POTENTIAL FLOWS AND TRANSFORMATION GROUPS

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand. It has not been submitted before for any degree or examination in any other University.

(Kevin Paul Pereira)
7th day of June 2013

Abstract

In this work we will consider the steady and two-dimensional potential flow of an incompressible fluid past a body without friction. Contrary to common experience, we will show that it is possible to calculate the Lie point symmetries that will leave the boundary value problem invariant. We are able to do this by solving the determining equation for the Lie point symmetries subject to a side condition. The side condition is a consequence of the boundary condition that occurs in the boundary value problem. We will show that solutions of the boundary value problem that were obtained previously using the method of conformal transformations are also group invariant solutions of the boundary value problem. We will also show that every group invariant solution of the boundary value problem can be used to generate new group invariant solutions of the same boundary value problem.

Dedicated to the dreams that come true.

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Contents

Chapter 1

Introduction

The problem of the steady and two-dimensional potential flow of an incompressible fluid past a body without friction is important in the history of fluid mechanics. The problem received great attention by mathematicians at the end of the nineteenth century because it provided the starting point for developing and understanding the basic principles in the emerging science of aerodynamics [1, 4]. By exploiting the close relationship that exists between the theories of fluid mechanics and functions of a complex variable, mathematicians at the time were able to use the method of conformal transformations to solve the problem. The most famous of these solutions was obtained by means of the Joukowski transformation. This transformation provided mathematicians with the solution of the problem for the flow of a fluid past an airfoil [20].

Earlier in the nineteenth century mathematicians were also making precise the notion of symmetry in science [25]. Their attempts to generalise the idea of symmetry in geometry as invariance under transformations were to prove invaluable in understanding the processes involved in solving both algebraic and differential equations. In this work we will use these symmetry methods to solve the problem of the flow of a fluid past a body.

This work can therefore be divided into three parts. In Chapter 2 we will discuss how the concept of symmetry can be used to solve differential equations. Following that, in Chapter 3, we will derive the boundary value problem for the flow of a fluid past a body without friction. We will then illustrate the method of conformal transformations by using it to determine the flow of a fluid past a wedge. In Chapter 4 we will develop a new method for solving the boundary value problem that is based on the discussion of symmetry in Chapter 2. Using this method we will provide group invariant solutions of the boundary value problem for the flow of a fluid past a circular cylinder, an elliptic cylinder and a parabolic cylinder. Since we are developing here a new approach to solving a well-known problem in fluid mechanics, the list of references at the end of this work includes mostly reference books and textbooks.

Chapter 2

Continuous Groups of Transformations

2.1 Introduction

The solutions of the polynomial equation of degree two,

$$x^2 + px + q = 0, \tag{2.1}$$

can be calculated using the quadratic formula,

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}. \tag{2.2}$$

Similar formulae exist that allow us to calculate the solutions of polynomial equations of degree three and four [3]. It is impossible, however, to derive a general formula such as (2.2) that only makes use of the four basic operations of arithmetic (addition, subtraction, multiplication, division) and the extraction of roots to calculate the solutions of polynomial equations of degree five and higher[1]. Nonetheless there are instances when it is possible to calculate the solutions of polynomial equations of degree five and higher subject to the aforementioned restrictions (so-called radical solutions). The criteria that must be satisfied for us to know that it is possible to calculate radical solutions of polynomial equations of degree five and higher

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[1] Niels Henrik Abel (1802- 1829) is credited with proving this result [3, 25].

were first given by Galois[2]. For example, if we apply these criteria to the equations

$$x^5 - x - 1 = 0 \text{ and } x^5 + 15x - 44 = 0, \tag{2.3}$$

then it would indicate to us that is possible to calculate radical solutions of the second equation only [3].

As a result of the rise to prominence of the theory of groups due to the work of Galois, the idea came to Lie[3] to try to use groups to put to order all the various methods for solving differential equations that existed at his time. Lie was able to achieve this by developing a theory of continuous groups of transformations. In this work our interest is with one-parameter continuous groups of transformations and in what follows we will give an overview of the theory of these groups.

2.2 One-Parameter Continuous Groups of Transformations

In setting up a one-parameter continuous group of transformations, the first requirement is to construct a space on which the group will act. As the main aim of this work is to use a group of transformations to solve a partial differential equation, the space will initially be made up of the dependent variable ϕ and the independent variables $x_1, \ldots, x_n$ that occur in the partial differential equation. Let the dependent variable ϕ be denoted as x_{n+1} so that for ease of discussion we may refer to a point in the space of the dependent and independent variables by means of the vector $\underline{x} = (x_1, \ldots, x_{n+1})$.

Next we introduce $n + 1$ functions that depend on the variables $x_1, \ldots, x_{n+1}$ and a

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[2] Evariste Galois (1811 - 1832) [3, 25].

[3] Marius Sophus Lie (1842 - 1899) [25].

parameter a; that is,

$$f_i\left(x_1, \ldots, x_{n+1}; a\right) = f_i\left(\underline{x}; a\right) \text{ where } i = 1, \ldots, n+1. \tag{2.4}$$

The functions (2.4) must be differentiable with respect to the variables $x_1, \ldots, x_{n+1}$ and the parameter a. Furthermore, the functions (2.4) must be independent. By this we mean that it must be impossible to determine a relation $\mathcal{F}$ between the functions (2.4) such that

$$\mathcal{F}\left(f_1, \ldots, f_{n+1}\right) = 0. \tag{2.5}$$

The functions (2.4) will be independent if the determinant of the matrix

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{n+1}} \\ \vdots & & \vdots \\ \dfrac{\partial f_{n+1}}{\partial x_1} & \cdots & \dfrac{\partial f_{n+1}}{\partial x_{n+1}} \end{bmatrix}, \tag{2.6}$$

otherwise known as the Jacobian determinant, is not identically equal to zero [7].

The functions (2.4) will now be made to act on the space of the dependent and independent variables by transforming the point $\underline{x}$ into the point $\underline{x}^*$ such that

$$x_i^* = f_i\left(\underline{x}; a\right). \tag{2.7}$$

For the transformation (2.7) to be continuous, the functions (2.4) must have the property that for small changes in the value of the parameter a they will transform the point $\underline{x}$ into points in the neighbourhood of the point $\underline{x}^*$ [8]. In this case the transformation (2.7) can be represented as a continuous curve that passes through the point $\underline{x}$ and this curve is called the path curve for the transformation (2.7) [13].

For the transformation (2.7) to form a group it must satisfy the following three conditions [6, 8, 13].

Condition 1. The existence of the inverse transformation.

For every value a of the parameter that occurs in the transformation (2.7) there must exist a corresponding unique value a_{-1} of the parameter that will allow the functions (2.4) to transform the point $\underline{x}^*$ back into the point $\underline{x}$; that is,

$$x_i = f_i\left(\underline{x}^*; a_{-1}\right). \tag{2.8}$$

Condition 2. The existence of the identity transformation.

There must exist a unique value a_0 of the parameter such that the functions (2.4) will transform the point $\underline{x}$ into itself; that is,

$$x_i = f_i\left(\underline{x}; a_0\right). \tag{2.9}$$

Condition 3. The existence of a composite transformation.

Given the transformation (2.7) of a point $\underline{x}$ into the point $\underline{x}^*$ and the transformation of the point $\underline{x}^*$ into the point $\underline{x}^{**}$ where

$$x_i^{**} = f_i\left(\underline{x}^*; b\right), \tag{2.10}$$

then there must exist a value c of the parameter that will allow the functions (2.4) to transform the point $\underline{x}$ directly into the point $\underline{x}^{**}$; that is,

$$x_i^{**} = f_i\left(\underline{x}; c\right). \tag{2.11}$$

The existence of the composite transformation (2.11) implies that we must be able to determine a relation between the values a and b of the parameter that occur in the transformations (2.7) and (2.10) such that

$$c = \delta\left(a, b\right). \tag{2.12}$$

It has been proven in turn [6, 13] that when a relation (2.12) exists, it is possible to reparameterise the functions (2.4) such that the relation (2.12) can be written in the form

$$c = a + b. \tag{2.13}$$

The consequence of this result is that the values a_{-1} and a_0 of the parameter that occur in the inverse transformation (2.8) and the identity transformation (2.9) must satisfy the equations

$$a_0 = a + a_{-1} \text{ and } a_{-1} = a_{-1} + a_0. \tag{2.14}$$

The solution of the system of equations (2.14) is $a_{-1} = -a$ and $a_0 = 0$. Therefore, for the remainder of this chapter the discussion will refer to a one-parameter continuous group of transformations (2.7) for which the inverse transformation is obtained by changing the sign of the parameter a,

$$x_i = f_i\left(\underline{x}^*, -a\right), \tag{2.15}$$

and the identity transformation corresponds to $a = 0$,

$$x_i = f_i\left(\underline{x}, 0\right). \tag{2.16}$$

2.2.1 Approximating a One-Parameter Continuous Group of Transformations

Expand each of the functions (2.4) as a Taylor series about the value $a = 0$ of the parameter so that

$$f_i\left(\underline{x}; a\right) \approx f_i\left(\underline{x}; 0\right) + a \left.\frac{\partial f_i\left(\underline{x}; a\right)}{\partial a}\right|_{a=0}. \tag{2.17}$$

If we substitute for the values of the functions given by the transformations (2.7) and (2.16) into the expansion (2.17), then we can approximate the transformation (2.7) by means of the infinitesimal transformation,

$$x_i^* = x_i + a\xi_i\left(\underline{x}\right), \tag{2.18}$$

where

$$\xi_i\left(\underline{x}\right) = \left.\frac{\partial f_i\left(\underline{x}; a\right)}{\partial a}\right|_{a=0}. \tag{2.19}$$

The quantities $\xi_i\left(\underline{x}\right)$ are called infinitesimals and geometrically they are the components of the vector

$$\underline{\xi}\left(\underline{x}\right) = \left(\xi_1\left(\underline{x}\right), \ldots, \xi_{n+1}\left(\underline{x}\right)\right) \tag{2.20}$$

that is tangent to the path curves of the transformation (2.7) [13].

2.2.2 Extending a One-Parameter Continuous Group of Transformations

Given a relation between the dependent and independent variables, namely,

$$x_{n+1} = \phi\left(x_1, \ldots, x_n\right). \tag{2.21}$$

The relation (2.21) represents a surface in the space of the dependent and independent variables. Let $\underline{x}$ and $\underline{x} + d\underline{x} = \left(x_1 + dx_1, \ldots, x_{n+1} + dx_{n+1}\right)$ be two points on the surface where, by virtue of the relation (2.21),

$$dx_{n+1} = \frac{\partial \phi}{\partial x_1} dx_1 + \cdots + \frac{\partial \phi}{\partial x_n} dx_n. \tag{2.22}$$

In the same way that the transformation of the point $\underline{x}$ into the point $\underline{x}^*$ is given by (2.7), let the point $\underline{x} + d\underline{x}$ be transformed into the point $\left(\underline{x} + d\underline{x}\right)^*$ where

$$\left(x_i + dx_i\right)^* = f_i(\underline{x} + d\underline{x}; a). \tag{2.23}$$

Hence, under the action of the transformations (2.7) and (2.23), the components of the displacement vector between the points $\underline{x}$ and $\underline{x} + d\underline{x}$ will transform according to the equation

$$\begin{aligned} dx_i^* &= \left(x_i + dx_i\right)^* - x_i^* \\ &= f_i\left(\underline{x} + d\underline{x}; a\right) - f_i\left(\underline{x}; a\right). \end{aligned} \tag{2.24}$$

If we keep the value of the parameter a fixed and we expand each of the functions $f_i\left(\underline{x} + d\underline{x}; a\right)$ as a Taylor series about the point $\underline{x}$, then we may approximate the dx_i^* in equation (2.24) as

$$dx_i^* \approx \frac{\partial f_i(\underline{x}; a)}{\partial x_1} dx_1 + \cdots + \frac{\partial f_i(\underline{x}; a)}{\partial x_{n+1}} dx_{n+1}. \tag{2.25}$$

If we next substitute the relations (2.21) and (2.22) into the approximation (2.25), then

$$\begin{aligned} dx_i^* \approx & \left[\frac{\partial f_i(x_1,\ldots,x_n,\phi;a)}{\partial x_1} + \frac{\partial f_i(x_1,\ldots,x_n,\phi;a)}{\partial \phi}\frac{\partial \phi}{\partial x_1}\right] dx_1 + \cdots \\ & + \left[\frac{\partial f_i(x_1,\ldots,x_n,\phi;a)}{\partial x_n} + \frac{\partial f_i(x_1,\ldots,x_n,\phi;a)}{\partial \phi}\frac{\partial \phi}{\partial x_n}\right] dx_n, \end{aligned} \tag{2.26}$$

where we have dropped the vector notation to emphasise the fact that the approximation (2.26) applies to the change in the displacement vector between two points on a surface in the space of the dependent and independent variables and not between two points in the space in general.

As a result of the transformation of the point $\underline{x}$ into the point $\underline{x}^*$ via (2.7), the relation (2.21) will likewise be transformed into a relation between the transformed variables, namely,

$$x_{n+1}^* = \phi^*\left(x_1^*,\ldots,x_n^*\right). \tag{2.27}$$

We now establish a connection between the relation (2.27) and the changes in the displacement vector (2.26) by insisting that

$$dx_{n+1}^* = \frac{\partial \phi^*}{\partial x_1^*}dx_1^* + \cdots + \frac{\partial \phi^*}{\partial x_n^*}dx_n^*. \tag{2.28}$$

The assumption (2.28) is referred to as the contact condition [6, 16].

Introducing the total differential operator,

$$D_j \equiv \frac{\partial}{\partial x_j} + \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial \phi}, \tag{2.29}$$

where $j = 1,\ldots,n$, and substituting for the dx_i^* with its approximation (2.26), the contact condition (2.28) can be expressed as

$$\begin{aligned}
&[D_1 f_{n+1}(x_1,\dots,x_n,\phi;a)]\,dx_1 + \cdots + [D_n f_{n+1}(x_1,\dots,x_n,\phi;a)]\,dx_n \\
&\quad = \{[D_1 f_1(x_1,\dots,x_n,\phi;a)]\,dx_1 \quad + \quad \cdots \\
&\qquad\qquad + \quad [D_n f_1(x_1,\dots,x_n,\phi;a)]\,dx_n\}\,\frac{\partial\phi^*}{\partial x_1^*} \\
&\qquad + \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.30) \\
&\qquad + \{[D_1 f_n(x_1,\dots,x_n,\phi;a)]\,dx_1 \quad + \quad \cdots \\
&\qquad\qquad + \quad [D_n f_n(x_1,\dots,x_n,\phi;a)]\,dx_n\}\,\frac{\partial\phi^*}{\partial x_n^*}.
\end{aligned}$$

Since the $dx_1,\dots,dx_n$ that occur in the contact condition (2.30) are arbitrary changes in the independent variables, it can be resolved into the following system of equations,

$$\begin{aligned}
D_j f_{n+1}(x_1,\dots,x_n,\phi;a) \;&=\; [D_j f_1(x_1,\dots,x_n,\phi;a)]\,\frac{\partial\phi^*}{\partial x_1^*} + \cdots \\
&\quad + [D_j f_n(x_1,\dots,x_n,\phi;a)]\,\frac{\partial\phi^*}{\partial x_n^*}. \qquad (2.31)
\end{aligned}$$

Proceeding as in [6], the system of equations (2.31) is written as the matrix equation,

$$\begin{bmatrix} D_1 f_{n+1}(x_1,\dots,x_n,\phi;a) \\ \vdots \\ D_n f_{n+1}(x_1,\dots,x_n,\phi;a) \end{bmatrix} = A \begin{bmatrix} \dfrac{\partial\phi^*}{\partial x_1^*} \\ \vdots \\ \dfrac{\partial\phi^*}{\partial x_n^*} \end{bmatrix}, \qquad (2.32)$$

where A is the $n \times n$ matrix,

$$A = \begin{bmatrix} D_1 f_1(x_1,\dots,x_n,\phi;a) & \cdots & D_1 f_n(x_1,\dots,x_n,\phi;a) \\ \vdots & & \vdots \\ D_n f_1(x_1,\dots,x_n,\phi;a) & \cdots & D_n f_n(x_1,\dots,x_n,\phi;a) \end{bmatrix}. \qquad (2.33)$$

It follows from the requirement of the functions (2.4) that the Jacobian determinant is not identically equal to zero that the matrix A will have a non-zero determinant and hence be invertible [8]. Thus, multiplying both sides of the matrix equation (2.32) with the inverse of the matrix A, the partial derivatives will transform according to the equation

$$\frac{\partial\phi^*}{\partial x_j^*} = \sum_{k=1}^{n} a_{jk}^{-1} \cdot D_k f_{n+1}(x_1,\dots,x_n,\phi;a) \qquad (2.34)$$

where a^{-1}_{jk} refers to the element in the jth row and kth column of the inverse of the matrix A. The value of the right-hand side of equation (2.34) is a function of the dependent variable ϕ, the independent variables $x_1, \ldots, x_n$, the partial derivatives $\partial\phi/\partial x_1, \ldots, \partial\phi/\partial x_n$ and the parameter a. If we denote the partial derivatives $\partial\phi/\partial x_j$ and $\partial\phi^*/\partial x^*_j$ by ϕ_j and ϕ^*_j, then equation (2.34) can be expressed as

$$\phi^*_j = h_j(x_1, \ldots, x_n, \phi, \phi_1, \ldots, \phi_n; a)\,. \tag{2.35}$$

It is shown in the appendix that the transformations of the partial derivatives given by the functions (2.35) also forms a group with respect to the parameter a. We may therefore create an extended one-parameter continuous group of transformations by combining the transformations (2.7) with the transformations (2.35).

Given that we are able to approximate the transformations (2.7) by means of the infinitesimal transformation (2.18), we now want to determine the corresponding approximation to the transformations (2.35). This will require us to distinguish more clearly the function that transforms the dependent variable from the functions that transform the independent variables. For this reason let

$$f_{n+1}(x_1, \ldots, x_n, \phi; a) = g(x_1, \ldots, x_n, \phi; a) \tag{2.36}$$

and denote the infinitesimal transformation (2.18) of the dependent variable as

$$\phi^* = \phi + a\eta\,(x_1, \ldots, x_n, \phi; a) \tag{2.37}$$

where

$$\eta\,(x_1, \ldots, x_n, \phi) = \left.\frac{\partial g(x_1, \ldots, x_n, \phi; a)}{\partial a}\right|_{a=0}. \tag{2.38}$$

Substituting the infinitesimal transformations (2.18) of the independent variables into the matrix A reduces the matrix to

$$A = I + aB \tag{2.39}$$

where B is the $n \times n$ matrix,

$$B = \begin{bmatrix} D_1\left[\xi_1\left(x_1,\ldots,x_n,\phi;a\right)\right] & \cdots & D_1\left[\xi_n\left(x_1,\ldots,x_n,\phi;a\right)\right] \\ \vdots & & \vdots \\ D_n\left[\xi_1\left(x_1,\ldots,x_n,\phi;a\right)\right] & \cdots & D_n\left[\xi_n\left(x_1,\ldots,x_n,\phi;a\right)\right] \end{bmatrix}, \tag{2.40}$$

and I is the $n \times n$ identity matrix. Consequently, if we approximate the inverse of the matrix A as

$$A^{-1} = (I + aB)^{-1} \approx I - aB \tag{2.41}$$

and we ignore terms of order a^2 in the matrix equation (2.32), then it follows that we may approximate the transformations (2.35) of the partial derivatives as

$$\phi_j^* \approx \phi_j + a\eta_j\left(x_1,\ldots,x_n,\phi,\phi_1,\ldots,\phi_n\right) \tag{2.42}$$

where

$$\eta_j\left(x_1,\ldots,x_n,\phi,\phi_1,\ldots,\phi_n\right) = D_j\left[\eta\left(x_1,\ldots,x_n,\phi\right)\right] - \sum_{k=1}^{n} D_j\left[\xi_k\left(x_1,\ldots,x_n,\phi\right)\right]\phi_k. \tag{2.43}$$

This process by which we are able to extend a one-parameter continuous group of transformations and then derive an approximation for the transformations may be continued as often as is necessary. Thus, if we let

$$\phi_{(r)} = \left\{ \frac{\partial^r \phi}{\partial x_j x_k \cdots x_l x_m} \text{ where } j,k,\ldots,l,m = 1,\ldots,n \right\} \tag{2.44}$$

be the set of all rth-order partial derivatives and we extend the total differential operator (2.29) to include these higher-order partial derivatives as follows,

$$D_j \equiv \frac{\partial}{\partial x_j} + \phi_j \frac{\partial}{\partial \phi} + \sum_{k=1}^{n} \phi_{jk} \frac{\partial}{\partial \phi_k} + \cdots + \sum_{m=1}^{n}\sum_{l=1}^{n}\cdots\sum_{k=1}^{n} \phi_{jk\ldots lm} \frac{\partial}{\partial \phi_{k\ldots lm}}, \tag{2.45}$$

then the infinitesimal transformations corresponding to a group that has been extended r times, namely,

$$\begin{aligned}
x_j^* &= f_j(x_1,\ldots,x_n,\phi)\,,\\
\phi^* &= g(x_1,\ldots,x_n,\phi)\,,\\
\phi_j^* &= h_j\Big(x_1,\ldots,x_n,\phi,\phi_{(1)}\Big)\,,\\
&\vdots\\
\phi_{jk\ldots l}^* &= h_{jk\ldots l}\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r-1)}\Big)\,,\\
\phi_{jk\ldots lm}^* &= h_{jk\ldots lm}\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r-1)},\phi_{(r)}\Big)
\end{aligned} \tag{2.46}$$

are

$$\begin{aligned}
x_j^* &= x_j + a\xi_j\left(x_1,\ldots,x_n,\phi\right),\\
\phi^* &= \phi + a\eta\left(x_1,\ldots,x_n,\phi\right),\\
\phi_j^* &= \phi_j + a\eta_j\Big(x_1,\ldots,x_n,\phi,\phi_{(1)}\Big)\,,\\
&\vdots\\
\phi_{jk\ldots l}^* &= \phi_{jk\ldots l} + a\eta_{jk\ldots l}\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r-1)}\Big)\,,\\
\phi_{jk\ldots lm}^* &= \phi_{jk\ldots lm} + a\eta_{jk\ldots lm}\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r-1)},\phi_{(r)}\Big)
\end{aligned} \tag{2.47}$$

where

$$\begin{aligned}
&\eta_{jk\ldots lm}\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r-1)},\phi_{(r)}\Big)\\
&\quad= D_j\Big[\eta_{k\ldots l}\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r-1)}\Big)\Big] - \sum_{s=1}^{n} D_j\left[\xi_s\left(x_1,\ldots,x_n,\phi\right)\right]\phi_{sk\ldots lm}.
\end{aligned} \tag{2.48}$$

In the next section we will describe how the creation of an extended one-parameter continuous group of transformations (2.46) leads to a method of solving partial differential equations that uses the corresponding infinitesimal transformations (2.47).

2.3 Group Invariant Solutions of Partial Differential Equations

Given an rth-order partial differential equation,

$$F\Big(x_1,\ldots,x_n,\phi,\phi_{(1)},\ldots,\phi_{(r)}\Big) = 0. \tag{2.49}$$

The partial differential equation (2.49) defines a surface in the space that is made up of the dependent variable ϕ, the independent variables $x_1, \ldots, x_n$ and the partial derivatives $\phi_{(1)}, \ldots, \phi_{(r)}$. If an extended one-parameter continuous group of transformations (2.46) transforms a point on the surface into another point on the surface, that is,

$$F\Big(x_1^*, \ldots, x_n^*, \phi^*, \phi_{(1)}^*, \ldots, \phi_{(r)}^*\Big) = 0 \text{ when } F\Big(x_1, \ldots, x_n, \phi, \phi_{(1)}, \ldots, \phi_{(r)}\Big) = 0, \tag{2.50}$$

then the partial differential equation is said to be invariant with respect to the transformations. The following theorem is a consequence of the invariant surface condition (2.50) [6, 13].

Theorem. Let

$$\begin{aligned} X^{(r)} \equiv\ & \sum_{j=1}^{n} \xi_j\left(x_1, \ldots, x_n, \phi\right) \frac{\partial}{\partial x_j} + \eta\left(x_1, \ldots, x_n, \phi\right) \frac{\partial}{\partial \phi} \\ & + \sum_{j=1}^{n} \eta_j \Big(x_1, \ldots, x_n, \phi, \phi_{(1)}\Big) \frac{\partial}{\partial \phi_j} \\ & + \cdots \\ & + \sum_{m=1}^{n} \sum_{l=1}^{n} \cdots \sum_{k=1}^{n} \sum_{j=1}^{n} \eta_{jk\ldots lm} \Big(x_1, \ldots, x_n, \phi, \phi_{(1)}, \ldots, \phi_{(r)}\Big) \frac{\partial}{\partial \phi_{jk\ldots lm}} \end{aligned} \tag{2.51}$$

be the differential operator associated with an extended one-parameter continuous group of transformations (2.46) via its infinitesimal transformations (2.47). The partial differential equation (2.49) is invariant with respect to the aforementioned extended one-parameter continuous group of transformations if and only if

$$X^{(r)} F\Big(x_1, \ldots, x_n, \phi, \phi_{(1)}, \ldots, \phi_{(r)}\Big) = 0 \text{ when } F\Big(x_1, \ldots, x_n, \phi, \phi_{(1)}, \ldots, \phi_{(r)}\Big) = 0. \tag{2.52}$$

For the relation

$$\phi = \vartheta\left(x_1, \ldots, x_n\right) \tag{2.53}$$

to be a solution of the partial differential equation (2.49) means that it satisfies the equation

$$F\Big(x_1,\ldots,x_n,\vartheta\left(x_1,\ldots,x_n\right),\vartheta_{(1)}\left(x_1,\ldots,x_n\right),\ldots,\vartheta_{(r)}\left(x_1,\ldots,x_n\right)\Big)=0. \tag{2.54}$$

However, if the partial differential equation (2.49) is invariant with respect to an extended one-parameter continuous group of transformations (2.46), then it follows from the invariant surface condition (2.50) that

$$F\Big(x_1^*,\ldots,x_n^*,\vartheta^*\left(x_1^*,\ldots,x_n^*\right),\vartheta^*_{(1)}\left(x_1^*,\ldots,x_n^*\right),\ldots,\vartheta^*_{(r)}\left(x_1^*,\ldots,x_n^*\right)\Big)=0. \tag{2.55}$$

Therefore, as a consequence of the invariant surface condition (2.50), a solution (2.53) of the partial differential equation (2.49) can be transformed into another solution of the partial differential equation. In particular, suppose that the solution (2.53) of the partial differential equation (2.49) is itself invariant with respect to the transformations that leave the partial differential equation invariant; that is,

$$\phi^*=\vartheta\left(x_1^*,\ldots,x_n^*\right)\ \text{ when } \phi=\vartheta\left(x_1,\ldots,x_n\right). \tag{2.56}$$

Then, by the above theorem,

$$X\left[\phi-\vartheta\left(x_1,\ldots,x_n\right)\right]=0 \text{ when } \phi=\vartheta\left(x_1,\ldots,x_n\right). \tag{2.57}$$

This leads on to Lie's method for solving partial differential equations.

Step 1. Given a partial differential equation (2.49). Solve equation (2.52) for the infinitesimals $\xi_j\left(x_1,\ldots,x_n,\phi\right)$ and $\eta\left(x_1,\ldots,x_n,\phi\right)$ that occur in the operator (2.51). Equation (2.52) is called the determining equation.

Step 2. Substituting the infinitesimals $\xi_j\left(x_1,\ldots,x_n,\phi\right)$ and $\eta\left(x_1,\ldots,x_n,\phi\right)$ that were determined in Step 1 into equation (2.57), we obtain the following partial differential equation,

$$\sum_{j=1}^{n}\xi_j\left(x_1,\ldots,x_n,\phi\right)\frac{\partial\phi}{\partial x_j}=\eta\left(x_1,\ldots,x_n,\phi\right). \tag{2.58}$$

The solution $\phi=\phi\left(x_1,\ldots,x_n\right)$ of the partial differential equation (2.58) represents the group invariant solution of the partial differential equation (2.49).

In recognition of Lie's work, continuous groups of transformations (2.7) that leave differential equations invariant are called Lie point symmetries.

Chapter 3

Potential Flows

3.1 Introduction

Since the rediscovery in the late 1950s of Lie's method of solving partial differential equations, much progress has been made in determining the general solution of a wide range of partial differential equations [14, 15]. Lie's method can be extended in a straightforward manner to determine the solution of a partial differential equation subject to given boundary conditions. In the case of such boundary value problems, one begins as is usual by determining all the Lie point symmetries of the partial differential equation. The next step is to check which of these Lie point symmetries will also leave the boundary conditions invariant. If there exist Lie point symmetries that leave the partial differential equation and the boundary conditions invariant, then the solution of the boundary value problem will be a function that is itself invariant with respect to these symmetries [6, 13]. The limited success of this approach, however, is the source of persistent criticism that it is not worthwhile to attempt to solve a boundary value problem using Lie point symmetries [10, 11]. In this chapter and the next we want to change this point of view by describing a different approach that will allow us to determine the Lie point symmetries that leave a boundary value problem invariant. Although this approach applies to a particular boundary value problem, we hope that its success will spur the development of other similar approaches that will allow us to use Lie point symmetries to solve a greater variety of boundary value problems.

The boundary value problem that we will consider in this work relates to the two-dimensional and steady potential flow of an incompressible fluid. A brief description of this boundary value problem is given in the next section. Following that we will consider one example of this boundary value problem, namely, the flow of a fluid past a wedge. This example will serve to illustrate the method of conformal transformations that is commonly used to obtain analytical solutions of this boundary value problem.

We will then conclude this chapter by showing that the boundary value problem for the flow of a fluid past a wedge admits a Lie point symmetry. This implies that the method of conformal transformations is providing us indirectly with the group invariant solution of the boundary value problem. Consequently, in the next chapter, we will consider a direct method of determining group invariant solutions of the boundary value problem.

3.2 The Equations of Motion of a Fluid

We are interested in solving the problem of the flow of a fluid past a body subject to certain assumptions. We will assume that the fluid is constrained to flow past the body within a series of parallel planes and that the flow of the fluid past the body in one plane is identical to the flow of the fluid past the body in every other plane. To satisfy this assumption we will assume that the body extends to infinity in a direction that is perpendicular to the planes of motion and the cross section of the body is the same in every plane. Therefore, if we introduce a Cartesian coordinate system that is fixed to the body and orientate its unit basis vectors $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ in such a way that the flow of the fluid past the body occurs within the planes $z =$ constant, then it will be sufficient to consider the flow of the fluid past the body in only one plane, for example, $z = 0$. In this way the flow of the fluid past the body is reduced to a problem in only two dimensions, namely, x and y.

As the fluid flows past the body we will assume that for an observer at the origin of the Cartesian coordinate system the value of the velocity vector $\underline{v}$ at every point in the fluid is independent of time; thus,

$$\underline{v} = \underline{v}\,(x, y)\,. \tag{3.1}$$

Under this assumption the flow of the fluid past the body is said to be steady. We will make the following additional assumptions regarding the fluid and its flow past the body.

We will assume that the flow of the fluid past the body is frictionless. The resulting slip of the fluid along the surface of the body due to the lack of friction means that everywhere along the surface of the body the velocity vector (3.1) will be tangent to the surface of the body. Thus, if the equation

$$F\,(x, y) = c \tag{3.2}$$

represents the surface of the body and

$$\underline{\nabla} F\,(x, y) \tag{3.3}$$

is a vector that is perpendicular to the surface of the body where $\underline{\nabla}$ is the del operator that occurs in vector analysis [17], then the condition that the velocity vector (3.1) is tangent to the surface of the body implies that the dot product of the velocity vector (3.1) with the vector (3.3) must be equal to zero everywhere along the surface of the body; that is,

$$\underline{v}\,(x, y) \cdot \underline{\nabla} F\,(x, y) = 0 \text{ for all } x \text{ and } y \text{ such that } F\,(x, y) = c. \tag{3.4}$$

We will assume that the fluid is incompressible. A fluid is said to be incompressible if the density of the fluid remains constant no matter the change in pressure acting on the fluid. Hence, if the density of a fluid element remains constant following the motion of the fluid element, then it is a consequence of the conservation of mass of the fluid element that the divergence of the velocity vector must be set equal to zero [2, 12, 20]; that is,

$$\underline{\nabla} \cdot \underline{v}\,(x, y) = 0. \tag{3.5}$$

Our final assumption is that the velocity vector (3.1) can be expressed as the gradient of a function $\phi(x, y)$; that is,

$$\underline{v}\,(x, y) = \underline{\nabla}\phi\,(x, y)\,. \tag{3.6}$$

In this case the flow of the fluid is said to be a potential flow and the function ϕ is called the potential function.

If we now take into consideration all the assumptions that we have made, then to determine the steady and two-dimensional potential flow of an incompressible fluid past a body in the absence of friction along the surface of the body we must solve Laplace's equation,

$$\underline{\nabla} \cdot \underline{\nabla}\phi\,(x, y) = 0, \tag{3.7}$$

subject to the boundary condition,

$$\underline{\nabla}F\,(x, y) \cdot \underline{\nabla}\phi\,(x, y) = 0 \text{ for all } x \text{ and } y \text{ such that } F\,(x, y) = c. \tag{3.8}$$

Laplace's equation (3.7) and the boundary condition (3.8) make up the boundary value problem that is of interest to us in this work. In the next section we will give a brief description of the method of conformal transformations that is commonly used to solve the boundary value problem (3.7)-(3.8).

3.3 Conformal Transformations

A picture of the flow of the fluid past a body $\mathcal{B}$ that satisfies the boundary value problem (3.7)-(3.8) is given by a plot of the lines that have the property that the velocity vector (3.1) at a point in the fluid is tangent to the line passing through the point. If we denote by $\psi(x,y)$ the function that will generate these lines, then the gradient of this function must be perpendicular to the velocity vector (3.1). For the potential flow of a fluid this requirement implies that

$$\underline{\nabla}\phi(x,y)\cdot\underline{\nabla}\psi(x,y)=0. \tag{3.9}$$

The function $\psi(x,y)$ is called the stream function and the lines $\psi(x,y)=$ constant are called streamlines.

If the del operator in a Cartesian coordinate system,

$$\underline{\nabla}\equiv\underline{e}_x\frac{\partial}{\partial x}+\underline{e}_y\frac{\partial}{\partial y}+\underline{e}_z\frac{\partial}{\partial z}, \tag{3.10}$$

is applied to the functions $\phi(x,y)$ and $\psi(x,y)$, then in a Cartesian coordinate system equation (3.9) states that

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}=0. \tag{3.11}$$

Equation (3.11) will be identically satisfied if

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\text{ and }\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}. \tag{3.12}$$

Note that the system of partial differential equations (3.12) is identical to the Cauchy-Riemann conditions that occur in the analysis of functions of a complex variable $\mathbb{z}=x+iy$ [19]. Therefore, given the potential function and the stream function for the flow of a fluid past a body $\mathcal{B}$ that satisfy the Cauchy-Riemann conditions (3.12) we may form the differentiable complex function,

$$\mathbb{w}(\mathbb{z})=\phi(x,y)+i\psi(x,y). \tag{3.13}$$

The complex function (3.13) is called a complex potential.

Suppose now that there exists a transformation of the complex variable,

$$\mathbb{z} = T\left(\mathbb{z}^*\right), \tag{3.14}$$

that has the effect of transforming a body $\mathcal{B}^*$ into the body $\mathcal{B}$. Substituting the transformation (3.14) into the complex potential (3.13) we will then obtain the complex potential for the flow of a fluid past the body $\mathcal{B}^*$; that is,

$$\mathbb{w}\left(T\left(\mathbb{z}^*\right)\right) = \mathbb{w}^*\left(\mathbb{z}^*\right). \tag{3.15}$$

Given the complex potential (3.15) we can then determine the corresponding potential function $\phi^*\left(x^*, y^*\right)$ and stream function $\psi^*\left(x^*, y^*\right)$ for the flow of the fluid past the body $\mathcal{B}^*$ since

$$\mathbb{w}^*\left(\mathbb{z}^*\right) = \phi^*\left(x^*, y^*\right) + i\psi^*\left(x^*, y^*\right). \tag{3.16}$$

This describes the basic idea of the method of conformal transformations. We will now use it to determine the potential function and the stream function for the flow past a wedge. In this case let the wedge with an interior angle equal to 2ω represent the body $\mathcal{B}^*$ and let the real and imaginary parts of the complex variable $\mathbb{z}^*$, namely, x^* and y^* be the axes of a Cartesian coordinate system as indicated in figure 3.1.

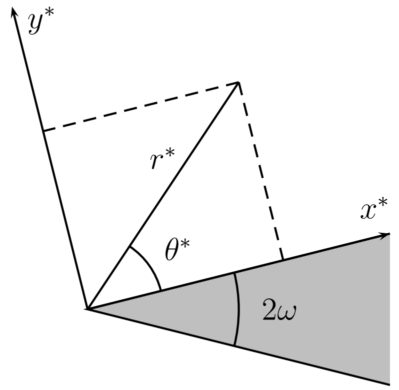

Figure 3.1: The Cartesian coordinate system for the flow past a wedge using the method of conformal transformations.

Consider now the transformation of the complex variable [20],

$$\mathbb{z} = (\mathbb{z}^*)^{\pi/(\pi-\omega)} . \tag{3.17}$$

Using Euler's formula [19], the transformation (3.17) expressed in terms of the polar coordinates

$$r^* = \sqrt{(x^*)^2 + (y^*)^2} \text{ and } \theta^* = \arctan\left(\frac{y^*}{x^*}\right) \tag{3.18}$$

will be

$$\mathbb{z} = (r^*)^{\pi/(\pi-\omega)} \left[\cos\left(\frac{\pi\theta^*}{\pi-\omega}\right) + i\sin\left(\frac{\pi\theta^*}{\pi-\omega}\right)\right]. \tag{3.19}$$

It can be seen in figure 3.1 that for points on the top surface of the wedge, $r^* > 0$ and $\theta^* = 0$, whereas for points on the bottom surface of the wedge, $r^* > 0$ and $\theta^* = 2\pi - 2\omega$. Hence, if we substitute $\theta^* = 0$ and $\theta^* = 2\pi - 2\omega$ into the transformation (3.19) and we equate the real and imaginary parts of the complex variable $\mathbb{z}$ with the corresponding real and imaginary parts of the transformation, then points on the surface of the wedge will transform into points on the surface of a body with coordinates $x = (r^*)^{\pi/(\pi-\omega)}$ and $y = 0$. In other words, the transformation (3.19) applied to the wedge has the effect of transforming it into a semi-infinite flat plate. The equipotential lines and the streamlines for the flow past a semi-infinite flat plate are shown in figure 3.2 (the thickness of the flat plate has been exaggerated).

It is apparent from figure 3.2 that the potential function and the stream function for a fluid flowing past a semi-infinite flat plate are

$$\phi(x, y) = x \text{ and } \psi(x, y) = y. \tag{3.20}$$

Consequently, the complex potential for the flow of a fluid past a semi-infinite flat plate is

$$\mathbb{w} = x + iy = \mathbb{z}. \tag{3.21}$$

Substituting the transformation (3.19) into the complex potential (3.21) the complex potential for the flow past the wedge is

$$\mathbb{w}^* = (r^*)^{\pi/(\pi-\omega)} \left[\cos\left(\frac{\pi\theta^*}{\pi-\omega}\right) + i\sin\left(\frac{\pi\theta^*}{\pi-\omega}\right)\right]. \tag{3.22}$$

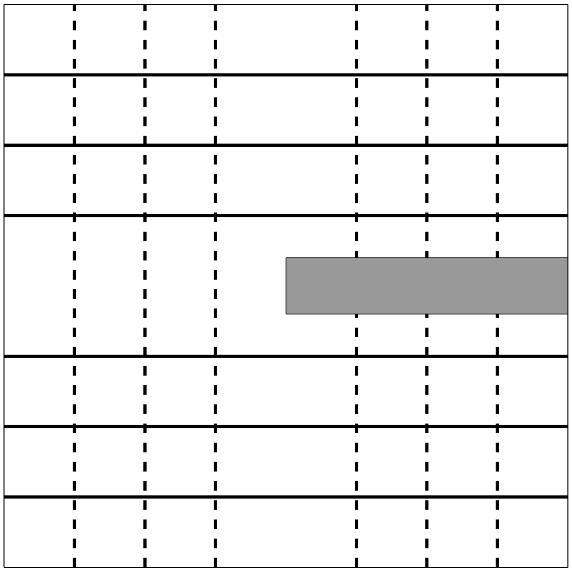

Figure 3.2: A plot of the streamlines (—) and the equipotential lines (- -) for the flow past a semi-infinite flat plate.

It follows from the above complex potential that the potential function and the stream function for the flow past the wedge are

$$\phi^*(x^*,y^*) = (r^*)^{\pi/(\pi-\omega)} \cos\left(\frac{\pi\theta^*}{\pi-\omega}\right), \tag{3.23}$$

$$\psi^*(x^*,y^*) = (r^*)^{\pi/(\pi-\omega)} \sin\left(\frac{\pi\theta^*}{\pi-\omega}\right). \tag{3.24}$$

A plot of the equipotential lines and the streamlines for the flow of a fluid past a wedge is given below in figure 3.3.

Comparing figures 3.2 and 3.3 we note that the equipotential lines remain perpendicular to the streamlines when the flow of a fluid past the semi-infinite flat plate is transformed into the flow of a fluid past the wedge. Transformations such as (3.19) that leave the angles between lines unchanged are called conformal transformations.

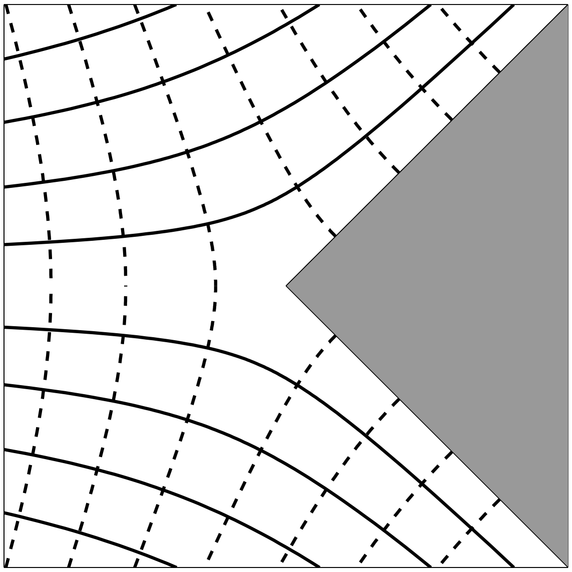

Figure 3.3: A plot of the streamlines (—) and the equipotential lines (- -) for the flow past a wedge.

3.4 Lie Point Symmetries of the Boundary Value Problem

The main advantage of the method of conformal transformations is that the potential function (3.23) for the flow of a fluid past the wedge was obtained without having to solve the corresponding boundary value problem (3.7)-(3.8). Instead, knowing the complex potential (3.21) for the flow of a fluid past a semi-infinite flat plate and knowing how to transform the wedge into the semi-infinite flat plate, we were able to determine the potential function (3.23). In this section we will determine a Lie point symmetry of the boundary value problem for the flow of a fluid past the wedge and hence show that the potential function (3.23) is the corresponding group invariant solution of the boundary value problem.

That we are confident that we should be able to determine a Lie point symmetry of the the boundary value problem (3.7)-(3.8) for the flow of a fluid past the wedge can be justified as follows. The solution of the boundary value problem can be represented as a surface in the space of the independent variables x, y and the dependent variable ϕ. The contours of this surface, $\phi\left(x,y\right) = \text{ constant}$, are the equipotential lines by another name. This observation, combined with the condition (3.9) that the equipotential lines must be perpendicular to the streamlines, $\psi\left(x,y\right) = \text{ constant}$, implies that as we move along a streamline we are moving from one solution of the boundary value problem to another solution of the boundary value problem. However, given that it is a property of a Lie point symmetry of a differential equation to transform a solution of the differential equation into another solution of the differential equation, it follows that the boundary value problem must admit a Lie point symmetry. Note that the projection in the xy-plane of the path curves and the vector (2.20) for this Lie point symmetry will then coincide with the streamlines and the velocity vector for the flow of a fluid past the wedge. Consequently, from equation (3.4), the vector (2.20) must likewise satisfy the equation

$$\underline{\xi}\left(x,y\right)\cdot\underline{\nabla}F\left(x,y\right) = 0 \text{ for all } x \text{ and } y \text{ such that } F\left(x,y\right) = c. \tag{3.25}$$

Whereas the method of conformal transformations depends on our ability to determine a suitable transformation (3.14), the method of Lie point symmetries requires us to solve differential equations. In this regard we will often find it necessary to make use of the following results (see, for example, [9]). The general solution of the partial differential equation,

$$P\left(x,y,\phi\right)\frac{\partial\phi}{\partial x} + Q\left(x,y,\phi\right)\frac{\partial\phi}{\partial y} = R\left(x,y,\phi\right), \tag{3.26}$$

may be written as

$$\nu = \Phi\left(\mu\right) \tag{3.27}$$

where Φ is an arbitrary function and

$$\mu\left(x,y,\phi\right) = \text{constant}, \tag{3.28}$$

$$\nu\left(x,y,\phi\right) = \text{constant} \tag{3.29}$$

are two independent solutions of the system of ordinary differential equations

$$\frac{dx}{P\left(x,y,\phi\right)}=\frac{dy}{Q\left(x,y,\phi\right)}=\frac{d\phi}{R\left(x,y,\phi\right)}. \tag{3.30}$$

The system of ordinary differential equations (3.30) are called the characteristic equations of the partial differential equation and the solutions (3.28)-(3.29) of the characteristic equations are called characteristic curves of the partial differential equation. It is worth noting that if $d\lambda$ is the change in the value of an arbitrary parameter such that

$$\frac{dx}{P\left(x,y,\phi\right)}=\frac{dy}{Q\left(x,y,\phi\right)}=\frac{d\phi}{R\left(x,y,\phi\right)}=d\lambda \tag{3.31}$$

and if $U=U\left(x,y,\phi\right)$, $V=V\left(x,y,\phi\right)$ and $W=W\left(x,y,\phi\right)$ are arbitrary functions of x, y and ϕ, then

$$Udx+Vdy+Wd\phi=\left[UP\left(x,y,\phi\right)+VQ\left(x,y,\phi\right)+WR\left(x,y,\phi\right)\right]d\lambda \tag{3.32}$$

from which it follows that the characteristic curves (3.28)-(3.29) of the partial differential equation will also satisfy the extended system of ordinary differential equations,

$$\frac{dx}{P\left(x,y,\phi\right)}=\frac{dy}{Q\left(x,y,\phi\right)}=\frac{d\phi}{R\left(x,y,\phi\right)}=\frac{Udx+Vdy+Wd\phi}{UP\left(x,y,\phi\right)+VQ\left(x,y,\phi\right)+WR\left(x,y,\phi\right)}. \tag{3.33}$$

Suppose now that from the general solution (3.27) of the partial differential equation (3.26) we are led to the ordinary differential equation

$$p\left(\mu\right)\frac{d^2\Phi}{d\mu^2}+q\left(\mu\right)\frac{d\Phi}{d\mu}+r\left(\mu\right)\Phi=0. \tag{3.34}$$

A variety of techniques exist for solving ordinary differential equations such as (3.34). However, if the coefficients $p=p\left(\mu\right)$, $q=q\left(\mu\right)$ and $r=r\left(\mu\right)$ that occur in the ordinary differential equation are such that

$$\frac{re^{2\int(q/p)d\mu}}{p}=\text{ constant}, \tag{3.35}$$

then by a change of variable from μ to

$$\theta = \int e^{-\int (q/p) d\mu} d\mu \tag{3.36}$$

we can transform the ordinary differential equation (3.34) into an ordinary differential equation that is easily solved, namely,

$$\frac{d^2\Phi}{d\theta^2} + k\Phi = 0, \tag{3.37}$$

where k is a constant. Consequently, by the change of variable (3.36), any solution $\Phi = \Phi(\theta)$ of the ordinary differential equation (3.37) can be transformed back into a solution $\Phi = \Phi(\mu)$ of the ordinary differential equation (3.34).

3.4.1 The Group Invariant Solution for the Flow Past a Wedge

If a Cartesian coordinate system is attached to the wedge as shown in figure 3.4, then figure 3.3 shows that the flow of a fluid past the wedge will be symmetrical with respect to the x-axis of the Cartesian coordinate system. It will therefore be sufficient to consider the flow of a fluid past the top surface of the wedge only. Thus, if the equation of the top surface of a wedge is

$$F(x, y) = \frac{y}{x} = c \tag{3.38}$$

where $c = \tan\omega$, then in a Cartesian coordinate system the boundary value problem (3.7)-(3.8) for the flow of a fluid past the wedge consists of solving Laplace's equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{3.39}$$

subject to the boundary condition,

$$-y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y} = 0 \text{ for all } x \text{ and } y \text{ such that } \frac{y}{x} = c. \tag{3.40}$$

Recall from the previous chapter that to calculate the Lie point symmetries of Laplace's equation (3.39) we must solve the determining equation

$$X^{(2)}(\phi_{xx} + \phi_{yy}) = 0 \text{ when } \phi_{xx} + \phi_{yy} = 0 \tag{3.41}$$

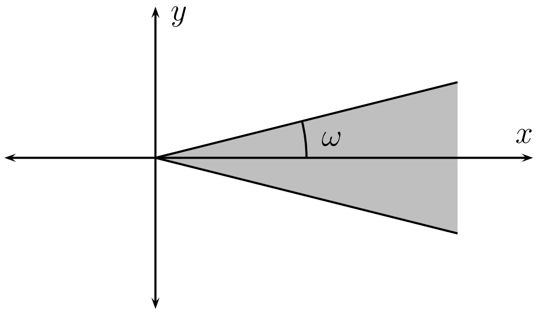

Figure 3.4: The Cartesian coordinate system for the flow past a wedge using the method of Lie point symmetries.

for the infinitesimals

$$\xi_1 = \xi_1(x, y, \phi), \tag{3.42}$$

$$\xi_2 = \xi_2(x, y, \phi), \tag{3.43}$$

$$\eta = \eta(x, y, \phi). \tag{3.44}$$

In the determining equation (3.41) we have written the partial derivatives $\partial^2\phi/\partial x^2$ and $\partial^2\phi/\partial y^2$ as ϕ_{xx} and ϕ_{yy} to emphasise the fact that all the partial derivatives of ϕ that occur in the determining equation must be regarded as independent variables. Hence, if we apply the operator

$$\begin{aligned} X^{(2)} \equiv\ & \xi_1(x, y, \phi)\frac{\partial}{\partial x} + \xi_2(x, y, \phi)\frac{\partial}{\partial y} + \eta(x, y, \phi)\frac{\partial}{\partial \phi} \\ & + \eta_1\left(x, y, \phi, \phi_{(1)}\right)\frac{\partial}{\partial \phi_x} + \eta_2\left(x, y, \phi, \phi_{(1)}\right)\frac{\partial}{\partial \phi_y} \\ & + \eta_{11}\left(x, y, \phi, \phi_{(1)}, \phi_{(2)}\right)\frac{\partial}{\partial \phi_{xx}} + \eta_{12}\left(x, y, \phi, \phi_{(1)}, \phi_{(2)}\right)\frac{\partial}{\partial \phi_{xy}} \\ & + \eta_{22}\left(x, y, \phi, \phi_{(1)}, \phi_{(2)}\right)\frac{\partial}{\partial \phi_{yy}} \end{aligned} \tag{3.45}$$

to Laplace's equation (3.39) and we replace ϕ_{xx} with $-\phi_{yy}$, then the determining equation (3.41) in its expanded form is

$$\begin{aligned}
&\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\left(2\frac{\partial^2\eta}{\partial x\partial\phi}-\frac{\partial^2\xi_1}{\partial x^2}-\frac{\partial^2\xi_1}{\partial y^2}\right)\phi_x+\left(2\frac{\partial^2\eta}{\partial y\partial\phi}-\frac{\partial^2\xi_2}{\partial x^2}-\frac{\partial^2\xi_2}{\partial y^2}\right)\phi_y\\
+&\left(\frac{\partial^2\eta}{\partial\phi^2}-2\frac{\partial^2\xi_1}{\partial x\partial\phi}\right)\phi_x^2-2\left(\frac{\partial^2\xi_2}{\partial x\partial\phi}+\frac{\partial^2\xi_1}{\partial y\partial\phi}\right)\phi_x\phi_y+\left(\frac{\partial^2\eta}{\partial\phi^2}-2\frac{\partial^2\xi_2}{\partial y\partial\phi}\right)\phi_y^2\\
-&\frac{\partial^2\xi_1}{\partial\phi^2}\phi_x^3-\frac{\partial^2\xi_2}{\partial\phi^2}\phi_x^2\phi_y-\frac{\partial^2\xi_1}{\partial\phi^2}\phi_x\phi_y^2-\frac{\partial^2\xi_2}{\partial\phi^2}\phi_y^3\\
+&2\left(\frac{\partial\xi_1}{\partial x}-\frac{\partial\xi_2}{\partial y}\right)\phi_{yy}-2\left(\frac{\partial\xi_2}{\partial x}+\frac{\partial\xi_1}{\partial y}\right)\phi_{xy}\\
+&2\frac{\partial\xi_1}{\partial\phi}\phi_x\phi_{yy}-2\frac{\partial\xi_2}{\partial\phi}\phi_y\phi_{yy}-2\frac{\partial\xi_2}{\partial\phi}\phi_x\phi_{xy}-2\frac{\partial\xi_1}{\partial\phi}\phi_y\phi_{xy}=0.
\end{aligned} \tag{3.46}$$

Since the infinitesimals (3.42)-(3.44) that occur in the determining equation (3.46) do not depend on the partial derivatives of ϕ, to satisfy the determining equation we must set the coefficients of each of the partial derivatives of ϕ equal to zero. The result thereof will be an over-determined system of partial differential equations for the infinitesimals (3.42)-(3.44). This system of partial differential equations is easily solved and the infinitesimals that are obtained are

$$\xi_1 = \xi_1(x,y), \tag{3.47}$$

$$\xi_2 = \xi_2(x,y), \tag{3.48}$$

$$\eta = \alpha\phi+\beta(x,y) \tag{3.49}$$

where the infinitesimals $\xi_1(x,y)$ and $\xi_2(x,y)$ must satisfy the system of partial differential equations,

$$\frac{\partial\xi_1}{\partial x}=\frac{\partial\xi_2}{\partial y}\text{ and }\frac{\partial\xi_1}{\partial y}=-\frac{\partial\xi_2}{\partial x}, \tag{3.50}$$

α is an arbitrary constant and the function $\beta(x,y)$ is itself any solution of Laplace's equation (3.39).

Using the infinitesimals (3.47)-(3.49) it is possible calculate infinitely many Lie point symmetries for Laplace's equation (3.39). However, there is no indication as to which Lie point symmetry to calculate that will also leave the boundary condition (3.40) invariant. This is the kind of impasse that frequently hampers our ability to determine the group invariant solutions of boundary value problems. In this work we will overcome this obstacle by insisting that a Lie point symmetry of Laplace's equation (3.39) also satisfies equation (3.25). The approach of using a side condition to help solve the determining equation is not presented here for the first time (see, for example, [10, 21]). However, our use of the boundary condition (3.4) to derive the side condition (3.25) is given here for the first time.

If we substitute the equation of the top surface of the wedge (3.38) into the side condition (3.25), then the infinitesimals must also satisfy the equation

$$-y\xi_1(x,y) + x\xi_2(x,y) = 0 \text{ for all } x \text{ and } y \text{ such that } \frac{y}{x} = c. \tag{3.51}$$

By inspection, the infinitesimals

$$\xi_1(x,y) = x \text{ and } \xi_2(x,y) = y \tag{3.52}$$

will satisfy the system of partial differential equations (3.50) and the side condition (3.51). We can now proceed to determine the group invariant solution of the boundary value problem.

It follows from the previous chapter that to determine the group invariant solution corresponding to the infinitesimals (3.49) and (3.52) we must to solve the partial differential equation

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = [\alpha\phi + \beta(x,y)] \tag{3.53}$$

for $\phi = \phi(x,y)$. For a first attempt at solving the partial differential equation, let $\beta(x,y) = 0$. In this case the characteristic equations for the partial differential equation are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{d\phi}{\alpha\phi}. \tag{3.54}$$

Integrating the ordinary differential equation,

$$\frac{dx}{x} = \frac{dy}{y}, \tag{3.55}$$

we obtain the first characteristic curve

$$\mu = \frac{y}{x}. \tag{3.56}$$

Although the second characteristic curve can easily be obtained by integrating one of the ordinary differential equations,

$$\frac{dx}{x} = \frac{d\phi}{\alpha\phi} \text{ or } \frac{dy}{y} = \frac{d\phi}{\alpha\phi}, \tag{3.57}$$

by trial and error we find that the subsequent calculations are less tedious if we make use of the result (3.33) with $U = 2x$, $V = 2y$, $W = 0$ and instead solve the ordinary differential equation

$$\frac{d\phi}{\alpha\phi} = \frac{2xdx + 2ydy}{2x^2 + 2y^2}. \tag{3.58}$$

In this way we obtain the second characteristic curve,

$$\nu = \frac{\phi}{\left(x^2 + y^2\right)^{\alpha/2}}. \tag{3.59}$$

Therefore, by equation (3.27), the general form of the group invariant solution of the boundary value problem is

$$\phi\left(x, y\right) = \left(x^2 + y^2\right)^{\alpha/2} \Phi\left(\mu\right). \tag{3.60}$$

Substituting the group invariant solution (3.60) into Laplace's equation (3.39) we obtain the ordinary differential equation,

$$\left(1 + \mu^2\right)^2 \frac{d^2\Phi}{d\mu^2} + 2\mu\left(1 + \mu^2\right)\frac{d\Phi}{d\mu} + \alpha^2\Phi = 0. \tag{3.61}$$

Since this ordinary differential equation has the property (3.35), the change of variable (3.36) transforms it into the ordinary differential equation

$$\frac{d^2\Phi}{d\theta^2} + \alpha^2\Phi = 0 \tag{3.62}$$

where $\theta = \arctan\mu$. Given that the solution of the differential equation (3.62) is

$$\Phi\left(\theta\right) = k_1\cos\left(\alpha\theta\right) + k_2\sin\left(\alpha\theta\right) \tag{3.63}$$

where k_1 and k_2 are arbitrary constants, it follows that the exact form of the group invariant solution (3.60) of the boundary value problem is

$$\phi\left(x,y\right)=\left(x^2+y^2\right)^{\alpha/2}\left[k_1\cos\left(\alpha\arctan\frac{y}{x}\right)+k_2\sin\left(\alpha\arctan\frac{y}{x}\right)\right]. \tag{3.64}$$

However, if we impose the boundary condition (3.40), then the arbitrary constants must satisfy the equation

$$-k_1\sin\left(\alpha\omega\right)+k_2\cos\left(\alpha\omega\right)=0. \tag{3.65}$$

If we solve equation (3.65) for the arbitrary constant k_2, then the group invariant solution of the boundary value problem expressed in terms of the arbitrary constant k_1 only is

$$\phi\left(x,y\right)=\frac{k_1}{\cos\left(\alpha\omega\right)}r^{\alpha}\cos\left[\alpha\left(\theta-\omega\right)\right] \tag{3.66}$$

where

$$r=\sqrt{x^2+y^2}\text{ and }\theta=\arctan\frac{y}{x}. \tag{3.67}$$

If we compare figure 3.1 with figure 3.4 and we observe that

$$r^*=r\text{ and }\theta^*=\theta-\omega, \tag{3.68}$$

then it is apparent that the group invariant solution (3.66) will coincide with the potential function (3.23) that was obtained earlier by the method of conformal transformations if

$$\alpha=\frac{\pi}{\pi-\omega}\text{ and }\frac{k_1}{\cos\left(\alpha c\right)}=1. \tag{3.69}$$

We have therefore been able to show that is possible to calculate a group invariant solution of the boundary value problem (3.7)-(3.8). In the next chapter we will use Lie point symmetries to solve the boundary value problem for the flow of a fluid past a circular cylinder, an elliptic cylinder and a parabolic cylinder. In doing so we will suggest a method that can be applied to solve the boundary value problem in general. An alternative method that uses Lie point symmetries in a nonclassical sense [5, 11, 21] to solve the problem for the flow of a fluid past a circular cylinder is given in [22].

Chapter 4

Group Invariant Solutions

4.1 Introduction

In the previous chapter we derived the boundary value problem for the steady and two-dimensional potential flow of an incompressible fluid past a body without friction. Expressed in terms of the Cartesian coordinates x and y the boundary value problem consists of solving Laplace's equation

$$\underline{\nabla} \cdot \underline{\nabla} \phi\,(x, y) = 0 \tag{4.1}$$

subject to the boundary condition,

$$\underline{\nabla} F\,(x, y) \cdot \underline{\nabla} \phi\,(x, y) = 0 \text{ for all } x \text{ and } y \text{ such that } F\,(x, y) = c, \tag{4.2}$$

where $F\,(x, y) = c$ is the equation of the surface of the body and

$$\underline{\nabla} \equiv \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} \tag{4.3}$$

is the del operator. Furthermore, in the previous chapter we showed that in a Cartesian coordinate system the infinitesimals of the Lie point symmetries of Laplace's equation are of the form

$$\xi_1 = \xi_1\,(x, y)\,, \tag{4.4}$$

$$\xi_2 = \xi_2\,(x, y)\,, \tag{4.5}$$

$$\eta = \alpha\phi + \beta\,(x, y) \tag{4.6}$$

where the infinitesimals $\xi_1(x,y)$ and $\xi_2(x,y)$ must satisfy the system of partial differential equations

$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial y} \text{ and } \frac{\partial \xi_1}{\partial y} = -\frac{\partial \xi_2}{\partial x}, \tag{4.7}$$

α is an arbitrary constant and the function $\beta(x,y)$ is itself any solution of Laplace's equation. Therefore, in a Cartesian coordinate system, Laplace's equation admits infinitely many Lie point symmetries. However, by insisting that the infinitesimals $\xi_1(x,y)$ and $\xi_2(x,y)$ satisfy the side condition,

$$\underline{\xi}(x,y) \cdot \underline{\nabla} F(x,y) = 0 \text{ for all } x \text{ and } y \text{ such that } F(x,y) = c \tag{4.8}$$

where

$$\underline{\xi}(x,y) = \xi_1(x,y)\,\underline{e}_x + \xi_2(x,y)\,\underline{e}_y, \tag{4.9}$$

we showed in the previous chapter that it is possible to determine the Lie point symmetries that will leave invariant Laplace's equation and the boundary condition. The form of the group invariant solution of the boundary value problem was then obtained by solving the partial differential equation

$$\xi_1(x,y)\frac{\partial \phi}{\partial x} + \xi_2(x,y)\frac{\partial \phi}{\partial y} = \alpha\phi + \beta(x,y) \tag{4.10}$$

for $\phi = \phi(x,y)$.

We demonstrated the viability of this approach in the previous chapter by using it to determine the group invariant solution of the boundary value problem for the flow of a fluid past a wedge in the case $\beta(x,y) = 0$. We will develop this approach further in this chapter but apply it instead in an orthogonal curvilinear coordinate system to determine the group invariant solutions of the boundary value problems for the flow of a fluid past a circular cylinder, an elliptic cylinder and a parabolic cylinder. This necessitates that we give a brief overview in the next section of the theory of orthogonal curvilinear coordinate systems that is required in this chapter.

4.2 Orthogonal Curvilinear Coordinate Systems

4.2.1 The General Laplace Equation

Introduce the functions

$$\bar{x} = \bar{x}(x, y), \tag{4.11}$$

$$\bar{y} = \bar{y}(x, y) \tag{4.12}$$

so that we may change the coordinates of a point from x and y in a Cartesian coordinate system to $\bar{x}$ and $\bar{y}$ in a curvilinear coordinate system as shown in figure 4.1.

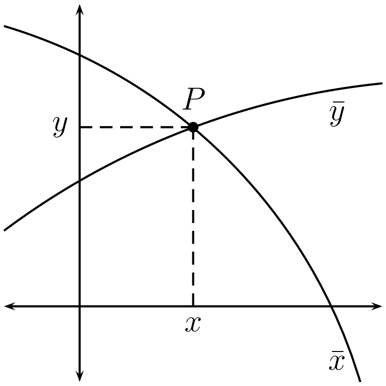

Figure 4.1: The point P has Cartesian coordinates (x, y) and curvilinear coordinates $(\bar{x}, \bar{y})$.

If we apply the del operator (4.3) to the function (4.11), then the vector

$$\underline{g}_{\bar{x}} = \frac{\partial \bar{x}}{\partial x}\underline{e}_x + \frac{\partial \bar{x}}{\partial y}\underline{e}_y \tag{4.13}$$

will be perpendicular to the coordinate lines $\bar{x} =$ constant of the curvilinear coordinate system [17]. Similarly, the vector

$$\underline{g}_{\bar{y}} = \frac{\partial \bar{y}}{\partial x}\underline{e}_x + \frac{\partial \bar{y}}{\partial y}\underline{e}_y \tag{4.14}$$

will be perpendicular to the coordinate lines $\bar{y} =$ constant. The vectors (4.13) and (4.14) are referred to in the literature as the contravariant basis vectors for a curvilinear coordinate system (see, for example, [2, 24]). Note that the literature also makes reference to the covariant basis vectors for a curvilinear coordinate system. These basis vectors have the property that they are parallel to the coordinate lines of a curvilinear coordinate system. However, for our purposes in this chapter it will be sufficient to work with the contravariant basis vectors only. The contravariant basis vectors for the curvilinear coordinate system illustrated in figure 4.1 are shown in figure 4.2.

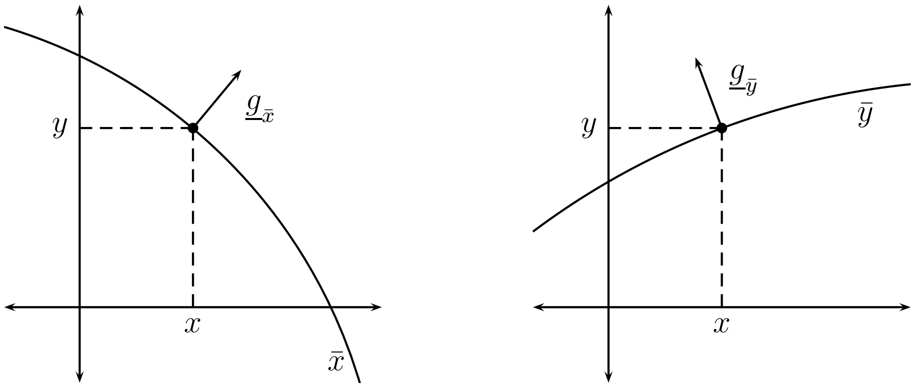

Figure 4.2: The contravariant basis vectors for a curvilinear coordinate system.

The del operator (4.3) in a Cartesian coordinate system is expressed in terms of the constant unit basis vectors $\underline{e}_x$ and $\underline{e}_y$ of the Cartesian coordinate system. In a curvilinear coordinate system the del operator may be similarly expressed in terms of the contravariant basis vectors; that is,

$$\underline{\nabla} \equiv \underline{g}_{\bar{x}} \frac{\partial}{\partial \bar{x}} + \underline{g}_{\bar{y}} \frac{\partial}{\partial \bar{y}}. \tag{4.15}$$

However, the contravariant basis vectors are not constant unit vectors in general. Hence, if we substitute the del operator (4.15) into the boundary value problem (4.1)-(4.2), then in a curvilinear coordinate system we have to solve the general Laplace equation

$$a\left(\bar{x},\bar{y}\right)\frac{\partial^2 \phi}{\partial \bar{x}^2} + b\left(\bar{x},\bar{y}\right)\frac{\partial^2 \phi}{\partial \bar{y}^2} + 2k\left(\bar{x},\bar{y}\right)\frac{\partial^2 \phi}{\partial \bar{x}\partial \bar{y}} + d\left(\bar{x},\bar{y}\right)\frac{\partial \phi}{\partial \bar{x}} + e\left(\bar{x},\bar{y}\right)\frac{\partial \phi}{\partial \bar{y}} = 0 \tag{4.16}$$

subject to the boundary condition,

$$a\left(\bar{x},\bar{y}\right)\frac{\partial F}{\partial \bar{x}}\frac{\partial \phi}{\partial \bar{x}}+b\left(\bar{x},\bar{y}\right)\frac{\partial F}{\partial \bar{y}}\frac{\partial \phi}{\partial \bar{y}}+k\left(\bar{x},\bar{y}\right)\left(\frac{\partial F}{\partial \bar{x}}\frac{\partial \phi}{\partial \bar{y}}+\frac{\partial F}{\partial \bar{y}}\frac{\partial \phi}{\partial \bar{x}}\right)=0 \qquad (4.17)$$
$$\text{for all } \bar{x} \text{ and } \bar{y} \text{ such that } F\left(\bar{x},\bar{y}\right)=c,$$

where

$$a\left(\bar{x},\bar{y}\right)=\underline{g}_{\bar{x}}\cdot\underline{g}_{\bar{x}},\; b\left(\bar{x},\bar{y}\right)=\underline{g}_{\bar{y}}\cdot\underline{g}_{\bar{y}},\; k\left(\bar{x},\bar{y}\right)=\underline{g}_{\bar{x}}\cdot\underline{g}_{\bar{y}} \qquad (4.18)$$

and

$$d\left(\bar{x},\bar{y}\right)=\underline{g}_{\bar{x}}\cdot\frac{\partial \underline{g}_{\bar{x}}}{\partial \bar{x}}+\underline{g}_{\bar{y}}\cdot\frac{\partial \underline{g}_{\bar{x}}}{\partial \bar{y}},\; e\left(\bar{x},\bar{y}\right)=\underline{g}_{\bar{x}}\cdot\frac{\partial \underline{g}_{\bar{y}}}{\partial \bar{x}}+\underline{g}_{\bar{y}}\cdot\frac{\partial \underline{g}_{\bar{y}}}{\partial \bar{y}}. \qquad (4.19)$$

If we substitute the contravariant basis vectors into the equations (4.18) for the coefficients of the partial derivatives, then

$$a\left(\bar{x},\bar{y}\right)=\left(\frac{\partial \bar{x}}{\partial x}\right)^2+\left(\frac{\partial \bar{x}}{\partial y}\right)^2,\; b\left(\bar{x},\bar{y}\right)=\left(\frac{\partial \bar{y}}{\partial x}\right)^2+\left(\frac{\partial \bar{y}}{\partial y}\right)^2 \qquad (4.20)$$

and

$$k\left(\bar{x},\bar{y}\right)=\frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial x}+\frac{\partial \bar{x}}{\partial y}\frac{\partial \bar{y}}{\partial y}. \qquad (4.21)$$

To be able to express the coefficients (4.19) of the partial derivatives in terms of the functions (4.11)-(4.12) also, we note that if the inverse of the functions (4.11)-(4.12) are

$$\begin{aligned} x &= x\left(\bar{x},\bar{y}\right), && (4.22)\\ y &= y\left(\bar{x},\bar{y}\right), && (4.23)\end{aligned}$$

then by the chain rule,

$$\begin{aligned} \frac{\partial x}{\partial x} &= \frac{\partial x}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x}+\frac{\partial x}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x}=1, && (4.24)\\ \frac{\partial x}{\partial y} &= \frac{\partial x}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y}+\frac{\partial x}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y}=0 && (4.25)\end{aligned}$$

and

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x}+\frac{\partial y}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x}=0, && (4.26)\\ \frac{\partial y}{\partial y} &= \frac{\partial y}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y}+\frac{\partial y}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y}=1. && (4.27)\end{aligned}$$

As a consequence of the partial derivatives (4.24)-(4.27) of the inverse functions (4.22)-(4.23), equations (4.19) will reduce to

$$\begin{aligned} d\left(\bar{x},\bar{y}\right) &= \left(\frac{\partial\bar{x}}{\partial x}\frac{\partial x}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial x}\frac{\partial x}{\partial\bar{y}}\right)\frac{\partial^2\bar{x}}{\partial x^2}+\left(\frac{\partial\bar{x}}{\partial y}\frac{\partial y}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial y}\frac{\partial y}{\partial\bar{y}}\right)\frac{\partial^2\bar{x}}{\partial y^2} \\ &\quad +\left(\frac{\partial\bar{x}}{\partial x}\frac{\partial y}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial x}\frac{\partial y}{\partial\bar{y}}\right)\frac{\partial^2\bar{x}}{\partial x\partial y}+\left(\frac{\partial\bar{x}}{\partial y}\frac{\partial x}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial y}\frac{\partial x}{\partial\bar{y}}\right)\frac{\partial^2\bar{x}}{\partial x\partial y} \\ &= \frac{\partial^2\bar{x}}{\partial x^2}+\frac{\partial^2\bar{x}}{\partial y^2} \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} e\left(\bar{x},\bar{y}\right) &= \left(\frac{\partial\bar{x}}{\partial x}\frac{\partial x}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial x}\frac{\partial x}{\partial\bar{y}}\right)\frac{\partial^2\bar{y}}{\partial x^2}+\left(\frac{\partial\bar{x}}{\partial y}\frac{\partial y}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial y}\frac{\partial y}{\partial\bar{y}}\right)\frac{\partial^2\bar{y}}{\partial y^2} \\ &\quad +\left(\frac{\partial\bar{x}}{\partial y}\frac{\partial x}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial y}\frac{\partial x}{\partial\bar{y}}\right)\frac{\partial^2\bar{x}}{\partial x\partial y}+\left(\frac{\partial\bar{x}}{\partial x}\frac{\partial y}{\partial\bar{x}}+\frac{\partial\bar{y}}{\partial x}\frac{\partial y}{\partial\bar{y}}\right)\frac{\partial^2\bar{x}}{\partial x\partial y} \\ &= \frac{\partial^2\bar{y}}{\partial x^2}+\frac{\partial^2\bar{y}}{\partial y^2}. \end{aligned} \tag{4.29}$$

4.2.2 The Determining Equation

It is important to note that unlike the del operator that changes from (4.3) in a Cartesian coordinate system to (4.15) in a curvilinear coordinate system, the form of the operator $X^{(r)}$ that occurs in the determining equation (2.52) does not change. Thus, by making a change of variables from x and y to $\bar{x}$ and $\bar{y}$ only, it follows from Chapter 2 that to calculate the infinitesimals

$$\xi_1 = \xi_1\left(\bar{x},\bar{y},\phi\right), \tag{4.30}$$

$$\xi_2 = \xi_2\left(\bar{x},\bar{y},\phi\right), \tag{4.31}$$

$$\eta = \eta\left(\bar{x},\bar{y},\phi\right) \tag{4.32}$$

of the Lie point symmetries of the general Laplace equation we must solve the determining equation,

$$\begin{gathered} X^{(2)}\left[a\left(\bar{x},\bar{y}\right)\phi_{\bar{x}\bar{x}}+b\left(\bar{x},\bar{y}\right)\phi_{\bar{y}\bar{y}}+2k\left(\bar{x},\bar{y}\right)\phi_{\bar{x}\bar{y}}+d\left(\bar{x},\bar{y}\right)\phi_{\bar{x}}+e\left(\bar{x},\bar{y}\right)\phi_{\bar{y}}\right]=0 \\ \text{when} \\ a\left(\bar{x},\bar{y}\right)\phi_{\bar{x}\bar{x}}+b\left(\bar{x},\bar{y}\right)\phi_{\bar{y}\bar{y}}+2k\left(\bar{x},\bar{y}\right)\phi_{\bar{x}\bar{y}}+d\left(\bar{x},\bar{y}\right)\phi_{\bar{x}}+e\left(\bar{x},\bar{y}\right)\phi_{\bar{y}}=0. \end{gathered} \tag{4.33}$$

The coefficients of the partial derivatives that occur in the operator

$$\begin{aligned} X^{(2)} \equiv\; & \xi_1(\bar{x},\bar{y},\phi)\frac{\partial}{\partial \bar{x}} + \xi_2(\bar{x},\bar{y},\phi)\frac{\partial}{\partial \bar{y}} + \eta(\bar{x},\bar{y},\phi)\frac{\partial}{\partial \phi} \\ & +\eta_1\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right)\frac{\partial}{\partial \phi_{\bar{x}}} + \eta_2\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right)\frac{\partial}{\partial \phi_{\bar{y}}} \\ & +\eta_{11}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right)\frac{\partial}{\partial \phi_{\bar{x}\bar{x}}} + \eta_{12}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right)\frac{\partial}{\partial \phi_{\bar{x}\bar{y}}} \\ & +\eta_{22}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right)\frac{\partial}{\partial \phi_{\bar{y}\bar{y}}} \end{aligned} \tag{4.34}$$

are given by the formulae (2.43) and (2.48); that is,

$$\eta_1\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right) = D_{\bar{x}}(\eta) - \phi_{\bar{x}}D_{\bar{x}}(\xi_1) - \phi_{\bar{y}}D_{\bar{x}}(\xi_2), \tag{4.35}$$

$$\eta_2\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right) = D_{\bar{y}}(\eta) - \phi_{\bar{x}}D_{\bar{y}}(\xi_1) - \phi_{\bar{y}}D_{\bar{y}}(\xi_2), \tag{4.36}$$

$$\eta_{11}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right) = D_{\bar{x}}(\eta_1) - \phi_{\bar{x}\bar{x}}D_{\bar{x}}(\xi_1) - \phi_{\bar{x}\bar{y}}D_{\bar{x}}(\xi_2), \tag{4.37}$$

$$\eta_{12}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right) = D_{\bar{x}}(\eta_2) - \phi_{\bar{x}\bar{y}}D_{\bar{x}}(\xi_1) - \phi_{\bar{y}\bar{y}}D_{\bar{x}}(\xi_2), \tag{4.38}$$

$$\eta_{22}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right) = D_{\bar{y}}(\eta_2) - \phi_{\bar{x}\bar{y}}D_{\bar{y}}(\xi_1) - \phi_{\bar{y}\bar{y}}D_{\bar{y}}(\xi_2) \tag{4.39}$$

where $D_{\bar{x}}$ and $D_{\bar{y}}$ are the total differential operators

$$D_{\bar{x}} \equiv \frac{\partial}{\partial \bar{x}} + \phi_{\bar{x}}\frac{\partial}{\partial \phi} + \phi_{\bar{x}\bar{x}}\frac{\partial}{\partial \phi_{\bar{x}}} + \phi_{\bar{x}\bar{y}}\frac{\partial}{\partial \phi_{\bar{y}}}, \tag{4.40}$$

$$D_{\bar{y}} \equiv \frac{\partial}{\partial \bar{y}} + \phi_{\bar{y}}\frac{\partial}{\partial \phi} + \phi_{\bar{x}\bar{y}}\frac{\partial}{\partial \phi_{\bar{x}}} + \phi_{\bar{y}\bar{y}}\frac{\partial}{\partial \phi_{\bar{y}}}. \tag{4.41}$$

In Appendix B we show by solving the determining equation (4.33) that the infinitesimals of the Lie point symmetries of the general Laplace equation can always be written in the form

$$\xi_1 = \xi_1(\bar{x},\bar{y}), \tag{4.42}$$

$$\xi_2 = \xi_2(\bar{x},\bar{y}), \tag{4.43}$$

$$\eta = \alpha(\bar{x},\bar{y})\phi + \beta(\bar{x},\bar{y}). \tag{4.44}$$

4.2.3 The Side Condition

It now remains to choose an appropriate curvilinear coordinate system for the boundary value problem. We begin by noting that if the change to a curvilinear coordinate system is given by the equations

$$\begin{aligned} \bar{x} &= F(x,y), & (4.45)\\ \bar{y} &= G(x,y) & (4.46) \end{aligned}$$

where $F(x,y) = c$ is the equation of the surface of the body in a Cartesian coordinate system and $G(x,y)$ is some other function yet to be determined, then the equation of the surface of the body in this curvilinear coordinate system will be

$$F(\bar{x}) = \bar{x} = c. \tag{4.47}$$

If we apply the del operator (4.15) to the function (4.47), then

$$\underline{\nabla} F(\bar{x}) = \underline{g}_{\bar{x}}. \tag{4.48}$$

Consequently, if we substitute the vectors (4.48) and

$$\underline{\xi}(\bar{x},\bar{y}) = \xi_1(\bar{x},\bar{y})\,\underline{g}_{\bar{x}} + \xi_2(\bar{x},\bar{y})\,\underline{g}_{\bar{y}} \tag{4.49}$$

into the side condition (4.8), then a Lie point symmetry of the general Laplace equation that is expressed in terms of the coordinates of the curvilinear coordinate system (4.45)-(4.46) will also leave the boundary condition invariant if its infinitesimals satisfy the equation

$$a(\bar{x},\bar{y})\,\xi_1(\bar{x},\bar{y}) + k(\bar{x},\bar{y})\,\xi_2(\bar{x},\bar{y}) \text{ for all } \bar{x} = c. \tag{4.50}$$

Suppose now that the curvilinear coordinate system (4.45)-(4.46) is orthogonal; that is, $k(\bar{x},\bar{y}) = 0$. Since

$$a(\bar{x},\bar{y}) = \left(\frac{\partial \bar{x}}{\partial x}\right)^2 + \left(\frac{\partial \bar{x}}{\partial y}\right)^2 \neq 0 \text{ for all } \bar{x} = c, \tag{4.51}$$

the solution of the side condition (4.50) in this case is $\xi_1(\bar{x},\bar{y}) = 0$. This suggests that we apply the following procedure to determine the group invariant solution of the boundary value problem.

Step 1. Working in a Cartesian coordinate system, derive an equation for the surface of the body that can be written in the form $F(x,y) = c$.

Step 2. Determine another function $G(x,y)$ such that

$$\underline{\nabla} F(x,y) \cdot \underline{\nabla} G(x,y) = 0. \tag{4.52}$$

Use the functions $F(x,y)$ and $G(x,y)$ to transform the Cartesian coordinate system to an orthogonal curvilinear coordinate system according to the equations (4.45)-(4.46).

Step 3. Calculate the coefficients (4.20), (4.28) and (4.29) of the partial derivatives that occur in the general Laplace equation and hence solve the determining equation (4.33) by looking for a solution of the form

$$\xi_1 = 0, \tag{4.53}$$

$$\xi_2 = \xi_2(\bar{x},\bar{y}), \tag{4.54}$$

$$\eta = \alpha(\bar{x},\bar{y})\,\phi + \beta(\bar{x},\bar{y}). \tag{4.55}$$

Step 4. For the infinitesimals (4.53)-(4.55) that are obtained in the previous step determine the form of the group invariant solution by solving the partial differential equation

$$\xi_2(\bar{x},\bar{y})\frac{\partial \phi}{\partial \bar{y}} = \alpha(\bar{x},\bar{y})\,\phi + \beta(\bar{x},\bar{y}) \tag{4.56}$$

for $\phi = \phi(\bar{x},\bar{y})$.

Step 5. Substitute the form of the group invariant solution obtained in the previous step into Laplace's equation and then solve the subsequent ordinary differential equation. In this way the group invariant solution will be reduced to a function that involves two arbitrary constants.

Step 6. The last step is to verify that the group invariant solution satisfies the boundary condition. In this regard it is worth noting that if we substitute the vector (4.48) into the boundary condition (4.2), then the boundary condition may be simply stated as

$$\frac{\partial \phi}{\partial \bar{x}} = 0 \text{ when } \bar{x} = c. \tag{4.57}$$

Application of the boundary condition (4.57) will, however, fix the value of only one of the arbitrary constants that occurs in the group invariant solution. A second boundary condition that describes the flow of the fluid at some distance away from the body is needed to determine the value of the other arbitrary constant. Instead of imposing this second boundary condition, however, we will show that for various values of the undetermined arbitrary constant we obtain a variety of different flows.

We will now apply this procedure to determine the group invariant solutions of the boundary value problems for the flow of a fluid past a circular cylinder, an elliptic cylinder and a parabolic cylinder.

4.3 The Flow Past a Circular Cylinder

4.3.1 Laplace's Equation in a Polar Coordinate System

If the radius of a circular cylinder is c, then the equation of the surface of the cylinder is

$$F(x, y) = \sqrt{x^2 + y^2} = c. \tag{4.58}$$

If we apply the del operator (4.3) to the functions (4.58) and $G(x, y) = \arctan(y/x)$, then it is easily verified that the functions satisfy equation (4.52). This suggests that a convenient choice for an orthogonal curvilinear coordinate system will be the polar coordinates

$$\begin{aligned} \bar{x} &= F(x, y) = \sqrt{x^2 + y^2}, && (4.59) \\ \bar{y} &= G(x, y) = \arctan \frac{y}{x}. && (4.60) \end{aligned}$$

If we substitute the functions (4.59)-(4.60) into the equations (4.20), (4.28) and (4.29), then the coefficients of the partial derivatives are

$$a\left(\bar{x},\bar{y}\right)=1,\ b\left(\bar{x},\bar{y}\right)=\frac{1}{\bar{x}^{2}},\ d\left(\bar{x},\bar{y}\right)=\frac{1}{\bar{x}}\text{ and }e\left(\bar{x},\bar{y}\right)=0. \tag{4.61}$$

Hence, Laplace's equation in a polar coordinate system is

$$\frac{\partial^{2}\phi}{\partial\bar{x}^{2}}+\frac{1}{\bar{x}^{2}}\frac{\partial^{2}\phi}{\partial\bar{y}^{2}}+\frac{1}{\bar{x}}\frac{\partial\phi}{\partial\bar{x}}=0. \tag{4.62}$$

4.3.2 The General Form of the Group Invariant Solution

The determining equation,

$$X^{(2)}\left(\phi_{\bar{x}\bar{x}}+\frac{1}{\bar{x}^{2}}\phi_{\bar{y}\bar{y}}+\frac{1}{\bar{x}}\phi_{\bar{x}}\right)=0\text{ when }\phi_{\bar{x}\bar{x}}+\frac{1}{\bar{x}^{2}}\phi_{\bar{y}\bar{y}}+\frac{1}{\bar{x}}\phi_{\bar{x}}=0, \tag{4.63}$$

will admit a solution of the form (4.53)-(4.55) provided that

$$\begin{aligned}\frac{\partial\xi_{2}}{\partial\bar{x}} &= 0, & \frac{\partial\xi_{2}}{\partial\bar{y}} &= 0,\\ \frac{\partial\alpha}{\partial\bar{x}} &= 0, & \frac{\partial\alpha}{\partial\bar{y}} &= 0\end{aligned} \tag{4.64}$$

and

$$\frac{\partial^{2}\beta}{\partial\bar{x}^{2}}+\frac{1}{\bar{x}^{2}}\frac{\partial^{2}\beta}{\partial\bar{y}^{2}}+\frac{1}{\bar{x}}\frac{\partial\beta}{\partial\bar{x}}=0. \tag{4.65}$$

Solving the system of partial differential equations (4.64), the infinitesimals of the Lie point symmetries of the boundary value problem are

$$\xi_{1} = 0, \tag{4.66}$$

$$\xi_{2} = \gamma, \tag{4.67}$$

$$\eta = \alpha\phi+\beta\left(\bar{x},\bar{y}\right) \tag{4.68}$$

where α and γ are arbitrary constants and the function $\beta\left(\bar{x},\bar{y}\right)$ is itself any solution of Laplace's equation in a polar coordinate system.

To calculate the general form of the group invariant solution of the boundary value problem given the infinitesimals (4.66)-(4.68) we must solve the partial differential equation

$$\gamma \frac{\partial \phi}{\partial \bar{y}} = \alpha \phi + \beta \left(\bar{x}, \bar{y} \right) \tag{4.69}$$

for $\phi = \phi \left(\bar{x}, \bar{y} \right)$. The characteristic equations for the partial differential equation are

$$\frac{d\bar{x}}{0} = \frac{d\bar{y}}{\gamma} = \frac{d\phi}{\alpha \phi + \beta \left(\bar{x}, \bar{y} \right)}. \tag{4.70}$$

From the characteristic equations (4.70) we will consider the following two ordinary differential equations, namely,

$$d\bar{x} = 0 \tag{4.71}$$

and

$$\frac{d\bar{y}}{\gamma} = \frac{d\phi}{\alpha \phi + \beta \left(\bar{x}, \bar{y} \right)}. \tag{4.72}$$

Integrating the ordinary differential equation (4.71) the first characteristic curve is

$$\mu = \bar{x}. \tag{4.73}$$

The ordinary differential equation (4.72) expressed in terms of the first characteristic curve is

$$\frac{d\phi}{d\bar{y}} - \frac{\alpha \phi}{\gamma} = \frac{\beta \left(\mu, \bar{y} \right)}{\gamma}. \tag{4.74}$$

If we multiply both sides of this ordinary differential equation with the integrating factor, $\exp \left(-\alpha \bar{y} / \gamma \right)$, then it can otherwise be written as

$$\frac{d}{d\bar{y}} \left(e^{-\alpha \bar{y} / \gamma} \phi \right) = \frac{\beta \left(\mu, \bar{y} \right) e^{-\alpha \bar{y} / \gamma}}{\gamma}. \tag{4.75}$$

Integrating the ordinary differential equation along the first characteristic curve $\mu =$ constant we obtain the second characteristic curve,

$$\nu = e^{-\alpha \bar{y} / \gamma} \phi - \int \frac{\beta \left(\bar{x}, \bar{y} \right) e^{-\alpha \bar{y} / \gamma}}{\gamma} d\bar{y}. \tag{4.76}$$

Therefore, by equation (3.27) in the previous chapter, the general form of the group invariant solution of the boundary value problem is

$$\phi \left(\bar{x}, \bar{y} \right) = e^{a \bar{y}} \int \frac{\beta \left(\bar{x}, \bar{y} \right) e^{-a \bar{y}}}{\gamma} d\bar{y} + e^{a \bar{y}} \Phi \left(\bar{x} \right) \tag{4.77}$$

where $a = \alpha / \gamma$.

The case $a = 0$

If we let $a = 0$ and $\beta(x, y) = 0$ and then we substitute the general form (4.77) of the group invariant solution into Laplace's equation (4.62), we obtain the following ordinary differential equation for the function $\Phi = \Phi(\bar{x})$,

$$\frac{d^2\Phi}{d\bar{x}^2} + \frac{1}{\bar{x}}\frac{d\Phi}{d\bar{x}} = 0. \tag{4.78}$$

The solution of this ordinary differential equation is

$$\Phi(\bar{x}) = k_1 \ln \bar{x} + k_2 \tag{4.79}$$

where k_1 and k_2 are arbitrary constants. If we substitute the solution (4.79) of the ordinary differential equation back into the general form (4.77) of the group invariant solution, then to satisfy the boundary condition (4.57) we must set $k_1 = 0$. We thus obtain the trivial group invariant solution of the boundary value problem,

$$\phi(\bar{x}, \bar{y}) = k_2. \tag{4.80}$$

However, (4.80) is not a realistic solution of the boundary value problem because according to definition (3.6) it implies that the fluid is motionless. Nonetheless, rather than discard this group invariant solution we can use it to determine a new group invariant solution as follows. Recall that in the general form (4.77) of the group invariant solution the function $\beta(\bar{x}, \bar{y})$ must be a solution of Laplace's equation. Hence, if we substitute $a = 0$ and $\beta(\bar{x}, \bar{y}) = k_2$ into the general form (4.77) of the group invariant solution, then a new group invariant solution will be

$$\phi(\bar{x}, \bar{y}) = \frac{k_2 \bar{y}}{\gamma} + \Phi(\bar{x}). \tag{4.81}$$

Substituting the new group invariant solution (4.81) into Laplace's equation (4.62) will again yield the ordinary differential equation (4.78). Hence, the new group invariant solution (4.81) expressed in terms of the solution (4.79) of the ordinary differential equation (4.78) is

$$\phi(\bar{x}, \bar{y}) = \frac{k_2 \bar{y}}{\gamma} + k_3 \ln \bar{x} + k_4 \tag{4.82}$$

where k_3 and k_4 are arbitrary constants. The new group invariant solution (4.82) will satisfy the boundary condition (4.57) provided we set $k_3 = 0$. Therefore, a new group invariant solution of the boundary value problem is

$$\phi\left(\bar{x}, \bar{y}\right) = \frac{k_2 \bar{y}}{\gamma} + k_4. \tag{4.83}$$

A physical interpretation of the flow of a fluid that is represented by the potential function (4.83) will be provided by the corresponding stream function. In general, to derive the stream function $\psi = \psi\left(\bar{x}, \bar{y}\right)$ that corresponds to a potential function $\phi = \phi\left(\bar{x}, \bar{y}\right)$ we must solve equation (3.9). If we apply the del operator (4.15) to the functions in equation (3.9), then in an orthogonal curvilinear coordinate system the equation may be stated as

$$a\left(\bar{x}, \bar{y}\right) \frac{\partial \phi}{\partial \bar{x}} \frac{\partial \psi}{\partial \bar{x}} + b\left(\bar{x}, \bar{y}\right) \frac{\partial \phi}{\partial \bar{y}} \frac{\partial \psi}{\partial \bar{y}} = 0 \tag{4.84}$$

where the coefficients $a\left(\bar{x}, \bar{y}\right)$ and $b\left(\bar{x}, \bar{y}\right)$ of the partial derivatives are calculated according to the equations (4.20). Hence, in a polar coordinate system, to determine the stream function that corresponds to the potential function (4.83) we must solve the equation

$$\frac{\partial \phi}{\partial \bar{x}} \frac{\partial \psi}{\partial \bar{x}} + \frac{1}{\bar{x}^2} \frac{\partial \phi}{\partial \bar{y}} \frac{\partial \psi}{\partial \bar{y}} = 0. \tag{4.85}$$

Given that equation (4.85) will be identically satisfied if

$$\frac{\partial \phi}{\partial \bar{x}} = \frac{1}{\bar{x}} \frac{\partial \psi}{\partial \bar{y}} \text{ and } \frac{1}{\bar{x}} \frac{\partial \phi}{\partial \bar{y}} = -\frac{\partial \psi}{\partial \bar{x}}, \tag{4.86}$$

then substituting the potential function (4.83) into the system of partial differential equations (4.86) and after performing the necessary integrations we obtain the stream function

$$\psi\left(\bar{x}, \bar{y}\right) = \psi_0 - \frac{k_2 \ln \bar{x}}{\gamma} \tag{4.87}$$

where ψ_0 is a constant of integration.

In the literature the convention is often used that the streamline $\psi\left(\bar{x}, \bar{y}\right) = 0$ is made to coincide with the equation of the surface of the body (see, for example, [12, 20]). For a stream function to satisfy this convention in orthogonal curvilinear coordinate systems of the kind considered here implies that

$$\psi\left(\bar{x},\bar{y}\right)=0 \text{ when } \bar{x}=c. \tag{4.88}$$

If we apply this convention to the stream function (4.87), then $\psi_0=(k_2/\gamma)\ln c$. Hence, the corresponding stream function for the potential function (4.83) is

$$\psi\left(\bar{x},\bar{y}\right)=\frac{k_2}{\gamma}\ln\left(\frac{c}{\bar{x}}\right). \tag{4.89}$$

Note that the value of the stream function (4.89) will be constant along the coordinate lines $\bar{x}=$ constant. However, according to the coordinate transformation (4.59)-(4.60), the coordinate lines $\bar{x}=$ constant in a polar coordinate system correspond to concentric circles in a Cartesian coordinate system. Therefore, the stream function (4.89) represents the flow of a fluid around a circular cylinder as shown in figure 4.3.

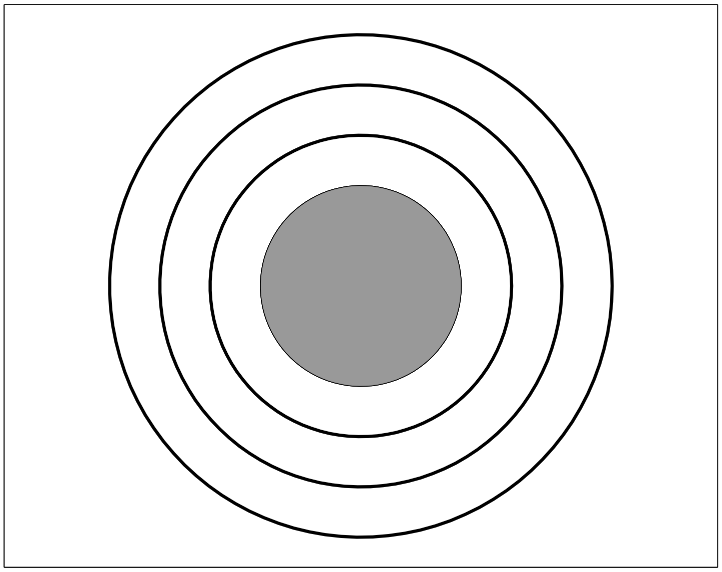

Figure 4.3: A plot of the streamlines for the stream function (4.89).

The case $a \neq 0$

If we let $\beta(\bar{x},\bar{y}) = 0$, then substituting the general form (4.77) of the group invariant solution into Laplace's equation (4.62) we obtain the following ordinary differential equation for the function $\Phi = \Phi(\bar{x})$,

$$\bar{x}^2 \frac{d^2\Phi}{d\bar{x}^2} + \bar{x}\frac{d\Phi}{d\bar{x}} + a^2\Phi = 0. \tag{4.90}$$

This ordinary differential equation is called the Cauchy-Euler equation [26]. The solution of the Cauchy-Euler equation will be $\Phi(\bar{x}) = \bar{x}^m$ provided that m satisfies the quadratic equation

$$m^2 + a^2 = 0. \tag{4.91}$$

Solving the quadratic equation (4.91) it follows that the group invariant solution in this case is

$$\phi(\bar{x},\bar{y}) = \left(k_1\bar{x}^{ia} + k_2\bar{x}^{-ia}\right)e^{a\bar{y}} \tag{4.92}$$

where k_1 and k_2 are arbitrary constants. If we apply the identities

$$\cos(a\ln\bar{x}) = \cosh(ia\ln\bar{x}),\ i\sin(a\ln\bar{x}) = \sinh(ia\ln\bar{x}) \tag{4.93}$$

and

$$\bar{x} = e^{\ln\bar{x}}, \tag{4.94}$$

then the group invariant solution (4.92) may otherwise be written as

$$\phi(\bar{x},\bar{y}) = \left[k_1\cosh(m\ln\bar{x}) + k_2\sinh(m\ln\bar{x})\right]e^{a\bar{y}}. \tag{4.95}$$

For the group invariant solution (4.95), the boundary condition (4.57) implies that the constants k_1 and k_2 must satisfy the equation

$$k_1\sinh(m\ln c) + k_2\cosh(m\ln c) = 0. \tag{4.96}$$

Furthermore, solving the system of partial differential equations (4.86) provides us with the stream function,

$$\psi(\bar{x},\bar{y}) = \frac{m}{a}\left[k_1\sinh(m\ln\bar{x}) + k_2\cosh(m\ln\bar{x})\right]e^{a\bar{y}}, \tag{4.97}$$

where we have set the constant of integration equal to zero to obey the convention (4.88).

Listed below are the potential functions and stream functions that were obtained from (4.95) and (4.97) for various values of a and m.

- $a = i$ and $m = 1$

$$\begin{aligned} \phi\left(\bar{x},\bar{y}\right) &= \frac{ck_1}{c^2+1}\left(\frac{\bar{x}}{c}+\frac{c}{\bar{x}}\right)\left(\cos\bar{y}+i\sin\bar{y}\right) && (4.98)\\ \psi\left(\bar{x},\bar{y}\right) &= \frac{ck_1}{c^2+1}\left(\frac{\bar{x}}{c}-\frac{c}{\bar{x}}\right)\left(\sin\bar{y}-i\cos\bar{y}\right) && (4.99)\end{aligned}$$

- $a = 2i$ and $m = 2$

$$\begin{aligned} \phi\left(\bar{x},\bar{y}\right) &= \frac{c^2k_1}{c^4+1}\left[\left(\frac{\bar{x}}{c}\right)^2+\left(\frac{c}{\bar{x}}\right)^2\right]\left[\cos\left(2\bar{y}\right)+i\sin\left(2\bar{y}\right)\right] && (4.100)\\ \psi\left(\bar{x},\bar{y}\right) &= \frac{c^2k_1}{c^4+1}\left[\left(\frac{\bar{x}}{c}\right)^2-\left(\frac{c}{\bar{x}}\right)^2\right]\left[\sin\left(2\bar{y}\right)-i\cos\left(2\bar{y}\right)\right] && (4.101)\end{aligned}$$

- $a = 1$ and $m = i$

$$\begin{aligned} \phi\left(\bar{x},\bar{y}\right) &= \frac{k_1}{\cos\left(\ln c\right)}\cos\left[\ln\left(\frac{\bar{x}}{c}\right)\right]e^{\bar{y}} && (4.102)\\ \psi\left(\bar{x},\bar{y}\right) &= \frac{-k_1}{\cos\left(\ln c\right)}\sin\left[\ln\left(\frac{\bar{x}}{c}\right)\right]e^{\bar{y}} && (4.103)\end{aligned}$$

Figures 4.4, 4.5 and 4.6 are plots of the streamlines for the stream functions (4.99), (4.101) and (4.103). Note, however, that figures 4.4 and 4.5 are plots of the real part only of the stream functions (4.99) and (4.101). This is because if

$$\psi_R\left(\bar{x},\bar{y}\right) = \frac{ck_1}{c^2+1}\left(\frac{\bar{x}}{c}-\frac{c}{\bar{x}}\right)\sin\bar{y} \tag{4.104}$$

is the real part of the stream function (4.99) and

$$\psi_I\left(\bar{x},\bar{y}\right) = \frac{-ck_1}{c^2+1}\left(\frac{\bar{x}}{c}-\frac{c}{\bar{x}}\right)\cos\bar{y} \tag{4.105}$$

is the imaginary part of the stream function, then

$$\psi_R\left(\bar{x},\bar{y}\right) = \psi_I\left(\bar{x},\bar{y}+\frac{\pi}{2}\right). \tag{4.106}$$

Therefore, given that figure 4.4 is a plot of the streamlines $\psi_R(\bar{x}, \bar{y}) =$ constant, a plot of the streamlines $\psi_I(\bar{x}, \bar{y}) =$ constant can be obtained by rotating figure 4.4 by $\pi/2$ radians about the centre of the circle. Similarly, a plot of the streamlines for the imaginary part of the stream function (4.101) can be obtained by rotating figure 4.5 by $\pi/4$ radians about the centre of the circle.

In figure 4.4 the streamlines represent the flow of a fluid past a circular cylinder in the case that the fluid approaches the cylinder from the left or from the right. On the other hand, the streamlines in figure 4.5 represent the flow of a fluid past a circular cylinder in the case that the fluid approaches the cylinder from the left and from the right. As the fluid approaches the cylinder from both directions in figure 4.5 it will then flow away from cylinder in the direction of the dashed line. The streamlines in figure 4.6 represent the swirling flow of a fluid around a circular cylinder that is similar to motion that is observed in whirlpools.

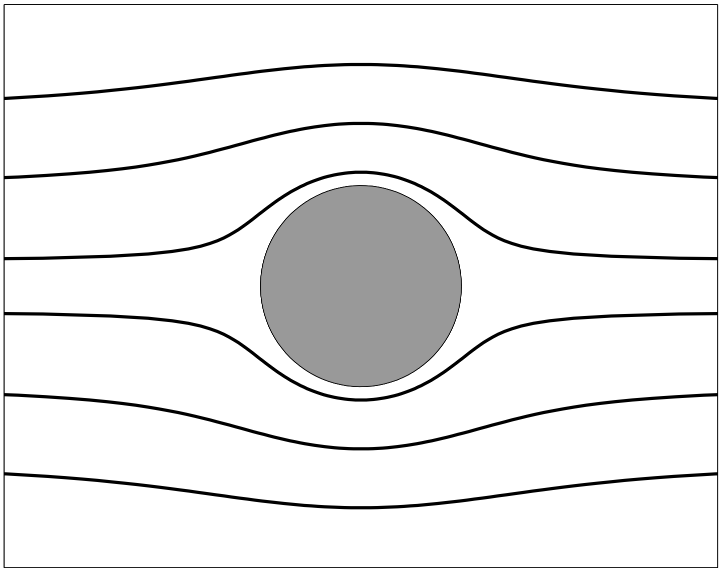

Figure 4.4: A plot of the streamlines for the real part of the stream function (4.99).

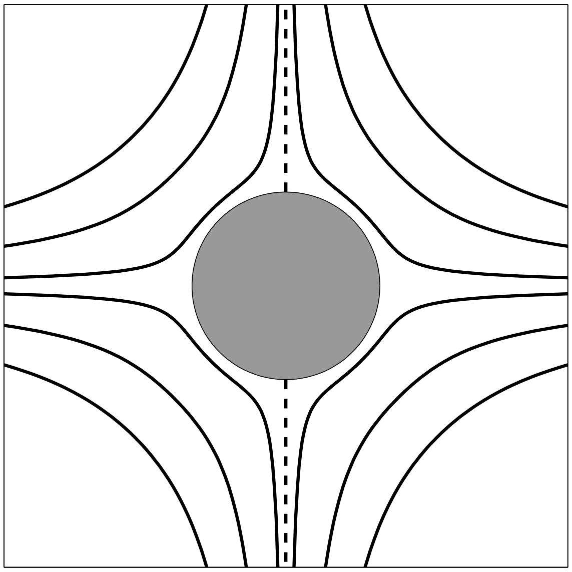

Figure 4.5: A plot of the streamlines for the real part of the stream function (4.101).

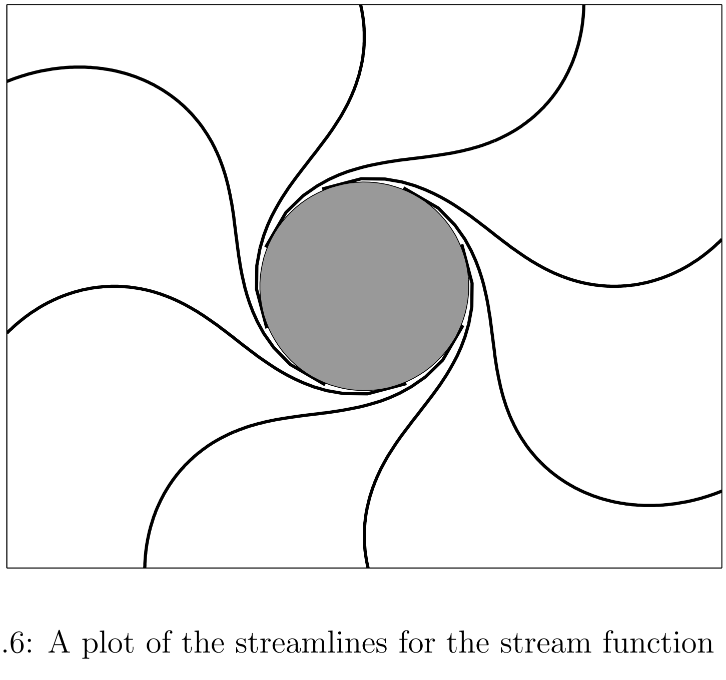

Figure 4.6: A plot of the streamlines for the stream function (4.103).

4.3.3 The Principle of Linear Superposition

Laplace's equation (4.62) in a polar coordinate system is a linear partial differential equation. Consequently, if $\phi_1 = \phi_1(\bar{x}, \bar{y})$ and $\phi_2 = \phi_2(\bar{x}, \bar{y})$ are two independent solutions of Laplace's equation that also satisfy the boundary condition (4.57), then it is easily verified by a direct substitution that a linear combination of these solutions, namely,

$$\phi(\bar{x}, \bar{y}) = \phi_1(\bar{x}, \bar{y}) + \lambda \phi_2(\bar{x}, \bar{y}) \tag{4.107}$$

where λ is an arbitrary constant, will also satisfy Laplace's equation and the boundary condition. Furthermore, it is also easily verified that if $\psi_1 = \psi_1(\bar{x}, \bar{y})$ and $\psi_2 = \psi_2(\bar{x}, \bar{y})$ are two independent stream functions that satisfy the system of partial differential equations (4.86), that is,

$$\begin{aligned} \frac{\partial \phi_1}{\partial \bar{x}} &= \frac{1}{\bar{x}} \frac{\partial \psi_1}{\partial \bar{y}}, & \frac{1}{\bar{x}} \frac{\partial \phi_1}{\partial \bar{y}} &= -\frac{\partial \psi_1}{\partial \bar{x}} \\ \frac{\partial \phi_2}{\partial \bar{x}} &= \frac{1}{\bar{x}} \frac{\partial \psi_2}{\partial \bar{y}}, & \frac{1}{\bar{x}} \frac{\partial \phi_2}{\partial \bar{y}} &= -\frac{\partial \psi_2}{\partial \bar{x}}, \end{aligned} \tag{4.108}$$

then the corresponding stream function for the potential function (4.107) is

$$\psi(\bar{x}, \bar{y}) = \psi_1(\bar{x}, \bar{y}) + \lambda \psi_2(\bar{x}, \bar{y}). \tag{4.109}$$

As an example of the principle of linear superposition, consider a linear combination of real parts of the potential functions (4.98) and (4.100); that is,

$$\phi(\bar{x}, \bar{y}) = \left[\left(\frac{\bar{x}}{c}\right)^2 + \left(\frac{c}{\bar{x}}\right)^2\right] \cos(2\bar{y}) + \lambda \left(\frac{\bar{x}}{c} + \frac{c}{\bar{x}}\right) \cos \bar{y}. \tag{4.110}$$

The streamlines for the corresponding stream function,

$$\psi(\bar{x}, \bar{y}) = \left[\left(\frac{\bar{x}}{c}\right)^2 - \left(\frac{c}{\bar{x}}\right)^2\right] \sin(2\bar{y}) + \lambda \left(\frac{\bar{x}}{c} - \frac{c}{\bar{x}}\right) \sin \bar{y}, \tag{4.111}$$

are shown in figures 4.7 and 4.8. As in figure 4.5, figures 4.7 and 4.8 show the flow of a fluid towards a cylinder from the left and the right that then flows away from the cylinder in the direction of the dashed line as the fluid approaches the cylinder. However, unlike figure 4.5, in figures 4.7 and 4.8 the fluid flows towards the cylinder with a greater speed from the right. If $0 < \lambda \leq 4$, then the faster moving fluid from the right will be diverted by the slower moving fluid from the left along the dashed line that is attached to cylinder as shown in figure 4.7. If $\lambda > 4$ then the fast moving fluid from the right is deflected by the slow moving fluid on the left only after it has flowed past the cylinder as shown in figure 4.8.

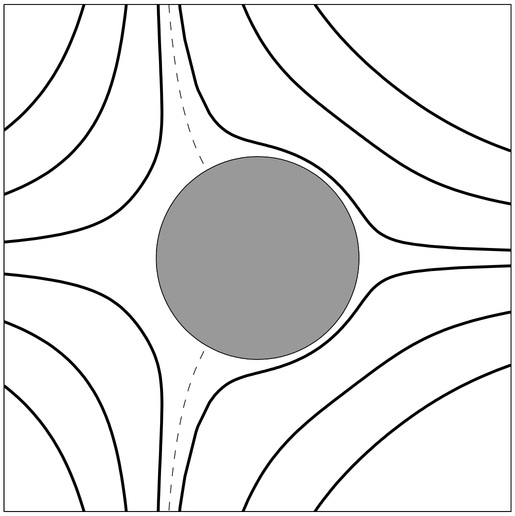

Figure 4.7: A plot of the streamlines for the stream function (4.111) for $\lambda = 2$.

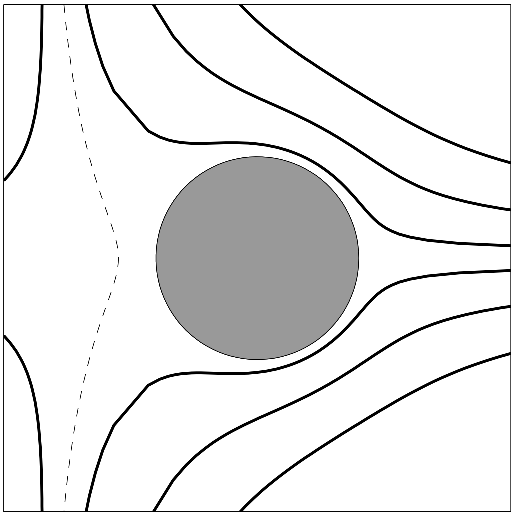

Figure 4.8: A plot of the streamlines for the stream function (4.111) for $\lambda = 4, 3$.

In this section we have shown in detail how to use Lie point symmetries to determine group invariant solutions of the boundary value problem and in a few instances we showed that these group invariant solutions represent the flow of a fluid past a circular cylinder. In the next two sections we will apply the same procedure to determine the group invariant solutions of the boundary value problems for the flow of a fluid past an elliptic cylinder and a parabolic cylinder.

4.4 The Flow Past an Elliptic Cylinder

4.4.1 Laplace's Equation in an Elliptic Coordinate System

To write down the equation of the ellipse we begin with the points $(-\epsilon, 0)$ and $(\epsilon, 0)$ on the x-axis of a Cartesian coordinate system. These points are called the foci of the ellipse. If the distances from the foci of the ellipse to a point (x, y) are $\sqrt{(x+\epsilon)^2 + y^2}$ and $\sqrt{(x-\epsilon)^2 + y^2}$, then the set of all points (x, y) such that

$$F(x,y) = \sqrt{(x+\epsilon)^2 + y^2} + \sqrt{(x-\epsilon)^2 + y^2} = c \text{ where } c > 2\epsilon \tag{4.112}$$

is called an ellipse [23]. On the other hand, the set of all the points (x, y) such that

$$G(x,y) = \sqrt{(x+\epsilon)^2 + y^2} - \sqrt{(x-\epsilon)^2 + y^2} = c \text{ where } |c| < 2\epsilon \tag{4.113}$$

is called a hyperbola [23]. Ellipses and hyperbolae that share the same foci are orthogonal to each other as can easily be verified by substituting the functions (4.112) and (4.113) into equation (4.52). Hence, let the equations

$$\begin{aligned} \bar{x} &= \sqrt{(x+\epsilon)^2 + y^2} + \sqrt{(x-\epsilon)^2 + y^2}, &(4.114)\\ \bar{y} &= \sqrt{(x+\epsilon)^2 + y^2} - \sqrt{(x-\epsilon)^2 + y^2} &(4.115) \end{aligned}$$

define a coordinate transformation from a Cartesian coordinate system to an orthogonal curvilinear coordinate system wherein the coordinate lines $\bar{x} =$ constant are ellipses and $\bar{y} =$ constant are hyperbolae. Let this orthogonal curvilinear coordinate system be referred to as an elliptic coordinate system. Note that it is a consequence of the definitions (4.112) and (4.113) of an ellipse and a hyperbola that the elliptic coordinates $\bar{x}$ and $\bar{y}$ must obey the inequalities

$$\bar{x}^2 - 4\epsilon^2 > 0 \text{ and } 4\epsilon^2 - \bar{y}^2 < 0. \tag{4.116}$$

If we substitute the functions (4.114)-(4.115) into the equations (4.20), (4.28) and (4.29), then the coefficients of the partial derivatives are

$$\begin{aligned} a(\bar{x},\bar{y}) &= \frac{4(\bar{x}^2-4\epsilon^2)}{\bar{x}^2-\bar{y}^2}, & b(\bar{x},\bar{y}) &= \frac{4(4\epsilon^2-\bar{y}^2)}{\bar{x}^2-\bar{y}^2}, \\ d(\bar{x},\bar{y}) &= \frac{4\bar{x}}{\bar{x}^2-\bar{y}^2}, & e(\bar{x},\bar{y}) &= \frac{-4\bar{y}}{\bar{x}^2-\bar{y}^2}. \end{aligned} \tag{4.117}$$

Hence, Laplace's equation in an elliptic coordinate system is

$$\left(\bar{x}^2 - 4\epsilon^2\right)\frac{\partial^2\phi}{\partial\bar{x}^2} + \left(4\epsilon^2 - \bar{y}^2\right)\frac{\partial^2\phi}{\partial\bar{y}^2} + \bar{x}\frac{\partial\phi}{\partial\bar{x}} - \bar{y}\frac{\partial\phi}{\partial\bar{x}} = 0. \tag{4.118}$$

4.4.2 The General Form of the Group Invariant Solution

If we solve the determining equation

$$\begin{gathered} X^{(2)}\left[\left(\bar{x}^2-4\epsilon^2\right)\phi_{\bar{x}\bar{x}} + \left(4\epsilon^2-\bar{y}^2\right)\phi_{\bar{y}\bar{y}} + \bar{x}\phi_{\bar{x}} - \bar{y}\phi_{\bar{y}}\right] = 0 \\ \text{when} \\ \left(\bar{x}^2-4\epsilon^2\right)\phi_{\bar{x}\bar{x}} + \left(4\epsilon^2-\bar{y}^2\right)\phi_{\bar{y}\bar{y}} + \bar{x}\phi_{\bar{x}} - \bar{y}\phi_{\bar{y}} = 0 \end{gathered} \tag{4.119}$$

by looking for a solution that is of the form (4.53)-(4.55), then we are able to show that the infinitesimals of the Lie point symmetries that will leave the boundary value problem invariant are

$$\xi_1 = 0, \tag{4.120}$$

$$\xi_2 = \gamma\sqrt{4\epsilon^2 - \bar{y}^2}, \tag{4.121}$$

$$\eta = \alpha\phi + \beta(\bar{x},\bar{y}) \tag{4.122}$$

where α and γ are arbitrary constants and the function $\beta(\bar{x},\bar{y})$ is itself any solution of Laplace's equation (4.118). Consequently, the general form of the group invariant solution of the boundary value problem is

$$\phi(\bar{x},\bar{y}) = e^{a\bar{z}}\int\left[\frac{\beta(\bar{x},\bar{y})}{\gamma}\frac{e^{-a\bar{z}}}{\sqrt{4\epsilon^2-\bar{y}^2}}\right]d\bar{y} + e^{a\bar{z}}\Phi(\bar{x}) \tag{4.123}$$

where $a = \alpha/\gamma$ and

$$\bar{z} = \arcsin\left(\frac{\bar{y}}{2\epsilon}\right). \tag{4.124}$$

If we let $\beta\left(\bar{x}, \bar{y}\right) = 0$, then substituting the general form (4.123) of the group invariant solution into Laplace's equation (4.118) we obtain the following ordinary differential equation for the function $\Phi = \Phi\left(\bar{x}\right)$,

$$\left(\bar{x}^2 - 4\epsilon^2\right)\frac{d^2\Phi}{d\bar{x}^2} + \bar{x}\frac{d\Phi}{d\bar{x}} + a^2\Phi = 0. \tag{4.125}$$

Since this ordinary differential equation has the property (3.35), if we make the change of variable (3.36) then we are able to transform it into the simpler ordinary differential equation

$$\frac{d^2\Phi}{d\bar{w}^2} + a^2\Phi = 0 \tag{4.126}$$

where

$$\bar{w} = \ln\left(\bar{x} - \sqrt{\bar{x}^2 - 4\epsilon^2}\right). \tag{4.127}$$

The case $a = 0$

If $a = 0$, then it follows from the solution of the ordinary differential equation (4.126) that the group invariant of the boundary value problem is

$$\phi\left(\bar{x}, \bar{y}\right) = k_1 \ln\left(\bar{x} - \sqrt{\bar{x}^2 - 4\epsilon^2}\right) + k_2 \tag{4.128}$$

where k_1 and k_2 are arbitrary constants. If we apply the boundary condition (4.57) to the above group invariant solution, then we must set $k_1 = 0$. Therefore, we obtain the trivial group invariant solution of the boundary value problem, namely,

$$\phi\left(\bar{x}, \bar{y}\right) = k_2. \tag{4.129}$$

The case $a \neq 0$

In the case that $a \neq 0$, if we look for a solution of the ordinary differential equation (4.126) that is of the form $\Phi\left(\bar{w}\right) = e^{m\bar{w}}$, then the group invariant solution of the boundary value problem expressed in terms of the arbitrary constants k_1 and k_2 is

$$\begin{aligned}\phi\left(\bar{x},\bar{y}\right) &= k_1 \exp\left[a \arcsin\left(\frac{\bar{y}}{2\epsilon}\right) + m \ln\left(\bar{x} - \sqrt{\bar{x}^2 - 4\epsilon^2}\right)\right] \\ &\quad + k_2 \exp\left[a \arcsin\left(\frac{\bar{y}}{2\epsilon}\right) - m \ln\left(\bar{x} - \sqrt{\bar{x}^2 - 4\epsilon^2}\right)\right] \qquad (4.130)\end{aligned}$$

where the constants a and m must also satisfy the quadratic equation (4.91). If we apply the boundary condition (4.57) to the above group invariant solution, then the constants k_1 and k_2 must satisfy the equation

$$k_1 \exp\left[m \ln\left(c - \sqrt{c^2 - 4\epsilon^2}\right)\right] - k_2 \exp\left[-m \ln\left(c - \sqrt{c^2 - 4\epsilon^2}\right)\right] = 0. \qquad (4.131)$$

To determine the corresponding stream function for the potential function (4.130) we note that in an elliptic coordinate system equation (4.84) will be identically satisfied if

$$\frac{\partial \phi}{\partial \bar{x}} = \frac{\sqrt{4\epsilon^2 - \bar{y}^2}}{\sqrt{\bar{x}^2 - 4\epsilon^2}} \frac{\partial \psi}{\partial \bar{y}} \quad \text{and} \quad \frac{\partial \phi}{\partial \bar{y}} = -\frac{\sqrt{\bar{x}^2 - 4\epsilon^2}}{\sqrt{4\epsilon^2 - \bar{y}^2}} \frac{\partial \psi}{\partial \bar{x}}. \qquad (4.132)$$

Solving the above system of partial differential equations given the potential function (4.130), the corresponding stream function that is obtained is

$$\begin{aligned}\psi\left(\bar{x},\bar{y}\right) &= \frac{a}{m}\left\{k_1 \exp\left[a \arcsin\left(\frac{\bar{y}}{2\epsilon}\right) + m \ln\left(\bar{x} - \sqrt{\bar{x}^2 - 4\epsilon^2}\right)\right]\right. \\ &\quad \left. -k_2 \exp\left[a \arcsin\left(\frac{\bar{y}}{2\epsilon}\right) - m \ln\left(\bar{x} - \sqrt{\bar{x}^2 - 4\epsilon^2}\right)\right]\right\}. \qquad (4.133)\end{aligned}$$

We conclude this section by noting that if we substitute the group invariant solutions (4.129) and (4.130) of the boundary value problem into the general form (4.123) of the group invariant solution, we can proceed to determine new group invariant solutions as was demonstrated in the previous section.

4.5 The Flow Past a Parabolic Cylinder

4.5.1 Laplace's Equation in a Parabolic Coordinate System

Let the equation $x = c$ define a vertical line in a Cartesian coordinate system and let $(\epsilon, 0)$ be a point on the x-axis of the Cartesian coordinate system where $c < \epsilon$. The

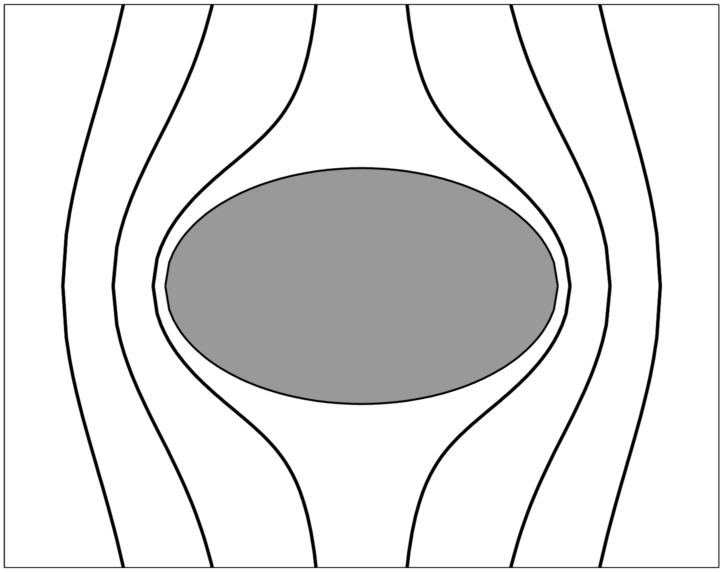

Figure 4.9: A plot of the streamlines for the real part of the stream function (4.133) in the case that $a = i$ and $m = 1$.

vertical line is called the directrix of the parabola and the point is called the focus of the parabola. Let $\sqrt{(x-\epsilon)^2 + y^2}$ be the distance from the focus of the parabola to a point (x, y) and let $x - c$ be the distance between the point (x, y) and the directrix of the parabola along a line that is perpendicular to the directrix. The set of all the points (x, y) such that

$$\sqrt{(x-\epsilon)^2 + y^2} = x - c \tag{4.134}$$

is called a parabola [23]. In the above definition of a parabola the directrix is located to the left of the focus. If we shift the directrix to the right of the focus, then the equation of the parabola is

$$\sqrt{(x-\epsilon)^2 + y^2} = c - x. \tag{4.135}$$

From the definitions (4.134) and (4.135) of a parabola it can be shown that the equations

$$\bar{x} = \sqrt{(x-\epsilon)^2 + y^2} - x, \tag{4.136}$$

$$\bar{y} = \sqrt{(x-\epsilon)^2 + y^2} + x \tag{4.137}$$

define a coordinate transformation from a Cartesian coordinate system to an orthogonal curvilinear coordinate system. This orthogonal curvilinear coordinate system will be referred to as a parabolic coordinate system.

If we substitute the functions (4.136)-(4.137) into the equations (4.20), (4.28) and (4.29), then the coefficients of the partial derivatives of Laplace's equation in a parabolic system are

$$\begin{aligned} a(\bar{x},\bar{y}) &= \frac{2(\bar{x}+\epsilon)}{\bar{x}+\bar{y}}, \qquad & b(\bar{x},\bar{y}) &= \frac{2(\bar{y}-\epsilon)}{\bar{x}+\bar{y}}, \\ d(\bar{x},\bar{y}) &= \frac{1}{\bar{x}+\bar{y}}, \qquad & e(\bar{x},\bar{y}) &= \frac{1}{\bar{x}+\bar{y}}. \end{aligned} \tag{4.138}$$

Hence, Laplace's equation in a parabolic coordinate system is

$$2(\bar{x}+\epsilon)\frac{\partial^2\phi}{\partial\bar{x}^2} + 2(\bar{y}-\epsilon)\frac{\partial^2\phi}{\partial\bar{y}^2} + \frac{\partial\phi}{\partial\bar{x}} + \frac{\partial\phi}{\partial\bar{y}} = 0. \tag{4.139}$$

Having thus determined Laplace's equation in a parabolic coordinate system we can now proceed as in the previous two examples and calculate the group invariant solutions of the boundary value problem.

4.5.2 The General Form of the Group Invariant Solution

Solving the determining equation,

$$\begin{gathered} X^{(2)}\left[2(\bar{x}+\epsilon)\phi_{\bar{x}\bar{x}} + 2(\bar{y}-\epsilon)\phi_{\bar{y}\bar{y}} + \phi_{\bar{x}} + \phi_{\bar{y}}\right] = 0 \\ \text{when} \\ 2(\bar{x}+\epsilon)\phi_{\bar{x}\bar{x}} + 2(\bar{y}-\epsilon)\phi_{\bar{y}\bar{y}} + \phi_{\bar{x}} + \phi_{\bar{y}} = 0, \end{gathered} \tag{4.140}$$

the infinitesimals of the Lie point symmetries that will leave the boundary value problem invariant are

$$\xi_1 = 0, \tag{4.141}$$

$$\xi_2 = \gamma\sqrt{\bar{y}-\epsilon}, \tag{4.142}$$

$$\eta = \alpha\phi + \beta(\bar{x},\bar{y}) \tag{4.143}$$

where α and γ are arbitrary constants and the function $\beta(\bar{x},\bar{y})$ is itself any solution of Laplace's equation (4.139). Consequently, the general form of the group invariant solution of the boundary value problem is

$$\phi(\bar{x},\bar{y}) = e^{2a\sqrt{\bar{y}-\epsilon}} \int \left[\frac{\beta(\bar{x},\bar{y})}{\gamma} \frac{e^{-2a\sqrt{\bar{y}-\epsilon}}}{\sqrt{\bar{y}-\epsilon}} \right] d\bar{y} + e^{2a\sqrt{\bar{y}-\epsilon}}\Phi(\bar{x}) \tag{4.144}$$

where $a = \alpha/\gamma$ as before.

If we let $\beta(\bar{x},\bar{y}) = 0$, then substituting the form (4.144) of the group invariant solution into Laplace's equation (4.139) we obtain the following ordinary differential equation for the function $\Phi = \Phi(\bar{x})$,

$$2(\bar{x}+\epsilon)\frac{d^2\Phi}{d\bar{x}^2} + \frac{d\Phi}{d\bar{x}} + 2a^2\Phi = 0. \tag{4.145}$$

Since this ordinary differential equation has the property (3.35), the change of variable

$$\bar{w} = 2\sqrt{\bar{x}+\epsilon} \tag{4.146}$$

will transform it into the simpler ordinary differential equation (4.126).

The case $a = 0$

If $a = 0$ then the resulting group invariant solution of the boundary value problem is

$$\phi(\bar{x},\bar{y}) = k_2 \tag{4.147}$$

where k_2 is an arbitrary constant. Replacing the function $\beta(\bar{x},\bar{y})$ in the general form (4.144) of the group invariant solution with the group invariant solution (4.147), the new group invariant solution that is obtained is

$$\phi(\bar{x},\bar{y}) = \frac{2k_2}{\gamma}\sqrt{\bar{y}-\epsilon} + k_4 \tag{4.148}$$

where k_4 is another arbitrary constant. If we repeat this procedure and we replace the function $\beta(\bar{x},\bar{y})$ in the general form (4.144) of the group invariant solution with the new group invariant solution (4.148), then another new group invariant solution that is obtained is

$$\phi\left(\bar{x},\bar{y}\right)=\frac{2k_2}{\gamma^2}\left(2\sqrt{\bar{x}+\epsilon}\sqrt{c+\epsilon}-\bar{x}+\bar{y}\right)+\frac{2k_4}{\gamma}\sqrt{\bar{y}-\epsilon}+k_6. \tag{4.149}$$

In a parabolic coordinate system equation (4.84) will be identically satisfied if

$$\frac{\partial\phi}{\partial\bar{x}}=\frac{\sqrt{\bar{y}-\epsilon}}{\sqrt{\bar{x}+\epsilon}}\frac{\partial\psi}{\partial\bar{y}}\text{ and }\frac{\partial\phi}{\partial\bar{y}}=-\frac{\sqrt{\bar{x}+\epsilon}}{\sqrt{\bar{y}-\epsilon}}\frac{\partial\psi}{\partial\bar{x}}. \tag{4.150}$$

Solving the system of partial differential equations (4.150) for each of the potential functions (4.148) and (4.149) we obtain the following stream functions:

$$\psi\left(\bar{x},\bar{y}\right)=\frac{2k_2}{\gamma}\left(\sqrt{c+\epsilon}-\sqrt{\bar{x}+\epsilon}\right) \tag{4.151}$$

and

$$\psi\left(\bar{x},\bar{y}\right)=\frac{4k_2}{\gamma^2}\sqrt{\bar{y}-\epsilon}\left(\sqrt{c+\epsilon}-\sqrt{\bar{x}+\epsilon}\right)+\frac{2k_4}{\gamma}\left(\sqrt{c+\epsilon}-\sqrt{\bar{x}+\epsilon}\right). \tag{4.152}$$

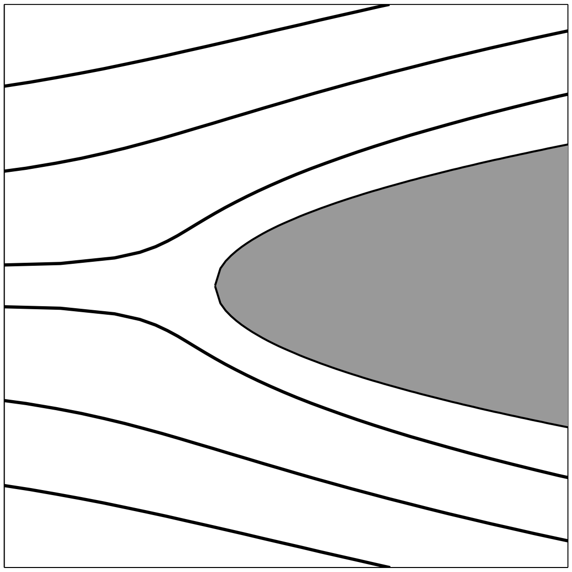

Figure 4.10: A plot of the streamlines for the stream function (4.152).

The case $a \neq 0$

If $a \neq 0$ then we obtain the group invariant solution

$$\phi(\bar{x}, \bar{y}) = k_1 \exp\left(2a\sqrt{\bar{y}-\epsilon} + 2m\sqrt{\bar{x}+\epsilon}\right) + k_2 \exp\left(2a\sqrt{\bar{y}-\epsilon} - 2m\sqrt{\bar{x}+\epsilon}\right). \tag{4.153}$$

The corresponding stream function for the potential function (4.153) is

$$\psi(\bar{x}, \bar{y}) = \frac{m}{a}\left[k_1 \exp\left(2a\sqrt{\bar{y}-\epsilon} + 2m\sqrt{\bar{x}+\epsilon}\right) - k_2 \exp\left(2a\sqrt{\bar{y}-\epsilon} - 2m\sqrt{\bar{x}+\epsilon}\right)\right] \tag{4.154}$$

where the constants k_1 and k_2 must satisfy the equation

$$k_1 \exp\left(2m\sqrt{c+\epsilon}\right) - k_2 \exp\left(-2m\sqrt{c+\epsilon}\right) = 0. \tag{4.155}$$

Chapter 5

Conclusions

In this work we considered the boundary value problem that describes the steady and two-dimensional potential flow of an incompressible fluid past a body without friction. A common and well-known technique for solving this boundary value problem is the method of conformal transformations. We showed in this work, however, that solutions of the boundary value problem that can be obtained by means of suitable conformal transformations can also be obtained by considering the invariance of the boundary value problem with respect to a one-parameter continuous group of transformations (the Lie point symmetries of the boundary value problem). This is a significant result because it seldom happens that we are able to calculate group invariant solutions of boundary value problems.

We were able to overcome the obstacles that usually occur when we look for group invariant solutions of a boundary value problem by solving the determining equation for the Lie point symmetries subject to a side condition. This side condition is a consequence of the boundary condition that occurs in the boundary value problem. We were thus able to develop a new method for solving the boundary value problem.

The first step of this method is to express the boundary value problem in terms of the coordinates $\bar{x}$ and $\bar{y}$ of an appropriate orthogonal curvilinear coordinate system. Solving the determining equation subject to the side condition, it will always be the case that the general form of the group invariant solution of the boundary value problem depends on a function $\beta(\bar{x}, \bar{y})$. This function must itself be a solution of the boundary value problem. If we let $\beta(\bar{x}, \bar{y}) = 0$ and we determine the corresponding group invariant solution, then we can replace the function $\beta(\bar{x}, \bar{y})$ with the group invariant solution thus found and proceed to calculate a new group invariant solution. Provided that we are always able to solve the resulting ordinary differential equations, this procedure can be continued indefinitely. Furthermore, any linear combination of these group invariant solutions will also be a group invariant solution of the boundary value problem. This is because Laplace's equation that occurs in the boundary value problem is a linear partial differential equation. Given our ability now to construct an almost limitless number of group invariant solutions of the boundary value problem, it seems reasonable to suppose that if we keep within the bounds of the assumptions made in the statement of the problem, group invariant solutions can be found that are capable of describing every feature associated with the flow of a fluid past a body.

We conclude this work by reiterating our earlier statement that although the method developed here applies to a particular boundary value problem, we hope that its success will spur on further developments that will allow us to use Lie point symmetries to calculate group invariant solutions of boundary value problems in general.

Appendix A

Given that the functions

$$\begin{aligned} x_j^* &= f_j\left(x_1,\ldots,x_n,\phi;a\right), && \text{(A.1)} \\ \phi^* &= g\left(x_1,\ldots,x_n,\phi;a\right) && \text{(A.2)} \end{aligned}$$

define a one-parameter continuous group of transformations of the dependent variable ϕ and the independent variables x_j where $j = 1, \ldots, n.$ In Chapter 2 we showed that the functions for transforming the partial derivatives $\phi_j \equiv \partial\phi/\partial x_j$, namely,

$$\phi_j^* = h_j\left(x_1,\ldots,x_n,\phi,\phi_1,\ldots,\phi_n;a\right), \tag{A.3}$$

are solutions of the matrix equation,

$$\begin{aligned} &\begin{bmatrix} D_1 f_1\left(x_1,\ldots,x_n,\phi;a\right) & \cdots & D_1 f_n\left(x_1,\ldots,x_n,\phi;a\right) \\ \vdots & & \vdots \\ D_n f_1\left(x_1,\ldots,x_n,\phi;a\right) & \cdots & D_n f_n\left(x_1,\ldots,x_n,\phi;a\right) \end{bmatrix} \begin{bmatrix} \phi_1^* \\ \vdots \\ \phi_n^* \end{bmatrix} \\ &= \begin{bmatrix} D_1 g\left(x_1,\ldots,x_n,\phi;a\right) \\ \vdots \\ D_n g\left(x_1,\ldots,x_n,\phi;a\right) \end{bmatrix}, \end{aligned} \tag{A.4}$$

where D_j is the total differential operator

$$D_j \equiv \frac{\partial}{\partial x_j} + \phi_j \frac{\partial}{\partial \phi}. \tag{A.5}$$

In this appendix we will show that functions (A.3) inherit the group property of the functions (A.1)-(A.2).

If

$$
\begin{aligned}
x_j^{**} &= f_j\left(x_1^*, \ldots, x_n^*, \phi^*; b\right), \qquad \text{(A.6)}\\
\phi^{**} &= g\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) \qquad \text{(A.7)}
\end{aligned}
$$

then it follows in an identical manner from Chapter 2 that the functions that will transform the partial derivatives $\phi_j^* \equiv \partial\phi^*/\partial x_j^*$, namely,

$$\phi_j^{**} = h_j\left(x_1^*, \ldots, x_n^*, \phi^*, \phi_1^*, \ldots, \phi_n^*; b\right), \qquad \text{(A.8)}$$

can be obtained by solving the matrix equation

$$
\begin{aligned}
&\begin{bmatrix} D_1^* f_1\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) & \cdots & D_1^* f_n\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) \\ \vdots & & \vdots \\ D_n^* f_1\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) & \cdots & D_n^* f_n\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) \end{bmatrix} \begin{bmatrix} \phi_1^{**} \\ \vdots \\ \phi_n^{**} \end{bmatrix} \\
&= \begin{bmatrix} D_1^* g\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) \\ \vdots \\ D_n^* g\left(x_1^*, \ldots, x_n^*, \phi^*; b\right) \end{bmatrix},
\end{aligned} \qquad \text{(A.9)}
$$

where D_j^* is the total differential operator

$$D_j^* \equiv \frac{\partial}{\partial x_j^*} + \phi_j^* \frac{\partial}{\partial \phi^*}. \qquad \text{(A.10)}$$

By the chain rule,

$$D_k x_j^{**} = \left(D_1^* x^{**}\right)\left(D_k x_1^*\right) + \cdots + \left(D_n^* x^{**}\right)\left(D_k x_n^*\right), \qquad \text{(A.11)}$$

where $k = 1, \ldots, n$. Therefore, multiplying both sides of the matrix equation (A.9) with the matrix

$$\begin{bmatrix} D_1 x_1^* & \cdots & D_1 x_n^* \\ \vdots & & \vdots \\ D_n x_1^* & \cdots & D_n x_n^* \end{bmatrix}, \qquad \text{(A.12)}$$

it follows that

$$\begin{aligned}&\begin{bmatrix} D_1 f_1\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) & \cdots & D_1 f_n\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) \\ \vdots & & \vdots \\ D_n f_1\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) & \cdots & D_n f_n\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) \end{bmatrix} \begin{bmatrix} \phi_1^{**} \\ \vdots \\ \phi_n^{**} \end{bmatrix} \\ &= \begin{bmatrix} D_1 g\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) \\ \vdots \\ D_n g\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) \end{bmatrix}, \end{aligned} \tag{A.13}$$

Suppose now that the functions (A.1)-(A.2) have the property that

$$\begin{aligned} x_j^{**} &= f_j\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) \\ &= f_j\left(f_1\left(x_1,\ldots,x_n,\phi;a\right),\ldots,f_n\left(x_1,\ldots,x_n,\phi;a\right),g\left(x_1,\ldots,x_n,\phi;a\right);b\right) \\ &= f_j\left(x_1,\ldots,x_n,\phi;a+b\right) \end{aligned} \tag{A.14}$$

and

$$\begin{aligned} \phi^{**} &= g\left(x_1^*,\ldots,x_n^*,\phi^*;b\right) \\ &= g\left(f_1\left(x_1,\ldots,x_n,\phi;a\right),\ldots,f_n\left(x_1,\ldots,x_n,\phi;a\right),g\left(x_1,\ldots,x_n,\phi;a\right);b\right) \\ &= g\left(x_1,\ldots,x_n,\phi;a+b\right). \end{aligned} \tag{A.15}$$

If the functions (A.14)-(A.15) are substituted into the matrix equation (A.13), then

$$\begin{aligned}&\begin{bmatrix} D_1 f_1\left(x_1,\ldots,x_n,\phi;a+b\right) & \cdots & D_1 f_n\left(x_1,\ldots,x_n,\phi;a+b\right) \\ \vdots & & \vdots \\ D_n f_1\left(x_1,\ldots,x_n,\phi;a+b\right) & \cdots & D_n f_n\left(x_1,\ldots,x_n,\phi;a+b\right) \end{bmatrix} \begin{bmatrix} \phi_1^{**} \\ \vdots \\ \phi_n^{**} \end{bmatrix} \\ &= \begin{bmatrix} D_1 g\left(x_1,\ldots,x_n,\phi;a+b\right) \\ \vdots \\ D_n g\left(x_1,\ldots,x_n,\phi;a+b\right) \end{bmatrix}. \end{aligned} \tag{A.16}$$

Comparing the matrix equation (A.16) with the matrix equation (A.4) it follows at once from the solution (A.3) of the matrix equation (A.4) that the solution of the matrix equation (A.16) is

$$\phi_j^{**} = h_j\left(x_1,\ldots,x_n,\phi,\phi_1,\ldots,\phi_n;a+b\right). \tag{A.17}$$

Hence, the function (A.3) inherits the group property of the functions (A.1)-(A.2).

Appendix B

It was shown in Chapter 3 that the general Laplace equation in a curvilinear coordinate system is

$$a(\bar{x},\bar{y})\frac{\partial^2\phi}{\partial\bar{x}^2}+b(\bar{x},\bar{y})\frac{\partial^2\phi}{\partial\bar{y}^2}+2k(\bar{x},\bar{y})\frac{\partial^2\phi}{\partial\bar{x}\partial\bar{y}}+d(\bar{x},\bar{y})\frac{\partial\phi}{\partial\bar{x}}+e(\bar{x},\bar{y})\frac{\partial\phi}{\partial\bar{y}}=0. \tag{B.1}$$

In this appendix we will calculate the Lie point symmetries of the general Laplace equation. However, the calculations will be less cumbersome if we write the general Laplace equation in the form

$$\frac{\partial^2\phi}{\partial\bar{x}^2}+B(\bar{x},\bar{y})\frac{\partial^2\phi}{\partial\bar{y}^2}+C(\bar{x},\bar{y})\frac{\partial^2\phi}{\partial\bar{x}\partial\bar{y}}+D(\bar{x},\bar{y})\frac{\partial\phi}{\partial\bar{x}}+E(\bar{x},\bar{y})\frac{\partial\phi}{\partial\bar{y}}=0 \tag{B.2}$$

where

$$\begin{aligned}B(\bar{x},\bar{y}) &= \frac{b(\bar{x},\bar{y})}{a(\bar{x},\bar{y})}, & C(\bar{x},\bar{y}) &= \frac{2k(\bar{x},\bar{y})}{a(\bar{x},\bar{y})},\\ D(\bar{x},\bar{y}) &= \frac{d(\bar{x},\bar{y})}{a(\bar{x},\bar{y})}, & E(\bar{x},\bar{y}) &= \frac{e(\bar{x},\bar{y})}{a(\bar{x},\bar{y})}.\end{aligned} \tag{B.3}$$

To calculate the Lie point symmetries of the general Laplace equation we must solve the determining equation

$$\begin{gathered}X^{(2)}\left[\phi_{\bar{x}\bar{x}}+B(\bar{x},\bar{y})\phi_{\bar{y}\bar{y}}+C(\bar{x},\bar{y})\phi_{\bar{x}\bar{y}}+D(\bar{x},\bar{y})\phi_{\bar{x}}+E(\bar{x},\bar{y})\phi_{\bar{y}}\right]=0\\ \text{when}\\ \phi_{\bar{x}\bar{x}}+B(\bar{x},\bar{y})\phi_{\bar{y}\bar{y}}+C(\bar{x},\bar{y})\phi_{\bar{x}\bar{y}}+D(\bar{x},\bar{y})\phi_{\bar{x}}+E(\bar{x},\bar{y})\phi_{\bar{y}}=0\end{gathered} \tag{B.4}$$

for the infinitesimals

$$\xi_1 = \xi_1(\bar{x},\bar{y},\phi), \tag{B.5}$$

$$\xi_2 = \xi_2(\bar{x},\bar{y},\phi), \tag{B.6}$$

$$\eta = \eta(\bar{x},\bar{y},\phi) \tag{B.7}$$

where the operator $X^{(2)}$ is defined to be

$$\begin{aligned}
X^{(2)} \equiv\; & \xi_1\left(\bar{x},\bar{y},\phi\right)\frac{\partial}{\partial\bar{x}}+\xi_2\left(\bar{x},\bar{y},\phi\right)\frac{\partial}{\partial\bar{y}}+\eta\left(\bar{x},\bar{y},\phi\right)\frac{\partial}{\partial\phi}\\
&+\eta_1\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right)\frac{\partial}{\partial\phi_{\bar{x}}}+\eta_2\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right)\frac{\partial}{\partial\phi_{\bar{y}}}\\
&+\eta_{11}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right)\frac{\partial}{\partial\phi_{\bar{x}\bar{x}}}+\eta_{12}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right)\frac{\partial}{\partial\phi_{\bar{x}\bar{y}}}\\
&+\eta_{22}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right)\frac{\partial}{\partial\phi_{\bar{y}\bar{y}}}
\end{aligned}\tag{B.8}$$

and the coefficients of the partial derivatives that occur in the operator $X^{(2)}$ are calculated according to the formula (2.48) that is given in Chapter 2; that is,

$$\eta_1\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right)=\frac{\partial\eta}{\partial\bar{x}}+\phi_{\bar{x}}\frac{\partial\eta}{\partial\phi}-\phi_{\bar{x}}\frac{\partial\xi_1}{\partial\bar{x}}-\phi_{\bar{x}}^2\frac{\partial\xi_1}{\partial\phi}-\phi_{\bar{y}}\frac{\partial\xi_2}{\partial\bar{x}}-\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial\xi_2}{\partial\phi},\tag{B.9}$$

$$\eta_2\left(\bar{x},\bar{y},\phi,\phi_{(1)}\right)=\frac{\partial\eta}{\partial\bar{y}}+\phi_{\bar{y}}\frac{\partial\eta}{\partial\phi}-\phi_{\bar{x}}\frac{\partial\xi_1}{\partial\bar{y}}-\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial\xi_1}{\partial\phi}-\phi_{\bar{y}}\frac{\partial\xi_2}{\partial\bar{y}}-\phi_{\bar{y}}^2\frac{\partial\xi_2}{\partial\phi},\tag{B.10}$$

$$\begin{aligned}
\eta_{11}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right) =\; & \frac{\partial^2\eta}{\partial\bar{x}^2}+2\phi_{\bar{x}}\frac{\partial^2\eta}{\partial\bar{x}\partial\phi}+\phi_{\bar{x}\bar{x}}\frac{\partial\eta}{\partial\phi}+\phi_{\bar{x}}^2\frac{\partial^2\eta}{\partial\phi^2}-2\phi_{\bar{x}\bar{x}}\frac{\partial\xi_1}{\partial\bar{x}}\\
&-\phi_{\bar{x}}\frac{\partial^2\xi_1}{\partial\bar{x}^2}-2\phi_{\bar{x}}^2\frac{\partial^2\xi_1}{\partial\bar{x}\partial\phi}-3\phi_{\bar{x}}\phi_{\bar{x}\bar{x}}\frac{\partial\xi_1}{\partial\phi}-\phi_{\bar{x}}^3\frac{\partial^2\xi_1}{\partial\phi^2}\\
&-2\phi_{\bar{x}\bar{y}}\frac{\partial\xi_2}{\partial\bar{x}}-\phi_{\bar{y}}\frac{\partial^2\xi_2}{\partial\bar{x}^2}-2\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial^2\xi_2}{\partial\bar{x}\partial\phi}-\phi_{\bar{y}}\phi_{\bar{x}\bar{x}}\frac{\partial\xi_2}{\partial\phi}\\
&-2\phi_{\bar{x}}\phi_{\bar{x}\bar{y}}\frac{\partial\xi_2}{\partial\phi}-\phi_{\bar{x}}^2\phi_{\bar{y}}\frac{\partial^2\xi_2}{\partial\phi^2},
\end{aligned}\tag{B.11}$$

$$\begin{aligned}
\eta_{12}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right) =\; & \frac{\partial^2\eta}{\partial\bar{x}\partial\bar{y}}+\phi_{\bar{y}}\frac{\partial^2\eta}{\partial\bar{x}\partial\phi}+\phi_{\bar{x}\bar{y}}\frac{\partial\eta}{\partial\phi}+\phi_{\bar{x}}\frac{\partial^2\eta}{\partial\bar{y}\partial\phi}+\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial^2\eta}{\partial\phi^2}\\
&-\phi_{\bar{x}\bar{y}}\frac{\partial\xi_1}{\partial\bar{x}}-\phi_{\bar{x}}\frac{\partial^2\xi_1}{\partial\bar{x}\partial\bar{y}}-\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial^2\xi_1}{\partial\bar{x}\partial\phi}-2\phi_{\bar{x}}\phi_{\bar{x}\bar{y}}\frac{\partial\xi_1}{\partial\phi}\\
&-\phi_{\bar{x}}^2\frac{\partial^2\xi_1}{\partial\bar{y}\partial\phi}-\phi_{\bar{x}}^2\phi_{\bar{y}}\frac{\partial^2\xi_1}{\partial\phi^2}-\phi_{\bar{x}\bar{x}}\frac{\partial\xi_1}{\partial\bar{y}}-\phi_{\bar{y}}\phi_{\bar{x}\bar{x}}\frac{\partial\xi_1}{\partial\phi}\\
&-\phi_{\bar{y}\bar{y}}\frac{\partial\xi_2}{\partial\bar{x}}-\phi_{\bar{y}}\frac{\partial^2\xi_2}{\partial\bar{x}\partial\bar{y}}-\phi_{\bar{y}}^2\frac{\partial^2\xi_2}{\partial\bar{x}\partial\phi}-\phi_{\bar{x}}\phi_{\bar{y}\bar{y}}\frac{\partial\xi_2}{\partial\phi}\\
&-\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial^2\xi_2}{\partial\bar{y}\partial\phi}-\phi_{\bar{x}}\phi_{\bar{y}}^2\frac{\partial^2\xi_2}{\partial\phi^2}-\phi_{\bar{x}\bar{y}}\frac{\partial\xi_2}{\partial\bar{y}}\\
&-2\phi_{\bar{x}\bar{y}}\phi_{\bar{y}}\frac{\partial\xi_2}{\partial\phi},
\end{aligned}\tag{B.12}$$

and

$$\begin{aligned}\eta_{22}\left(\bar{x},\bar{y},\phi,\phi_{(1)},\phi_{(2)}\right) &= \frac{\partial^2\eta}{\partial\bar{y}^2}+2\phi_{\bar{y}}\frac{\partial^2\eta}{\partial\bar{y}\partial\phi}+\phi_{\bar{y}\bar{y}}\frac{\partial\eta}{\partial\phi}+\phi_{\bar{y}}^2\frac{\partial^2\eta}{\partial\phi^2}-2\phi_{\bar{x}\bar{y}}\frac{\partial\xi_1}{\partial\bar{y}}\\ &\quad -\phi_{\bar{x}}\frac{\partial^2\xi_1}{\partial\bar{y}^2}-2\phi_{\bar{x}}\phi_{\bar{y}}\frac{\partial^2\xi_1}{\partial\bar{y}\partial\phi}-2\phi_{\bar{y}}\phi_{\bar{x}\bar{y}}\frac{\partial\xi_1}{\partial\phi}-\phi_{\bar{x}}\phi_{\bar{y}\bar{y}}\frac{\partial\xi_1}{\partial\phi}\\ &\quad -\phi_{\bar{x}}\phi_{\bar{y}}^2\frac{\partial^2\xi_1}{\partial\phi^2}-2\phi_{\bar{y}\bar{y}}\frac{\partial\xi_2}{\partial\bar{y}}-\phi_{\bar{y}}\frac{\partial^2\xi_2}{\partial\bar{y}^2}-2\phi_{\bar{y}}^2\frac{\partial^2\xi_2}{\partial\bar{y}\partial\phi}\\ &\quad -3\phi_{\bar{y}}\phi_{\bar{y}\bar{y}}\frac{\partial\xi_2}{\partial\phi}-\phi_{\bar{y}}^3\frac{\partial^2\xi_2}{\partial\phi^2}. \end{aligned} \tag{B.13}$$

In the determining equation (B.4) subscripts are used to denote the partial derivatives of ϕ; for example, the partial derivative $\partial^2\phi/\partial\bar{x}^2$ is denoted as $\phi_{\bar{x}\bar{x}}$. The purpose of this notation is to emphasise the fact that the partial derivatives of ϕ that occur in the determining equation are to be regarded as independent variables. Consequently, if we apply the operator (B.8) to the general Laplace equation and we replace $\phi_{\bar{x}\bar{x}}$ with

$$-B\left(\bar{x},\bar{y}\right)\phi_{\bar{y}\bar{y}}-C\left(\bar{x},\bar{y}\right)\phi_{\bar{x}\bar{y}}-D\left(\bar{x},\bar{y}\right)\phi_{\bar{x}}-E\left(\bar{x},\bar{y}\right)\phi_{\bar{y}},$$

then to satisfy the determining equation the coefficients of all the partial derivatives of ϕ must be set equal to zero. The result thereof is the following over-determined system of partial differential equations for the infinitesimals (B.5)-(B.7):

$$\frac{\partial^2\xi_1}{\partial\phi^2} = 0, \tag{B.14}$$

$$B\frac{\partial^2\xi_2}{\partial\phi^2} = 0, \tag{B.15}$$

$$C\frac{\partial^2\xi_1}{\partial\phi^2}+\frac{\partial^2\xi_2}{\partial\phi^2} = 0, \tag{B.16}$$

$$B\frac{\partial^2\xi_1}{\partial\phi^2}+C\frac{\partial^2\xi_2}{\partial\phi^2} = 0, \tag{B.17}$$

$$C\frac{\partial\xi_1}{\partial\phi}-2\frac{\partial\xi_2}{\partial\phi} = 0, \tag{B.18}$$

$$2B\frac{\partial\xi_1}{\partial\phi}-C\frac{\partial\xi_2}{\partial\phi} = 0, \tag{B.19}$$

$$\left(-2B+C^2\right)\frac{\partial\xi_1}{\partial\phi}-C\frac{\partial\xi_2}{\partial\phi} = 0, \tag{B.20}$$

$$BC\frac{\partial\xi_1}{\partial\phi}-2B\frac{\partial\xi_2}{\partial\phi} = 0, \tag{B.21}$$

$$\begin{aligned}
-2\frac{\partial^2\xi_1}{\partial\bar{x}\partial\phi} - C\frac{\partial^2\xi_1}{\partial\bar{y}\partial\phi} + 2D\frac{\partial\xi_1}{\partial\phi} + \frac{\partial^2\eta}{\partial\phi^2} &= 0, & \text{(B.22)}\\
-C\frac{\partial^2\xi_1}{\partial\bar{x}\partial\phi} - 2B\frac{\partial^2\xi_1}{\partial\bar{y}\partial\phi} + (CD + 2E)\frac{\partial\xi_1}{\partial\phi} & & \\
-2\frac{\partial^2\xi_2}{\partial\bar{x}\partial\phi} - C\frac{\partial^2\xi_2}{\partial\bar{y}\partial\phi} + C\frac{\partial^2\eta}{\partial\phi^2} &= 0, & \text{(B.23)}\\
CE\frac{\partial\xi_1}{\partial\phi} - C\frac{\partial^2\xi_2}{\partial\bar{x}\partial\phi} - 2B\frac{\partial^2\xi_2}{\partial\bar{y}\partial\phi} + B\frac{\partial^2\eta}{\partial\phi^2} &= 0, & \text{(B.24)}\\
C\frac{\partial\xi_1}{\partial\bar{x}} + \left(-2B + C^2\right)\frac{\partial\xi_1}{\partial\bar{y}} + \xi_1\frac{\partial C}{\partial\bar{x}} - 2\frac{\partial\xi_2}{\partial\bar{x}} - C\frac{\partial\xi_2}{\partial\bar{y}} + \xi_2\frac{\partial C}{\partial\bar{y}} &= 0, & \text{(B.25)}\\
2B\frac{\partial\xi_1}{\partial\bar{x}} + BC\frac{\partial\xi_1}{\partial\bar{y}} + \xi_1\frac{\partial B}{\partial\bar{x}} - C\frac{\partial\xi_2}{\partial\bar{x}} - 2B\frac{\partial\xi_2}{\partial\bar{y}} + \xi_2\frac{\partial B}{\partial\bar{y}} &= 0, & \text{(B.26)}\\
-\frac{\partial^2\xi_1}{\partial\bar{x}^2} - C\frac{\partial^2\xi_1}{\partial\bar{x}\partial\bar{y}} - B\frac{\partial^2\xi_1}{\partial\bar{y}^2} + D\frac{\partial\xi_1}{\partial\bar{x}} + (CD - E)\frac{\partial\xi_1}{\partial\bar{y}} + \xi_1\frac{\partial D}{\partial\bar{x}} & & \\
+\xi_2\frac{\partial D}{\partial\bar{y}} + 2\frac{\partial^2\eta}{\partial\bar{x}\partial\phi} + C\frac{\partial^2\eta}{\partial\bar{y}\partial\phi} &= 0, & \text{(B.27)}\\
-\frac{\partial^2\xi_2}{\partial\bar{x}^2} - C\frac{\partial^2\xi_2}{\partial\bar{x}\partial\bar{y}} - B\frac{\partial^2\xi_2}{\partial\bar{y}^2} - D\frac{\partial\xi_2}{\partial\bar{x}} - E\frac{\partial\xi_2}{\partial\bar{y}} + \xi_2\frac{\partial E}{\partial\bar{y}} & & \\
+2E\frac{\partial\xi_1}{\partial\bar{x}} + CE\frac{\partial\xi_1}{\partial\bar{y}} + \xi_1\frac{\partial E}{\partial\bar{x}} + C\frac{\partial^2\eta}{\partial\bar{x}\partial\phi} + 2B\frac{\partial^2\eta}{\partial\bar{y}\partial\phi} &= 0, & \text{(B.28)}\\
\frac{\partial^2\eta}{\partial\bar{x}^2} + B\frac{\partial^2\eta}{\partial\bar{y}^2} + C\frac{\partial^2\eta}{\partial\bar{x}\partial\bar{y}} + D\frac{\partial\eta}{\partial\bar{x}} + E\frac{\partial\eta}{\partial\bar{y}} &= 0. & \text{(B.29)}
\end{aligned}$$

If we subtract the partial differential equation (B.19) from the partial differential equation (B.20), then we obtain the partial differential equation

$$\left(C^2 - 4B\right)\frac{\partial\xi_1}{\partial\phi} = 0. \qquad \text{(B.30)}$$

Since Laplace's equation is an elliptic partial differential equation, we must have that $C^2 - 4B < 0$ [18]. Therefore, from equation (B.30), $\partial\xi_1/\partial\phi = 0$, and then from equations (B.19) and (B.20), $\partial\xi_2/\partial\phi = 0$; thus,

$$\begin{aligned}
\xi_1 &= \xi_1(\bar{x}, \bar{y}), & \text{(B.31)}\\
\xi_2 &= \xi_2(\bar{x}, \bar{y}). & \text{(B.32)}
\end{aligned}$$

As a consequence of this result the partial differential equations (B.14), (B.15), (B.16), (B.17), (B.18) and (B.21) are identically satisfied and the partial differential equations (B.22), (B.23) and (B.24) reduce to $\partial^2\eta/\partial\phi^2 = 0$; thus,

$$\eta = \alpha\left(\bar{x},\bar{y}\right)\phi + \beta\left(\bar{x},\bar{y}\right). \tag{B.33}$$

Since the functions $\alpha\left(\bar{x},\bar{y}\right)$ and $\beta\left(\bar{x},\bar{y}\right)$ are independent of ϕ, it follows from the partial differential equation (B.29) that both these functions must satisfy the general Laplace equation; that is,

$$\frac{\partial^2\alpha}{\partial\bar{x}^2} + B\frac{\partial^2\alpha}{\partial\bar{y}^2} + C\frac{\partial^2\alpha}{\partial\bar{x}\partial\bar{y}} + D\frac{\partial\alpha}{\partial\bar{x}} + E\frac{\partial\alpha}{\partial\bar{y}} = 0, \tag{B.34}$$

$$\frac{\partial^2\beta}{\partial\bar{x}^2} + B\frac{\partial^2\beta}{\partial\bar{y}^2} + C\frac{\partial^2\beta}{\partial\bar{x}\partial\bar{y}} + D\frac{\partial\beta}{\partial\bar{x}} + E\frac{\partial\beta}{\partial\bar{y}} = 0. \tag{B.35}$$

Given a function $\alpha\left(\bar{x},\bar{y}\right)$ that solves the general Laplace equation (B.34), we must then solve the remaining partial differential equations (B.25), (B.26), (B.27) and (B.28) to determine the exact form of the infinitesimals $\xi_1\left(\bar{x},\bar{y}\right)$ and $\xi_2\left(\bar{x},\bar{y}\right)$.

Bibliography

[1] Anderson Jr, J.D., *A History of Aerodynamics.* Cambridge: Cambridge University Press, 1997.

[2] Aris, R., *Vectors, Tensors and the Basic Equations of Fluid Mechanics.* New York: Dover Publications, 1989.

[3] Bewersdorff, J., *Galois Theory for Beginners. A Historical Perspective.* Translated by D. Kramer, Providence: American Mathematical Society, 2006.

[4] Bloor, D., *The Enigma of the Aerofoil. Rival Theories in Aerodynamics, 1909-1930.* Chicago: The University of Chicago Press, 2011.

[5] Bluman, G.W. and Cole, J.D., The General Similarity Solution of the Heat Equation. *Journal of Mathematics and Mechanics,* **18**, 1969, pp. 1025-1042.

[6] Bluman G.W. and Kumei, S., *Symmetries and Differential Equations.* New York: Springer-Verlag, 1989.

[7] Courant, R., *Differential and Integral Calculus.* Translated by E. J. McShane, London: Blackie and Son Limited, 1964.

[8] Eisenhart, L.P., *Continuous Groups of Transformations.* New York: Dover Publications, 2003.

[9] Forsyth, A.R., *A Treatise on Differential Equations. 6th Edition.* London: Macmillin and Co., 1933.

[10] Goard, J., Noninvariant Boundary Conditions. *Applicable Analysis,* **82**, 2003, pp. 473-481.

[11] Goard, J., Finding Symmetries by Incorporating Initial Conditions as Side Conditions. *European Journal of Applied Mathematics,* **19**, 2008, pp. 701-715.

[12] Granger, R.A., *Fluid Mechanics.* New York: Dover Publications, 1995.

[13] Ibragimov, N.H., *Elementary Lie Group Analysis and Ordinary Differential Equations.* Chichester: John Wiley and Sons, 1999.

[14] Ibragimov, N.H. ed, *CRC Handbook of Lie Group Analysis of Differential Equations. Volume 1.* Boca Raton: CRC Press, 1994.

[15] Ibragimov, N.H. ed, *CRC Handbook of Lie Group Analysis of Differential Equations. Volume 2.* Boca Raton: CRC Press, 1995.

[16] Klein, F., *Elementary Mathematics from an Advanced Standpoint. Geometry.* Translated by E.R. Hedrick and C.A. Noble, New York: Dover Publications, 1939.

[17] Marsden, J.E. and Tromba A.J., *Vector Calculus. 4th Edition.* New York: W.H. Freeman and Company, 1996.

[18] Mason, D.P., *Lecture Notes on Partial Differential Equations.* University of the Witwatersrand, Johannesburg, 1997.

[19] Mathews, J.H. and Howell, R.W., *Complex Analysis for Mathematics and Engineering. 4th Edition.* Boston: Jones and Bartlett Publishers, 2001.

[20] Milne-Thomson, L.M., *Theoretical Hydrodynamics. 5th Edition.* New York: Dover Publications, 1996.

[21] Olver P.J. and Rosenau P., Group-Invariant Solutions of Differential Equations. *SIAM Journal on Applied Mathematics,* **47**, 1987, pp. 263-278.

[22] Pereira, K.P., Transformation Groups Applied to Two-Dimensional Boundary Value Problems in Fluid Mechanics. *Journal of Nonlinear Mathematical Physics,* **15**, Supplement 1, 2008, pp. 192-202.

[23] Salas, S.L. and Hille, E., *Calculus. 6th Edition.* New York: John Wiley and Sons, 1990.

[24] Sokolnikoff, I.S., *Tensor Analysis. Theory and Applications.* New York: John Wiley and Sons, 1951.

[25] Yaglom, I.M., *Felix Klein and Sophus Lie. Evolution of the Idea of Symmetry in the Nineteenth Century.* Translated by S. Sossinsky and edited by H. Grant and A. Shenitzer, Boston: Birkhäuser, 1988.

[26] Zill, D.G., *A First Course in Differential Equations with Applications. 3rd Edition.* Boston: Prindle, Weber and Schmidt, 1986.