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Research paper

A new inertial condition on the subgradient extragradient method for solving pseudomonotone equilibrium problem

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ABSTRACT

In this paper, we study the pseudomonotone equilibrium problem. We consider a new inertial condition for the subgradient extragradient method with self-adaptive step size for approximating a solution of the equilibrium problem in a real Hilbert space. Our proposed method contains inertial factor with new conditions that only depend on the iteration coefficient. We obtain a weak convergence result of the proposed method under weaker conditions on the inertial factor than many existing conditions in the literature. Finally, we present some numerical experiments for our proposed method and compared it to existing methods in the literature. Our result improves, extends, and generalizes several existing results in the literature.

1. Introduction

Let C be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. The equilibrium problem (EP) introduced by Blum and Oettli [1] is the problem of finding a point $x^* \in C$ such that

$$F(x^*, y) \geq 0, \ \forall \ y \in C, \tag{1.1}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. Any point $x^* \in C$ that solves this problem is called an equilibrium point of F. We denote by $EP(F, C)$ the solution set of Problem (1.1).

Definition 1.1. A bifunction $F : C \times C \to \mathbb{R}$ is said to be

(i) *strongly monotone on* C, if there exists a constant $c > 0$ such that

$$F(x, y) + F(y, x) \leq -c\|x - y\|^2, \quad \forall \ x, y \in C,$$

(ii) *monotone on* C, if

$$F(x, y) + F(y, x) \leq 0, \ \forall \ x, y \in C,$$

(iii) *pseudomonotone on* C, if

$$F(x, y) \geq 0 \implies F(y, x) \leq 0, \ \forall \ x, y \in C,$$

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(iv) satisfying a Lipschitz-like condition on C if there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(x, y) + F(y, w) \geq F(x, w) - a_1\|x - y\|^2 - a_2\|y - w\|^2, \quad \forall x, y, w \in C.$$

We observe that $(i) \implies (ii) \implies (iii)$ but the converses are not always true (see [2] and other references therein).

The EP (1.1) has received a lot of attention from several researchers due to the fact that it unifies in a simple form several mathematical models such as optimization problem, fixed point problem, convex minimization problem, Nash equilibrium, variational inequality problem, saddle point problem, among others (see [3–6] and other references therein). Many authors have proposed and studied several iterative methods for approximating solutions of EP (1.1) and other related optimization problems (see [7–12] and other references therein). In 1976, Kopelevich [13] introduced the extragradient method for solving saddle point problems. Quoc et al. [14] extended the extragradient method to solve the EP (1.1) in a finite dimensional space. This result was later extended to an infinite dimensional Hilbert space by Vinh and Muu [15]. They obtained a weak convergence result under the assumption that the equilibrium bifunction is pseudomonotone and satisfies the Lipschitz-like condition. When using this method, one needs to solve two strongly convex optimization problems in the feasible set C per iteration. This is a major drawback of the extragradient method and could cause the method to be computationally expensive if the set C is not simple. To circumvent this limitation, Rehman et al. [16] extended the subgradient extragradient method in [17] from solving variational inequalities to solving EP (1.1). The major advantage of the subgradient extragradient method over the extragradient method is that the second convex optimization problem is onto a half-space which has a closed form formula. Thus, its computational complexity is less expensive than the extragradient method.

The inertial technique which originated from the heavy ball method of a second order dissipative dynamical system in time was derived by Polyak [18]. It is one of the techniques often employed by authors to improve the convergence speed of iterative methods when solving optimization problems. This is because it increases the rate of convergence of iterative schemes. There has been an increased interest in studying inertial type algorithms for solving optimization problems (see [19–24]), and one key interest in these studies is how to improve the conditions on the inertial factor [19]. In 2003, Moudafi [24] proposed an inertial algorithm for solving the EP (1.1): Find $x_{n+1} \in C$ such that

$$F(x_{n+1}, x) + \lambda_n^{-1}\langle x_{n+1} - y_n, x - x_{n+1}\rangle \geq -\epsilon_n, \quad \forall \ \ x \in C,$$

where $y_n := x_n + \theta_n(x_n - x_{n-1})$, $\{\lambda_n\}, \{\epsilon_n\}$ are sequences of nonnegative real numbers and the inertial factor θ_n satisfies

$$0 \leq \theta_n \leq \theta < 1 \ \ \forall n \geq 1, \qquad \sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\|^2 < \infty. \tag{1.2}$$

Note that condition (1.2) involves the knowledge of the iterates x_n and x_{n-1} that are a priori unknown. However, it can be ensured in practice by using the suitable on-line rule: $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1} \\ \theta & \text{otherwise,} \end{cases} \tag{1.3}$$

with $\sum_{n=1}^{\infty} \epsilon_n < \infty$ and $\theta \in [0, 1)$. The on-line rule (1.3) which also depends on the knowledge of the iterates x_n and x_{n-1}, was considered in [15,16] for solving EP (1.1).

The relaxation technique is another essential technique used by authors to improve the convergence speed of iterative methods when solving optimization problems [22]. In other words, the relaxation and inertial techniques are the two very important ways for improving the convergence speed of iterative methods [23,25]. Both techniques arise naturally from an explicit time discretization of a dynamical system (see, for example, [26,27]). Their influence on the numerical performance of iterative methods was studied in [25]. Moreover, these two techniques have recently been incorporated into known methods in order to achieve a high convergence speed of the resulting modified methods (see [22,23,25,26,28]). It is shown in these papers that combining the relaxation and the inertial techniques gives a better rate of convergence when compared with only the relaxation technique or the inertial technique. In [29] (see also [30]), the authors introduced the following condition on the inertial factor θ_n and relaxation factor ϕ_n:

$$0 = \theta_1 \leq \theta_n \leq \theta_{n+1} \leq \theta < 1, \quad \forall n \geq 1,$$

$$\tau > \frac{\theta^2(1+\theta) + \theta\sigma}{1 - \theta^2}, \quad 0 \leq \phi \leq \phi_n \leq \frac{\tau - \theta[\theta(1+\theta) + \theta\tau + \sigma]}{\tau[1 + \theta(1+\theta) + \theta\tau + \sigma]}, \tag{1.4}$$

where $\sigma, \tau > 0$. Unlike in (1.2) and (1.3), condition (1.4) does not require any information on the iterates but on the coefficient ϕ_n (the relaxation factor) and other parameters. However, it is complicated to get the upper bound of the inertial sequence even if ϕ_n is known. We can also see that the inertial factor is restrictive in (1.4). For more relaxed inertial techniques of existing algorithms in the literature, see [19,22,23,26,28,31–35].

The main purpose of this paper is to consider an inertial factor with new conditions that only depend on the iteration coefficient ϕ_n, and where the upper bound of the inertial sequence is easy to determine. Combining these relaxed inertial terms (i.e, the terms with θ_n and ϕ_n) with the subgradient extragradient method, we propose a new method for solving the EP (1.1) when F is pseudomonotone. We prove that the proposed method converges weakly to a solution of EP (1.1). Furthermore, we present some numerical experiments for our proposed method and compare it with other related methods in the literature.

The rest of the paper is organized as follows: In Section 2 we recall some basic definitions and results required for our convergence analysis. Section 3 presents and discusses the features of our proposed method. In Section 4, we study the convergence of this method. In Section 5, we carry out some numerical experiments of our method in comparison with other methods in the literature. We conclude in Section 6.

2. Preliminaries

In this section, we recall some lemmas and definitions that will be needed in the subsequent sections. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and associated norm $\| \cdot \|$ defined by $\|x\| = \sqrt{\langle x, x \rangle}, \quad \forall\, x \in \mathcal{H}$. We denote the weak convergence of a sequence $\{x_n\}$ to x^* by $x_n \rightharpoonup x^*$.

Definition 2.1. The domain of a function $F : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is defined by $D(F) = \{x \in \mathcal{H} : F(x) < \infty\}$. The function $F : D(F) \subseteq \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is said to be *lower semicontinuous at a point* $x \in D(F)$, if

$$F(x) \leq \liminf_{x_n \to x} F(x_n).$$

Definition 2.2. Let $F : \mathcal{H} \to (-\infty, \infty]$ be proper. The subdifferential of F at $x \in \mathcal{H}$ is

$$\partial F(x) = \{u \in \mathcal{H} |\ F(y) \geq F(x) + \langle y - x,\ u \rangle,\ \forall\ y \in \mathcal{H}\}.$$

The normal cone N_C of C at $x \in C$ is defined by

$$N_C(x) = \{w \in \mathcal{H} : \langle w, y - x \rangle \leq 0, \quad \forall\ y \in C\}.$$

Lemma 2.3 ([36]). *Let C be a nonempty closed and convex subset of $\mathcal{H}$ and $g : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous functions on $\mathcal{H}$. Assume either that g is continuous at some point of C, or that there is an interior point of C where g is finite. Then, $\bar{x}$ is a solution to the following convex problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(\bar{x}) + N_C(\bar{x})$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(\bar{x})$ is the normal cone of C at $\bar{x}$.*

Lemma 2.4 ([37]). *Let $\mathcal{H}$ be a real Hilbert space, then the following assertions hold:*

(1) $2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in \mathcal{H}$;
(2) $\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \quad \forall x, y \in \mathcal{H},\ \alpha \in \mathbb{R}$.

Lemma 2.5 ([38]). *Let C be a nonempty subset of $\mathcal{H}$ and let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that the following two conditions hold:*

(a) *for each $p \in C$, $\lim_{n\to\infty} \|x_n - p\|$ exists;*
(b) *every sequential weak cluster point of $\{x_n\}$ belongs to C.*

Then, $\{x_n\}$ converges weakly to a point in C.

Lemma 2.6 ([39]). *Let $\{\gamma_n\}, \{\psi_n\}$ and $\{t_n\}$ be nonnegative sequences. Assume that*

$$\gamma_{n+1} \leq \gamma_n + \psi_n(\gamma_n - \gamma_{n-1}) + t_n,$$

and $0 \leq \psi_n \leq \psi < 1$ and $\sum_{n=1}^{+\infty} t_n < +\infty$. Then, $\lim_{n\to+\infty} \gamma_n$ exists.

3. Proposed method

In this section, we present our proposed method. We begin by giving the following assumptions under which our weak convergence result is obtained.

Assumption 3.1. Let $F : C \times C \to \mathbb{R}$ be a function satisfying the following assumptions

(1) $F(x, x) = 0,\ \forall\ x \in C$;
(2) F is pseudomonotone on C;
(3) F satisfies the Lipschitz-like condition on $\mathcal{H}$ with constants a_1 and a_2;
(4) $F(x, \cdot)$ is convex, lower semicontinuous and subdifferential on C for every $x \in C$;
(5) $F(\cdot, y)$ is continuous on C for every $y \in C$.

Assumption 3.2. For all $n \geq 1$ and sufficiently small $\epsilon > 0$, let $0 = \theta_1 \leq \theta_n \leq \theta_{n+1}$ and:

(i) $\theta_{n+1} \le \beta_n$, if $\phi_n \in (0, 0.5)$, $\phi_n^{-1} - \phi_{n+1}^{-1} + 3 > 0$, where

$$\beta_n := \frac{1}{2}\frac{1}{\phi_{n+1}^{-1} - 2}\Big(\phi_n^{-1} + \phi_{n+1}^{-1} - 1 - \triangle_n\Big) \tag{3.1}$$

with

$$\triangle_n := \sqrt{(\phi_n^{-1} + \phi_{n+1}^{-1} - 1)^2 - 4(\phi_n^{-1} - 1 - \epsilon)(\phi_{n+1}^{-1} - 2)}. \tag{3.2}$$

(ii) $\theta_{n+1} \le \frac{1-\epsilon}{3}$, if $\phi_n \equiv 0.5$.

(iii) $\theta_{n+1} \le \sqrt{p_n^2 + q_n} - p_n$, if $\phi_n \in (0.5, 1-\epsilon]$, where

$$p_n := \frac{1}{2}\frac{1}{2 - \phi_{n+1}^{-1}}\Big(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\Big) \tag{3.3}$$

and

$$q_n := \frac{1}{2 - \phi_{n+1}^{-1}}\Big(\phi_n^{-1} - 1 - \epsilon\Big). \tag{3.4}$$

Algorithm 3.3. Relaxed inertial subgradient extragradient method with adaptive step size strategy.

Step 0: Choose initial points $x_0, x_1 \in \mathcal{H}$, let $\lambda_1 > 0, \mu \in (0,1)$ and set $n = 1$.

Step 1: Given the current iterates x_{n-1} and x_n $(n \ge 1)$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

and

$$y_n = \arg\min\Big\{\lambda_n F(w_n, y) + \frac{1}{2}\|w_n - y\|^2 : \ y \in C\Big\}.$$

If $y_n = w_n$: STOP. Otherwise, go to **Step 2.**

Step 2: Choose $\omega_n \in \partial F(w_n, y_n)$ and $w^* \in N_C(y_n)$ such that $w^* = w_n - \lambda_n\omega_n - y_n$ and construct the half-space

$$T_n = \{x \in \mathcal{H} : \langle w_n - \lambda_n\omega_n - y_n,\ x - y_n\rangle \le 0\}.$$

Then, compute

$$z_n = \arg\min\Big\{\lambda_n F(y_n, y) + \frac{1}{2}\|w_n - y\|^2 : \ y \in T_n\Big\}.$$

STEP 3: Compute

$$x_{n+1} = (1-\phi_n)w_n + \phi_n z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\Big\{\frac{\mu(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(F(w_n,z_n) - F(w_n,y_n) - F(y_n,z_n))},\ \lambda_n\Big\}, & \text{if } F(w_n,z_n) - F(w_n,y_n) - F(y_n,z_n) > 0,\\ \lambda_n, & \text{otherwise.}\end{cases} \tag{3.5}$$

Set $n := n + 1$ and return to **Step 1.**

Remark 3.4.

(a). $\phi_{n+1}^{-1} - 2 > 0$ in (i) since $\theta_{n+1} < \frac{1}{2}$. Similarly, $2 - \phi_{n+1}^{-1} > 0$ in (iii).

(b). The sequence $\{\triangle_n\}$ defined in (3.2) is well-defined. Indeed,

$$\begin{aligned} &(\phi_n^{-1} + \phi_{n+1}^{-1} - 1)^2 - 4(\phi_{n+1}^{-1} - 2)(\phi_n^{-1} - 1 - \epsilon)\\ =\ &\Big(\phi_n^{-1} - \phi_{n+1}^{-1}\Big)^2 + 6\phi_n^{-1} + 2\phi_{n+1}^{-1} + 4\Big(\phi_{n+1}^{-1} - 2\Big)\epsilon - 7\\ >\ &0. \end{aligned}$$

Therefore, Assumption 3.2 is valid.

Remark 3.5. In contrast to the assumptions in [15,16,24], (3.2) does not require the knowledge of the iterates. Also, unlike in [29], the choice of θ_n is relaxed and its upper bound is easy to obtain; once ϕ_n is chosen, it becomes very easy to compute θ_n.

Lemma 3.6 (*[40]*). *The sequence $\{\lambda_n\}$ generated by Algorithm 3.3 is a monotonically decreasing sequence with* $\min\left\{\frac{\mu}{2\max\{a_1,a_2\},\lambda_1}\right\}$ *as lower bound.*

Lemma 3.7 (*[41]*). *Let $\{z_n\}$ be a sequence generated by Algorithm 3.3 under Assumption 3.1. Then, for each $\bar{w} \in EP(F,C)$, the following inequality holds:*

$$\|z_n - \bar{w}\|^2 \leq \|w_n - \bar{w}\|^2 - \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}}\right)\left[\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right].$$

4. Convergence analysis

Lemma 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 under Assumptions 3.1 and 3.2. Then, for $\bar{w} \in EP(F,C)$,*

$$\Gamma_{n+1} \leq \Gamma_n - \epsilon\|x_{n+1} - x_n\|^2,$$

where $\Gamma_n = \|x_n - \bar{w}\|^2 - \theta_n\|x_{n-1} - \bar{w}\|^2 + \delta_n\|x_n - x_{n-1}\|^2$ and $\delta_n = (1+\theta_n)\theta_n + \phi_n^{-1}(1-\phi_n)(1-\theta_n)\theta_n$.

Proof. Let $\bar{w} \in EP(F,C)$. From the definition of w_n in Step 1 and Lemma 2.4 (2), we have

$$\begin{aligned}\|w_n - \bar{w}\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - \bar{w}\|^2\\ &= \|(1+\theta_n)(x_n - \bar{w}) - \theta_n(x_{n-1} - \bar{w})\|^2\\ &= (1+\theta_n)\|x_n - \bar{w}\|^2 - \theta_n\|x_{n-1} - \bar{w}\|^2 + (1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{4.1}$$

Also, from the definition of x_{n+1} and Lemma 3.7, we have

$$\begin{aligned}\|x_{n+1} - \bar{w}\|^2 &= \|(1-\phi_n)w_n + \phi_n z_n - \bar{w}\|^2\\ &= \|(1-\phi_n)(w_n - \bar{w}) + \phi_n(z_n - \bar{w})\|^2\\ &= (1-\phi_n)\|w_n - \bar{w}\|^2 + \phi_n\|z_n - \bar{w}\|^2 - \phi_n(1-\phi_n)\|z_n - w_n\|^2\\ &\leq (1-\phi_n)\|w_n - \bar{w}\|^2 + \phi_n\|w_n - \bar{w}\|^2 - \phi_n\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\left[\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right]\\ &\quad - \phi_n(1-\phi_n)\|z_n - w_n\|^2\\ &= \|w_n - \bar{w}\|^2 - \phi_n\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\left[\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right] - \phi_n(1-\phi_n)\|z_n - w_n\|^2.\end{aligned}$$

By Lemma 3.6, we have

$$\lim_{n\to\infty}\left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right) = 1 - \mu > 0. \tag{4.2}$$

Thus, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $\left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right) > 0$. Hence,

$$\|x_{n+1} - \bar{w}\|^2 \leq \|w_n - \bar{w}\|^2 - \phi_n(1-\phi_n)\|z_n - w_n\|^2. \tag{4.3}$$

From the definition of x_{n+1}, we have

$$z_n - w_n = \phi_n{}^{-1}(x_{n+1} - w_n).$$

Thus,

$$\begin{aligned}\|z_n - w_n\|^2 &= \phi_n^{-2}\|x_{n+1} - w_n\|^2\\ &= \phi_n^{-2}\|x_{n+1} - x_n - (x_n - x_{n-1}) + (1-\theta_n)(x_n - x_{n-1})\|^2\\ &= \phi_n^{-2}\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \phi_n^{-2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2\\ &\quad + 2\phi_n^{-2}(1-\theta_n)\langle x_{n+1} - 2x_n + x_{n-1}, x_n - x_{n-1}\rangle\\ &= \phi_n^{-2}\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \phi_n^{-2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2\\ &\quad + \phi_n^{-2}(1-\theta_n)\left[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2 - \|x_{n+1} - 2x_n + x_{n-1}\|^2\right]\\ &= \phi_n^{-2}\theta_n\|x_{n+1} - 2x_n + x_{n-1}\|^2 + \phi_n^{-2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2\\ &\quad + \phi_n^{-2}(1-\theta_n)\left[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2\right]\\ &\geq \phi_n^{-2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 + \phi_n^{-2}(1-\theta_n)\Big[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2\Big].\end{aligned} \tag{4.4}$$

Substituting (4.1) and (4.4) into (4.3), we have

$$\begin{aligned}\|x_{n+1} - \bar{w}\|^2 &\leq (1+\theta_n)\|x_n - \bar{w}\|^2 - \theta_n\|x_{n-1} - \bar{w}\|^2 + (1+\theta_n)\theta_n\|x_n - x_{n-1}\|^2\\ &\qquad - \phi_n(1-\phi_n)\Big(\phi_n^{-2}(1-\theta_n)^2\|x_n - x_{n-1}\|^2 + \phi_n^{-2}(1-\theta_n)\Big[\|x_{n+1} - x_n\|^2 - \|x_n - x_{n-1}\|^2\Big]\Big)\end{aligned}$$

$$= \|x_n - \bar{w}\|^2 + \theta_n(\|x_n - \bar{w}\|^2 - \|x_{n-1} - \bar{w}\|^2) - \phi_n^{-1}(1-\phi_n)(1-\theta_n)\|x_{n+1} - x_n\|^2 + \delta_n\|x_n - x_{n-1}\|^2, \quad (4.5)$$

where $\delta_n = (1+\theta_n)\theta_n + \phi_n^{-1}(1-\phi_n)(1-\theta_n)\theta_n$.

This implies that

$$\|x_{n+1} - \bar{w}\|^2 - \|x_n - \bar{w}\|^2 - \theta_n(\|x_n - \bar{w}\|^2 - \|x_{n-1} - \bar{w}\|^2) - \delta_n\|x_n - x_{n-1}\|^2 + \delta_{n+1}\|x_{n+1} - x_n\|^2 \leq -\left(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\right)\|x_{n+1} - x_n\|^2. \quad (4.6)$$

Using the fact that $\theta_n \leq \theta_{n+1}$ and (4.6), we have

$$\begin{aligned} -\left(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\right)\|x_{n+1} - x_n\|^2 &\geq \|x_{n+1} - \bar{w}\|^2 - \|x_n - \bar{w}\|^2 - \theta_n(\|x_n - \bar{w}\|^2 - \|x_{n-1} - \bar{w}\|^2) \\ &\quad - \delta_n\|x_n - x_{n-1}\|^2 + \delta_{n+1}\|x_{n+1} - x_n\|^2 \\ &\geq \|x_{n+1} - \bar{w}\|^2 - \|x_n - \bar{w}\|^2 - \theta_{n+1}\|x_n - \bar{w}\|^2 \\ &\quad + \theta_n\|x_{n-1} - \bar{w}\|^2 - \delta_n\|x_n - x_{n-1}\|^2 + \delta_{n+1}\|x_{n+1} - x_n\|^2, \end{aligned}$$

which implies that

$$\Gamma_{n+1} \leq \Gamma_n - \left(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\right)\|x_{n+1} - x_n\|^2, \quad (4.7)$$

where $\Gamma_n = \|x_n - \bar{w}\|^2 - \theta_n\|x_{n-1} - \bar{w}\|^2 + \delta_n\|x_n - x_{n-1}\|^2$.

Now, observe for $\epsilon > 0$, we have

$$\begin{aligned} \phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1} - \epsilon &= \phi_n^{-1}(1-\phi_n)(1-\theta_n) - \theta_{n+1} - \theta_{n+1}^2 \\ &\quad + \phi_{n+1}^{-1}(1-\phi_{n+1})(\theta_{n+1} - 1)\theta_{n+1} - \epsilon \\ &\geq \phi_n^{-1}(1-\phi_n)(1-\theta_{n+1}) - \theta_{n+1} - \theta_{n+1}^2 \\ &\quad + \phi_{n+1}^{-1}(1-\phi_{n+1})(\theta_{n+1} - 1)\theta_{n+1} - \epsilon \\ &= -\left(2 - \phi_{n+1}^{-1}\right)\theta_{n+1}^2 - \left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\right)\theta_{n+1} + \phi_n^{-1} - 1 - \epsilon. \end{aligned} \quad (4.8)$$

Now, we consider three cases.

Case 1: Suppose $\phi_n \in (0, 0.5)$. Then, from the condition $\theta_{n+1} \leq \frac{1}{2}\frac{1}{\phi_{n+1}^{-1} - 2}\left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1 - \triangle_n\right)$ in (3.1), we get

$$\triangle_n \leq \phi_n^{-1} + \phi_{n+1}^{-1} - 1 - 2(\phi_{n+1}^{-1} - 2)\theta_{n+1}.$$

By Remark 3.4(b), we have that $\triangle_n \geq 0$. Hence,

$$\begin{aligned} \triangle_n^2 &\leq \left[\left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\right) - \left(2(\phi_{n+1}^{-1} - 2)\theta_{n+1}\right)\right]^2 \\ &= (\phi_n^{-1} + \phi_{n+1}^{-1} - 1)^2 - 4(\phi_n^{-1} + \phi_{n+1}^{-1} - 1)(\phi_{n+1}^{-1} - 2)\theta_{n+1} + 4(\phi_{n+1}^{-1} - 2)^2\theta_{n+1}^2. \end{aligned}$$

That is,

$$\begin{aligned} &(\phi_n^{-1} + \phi_{n+1}^{-1} - 1)^2 - 4(\phi_n^{-1} - 1 - \epsilon)(\phi_{n+1}^{-1} - 2) \\ &\leq (\phi_n^{-1} + \phi_{n+1}^{-1} - 1)^2 - 4(\phi_n^{-1} + \phi_{n+1}^{-1} - 1)(\phi_{n+1}^{-1} - 2)\theta_{n+1} + 4(\phi_{n+1}^{-1} - 2)^2\theta_{n+1}^2. \end{aligned}$$

Hence,

$$-\left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\right)(\phi_{n+1}^{-1} - 2)\theta_{n+1} + (\phi_{n+1}^{-1} - 2)^2\theta_{n+1}^2 + (\phi_n^{-1} - 1 - \epsilon)(\phi_{n+1}^{-1} - 2) \geq 0.$$

Since $\phi_{n+1}^{-1} - 2 > 0$, we obtain

$$-\left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\right)\theta_{n+1} + (\phi_{n+1}^{-1} - 2)\theta_{n+1}^2 + (\phi_n^{-1} - 1 - \epsilon) \geq 0.$$

That is,

$$-(2 - \phi_{n+1}^{-1})\theta_{n+1}^2 - \left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\right)\theta_{n+1} + \phi_n^{-1} - 1 - \epsilon \geq 0. \quad (4.9)$$

Using (4.9) in (4.8), we get

$$\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1} - \epsilon \geq -(2 - \phi_{n+1}^{-1})\theta_{n+1}^2 - \left(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\right)\theta_{n+1} + \phi_n^{-1} - 1 - \epsilon \geq 0,$$

which implies

$$-\left(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\right) \leq -\epsilon.$$

Case 2: Suppose $\phi_n \equiv 0.5$. Then from (4.8), we obtain

$$\begin{aligned}\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1} - \epsilon &\geq -3\theta_{n+1} + 1 - \epsilon \\ &\geq 0,\end{aligned}$$

since by Assumption 3.2 (ii), $\theta_{n+1} \leq \frac{1-\epsilon}{3}$. Hence,

$$-\Big(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\Big) \leq -\epsilon.$$

Case 3: Suppose $\phi_n \in (0.5, 1-\epsilon]$. Then from the condition $\theta_{n+1} \leq \sqrt{p_n^2 + q_n} - p_n$, we have

$$\Big(\theta_{n+1} + p_n\Big)^2 \leq p_n^2 + q_n,$$

which implies that

$$\theta_{n+1}^2 + 2p_n\theta_{n+1} - q_n \leq 0.$$

Now, using (3.3) and (3.4), we get

$$\theta_{n+1}^2 + \frac{1}{2-\phi_{n+1}^{-1}}\Big(\phi_n^{-1} + \phi_{n+1}^{-1} - 1\Big)\theta_{n+1} - \frac{1}{2-\phi_{n+1}^{-1}}(\phi_n^{-1} - 1 - \epsilon) \leq 0.$$

Hence,

$$(2-\phi_{n+1}^{-1})\theta_{n+1}^2 + (\phi_n^{-1} + \phi_{n+1}^{-1} - 1)\theta_{n+1} - (\phi_n^{-1} - 1 - \epsilon) \leq 0,$$

which implies

$$-(2-\phi_{n+1}^{-1})\theta_{n+1}^2 - (\phi_n^{-1} - \phi_{n+1}^{-1} - 1)\theta_{n+1} + (\phi_n^{-1} - 1 - \epsilon) \geq 0.$$

Thus, by (4.8), we obtain

$$-\Big(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\Big) \leq -\epsilon.$$

Therefore, in all cases, we have established that

$$-\Big(\phi_n^{-1}(1-\phi_n)(1-\theta_n) - \delta_{n+1}\Big) \leq -\epsilon.$$

Now, using this and (4.7), we get

$$\Gamma_{n+1} \leq \Gamma_n - \epsilon\|x_{n+1} - x_n\|^2. \quad \square$$

Lemma 4.2. *Let $\{x_n\}$ be generated by Algorithm 3.3 under Assumptions 3.1 and 3.2. Then $\lim_{n\to\infty}\|x_n - \bar{w}\|$ exists for all $\bar{w} \in EP(F,C)$.*

Proof. By Lemma 4.1, we see that $\{\Gamma_n\}$ is nonincreasing. Now, let $\gamma_n := \|x_n - \bar{w}\|^2$. Then, $\Gamma_n := \gamma_n - \theta_n\gamma_{n-1} + \delta_n\|x_n - x_{n-1}\|^2$. Thus,

$$\gamma_n - \theta_n\gamma_{n-1} \leq \Gamma_n \leq \Gamma_1,$$

which implies that

$$\begin{aligned}\gamma_n &\leq \theta_n\gamma_{n-1} + \Gamma_1 \\ &\leq \theta\gamma_{n-1} + \Gamma_1 \qquad (\text{for some} \quad \theta < 1, \text{since} \quad \theta_n < 1) \\ &\leq \theta(\theta\gamma_{n-2} + \Gamma_1) + \Gamma_1 \\ &= \theta^2\gamma_{n-2} + \theta\Gamma_1 + \Gamma_1 \\ &\vdots \\ &\leq \theta^n\gamma_1 + \theta^{n-1}\Gamma_1 + \cdots + \theta\Gamma_1 + \Gamma_1 \leq \theta^n\gamma_1 + \frac{\Gamma_1}{1-\theta}.\end{aligned}$$

Hence, for $j \leq n-1$, we obtain from Lemma 4.1 that

$$\epsilon\sum_{j=1}^{n-1}\|x_{j+1} - x_j\|^2 \leq \Gamma_1 - \Gamma_n \leq \Gamma_1 + \theta_n\gamma_{n-1} \leq \Gamma_1 + \theta^{n-1}\gamma_1 + \frac{\Gamma_1}{1-\theta}.$$

Thus we have that $\{\sum_{j=1}^{n-1}\|x_{j+1} - x_j\|^2\}$ is bounded for all n. Hence,

$$\sum_{n=1}^{+\infty}\|x_{n+1} - x_n\|^2 < +\infty. \tag{4.10}$$

Now, from (4.5) we obtain

$$\|x_{n+1}-\bar{w}\|^2 \leq \|x_n-\bar{w}\|^2+\theta_n\Big(\|x_n-\bar{w}\|^2-\|x_{n-1}-\bar{w}\|^2\Big)+\delta_n\|x_n-x_{n-1}\|^2.$$

From the previous inequality, (4.10) and Lemma 2.6, we have that $\lim_{n\to+\infty}\|x_n-\bar{w}\|$ exists. This implies that $\{x_n\}$ is bounded. □

Theorem 4.3. *Let $\{x_n\}$ be generated by Algorithm 3.3 under Assumptions 3.1 and 3.2. Then $\{x_n\}$ converges weakly to $\bar{w}\in EP(F,C)$.*

Proof. From (4.10), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.11}$$

Hence, we have

$$\begin{aligned}\|x_n-w_n\| &= \|x_n-(x_n+\theta_n(x_n-x_{n-1}))\|\\ &\leq \|x_n-x_{n-1}\|\to 0, \quad \text{as} \quad n\to\infty.\end{aligned} \tag{4.12}$$

From (4.11) and (4.12), we get

$$\lim_{n\to\infty}\|x_{n+1}-w_n\|\to 0, \quad \text{as} \quad n\to\infty.$$

Hence,

$$\lim_{n\to\infty}\|z_n-w_n\|=\lim_{n\to\infty}\phi_n^{-1}\|x_{n+1}-w_n\|=0. \tag{4.13}$$

From Lemma 3.7, we get

$$\begin{aligned}\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|w_n-y_n\|^2 &\leq \|w_n-\bar{w}\|^2-\|z_n-\bar{w}\|^2\\ &\leq \Big(\|w_n-\bar{w}\|+\|z_n-\bar{w}\|\Big)\Big(\|w_n-\bar{w}\|-\|z_n-\bar{w}\|\Big)\\ &\leq \Big(\|w_n-\bar{w}\|+\|z_n-\bar{w}\|\Big)\|z_n-w_n\|\to 0, \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|w_n-y_n\|=0. \tag{4.14}$$

Furthermore, we have

$$0\leq\|x_n-y_n\|\leq\|x_n-w_n\|+\|w_n-y_n\|\to 0, \quad \text{as} \quad n\to\infty.$$

From (4.11) to (4.14) we have

$$\|x_{n+1}-y_n\|\to 0, \quad \text{as} \quad n\to\infty, \quad \|z_n-y_n\|\to 0, \quad \text{as} \quad n\to\infty. \tag{4.15}$$

Next, we show that the set of all sequentially weak limit points of the sequence $\{x_n\}$ belongs to $EP(F,C)$. Since $\{x_n\}$ is bounded we have that $\{x_n\}$ has at least one accumulation point, say $z\in\mathcal{H}$. Assume that $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\rightharpoonup z, k\to\infty$. Since $\|y_n-x_n\|\to 0,\ n\to\infty$, we have that $y_{n_k}\rightharpoonup z,\ k\to\infty$ for some $\{y_{n_k}\}\subset\{y_n\}$.

Now, from [41, Lemma 3.2, equation (14)] and [41, Lemma 3.2, equation (6)], we get

$$2F(y_n,z_n)\geq\frac{2}{\lambda_n}\langle w_n-y_n,z_n-y_n\rangle-\frac{\mu}{\lambda_{n+1}}\left[\|w_n-y_n\|^2+\|z_n-y_n\|^2\right] \tag{4.16}$$

and

$$\lambda_n F(y_n,y)\geq\lambda_n F(y_n,z_n)+\langle w_n-z_n,y-z_n\rangle, \quad \forall\ y\in C, \tag{4.17}$$

respectively. Combining (4.16) and (4.17), we obtain

$$\begin{aligned}\lambda_{n_k}F(y_{n_k},y) &\geq \langle w_{n_k}-y_{n_k},z_{n_k}-y_{n_k}\rangle-\frac{\mu}{2}\frac{\lambda_{n_k}}{\lambda_{n_{k+1}}}\left[\|w_{n_k}-y_{n_k}\|^2+\|z_{n_k}-y_{n_k}\|^2\right]\\ &\quad+\langle w_{n_k}-z_{n_k},y-z_{n_k}\rangle,\ \forall\ y\in C.\end{aligned} \tag{4.18}$$

Taking the limit in (4.18) as $k\to\infty$ (taking note of (4.13)–(4.15)), we obtain

$$F(z,y)\geq 0,\ \forall\ y\in C.$$

which implies that $z\in EP(F,C)$. Using this and Lemma 4.2 in Lemma 2.5, we have that $\{x_n\}$ converges weakly to an element in $EP(F,C)$. This completes the proof. □

5. Numerical experiments

The focus of this section is to provide some computational experiments to demonstrate the effectiveness, accuracy and easy-to-implement nature of our proposed algorithm. We compare our proposed algorithm (Algorithm 3.3) with Algorithm 1 in [16], Algorithm 1 in [15], Algorithm 2 in [42] and Algorithm 1 in [43]. Throughout this section, we shall name these algorithms RKSPW (Alg. 1), VM (Alg. 1), CS (Alg. 2) and SCJ (Alg. 1), respectively.

Example 5.1. We consider the Nash-Cournot oligopolistic equilibrium model in [44] where the bifunction F in $\mathbb{R}^N$ is of the form:

$$F(x, y) = \langle Px + Qy + q, y - x\rangle,$$

where $q \in \mathbb{R}^N$ and P, Q are two matrices of order N such that Q is symmetric positive semi-definite and $Q - P$ is symmetric negative semi-definite. The feasible set is defined as $C = \{x \in \mathbb{R}^N : -5 \le x_i \le 5\}$. The bifunction satisfies Assumption 3.1 with $a_1 = a_2 = \frac{1}{2}\|P - Q\|$ (see [14]). The vectors x_0, x_1, q are generated randomly and uniformly in $[-N, N]$ and the two matrices P, Q are generated randomly such that their properties are satisfied.

Example 5.2. Let $\mathcal{H} = (\ell_2(\mathbb{R}), \|\cdot\|_2)$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^{\infty}$ of scalars in $\mathbb{R}$, that is

$$\ell_2(\mathbb{R}) = \mathcal{H} = \left\{ x = (x_1, x_2, \dots, x_i, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ and norm $\|\cdot\| : \ell_2 \to \mathbb{R}$ defined by $\langle x, y\rangle := \sum_{j=1}^{\infty} x_i y_i$ and $\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$, for $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty} \in \ell_2(\mathbb{R})$. Let $C = \left\{ x \in \mathcal{H} : \|x\| \le 1 \right\}$. Define the bifunction $F : C \times C \to \mathbb{R}$ by $F(x, y) = (3 - \|x\|)\langle x, y - x\rangle, \forall\ x, y \in C$. It is easy to show that F is a pseudomonotone bifunction which is not monotone and F satisfies the Lipschitz-type condition with constants $a_1 = a_2 = \frac{5}{2}$. Also, F satisfies Assumption 3.1 ((4)-(5)). We consider the following cases for the numerical experiments of this example

Case 1: Take $x_1 = \left(\frac{5}{7}, \frac{1}{7}, \frac{1}{35}, \dots\right)$ and $x_0 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \dots\right)$.

Case 2: Take $x_1 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \dots\right)$ and $x_0 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\right)$.

Case 3: Take $x_1 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\right)$ and $x_0 = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{10}, \dots\right)$.

During the computation, we make use of the following:

- Algorithm 3.3: $\lambda_1 = 0.1, \mu = 0.5, \epsilon = 0.000001$, $\theta_n = \beta_n$ (when $\phi_n = \frac{n-0.5}{2n}$), where β_n is as defined in (3.1), $\theta_n = \frac{1-\epsilon}{3}$ (when $\phi_n = 0.5$) and $\theta_n = \sqrt{p_n^2 + q_n} - p_n$ (when $\phi_n = \frac{n-0.1}{n}$), where p_n and q_n are as defined in (3.3) and (3.4), respectively.
- RKSPW (Alg. 1) in [16]: $\lambda_1 = 0.1, \delta = 0.9, \sigma = 0.9\min\{1, 0.5a_1, 0.5a_2\}, \mu = 0.9\sigma$ and $\epsilon_n = \frac{100}{(n+1)^2}$.
- VM (Alg. 1) in [15]: $\lambda = 0.9\min\{0.5a_1, 0.5a_2\}, \theta = 0.9$ and $\epsilon_n = \frac{100}{(n+1)^2}$.
- CS (Alg. 2) in [42]: $\lambda_1 = 0.9\min\{0.5a_1, 0.5a_2\}, \alpha_n = \frac{1}{2n+1}, \mu = 0.1, \beta_n = \frac{n}{100n+1}$, $\nabla\varphi(x) = \rho x$ (where $\rho \in [0.1)$), and $\delta_n := \begin{cases} \frac{0.5}{n^2\|x_n - x_{n-1}\|}, & \text{if } x_n \ne x_{n-1} \\ 0.5, & \text{otherwise.} \end{cases}$
- SCJ (Alg. 1) in [43]: $\mu_n = \mu = 0.9\min\{0.5a_1, 0.5a_2\}, \eta_n = \frac{1}{n+1}, \nabla\phi(x) = \rho x$ (where $\rho \in [0.1)$), and $\delta_n := \begin{cases} \frac{1}{n\|x_n - x_{n-1}\|}, & \text{if } x_n \ne x_{n-1} \\ 0.5, & \text{otherwise.} \end{cases}$

We then use the stopping criterion; $\mathrm{TOL}_n := \|y_n - w_n\| < \varepsilon$, where ε is the predetermined error.

All the computations are performed using Matlab R2023b which is running on a personal computer with an Intel(R) Core(TM) i5-10210U CPU at 2.11 GHz and 8.00 Gb-RAM.

In the Tables 1 and 2, "Iter" means the number of iterations. Also, in the tables and figures, Alg. 3.3, RKSPW (Alg. 1), VM (Alg. 1), CS (Alg. 2) and SCJ (Alg. 1) represent Algorithm 3.3, Algorithm 1 in [16], Algorithm 1 in [15] Algorithm 2 in [42] and Algorithm 1 in [43], respectively (see Figs. 1 and 2).

6. Conclusion

We have considered in this paper, a new inertial condition for the subgradient extragradient method with self-adaptive step size for solving pseudomonotone equilibrium problem in a real Hilbert space. It was proved that the sequence of iterates generated by

Table 1
Numerical results for Example 5.1 with $\epsilon = 10^{-5}$.

Algorithms	N = 20		N = 50		N = 100	
	CPU time	Iter.	CPU time	Iter.	CPU time	Iter.
Alg. 3.3 $\left(\phi_n = \frac{n-0.5}{2n}\right)$	1.5845	172	2.7013	210	2.9610	288
Alg. 3.3 $\left(\phi_n = 0.5\right)$	1.4864	129	1.8413	144	2.5517	213
Alg. 3.3 $\left(\phi_n = \frac{n-0.1}{n}\right)$	0.9217	86	0.9687	89	1.6092	153
RKSPW (Alg. 1) in [16]	6.3664	676	5.2644	577	3.3365	350
VM (Alg. 1) in [15]	2.8233	243	3.4392	219	3.5181	312
CS (Alg. 2) in [42]	8.3144	832	3.8683	209	8.0181	843
SCJ (Alg. 1) in [43]	3.9099	299	3.5544	291	3.7348	331

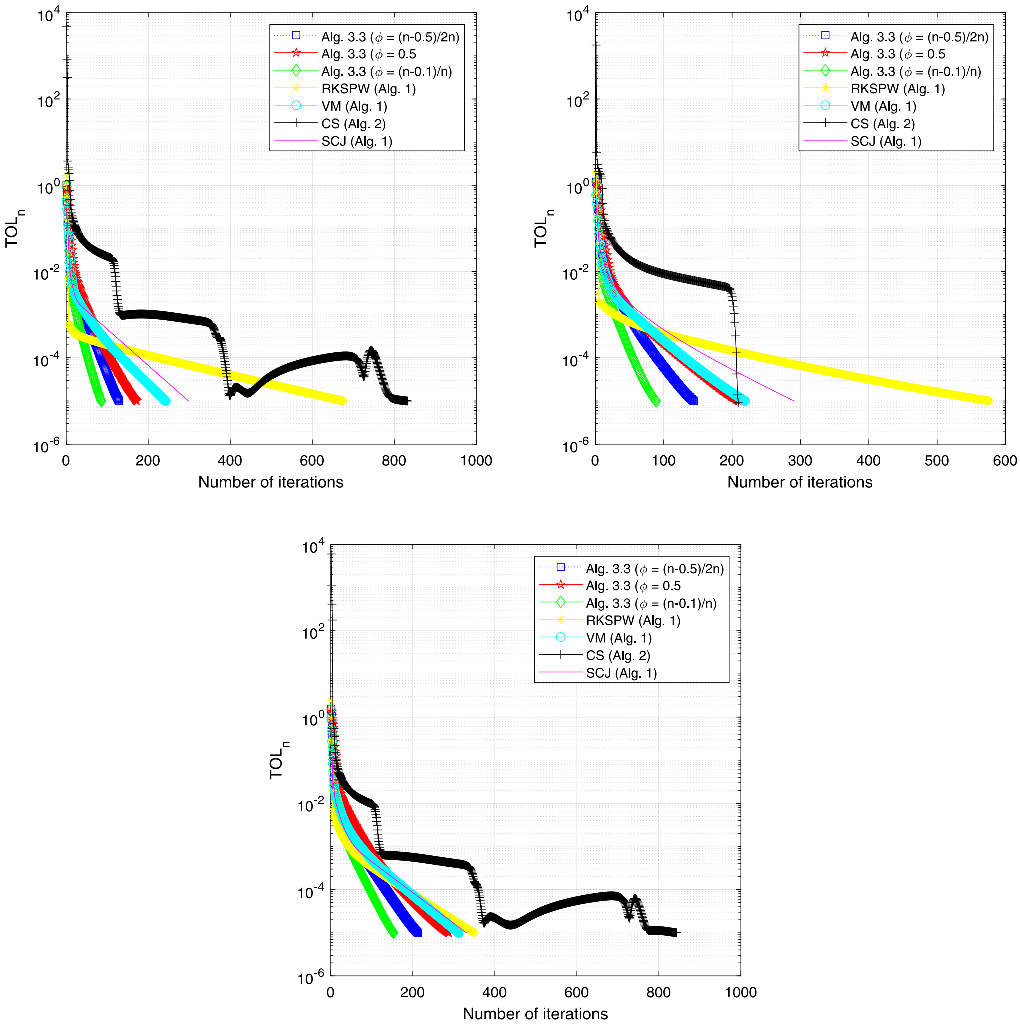

Fig. 1. The behavior of TOL_n with $\epsilon = 10^{-5}$: Top Left: $N = 20$; Top Right: $N = 50$; Bottom: $N = 100$.

our proposed method converges weakly to a solution of the equilibrium problem under improved conditions on the inertial factor than many existing conditions in the literature. The assumptions regarding the inertial relaxation are less restrictive compared to many other papers addressing pseudomonotone equilibrium problems in the literature. These conditions remain independent of the iterates, which are frequently unknown a priori. Additionally, determining the upper bound of the inertial sequence is straightforward. Numerical results are given to support our analysis. These results demonstrate the effectiveness, accuracy and easy-to-implement nature of our proposed method when compared to others in the literature.

Table 2
Numerical results for Example 5.2 $\epsilon = 10^{-5}$.

Algorithms	Case 1		Case 2		Case 3	
	CPU time	Iter.	CPU time	Iter.	CPU time	Iter.
Alg. 3.3 $\left(\phi_n = \frac{n-0.5}{2n}\right)$	0.0068	99	0.0098	106	0.0097	102
Alg. 3.3 $\left(\phi_n = 0.5\right)$	0.0030	63	0.0031	68	0.0022	66
Alg. 3.3 $\left(\phi_n = \frac{n-0.1}{n}\right)$	0.0021	47	0.0029	51	0.0021	49
RKSPW (Alg. 1 in [16])	0.0912	110	0.0951	110	0.0803	110
VM (Alg. 1 in [15])	0.0905	140	0.0833	162	0.0996	150
CS (Alg. 2) in [42]	0.0803	108	0.0811	114	0.0817	119
SCJ (Alg. 1) in [43]	0.0919	141	0.0919	208	0.0901	123

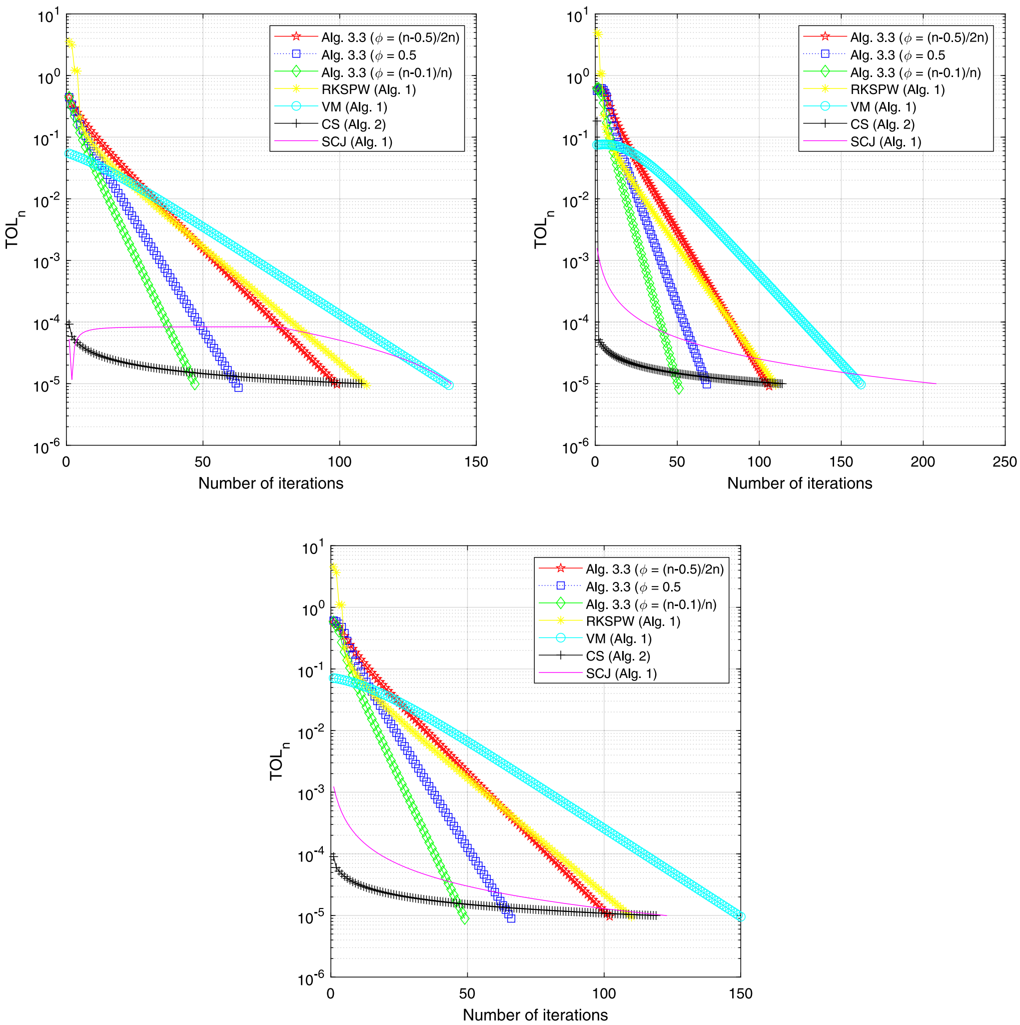

Fig. 2. The behavior of TOL_n with $\epsilon = 10^{-5}$: Top Left: **Case 1**; Top Right: **Case 2**; Bottom: **Case 3**.

CRediT authorship contribution statement

Chinedu Izuchukwu: Software, Methodology, Investigation, Conceptualization. **Grace Nnennaya Ogwo:** Writing – review & editing, Writing – original draft. **Bertin Zinsou:** Validation, Supervision, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Blum E, Oettli W. From optimization and variational inequalities to equilibrium problems. Math Stud 1994;63:123–45.
[2] Rehman H, Kumam P, Je Cho Y, Suleiman YI, Kumam W. Modified Popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems. Optim Methods Softw 2021;36(1):82–113.
[3] Iiduka H. A new iterative algorithm for the variational inequality problem over the fixed point set of a firmly nonexpansive mapping. Optimization 2010;59:873–85.
[4] Iiduka H, Yamada I. A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping. SIAM J Optim 2009;19:1881–93.
[5] Maingé PE. Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints. European J Oper Res 2010;205:501–6.
[6] Maingé PE. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal 2008;16:899–912.
[7] Shehu Y, Izuchukwu C, Yao J-C, Qin X. Strongly convergent inertial extragradient type method for equilibrium problems. Appl Anal 2023;102(8):2160–88.
[8] Hieu DV. Halpern subgradient extragradient method extended to equilibrium problems. Rev R Acad Cienc Exactas Fís Nat Ser A Mat 2017;111:823–40.
[9] Hieu DV. Hybrid projection methods for equilibrium problems with non-Lipschitz type bifunctions. Math Methods Appl Sci 2017;40:4065–79.
[10] Peng JW, Liou YC, Yao JC. An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions. Fixed Point Theory Appl 2009;794178. http://dx.doi.org/10.1155/2009/794178, 21p.
[11] Qin X, Cho YJ, Kang SM. Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J Comput Appl Math 2009;225:20–30.
[12] Moudafi A. Proximal point algorithm extended to equilibrum problem. J Nat Geom 1999;15:91–100.
[13] Korpelevich GM. An extragradient method for finding saddle points and for other problems. Ekon Mat Metody 1976;12:747–56.
[14] Quoc TD, Muu LD, Nguyen VH. Extragradient algorithms extended to equilibrium problems. Optimization 2008;57:749–76.
[15] Vinh NT, Muu LD. Inertial extragradient algorithms for solving equilibrium problems. Acta Math Vietnam 2019;44(3):639–63.
[16] Rehman HU, Kumam P, Shutaywi M, Pakkaranang N, Wairojjana N. An inertial extragradient method for iteratively solving equilibrium problems in real Hilbert spaces. Int J Comput Math 2020;99(6):1081–104.
[17] Censor Y, Gibali A, Reich S. The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl 2011;148:318–35.
[18] Polyak BT. Some methods of speeding up the convergence of iteration methods. USSR Comput Math Math Phys 1964;4(5):1–17.
[19] Dong Y. New inertial factors of the Krasnosel'skiĭ-Mann iteration. Set-Valued Var Anal 2021;29:145–61.
[20] Attouch H, Peypouquet J, Redont P. A dynamical approach to an inertial forward–backward algorithm for convex minimization. SIAM J Optim 2014;24(1):232–56.
[21] Beck A, Teboulle M. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM J Imaging Sci 2009;2(1):183–202.
[22] Izuchukwu C, Reich S, Shehu Y. Relaxed inertial methods for solving the split monotone variational inclusion problem beyond co-coerciveness. Optimization 2023;72:607–46.
[23] Ogwo GN, Izuchukwu C, Shehu Y, Mewomo OT. Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems. J Sci Comput 2022;90:1–36.
[24] Moudafi A. Second-order differential proximal methods for equilibrium problems. J Inequal Pure Appl Math 2003;4(1):1–7.
[25] Iutzeler F, Hendrickx JM. A generic online acceleration scheme for optimization algorithms via relaxation and inertia. Optim Methods Softw 2019;34:383–405.
[26] Attouch H, Cabot A. Convergence of a relaxed inertial forward–backward algorithm for structured monotone inclusions. Appl Math Optim 2019;80:547–98.
[27] Xia Y, Wang J. A general methodology for designing globally convergent optimization neural networks. IEEE Trans Neural Netw 1998;9(6):1331–43.
[28] Attouch H, Cabot A. Convergence of a relaxed inertial proximal algorithm for maximally monotone operators. Math Program 2020;184:243–87.
[29] Boţ RI, Csetnek ER, Hendrich C. Inertial Douglas–Rachford splitting for monotone inclusion problems. Appl Math Comput 2015;256:472–87.
[30] Shehu Y, Iyiola OS, Li X-H, Dong Q-L. Convergence analysis of projection method for variational inequalities. Set-Valued Var Anal 2021;29:145–61.
[31] Dong QL, Huang JZ, Li XH, Cho YJ, Rassias ThM. MiKM: multi-step inertial Krasnosel'skiĭ-Mann algorithm and its applications. J Global Optim 2019;73:801–24.

[32] Dong QL, Kazmi KR, Ali R, Li X-H. Inertial Krasnosel'skiĭ-Mann type hybrid algorithms for solving hierarchical fixed point problems. J Fixed Point Theory Appl 2019;21:57, 22 pp.
[33] Maingé PE. Convergence theorems for inertial KM-type algorithms. J Comput Appl Math 2008;219:223–36.
[34] Fierro I, Maulén JJ, Peypouquet J. Inertial Krasnoselskii-Mann iterations. Set-Valued Var Anal 2024;32.
[35] Vong S, Liu D. An inertial Mann algorithm for nonexpansive mappings. J Fixed Point Theory Appl 2018;20:102, 16 pp.
[36] Peypouquet J. Convex optimization in normed spaces, theory, methods and examples. Berlin: Springer; 2015.
[37] Bauschke HH, Combettes PL. Convex analysis and monotone operator theory in Hilbert spaces. New York: Springer; 2011.
[38] Opial Z. Weak convergence of successive approximations for nonexpansive mappings. Bull Amer Math Soc 1967;73:591–7.
[39] Alvarez F, Attouch H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Var Anal 2001;9:3–11.
[40] Yang J, Liu H. The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space. Optim Lett 2020;14:1803–16.
[41] Thong DV, Cholamjiak P, Rassias MT, Cho YJ. Strong convergence of inertial subgradient extragradient algorithm for solving pseudomonotone equilibrium problems. Optim Lett 2022;16(2):545–73.
[42] Cholamjiak W, Suparatulatorn R. Strong convergence of a modified extragradient algorithm to solve pseudomonotone equilibrium and application to classification of diabetes mellitus. Chaos Solit Fractals 2023;168:113108.
[43] Suparatulatorn R, Cholamjiak W, Jun-onc N. A modified subgradient extragradient method for equilibrium problems to predict prospective mathematics teachers' digital proficiency level. Res Nonlinear Anal 2023;6(3):1–18.
[44] Facchinei F, Pang JS. Finite-dimensional variational inequalities and complementarity problems. Springer Science and Business Media; 2007.