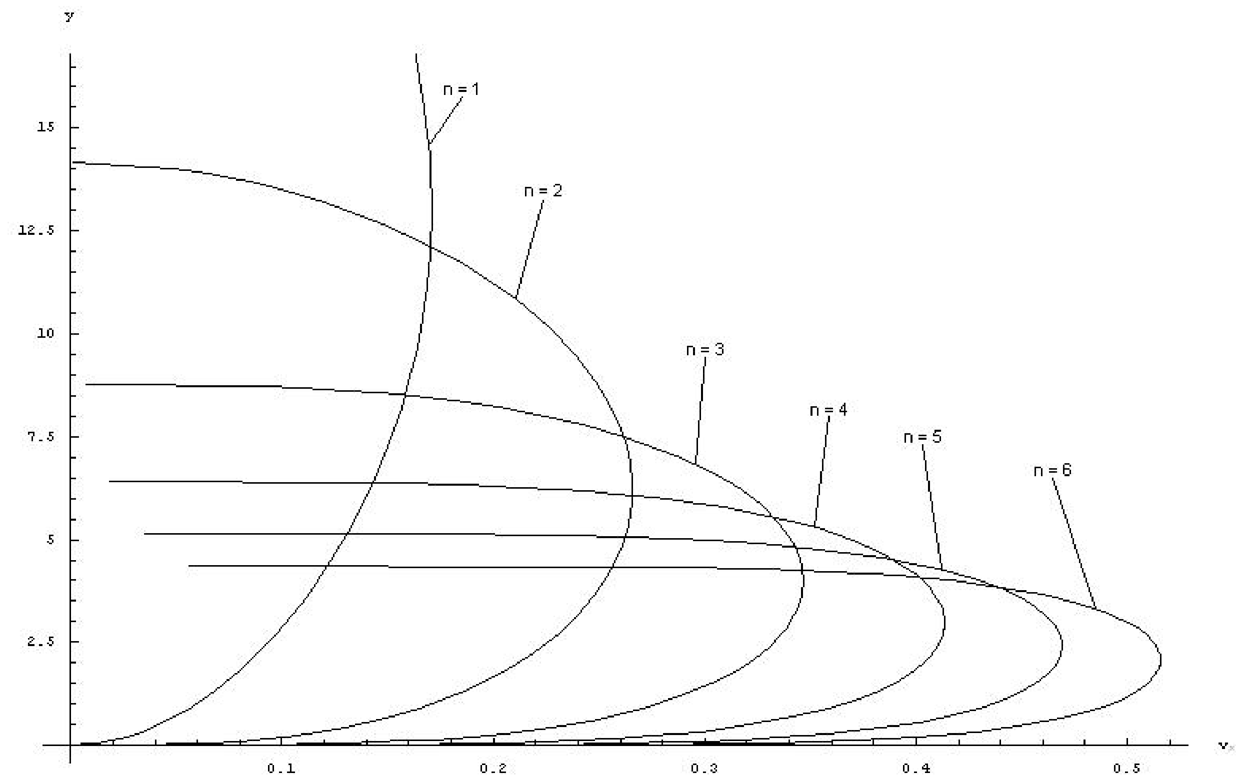

Figure 3.7: Velocity profile of two-dimensional wall jet varying n with $J = 1$, $E_0 = 100$, $x = 0.5$

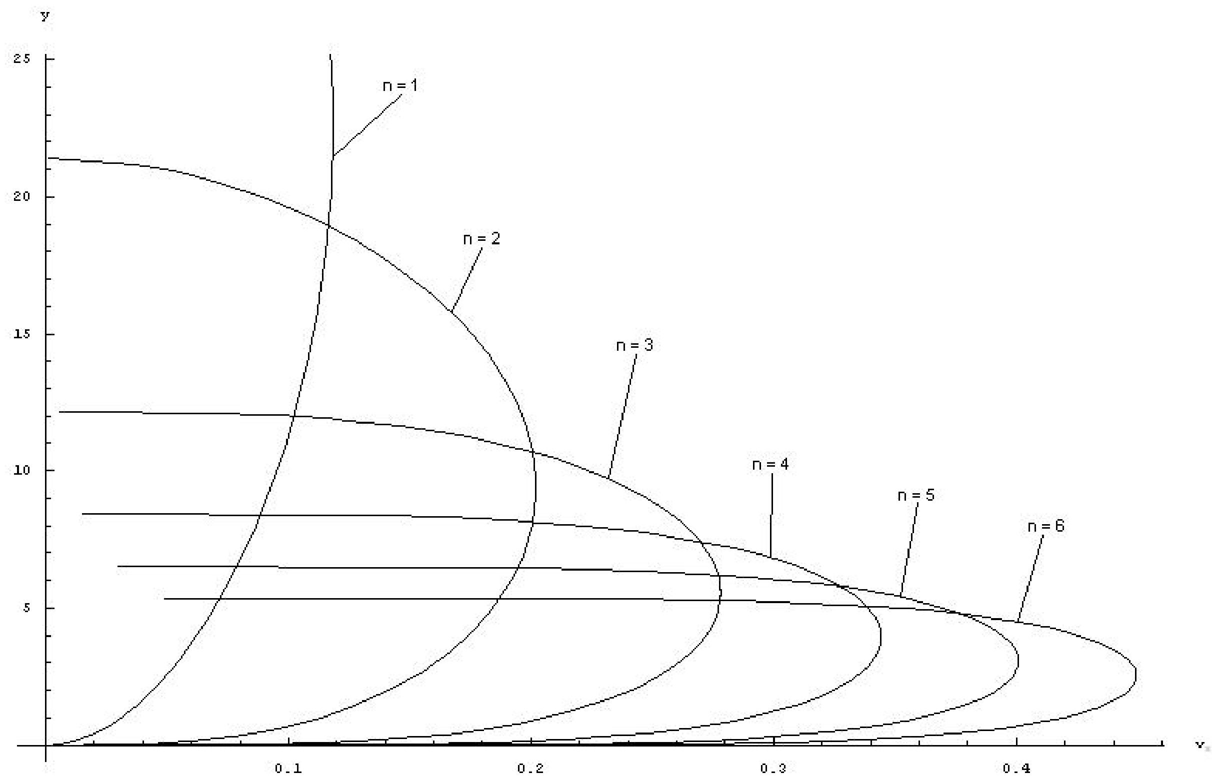

Figure 3.8: Velocity profile of two-dimensional wall jet varying n with $J = 1$, $E_0 = 100$, $x = 1.5$

In Figures 3.7 and 3.8, $\bar{v}_x(x, y)$ is plotted against y with J and E_0 kept fixed at $J = 1$ and $E_0 = 100$ while n takes the integer values $n = 1$ to $n = 6$. In Figure 3.7, $x = 0.5$ and in Figure 3.8, $x = 1.5$. As n increases the width of the jet decreases and the length increases. For $n > 1$ the velocity profile meets the axis at $\eta = \eta_0$ at right angles. The width of the jet increases as x increases because the jet has had more time to diffuse in the y-direction.

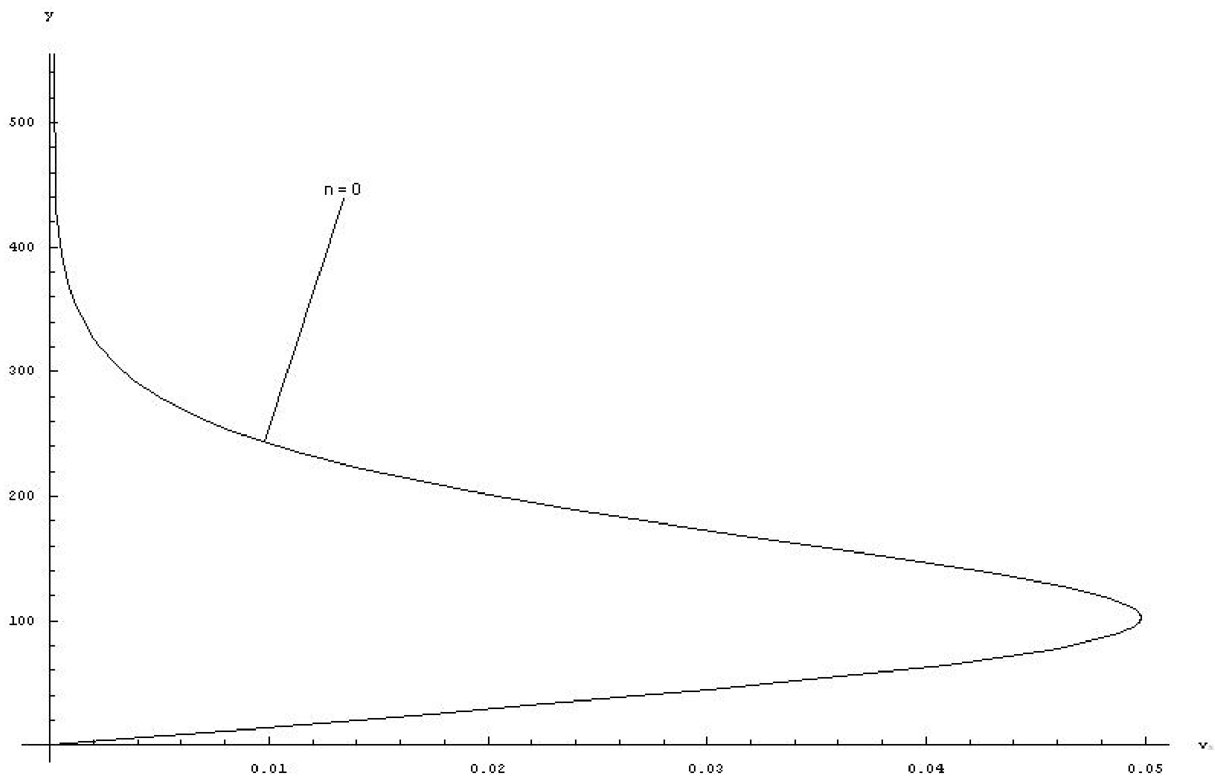

Figure 3.9: Velocity profile of two-dimensional laminar wall jet where $n = 0$, $J = 1$, $E_0 = 100$ and $x = 1$

In Figure 3.9 the graph of $\bar{v}_x(x, y)$ against y is plotted for a laminar jet for which $n = 0$ and $E_0 = 100$, $J = 1$ and $x = 1$. For the laminar jet the wall jet extends to $y = \infty$.

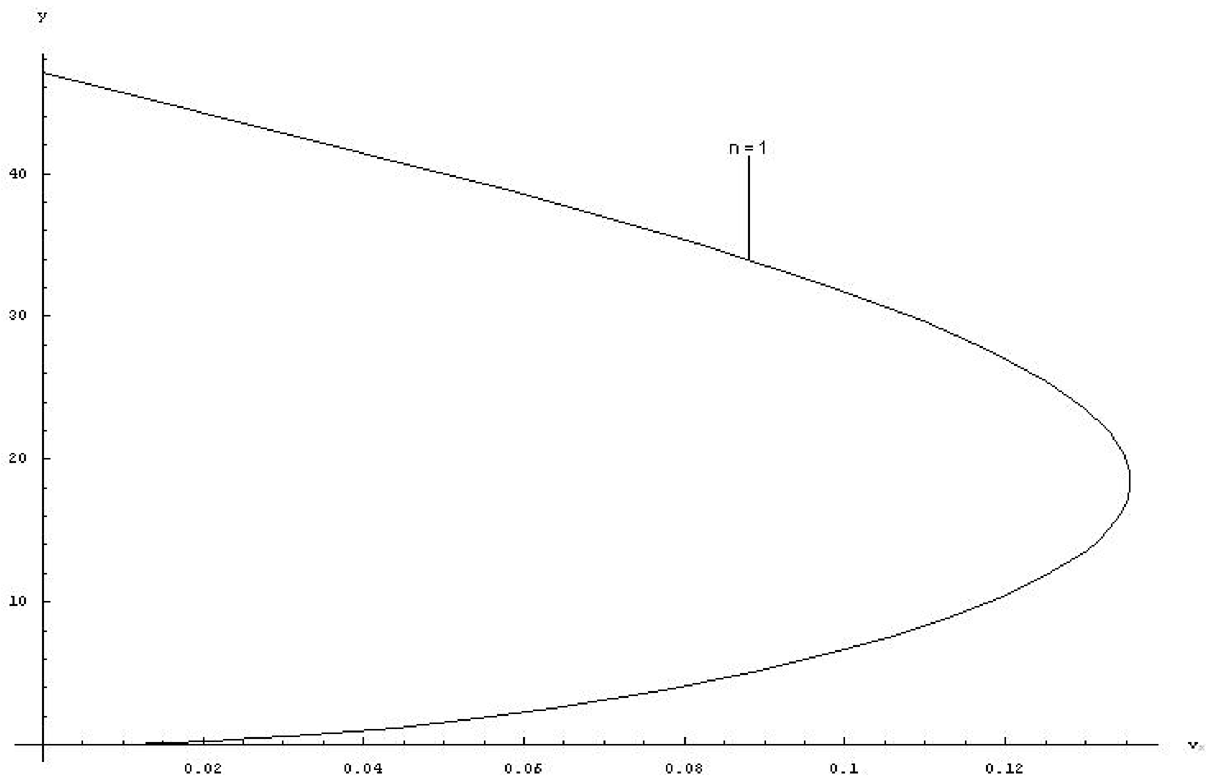

Figure 3.10: Velocity profile of two-dimensional wall jet where $n = 1$, $J = 1$, $E_0 = 100$ and $x = 1$

In Figure 3.10 the graph of $\bar{v}_x(x, y)$ is plotted against y with $J = 1$, $E_0 = 100$, $x = 1$ and $n = 1$. For $n = 1$, the velocity profile makes a finite non-zero angle with the y-axis at $\eta = \eta_0$.

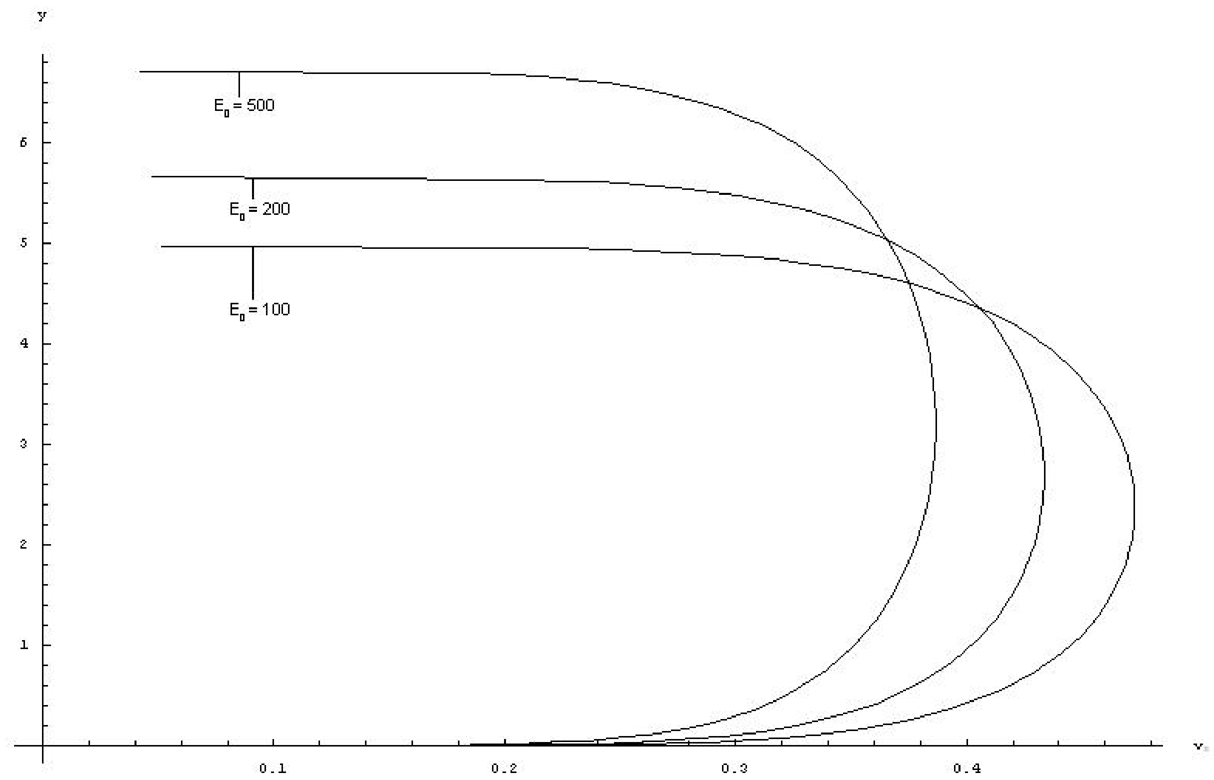

Figure 3.11: Velocity profile of two-dimensional wall jet varying E_0 with $J = 1$, $n = 6$, $x = 1$

In Figure 3.11, $\bar{v}_x(x, y)$ is plotted against y at $x = 1$ with J and n kept fixed at $J = 1$ and $n = 6$ and E_0 ranges from $E_0 = 100$, $E_0 = 200$ and $E_0 = 500$. As E_0 increase the maximum velocity decreases which agrees with the result that the maximum velocity is proportional to $E_0^{-\frac{1}{2+n}}$. Also as E_0 increases the width of the wall jet increases due to an increase in diffusion and the length decreases.

As pointed out by Glauert [11] the solution is satisfactory up to the maximum of the velocity. However the rapid decrease in the velocity after the velocity maximum is not realistic. A better model would be to impose Prandtl's hypothesis of constant eddy viscosity across the jet in the outer region of the jet. This was investigated by Glauert [11] by matching the solution for $n = 6$ with the solution for constant eddy viscosity.

3.6 Concluding remarks

For a turbulent wall jet with effective viscosity, $E(x) = E_0x^n$, in the form of a power law, Prandtl's hypothesis that the effective viscosity is constant across the jet is satisfied. We saw that Glauert's [11] analytical solution for a laminar wall jet could be extended to a turbulent wall jet. This allowed us to determine analytically the effect of the eddy viscosity, jet strength and exponent n on the wall jet. The behaviour was in agreement with the computer generated graphs.

For the turbulent wall jet with effective viscosity $E = M(x)(\bar{v}_x(x, y))^n$ the effective viscosity is not constant across the jet and Prandtl's hypothesis is not satisfied. This form was chosen because the conserved quantity is still valid and the effective viscosity does not diverge as $y \to \infty$. The value $n = 6$, considered by Glauert [11], was based on Blasius' empirical result for the stress at the wall in turbulent pipe flow. We were able to extend Glauert's analytical solutions for $n = 0$ and $n = 6$ to all values of $n > 0$. This allowed us to investigate the transition from a laminar wall jet with $n = 0$ to a turbulent wall jet with increasing values of n. Only the laminar wall jet extended to $y = \infty$. The decrease in the effective viscosity due to the decrease in $\bar{v}_x(x, y)$ as $y \to \infty$ is too rapid and the solution for $n > 0$ needs to be matched with an outer solution with constant eddy viscosity as Glauert did for $n = 6$.

Both the analytical solutions for a turbulent wall jet derived in this chapter are group invariant solutions. The results are quite general and further special solutions could be investigated. The effective viscosity $E(x, y)$ was not prescribed at the start of the analysis. A condition on $E(x, y)$ in the form of a first order partial differential equation was obtained for the Lie point symmetries to exist and the solution of this equation gave forms of the effective viscosity that could be investigated.

Chapter 4

Conclusions

In this chapter we will review what has been achieved and present some conclusions that we have found while studying turbulent boundary layers and wall jets.

In this dissertation we considered the mainstream matching boundary condition and the conditions at the wall. In Chapter 1, Figure 1.4, we saw that the turbulent boundary layer is made up of three different layers which are dependent on the relationship between the Reynold's stresses to the viscous stresses. In this dissertation we did not consider the laminar sublayer and the transition layer which were referred to in Chapter 1. We instead focussed on the outer turbulent layer but in order to obtain some results we considered the slip condition or suction/blowing condition at the wall. If we had considered the laminar sublayer or the transition layer then we would have had to match the velocities in the x-direction when moving from one layer to the next. For example, we would need to match the velocity at the inner edge of the turbulent layer with the velocity at the outer edge of the transition layer.

A difference between the method of solution for the turbulent boundary layer and the wall jet is that for the boundary layer the parameters are determined from mainstream matching while for the wall jet they are determined from the conserved quantity. Mainstream matching does not lead to results for the wall jet because the exterior velocity vanishes. The boundary layer is strongly influenced by the exterior flow. In the wall jet the conserved quantity defined the strength of the jet. It also places restrictions on the form of the effective viscosity, for example, $E = E(x)$ and $E = M(x)(\bar{v}_x(x,y))^n$, and determines parameters in the group invariant solution.

In Chapter 1 we introduced Prandtl's hypothesis but we did not impose it in general. We applied it to determine exponents in power laws for the effective viscosity which can be used for comparison. The wall jet with effective viscosity $E(x,y) = M(x)(\bar{v}_x(x,y))^n$ did not satisfy Prandtl's hypothesis. We found that the decrease in the velocity in the outer region was too rapid and the solution there was not satisfactory. Glauert [11] for $n = 6$ matched it with a solution with constant effective viscosity which satisfied Prandtl's hypothesis.

The results showed several differences between turbulent and laminar flows. In the solutions derived the results for laminar flow could be obtained by letting the parameters tend to special limits. In general we found that the turbulent boundary layers and wall jets were broader than the laminar flows due to the increase in diffusion because of the much larger effective viscosity. One exception was when $E = M(x)(\bar{v}_x(x,y))^n$. The width of the laminar wall jet, $n = 0$, extended to infinity but the width of the turbulent jet, $n > 0$, was finite and decreased as n increased. This was because of the very rapid decrease in the velocity $\bar{v}_x(x,y)$ in the outer regions. We also investigated the Blasius boundary layer for effective viscosity $E(x) = E_0 x^n$. We found that the growth of the boundary layer along the plate is $\delta(x) \propto x^{\frac{1}{2}(1+n)}$ while for laminar flow the profile is parabolic, $\delta(x) \propto x^{\frac{1}{2}}$.

The derivation of group invariant solutions using the Lie point symmetries of the partial differential equations was a powerful method of deriving solutions for the turbulent boundary layer and wall jet. The partial differential equations are reduced to ordinary differential equations which can be solved numerically if an analytical solution cannot be derived. The method is more powerful than the scaling transformation approach of Dresner [8] and used by Glauert [11], which can be applied only to power law effective viscosities. The approach was to keep the effective viscosity as general as possible, consistent with conservation laws, and to determine the condition on the effective viscosity for the Lie point symmetries to exist.

To put our results in context, we have obtained solutions to two particular problems, the two-dimensional turbulent boundary layer and the two-dimensional turbulent wall jet described by eddy viscosity. Many of the results in the literature on turbulence obtained by symmetry arguments are general results [5, 12, 14, 19]. Our contribution has been to look at specific problems and derive particular results.

Further investigations could be undertaken. Special cases not considered in this dissertation could be analysed, for example, boundary layer flow in a converging channel and the numerical solution of some of the ordinary differential equations derived in this dissertation. Glauert [11] considered both the two-dimensional and radial wall jets. The turbulent radial wall jet could be investigated by the methods of this dissertation.

Appendix A

Calculation of the Lie point symmetry for the two-dimensional turbulent boundary layer

In Appendix A the details of the calculation of the Lie point symmetry

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi} \tag{A.1}$$

of the partial differential equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - W(x) - \frac{\partial}{\partial y}\left(E(x, y)\frac{\partial^2 \psi}{\partial y^2}\right) = 0 \tag{A.2}$$

are presented. Details in the reduction of the partial differential equation (A.2) to an ordinary differential equation using the group invariant solution are also given.

A.1 Calculation of prolongation formulae

The prolongation formulae which were defined in Section 1.6.4 are used :

$$\zeta_i = D_i(\eta) - \psi_s\ D_i(\xi^s), \tag{A.3}$$

$$\zeta_{ij} = D_j(\zeta_i) - \psi_{is}\ D_j(\xi^s), \tag{A.4}$$

$$\zeta_{ijk} = D_k(\zeta_{ij}) - \psi_{ijs}\ D_k(\xi^s). \tag{A.5}$$

Now ζ_1, ζ_2, ζ_{12}, ζ_{22} and ζ_{222} are required. The index 1 refers to all calculations dealing with the variable x and the index 2 refers to all calculations dealing with the variable y. The total derivative formulae which were defined in Section 1.6.4 are used :

$$D_1 = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial \psi} + \psi_{xx}\frac{\partial}{\partial \psi_x} + \psi_{yx}\frac{\partial}{\partial \psi_y} + \cdots, \tag{A.6}$$

$$D_2 = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \cdots. \tag{A.7}$$

where D_1 and D_2 refer to total differentiation with respect to x and y, respectively. For example, since $\eta = \eta(x, y, \psi)$

$$D_1(\eta) = D_x(\eta) = \eta_x + \eta_\psi \psi_x, \tag{A.8}$$

and

$$D_2(\eta) = D_y(\eta) = \eta_y + \eta_\psi \psi_y, \tag{A.9}$$

The detailed calculations for ζ_1 and ζ_2 are given below. The remainder of the prolongation formulae will be listed.

$$\begin{aligned}
\zeta_1 &= D_1(\eta) - \psi_r D_1(\xi^r) \\
&= D_x(\eta) - \psi_x D_x(\xi^1) - \psi_y D_x(\xi^2) \\
&= \eta_x + \eta_\psi \psi_x - \psi_x(\xi^1_x + \xi^1_\psi \psi_x) - \psi_y(\xi^2_x + \xi^2_\psi \psi_x) \\
&= \eta_x + \eta_\psi \psi_x - \xi^1_x \psi_x - \xi^1_\psi (\psi_x)^2 - \xi^2_x \psi_y - \xi^2_\psi \psi_x \psi_y,
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
\zeta_2 &= D_2(\eta) - \psi_r D_2(\xi^r) \\
&= D_y(\eta) - \psi_x D_y(\xi^1) - \psi_y D_y(\xi^2) \\
&= \eta_y + \eta_\psi \psi_y - \psi_x(\xi^1_y + \xi^1_\psi \psi_y) - \psi_y(\xi^2_y + \xi^2_\psi \psi_y) \\
&= \eta_y + \eta_\psi \psi_y - \xi^1_y \psi_x - \xi^1_\psi \psi_x \psi_y - \xi^2_y \psi_y - \xi^2_\psi (\psi_y)^2.
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\zeta_{12} = {} & \eta_{xy} + \eta_{x\psi}\psi_y + \eta_{y\psi}\psi_x + \eta_{\psi\psi}\psi_x\psi_y + \eta_\psi \psi_{xy} - \xi^1_x \psi_{xy} - \xi^1_\psi \psi_x \psi_{xy} \\
& - \xi^1_{xy}\psi_x - \xi^1_{x\psi}\psi_x\psi_y - \xi^1_{y\psi}(\psi_x)^2 - \xi^1_{\psi\psi}(\psi_x)^2\psi_y - \xi^1_\psi \psi_x \psi_{xy} - \xi^2_x \psi_{yy} \\
& - \xi^2_\psi \psi_x \psi_{yy} - \xi^2_{xy}\psi_y - \xi^2_{x\psi}(\psi_y)^2 - \xi^2_{y\psi}\psi_x\psi_y - \xi^2_{\psi\psi}\psi_x(\psi_y)^2 - \xi^2_\psi \psi_y \psi_{xy} \\
& - \xi^1_y \psi_{xx} - \xi^1_\psi \psi_y \psi_{xx} - \xi^2_y \psi_{xy} - \xi^2_\psi \psi_{xy}\psi_y.
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\zeta_{22} = {} & \eta_{yy} + 2\eta_{y\psi}\psi_y + \eta_{\psi\psi}(\psi_y)^2 + \eta_\psi \psi_{yy} - 2\xi^1_y \psi_{xy} - 2\xi^1_\psi \psi_y \psi_{xy} - \xi^1_{yy}\psi_x \\
& - 2\xi^1_{y\psi}\psi_x\psi_y - \xi^1_{\psi\psi}\psi_x(\psi_y)^2 - \xi^1_\psi \psi_x \psi_{yy} - 2\xi^2_y \psi_{yy} - 2\xi^2_\psi \psi_y \psi_{yy} - \xi^2_{yy}\psi_y \\
& - 2\xi^2_{y\psi}(\psi_y)^2 - \xi^2_{\psi\psi}(\psi_y)^3 - \xi^2_\psi \psi_y \psi_{yy}.
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\zeta_{222} = {} & \eta_{yyy} + 3\eta_{yy\psi}\psi_y + 3\eta_{y\psi\psi}(\psi_y)^2 + 3\eta_{y\psi}\psi_{yy} + \eta_{\psi\psi\psi}(\psi_y)^3 + 3\eta_{\psi\psi}\psi_y\psi_{yy} \\
& + \eta_\psi \psi_{yyy} - 3\xi^1_y \psi_{xyy} - 3\xi^1_\psi \psi_y \psi_{xyy} - 3\xi^1_{yy}\psi_{xy} - 6\xi^1_{y\psi}\psi_y\psi_{xy} - 3\xi^1_\psi \psi_{yy}\psi_{xy} \\
& - 3\xi^1_{\psi\psi}(\psi_y)^2\psi_{xy} - \xi^1_{yyy}\psi_x - 3\xi^1_{yy\psi}\psi_x\psi_y - 3\xi^1_{y\psi\psi}\psi_x(\psi_y)^2 - 3\xi^1_{y\psi}\psi_x\psi_{yy} \\
& - \xi^1_{\psi\psi\psi}\psi_x(\psi_y)^3 - 3\xi^1_{\psi\psi}\psi_x\psi_y\psi_{yy} - \xi^1_\psi \psi_x \psi_{yyy} - 3\xi^2_y \psi_{yyy} - 3\xi^2_\psi \psi_y \psi_{yyy} \\
& - 3\xi^2_{yy}\psi_{yy} - 6\xi^2_{y\psi}\psi_y\psi_{yy} - 3\xi^2_{\psi\psi}(\psi_y)^2\psi_{yy} - 3\xi^2_\psi(\psi_{yy})^2 - \xi^2_{yyy}\psi_y
\end{aligned}$$

$$-3\xi^2_{yy\psi}(\psi_y)^2 - 3\xi^2_{y\psi\psi}(\psi_y)^3 - 3\xi^2_{y\psi}\psi_y\psi_{yy} - 3\xi^2_{\psi\psi}(\psi_y)^2\psi_{yy} - \xi^2_{\psi\psi\psi}(\psi_y)^4$$
$$-\xi^2_{\psi}\psi_y\psi_{yyy}. \qquad \text{(A.14)}$$

A.2 Determining equation

The determining equation (2.88) from Section 2.4 is as follows

$$-E_{xy}\psi_{yy}\xi^1 - E_x\psi_{yyy}\xi^1 - W_x\xi^1 - E_{yy}\psi_{yy}\xi^2 - E_y\psi_{yyy}\xi^2$$
$$-\psi_{yy}\zeta_1 + \psi_{xy}\zeta_2 + \psi_y\zeta_{12} - \psi_x\zeta_{22} - E_y\zeta_{22} - E\zeta_{222}|_{F=0} = 0. \qquad \text{(A.15)}$$

Substituting (A.10), (A.11), (A.12), (A.13) and (A.14) and (2.89) for ψ_{yyy},

$$\psi_{yyy} = \frac{1}{E}\psi_y\psi_{xy} - \frac{1}{E}\psi_x\psi_{yy} - \frac{1}{E}E_y\psi_{yy} - \frac{W}{E}, \qquad \text{(A.16)}$$

into (A.15), the determining equation in expanded form becomes

$$-\xi^1E_{xy}\psi_{yy} - \xi^1E_x\tfrac{1}{E}\psi_y\psi_{xy} + \xi^1E_x\tfrac{1}{E}\psi_x\psi_{yy} + \xi^1E_xE_y\tfrac{1}{E}\psi_{yy} + \xi^1E_x\tfrac{W}{E} - \xi^1W_x$$

$$-\xi^2E_{yy}\psi_{yy} - \xi^2E_y\tfrac{1}{E}\psi_y\psi_{xy} + \xi^2E_y\tfrac{1}{E}\psi_x\psi_{yy} + \xi^2(E_y)^2\tfrac{1}{E}\psi_{yy} + \xi^2E_y\tfrac{W}{E} - \eta_x\psi_{yy}$$

$$-\eta_\psi\psi_x\psi_{yy} + \xi^1_x\psi_x\psi_{yy} + \xi^1_\psi(\psi_x)^2\psi_{yy} + \xi^2_x\psi_y\psi_{yy} + \xi^2_\psi\psi_x\psi_y\psi_{yy} + \eta_y\psi_{xy}$$

$$+\eta_\psi\psi_y\psi_{xy} - \xi^1_y\psi_x\psi_{xy} - \xi^1_\psi\psi_x\psi_y\psi_{xy} - \xi^2_y\psi_y\psi_{xy} - \xi^2_\psi(\psi_y)^2\psi_{xy} - \xi^2_\psi(\psi_y)^2\psi_{xy}$$

$$+\eta_{xy}\psi_y + \eta_{x\psi}(\psi_y)^2 + \eta_{y\psi}\psi_x\psi_y + \eta_{\psi\psi}\psi_x(\psi_y)^2 + \eta_\psi\psi_y\psi_{xy} - \xi^1_x\psi_y\psi_{xy}$$

$$-\xi^1_\psi\psi_x\psi_y\psi_{xy} - \xi^1_{xy}\psi_x\psi_y - \xi^1_{x\psi}\psi_x(\psi_y)^2 - \xi^1_{y\psi}(\psi_x)^2\psi_y - \xi^1_{\psi\psi}(\psi_x)^2(\psi_y)^2$$

$$-\xi^1_\psi\psi_x\psi_y\psi_{xy} - \xi^2_x\psi_y\psi_{yy} - \xi^2_\psi\psi_x\psi_y\psi_{yy} - \xi^2_{xy}(\psi_y)^2 - \xi^2_{x\psi}(\psi_y)^3 - \xi^2_{y\psi}\psi_x(\psi_y)^2$$

$$-\xi^2_{\psi\psi}\psi_x(\psi_y)^3 - \xi^2_\psi(\psi_y)^2\psi_{xy} - \xi^1_y\psi_y\psi_{xx} - \xi^1_\psi(\psi_y)^2\psi_{xx} - \xi^2_y\psi_y\psi_{xy} + \xi^1_{yyy}E\psi_x$$

$$-\eta_{yy}\psi_x - 2\eta_{y\psi}\psi_x\psi_y - \eta_{\psi\psi}\psi_x(\psi_y)^2 - \eta_\psi\psi_x\psi_{yy} + 2\xi^1_y\psi_x\psi_{xy} + 2\xi^1_\psi\psi_x\psi_y\psi_{xy}$$

$$+\xi^1_{yy}(\psi_x)^2 + 2\xi^1_{y\psi}(\psi_x)^2\psi_y + \xi^1_{\psi\psi}(\psi_x)^2(\psi_y)^2 + \xi^1_\psi(\psi_x)^2\psi_{yy} + 2\xi^2_y\psi_x\psi_{yy} + 2\xi^2_\psi\psi_x\psi_y\psi_{yy}$$

$$+\xi^2_{yy}\psi_x\psi_y + 2\xi^2_{y\psi}\psi_x(\psi_y)^2 + \xi^2_{\psi\psi}\psi_x(\psi_y)^3 + \xi^2_\psi\psi_x\psi_y\psi_{yy} - \eta_{yy}E_y - 2\eta_{y\psi}E_y\psi_y$$

$$-\eta_{\psi\psi}E_y(\psi_y)^2 - \eta_\psi E_y\psi_{yy} + 2\xi^1_yE_y\psi_{xy} + 2\xi^1_\psi E_y\psi_y\psi_{xy} + \xi^1_{yy}E_y\psi_x + 2\xi^1_{y\psi}E_y\psi_x\psi_y$$

$$
\begin{aligned}
&+\xi^1_{\psi\psi}E_y\psi_x(\psi_y)^2+\xi^1_\psi E_y\psi_x\psi_{yy}+2\xi^2_yE_y\psi_{yy}+2\xi^2_\psi E_y\psi_y\psi_{yy}+\xi^2_{yy}E_y\psi_y+2\xi^2_{y\psi}E_y(\psi_y)^2\\
&+\xi^2_{\psi\psi}E_y(\psi_y)^3+\xi^2_\psi E_y\psi_y\psi_{yy}-\eta_{yyy}E-3\eta_{yy\psi}E\psi_y-3\eta_{\psi\psi y}E(\psi_y)^2-3\eta_{y\psi}E\psi_{yy}\\
&-\eta_{\psi\psi\psi}E(\psi_y)^3-3\eta_{\psi\psi}E\psi_y\psi_{yy}-\eta_\psi\psi_y\psi_{xy}+\eta_\psi\psi_x\psi_{yy}+\eta_\psi E_y\psi_{yy}+\eta_\psi W\\
&+3\xi^1_yE\psi_{xyy}+3\xi^1_\psi E\psi_y\psi_{xyy}+3\xi^1_{yy}E\psi_{xy}+6\xi^1_{y\psi}E\psi_y\psi_{xy}+3\xi^1_\psi E\psi_{yy}\psi_{xy}+3\xi^1_{\psi\psi}E(\psi_y)^2\psi_{xy}\\
&+3\xi^1_{yy\psi}E\psi_x\psi_y+3\xi^1_{y\psi\psi}E\psi_x(\psi_y)^2+3\xi^1_{y\psi}E\psi_x\psi_{yy}+\xi^1_{\psi\psi\psi}E\psi_x(\psi_y)^3+3\xi^1_{\psi\psi}E\psi_x\psi_y\psi_{yy}\\
&+\xi^1_\psi\psi_x\psi_y\psi_{xy}-\xi^1_\psi(\psi_x)^2\psi_{yy}-\xi^1_\psi E_y\psi_x\psi_{yy}-\xi^1_\psi W\psi_x+3\xi^2_y\psi_y\psi_{xy}-3\xi^2_y\psi_x\psi_{yy}\\
&-3\xi^2_yE_y\psi_{yy}-3\xi^2_yW+3\xi^2_\psi(\psi_y)^2\psi_{xy}-3\xi^2_\psi\psi_x\psi_y\psi_{yy}-3\xi^2_\psi E_y\psi_y\psi_{yy}-3\xi^2_\psi W\psi_y\\
&+3\xi^2_{yy}E\psi_{yy}+6\xi^2_{y\psi}E\psi_y\psi_{yy}+3\xi^2_{\psi\psi}E(\psi_y)^2\psi_{yy}+3\xi^2_\psi E(\psi_{yy})^2+\xi^2_{yyy}E\psi_y+3\xi^2_{yy\psi}E(\psi_y)^2\\
&+3\xi^2_{y\psi\psi}E(\psi_y)^3+3\xi^2_{y\psi}E\psi_y\psi_{yy}+3\xi^2_{\psi\psi}E(\psi_y)^2\psi_{yy}+\xi^2_{\psi\psi\psi}E(\psi_y)^4+\xi^2_\psi(\psi_y)^2\psi_{xy}-\xi^2_\psi\psi_x\psi_y\psi_{yy}\\
&-\xi^2_\psi E_y\psi_y\psi_{yy}-\xi^2_\psi W\psi_y \quad = \quad 0.
\end{aligned} \tag{A.17}
$$

A system of partial differential equations is found by equating the coefficients of the derivatives of ψ since $\xi^1(x,y,\psi)$, $\xi^2(x,y,\psi)$, $\eta(x,y,\psi)$, $E(x,y)$ and $W(x)$ do not depend on the derivatives of ψ.

$$
\begin{aligned}
\psi_{yy}: \quad & -E_{xy}\xi^1+E_xE_y\tfrac{1}{E}\xi^1+(E_y)^2\tfrac{1}{E}\xi^2-\eta_x-E_y\eta_\psi+2E_y\xi^2_y\\
& -3E\eta_{y\psi}+E_y\eta_\psi-3E_y\xi^2_y+3E\xi^2_{yy}-E_{yy}\xi^2=0
\end{aligned} \tag{A.18}
$$

$$
\begin{aligned}
\psi_y\psi_{xy}: \quad & -E_x\tfrac{1}{E}\xi^1-E_y\tfrac{1}{E}\xi^2+\eta_\psi-\xi^2_y+\eta_\psi-\xi^1_x-\xi^2_y+2E_y\xi^1_\psi\\
& -\eta_\psi+3\xi^2_y+6E\xi^1_{y\psi}=0
\end{aligned} \tag{A.19}
$$

$$
\begin{aligned}
\psi_x\psi_{yy}: \quad & E_x\tfrac{1}{E}\xi^1+E_y\tfrac{1}{E}\xi^2-\eta_\psi+\xi^1_x-\eta_\psi+2\xi^2_y+E_y\xi^1_\psi+\eta_\psi\\
& +3E\xi^1_{\psi y}-E_y\xi^1_\psi-3\xi^2_y=0
\end{aligned} \tag{A.20}
$$

$$
(\psi_x)^2\psi_{yy}: \quad \xi^1_\psi+\xi^1_\psi-\xi^1_\psi=0 \tag{A.21}
$$

$$
\psi_y\psi_{yy}: \quad \xi^2_x-\xi^2_x+2E_y\xi^2_\psi+E_y\xi^2_\psi-3E\eta_{\psi\psi}+6E\xi^2_{y\psi}-3E_y\xi^2_\psi
$$

$$+3E\xi^2_{y\psi} - E_y\xi^2_\psi = 0 \quad \text{(A.22)}$$

$$\psi_x\psi_y\psi_{yy}: \quad \xi^2_\psi - \xi^2_\psi + 2\xi^2_\psi + \xi^2_\psi - 3\xi^2_\psi - \xi^2_\psi + 3E\xi^1_{\psi\psi} = 0 \quad \text{(A.23)}$$

$$\psi_{xy}: \quad \eta_y + 2E_y\xi^1_y + 3E\xi^1_{yy} = 0 \quad \text{(A.24)}$$

$$\psi_x\psi_{xy}: \quad -\xi^1_y + 2\xi^1_y = 0 \quad \text{(A.25)}$$

$$\psi_x\psi_y\psi_{xy}: \quad -\xi^1_\psi - \xi^1_\psi - \xi^1_\psi + 2\xi^1_\psi + \xi^1_\psi = 0 \quad \text{(A.26)}$$

$$(\psi_y)^2\psi_{xy}: \quad -\xi^2_\psi - \xi^2_\psi - \xi^2_\psi + 3E\xi^1_{\psi\psi} + 3\xi^2_\psi + \xi^2_\psi = 0 \quad \text{(A.27)}$$

$$\psi_{xy}\psi_{yy}: \quad 3E\xi^1_\psi = 0 \quad \text{(A.28)}$$

$$\psi_y: \quad \eta_{xy} - 2E_y\eta_{y\psi} + E_y\xi^2_{yy} - 3E\eta_{yy\psi} - 3W\xi^2_\psi - W\xi^2_\psi$$
$$+E\xi^2_{yyy} = 0 \quad \text{(A.29)}$$

$$(\psi_y)^2: \quad \eta_{x\psi} - \xi^2_{xy} - E_y\eta_{\psi\psi} + 2E_y\xi^2_{y\psi} - 3E\eta_{y\psi\psi} + 3E\xi^2_{yy\psi} = 0 \quad \text{(A.30)}$$

$$\psi_x\psi_y: \quad 3E\xi^1_{yy\psi} + \eta_{y\psi} - \xi^1_{xy} - 2\eta_{y\psi} + \xi^2_{yy} + 2E_y\xi^1_{y\psi} = 0 \quad \text{(A.31)}$$

$$\psi_x(\psi_y)^2: \quad \eta_{\psi\psi} - \xi^1_{x\psi} - \xi^2_{y\psi} - \eta_{\psi\psi} + 2\xi^2_{y\psi} + E_y\xi^1_{\psi\psi} + 3E\xi^1_{y\psi\psi} = 0 \quad \text{(A.32)}$$

$$(\psi_x)^2\psi_y: \quad -\xi^1_{y\psi} + 2\xi^1_{y\psi} = 0 \quad \text{(A.33)}$$

$$(\psi_x)^2(\psi_y)^2: \quad -\xi^1_{\psi\psi} + \xi^1_{\psi\psi} = 0 \quad \text{(A.34)}$$

$$(\psi_y)^3: \quad 3E\xi^2_{y\psi\psi} - E\eta_{\psi\psi\psi} - \xi^2_{x\psi} + E_y\xi^2_{\psi\psi} = 0 \quad \text{(A.35)}$$

$$\psi_x(\psi_y)^3: \quad -\xi^2_{\psi\psi} + \xi^2_{\psi\psi} + E\xi^1_{\psi\psi\psi} = 0 \quad \text{(A.36)}$$

$$\psi_y\psi_{xx}: \quad -\xi^1_y = 0 \quad \text{(A.37)}$$

$$(\psi_y)^2\psi_{xx}: \quad -\xi^1_\psi = 0 \quad \text{(A.38)}$$

$$\psi_x : \qquad -\eta_{yy} + E_y\xi^1_{yy} + E\xi^1_{yyy} - W\xi^1_\psi = 0 \tag{A.39}$$

$$(\psi_x)^2 : \qquad \xi^1_{yy} = 0 \tag{A.40}$$

$$\psi_{xyy} : \qquad 3E\xi^1_y = 0 \tag{A.41}$$

$$\psi_y\psi_{xyy} : \qquad 3E\xi^1_\psi = 0 \tag{A.42}$$

$$(\psi_y)^2\psi_{yy} : \qquad 3E\xi^2_{\psi\psi} + 3E\xi^2_{\psi\psi} = 0 \tag{A.43}$$

$$(\psi_{yy})^2 : \qquad 3E\xi^2_\psi = 0 \tag{A.44}$$

$$(\psi_y)^4 : \qquad E\xi^2_{\psi\psi\psi} = 0 \tag{A.45}$$

$$\begin{aligned} remainder : \qquad & E_x\tfrac{W}{E}\xi^1 - W_x\xi^1 + E_y\tfrac{W}{E}\xi^2 - E_y\eta_{yy} - E\eta_{yyy} + W\eta_\psi \\ & -3W\xi^2_y = 0 \end{aligned} \tag{A.46}$$

Equations (A.26) and (A.34) are identically satisfied. From (A.37) and (A.38) it is found that

$$\xi^1 = \xi^1(x) = a(x), \tag{A.47}$$

where $a(x)$ is an arbitrary function of x. Equations (A.21), (A.25), (A.28), (A.33), (A.36), (A.40), (A.41) and (A.42) are identically satisfied when we substitute (A.47) into them. Using (A.47) in (A.24) we find that

$$\eta_y = 0 \tag{A.48}$$

and therefore

$$\eta = \eta(x, \psi). \tag{A.49}$$

From (A.47) and (A.49) it follows that (A.39) is identically satisfied. Equation (A.44) implies that

$$\xi^2 = \xi^2(x, y). \tag{A.50}$$

Using (A.47), (A.49) and (A.50) we see that (A.23), (A.27), (A.32), (A.43) and (A.45) are all identically satisfied and (A.22) becomes

$$\eta_{\psi\psi}(x, \psi) = 0, \tag{A.51}$$

which implies that

$$\eta(x,\psi) = b(x)\psi + c(x), \tag{A.52}$$

where $b(x)$ and $c(x)$ are arbitrary functions of x. Using (A.50) and (A.52) it follows that (A.35) is identically satisfied. Using (A.47), (A.49) and (A.50) once again, we see (A.31) reduces to

$$\xi^2_{yy}(x,y) = 0, \tag{A.53}$$

and therefore

$$\xi^2(x,y) = d(x)y + e(x), \tag{A.54}$$

where $d(x)$ and $e(x)$ are arbitrary functions of x. Equation (A.29) is identically satisfied when we substitute (A.49), (A.50) and (A.54). Thus far we have the following expressions for ξ^1, ξ^2 and η :

$$\xi^1(x) = a(x), \quad \xi^2(x,y) = d(x)y + e(x), \quad \eta(x,\psi) = b(x)\psi + c(x). \tag{A.55}$$

Equation (A.55) reduces (A.30) to

$$b'(x) - d'(x) = 0, \tag{A.56}$$

and hence

$$b(x) = d(x) + C_1, \tag{A.57}$$

where C_1 is an arbitrary constant. The only equations we have not used so far are (A.18), (A.19), (A.20) and (A.46) :

$$\begin{aligned} -E_{xy}\xi^1 + E_xE_y\tfrac{1}{E}\xi^1 + (E_y)^2\tfrac{1}{E}\xi^2 - \eta_x - E_y\xi^2_y \\ -3E\eta_{y\psi} + 3E\xi^2_{yy} - E_{yy}\xi^2 = 0, \end{aligned} \tag{A.58}$$

$$\begin{aligned} -E_x\tfrac{1}{E}\xi^1 - E_y\tfrac{1}{E}\xi^2 + \eta_\psi - \xi^1_x + \xi^2_y + 2E_y\xi^1_\psi \\ +6E\xi^1_{y\psi} = 0, \end{aligned} \tag{A.59}$$

$$E_x\tfrac{1}{E}\xi^1 + E_y\tfrac{1}{E}\xi^2 - \eta_\psi + \xi^1_x + 3E\xi^1_{\psi y} - \xi^2_y = 0, \tag{A.60}$$

$$\begin{aligned} E_x\tfrac{W}{E}\xi^1 - W_x\xi^1 + E_y\tfrac{W}{E}\xi^2 - E_y\eta_{yy} - E\eta_{yyy} + W\eta_\psi \\ -3W\xi^2_y = 0. \end{aligned} \tag{A.61}$$

Substituting (A.55) and (A.57) into (A.58) to (A.61) gives

$$-E_{xy}a(x) + E_xE_y\tfrac{1}{E}a(x) + (E_y)^2\tfrac{1}{E}\left(d(x)y + e(x)\right) - E_yd(x)$$

$$-(d'(x)\psi + c'(x)) - E_{yy}\left(d(x)y + e(x)\right) = 0 \tag{A.62}$$

$$-E_x \tfrac{1}{E} a(x) - E_y \tfrac{1}{E}\left(d(x)y + e(x)\right) + C_1 - a'(x) + 2d(x) = 0, \tag{A.63}$$

$$E_x \tfrac{1}{E} a(x) + E_y \tfrac{1}{E}\left(d(x)y + e(x)\right) - C_1 + a'(x) - 2d(x) = 0, \tag{A.64}$$

$$E_x \tfrac{W}{E} a(x) - W_x a(x) + E_y \tfrac{W}{E}\left(d(x)y + e(x)\right) + WC_1 - 2Wd(x) = 0. \tag{A.65}$$

Equation (A.63) is (A.64) multiplied through by -1. Differentiating (A.62) with respect to ψ gives

$$d'(x) = 0, \tag{A.66}$$

and therefore

$$d(x) = C_2, \tag{A.67}$$

where C_2 is an arbitrary constant. Hence substituting (A.67) into (A.62), (A.63) and (A.65) gives

$$-E_{xy}a(x) + E_x E_y \frac{1}{E} a(x) + (E_y)^2 \frac{1}{E}(C_2 y + e(x)) - E_y C_2 - E_{yy}(C_2 y + e(x))$$

$$-c'(x) = 0, \tag{A.68}$$

$$-E_x \frac{1}{E} a(x) - E_y \frac{1}{E}(C_2 y + e(x)) + C_1 + 2C_2 - a'(x) = 0, \tag{A.69}$$

$$E_x \frac{W}{E} a(x) - W_x a(x) + E_y \frac{W}{E}(C_2 y + e(x)) + WC_1 - 2WC_2 = 0. \tag{A.70}$$

Differentiate (A.69) with respect to y and multiply through by E :

$$-E_{xy}a(x) + E_x E_y \frac{1}{E} a(x) + (E_y)^2 \frac{1}{E}(C_2 y + e(x)) - E_y C_2 - E_{yy}(C_2 y + e(x)) = 0. \tag{A.71}$$

Now subtract (A.68) from (A.71). This gives

$$c'(x) = 0, \tag{A.72}$$

and hence

$$c(x) = C_3, \tag{A.73}$$

where C_3 is an arbitrary constant. Using (A.57), (A.67) and (A.73), equations (A.55) for ξ^1, ξ^2 and η become :

$$\xi^1(x) = a(x), \quad \xi^2(x,y) = C_2 y + e(x), \quad \eta(x,\psi) = (C_1 + C_2)\psi + C_3. \tag{A.74}$$

Now (A.69) can be written as

$$\frac{1}{E}E_x a(x) + \frac{1}{E}E_y(C_2 y + e(x)) = 2C_2 + C_1 - a'(x), \tag{A.75}$$

and (A.70) can be written as

$$\left(\frac{1}{E}E_x a(x) + \frac{1}{E}E_y(C_2 y + e(x))\right)W - a(x)W_x + (C_1 - 2C_2)W = 0. \tag{A.76}$$

By substituting (A.75) into (A.76) we obtain the following ordinary differential equation for $W(x)$:

$$a(x)\frac{dW}{dx} + (a'(x) - 2C_1)W(x) = 0, \tag{A.77}$$

which must be satisfied for the Lie point symmetry to exist. Rearranging (A.75), a first order linear partial differential equation for $E(x,y)$ is found :

$$a(x)E_x + [C_2 y + e(x)]E_y = [2C_2 + C_1 - a'(x)]E, \tag{A.78}$$

which must also be satisfied for the Lie point symmetry to exist.

The remaining equation (A.68) where $c(x)$ is given by (A.73), is not a further condition on $E(x,y)$ because it can be expressed in terms of equation (A.78) and the derivative of (A.78) with respect to y.

In summary the Lie point symmetry of the partial differential equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - W(x) - \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2\psi}{\partial y^2}\right) = 0, \tag{A.79}$$

is

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi}, \tag{A.80}$$

where

$$\xi^1 = a(x), \tag{A.81}$$

$$\xi^2 = C_2 y + e(x), \tag{A.82}$$

$$\eta = (C_1 + C_2)\psi + C_3, \tag{A.83}$$

where $a(x)$ and $e(x)$ are arbitrary functions of x and C_1, C_2 and C_3 are arbitrary constants, provided $W(x)$ satisfies the ordinary differential equation

$$a(x)\frac{dW}{dx} + (a'(x) - 2C_1)W(x) = 0 \tag{A.84}$$

and $E(x,y)$ satisfies the first order linear partial differential equation

$$a(x)\frac{\partial E}{\partial x} + [C_2 y + e(x)]\frac{\partial E}{\partial y} = [2C_2 + C_1 - a'(x)]E. \tag{A.85}$$

A.3 Calculations for Section 2.5 : Reducing the partial differential equation to an ordinary differential equation

Consider the third order partial differential equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - W(x) - \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2 \psi}{\partial y^2}\right) = 0. \tag{A.86}$$

In Section 2.5 it was shown that

$$\psi(x,y) = e^{(C_1+C_2)B(x)}F(\eta) - \frac{C_3}{C_1+C_2}, \tag{A.87}$$

$$E(x,y) = \frac{1}{a(x)}\, e^{(2C_2+C_1)\; B(x)}\; G(\eta), \tag{A.88}$$

$$W(x) = \frac{C_7}{a(x)}e^{2C_1 B(x)}, \tag{A.89}$$

where

$$\eta = e^{-C_2 B(x)}y - D(x). \tag{A.90}$$

The functions $B(x)$ and $D(x)$ are defined by (2.103). An ordinary differential equation is derived for $F(\eta)$ by substituting (A.87), (A.88), (A.89) and (A.90) into (A.86). Equation (A.90) can be rearranged so that

$$y = (\eta + D(x))e^{C_2 B(x)}. \tag{A.91}$$

Using (A.91) we eliminate y from the partial differential equation (A.86).

Differentiating (A.90) with respect to x and y gives

$$\frac{\partial \eta}{\partial x} = -C_2 B'(x)(\eta + D(x)) - D'(x) \tag{A.92}$$

and

$$\frac{\partial \eta}{\partial y} = e^{-C_2 B(x)}. \tag{A.93}$$

The derivatives of $\psi(x,y)$ are as follows :

$$\frac{\partial \psi}{\partial x} = e^{(C_1+C_2)B(x)}\left[(C_1+C_2)B'(x)F - [C_2 B'(x)(\eta + D(x)) + D'(x)]\frac{dF}{d\eta}\right), \tag{A.94}$$

$$\frac{\partial \psi}{\partial y} = e^{C_1 B(x)} \frac{dF}{d\eta}, \tag{A.95}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = C_1 B'(x) e^{C_1 B(x)} \frac{dF}{d\eta} - [C_2 B'(x)(\eta + D(x)) + D'(x)] e^{C_1 B(x)} \frac{d^2 F}{d\eta^2}, \tag{A.96}$$

$$\frac{\partial^2 \psi}{\partial y^2} = e^{(C_1 - C_2) B(x)} \frac{d^2 F}{d\eta^2}, \tag{A.97}$$

$$\frac{\partial^3 \psi}{\partial y^3} = e^{(C_1 - 2C_2) B(x)} \frac{d^3 F}{d\eta^3}. \tag{A.98}$$

Substituting (A.94) to (A.98) into (A.86) gives

$$\begin{aligned} &\frac{C_1}{a(x)} e^{2C_1 B(x)} \left(\frac{dF}{d\eta}\right)^2 - \left[\frac{C_2}{a(x)}(\eta + D(x)) + D'(x)\right] e^{2C_1 B(x)} \frac{dF}{d\eta} \frac{d^2 F}{d\eta^2} \\ &- \frac{(C_1 + C_2)}{a(x)} e^{2C_1 B(x)} F \frac{d^2 F}{d\eta^2} + \left[\frac{C_2}{a(x)}(\eta + D(x)) + D'(x)\right] e^{2C_1 B(x)} \frac{dF}{d\eta} \frac{d^2 F}{d\eta^2} \\ &= \frac{C_7}{a(x)} e^{2C_1 B(x)} + \frac{1}{a(x)} e^{2C_1 B(x)} \frac{dG}{d\eta} \frac{d^2 F}{d\eta^2} + \frac{1}{a(x)} e^{2C_1 B(x)} G \frac{d^3 F}{d\eta^3}. \end{aligned} \tag{A.99}$$

Simplifying (A.99) leads to the following third order ordinary differential equation :

$$C_1 \left(\frac{dF}{d\eta}\right)^2 - (C_1 + C_2) F \frac{d^2 F}{d\eta^2} = C_7 + \frac{dG}{d\eta} \frac{d^2 F}{d\eta^2} + G \frac{d^3 F}{d\eta^3}. \tag{A.100}$$

We rearrange (A.100) by adding and subtracting $(C_1 + C_2) \left(\frac{dF}{d\eta}\right)^2$:

$$\frac{d}{d\eta}\left(G \frac{d^2 F}{d\eta^2}\right) + (C_1 + C_2) \frac{d}{d\eta}\left(F \frac{dF}{d\eta}\right) - (2C_1 + C_2) \left(\frac{dF}{d\eta}\right)^2 + C_7 = 0. \tag{A.101}$$

Appendix B

Calculation of the Lie point symmetry for the two-dimensional turbulent wall jet

In Appendix B the details of the calculation of the Lie point symmetry

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi} \tag{B.1}$$

of the partial differential equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - M(x)\frac{df}{d\psi_y}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 - M(x)f(\psi_y)\frac{\partial^3 \psi}{\partial y^3} = 0 \tag{B.2}$$

are presented.

B.1 Calculation of prolongation formulae

We will require only ζ_1, ζ_2, ζ_{12}, ζ_{22} and ζ_{222}. The detailed calculations of the prolongation formulae are found in Appendix A.1.

$$\zeta_1 = \eta_x + \eta_\psi \psi_x - \xi^1_x \psi_x - \xi^1_\psi (\psi_x)^2 - \xi^2_x \psi_y - \xi^2_\psi \psi_x \psi_y, \tag{B.3}$$

$$\zeta_2 = \eta_y + \eta_\psi \psi_y - \xi^1_y \psi_x - \xi^1_\psi \psi_x \psi_y - \xi^2_y \psi_y - \xi^2_\psi (\psi_y)^2, \tag{B.4}$$

$$\begin{aligned}\zeta_{12} = {} & \eta_{xy} + \eta_{x\psi}\psi_y + \eta_{y\psi}\psi_x + \eta_{\psi\psi}\psi_x\psi_y + \eta_\psi \psi_{xy} - \xi^1_x \psi_{xy} - \xi^1_\psi \psi_x \psi_{xy} \\ & -\xi^1_{xy}\psi_x - \xi^1_{x\psi}\psi_x\psi_y - \xi^1_{y\psi}(\psi_x)^2 - \xi^1_{\psi\psi}(\psi_x)^2\psi_y - \xi^1_\psi \psi_x \psi_{xy} - \xi^2_x \psi_{yy} \\ & -\xi^2_\psi \psi_x \psi_{yy} - \xi^2_{xy}\psi_y - \xi^2_{x\psi}(\psi_y)^2 - \xi^2_{y\psi}\psi_x\psi_y - \xi^2_{\psi\psi}\psi_x(\psi_y)^2 - \xi^2_\psi \psi_y \psi_{xy} \\ & -\xi^1_y \psi_{xx} - \xi^1_\psi \psi_y \psi_{xx} - \xi^2_y \psi_{xy} - \xi^2_\psi \psi_{xy}\psi_y,\end{aligned} \tag{B.5}$$

$$
\begin{aligned}
\zeta_{22} &= \eta_{yy} + 2\eta_{y\psi}\psi_y + \eta_{\psi\psi}(\psi_y)^2 + \eta_\psi\psi_{yy} - 2\xi^1_y\psi_{xy} - 2\xi^1_\psi\psi_y\psi_{xy} - \xi^1_{yy}\psi_x \\
&-2\xi^1_{y\psi}\psi_x\psi_y - \xi^1_{\psi\psi}\psi_x(\psi_y)^2 - \xi^1_\psi\psi_x\psi_{yy} - 2\xi^2_y\psi_{yy} - 2\xi^2_\psi\psi_y\psi_{yy} - \xi^2_{yy}\psi_y \\
&-2\xi^2_{y\psi}(\psi_y)^2 - \xi^2_{\psi\psi}(\psi_y)^3 - \xi^2_\psi\psi_y\psi_{yy}, \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{222} &= \eta_{yyy} + 3\eta_{yy\psi}\psi_y + 3\eta_{y\psi\psi}(\psi_y)^2 + 3\eta_{y\psi}\psi_{yy} + \eta_{\psi\psi\psi}(\psi_y)^3 + 3\eta_{\psi\psi}\psi_y\psi_{yy} \\
&+\eta_\psi\psi_{yyy} - 3\xi^1_y\psi_{xyy} - 3\xi^1_\psi\psi_y\psi_{xyy} - 3\xi^1_{yy}\psi_{xy} - 6\xi^1_{y\psi}\psi_y\psi_{xy} - 3\xi^1_\psi\psi_{yy}\psi_{xy} \\
&-3\xi^1_{\psi\psi}(\psi_y)^2\psi_{xy} - \xi^1_{yyy}\psi_x - 3\xi^1_{yy\psi}\psi_x\psi_y - 3\xi^1_{y\psi\psi}\psi_x(\psi_y)^2 - 3\xi^1_{y\psi}\psi_x\psi_{yy} \\
&-\xi^1_{\psi\psi\psi}\psi_x(\psi_y)^3 - 3\xi^1_{\psi\psi}\psi_x\psi_y\psi_{yy} - \xi^1_\psi\psi_x\psi_{yyy} - 3\xi^2_y\psi_{yyy} - 3\xi^2_\psi\psi_y\psi_{yyy} \\
&-3\xi^2_{yy}\psi_{yy} - 6\xi^2_{y\psi}\psi_y\psi_{yy} - 3\xi^2_{\psi\psi}(\psi_y)^2\psi_{yy} - 3\xi^2_\psi(\psi_{yy})^2 - \xi^2_{yyy}\psi_y \\
&-3\xi^2_{yy\psi}(\psi_y)^2 - 3\xi^2_{y\psi\psi}(\psi_y)^3 - 3\xi^2_{y\psi}\psi_y\psi_{yy} - 3\xi^2_{\psi\psi}(\psi_y)^2\psi_{yy} - \xi^2_{\psi\psi\psi}(\psi_y)^4 \\
&-\xi^2_\psi\psi_y\psi_{yyy}. \qquad \text{(B.7)}
\end{aligned}
$$

B.2 Determining equation

The determining equation (3.186) from Section 3.5.1 is as follows :

$$
\begin{aligned}
&-\frac{dM}{dx}\frac{df}{d\psi_y}\psi^2_{yy}\xi^1 - \frac{dM}{dx}f(\psi_y)\psi_{yyy}\xi^1 - \psi_{yy}\zeta_1 + \psi_{xy}\zeta_2 \\
&\quad -M(x)\frac{d^2f}{d\psi^2_y}\psi^2_{yy}\zeta_2 - M(x)\frac{df}{d\psi_y}\psi_{yyy}\zeta_2 + \psi_y\zeta_{12} - \psi_x\zeta_{22} \\
&\qquad -2M(x)\frac{df}{d\psi_y}\psi_{yy}\zeta_{22} - M(x)f(\psi_y)\zeta_{222}|_{F=0} = 0. \qquad \text{(B.8)}
\end{aligned}
$$

Substituting (B.3), (B.4), (B.5), (B.6) and (B.7) and (3.187) for $M(x)f(\psi_y)\psi_{yyy}$ which is

$$
M(x)f(\psi_y)\psi_{yyy} = \psi_y\psi_{xy} - \psi_x\psi_{yy} - W(x) - M(x)\frac{df}{d\psi_y}\psi^2_{yy}, \qquad \text{(B.9)}
$$

into (B.8), the determining equation in expanded form becomes

$$
-\tfrac{dM}{dx}\tfrac{df}{d\psi_y}\xi^1\psi^2_{yy} - \tfrac{1}{M(x)}\tfrac{dM}{dx}\xi^1\psi_y\psi_{xy} + \tfrac{1}{M(x)}\tfrac{dM}{dx}\xi^1\psi_x\psi_{yy} + \tfrac{dM}{dx}\tfrac{df}{d\psi_y}\xi^1\psi^2_{yy} - \eta_x\psi_{yy}
$$

$$
-\eta_\psi\psi_x\psi_{yy} + \xi^1_x\psi_x\psi_{yy} + \xi^1_\psi(\psi_x)^2\psi_{yy} + \xi^2_x\psi_y\psi_{yy} + \xi^2_\psi\psi_x\psi_y\psi_{yy} + \eta_y\psi_{xy}
$$

$$
+\eta_\psi\psi_y\psi_{xy} - \xi^1_y\psi_x\psi_{xy} - \xi^1_\psi\psi_x\psi_y\psi_{xy} - \xi^2_y\psi_y\psi_{xy} - \xi^2_\psi(\psi_y)^2\psi_{xy}
$$

$$
-M(x)\tfrac{d^2f}{d\psi^2_y}\eta_y\psi^2_{yy} - M(x)\tfrac{d^2f}{d\psi^2_y}\eta_\psi\psi_y\psi^2_{yy} + M(x)\tfrac{d^2f}{d\psi^2_y}\xi^1_y\psi_x\psi^2_{yy} + M(x)\tfrac{d^2f}{d\psi^2_y}\xi^1_\psi\psi_x\psi_y\psi^2_{yy}
$$

$$
+M(x)\tfrac{d^2f}{d\psi^2_y}\xi^2_y\psi_y\psi^2_{yy} + M(x)\tfrac{d^2f}{d\psi^2_y}\xi^2_\psi(\psi_y)^2\psi^2_{yy} - \tfrac{1}{f}\tfrac{df}{d\psi_y}\eta_y\psi_y\psi_{xy} + \tfrac{1}{f}\tfrac{df}{d\psi_y}\eta_y\psi_x\psi_{yy}
$$

$$+M(x)\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\eta_y\psi_{yy}^2 - \frac{1}{f}\frac{df}{d\psi_y}\eta_\psi\psi_y^2\psi_{xy} + \frac{1}{f}\frac{df}{d\psi_y}\eta_\psi\psi_x\psi_y\psi_{yy} + M(x)\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\eta_\psi\psi_y\psi_{yy}^2$$

$$+\frac{1}{f}\frac{df}{d\psi_y}\xi_y^1\psi_x\psi_y\psi_{xy} - \frac{1}{f}\frac{df}{d\psi_y}\xi_y^1\psi_x^2\psi_{yy} - M(x)\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\xi_y^1\psi_x\psi_{yy}^2 + \frac{1}{f}\frac{df}{d\psi_y}\xi_\psi^1\psi_x\psi_y^2\psi_{xy}$$

$$-\frac{1}{f}\frac{df}{d\psi_y}\xi_\psi^1\psi_x^2\psi_y\psi_{yy} - M(x)\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\xi_\psi^1\psi_x\psi_y\psi_{yy}^2 + \frac{1}{f}\frac{df}{d\psi_y}\xi_y^2\psi_y^2\psi_{xy} - \frac{1}{f}\frac{df}{d\psi_y}\xi_y^2\psi_x\psi_y\psi_{yy}$$

$$-M(x)\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\xi_y^2\psi_y\psi_{yy}^2 + \frac{1}{f}\frac{df}{d\psi_y}\xi_\psi^2\psi_y^3\psi_{xy} - \frac{1}{f}\frac{df}{d\psi_y}\xi_\psi^2\psi_x\psi_y^2\psi_{yy}$$

$$-M(x)\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\xi_\psi^2\psi_y^2\psi_{yy}^2 + \eta_{xy}\psi_y + \eta_{x\psi}\psi_y^2 + \eta_{y\psi}\psi_x\psi_y + \eta_{\psi\psi}\psi_x\psi_y^2 + \eta_\psi\psi_y\psi_{xy}$$

$$-\xi_x^1\psi_y\psi_{xy} - \xi_\psi^1\psi_x\psi_y\psi_{xy} - \xi_{xy}^1\psi_x\psi_y - \xi_{x\psi}^1\psi_x\psi_y^2 - \xi_{y\psi}^1(\psi_x)^2\psi_y - \xi_{\psi\psi}^1(\psi_x)^2\psi_y^2$$

$$-\xi_\psi^1\psi_x\psi_y\psi_{xy} - \xi_x^2\psi_y\psi_{yy} - \xi_\psi^2\psi_x\psi_y\psi_{yy} - \xi_{xy}^2\psi_y^2 - \xi_{x\psi}^2(\psi_y)^3 - \xi_{y\psi}^2\psi_x\psi_y^2 - \xi_{\psi\psi}^2\psi_x(\psi_y)^3$$

$$-\xi_\psi^2\psi_y^2\psi_{xy} - \xi_y^1\psi_y\psi_{xx} - \xi_\psi^1\psi_y^2\psi_{xx} - \xi_y^2\psi_y\psi_{xy} - \xi_\psi^2\psi_y^2\psi_{xy} - \eta_{yy}\psi_x - 2\eta_{y\psi}\psi_x\psi_y$$

$$-\eta_{\psi\psi}\psi_x(\psi_y)^2 - \eta_\psi\psi_x\psi_{yy} + 2\xi_y^1\psi_x\psi_{xy} + 2\xi_\psi^1\psi_x\psi_y\psi_{xy} + \xi_{yy}^1\psi_x^2 + 2\xi_{y\psi}^1\psi_x^2\psi_y + \xi_{\psi\psi}^1\psi_x^2(\psi_y)^2$$

$$+\xi_\psi^1\psi_x^2\psi_{yy} + 2\xi_y^2\psi_x\psi_{yy} + 2\xi_\psi^2\psi_x\psi_y\psi_{yy} + \xi_{yy}^2\psi_x\psi_y + 2\xi_{y\psi}^2\psi_x(\psi_y)^2 + \xi_{\psi\psi}^2\psi_x(\psi_y)^3 + \xi_\psi^2\psi_x\psi_y\psi_{yy}$$

$$-2M(x)\frac{df}{d\psi_y}\eta_{yy}\psi_{yy} - 4M(x)\frac{df}{d\psi_y}\eta_{y\psi}\psi_y\psi_{yy} - 2M(x)\frac{df}{d\psi_y}\eta_{\psi\psi}(\psi_y)^2\psi_{yy} - 2M(x)\frac{df}{d\psi_y}\eta_\psi\psi_{yy}^2$$

$$+4M(x)\frac{df}{d\psi_y}\xi_y^1\psi_{xy}\psi_{yy} + 4M(x)\frac{df}{d\psi_y}\xi_\psi^1\psi_y\psi_{xy}\psi_{yy} + 2M(x)\frac{df}{d\psi_y}\xi_{yy}^1\psi_x\psi_{yy} + 4M(x)\frac{df}{d\psi_y}\xi_{y\psi}^1\psi_x\psi_y\psi_{yy}$$

$$+2M(x)\frac{df}{d\psi_y}\xi_{\psi\psi}^1\psi_x(\psi_y)^2\psi_{yy} + 2M(x)\frac{df}{d\psi_y}\xi_\psi^1\psi_x\psi_{yy}^2 + 4M(x)\frac{df}{d\psi_y}\xi_y^2\psi_{yy}^2 + 4M(x)\frac{df}{d\psi_y}\xi_\psi^2\psi_y\psi_{yy}^2$$

$$+2M(x)\frac{df}{d\psi_y}\xi_{yy}^2\psi_y\psi_{yy} + 4M(x)\frac{df}{d\psi_y}\xi_{y\psi}^2(\psi_y)^2\psi_{yy} + 2M(x)\frac{df}{d\psi_y}\xi_{\psi\psi}^2(\psi_y)^3\psi_{yy} + 2M(x)\frac{df}{d\psi_y}\xi_\psi^2\psi_y\psi_{yy}^2$$

$$-M(x)f(\psi_y)\eta_{yyy} - 3M(x)f(\psi_y)\eta_{yy\psi}\psi_y - 3M(x)f(\psi_y)\eta_{y\psi\psi}(\psi_y)^2 - 3M(x)f(\psi_y)\eta_{y\psi}\psi_{yy}$$

$$-M(x)f(\psi_y)\eta_{\psi\psi\psi}(\psi_y)^3 - 3M(x)f(\psi_y)\eta_{\psi\psi}\psi_y\psi_{yy} - \eta_\psi\psi_y\psi_{xy} + \eta_\psi\psi_x\psi_{yy}$$

$$+M(x)\frac{df}{d\psi_y}\eta_\psi\psi_{yy}^2 + 3M(x)f(\psi_y)\xi_y^1\psi_{xyy} + 3M(x)f(\psi_y)\xi_\psi^1\psi_y\psi_{xyy} + 3M(x)f(\psi_y)\xi_{yy}^1\psi_{xy}$$

$$+6M(x)f(\psi_y)\xi_{y\psi}^1\psi_y\psi_{xy} + 3M(x)f(\psi_y)\xi_\psi^1\psi_{yy}\psi_{xy} + 3M(x)f(\psi_y)\xi_{\psi\psi}^1(\psi_y)^2\psi_{xy}$$

$$+M(x)f(\psi_y)\xi_{yyy}^1\psi_x + 3M(x)f(\psi_y)\xi_{yy\psi}^1\psi_x\psi_y + 3M(x)f(\psi_y)\xi_{y\psi\psi}^1\psi_x(\psi_y)^2$$

$$+3M(x)f(\psi_y)\xi^1_{y\psi}\psi_x\psi_{yy}+M(x)f(\psi_y)\xi^1_{\psi\psi\psi}\psi_x(\psi_y)^3+3M(x)f(\psi_y)\xi^1_{\psi\psi}\psi_x\psi_y\psi_{yy}$$

$$+\xi^1_\psi\psi_x\psi_y\psi_{xy}-\xi^1_\psi\psi_x^2\psi_{yy}-M(x)\tfrac{df}{d\psi_y}\xi^1_\psi\psi_x\psi_{yy}^2+3\xi^2_y\psi_y\psi_{xy}-3\xi^2_y\psi_x\psi_{yy}-3M(x)\tfrac{df}{d\psi_y}\xi^2_y\psi_{yy}^2$$

$$+3\xi^2_\psi\psi_y^2\psi_{xy}-3\xi^2_\psi\psi_x\psi_y\psi_{yy}-3M(x)\tfrac{df}{d\psi_y}\xi^2_\psi\psi_y\psi_{yy}^2+3M(x)f(\psi_y)\xi^2_{yy}\psi_{yy}$$

$$+9M(x)f(\psi_y)\xi^2_{y\psi}\psi_y\psi_{yy}+3M(x)f(\psi_y)\xi^2_{\psi\psi}(\psi_y)^2\psi_{yy}+3M(x)f(\psi_y)\xi^2_\psi(\psi_{yy})^2 \quad \text{(B.10)}$$

$$+M(x)f(\psi_y)\xi^2_{yyy}\psi_y+3M(x)f(\psi_y)\xi^2_{yy\psi}(\psi_y)^2+3M(x)f(\psi_y)\xi^2_{y\psi\psi}(\psi_y)^3$$

$$+3M(x)f(\psi_y)\xi^2_{\psi\psi}(\psi_y)^2\psi_{yy}+M(x)f(\psi_y)\xi^2_{\psi\psi\psi}(\psi_y)^4+\xi^2_\psi\psi_y^2\psi_{xy}-\xi^2_\psi\psi_x\psi_y\psi_{yy}$$

$$-M(x)\tfrac{df}{d\psi_y}\xi^2_\psi\psi_y\psi_{yy}^2=0$$

A system of partial differential equations is found by equating the coefficients of the derivatives of ψ. Now

$$\xi^1=\xi^1(x,y,\psi),\quad \xi^2=\xi^2(x,y,\psi),\quad M=M(x),\quad f=f(\psi_y). \quad \text{(B.11)}$$

We therefore separate according to second derivatives of ψ, third derivatives of ψ and ψ_x :

$$\psi_{xyy},\quad \psi^2_{yy},\quad \psi_{xy}\psi_{yy},\quad \psi_{xx},\quad \psi_{xy},\quad \psi_{yy},\quad \psi_x\psi_{yy},\quad \psi_x,\quad \textit{Remainder}$$

First consider the coefficients of ψ_{xyy}. Since $M(x)\neq 0$ and $f(\psi_y)\neq 0$ we obtain

$$\xi^1_y+\psi_y\xi^1_\psi=0. \quad \text{(B.12)}$$

Separating (B.12) according to powers of ψ_y gives

$$\xi^1_y=0,\qquad\qquad \xi^1_\psi=0. \quad \text{(B.13)}$$

Thus

$$\xi^1(x,y,\psi)=A(x), \quad \text{(B.14)}$$

where $A(x)$ is an arbitrary function of x.

Using (B.13) we see that the coefficient of $\psi_{xy}\psi_{yy}$ in the determining equation (B.10) is zero. The coefficients of ψ^2_{yy} lead to the following equation :

$$\frac{d^2f}{d\psi_y^2}\left[-\eta_y-\eta_\psi\psi_y+\xi^2_y\psi_y+\xi^2_\psi\psi_y^2\right]-\frac{1}{f}\left(\frac{df}{d\psi_y}\right)^2\left[-\eta_y-\eta_\psi\psi_y+\xi^2_y\psi_y+\xi^2_\psi\psi_y^2\right]$$

$$+\frac{1}{\psi_y}\frac{df}{d\psi_y}\left[-\eta_\psi\psi_y+\xi^2_y\psi_y+2\xi^2_\psi\psi_y^2\right]+f(\psi_y)\left[3\xi^2_\psi\right]=0, \qquad \text{(B.15)}$$

where the common factor of $M(x)$ has been cancelled and the terms have been grouped according to their dependence on $f(\psi_y)$. There are no terms involving ψ_{xx} which remain once we have substituted (B.13) into the determining equation (B.10). The coefficients of $\psi_x\psi_{yy}$ yield

$$\frac{1}{M(x)}\frac{dM}{dx}\xi^1+\xi^1_x-\eta_\psi-\xi^2_y-\xi^2_\psi\psi_y+\frac{1}{f}\frac{df}{d\psi_y}\left[\eta_y+\eta_\psi\psi_y-\xi^2_y\psi_y-\xi^2_\psi\psi_y^2\right]=0.\text{(B.16)}$$

The coefficients of ψ_{yy} give

$$\begin{aligned}-\eta_x+2M(x)\frac{df}{d\psi_y}&\left[-\eta_{yy}-2\eta_{y\psi}\psi_y-\eta_{\psi\psi}\psi_y^2+\xi^2_{yy}\psi_y+2\xi^2_{y\psi}\psi_y^2+\xi^2_{\psi\psi}\psi_y^3\right]\\ &+3M(x)f(\psi_y)\left[-\eta_{y\psi}-\eta_{\psi\psi}\psi_y+\xi^2_{yy}+3\xi^2_{y\psi}\psi_y+2\xi^2_{\psi\psi}\psi_y^2\right]=0. \qquad \text{(B.17)}\end{aligned}$$

The coefficients of ψ_{xy}, divided by ψ_y which is a common factor, leads to

$$-\frac{1}{M(x)}\frac{dM}{dx}\xi^1-\xi^1_x+\frac{1}{\psi_y}\eta_y+\eta_\psi+\xi^2_y+\xi^2_\psi\psi_y+\frac{1}{f}\frac{df}{d\psi_y}\left[-\eta_y-\eta_\psi\psi_y+\xi^2_y\psi_y+\xi^2_\psi\psi_y^2\right]=0. \qquad \text{(B.18)}$$

Separating by ψ_x we obtain

$$\xi^2_{y\psi}\psi_y^2+(\xi^2_{yy}-\eta_{y\psi})\psi_y-\eta_{yy}=0. \qquad \text{(B.19)}$$

Equation (B.19) does not depend on $f(\psi_y)$. We can therefore separate according to powers of ψ_y.

$$\psi_y^2: \qquad \xi^2_{y\psi}=0, \qquad \text{(B.20)}$$

$$\psi_y^1: \quad \xi^2_{yy}-\eta_{y\psi}=0, \qquad \text{(B.21)}$$

$$\psi_y^0: \qquad \eta_{yy}=0. \qquad \text{(B.22)}$$

The remaining terms in the determining equation (B.10) are

$$\begin{aligned}&\eta_{xy}\psi_y+\eta_{x\psi}\psi_y^2-\xi^2_{xy}\psi_y^2-\xi^2_{x\psi}\psi_y^3+M(x)f(\psi_y)\Big[-\eta_{yyy}-3\eta_{yy\psi}\psi_y\\ &-3\eta_{y\psi\psi}\psi_y^2-\eta_{\psi\psi\psi}\psi_y^3+\xi^2_{yyy}\psi_y+3\xi^2_{yy\psi}\psi_y^2+3\xi^2_{y\psi\psi}\psi_y^3+\xi^2_{\psi\psi\psi}\psi_y^4\Big]. \qquad \text{(B.23)}\end{aligned}$$

We now use equations (B.15), (B.16), (B.17), (B.18), (B.20), (B.21), (B.22) and (B.23) to solve for ξ^1, ξ^2 and η.

Adding equations (B.16) and (B.18) leads to

$$\eta_y = 0, \tag{B.24}$$

and hence

$$\eta(x, y, \psi) = C(x, \psi), \tag{B.25}$$

where $C(x, \psi)$ is an arbitrary function of x and ψ. Substituting (B.25) into (B.20), (B.21) and (B.22) leaves us with

$$\xi^2_{y\psi} = 0, \tag{B.26}$$

$$\xi^2_{yy} = 0. \tag{B.27}$$

From (B.27) we have that

$$\xi^2 = yB(x, \psi) + K(x, \psi), \tag{B.28}$$

where $B(x, \psi)$ and $K(x, \psi)$ are both arbitrary functions of x and ψ. Substituting (B.28) into (B.26) it follows that

$$B = B(x), \tag{B.29}$$

and therefore

$$\xi^2 = yB(x) + K(x, \psi). \tag{B.30}$$

Thus far we have from (B.14),(B.25) and (B.30) that

$$\xi^1 = A(x), \qquad \xi^2 = yB(x) + K(x, \psi), \qquad \eta = C(x, \psi). \tag{B.31}$$

To proceed further with the form of $f(\psi_y)$ unspecified would be difficult. We now specify $f(\psi_y)$ to be in the form of a power law :

$$f(\psi_y) = \psi_y^n, \quad n > 0. \tag{B.32}$$

Since $\psi_y \to 0$ as $y \to \infty$ we require $n > 0$ to keep $f(\psi_y)$ finite as $y \to \infty$.

Substituting (B.31) and (B.32) into (B.15) gives

$$(3 + n)\frac{\partial K(x, \psi)}{\partial \psi} = 0, \tag{B.33}$$

and therefore

$$\frac{\partial K}{\partial \psi} = 0, \;\; if \;\; n \neq -3. \tag{B.34}$$

Substituting (B.31) and (B.32) into (B.16) we obtain

$$\frac{A(x)}{M(x)}\frac{dM}{dx} + \frac{dA}{dx} + (n-1)\frac{\partial C(x,\psi)}{\partial \psi} - (n+1)B(x) - (n+1)\frac{\partial K(x,\psi)}{\partial \psi}\psi_y = 0. \quad \text{(B.35)}$$

Separate (B.35) according to powers of ψ_y :

$$\psi_y^1 : \ (n+1)\frac{\partial K(x,\psi)}{\partial \psi} = 0, \quad \text{(B.36)}$$

$$\psi_y^0 : \ \frac{A(x)}{M(x)}\frac{dM}{dx} + \frac{dA}{dx} + (n-1)\frac{\partial C(x,\psi)}{\partial \psi} - (n+1)B(x) = 0. \quad \text{(B.37)}$$

From (B.36) it follows that

$$\frac{\partial K}{\partial \psi} = 0 \quad if \quad n \neq -1. \quad \text{(B.38)}$$

Thus, combining (B.34) and (B.38), for all values of n,

$$\frac{\partial K}{\partial \psi} = 0 \quad \text{(B.39)}$$

and therefore

$$K = K(x). \quad \text{(B.40)}$$

Substituting (B.40) into (B.31) and (B.35) we have

$$\xi^1 = A(x), \qquad \xi^2 = yB(x) + K(x), \qquad \eta = C(x,\psi), \quad \text{(B.41)}$$

$$\frac{A(x)}{M(x)}\frac{dM}{dx} + \frac{dA}{dx} + (n-1)\frac{\partial C(x,\psi)}{\partial \psi} - (n+1)B(x) = 0. \quad \text{(B.42)}$$

Now substitute (B.41) into (B.17) :

$$\frac{\partial C(x,\psi)}{\partial x} + 2M(x)\psi_y^2\frac{df}{d\psi_y}\frac{\partial^2 C(x,\psi)}{\partial \psi^2} + 3M(x)\psi_y f(\psi_y)\frac{\partial^2 C(x,\psi)}{\partial \psi^2} = 0. \quad \text{(B.43)}$$

Substituting (B.32) into (B.43) we have

$$\frac{\partial C(x,\psi)}{\partial x} + (2n+3)M(x)\psi_y^{n+1}\frac{\partial^2 C(x,\psi)}{\partial \psi^2} = 0. \quad \text{(B.44)}$$

We are assuming that $n > 0$ and therefore $n \neq -1$. Separate (B.44) according to powers of ψ_y :

$$\psi_y^0 : \ \frac{\partial C(x,\psi)}{\partial x} = 0 \quad \text{(B.45)}$$

$$\psi_y^{n+1} : \ (2n+3)\frac{\partial^2 C(x,\psi)}{\partial \psi^2} = 0. \quad \text{(B.46)}$$

Thus from (B.45), if $n \neq -1$, $C = C(\psi)$ and from (B.46), if further, $n \neq -\frac{3}{2}$, then

$$C(\psi) = c_1\psi + c_2, \tag{B.47}$$

where c_1 and c_2 are arbitrary constants. Substituting (B.47) into (B.41) gives

$$\xi^1 = A(x), \qquad \xi^2 = yB(x) + K(x), \qquad \eta = c_1\psi + c_2, \tag{B.48}$$

provided $n \neq -1$ and $n \neq -\frac{3}{2}$.

Substituting (B.48) and (B.32) into (B.23) we obtain

$$\frac{dB}{dx}\psi_y^2 = 0. \tag{B.49}$$

Thus

$$B(x) = c_3 \tag{B.50}$$

where c_3 is an arbitrary constant.

Substituting (B.50) into (B.48) and (B.42) we obtain

$$\xi^1 = A(x), \tag{B.51}$$

$$\xi^2 = c_3 y + K(x), \tag{B.52}$$

$$\eta = c_1\psi + c_2, \tag{B.53}$$

$$\frac{A(x)}{M(x)}\frac{dM}{dx} + \frac{dA}{dx} + (n-1)c_1 - (n+1)c_3 = 0, \tag{B.54}$$

provided that

$$n \neq -1, \qquad n \neq -\frac{3}{2}. \tag{B.55}$$

In summary the Lie point symmetry of the partial differential equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - M(x)\frac{df}{d\psi_y}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2 - M(x)f(\psi_y)\frac{\partial^3\psi}{\partial y^3} = 0, \tag{B.56}$$

where

$$f(\psi_y) = \psi_y^n, \qquad n \neq -1,\ n \neq -\frac{3}{2}, \tag{B.57}$$

is

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial\psi}, \tag{B.58}$$

where

$$\xi^1 = A(x), \tag{B.59}$$

$$\xi^2 = c_3 y + K(x), \tag{B.60}$$

$$\eta = c_1\psi + c_2, \tag{B.61}$$

$A(x)$ and $K(x)$ are arbitrary functions of x and c_1, c_2 and c_3 are arbitrary constants, provided $M(x)$ satisfies the ordinary differential equation

$$\frac{dM}{dx} + \left[\frac{1}{A(x)}\frac{dA}{dx} + ((n-1)c_1 - (n+1)c_3)\,\frac{1}{A(x)}\right] M = 0. \tag{B.62}$$

B.3 Reduction of the partial differential equation to an ordinary differential equation

Consider the third order partial differential equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - M(x)\frac{df}{d\psi_y}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 - M(x) f(\psi_y)\frac{\partial^3 \psi}{\partial y^3} = 0. \tag{B.63}$$

In Section 3.5.2 it was shown that

$$\psi(x,y) = e^{c_1 L(x)} F(\eta) - \frac{c_2}{c_1}, \tag{B.64}$$

$$E(x,y) = M(x) f(\psi_y), \tag{B.65}$$

$$M(x) = \frac{M_0}{A(x)} exp[((n+1)c_3 - (n-1)c_1)\, L(x)], \tag{B.66}$$

where

$$\eta = e^{-c_3 L(x)} y - H(x), \qquad f(\psi_y) = \psi_y^n, \qquad n \neq -1, n \neq -\frac{3}{2}. \tag{B.67}$$

The functions $L(x)$ and $H(x)$ are defined by (3.203). An ordinary differential equation is derived for $F(\eta)$ by substituting (B.64), (B.65), (B.66) and (B.67) into (B.63).

From equation (B.67)

$$y = (\eta + H(x)) e^{c_3 L(x)}. \tag{B.68}$$

Using (B.68) we eliminate y from the partial differential equation (B.63). Differentiating η with respect to x and y gives

$$\frac{\partial \eta}{\partial x} = -\frac{c_3}{A(x)}(\eta + H(x)) - H'(x), \tag{B.69}$$

and

$$\frac{\partial \eta}{\partial y} = e^{-c_3 L(x)}. \tag{B.70}$$

The derivatives of $\psi(x,y)$ are as follows :

$$\frac{\partial \psi}{\partial x} = e^{c_1 L(x)}\left[\frac{c_1}{A(x)} F - \left[\frac{c_3}{A(x)}(\eta + H(x)) + H'(x)\right]\frac{dF}{d\eta}\right], \tag{B.71}$$

$$\frac{\partial \psi}{\partial y} = e^{(c_1-c_3)L(x)}\frac{dF}{d\eta}, \tag{B.72}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{(c_1-c_3)}{A(x)}e^{(c_1-c_3)L(x)}\frac{dF}{d\eta} - \left[\frac{c_3}{A(x)}(\eta + H(x)) + H'(x)\right]e^{(c_1-c_3)L(x)}\frac{d^2F}{d\eta^2}, \tag{B.73}$$

$$\frac{\partial^2 \psi}{\partial y^2} = e^{(c_1-2c_3)L(x)}\frac{d^2F}{d\eta^2}, \tag{B.74}$$

$$\frac{\partial^3 \psi}{\partial y^3} = e^{(c_1-3c_3)L(x)}\frac{d^3F}{d\eta^3}. \tag{B.75}$$

Substituting (B.71) to (B.75) into (B.63) gives

$$\begin{aligned}&\frac{(c_1-c_3)}{A(x)}e^{2(c_1-c_3)L(x)}\left(\frac{dF}{d\eta}\right)^2 - \left[\frac{c_3}{A(x)}(\eta + H(x)) + H'(x)\right]e^{2(c_1-c_3)L(x)}\frac{dF}{d\eta}\frac{d^2F}{d\eta^2}\\ &-\frac{c_1}{A(x)}e^{2(c_1-c_3)L(x)}F\frac{d^2F}{d\eta^2} + \left[\frac{c_3}{A(x)}(\eta + H(x)) + H'(x)\right]e^{2(c_1-c_3)L(x)}\frac{dF}{d\eta}\frac{d^2F}{d\eta^2}\\ &-\frac{nM_0}{A(x)}e^{2(c_1-c_3)L(x)}\left(\frac{dF}{d\eta}\right)^{n-1}\left(\frac{d^2F}{d\eta^2}\right)^2 - \frac{M_0}{A(x)}e^{2(c_1-c_3)L(x)}\left(\frac{dF}{d\eta}\right)^n\frac{d^3F}{d\eta^3} = 0.\end{aligned} \tag{B.76}$$

Simplifying (B.76) leads to the following third order ordinary differential equation :

$$M_0\left(\frac{dF}{d\eta}\right)^n\frac{d^3F}{d\eta^3} + M_0 n\left(\frac{dF}{d\eta}\right)^{n-1}\left(\frac{d^2F}{d\eta^2}\right)^2 + c_1 F\frac{d^2F}{d\eta^2} - (c_1-c_3)\left(\frac{dF}{d\eta}\right)^2 = 0. \tag{B.77}$$

We rearrange (B.77) by adding and subtracting $c_1\left(\frac{dF}{d\eta}\right)^2$:

$$\frac{d}{d\eta}\left[\left(\frac{dF}{d\eta}\right)^n\frac{d^2F}{d\eta^2}\right] + \frac{c_1}{M_0}\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right) - \frac{(2c_1-c_3)}{M_0}\left(\frac{dF}{d\eta}\right)^2 = 0. \tag{B.78}$$

Bibliography

[1] D.J. Acheson. Elementary Fluid Dynamics, Clarendon Press, Oxford (1990), 275-278, 292-293.

[2] M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions, Dover Publications, New York (1965), 80.

[3] A.A. Avramenko, S.G. Kobzar, I.V. Shevchuk, A.V. Kuznetsov, L.T. Iwanisov. Symmetry of turbulent boundary-layer flows : Investigation of different eddy viscosity models, *Acta Mechanica*, **151** (2001), 1-14.

[4] M.S. Bruzon, P.A. Clarkson, M.L. Gandarias and E Medina. The symmetry reductions of a turbulence model, *J. Phys. A: Math. Gen.*, **34** (2001), 3751-3760.

[5] B.J. Cantwell. Introduction to Symmetry Analysis, Cambridge University Press, Cambridge (2002), 395-445.

[6] M.Y. Chung and D.P. Mason. Unpublished manuscript.

[7] L.J. Crane. Flow past a stretching plate, *Brief reports*, **21** (1970), 645-647.

[8] L. Dresner. *Similarity solutions of nonlinear partial differential equations*, Pitman Advanced Publishing Program, London (1983).

[9] M.L. Gandarias, E. Medina and M.S. Bruzon. An approach to the b-ϵ model for turbulence through symmetry reductions, *Europhys. Lett.*, **44** (1998), 679-685.

[10] K. Gersten and H. Schlichting. *Boundary Layer Theory*, Springer, Berlin (2000), pp. 38, 41.

[11] M.B. Glauert. The wall jet, *J. Fluid Mech.*, **1** (1956), 625-643.

[12] V.N. Grebenev, M. Oberlack, A.N. Grishkov. Lie Algebra methods in turbulence, *Proc. App. Math. Mech.*, **8** (2009), 10593-10594

[13] D.L. Hill and D.P. Mason. Group invariant solution for a two–dimensional turbulent free jet described by eddy viscosity. *J. Nonlinear Math. Phys.*, **15**, Supplement 1 (2008), 146-160.

[14] P. Holmes, J.L. Lumley, G. Berkooz. Turbulence, Coherent Structures, Dynamical Systems and Symmetry, Cambridge University Press, Cambridge (1996), Parts 1 and 3.

[15] N.H. Ibragimov and R.L. Anderson, Lie theory of differential equations, *In Lie Group Analysis of Differential Equations*, Vol.**1**, Symmetries, Exact Solutions and Conservation Laws (N.H. Ibragimov, Ed.), CRC Press, Boca Raton, (1994), pp 7-14.

[16] A.H. Kara and F.M. Mahomed. Relationship between symmetries and conservation laws, *Inter. J. Theoret. Phys.*, **39** (2000), 23-40.

[17] M.S. Klamkin. On the transformation of a class of boundary value problems into initial value problems for ordinary differential equations. *SIAM Review*, **4** (1962), 43-47.

[18] M.S. Klamkin. Transformation of boundary value problems into initial value problems. *J. Math. Anal. Applic*, **32** (1970), 308-330.

[19] B. Lindgren, J.M. Osterlund and A.V. Johansson, Evaluation of scaling laws derived from Lie group symmetry methods in turbulent boundary layers, 40th AIAA Aerospace Sciences Meeting and Exhibit, Reno, NV; United States; 14-17 Jan. 2002.

[20] D.P. Mason. Group invariant solution and conservation law for a free laminar two–dimensional jet, *J. Nonlinear Math. Phys.*, **9** (2002), 92-101.

[21] J.H. Merkin and D.J. Needham. A note on the wall-jet problem, *Journal of Engineering Mathematics*, **20** (1986), 21-26.

[22] T.Y. Na. Further extensions on transforming from boundary value to initial value problems. *SIAM Review*, **10** (1968), 85-87.

[23] T.Y. Na. Transforming boundary conditions into initial conditions for ordinary differential equations. *SIAM Review*, **10** (1968), 85-87.

[24] M.A. Oberlack. A unified approach for symmetries in plane parallel turbulent shear flows. *J. Fluid Mech.*, **427** (2001), 299-328.

[25] M.A. Oberlack and G. Khujadze. Symmetry methods in turbulent boundary layer theory. In One Hundred Years of Boundary Layer Research, Proceedings of the IUTAM Symposium held at DLR-Gottingen, Germany, August 12-14, 2004, Editors: Meier G E A and Sreenwasan K R, Springer, (2006), 39-48.

[26] S.B. Pope. *Turbulent Flows*, Cambridge University Press, Cambridge (2000), p. 93.

[27] L. Rosenhead. *Laminar Boundary Layers*, Clarendon Press, Oxford (1963), 243-252.

[28] I. Ruscic. Group invariant solutions for boundary layer theory of steady laminar jets, MSc Research Report, University of Witwatersrand, (2003).

[29] B.C. Sakiadis. Boundary layer behaviour on continuous solid surfaces: I Boundary layer equations for two dimensional and axisymmetric flow, *American Inst. Chemical Eng. J.*, **7** (1961), 26-28.

[30] H. Schlichting. *Boundary Layer Theory*, McGraw-Hill, New York (1968), pp. 523-531, 544-545.

[31] G. *Ü*nal. Application of equivalence transformations to inertial subrange of turbulence, *Lie Groups and their Applications* **1**, No 1 (1994), 232-240.

[32] F.M. White. *Viscous Fluid Flow*, McGraw-Hill, New York (1974), p. 468.

[33] A.D. Young. *Boundary Layers*, BSP Professional Books, London (1989), pp. 12, 236-242.