Z. Angew. Math. Phys. (2025) 76:142

https://doi.org/10.1007/s00033-025-02527-5

Zeitschrift für angewandte Mathematik und Physik ZAMP

Traveling waves for a general diffusive virus infection model with both virus-to-cell and cell-to-cell transmissions and adaptive immunity

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Abstract. The main purpose of this work is to investigate the existence and nonexistence of traveling waves for a virus infection model with adaptive immunity, virus-to-cell infection and cell-to-cell transmission. The virus-to-cell and cell-to-cell incidence rates are modeled by general nonlinear functions. The basic reproduction numbers are calculated for virus infection, antibody immune response, cytotoxic T lymphocytes (CTL) immune response, CTL immune competition, and antibody immune competition. By introducing an auxiliary nonlinear differential system and applying the Schauder's fixed point theorem, combined with the method of upper-lower solutions, we prove the existence of traveling waves dependent not only on the five reproduction numbers but also on the critical wave speed. Moreover, we show that these traveling waves connect the infection-free equilibrium to each of the other four equilibria, namely the immune-free infection equilibrium, the infection equilibrium with only antibody immune defense, the infection equilibrium with only CTL immune response and the CTL-antibody-present infection equilibrium. Finally, an application is provided and some numerical simulations are performed to illustrate the theoretical results obtained.

Mathematics Subject Classification. 92D30, 35B40, 35C07, 35K57.

Keywords. Diffusive virus infection model, Boundary asymptotic behavior, Minimal wave speed, Upper–lower solutions, Traveling waves.

1. Introduction

According to several reports from various health organizations, such as the National Notifiable Diseases Surveillance System and the Center for Disease Control and Prevention, the leading causes for the death of children, adolescents and adults are infectious diseases. In recent decades, the study of population dynamics of infectious diseases has attracted considerable attention. In this regard, there has been a tremendous effort in the mathematical formulation of within-host virus dynamics models. These models have been exploited to describe the dynamics inside the host of various infectious diseases, particularly the in vivo infection process with adaptive immune response such as hepatitis B virus (HBV), human immunodeficiency virus (HIV), hepatitis C virus (HCV), and human T-cell leukemia virus type 1 (HTLV-1) (see, for instance [10–12,27,43]). Note that the role of the immune response in controlling these diseases in the host is momentous, as it protects the host against pathogens.

The adaptive immunity consists of CTLs and antibody B cells. These two components play a crucial role in preventing and modulating infections. The antibody B cells response is carried out by functional immunocompetent B lymphocytes [6]. They migrate from the bone marrow to other lymphatic organs, where they begin to generate antibodies to attack, neutralize and remove viruses, and prevent reinfection. CTLs attack and kill infected cells to reduce viral load [6,12,31]. Several works have proposed some mathematical models to explain how these two immune responses are important in curing the viral infection [12,43,49].

Published online: 15 June 2025

The aforementioned models only focus on virus-to-cell spread in the bloodstream despite the fact that some works reveal that cell-to-cell transmission is vital to spread of virus in vivo [1,9,39]. In fact, the direct cell-to-cell transmission mode has been pointed out in many infections such as HBV, HTLV, HCV, HIV, murine leukemia virus (MLV), and many others [25,33,46]. In [24,38], the authors reported that cell-to-cell transmission seems to be a more potent and efficient means of virus propagation than the virus-to-cell transmission. Different models of viral infection with two modes of transmission have been proposed. In [26], the authors investigated the global dynamics of HIV infection which included a direct cell-to-cell transmission. Lai and Zou discussed the effect of cell-to-cell transfer of HIV-1 on the virus dynamics [25].

The spatial mobility of cells and viruses has been ignored in most of the above models. To investigate the influence of spatial structures on virus dynamics, the authors in [41] developed a diffusive HBV model and assumed that the mobility of free virus follows the Fickian diffusion. However, biological motion of cells and viruses have a crucial role in biological systems [3]. Moreover, from a biological viewpoint, cells are distributed in space and typically interact with the physical environment and other organisms in their spatial neighborhood [5,31]. Specifically, in the case of HIV-1, the spatial variation is not negligible, since uninfected cells are densely packed in lymphoid tissues, and the spread and replication of the virus may be different in distinct positions. Target cells tend to stay away from infected cells; meanwhile, infected cells would get closer to target cells.

The mathematical analysis of the above models is needful to obtain an integrated view for the virus dynamics in vivo. One analytical approach that is becoming increasingly fascinating in the study of infectious diseases is the traveling waves method, because various infectious diseases can be well described by a mathematical model with spatial effects that can give rise to a moving zone of transition from an infective state to a disease-free steady state in general [51]. It is therefore essential to analyze the epidemic wave, which is described by traveling waves solutions propagating at a certain speed. Traveling waves are waves that move in a specific direction with a constant propagation speed while retaining a fixed shape [8]. The study of traveling waves solutions to nonlinear PDE plays a crucial role in the modeling of nonlinear phenomena. The existence of such traveling waves is generally a consequence of the coupling of various effects such as convection or diffusion or chemotaxis. In [41], the authors extended the basic HBV infection model designed in [36], and included the spatial mobility of free viruses. They assumed that the viruses move in an infinite one-dimensional spatial domain $(-\infty, \infty)$, and they studied the existence of traveling waves solutions by using the geometric singular perturbation method. In the meantime, progress has been made in the study of traveling waves solutions to nonlinear reaction-diffusion equations. Ge and collaborators [15,16] employed the iteration technique designed in [44] to study the existence of traveling waves solutions for a diffusive two-species predator–prey system with stage structure. In [21], the authors used the Schauder's fixed point theorem to discuss the existence of traveling waves solutions of a two delayed reaction-diffusion system. Gan an coworkers [14] investigated the existence of traveling waves for a HBV infection model with spatial diffusion and discrete time delay. By applying the cross-iteration method and the Schauder's fixed point theorem, they reduced the existence of traveling waves to the existence of a pair of upper-lower solutions. Very recently, Wu and collaborators [45] studied the existence and nonexistence of traveling waves solutions for a general diffusive virus infection model with humoral immunity and cell-to-cell transmission. They established the upper-lower solutions with the aid of an auxiliary system. Then, by applying the Schauder's fixed point theorem and Lyapunov methods, they obtained sufficient conditions for the existence of traveling waves solutions.

We found that the models above, studying infectious diseases such as HIV, HCV, HTLV-1 and many others, ignore the role played by virus DNA-containing capsid. Note that the viral capsid is a structural protein that encloses and protects the genetic material of the virus during the viral replication process [22]. On the other hand, it plays an important role in virus formation and replication during the maturation phase of the virus [4,12]. Miao and collaborators [31] developed and studied a virus dynamics model with only humoral impairment, and ignored the cell-to-cell transmission mode and virus DNA-containing capsid.

Motivated by the above works, particularly [31,43,45], we propose the following virus infection model with spatial diffusion, both virus-to-cell and cell-to-cell transmissions, capsid, adaptive immunity and general nonlinear incidence functions:

$$\begin{cases} \frac{\partial H}{\partial t} = d_1\Delta H(x,t) + s_0 - dH(x,t) - \beta_1 f(H(x,t),V(x,t)) - \beta_2 \mathrm{g}(H(x,t),I(x,t)), \\ \frac{\partial I}{\partial t} = d_2\Delta I(x,t) + \beta_1 f(H(x,t),V(x,t)) + \beta_2 \mathrm{g}(H(x,t),I(x,t)) - \delta I(x,t) - pI(x,t)Z(x,t), \\ \frac{\partial D}{\partial t} = d_3\Delta D(x,t) + kI(x,t) - (\alpha+\delta)D(x,t), \\ \frac{\partial V}{\partial t} = d_4\Delta V(x,t) + \alpha D(x,t) - \mu V(x,t) - rV(x,t)W(x,t), \\ \frac{\partial W}{\partial t} = d_5\Delta W(x,t) + bV(x,t)W(x,t) - \sigma W(x,t), \\ \frac{\partial Z}{\partial t} = d_6\Delta Z(x,t) + aI(x,t)Z(x,t) - qZ(x,t), \end{cases} \tag{1.1}$$

where $t \in [0,\infty)$, $x \in \Omega$ with Ω a suitable subset of $\mathbb{R}^n$, n a given positive integer. In this work, we deal with the one-dimensional case (n=1) and take $\Omega = (-\infty,\infty)$ [41] so that the Laplacian applied to a function, Δu, can be written $\frac{\partial^2 u}{\partial x^2}$ or u_{xx}. The biological terms $H(x,t)$, $I(x,t)$, $D(x,t)$ $V(x,t)$, $W(x,t)$ and $Z(x,t)$ denote the density of uninfected target cells, infected cells, virus DNA-containing capsids, virus, magnitudes of B cells and CTL cells at time t and location x, respectively.

The uninfected cells are fabricated from precursors in the bone marrow and thymus in the body at a constant rate s_0 [31] and die at rate d. Uninfected cell become actively infected at rates β_1 and β_2, resulting from virus-to-cell and cell-to-cell transmission, respectively. The nonlinear incidence functions $f(H(x,t),V(x,t))$ and $\mathrm{g}(H(x,t),V(x,t))$ stand for the influence of infection transmission through the virus-to-cell and cell-to-cell modes, respectively. Thus, the term $\beta_1 f(H(x,t),V(x,t))+\beta_2\mathrm{g}(H(x,t),I(x,t))$ represents the total infection rate of uninfected cells. Infected cells perish at the per capita rate δ. Virus DNA-containing capsids are fabricate from infected cells at rate k and removed at rate δ. They are transmitted to blood in order to convert into virus at rate α. The biological justification for adopting the same death rate for infected cells and virus DNA-containing capsids is given in [12,34,35]. The biological parameter μ represents the clearance rate of free virions in the plasma. The model considers that infected cells are eliminated at rate p by CTL cells, whereas free virions are neutralized by antibodies at rate r. Antivirus CTL cells and B cells reduce proviral and viral loads, respectively. However, these reductions would imply less stimulation of CTL cells and B cells proliferation [37]. As a result, it is reasonable to consider that CTL cells and B cells stimulation have the densities-dependent form $aI(x,t)Z(x,t)$ and $bV(x,t)W(x,t)$, respectively. This indicates that B cells expand in response to the viruses at rate b and decay at rate c, and CTL cells are activated by infected cells at rate a (also denotes the cytotoxic responsiveness) and are destroyed in the absence of antigenic stimulation at rate q.

The spatial mobility is described by the diffusion terms $d_1\Delta H$, $d_2\Delta I$, $d_3\Delta D$, $d_4\Delta V$, $d_5\Delta W$ and $d_6\Delta Z$, where Δ denotes the one-dimensional Laplace operator (the second derivative with respect to the one-dimensional space variable), and d_i, $i = 1,\cdots,6$ are the diffusion coefficients. Here, we assume that d_i, $i = 1,\cdots,6$, are positive real constants as in [47].

Finally, although we are dealing with a spatial model, all biological parameters of system (1.1), described in Table 1, are assumed to be positive and independent of the position in the one-dimensional space.

Miao and Jiao [30] studied model (1.1) with one branch of adaptive immunity, in the presence of two discrete delays, when $d_1 = d_2 = 0$. Specifically, they discussed the positivity and boundedness of the solution, and established the global dynamical behavior. In fact, model (1.1) gives an insight into the intra-host models of infectious diseases.

As a particular type of solutions to the reaction-diffusion equation, traveling waves solutions, as presented above, can well describe the process of matter transfer from one equilibrium state to another. For model (1.1), traveling waves solutions may reflect the development of virus from infection-free equilibrium to immune-free equilibrium, or from infection-free equilibrium to only antibody immune equilibrium, or from infection-free equilibrium to only CTL immune equilibrium, or from infection-free equilibrium to both CTL-antibody immune equilibrium, at a certain wave speed. To the best of our knowledge, there does not exist any work in the literature which investigates the traveling waves solutions for model (1.1), which satisfy different appropriate boundary conditions.

We aim in this paper to study sufficient conditions for the existence of traveling waves solutions of system (1.1) connecting infection-free equilibrium E_0 and immune-free infection equilibrium E_1, connecting infection-free equilibrium E_0 and infection equilibrium with only antibody immune defense E_2, connecting infection-free equilibrium E_0 and infection equilibrium with only CTL immune response E_3 and connecting infection-free equilibrium E_0 and CTL-antibody-present infection equilibrium E_4, and draw the critical wave speed from the characteristic equation. For system (1.1), we introduce an auxiliary system by which a bounded cone is constructed, and we apply the Schauder's fixed point theorem.

This paper is organized as follows. In Sect. 2, we study the existence of feasible equilibria, and the minimal wave speed c^* is studied by the linearization method when the antibody immune response reproduction number $\mathcal{R}_1$ is less than one. In Sect. 3, a pair of upper and lower solutions is constructed. In Sect. 4, a closed convex set is built and the Schauder's fixed point theorem is applied to derive the existence of traveling waves solutions of model (1.1). In Sect. 5, we show how these traveling waves connect the infection-free equilibrium to each of the other four equilibria. In Sect. 6, an application is given and numerical simulations are performed to confirm the theoretical results obtained. Finally, the conclusion is drawn in Sect. 7.

2. Preliminaries

For any integer $m > 0$, we denote $\mathbb{R}_+^m = \{x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : x_i \geq 0,\ i = 1, 2, \cdots, m\}$. For convenience and for a continuously differentiable function f, we denote $f_{x_i}(x) = \frac{\partial f(x)}{\partial x_i}$. In model (1.1), the incidence functions $f(H, V)$ and $\mathrm{g}(T, I)$ are assumed to satisfy the following assumption:

(Q1) Functions $f,\ \mathrm{g} : \mathbb{R}_+^2 \to \mathbb{R}_+$ are continuously differentiable; $f(H, 0) = f(0, V) = \mathrm{g}(H, 0) = \mathrm{g}(0, I) = 0$; $f_H(H, V) > 0$, for all H, V, I; $f(H, V)$ and $\mathrm{g}(H, I)$ are nondecreasing for all $H, V, I > 0$; $f_V(H, V)$ is nonincreasing for all $V \geq 0$; $\mathrm{g}_I(H, I)$ is nonincreasing for all $I \geq 0$.

Furthermore, one can check that the class of general incidence functions $f(H, V)$ and $\mathrm{g}(H, I)$ satisfying hypothesis (Q1) includes incidence functions such as $f(H, V) = HV$ [41], $f(H, V) = \frac{HV}{H+V}$ [17], $f(H, V) = \frac{HV}{1+\omega V}$ [47], $f(H, V) = \frac{HV}{1+\gamma H+\omega V}$ [2,7], $f(H, V) = \frac{HV}{1+\gamma H+\omega V+\gamma\omega HV}$ [23], where $\gamma, \quad \omega > 0$.

Model (1.1) always has an infection-free equilibrium $E_0 = (H_0, 0, 0, 0, 0, 0)$, where $H_0 = s_0/d$. The immune-free infection equilibrium, the infection equilibrium with only antibody immune defense, the infection equilibrium with only CTL immune response and the CTL-antibody-present infection equilibrium are determined by the parameters of model (1.1). To obtain the immune-free infection equilibrium, we first compute the basic reproduction number for virus infection $\mathcal{R}_0$. Using the method developed in [40,42], we have

$$\mathcal{R}_0 = \frac{k\alpha\beta_1 f_V(H_0, 0) + \beta_2 \mathrm{g}_I(H_0, 0)\mu(\alpha+\delta)}{\delta\mu(\alpha+\delta)}.$$

When $\mathcal{R}_0 > 1$, model (1.1) has a unique immune-free equilibrium $E_1 = (H_1, I_1, D_1, V_1, 0, 0)$, where

TABLE 1. Biological description, set of parameter values and unit of the parameters of model (2.1)

Parameters	Biological description	Case I	Case II	Case III	Unit	Source
s_0	Production rate of uninfected cells	1	10	1	cells ml^{-1} day^{-1}	[45]
d	Death rate of uninfected cells	0.1	0.03	0.1	day^{-1}	[12, 45]
β_1	Rate of infection of target cells by virus	0.06	0.3	0.07	ml virus^{-1} day^{-1}	[45]
β_2	Rate of infection of target cells by infected cells	0.06	0.3	0.07	ml cell^{-1} day^{-1}	[45]
δ	Death rate of infected cells and capsid	0.5	0.5	0.2	day^{-1}	[45]
k	Capsids production rate	0.4	0.4	1	capsids cells^{-1} day^{-1}	[45]
μ	Decay rate of free virus	3	3	1	day^{-1}	[12]
r	Neutralizing rate of antibody	1	1	1	ml virus^{-1} day^{-1}	[45]
p	CTL effectiveness	0.95	0.95	0.95	ml cells^{-1} day^{-1}	[12]
α	Virus production rate	0.87	0.87	0.87	day^{-1}	[12, 18]
σ	Antibody death rate	1	1	1	day^{-1}	[12]
q	CTL death rate	0.05	0.05	0.05	day^{-1}	[12]
a	CTL activation rate	0.02	0.2	0.01	ml cells^{-1} day^{-1}	[12]
b	Antibody activation rate	1.5	1.5	1.5	ml virus^{-1} day^{-1}	[12]
d_1	Diffusion coefficient of uninfected cells	0.01	0.1	0.01	cm^2 day^{-1}	[29]
d_2	Diffusion coefficient of infected cells	0.01	0.1	0.01	cm^2 day^{-1}	[29]
d_3	Diffusion coefficient of capsids	0.75	0.1	0.75	cm^2 day^{-1}	[28]
d_4	Diffusion coefficient of viruses	0.75	0.75	0.01	cm^2 day^{-1}	[18]
d_5	Diffusion coefficient of B cells	0.75	0.07	0.75	cm^2 day^{-1}	[32]
d_6	Diffusion coefficient of CTL cells	0.75	0.1	0.75	cm^2 day^{-1}	[32]

$H_1 = \dfrac{s_0 k\alpha - \delta\mu(\alpha+\delta)V_1}{dk\alpha}$, $I_1 = \dfrac{\mu(\alpha+\delta)}{k\alpha}V_1$, $D_1 = \dfrac{\mu}{\alpha}V_1$ and $V_1 \in \left(0, \dfrac{s_0 k\alpha}{\delta\mu(\alpha+\delta)}\right)$ is the unique solution of equation

$$\beta_1 \frac{f\left(\dfrac{s_0 k\alpha - \delta\mu(\alpha+\delta)V_1}{dk\alpha}, v\right)}{v} + \beta_2 \frac{\mathrm{g}\left(\dfrac{s_0 k\alpha - \delta\mu(\alpha+\delta)V_1}{dk\alpha}, \dfrac{\mu(\alpha+\delta)}{k\alpha}v\right)}{v} - \frac{\delta\mu(\alpha+\delta)}{k\alpha} = 0.$$

To obtain the existence of the infection equilibrium with only antibody immune defense $E_2 = (H_2, I_2, D_2, V_2, W_2, 0)$ and the infection equilibrium with only CTL immune response $E_3 = (H_3, I_3, D_3, V_3, 0, Z_3)$, we define the antibody immune response reproduction number $\mathcal{R}_1$ and the CTL immune defense reproduction number $\mathcal{R}_2$, as follows:

$$\mathcal{R}_1 = \frac{bV_1}{\sigma} = \frac{kb\alpha\left[\beta_1 f(H_1, V_1) + \beta_2 \mathrm{g}\left(H_1, \dfrac{\mu(\alpha+\delta)}{k\alpha}V_1\right)\right]}{\delta\mu\sigma} \quad \text{and}$$
$$\mathcal{R}_2 = \frac{aI_1}{q} = \frac{a\alpha\left[\beta_1 f(H_1, V_1) + \beta_2 \mathrm{g}\left(H_1, \dfrac{\mu(\alpha+\delta)}{k\alpha}V_1\right)\right]}{q\delta(\alpha+\delta)}. \tag{2.1}$$

When $\mathcal{R}_0 > 1$ and $\mathcal{R}_1 > 1$, model (1.1) has a unique infection equilibrium with only antibody immune defense $E_2 = (H_2, I_2, D_2, V_2, W_2, 0)$, where $V_2 = \dfrac{\sigma}{b}$, $I_2 = \dfrac{s_0 - dH_2}{\delta}$, $D_2 = \dfrac{k(s_0 - dH_2)}{\delta(\alpha+\delta)}$, $W_2 = \dfrac{bk\alpha(s_0 - dH_2)}{\delta\sigma r(\alpha+\delta)} - \dfrac{\mu}{r}$ and $H_2 \in \left(0, \dfrac{s_0}{d} - \dfrac{\delta\mu\sigma(\alpha+\delta)}{dbk\alpha}\right)$ is the unique solution of equation

$$s_0 - dH - \beta_1 f\left(H, \frac{\sigma}{b}\right) - \beta_2 \mathrm{g}\left(H, \frac{s_0 - dH}{\delta}\right) = 0.$$

When $\mathcal{R}_0 > 1$ and $\mathcal{R}_2 > 1$, model (1.1) has a unique infection equilibrium with only CTL immune response $E_3 = (H_3, I_3, D_3, V_3, 0, Z_3)$, where $I_3 = \dfrac{q}{a}$, $D_3 = \dfrac{kq}{a(\alpha+\delta)}$, $V_3 = \dfrac{k\alpha q}{\mu a(\alpha+\delta)}$, $Z_3 = \dfrac{a(s_0 - dH_3)}{pq} - \dfrac{\delta}{p}$ and $H_3 \in \left(0, \dfrac{s_0}{d} - \dfrac{\delta q}{ad}\right)$ is the unique solution of equation

$$s_0 - dH - \beta_1 f\left(H, \frac{k\alpha q}{\mu a(\alpha+\delta)}\right) - \beta_2 \mathrm{g}\left(H, \frac{q}{a}\right) = 0.$$

To obtain the existence of the CTL-antibody-present infection equilibrium, $E_4 = (H_4, I_4, D_4, V_4, W_4, Z_4)$, we define the competitive CTL immune response reproduction number $\mathcal{R}_3$ and the competitive antibody immune response reproduction number $\mathcal{R}_4$, as follows:

$$\mathcal{R}_3 = \frac{aI_2}{q} = \frac{a(s_0 - dH_2)}{q\delta} \quad \text{and} \quad \mathcal{R}_4 = \frac{bV_3}{\sigma} = \frac{kb\alpha q}{\mu a(\alpha+\delta)\sigma}. \tag{2.2}$$

When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, model (1.1) has a unique CTL-antibody-present infection equilibrium, $E_4 = (H_4, I_4, D_4, V_4, W_4, Z_4)$, where $I_4 = \dfrac{q}{a}$, $D_4 = \dfrac{kq}{a(\alpha+\delta)}$, $V_4 = \dfrac{\sigma}{b}$, $W_4 = \dfrac{\mu}{r}(\mathcal{R}_4 - 1)$, $Z_4 = \dfrac{a(s_0 - dH_4)}{pq} - \dfrac{\delta}{p}$ and $H_4 \in \left(0, \dfrac{s_0}{d} - \dfrac{\delta q}{ad}\right)$ is the unique solution of equation

$$s_0 - dH - \beta_1 f\left(H, \frac{\sigma}{b}\right) - \beta_2 \mathrm{g}\left(H, \frac{q}{a}\right) = 0.$$

Based on the above discussion, we have the following result.

Lemma 2.1. *(a) When $\mathcal{R}_0 \le 1$, model (1.1) has only one equilibrium point E_0.*
(b) When $\mathcal{R}_0 > 1$ and $\mathcal{R}_1 \le 1$, model (1.1) has two equilibria E_0 and E_1.
(c) When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 \le 1$ and $\mathcal{R}_4 \le 1$, model (1.1) has four equilibria E_0, E_1, E_2 and E_3.
(d) When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, model (1.1) has five equilibria E_0, E_1, E_2, E_3 and E_4.

Now, the traveling waves solution of model (1.1) is defined as

$$\begin{aligned}&(H(x,t), I(x,t), D(x,t), V(x,t), W(x,t), Z(x,t))\\ &= (H(x+ct), I(x+ct), D(x+ct), V(x+ct), W(x+ct), Z(x+ct)),\end{aligned}$$

where $c > 0$ is the waves speed. Let $s = x + ct$, then from system (1.1), we get the following corresponding waves system:

$$\begin{cases} d_1 H_{ss} - cH_s + s_0 - dH - \beta_1 f(H,V) - \beta_2 \mathrm{g}(H,I) = 0, \\ d_2 I_{ss} - cI_s + \beta_1 f(H,V) + \beta_2 \mathrm{g}(H,I) - \delta I - pIZ = 0, \\ d_3 D_{ss} - cD_s + kI - (\alpha+\delta)D = 0, \\ d_4 V_{ss} - cV_s + \alpha D - \mu V - rVW = 0, \\ d_5 W_{ss} - cW_s + bVW - \sigma W = 0, \\ d_6 Z_{ss} - cZ_s + aIZ - qZ = 0. \end{cases} \tag{2.3}$$

In this work, we will investigate the traveling waves solution $(H(s), I(s), D(s), V(s), W(s), Z(s))$ of system (2.3) with the following nine cases of boundary conditions:

Case 1: When $\mathcal{R}_0 > 1$ and $\mathcal{R}_1 \le 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_0, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_1; \tag{2.4}$$

Case 2: When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$ and $\mathcal{R}_3 \le 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_0, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_2; \tag{2.5}$$

Case 3: When $\mathcal{R}_0 > 1$, $\mathcal{R}_2 > 1$ and $\mathcal{R}_4 \le 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_0, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_3; \tag{2.6}$$

Case 4: When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$ and $\mathcal{R}_3 \le 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_1, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_2; \tag{2.7}$$

Case 5: When $\mathcal{R}_0 > 1$, $\mathcal{R}_2 > 1$ and $\mathcal{R}_4 \le 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_1, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_3; \tag{2.8}$$

Case 6: When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_0, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_4; \tag{2.9}$$

Case 7: When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_1, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_4; \tag{2.10}$$

Case 8: When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_2, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_4; \tag{2.11}$$

Case 9: When $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$,

$$\lim_{s\to-\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_3, \quad \lim_{s\to\infty}(H(s), I(s), D(s), V(s), W(s), Z(s)) = E_4. \tag{2.12}$$

Linearizing system (2.3) at the infection-free equilibrium $E_0 = (H_0, 0, 0, 0, 0, 0)$, from the last five equations of this system, we get

$$\begin{cases} d_2 I_{ss} - cI_s + \beta_1 f_V(H_0, 0)V + \beta_2 \mathrm{g}_I(H_0, 0)I - \delta I = 0, \\ d_3 D_{ss} - cD_s + kI - (\alpha + \delta)D = 0, \\ d_4 V_{ss} - cV_s + \alpha D - \mu V = 0, \\ d_5 W_{ss} - cW_s - \sigma W = 0, \\ d_6 Z_{ss} - cZ_s - qZ = 0. \end{cases} \tag{2.13}$$

Substituting $(I(s), D(s), V(s), W(s), Z(s)) = e^{\gamma s}(k_2, k_3, k_4, k_5, k_6)$ into system (2.13) gives

$$\begin{cases} c\gamma k_2 = (d_2\gamma^2 + \beta_2 \mathrm{g}_I(H_0, 0) - \delta)k_2 + \beta_1 f_V(H_0, 0)k_4, \\ c\gamma k_3 = kk_2 + (d_3\gamma^2 - (\alpha + \delta))k_3, \\ c\gamma k_4 = \alpha k_3 + (d_4\gamma^2 - \mu)k_4, \\ c\gamma k_5 = (d_5\gamma^2 - \sigma)k_5, \\ c\gamma k_6 = (d_6\gamma^2 - q)k_6. \end{cases} \tag{2.14}$$

System (2.14) can be rewritten in the following matrix form

$$c\gamma(k_2, k_3, k_4, k_5, k_6)^T = A(\gamma)(k_2, k_3, k_4, k_5, k_6)^T, \tag{2.15}$$

where

$$A(\gamma) = \begin{pmatrix} d_2\gamma^2 + \beta_2 \mathrm{g}_I(H_0, 0) - \delta & 0 & \beta_1 f_V(H_0, 0) & 0 & 0 \\ k & d_3\gamma^2 - (\alpha + \delta) & 0 & 0 & 0 \\ 0 & \alpha & d_4\gamma^2 - \mu & 0 & 0 \\ 0 & 0 & 0 & d_5\gamma^2 - \sigma & 0 \\ 0 & 0 & 0 & 0 & d_6\gamma^2 - q \end{pmatrix}.$$

In the sequel, we will need the following matrix

$$B(\gamma) = \begin{pmatrix} d_2\gamma^2 + \beta_2 \mathrm{g}_I(H_0, 0) - \delta & 0 & \beta_1 f_V(H_0, 0) \\ k & d_3\gamma^2 - (\alpha + \delta) & 0 \\ 0 & \alpha & d_4\gamma^2 - \mu \end{pmatrix}.$$

The characteristic polynomial of the matrix $A(\gamma)$ is

$$\begin{aligned} P_\gamma(\lambda) &= \det(A(\gamma) - \lambda\mathcal{I}) \\ &= (d_5\gamma^2 - \sigma - \lambda)(d_6\gamma^2 - q - \lambda)\left[(h_1(\gamma) - \lambda)(h_2(\gamma) - \lambda)(h_3(\gamma) - \lambda) + k\alpha\beta_1 f_V(H_0, 0)\right] \\ &= (d_5\gamma^2 - \sigma - \lambda)(d_6\gamma^2 - q - \lambda) \\ &\left[-\lambda^3 + (h_1 + h_2 + h_3)\lambda^2 - (h_1h_2 + h_1h_3 + h_2h_3)\lambda + h_1h_2h_3 + k\alpha\beta_1 f_V(H_0, 0)\right]. \end{aligned}$$

where $h_1(\gamma) = d_2\gamma^2 + \beta_2 g_I(H_0,0) - \delta$, $h_2(\gamma) = d_3\gamma^2 - (\alpha+\delta)$, $h_3(\gamma) = d_4\gamma^2 - \mu$ and $\mathcal{I}$ is the square identity matrix. Let

$$Q_\gamma(\lambda) = \lambda^3 - (h_1+h_2+h_3)\lambda^2 + (h_1h_2+h_1h_3+h_2h_3)\lambda - h_1h_2h_3 - k\alpha\beta_1 f_V(H_0,0). \tag{2.16}$$

Let $\lambda_1(\gamma)$, $\lambda_2(\gamma)$, $\lambda_3(\gamma)$, $\lambda_4(\gamma)$ and $\lambda_5(\gamma)$ be the roots of equation $P_\gamma(\lambda)=0$, where $\lambda_4(\gamma) = d_5\gamma^2 - \sigma$, $\lambda_5(\gamma) = d_6\gamma^2 - q$ and $\lambda_1(\gamma)$, $\lambda_2(\gamma)$ and $\lambda_3(\gamma)$ are the roots of equation $Q_\gamma(\lambda)=0$.

We have

$$Q_0(\lambda) = \lambda^3 + \left(2+\alpha+\delta+\mu+\frac{k\alpha\beta_1 f_V(H_0,0)}{\delta\mu(\alpha+\delta)} + (1-\mathcal{R}_0)\delta\right)\lambda^2 + \left(\mu(\alpha+\delta) + \frac{(\alpha+\delta+\mu)k\alpha\beta_1 f_V(H_0,0)}{\mu(\alpha+\delta)} + \delta(\alpha+\delta+\mu)(1-\mathcal{R}_0)\right)\lambda + \delta\mu(\alpha+\delta)(1-\mathcal{R}_0).$$

Since $\mathcal{R}_0 > 1$, by Descartes rule of signs, equation $Q_0(\lambda)=0$ has at least one positive root. That is, at least one of the $\lambda_i(0)$, $i=1,2,3$, is a positive real number whenever $\mathcal{R}_0>1$. Let $\lambda_1(0)$ denote this number. Setting $e_2(\gamma) = -(h_1+h_2+h_3)$, $e_1(\gamma) = h_1h_2+h_1h_3+h_2h_3$, $e_0(\gamma) = -h_1h_2h_3 - k\alpha\beta_1 f_V(H_0,0)$ and $\lambda(\gamma) = y - \frac{e_2(\gamma)}{3}$, equation (2.16) becomes

$$y^3 + B_1(\gamma)y + B_0(\gamma) = 0, \tag{2.17}$$

where

$$\begin{aligned} B_1(\gamma) &= e_1 - \tfrac{e_2^2}{3} = h_1h_2+h_1h_3+h_2h_3 - \frac{1}{3}(h_1+h_2+h_3)^2 \\ &= \left[d_2d_3+d_2d_4+d_3d_4-\tfrac{1}{3}(d_2+d_3+d_4)^2\right]\gamma^4 + \left[d_3(\beta_2 g_I(H_0,0)-\delta-\mu) + \tfrac{1}{3}(d_4-d_2-d_3)(\beta_2 g_I(H_0,0)\right. \\ &\quad \left. -\alpha-2\delta-\mu) - d_2(\alpha+\delta+\mu)\right]\gamma^2 + \mu^2 - (\alpha+\delta+\mu)(\beta_2 g_I(H_0,0)-\delta-\mu) - \tfrac{1}{3}(\beta_2 g_I(H_0,0) - \\ &\quad \alpha-2\delta-\mu)^2, \end{aligned}$$

$$\begin{aligned} B_0(\gamma) &= \tfrac{2e_2^3}{27} - \tfrac{e_1e_2}{3} + e_0 \\ &= \tfrac{1}{3}(h_1+h_2+h_3)[h_1h_2+h_1h_3+h_2h_3 - \tfrac{2}{9}(h_1+h_2+h_3)^2] - h_1h_2h_3 - k\alpha\beta_1 f_V(H_0,0) \\ &= \tfrac{1}{3}[(d_2+d_3+d_4)\gamma^2 + \beta_2 g_I(H_0,0) - \alpha - 2\delta - \mu]\left\{\left[d_2d_3+d_2d_4+d_3d_4-\tfrac{2}{9}(d_2+d_3+d_4)^2\right]\gamma^4\right. \\ &\quad + \left[d_3(\beta_2 g_I(H_0,0)-\delta-\mu) + \tfrac{4}{9}(5d_4-d_2-d_3)(\beta_2 g_I(H_0,0)-\alpha-2\delta-\mu) - d_2(\alpha+\delta+\mu)\right]\gamma^2 \\ &\quad \left. +\mu^2 - (\alpha+\delta+\mu)(\beta_2 g_I(H_0,0)-\delta-\mu) - \tfrac{2}{9}(\beta_2 g_I(H_0,0)-\alpha-2\delta-\mu)^2\right\} - d_2d_3d_4\gamma^6 - [d_3d_4(\beta_2 g_I(H_0,0) - \\ &\quad \delta) - d_2(d_3\mu + d_4(\alpha+\delta))]\gamma^4 - [d_2\mu(\alpha+\delta) - (\beta_2 g_I(H_0,0)-\delta)d_2(d_3\mu+d_4(\alpha+\delta))]\gamma^2 + \mu(\alpha+\delta)(1-\mathcal{R}_0). \end{aligned}$$

Since $\gamma \in \mathbb{R}_+$, all coefficients $e_2(\gamma)$, $e_1(\gamma)$ and $e_0(\gamma)$ in the equation $Q_\lambda(\gamma)=0$ are real numbers. Thus, equation $Q_\lambda(\gamma)=0$ has at least one real root. The discriminant of equation (2.17) is $\varDelta = -4B_1^3(\gamma) - 27B_0^2(\gamma)$. Recall that, since $\varDelta \neq 0$, if $\varDelta > 0$, equation $Q_\lambda(\gamma)=0$ has three distinct real roots; and if $\varDelta < 0$, equation $Q_\lambda(\gamma)=0$ has one real root and two non-real complex conjugate roots. In what follows, we assume that $\varDelta<0$. Then, using the Cardan formula, the real part of the roots of equation (2.16) are given by

$$\lambda_1(\gamma) = \frac{1}{3}[(d_2+d_3+d_4)\gamma^2 + \beta_2 g_I(H_0,0) - \alpha - 2\delta - \mu] + \left\{-\frac{B_0(\gamma)}{2} + \left\{\frac{B_0^2(\gamma)}{4} + \frac{B_1^3(\gamma)}{27}\right\}^{\frac{1}{2}}\right\}^{\frac{1}{3}} + \left\{-\frac{B_0(\gamma)}{2} - \left\{\frac{B_0^2(\gamma)}{4} + \frac{B_1^3(\gamma)}{27}\right\}^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

$$Re\{\lambda_2(\gamma)\} = Re\{\lambda_3(\gamma)\} = \frac{1}{3}[(d_2+d_3+d_4)\gamma^2 + \beta_2 g_I(H_0,0) - \alpha - 2\delta - \mu] - \frac{1}{2}\left\{-\frac{B_0(\gamma)}{2} + \left\{\frac{B_0^2(\gamma)}{4} + \frac{B_1^3(\gamma)}{27}\right\}^{\frac{1}{2}}\right\}^{\frac{1}{3}} - \frac{1}{2}\left\{-\frac{B_0(\gamma)}{2} - \left\{\frac{B_0^2(\gamma)}{4} + \frac{B_1^3(\gamma)}{27}\right\}^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

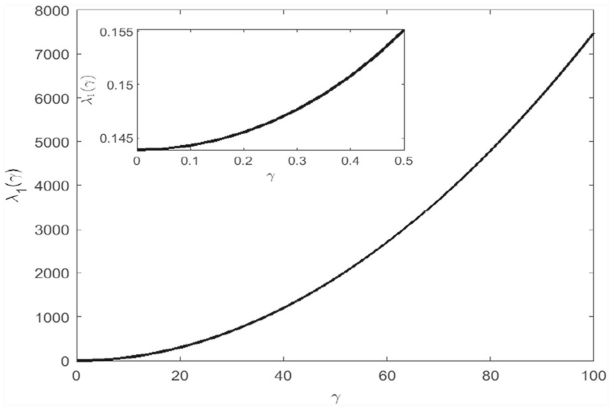

FIG. 1. Plot of function $\lambda_1(\gamma)$ with parameter values for Case I in Table 1

where $Re\{\lambda_i(\gamma)\}$ denotes the real part of $\lambda_i(\gamma)$. Define

$$\tilde{\lambda}(\gamma) = \max\{\lambda_1(\gamma), \lambda_4(\gamma), \lambda_5(\gamma)\}.$$

Hence, we have the following result.

Lemma 2.2. *Assume that for $\gamma > 0$, we have $B_0'(\gamma) < 0$ and $3B_1'(\gamma)B_0(\gamma) - 2B_0'(\gamma)B_1(\gamma) > 0$. Then, the following statements hold.*

(i) $\tilde{\lambda}(\gamma)$ is an even and strictly convex function for all $\gamma > 0$.
(ii) $\tilde{\lambda}(\gamma)$ is an increasing function for all $\gamma > 0$.

Proof. It is obvious that the functions $\lambda_4(\gamma) = d_5\gamma^2 - \sigma$ and $\lambda_5(\gamma) = d_6\gamma^2 - q$ are even, strictly convex and increasing for all $\gamma \in [0, \infty)$.

Consider the function $\lambda_1(\gamma)$. Then, it is clear from the expression of $\lambda_1(\gamma)$ that $\lambda_1(\gamma)$ is an even function.

The function $\lambda_1(\gamma)$ can be rewritten as

$$\lambda_1(\gamma) = \frac{1}{3}[(d_2 + d_3 + d_4)\gamma^2 + \beta_2 g_I(H_0, 0) - \alpha - 2\delta - \mu] - \frac{B_0^{1/3}(\gamma)}{2^{1/3}}\left[(1 - \xi(\gamma))^{1/3} + (1 + \xi(\gamma))^{1/3}\right],$$

where

$\xi(\gamma) = \sqrt{1 + \frac{4B_1^3(\gamma)}{27B_0^2(\gamma)}}$. Since we have assumed that $\Delta < 0$, then $1 + \frac{4B_1^3(\gamma)}{27B_0^2(\gamma)} > 0$ for all $\gamma > 0$. Thus $\xi(\gamma)$ exist and is a positive real function. Hence, since $(1 - \xi(\gamma))^{1/3} + (1 + \xi(\gamma))^{1/3} > 0$ and $(1 - \xi(\gamma))^{-2/3} - (1 + \xi(\gamma))^{-2/3} > 0$, differentiation of $\lambda_1(\gamma)$ with respect to γ yields

$$\begin{aligned}\lambda_1'(\gamma) = &\frac{2}{3}(d_2 + d_3 + d_4)\gamma - \frac{1}{3 \times 2^{1/3}}B_0'(\gamma)B_0^{-2/3}(\gamma)\left[(1 - \xi(\gamma))^{1/3} + (1 + \xi(\gamma))^{1/3}\right] \\ &+ \frac{1}{3 \times 2^{4/3}\xi(\gamma)}B_0^{-8/3}(\gamma)B_1^2(\gamma)\left(3B_1'(\gamma)B_0(\gamma) - 2B_0'(\gamma)B_1(\gamma)\right)\left[(1 - \xi(\gamma))^{-2/3} - (1 + \xi(\gamma))^{-2/3}\right] \\ &> 0, \text{ for } \gamma > 0,\end{aligned}$$

which indicates that $\lambda_1(\gamma)$ is increasing on $\gamma \in (0, \infty)$, provided that $B_0'(\gamma) < 0$ and $3B_1'(\gamma)B_0(\gamma) - 2B_0'(\gamma)B_1(\gamma) > 0$.

Note that the strict convexity of $\lambda_1(\gamma)$ is hard to prove analytically. Using the parameter values of Table 1, a case is presented graphically in Fig. 1, where these hypotheses of Lemma 2.2 hold.

Clearly, Fig. 1 shows that function $\lambda_1(\gamma)$, for the chosen parameter values for Case I in Table 1, is indeed convex and increasing for all $\gamma > 0$. Note that for the parameter values of Case II in Table 1, we obtain a graph close to that of Fig. 1. □

For $\gamma > 0$, we define the function

$$h(\gamma) = \frac{\tilde{\lambda}(\gamma)}{\gamma}, \tag{2.18}$$

and state the following result.

Lemma 2.3. *Let $\mathcal{R}_0 > 1$ and $B_0(\gamma) < 0$. Then, under the conditions of Lemma 2.2, one has:*

(1) Function $h(\gamma)$ is positive for all $\gamma > 0$.
(2) Function $h(\gamma)$ admits a minimum value c^*. Furthermore, there exists a unique $\gamma^* > 0$ such that $c^* = h(\gamma^*)$.
(3) For every $c > c^*$, equation $h(\gamma) = c$ has exactly two roots $\gamma_1(c)$ and $\gamma_2(c)$ satisfying $0 < \gamma_1(c) < \gamma^* < \gamma_2(c)$
(4) For every $c > c^*$, there exist $\gamma_c > 0$ and a positive eigenvector $\mathbf{w} = (k_2, k_3, k_4)^T$ satisfying

$$h(\gamma_c) = c \quad \text{and} \quad B(\gamma_c)\mathbf{w} = c\gamma_c\mathbf{w}.$$

Proof. Assume that $\mathcal{R}_0 > 1$ and $B_0(\gamma) < 0$.

(i) Then, if $\beta_2 g_I(H_0, 0) - \alpha - 2\delta - \mu \geq 0$, item (1) is valid. Now, we consider the case $\beta_2 g_I(H_0, 0) - \alpha - 2\delta - \mu < 0$. Then $\lambda_1(\gamma) > 0$ for all $\gamma \geq \gamma_-$, where

$$\gamma_- = \sqrt{-(\beta_2 g_I(H_0, 0) - \alpha - 2\delta - \mu)/(d_2 + d_3 + d_4)}.$$

For $\gamma < \gamma_-$, set $z(\gamma) = (d_2\gamma^2 + \beta_2 g_I(H_0, 0) - \delta)(d_3\gamma^2 - (\alpha + \delta))(d_4\gamma^2 - \mu)$. Then, one has

$z'(\gamma) = 2\gamma[3d_2d_3d_4\gamma^4 - 2(d_2d_3\mu + d_2d_4(\alpha+\delta) - d_3d_4(\beta_2 g_I(H_0, 0) - \delta))\gamma^2 + d_2\mu(\alpha+\delta) - (\beta_2 g_I(H_0, 0) - \delta)(d_3\mu + d_4(\alpha + \delta))]$ and $z(0) = \mu(\alpha + \delta)(\beta_2 g_I(H_0, 0) - \delta) > -k\alpha\beta_1 f_V(H_0, 0)$, since $\mathcal{R}_0 > 1$. Let $y = \gamma_-^2$ and $\ell(y) = 3d_2d_3d_4y^2 - 2(d_2d_3\mu + d_2d_4(\alpha + \delta) - d_3d_4(\beta_2 g_I(H_0, 0) - \delta))y + d_2\mu(\alpha + \delta) - (\beta_2 g_I(H_0, 0) - \delta)(d_3\mu + d_4(\alpha + \delta))$. By Descartes rule of signs, we have:

if $d_2d_3\mu + d_2d_4(\alpha+\delta) - d_3d_4(\beta_2 g_I(H_0, 0) - \delta) > 0$ and $d_2\mu(\alpha+\delta) - (\beta_2 g_I(H_0, 0) - \delta)(d_3\mu + d_4(\alpha+\delta)) > 0$, then $\ell(y)$ has exactly two positive roots. Let y_1 and y_2 be these two positive roots, with $y_1 < y_2$. Then $\max_{\gamma\in(0,\gamma_-]} z(\gamma) = \max\{z(\sqrt{y_1}), z(\gamma_-)\}$, since $z(\gamma)$ is increasing for $\gamma \in (0, \sqrt{y_1}]$, decreasing for $\gamma \in (\sqrt{y_1}, \sqrt{y_2}]$ and increasing for $\gamma \in (\sqrt{y_2}, \infty]$. Therefore, if $z(\gamma_-) > 0$ or $z(\sqrt{y_1}) > 0$, $\max_{\gamma\in(0,\gamma_-]} z(\gamma) > 0$ and if $z(\sqrt{y_1}) < z(\gamma_-) < 0$, or $z(\gamma_-) < z(\sqrt{y_1}) < 0$, $\max_{\gamma\in(0,\gamma_-]} z(\gamma) > z(0) > -k\alpha\beta_1 f_V(H_0, 0)$;

if $d_2d_3\mu + d_2d_4(\alpha+\delta) - d_3d_4(\beta_2 g_I(H_0, 0) - \delta) > 0$, $d_2\mu(\alpha+\delta) - (\beta_2 g_I(H_0, 0) - \delta)(d_3\mu + d_4(\alpha+\delta)) < 0$, $d_2d_3\mu + d_2d_4(\alpha+\delta) - d_3d_4(\beta_2 g_I(H_0, 0) - \delta) < 0$ and $d_2\mu(\alpha+\delta) - (\beta_2 g_I(H_0, 0) - \delta)(d_3\mu + d_4(\alpha+\delta)) < 0$, then $\ell(y)$ has one positive root. Let y_3 be this positive root. Then, $\max_{\gamma\in(0,\gamma_-]} z(\gamma) = \max\{z(0), z(\gamma_-)\} > -k\alpha\beta_1 f_V(H_0, 0)$, since $z(\gamma)$ is decreasing for $\gamma \in (0, \sqrt{y_3}]$ and increasing for $\gamma \in (\sqrt{y_3}, \infty]$.

if $d_2d_3\mu + d_2d_4(\alpha+\delta) - d_3d_4(\beta_2 g_I(H_0, 0) - \delta) < 0$ and $d_2\mu(\alpha+\delta) - (\beta_2 g_I(H_0, 0) - \delta)(d_3\mu + d_4(\alpha+\delta)) > 0$, then $z'(\gamma) > 0$. Thus $\max_{\gamma\in(0,\gamma_-]} z(\gamma) = z(\gamma_-) > z(0) > -k\alpha\beta_1 f_V(H_0, 0)$, since $z(\gamma)$ is increasing for $\gamma \in (0, \infty)$. As a result, we have $z(\gamma) > -k\alpha\beta_1 f_V(H_0, 0)$, for all $\gamma \in (0, \gamma_-]$.

It follows from the above discussion that $\lambda_1(\gamma) > 0$ for all $\gamma > 0$. Combining the definition of $h(\gamma) = \frac{\max\{\lambda_1(\gamma), \lambda_4(\gamma), \lambda_5(\gamma)\}}{\gamma}$, item (1) is proved.

(ii) From $\mathcal{R}_0 > 1$, it is known that $\lambda_1(0) > 0$, which implies $\lim_{\gamma\to 0^+} \frac{\lambda_1(\gamma)}{\gamma} = \infty$. On the other hand $\frac{\lambda_1(\gamma)}{\gamma} \geq \frac{1}{3\gamma}[(d_2 + d_3 + d_4)\gamma^2 + \beta_2 g_I(H_0, 0) - \alpha - 2\delta - \mu]$. Hence $\lim_{\gamma\to\infty^+} \frac{\lambda_1(\gamma)}{\gamma} = \infty$. Accordingly, there exists a minimum value c^* thanks to the continuity of the function $\frac{\lambda_1(\gamma)}{\gamma}$. So, using the definition of $h(\gamma)$ with $\lambda_4(\gamma) = d_5\gamma^2 - \sigma$ and $\lambda_5(\gamma) = d_6\gamma^2 - q$, it follows that item (2) is valid.

(iii) From (2) and the intermediate value theorem, it follows that there exists at least one $\gamma^* > 0$ such that $c^* = h(\gamma^*)$, and the equation $c = h(\gamma)$ has at least two solutions, $\gamma_1(c)$ and $\gamma_2(c)$, satisfying

$0 < \gamma_1(c) < \gamma^* < \gamma_2(c)$ for every $c > c^*$. We now prove the uniqueness of $\gamma_1(c)$ and $\gamma_2(c)$. By contradiction, assume that there is $\gamma_1(c)$, $\tilde{\gamma}_1(c)$ and $\gamma_2(c)$ such that

$$c = h(\gamma_1(c)) = h(\tilde{\gamma}_1(c)) = h(\gamma_2(c)), \quad 0 < \gamma_1(c) < \tilde{\gamma}_1(c) < \gamma^*(c) < \gamma_2(c).$$

This indicates that the straight line $y = c\gamma$ and the curve $y = h(\gamma)$ in the positive quadrant has three intersections $(c\gamma_1(c), h(\gamma_1(c)))$, $(c\tilde{\gamma}_1(c), h(\tilde{\gamma}_1(c)))$, and $(c\gamma_2(c), h(\gamma_2(c)))$. This contradicts the strict convexity of function $h(\gamma)$ for all $\gamma \in (0, \infty)$. Thus item (3) holds.

(iv) It is obvious that $\lambda_1(\gamma)$, $\lambda_4(\gamma)$ and $\lambda_5(\gamma)$ are also the eigenvalues of $B(\gamma)$, and $\lambda_1(\gamma)$ is the principle eigenvalue of $B(\gamma)$. Hence, the eigenvector corresponding to $\lambda_1(\gamma)$ is positive. By the definition of $h(\gamma)$, item (4) follows. This completes the proof. □

3. Upper and lower solutions

This section is devoted to the construction of suitable upper and lower solutions. To do this, we introduce the following auxiliary system

$$\begin{cases} d_1 H_{ss} - cH_s + s_0 - dH - \beta_1 f(H, V) - \beta_2 \mathrm{g}(H, I) = 0, \\ d_2 I_{ss} - cI_s + \beta_1 f(H, V) + \beta_2 \mathrm{g}(H, I) - \delta I - pIZ = 0, \\ d_3 D_{ss} - cD_s + kI - (\alpha + \delta)D = 0, \\ d_4 V_{ss} - cV_s + \alpha D - \mu V - rVW = 0, \\ d_5 W_{ss} - cW_s + bVW - \sigma W - \varepsilon W^2 = 0, \\ d_6 Z_{ss} - cZ_s + aIZ - qZ - \varepsilon Z^2 = 0, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a real constant. We define the following continuous and positive functions.

$$\overline{H} := H_0, \quad (\overline{I}, \overline{D}, \overline{V})^T := e^{\gamma_1 s}(k_2, k_3, k_4)^T, \quad \overline{W} := \frac{bk_4}{\varepsilon} e^{\gamma_1 s} \quad \text{and} \quad \overline{Z} := \frac{ak_2}{\varepsilon} e^{\gamma_1 s},$$

where $s \in \mathbb{R}$. The parameters k_2, k_3, k_4 and $\gamma_1 := \gamma_1(c)$ are defined in Lemma 2.3.

We now demonstrate in the following result that the function $(\overline{H}, \overline{I}, \overline{D}, \overline{V}, \overline{W}, \overline{Z})$ is an upper solution of system (3.1).

Lemma 3.1. *The functions* $(\overline{H}, \overline{I}, \overline{D}, \overline{V}, \overline{W}, \overline{Z})$ *satisfy*

$$d_1 \overline{H}_{ss} - c\overline{H}_s + s_0 - d\overline{H} - \beta_1 f(\overline{H}, V) - \beta_2 g(\overline{H}, I) \le 0, \tag{3.2}$$
$$d_2 \overline{I}_{ss} - c\overline{I}_s + \beta_1 f(\overline{H}, \overline{V}) + \beta_2 g(\overline{H}, \overline{I}) - \delta\overline{I} - p\overline{I}Z \le 0, \tag{3.3}$$
$$d_3 \overline{D}_{ss} - c\overline{D}_s + kI - (\alpha + \delta)\overline{D} \le 0, \tag{3.4}$$
$$d_4 \overline{V}_{ss} - c\overline{V}_s + \alpha D - \mu\overline{V} - r\overline{V}W \le 0, \tag{3.5}$$
$$d_5 \overline{W}_{ss} - c\overline{W}_s + b\overline{V}\overline{W} - \sigma\overline{W} - \varepsilon\overline{W}^2 \le 0, \tag{3.6}$$
$$d_6 \overline{Z}_{ss} - c\overline{Z}_s + a\overline{I}\overline{Z} - q\overline{Z} - \varepsilon\overline{Z}^2 \le 0. \tag{3.7}$$

Proof. Using the fact that $\beta_1 f(\overline{H}, V) + \beta_2 \mathrm{g}(\overline{H}, I) \ge 0$ for all $V \ge 0$ and $I \ge 0$, it follows that inequality (3.2) is valid.

From hypothesis (Q1), we have $f(H_0,\overline{V}) \le f_V(H_0,0)\overline{V}$ and $g(H_0,\overline{I}) \le g_I(H_0,0)\overline{I}$. Thus it follows that

$$
\begin{aligned}
&\begin{pmatrix} d_2\overline{I}_{ss} - c\overline{I}_s + \beta_1 f(\overline{H},\overline{V}) + \beta_2 g(\overline{H},\overline{I}) - \delta\overline{I} - p\overline{I}Z \\ d_3\overline{D}_{ss} - c\overline{D}_s + kI - (\alpha+\delta)\overline{D} \\ d_3\overline{V}_{ss} - c\overline{V}_s + \alpha D - \mu\overline{V} - r\overline{V}W \end{pmatrix} \\
&= \begin{pmatrix} d_2\gamma_1^2 k_2 e^{\gamma_1 s} - c\gamma_1 k_2 e^{\gamma_1 s} + \beta_1 f(H_0, k_4 e^{\gamma_1 s}) + \beta_2 g(H_0, k_2 e^{\gamma_1 s}) - \delta k_2 e^{\gamma_1 s} - p k_2 e^{\gamma_1 s} Z \\ d_3\gamma_1^2 k_3 e^{\gamma_1 s} - c\gamma_1 k_3 e^{\gamma_1 s} + k k_2 e^{\gamma_1 s} - (\alpha+\delta) k_3 e^{\gamma_1 s} \\ d_4\gamma_1^2 k_4 e^{\gamma_1 s} - c\gamma_1 k_4 e^{\gamma_1 s} + \alpha k_3 e^{\gamma_1 s} - \mu k_4 e^{\gamma_1 s} - r k_4 e^{\gamma_1 s} W \end{pmatrix} \\
&\le e^{\gamma_1 s} \begin{pmatrix} d_2\gamma_1^2 k_2 - c\gamma_1 k_2 + \beta_1 f(H_0, k_4 e^{\gamma_1 s}) e^{-\gamma_1 s} + \beta_2 g(H_0, k_2 e^{\gamma_1 s}) e^{-\gamma_1 s} - \delta k_2 \\ d_3\gamma_1^2 k_3 - c\gamma_1 k_3 + k k_2 - (\alpha+\delta) k_3 \\ d_4\gamma_1^2 k_3 - c\gamma_1 k_3 + \alpha k_3 - \mu k_4 \end{pmatrix} \\
&\le e^{\gamma_1 s} \begin{pmatrix} d_2\gamma_1^2 - c\gamma_1 + \beta_2 g_I(H_0,0) - \delta & 0 & \beta_1 f_V(H_0,0) \\ k & d_3\gamma_1^2 - c\gamma_1 - (\alpha+\delta) & \\ 0 & \alpha & d_4\gamma_1^2 - c\gamma_1 - \mu \end{pmatrix} \begin{pmatrix} k_2 \\ k_3 \\ k_4 \end{pmatrix} \\
&= 0.
\end{aligned}
$$

This indicates that inequalities (3.3), (3.4) and (3.5) hold.

Using the definition of γ_1 in Lemma 2.3, since $\overline{V} = k_4 e^{\gamma_1 s}$ and $\overline{W} = \dfrac{bk_4}{\varepsilon} e^{\gamma_1 s}$, we obtain

$$
\begin{aligned}
&d_5\overline{W}_{ss} - c\overline{W}_s + b\overline{V}\overline{W} - \sigma\overline{W} - \varepsilon\overline{W}^2 \\
&= d_5\gamma_1^2 \frac{bk_4}{\varepsilon} e^{\gamma_1 s} - c\gamma_1 \frac{bk_4}{\varepsilon} e^{\gamma_1 s} - \sigma \frac{bk_4}{\varepsilon} e^{\gamma_1 s} + \left(b\overline{V} - \varepsilon\overline{W}\right) \frac{bk_4}{\varepsilon} e^{\gamma_1 s} \\
&= \left(d_5\gamma_1^2 - c\gamma_1 - \sigma\right) \frac{bk_4}{\varepsilon} e^{\gamma_1 s} \le 0,
\end{aligned}
$$

which implies that inequality (3.6) holds.

Also, from the definition of γ_1 in Lemma 2.3, since $\overline{I} = k_2 e^{\gamma_1 s}$ and $\overline{Z} = \dfrac{ak_2}{\varepsilon} e^{\gamma_1 s}$, we have

$$
\begin{aligned}
&d_6\overline{Z}_{ss} - c\overline{Z}_s + a\overline{I}\overline{Z} - q\overline{Z} - \varepsilon\overline{Z}^2 \\
&= d_6\gamma_1^2 \frac{ak_2}{\varepsilon} e^{\gamma_1 s} - c\gamma_1 \frac{ak_2}{\varepsilon} e^{\gamma_1 s} - q \frac{ak_2}{\varepsilon} e^{\gamma_1 s} + \left(a\overline{I} - \varepsilon\overline{Z}\right) \frac{ak_2}{\varepsilon} e^{\gamma_1 s} \\
&= \left(d_6\gamma_1^2 - c\gamma_1 - q\right) \frac{ak_2}{\varepsilon} e^{\gamma_1 s} \le 0,
\end{aligned}
$$

which implies that inequality (3.7) holds. □

Now, define the following functions:

$$
\begin{aligned}
&\underline{H} := \max\{H_0 - M_1 e^{\gamma_0 s}, 0\}, \quad \underline{I} \\
&:= \max\{\overline{I} - M_2 e^{\tilde{\gamma} s}, 0\}, \quad \underline{D} := \max\{\overline{D} - M_3 e^{\tilde{\gamma} s}, 0\}, \quad \underline{V} := \max\{\overline{V} - M_4 e^{\tilde{\gamma} s}, 0\}, \\
&\underline{W} := \max\{e^{\alpha_1 s}, e^{\alpha_1(2s^* - s)}\} \quad \text{and} \quad \underline{Z} := \max\{e^{\alpha_2 s}, e^{\alpha_2(2s^{**} - s)}\},
\end{aligned}
$$

where $s \in \mathbb{R}$. The parameters γ_0, $\tilde{\gamma}$ satisfy $0 < \gamma_0 < \min\left\{\frac{c}{d_1}, \gamma_1\right\}$ and $0 < \tilde{\gamma} < \min\left\{\frac{c}{d_2}, \frac{c}{d_3}, \frac{c}{d_4}, \gamma_1\right\}$. There are positive constants M_i, $i = 1, \cdots, 4$, α_1, α_2, s^* and s^{**} such that a lower solution is guaranteed by the above functions. We now demonstrate that $(\underline{H}, \underline{I}, \underline{D}, \underline{V}, \underline{W}, \underline{Z})$ is a lower solution of system (3.1). We have the following result.

Lemma 3.2. *The functions* $(\underline{H}, \underline{I}, \underline{D}, \underline{V}, \underline{W}, \underline{Z})$ *satisfy the following inequalities for appropriate choices of positive parameters* M_i, $i = 1, \cdots, 4$, α_1, α_2, s^* *and* s^{**}.

$$d_1\underline{H}_{ss} - c\underline{H}_s + s_0 - d\underline{H} - \beta_1 f(\underline{H}, \overline{V}) - \beta_2 g(\underline{H}, \overline{I}) \geq 0,\ \text{for } s \in (-\infty, s_1^+) \cup (s_1^+, \infty),\ s_1^+ := \frac{1}{\gamma_0} \ln \frac{H_0}{M_1}. \tag{3.8}$$

$$d_2\underline{I}_{ss} - c\underline{I}_s + \beta_1 f(\underline{H}, \underline{V}) + \beta_2 g(\underline{H}, \underline{I}) - \delta\underline{I} - p\underline{I}\overline{Z} \geq 0,\ \text{for } s \in (-\infty, s_2^+) \cup (s_2^+, \infty),\ s_2^+ := \frac{1}{\tilde{\gamma} - \gamma_1} \ln \frac{k_2}{M_2}. \tag{3.9}$$

$$d_3\underline{D}_{ss} - c\underline{D}_s + k\underline{I} - (\alpha + \delta)\underline{D} \geq 0,\ \text{for } s \in (-\infty, s_3^+) \cup (s_3^+, \infty),\ s_3^+ := \frac{1}{\tilde{\gamma} - \gamma_1} \ln \frac{k_3}{M_3}. \tag{3.10}$$

$$d_4\underline{V}_{ss} - c\underline{V}_s + \alpha\underline{D} - \mu\underline{V} - r\underline{V}\overline{W} \geq 0,\ \text{for } s \in (-\infty, s_4^+) \cup (s_4^+, \infty),\ s_4^+ := \frac{1}{\tilde{\gamma} - \gamma_1} \ln \frac{k_4}{M_4}. \tag{3.11}$$

$$d_5\underline{W}_{ss} - c\underline{W}_s + b\underline{V}\underline{W} - \sigma\underline{W} - \varepsilon\underline{W}^2 \geq 0,\ \text{for } s \in (-\infty, s^*) \cup (s^*, \infty),\ s^* := s_4^+ \tag{3.12}$$

$$d_6\underline{Z}_{ss} - c\underline{Z}_s + a\underline{I}\underline{Z} - q\underline{Z} - \varepsilon\underline{Z}^2 \geq 0,\ \text{for } s \in (-\infty, s^{**}) \cup (s^{**}, \infty),\ s^{**} := s_2^+. \tag{3.13}$$

Proof. If $s \geq s_1^+$, then $\underline{H} = 0$, and thus inequality (3.8) holds. Assume now that $s < s_1^+$, then $\underline{H} = H_0 - M_1 e^{\gamma_0 s}$. For this case, employing the fact that $0 < \gamma_0 < \min\left\{\frac{c}{d_1}, \gamma_1\right\}$, and letting $M_1 > H_0$ such that $s_1^+ < 0$, we get

$$\begin{aligned}
&d_1\underline{H}_{ss} - c\underline{H}_s + s_0 - d\underline{H} - \beta_1 f(\underline{H}, \overline{V}) - \beta_2 \mathrm{g}(\underline{H}, \overline{I}) \\
&= -d_1\gamma_0^2 M_1 e^{\gamma_0 s} + c\gamma_0 M_1 e^{\gamma_0 s} + dM_1 e^{\gamma_0 s} - \beta_1 f(\underline{H}, \overline{V}) - \beta_2 \mathrm{g}(\underline{H}, \overline{I}) \\
&\geq \left[M_1(d + \gamma_0(c - d_1\gamma_0)) - (\beta_1 f_V(H_0, 0)k_4 + \beta_2 \mathrm{g}_I(H_0, 0)k_2)\, e^{(\gamma_1 - \gamma_0)s}\right] e^{\gamma_0 s} \\
&\geq \left[M_1 d - (\beta_1 f_V(H_0, 0)k_4 + \beta_2 \mathrm{g}_I(H_0, 0)k_2)\right] e^{\gamma_0 s}.
\end{aligned}$$

Then, taking $M_1 > \max\left\{H_0, \frac{\beta_1 f_V(H_0,0)k_4 + \beta_2 \mathrm{g}_I(H_0,0)k_2}{d}\right\}$ and returning to the calculation above, it follows that (3.8) holds.

If $s \geq s_2^+$, then $\underline{I} = 0$, and hence inequality (3.9) holds. Assume that $s < s_2^+$, and also consider the case $s < \min\left\{s_2^+, s_3^+, s_4^+\right\}$, then $\underline{I} = \overline{I} - M_2 e^{\tilde{\gamma}s}$, $\underline{V} = \overline{V} - M_4 e^{\tilde{\gamma}s}$ and $0 < \tilde{\gamma} < \frac{c}{d_2}$. Thus,

$$\begin{aligned}
&d_2\underline{I}_{ss} - c\underline{I}_s + \beta_1 f(\underline{H}, \underline{V}) + \beta_2 \mathrm{g}(\underline{H}, \underline{I}) - \delta\underline{I} - p\underline{I}\overline{Z} \\
&\geq d_2\left(\overline{I}_{ss} - M_2\tilde{\gamma}^2 e^{\tilde{\gamma}s}\right) - c\left(\overline{I}_s - M_2\tilde{\gamma}e^{\tilde{\gamma}s}\right) - \delta\left(\overline{I} - M_2 e^{\tilde{\gamma}s}\right) - p\overline{Z}\left(\overline{I} - M_2 e^{\tilde{\gamma}s}\right), \\
&= d_2\overline{I}_{ss} - c\overline{I}_s - \delta\overline{I} + \beta_1 f_V(H_0, 0)\overline{V} + \beta_2 \mathrm{g}_I(H_0, 0)\overline{I} + M_2[\delta + \tilde{\gamma}(c - d_2\tilde{\gamma})]e^{\tilde{\gamma}s} \\
&\quad - p\overline{Z}\left(\overline{I} - M_2 e^{\tilde{\gamma}s}\right) - \beta_1 f_V(H_0, 0)\overline{V} - \beta_2 \mathrm{g}_I(H_0, 0)\overline{I} \\
&\geq \left[M_2\delta - \frac{pak_2^2}{\varepsilon}e^{(2\gamma_1 - \tilde{\gamma})s} + \frac{pak_2M_2}{\varepsilon}e^{\gamma_1 s} - \beta_1 f_V(H_0, 0)k_4 e^{(\gamma_1 - \tilde{\gamma})s} - \beta_2 \mathrm{g}_I(H_0, 0)k_2 e^{(\gamma_1 - \tilde{\gamma})s}\right] \\
&\quad e^{\tilde{\gamma}s} := \Phi_1(s)e^{\tilde{\gamma}s}.
\end{aligned}$$

Since $0 < \tilde{\gamma} < \gamma_1$, thanks to the classical limit of the exponential function at $-\infty$, we have $\lim_{s\to-\infty}\Phi_1(s) = M_2\delta > 0$. This means that there exists $s_2^* < 0$ such that $\Phi_1(s) > 0$ for all $s < s_2^*$. Taking $s_2^+ = s_2^*$, we get $M_2 = k_2 e^{(\gamma_1 - \tilde{\gamma})s_2^*}$. It is not difficult to check that $0 < M_2 < k_2$ and the inequality (3.9) holds by returning to the calculation above.

If $s_4^+ \leq s < s_2^+$, then $\underline{I} = \overline{I} - M_2 e^{\tilde{\gamma}s}$, $\underline{V} = 0$ and inequality (3.9) can be established similarly as above.

If $s_3^+ \leq s < s_2^+$, then $\underline{D} = 0$, and hence inequality (3.10) holds. Next, we consider $s < s_3^+$. Let $0 < M_3 < \min\left\{k_3, \frac{k_3}{k_2}M_2\right\}$. It follows that $s_3^+ < s_2^+ < 0$. Then, $\underline{D} = \overline{D} - M_3 e^{\tilde{\gamma}s}$ and $\underline{I} = \overline{I} - M_2 e^{\tilde{\gamma}s}$.

Hence, from the third equation of system (2.14), and the facts that $s<0$ and $<0<\tilde\gamma<\frac{c}{d_3}$, we get

$$\begin{aligned} d_3\underline{D}_{ss}-c\underline{D}_s+k\underline{I}-(\alpha+\delta)\underline{D} &\geq d_3\left(\overline{D}_{ss}-M_3\tilde\gamma^2e^{\tilde\gamma s}\right)-c\left(\overline{D}_s-M_3\tilde\gamma e^{\tilde\gamma s}\right)-(\alpha+\delta)\left(\overline{D}-M_3e^{\tilde\gamma s}\right)\\ &=\left(d_3\overline{D}_{ss}-c\overline{D}_s+k\overline{I}-(\alpha+\delta)\overline{D}\right)-k\overline{I}+M_3\left((\alpha+\delta)+\tilde\gamma(c-d_3\tilde\gamma)\right)e^{\tilde\gamma s}\\ &\geq\left[M_3(\alpha+\delta)-kk_2e^{(\gamma_1-\tilde\gamma)s}\right]e^{\tilde\gamma s}. \end{aligned} \tag{3.14}$$

Since $0<\tilde\gamma<\gamma_1$, thanks to the classical limit of the exponential function at $-\infty$, we have $\lim_{s\to-\infty}\left[M_3(\alpha+\delta)-kk_2e^{(\gamma_1-\tilde\gamma)s}\right]=M_3(\alpha+\delta)>0$, which implies that there exists $s_3^*<0$ such that $\left[M_3(\alpha+\delta)-kk_2e^{(\gamma_1-\tilde\gamma)s}\right]>0$ for all $s<s_3^*$. Thus taking $0<M_3<\min\left\{k_3,\frac{k_3}{k_2}M_2\right\}$, the inequality (3.10) holds by repeating the above calculation.

If $s_4^+\leq s<s_3^+<s_2^+$, then $\underline{I}=\overline{I}-M_2e^{\tilde\gamma s}$, $\underline{D}=\overline{D}-M_3e^{\tilde\gamma s}$, $\underline{V}=0$ and inequality (3.10) can be obtained similarly.

Now, we focus on the inequality (3.11) about $\underline{V}$. For this, we first let $0<M_4<\min\left\{k_4,\frac{k_4}{k_3}M_3\right\}$. Then, it follows that $s_4^+<s_3^+<0$. Recall that when $s\geq s_4^+$, $\underline{V}=0$, which means that inequality (3.11) holds.

Now, we consider $s<s_4^+$. Then, from the above discussion, we get $s<s_4^+<s_3^+<0$. Thus $\underline{V}=\overline{V}-M_4e^{\tilde\gamma s}$ and $\underline{D}=\overline{D}-M_3e^{\tilde\gamma s}$. Then, from the third equation of system (2.14) and the facts that $s<0$ and $<0<\tilde\gamma<\frac{c}{d_4}$, we obtain

$$\begin{aligned} d_4\underline{V}_{ss}-c\underline{V}_s+k\underline{I}-\mu\underline{V}-r\underline{V}\overline{W} &\geq d_4\left(\overline{V}_{ss}-M_4\tilde\gamma^2e^{\tilde\gamma s}\right)-c\left(\overline{V}_s-M_4\tilde\gamma e^{\tilde\gamma s}\right)-\mu\left(\overline{V}-M_4e^{\tilde\gamma s}\right)-r\underline{V}\overline{W}\\ &=\left(d_4\overline{V}_{ss}-c\overline{V}_s+\alpha\overline{D}-\mu\overline{V}\right)-\alpha\overline{D}-r\underline{V}\overline{W}+M_4\left(\mu+\tilde\gamma(c-d_4\tilde\gamma)\right)e^{\tilde\gamma s}\\ &\geq\left[M_4\mu-\alpha k_3e^{(\gamma_1-\tilde\gamma)s}-\frac{brk_4^2}{\varepsilon}e^{(2\gamma_1-\tilde\gamma)s}+\frac{brk_4M_4}{\varepsilon}e^{\gamma_1 s}\right]e^{\tilde\gamma s}:=\Phi_2(s)e^{\tilde\gamma s}. \end{aligned}$$

Then, since $0<\tilde\gamma<\gamma_1$, thanks to the classical limit of the exponential function at $-\infty$, we have $\lim_{s\to-\infty}\Phi_2(s)=M_4\mu>0$, which implies that there exists a s_4^* such that $\Phi_2(s)>0$ for all $s<s_4^*$. So, taking $s_4^+=s_4^*$ and $0<M_4<\min\left\{k_4,\frac{k_4}{k_3}M_3,k_3e^{(\gamma_1-\tilde\gamma)s_3^*}\right\}$, it is not difficult to check that the inequality (3.11) holds by returning to the calculation above.

Next, we focus on the inequality (3.12) about $\underline{W}$. When $s<s_4^+$, we have $\underline{V}=k_4e^{\gamma s}-M_4e^{\tilde\gamma s}$ and $\underline{W}=e^{\alpha_1 s}$. Let $\alpha_1>0$, $\tilde\gamma>0$ and $s_4^+<0$. Then, after some computation, we get

$$\begin{aligned} d_5\underline{W}_{ss}-c\underline{W}_s+b\underline{V}\underline{W}-\sigma\underline{W}-\varepsilon\underline{W}^2 &=\left(d_5\alpha_1^2-c\alpha_1-\sigma\right)e^{\alpha_1 s}+b\left(k_4e^{\gamma s}-M_4e^{\tilde\gamma s}\right)e^{\alpha_1 s}-\varepsilon e^{2\alpha_1 s}\\ &\geq\left[\left(d_5\alpha_1^2-c\alpha_1-\sigma\right)-bM_4e^{\tilde\gamma s}-\varepsilon e^{\alpha_1 s}\right]e^{\alpha_1 s}\\ &\geq\left[\left(d_5\alpha_1^2-c\alpha_1-\sigma\right)-bM_4-\varepsilon\right]e^{\alpha_1 s}. \end{aligned}$$

By taking $\alpha_1>\frac{c+\sqrt{c^2+4d_5(\sigma+bM_4+\varepsilon)}}{2d_5}$, it follows that the inequality $d_5\underline{W}_{ss}-c\underline{W}_s+b\underline{V}\underline{W}-\sigma\underline{W}-\varepsilon\underline{W}^2\geq0$ holds for all $s<s_4^+$.

Now, if $s\geq s_4^+$, then $\underline{V}=0$ and $\underline{W}=e^{\alpha_1(2s^*-s)}$. Again, let $\alpha_1>0$ and $s_4^+<0$. Then, one has

$$\begin{aligned} d_5\underline{W}_{ss}-c\underline{W}_s+b\underline{V}\underline{W}-\sigma\underline{W}-\varepsilon\underline{W}^2 &=e^{2\alpha_1 s_3^+}\left[\left(d_5\alpha_1^2+c\alpha_1-\sigma\right)-\varepsilon e^{2\alpha_1 s_3^+}e^{-\alpha_1 s}\right]e^{-\alpha_1 s}\\ &\geq e^{2\alpha_1 s_3^+}\left[\left(d_5\alpha_1^2+c\alpha_1-\sigma\right)-\varepsilon e^{2\alpha_1 s_3^+}e^{-\alpha_1 s_3^+}\right]e^{\alpha_1 s}\\ &=e^{2\alpha_1 s_3^+}\left[\left(d_5\alpha_1^2+c\alpha_1-\sigma\right)-\varepsilon e^{\alpha_1 s_3^+}\right]e^{\alpha_1 s}\\ &\geq e^{2\alpha_1 s_3^+}\left[\left(d_5\alpha_1^2+c\alpha_1-\sigma\right)-\varepsilon\right]e^{\alpha_1 s}. \end{aligned}$$

Since $\alpha_1>\frac{c+\sqrt{c^2+4d_5(\sigma+bM_4+\varepsilon)}}{2d_5}$, it is clear that $\left(d_5\alpha_1^2+c\alpha_1-\sigma\right)-\varepsilon>0$. Thus the inequality (3.12) holds.

Finally, we focus on the inequality (3.13) about $\underline{Z}$. When $s < s_2^+$, we have $\underline{I} = k_2 e^{\gamma s} - M_2 e^{\tilde{\gamma} s}$ and $\underline{Z} = e^{\alpha_2 s}$. Let $\alpha_2 > 0$, $\tilde{\gamma} > 0$ and $s_2^+ < 0$, then after calculation, we obtain

$$\begin{aligned} d_6 \underline{Z}_{ss} - c\underline{Z}_s + a\underline{I}\underline{Z} - q\underline{Z} - \varepsilon \underline{Z}^2 &= \left(d_6 \alpha_2^2 - c\alpha_2 - q\right) e^{\alpha_2 s} + a\left(k_2 e^{\gamma s} - M_2 e^{\tilde{\gamma} s}\right) e^{\alpha_2 s} - \varepsilon e^{2\alpha_2 s} \\ &\geq \left[\left(d_6 \alpha_2^2 - c\alpha_2 - q\right) - aM_2 e^{\tilde{\gamma} s} - \varepsilon e^{\alpha_2 s}\right] e^{\alpha_2 s} \\ &\geq \left[\left(d_6 \alpha_2^2 - c\alpha_2 - q\right) - aM_2 - \varepsilon\right] e^{\alpha_2 s}. \end{aligned}$$

By taking $\alpha_2 > \frac{c + \sqrt{c^2 + 4d_6(q + aM_2 + \varepsilon)}}{2d_6}$, it follows that the inequality $d_6 \underline{Z}_{ss} - c\underline{Z}_s + a\underline{I}\underline{Z} - q\underline{Z} - \varepsilon \underline{Z}^2 \geq 0$ holds for all $s < s_2^+$.

Now, if $s \geq s_2^+$, then $\underline{I} = 0$ and $\underline{Z} = e^{\alpha_2(2s^* - s)}$. Again, let $\alpha_2 > 0$ and $s_2^+ < 0$. Then, one has

$$\begin{aligned} d_6 \underline{Z}_{ss} - c\underline{Z}_s + a\underline{I}\underline{Z} - q\underline{Z} - \varepsilon \underline{Z}^2 &= e^{2\alpha_2 s_2^+} \left[\left(d_6 \alpha_2^2 + c\alpha_2 - q\right) - \varepsilon e^{2\alpha_2 s_2^+} e^{-\alpha_2 s}\right] e^{-\alpha_2 s} \\ &\geq e^{2\alpha_2 s_2^+} \left[\left(d_6 \alpha_2^2 + c\alpha_2 - q\right) - \varepsilon e^{2\alpha_2 s_2^+} e^{-\alpha_2 s_2^+}\right] e^{\alpha_2 s} \\ &= e^{2\alpha_2 s_2^+} \left[\left(d_6 \alpha_2^2 + c\alpha_2 - q\right) - \varepsilon e^{\alpha_2 s_2^+}\right] e^{\alpha_2 s} \\ &\geq e^{2\alpha_2 s_2^+} \left[\left(d_6 \alpha_2^2 + c\alpha_2 - q\right) - \varepsilon\right] e^{\alpha_2 s}. \end{aligned}$$

Since $\alpha_2 > \frac{c + \sqrt{c^2 + 4d_6(q + aM_2 + \varepsilon)}}{2d_6}$, it is easy to see that $\left(d_6 \alpha_2^2 + c\alpha_2 - q\right) - \varepsilon > 0$. Thus the inequality (3.13) holds. This completes the proof. $\square$

4. Offshoot of the fixed point problem

To prove the existence of the traveling waves solutions, we need some preparation. For this, let $c > c^*$ and $l > \max\{|s_1^+|, |s_2^+|, |s_3^+|, |s_4^+|\}$. This section is devoted to the study of the following truncated boundary value problem:

$$\begin{cases} d_1 H_{ss} - cH_s + s_0 - dH - \beta_1 f(H, V) - \beta_2 \mathrm{g}(H, I) = 0, & s \in (-l, l), \\ d_2 I_{ss} - cI_s + \beta_1 f(H, V) + \beta_2 \mathrm{g}(H, I) - \delta I - pIZ = 0, & s \in (-l, l), \\ d_3 D_{ss} - cD_s + kI - (\alpha + \delta) D = 0, & s \in (-l, l), \\ d_4 V_{ss} - cV_s + \alpha D - \mu V - rVW = 0, & s \in (-l, l), \\ d_5 W_{ss} - cW_s + bVW - \sigma W - \varepsilon W^2 = 0, & s \in (-l, l), \\ d_6 Z_{ss} - cZ_s + aIZ - qZ - \varepsilon Z^2 = 0, & s \in (-l, l), \\ H(s) = \underline{H}(s), I(s) = \underline{I}(s), D(s) = \underline{D}(s), & \\ V(s) = \underline{V}(s), W(s) = \underline{W}(s), Z(s) = \underline{Z}(s), & s \in \{-l, l\}. \end{cases} \tag{4.1}$$

Let $I_l = [-l, l]$, $M = C(I_l) \times C(I_l) \times C(I_l) \times C(I_l) \times C(I_l) \times C(I_l)$ and

$$\begin{aligned} \Omega_l = \{(\phi, \varphi, \psi, \upsilon, \theta, \omega) \in M : \underline{H} \leq \phi \leq H_0, \underline{I} \leq \varphi \leq \overline{I}, \underline{D} \\ \leq \psi \leq \overline{D}, \underline{V} \leq \upsilon \leq \overline{V}, \underline{W} \leq \theta \leq \overline{W}, \underline{Z} \leq \omega \leq \overline{Z} \text{ in } I_l\}. \end{aligned} \tag{4.2}$$

Hence, Ω_l is a closed and convex set in M, with the norm

$$\|(\phi, \varphi, \psi, \upsilon, \theta, \omega)\|_M = \|\phi\|_{C(I_l)} + \|\varphi\|_{C(I_l)} + \|\psi\|_{C(I_l)} + \|\upsilon\|_{C(I_l)} + \|\theta\|_{C(I_l)} + \|\omega\|_{C(I_l)}.$$

Now, for any given $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) \in \Omega_l$, we consider the problem below

$$\begin{cases} c\phi_s = d_1\phi_{ss} + s_0 - d\phi - \beta_1 f(\phi_0, \upsilon_0) - \beta_2 \mathrm{g}(\phi_0, \varphi_0), \\ c\varphi_s = d_2\varphi_{ss} + \beta_1 f(\phi_0, \upsilon_0) + \beta_2 \mathrm{g}(\phi_0, \varphi_0) - \delta\varphi - p\omega_0\varphi, \\ c\psi_s = d_3\psi_{ss} + k\phi_0 - (\alpha+\delta)\psi, \\ c\upsilon_s = d_4\upsilon_{ss} + \alpha\psi_0 - \mu\upsilon - r\theta_0\upsilon, \\ c\theta_s = d_5\theta_{ss} + b\upsilon_0\theta - \sigma\theta - \varepsilon\theta_0\theta, \\ c\omega_s = d_6\omega_{ss} + a\varphi_0\omega - q\omega - \varepsilon\omega_0\omega, \end{cases} \tag{4.3}$$

with the boundary conditions

$$\phi(s) = \underline{H}(s), \varphi(s) = \underline{I}(s), \psi(s) = \underline{D}(s), \upsilon(s) = \underline{V}(s), \theta(s) = \underline{W}(s), \omega(s) = \underline{Z}(s), \ s \in \{-l, l\}.$$

We have the following result.

Theorem 4.1. *For any $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) \in \Omega_l$, the linear boundary value problem (4.3) admits a unique solution which belongs to Ω_l.*

Proof. The existence and uniqueness of a solution to the linear boundary value problem (4.3) follows from the general theory of second order systems of ordinary differential equations (see [20, Theorem 3.1 of Chapter 12]). It only remains to show that this solution is in Ω_l. Let $(\phi, \varphi, \psi, \upsilon, \theta, \omega)$ be the solution of the linear boundary value problem (4.3) with $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) \in \Omega_l$. By the second equation of (4.3), it is known that $d_2\varphi_{ss} - c\varphi_s - \delta\varphi - p\omega_0\varphi = -\beta_1 f(\phi_0, \upsilon_0) - \beta_2 \mathrm{g}(\phi_0, \varphi_0) \le 0$, for all $s \in (-l, l)$, and $\varphi(l) = \overline{I}(l) = 0$ and $\varphi(-l) = \overline{I}(-l) > 0$, because $l \ge s_2^+$. Then by the maximum principle, it follows that $\varphi(s) > 0$ for all $s \in (-l, l)$. In a similar way, one can demonstrate that $\phi(s) > 0$, $\psi(s) > 0$, $\upsilon(s) > 0$, $\theta(s) > 0$, $\omega(s) > 0$ for all $s \in (-l, l)$.

Next, we demonstrate that $\varphi(s) < \overline{I}(l)$ for all $s \in (-l, l)$. Owing to $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) \in \Omega_l$, from the second equation of (4.3), we get

$$d_2\varphi_{ss} - c\varphi_s + \beta_1 f(H_0, \overline{V}) + \beta_2 \mathrm{g}(H_0, \overline{I}) - \delta\varphi - pZ\varphi \ge 0, \tag{4.4}$$

$$d_2\varphi_{ss} - c\varphi_s + \beta_1 f(\underline{H}, \underline{V}) + \beta_2 \mathrm{g}(\underline{H}, \underline{I}) - \delta\varphi - p\overline{Z}\varphi \le 0, \tag{4.5}$$

for all $s \in (-l, l)$. Set $\varpi(s) = \overline{I}(s) - \varphi(s)$. Then by equations (3.3) and (4.4), we have

$$\begin{aligned} d_2\varpi_{ss} - c\varpi_s - \delta\varpi - pZ\varpi &= d_2\overline{I}_{ss} - d_2\varphi_{ss} - c\overline{I}_s + c\varphi_s - \delta\overline{I}(s) + \delta\varphi - pZ\overline{I}(s) + pZ\varphi \\ &\le -d_2\varphi_{ss} + c\varphi_s + \delta\varphi + pZ\varphi - \beta_1 f(H_0, \overline{V}) - \beta_2 \mathrm{g}(H_0, \overline{I}) \le 0. \end{aligned}$$

Combining the fact $\varpi(\pm l) = \overline{I}(\pm l) - \varphi(\pm l) = \overline{I}(\pm l) - \underline{I}(\pm l) > 0$ with the maximum principle, it follows that $\varpi(s) \ge 0$ for all $s \in (-l, , l)$. Therefore, $\varphi(s) < \overline{I}$ for all $s \in (-l, l)$.

Now, we prove that $\varphi(s) \ge \underline{I}$ for all $s \in (-l, l)$. To do this, we set $\widetilde{\varpi}(s) = \varphi(s) - \underline{I}(s)$. Then by (3.9) and (4.5), for $s \in (-l, s_2^+)$, we get

$$\begin{aligned} d_2\widetilde{\varpi}_{ss} - c\widetilde{\varpi}_s - \delta\widetilde{\varpi} - p\overline{Z}\widetilde{\varpi} &= d_2\varphi_{ss} - d_2\underline{I}_{ss} - c\varphi_s + c\underline{I}_s - \delta\varphi + \delta\underline{I}(s) - p\overline{Z}\varphi + p\overline{Z}\underline{I}(s) \\ &\le -d_2\underline{I}_{ss} + c\underline{I}_s + \delta\underline{I} + p\overline{Z}\underline{I} - \beta_1 f(H_0, \overline{V}) - \beta_2 \mathrm{g}(H_0, \overline{I}) \le 0. \end{aligned} \tag{4.6}$$

Moreover, combining $\widetilde{\varpi}(-l) = \varphi(-l) - \underline{I}(-l) = 0$, $\widetilde{\varpi}(s_2^+) = \varphi(s_2^+) - \underline{I}(s_2^+) > 0$ with the maximum principle, it follows that $\widetilde{\varpi}(s) \ge 0$, which implies $\varphi(s) \ge \underline{I}(s)$ for all $s \in (-l, s_2^+)$. In fact, we always have $\varphi(s) \ge \underline{I}(s)$ for all $s \in (-l, l)$ because $\underline{I}(s) = 0$ for $s \in [s_2^+, l)$. Accordingly, $\underline{I} \le \varphi(s) \le \overline{I}$ for all $s \in (-l, l)$.

Reasoning in the same way as above and combining Lemma 3.1 and Lemma 3.2, we show that $\phi \in [\underline{H}, H_0]$, $\psi \in [\underline{D}, \overline{D}]$, $\upsilon \in [\underline{V}, \overline{V}]$, $\theta \in [\underline{W}, \overline{W}]$, $\omega \in [\underline{Z}, \overline{Z}]$. This completes the proof. □

It follows from Theorem 4.1, that one can define a map $L : \Omega_l \to \Omega_l$ by

$$L(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) = (\phi, \varphi, \psi, \upsilon, \theta, \omega), \tag{4.7}$$

where $(\phi, \varphi, \psi, \upsilon, \theta, \omega)$ is the unique solution of problem (4.3). For proving that the nonlinear boundary value problem (4.2) has a unique solution, we need to show by relying on Schauder fixed point theorem that L is continuous and compact.

Lemma 4.2. *The mapping L is continuous.*

Proof. Let $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0), (\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0) \in \Omega_l$ such that

$$L(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) = (\phi, \varphi, \psi, \upsilon, \theta, \omega) \quad \text{and} \quad L(\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0) = (\tilde{\phi}, \tilde{\varphi}, \tilde{\psi}, \tilde{\upsilon}, \tilde{\theta}, \tilde{\omega}).$$

By the first equation of (4.3), we have

$$d_1\phi_{ss} - c\phi_s + s_0 - d\phi - \beta_1 f(\phi_0, \upsilon_0) - \beta_2 \mathrm{g}(\phi_0, \varphi_0) = 0 \tag{4.8}$$

and

$$d_1\tilde{\phi}_{ss} - c\tilde{\phi}_s + s_0 - d\tilde{\phi} - \beta_1 f(\tilde{\phi}_0, \tilde{\upsilon}_0) - \beta_2 \mathrm{g}(\tilde{\phi}_0, \tilde{\varphi}_0) = 0. \tag{4.9}$$

Let $\Psi = \phi - \tilde{\phi}$. Then, $\Psi(\pm l) = \phi(\pm l) - \tilde{\phi}(\pm l) = 0$ and from (4.8) and (4.9), we get

$$\Psi_{ss} - \frac{c}{d_1}\Psi_s + \eta(s)\Psi = u(s), \quad s \in (-l, l),$$

where $\eta(s) = -\frac{d}{d_1}$ and $u(s) = \frac{1}{d_1}\left[\beta_1 f(\phi_0, \upsilon_0) - \beta_1 f(\tilde{\phi}_0, \tilde{\upsilon}_0) + \beta_2 \mathrm{g}(\phi_0, \varphi_0) - \beta_2 \mathrm{g}(\tilde{\phi}_0, \tilde{\varphi}_0)\right]$. Since $(\phi, \varphi, \psi, \upsilon, \theta, \omega)$, $(\tilde{\phi}, \tilde{\varphi}, \tilde{\psi}, \tilde{\upsilon}, \tilde{\theta}, \tilde{\omega})$, $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0)$, $(\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0) \in \Omega_l$, there exist two positive constants K_1 and K_2 such that

$$-K_1 \le \eta(s) \le 0 \quad \text{and} \quad |u(s)| \le K_2\left[\|\tilde{\upsilon}_0 - \upsilon_0\|_{C(I_l)} + \|\tilde{\varphi}_0 - \varphi_0\|_{C(I_l)}\right], \text{ for } s \in (-l, l).$$

In fact, one can choose $K_1 = \frac{d}{d_1}$ and

$K_2 = \frac{1}{d_1}\left[\beta_1 f_{HV}(H_0, 0)k_4 e^{\gamma l} + \beta_2 \mathrm{g}_{HI}(H_0, 0)k_2 e^{\gamma l} + \beta_1 f_V(H_0, 0) + \beta_2 \mathrm{g}_I(H_0, 0)\right]$. Therefore, by [13, Lemma 3.2], there exists a positive constant K_3, depending only on K_1, c, d_1 and l, such that

$$\|\Psi(s)\|_{C(I_l)} = \|\phi(s) - \tilde{\phi}(s)\|_{C(I_l)} \le K_2K_3\left[\|\tilde{\upsilon}_0 - \upsilon_0\|_{C(I_l)} + \|\tilde{\varphi}_0 - \varphi_0\|_{C(I_l)}\right].$$

Reasoning similarly for the other equations, we show that, there exist some positive constants K_4, K_5, K_6, K_7 and K_8 depending on parameters $c, d_i, \beta_j, k, k_{i_1}, \alpha, \delta, b, a, \sigma, q, (i = 1, \cdots, 4; j = 1, 2; i_1 = 2, 4)$ and l such that

$$\begin{aligned}
&\|\varphi(s) - \tilde{\varphi}(s)\|_{C(I_l)} \le K_4\left[\|\tilde{\phi}_0 - \phi_0\|_{C(I_l)} + \|\tilde{\upsilon}_0 - \upsilon_0\|_{C(I_l)} + \|\tilde{\varphi}_0 - \varphi_0\|_{C(I_l)} + \|\tilde{\omega}_0 - \omega_0\|_{C(I_l)}\right],\\
&\|\psi(s) - \tilde{\psi}(s)\|_{C(I_l)} \le K_5\|\tilde{\phi}_0 - \phi_0\|_{C(I_l)},\\
&\|\upsilon(s) - \tilde{\upsilon}(s)\|_{C(I_l)} \le K_6\left[\|\tilde{\theta}_0 - \theta_0\|_{C(I_l)} + \|\tilde{\psi}_0 - \psi_0\|_{C(I_l)}\right],\\
&\|\theta(s) - \tilde{\theta}(s)\|_{C(I_l)} \le K_7\left[\|\tilde{\upsilon}_0 - \upsilon_0\|_{C(I_l)} + \|\tilde{\theta}_0 - \theta_0\|_{C(I_l)}\right],\\
&\|\omega(s) - \tilde{\omega}(s)\|_{C(I_l)} \le K_8\left[\|\tilde{\omega}_0 - \omega_0\|_{C(I_l)} + \|\tilde{\varphi}_0 - \varphi_0\|_{C(I_l)}\right].
\end{aligned}$$

Accordingly, $\|L(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) - L(\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0)\|_{C(I_l)} \le K^*\|(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) - (\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0)\|_{C(I_l)}$, where $K^* = K_2K_3 + \sum_{i=4}^{8} K_i$. Thus, for any given $\epsilon > 0$, one can choose $0 < \sigma_0 < \epsilon/K^*$ such that

$$\|L(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) - L(\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0)\|_{C(I_l)} \le \epsilon,$$

for any $(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0), (\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0) \in \Omega_l$ verifying $\|(\phi_0, \varphi_0, \psi_0, \upsilon_0, \theta_0, \omega_0) - (\tilde{\phi}_0, \tilde{\varphi}_0, \tilde{\psi}_0, \tilde{\upsilon}_0, \tilde{\theta}_0, \tilde{\omega}_0)\|_{C(I_l)} \le \sigma_0$. Whence, the mapping L is continuous. □

Lemma 4.3. *L is a compact mapping.*

Proof. Let $(\phi_n, \varphi_n, \psi_n, \upsilon_n, \theta_n, \omega_n) = L(\phi_{0,n}, \varphi_{0,n}, \psi_{0,n}, \upsilon_{0,n}, \theta_{0,n}, \omega_{0,n})$, for a given sequence $(\phi_{0,n}, \varphi_{0,n}, \psi_{0,n}, \upsilon_{0,n}, \theta_{0,n}, \omega_{0,n}) \in \Omega_l$. Then, by Theorem 4.1, it follows that $(\phi_n, \varphi_n, \psi_n, \upsilon_n, \theta_n, \omega_n) \in \Omega_l$. Since $\underline{H}$, $\underline{I}$, $\underline{D}$, $\underline{V}$, $\underline{W}$, $\underline{Z} \geq 0$ and $\overline{I}$, $\overline{D}$, $\overline{V}$, $\overline{W}$, $\overline{Z} \leq \left(k_2 + k_3 + k_4 + \frac{bk_4}{\varepsilon} + \frac{ak_2}{\varepsilon}\right) e^{\gamma l}$ in I_l, the sequences $\{\phi_{0,n}\}$, $\{\varphi_{0,n}\}$, $\{\psi_{0,n}\}$, $\{\upsilon_{0,n}\}$, $\{\theta_{0,n}\}$, $\{\omega_{0,n}\}$ are uniformly bounded in I_l. Thus, thanks to classical a priori estimates for second order linear ordinary differential equations (see Lemma 3.3 in [13]), the sequences of first order derivatives $(\phi_n)_s, (\varphi_n)_s, (\psi_n)_s, (\upsilon_n)_s, (\theta_n)_s, (\omega_n)_s$ are also uniformly bounded in I_l. Then, thanks to Arzelà-Ascoli Theorem, there exists a subsequence $\{(\phi_{nj}, \varphi_{nj}, \psi_{nj}, \upsilon_{nj}, \theta_{nj}, \omega_{nj})\}$ of $\{(\phi_n, \varphi_n, \psi_n, \upsilon_n, \theta_n, \omega_n)\}$ verifying

$$(\phi_{nj}, \varphi_{nj}, \psi_{nj}, \upsilon_{nj}, \theta_{nj}, \omega_{nj}) \to (\phi, \varphi, \psi, \upsilon, \theta, \omega),$$

uniformly in I_l as $j \to \infty$, for some $(\phi, \varphi, \psi, \upsilon, \theta, \omega) \in \Omega_l$. Consequently, the mapping L is compact. □

Combining Theorem 4.1, Lemmas 4.2 and 4.3, it follows from the Schauder's fixed point theorem that the mapping L has a unique fixed point, which is a nonnegative solution of the nonlinear boundary value problem (4.1). Therefore, we state the following existence and uniqueness result.

Theorem 4.4. *Let $l > 0$. Then the mappping $L : \Omega_l \to \Omega_l$ admits a unique fixed point $(\phi_l^*(s), \varphi_l^*(s), \psi_l^*(s), \upsilon_l^*(s), \theta_l^*(s), \omega_l^*(s)) \in \Omega_l$, which is the solution of the nonlinear boundary value problem (4.1) defined on $[-l, l]$.*

5. Existence of traveling waves

In this section, the nontrivial and nonnegative traveling waves solutions satisfying the boundary conditions are investigated. We have the following theorem.

Theorem 5.1. *Suppose that $\mathcal{R}_0 > 1$ and $c > c^*$. Then system (3.1) has a solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ defined for all $s \in \mathbb{R}$ and verifying*

$$\underline{H} \leq H^* \leq H_0, \ \ \underline{I} \leq I^* \leq \overline{I}, \ \ \underline{D} \leq D^* \leq \overline{D}, \ \ \underline{V} \leq V^* \leq \overline{V}, \ \ \underline{W} \leq W^* \leq \overline{W}, \ \ \underline{Z} \leq Z^* \leq \overline{Z},$$
$$(H^*(-\infty), I^*(-\infty), D^*(-\infty), V^*(-\infty), W^*(-\infty), Z^*(-\infty)) = (H_0, 0, 0, 0, 0, 0).$$

Moreover,

$$0 < H^*(s) < H_0, \ \ I^*(s) > 0, \ \ D^*(s) > 0, \ \ V^*(s) > 0, \ \ W^*(s) > 0 \ \ \text{and} \ \ Z^*(s) > 0, \text{ for all } s \in \mathbb{R}.$$

Proof. Consider an increasing sequence $\{l_n\}_{n=1}^{\infty}$ such that $l_n > \max\left\{|s_1^+|, |s_2^+|, |s_3^+|, |s_4^+|\right\}$ and $\lim_{n\to\infty} l_n = \infty$. Now, let $(H_n^*(s), I_n^*(s), D_n^*(s), V_n^*(s), W_n^*(s), Z_n^*(s))$ be the fixed point of the operator L on Ω_{l_n}, obtained in Theorem 4.4 with $l = l_n$. Clearly, the sequences $\{(H_n^*(s), I_n^*(s), D_n^*(s), V_n^*(s), W_n^*(s), Z_n^*(s))\}_{n\geq j}$ are uniformly bounded in the interval $[-l_j, l_j]$ for any positive integer j. Again, thanks to classical a priori estimates for second order linear ordinary differential equations (see Lemma 3.3 in [13]), the sequences $\{(H_{ns}^*(s), I_{ns}^*(s), D_{ns}^*(s), V_{ns}^*(s), W_{ns}^*(s), Z_{ns}^*(s))\}_{n\geq j}$ are uniformly bounded in the interval $[-l_j, l_j]$. Hence, by the Mean Value Theorem, the sequences $\{(H_n^*(s), I_n^*(s), D_n^*(s), V_n^*(s), W_n^*(s), Z_n^*(s))\}_{n\geq j}$ are equicontinuous. Employing (3.1), one can express $H_{nss}^*(s)$, $I_{nss}^*(s)$, $D_{nss}^*(s)$, $V_{nss}^*(s)$, $W_{nss}^*(s)$ and $Z_{nss}^*(s)$ in terms of $H_n^*(s)$, $I_n^*(s)$, $D_n^*(s)$, $V_n^*(s)$, $W_n^*(s)$, $Z_n^*(s)$, $(H_{ns}^*(s)$, $I_{ns}^*(s)$, $D_{ns}^*(s)$, $V_{ns}^*(s)$, $W_{ns}^*(s)$ and $Z_{ns}^*(s)$. Thus, the sequences $\{(H_{nss}^*(s),\ I_{nss}^*(s),\ D_{nss}^*(s),\ V_{nss}^*(s),\ W_{nss}^*(s)\ , Z_{nss}^*(s))\}_{n\geq j}$ are uniformly bounded in $[-l_j, l_j]$. Therefore, by the Mean Value Theorem, the sequences $\{(H_{ns}^*(s),\ I_{ns}^*(s),\ D_{ns}^*(s),\ V_{ns}^*(s),\ W_{ns}^*(s), Z_{ns}^*(s))\}_{n\geq j}$ are also equicontinuous. Now, differentiating (3.1), one can use the resulting equations to express $H_{nsss}^*(s)$, $I_{nsss}^*(s)$, $D_{nsss}^*(s)$, $V_{nsss}^*(s)$, $W_{nsss}^*(s)$ and $Z_{nsss}^*(s)$ in terms of $H_n^*(s)$, $I_n^*(s)$, $D_n^*(s)$, $V_n^*(s)$, $W_n^*(s)$, $Z_n^*(s)$ $H_{ns}^*(s)$, $I_{ns}^*(s)$, $D_{ns}^*(s)$, $V_{ns}^*(s)$, $W_{ns}^*(s)$, $Z_{ns}^*(s)$ $H_{nss}^*(s)$, $I_{nss}^*(s)$, $D_{nss}^*(s)$, $V_{nss}^*(s)$, $W_{nss}^*(s)$ and $Z_{nss}^*(s)$. Accordingly, the sequences $\{(H_{nsss}^*(s), I_{nsss}^*(s), D_{nsss}^*(s), V_{nsss}^*(s), W_{nsss}^*(s), Z_{nsss}^*(s))\}_{n\geq j}$ are uniformly bounded in $[-l_j, l_j]$. Hence, by the Mean Value Theorem, the sequences $\{(H^*_{nss}(s),\ I^*_{nss}(s),\ D^*_{nss}(s),\ V^*_{nss}(s),\ W^*_{nss}(s)\ ,Z^*_{nss}(s))\}_{n\geq j}$ are equicontinuous as well. Now, applying the diagonal extraction procedure and the Arzelà-Ascoli Theorem, there exists a subsequence $\{(H^*_{nm}(s), I^*_{nm}(s), D^*_{nm}(s), V^*_{nm}(s), W^*_{nm}(s), Z^*_{nm}(s))\}$ of $\{(H^*_n(s), I^*_n(s), D^*_n(s), V^*_n(s), W^*_n(s), Z^*_n(s))\}$ such that

$$\begin{aligned}
&(H^*_{nm}(s), I^*_{nm}(s), D^*_{nm}(s), V^*_{nm}(s), W^*_{nm}(s), Z^*_{nm}(s)) \to (H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s)),\\
&(H^*_{nms}(s), I^*_{nms}(s), D^*_{nms}(s), V^*_{nms}(s), W^*_{nms}(s), Z^*_{nms}(s)) \to (H^*_s(s), I^*_s(s), D^*_s(s), V^*_s(s), W^*_s(s), Z^*_s(s)),\\
&(H^*_{nmss}(s), I^*_{nmss}(s), D^*_{nmss}(s), V^*_{nmss}(s), W^*_{nmss}(s), Z^*_{nmss}(s))\\
&\to (H^*_{ss}(s), I^*_{ss}(s), D^*_{ss}(s), V^*_{ss}(s), W^*_{ss}(s), Z^*_{ss}(s)),
\end{aligned}$$

as $m \to \infty$, uniformly in any compact interval of $\mathbb{R}$, for $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s)) \in C^2(\mathbb{R})$. It is obvious that $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, being defined for all $s \in \mathbb{R}$, is a solution of system (3.1) and verifies $\underline{H} \leq H^* \leq H_0$, $\underline{I} \leq I^* \leq \overline{I}$, $\underline{D} \leq D^* \leq \overline{D}$, $\underline{V} \leq V^* \leq \overline{V}$, $\underline{W} \leq W^* \leq \overline{W}$, $\underline{Z} \leq Z^* \leq \overline{Z}$. Furthermore, from the definition of upper and lower solutions, we have $(H^*(-\infty), I^*(-\infty), D^*(-\infty), V^*(-\infty), W^*(-\infty), Z^*(-\infty)) = (H_0, 0, 0, 0, 0, 0)$.

Now, we demonstrate the last assertion of Theorem 5.1, i.e., $0 < H^*(s) < H_0$, $I^*(s) > 0$, $D^*(s) > 0$, $V^*(s) > 0$, $W^*(s) > 0$ and $Z^*(s) > 0$ for all $s \in \mathbb{R}$. Assume to obtain a contradiction that there exists $z_1 \in \mathbb{R}$ such that $H^*(z_1) = 0$ and $H^*_s(z_1) = 0$. Then, $H^*_{ss}(z_1) \geq 0$ because $H^*(s) \geq 0$ for all $s \in \mathbb{R}$. But, from the first equation of (3.1), one has $d_1H^*_{ss}(z_1) = -s_0 < 0$, which is a contradiction. Therefore, $H^*(s) > 0$ for all $s \in \mathbb{R}$.

In a similar way, we suppose that there exists $z_2 \in \mathbb{R}$ such that $D^*(z_2) = 0$ and $D^*_s(z_2) = 0$. Then, $D^*_{ss}(z_2)$ because $D^*(s) \geq 0$ for all $s \in \mathbb{R}$. But, from the third equation of (3.1), one gets $d_3D^*_{ss}(z_2) = -kI^*(z_2) \leq 0$, which means that $I^*(z_2) = 0$ and hence $I^*_s(z_2) = 0$. Let $x_1 = H$, $x_2 = H_s$, $x_3 = I$, $x_4 = I_s$, $x_5 = D$, $x_6 = D_s$, $x_7 = V$, $x_8 = V_s$, $x_9 = W$, $x_{10} = W_s$, $x_{11} = Z$ and $x_{12} = Z_s$. Then system (3.1) becomes

$$\begin{cases}
x_1' = x_2,\\
d_1x_2' = cx_2 - s_0 + dx_1 + \beta_1 f(x_1, x_7) + \beta_2 \mathrm{g}(x_1, x_3),\\
x_3' = x_4,\\
d_2x_4' = cx_4 - \beta_1 f(x_1, x_7) - \beta_2 \mathrm{g}(x_1, x_3) + \delta x_3 + px_3x_{11},\\
x_5' = x_6,\\
d_3x_6' = cx_6 - kx_3 + (\alpha + \delta)x_5,\\
x_7' = x_8,\\
d_4x_8' = cx_8 - \alpha x_5 + \mu x_7 + rx_7x_9,\\
x_9' = x_{10},\\
d_5x_{10}' = cx_{10} - bx_7x_9 + \sigma x_9 + \varepsilon x_9^2,\\
x_{11}' = x_{12},\\
d_6x_{12}' = cx_{12} - ax_3x_{11} + qx_{11} + \varepsilon x_{11}^2.
\end{cases} \tag{5.1}$$

Clearly, the region $\Gamma_1 = \{(x_1, x_2, 0, 0, 0, 0, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) : x_1, x_2, x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \in \mathbb{R}\}$ is an invariant set for system (5.1) and $(x_1(z_2), x_2(z_2), x_3(z_2), x_4(z_2), x_5(z_2), x_6(z_2), x_7(z_2), x_8(z_2), x_9(z_2), x_{10}(z_2), x_{11}(z_2), x_{12}(z_2)) \in \Gamma_1$. This implies that $I^*(s) = D^*(s) = V^*(s) = W^*(s) = Z^*(s) = 0$ for all $s \in \mathbb{R}$. This is a contradiction since from the definition of $\underline{I}$, $\underline{D}$ and $\underline{V}$, we have $I^*(s), D^*(s), V^*(s) > 0$ for all $s < \min\{s_2^+, s_3^+, s_4^+\}$. Thus $I^*(s) > 0$, $D^*(s) > 0$ and $V^*(s) > 0$ for all $s \in \mathbb{R}$. Also, from the definition of $\underline{W}$ and $\underline{Z}$, we also conclude that $W^*(s) > 0$ and $Z^*(s) > 0$ for all $s \in \mathbb{R}$.

Finally, we prove that $H^*(s) < H_0$. Suppose to obtain a contradiction that there exists $z_3 > 0$ such that $H^*(z_3) = H_0$ and $H^*_s(z_3) = 0$. Then, $H^*_{ss}(z_1) \leq 0$. By the first equation of (3.1), one has

$d_1 H^*_{ss}(z_1) = \beta_1 f(H^*(z_3), V^*(z_3)) + \beta_2 \mathrm{g}(H^*(z_3), I^*(z_3)) > 0$. This is a contradiction with $I^*(s), V^*(s) > 0$ whenever $s < \min\{s_2^+, s_3^+, s_4^+\}$. Hence, $H^*(s) < H_0$ for all $s \in \mathbb{R}$. This completes the proof. □

Based on the previous discussions, we now consider the original system (2.3), and have the following result.

Theorem 5.2. *Suppose that $\mathcal{R}_0 > 1$ and $c > c^*$. Then system (2.3) has a solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ defined for all $s \in \mathbb{R}$ and verifying*

$$\underline{H} \le H^* \le H_0,\ \ \underline{I} \le I^* \le \overline{I},\ \ \underline{D} \le D^* \le \overline{D},\ \ \underline{V} \le V^* \le \overline{V},\ \ \underline{W} \le W^* \le \overline{W},\ \ \underline{Z} \le Z^* \le \overline{Z},$$
$$(H^*(-\infty), I^*(-\infty), D^*(-\infty), V^*(-\infty), W^*(-\infty), Z^*(-\infty)) = (H_0, 0, 0, 0, 0, 0).$$

Furthermore,

$$0 < H^*(s) < H_0,\ \ I^*(s) > 0,\ \ D^*(s) > 0,\ \ V^*(s) > 0,\ \ W^*(s) > 0 \quad \text{and} \quad Z^*(s) > 0,\ \text{for all } s \in \mathbb{R}.$$

Proof. Let $\{\varepsilon_n\} \subset [0, 1)$ be a decreasing sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. From Theorem 5.1, it follows that for any given $n \in \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set positive integers, system (3.1) has a solution sequence $\{(H^*_n(s),\ I^*_n(s),\ D^*_n(s),\ V^*_n(s),\ W^*_n(s),\ Z^*_n(s))\}$ which verifies

$$\begin{cases} d_1 H^*_{nss} - cH^*_{ns} + s_0 - dH^*_n - \beta_1 f(H^*_n, V^*_n) - \beta_2 \mathrm{g}(H^*_n, I^*_n) = 0, \\ d_2 I_{nss} - cI^*_{ns} + \beta_1 f(H^*_n, V^*_n) + \beta_2 \mathrm{g}(H^*_n, I^*_n) - \delta I^*_n - pI^*_n Z^*_n = 0, \\ d_3 D_{nss} - cD^*_{ns} + kI - (\alpha + \delta) D^*_n = 0, \\ d_4 V_{nss} - cV^*_{ns} s + \alpha D^*_n - \mu V^*_n - rV^*_n W^*_n = 0, \\ d_5 W_{nss} - cW^*_{ns} + bVW - \sigma W^*_n - \varepsilon_n W^{*2}_n = 0, \\ d_6 Z_{nss} - cZ^*_{ns} + aI^*_n Z^*_n - qZ^*_n - \varepsilon_n Z^{*2}_n = 0, \end{cases} \tag{5.2}$$

and

$$\underline{H} \le H^*_n(s) \le H_0,\ \ \underline{I} \le I^*_n(s) \le \overline{I},\ \ \underline{D} \le D^*_n(s)$$
$$\le \overline{D},\ \ \underline{V} \le V^*_n(s) \le \overline{V},\ \ \underline{W} \le W^*_n(s) \le \overline{W},\ \ \underline{Z} \le Z^*_n(s) \le \overline{Z},$$
$$(H^*_n(-\infty), I^*_n(-\infty), D^*_n(-\infty), V^*_n(-\infty), W^*_n(-\infty), Z^*_n(-\infty)) = (H_0, 0, 0, 0, 0, 0).$$

Furthermore,

$$0 < H^*_n(s) < H_0,\ \ I^*_n(s) > 0,\ \ D^*_n(s) > 0,\ \ V^*_n(s) > 0,\ \ W^*_n(s) > 0 \quad \text{and} \quad Z^*_n(s) > 0,\ \text{for all } s \in \mathbb{R}.$$

Let $\nu_{\varepsilon_n} = \frac{k_2 + k_3 + k_4 + bk_4 + ak_2}{\varepsilon_n} \frac{1}{\varepsilon_n}$. Clearly $\nu_{\varepsilon_n} \to \infty$ as $n \to \infty$. Thus, for $s \in [-j, j]$, $j \in \mathbb{N}^*$, it follows from the definition of $\overline{I}$, $\overline{D}$, $\overline{V}$, $\overline{W}$, $\overline{Z}$ that there exists $n_j \in \mathbb{N}_+$ such that $\max\{\overline{I}, \overline{D}, \overline{V}, \overline{W}, \overline{Z}\} \le \frac{k_2 + k_3 + k_4 + bk_4 + ak_2}{\varepsilon_n} e^{\gamma s} \le \nu_{\varepsilon_n}$ for $n > n_j$ and $s \in [-j, j]$. Thus, using similar arguments as in the proof of Theorem 5.1, it follows that the sequences of solutions $\{(H^*_n(s), I^*_n(s), D^*_n(s), V^*_n(s), W^*_n(s), Z^*_n(s))\}$ are uniformly bounded and equicontinuous on $[-j, j]$. Therefore, by Arzelà-Ascoli Theorem, there exists a uniformly convergent subsequence $\{(H^*_{jm}(s), I^*_{jm}(s), D^*_{jm}(s), V^*_{jm}(s), W^*_{jm}(s), Z^*_{jm}(s))\}$ of $\{(H^*_n(s), I^*_n(s), D^*_n(s), V^*_n(s), W^*_n(s), Z^*_n(s))\}$ as $m \to \infty$.

Whence, applying the diagonal extraction procedure, we can choose a subsequence $\{(H^*_{mm}(s), I^*_{mm}(s), D^*_{mm}(s), V^*_{mm}(s), W^*_{mm}(s), Z^*_{mm}(s))\}$ of $\{(H^*_{jm}(s), I^*_{jm}(s), D^*_{jm}(s), V^*_{jm}(s), W^*_{jm}(s), Z^*_{jm}(s))\}$ which converges uniformly on every interval $[-j, j]$, $j \in \mathbb{N}^*$, and $\lim_{m\to\infty}(H^*_{mm}(s), I^*_{mm}(s), D^*_{mm}(s), V^*_{mm}(s), W^*_{mm}(s), Z^*_{mm}(s)) = (H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$.

For any $m \in \mathbb{N}_+$, it is known that

$$\begin{cases} d_1 H^*_{mmss} - cH^*_{mms} + s_0 - dH^*_{mm} - \beta_1 f(H^*_{mm}, V^*_{mm}) - \beta_2 \mathrm{g}(H^*_{mm}, I^*_{mm}) = 0, \\ d_2 I_{mmss} - cI^*_{mms} + \beta_1 f(H^*_{mm}, V^*_{mm}) + \beta_2 \mathrm{g}(H^*_{mm}, I^*_{mm}) - \delta I^*_{mm} - pI^*_{mm} Z^*_{mm} = 0, \\ d_3 D_{mmss} - cD^*_{mms} + kI - (\alpha+\delta) D^*_{mm} = 0, \\ d_4 V_{mmss} - cV^*_{mms} s + \alpha D^*_{mm} - \mu V^*_{mm} - rV^*_{mm} W^*_{mm} = 0, \\ d_5 W_{mmss} - cW^*_{mms} + bVW - \sigma W^*_{mm} - \varepsilon_{mm} W^{*2}_{mm} = 0, \\ d_6 Z_{mmss} - cZ^*_{mms} + aI^*_{mm} Z^*_{mm} - qZ^*_{mm} - \varepsilon_{mm} Z^{*2}_{mm} = 0, \end{cases} \tag{5.3}$$

where (ε_{mm}) is a subsequence of the sequence ε_{nm}. Letting $m \to \infty$ in (5.3) we readily deduce that $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ satisfies system (2.3). Repeating a similar proof as in Theorem 5.1, we get $0 < H^*(s) < H_0$, $I^*(s) > 0$, $D^*(s) > 0$, $V^*(s) > 0$, $W^*(s) > 0$ and $Z^*(s) > 0$, for all $s \in \mathbb{R}$. This completes the proof. □

Now, we investigate the asymptotic behavior of the solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ when $s \to \infty$. To do this, we need the following additional assumption.

(Q2) There exists a continuous, increasing and differentiable function $\ell(H)$ defined for all $H \geq 0$, satisfying $\ell(0) = 0$,

$$\left(1 - \frac{f(H_i, V_i)\ell(H)}{f(H, V)\ell(H_i)}\right)\left(\frac{f(H, V)\ell(H_i)}{f(H_i, V_i)\ell(H)} - \frac{V}{V_i}\right) \leq 0, \quad i = 1, 2, 3, 4,$$

and

$$\left(1 - \frac{\mathrm{g}(H_i, I_i)\ell(H)}{\mathrm{g}(H, I)\ell(H_i)}\right)\left(\frac{\mathrm{g}(H, I)\ell(H_i)}{\mathrm{g}(H_i, I_i)\ell(H)} - \frac{I}{I_i}\right) \leq 0, \quad i = 1, 2, 3, 4.$$

Theorem 5.3. *Suppose that $\mathcal{R}_0 > 1$, $c > c^*$ and assumption (Q1) hold. Then system (2.3) has a solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ defined for all $s \in \mathbb{R}$ and verifying*

$$0 < H^*(s) < H_0, \quad I^*(s) > 0, \quad D^*(s) > 0, \quad V^*(s) > 0, \quad W^*(s) > 0 \quad \textit{and} \quad Z^*(s) > 0, \textit{ for all } s \in \mathbb{R}.$$
$$(H^*(-\infty), I^*(-\infty), D^*(-\infty), V^*(-\infty), W^*(-\infty), Z^*(-\infty)) = (H_0, 0, 0, 0, 0, 0).$$

Moreover,

(i) If (Q2) holds, $\mathcal{R}_1 \leq 1$ and $\mathcal{R}_2 \leq 1$, then $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_1, I_1, D_1, V_1, 0, 0)$.
(ii) If (Q2) holds, $\mathcal{R}_1 > 1$ and $\mathcal{R}_3 \leq 1$, then $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_2, I_2, D_2, V_2, W_2, 0)$.
(iii) If (Q2) holds, $\mathcal{R}_2 > 1$ and $\mathcal{R}_4 \leq 1$, then $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_3, I_3, D_3, V_3, 0, Z_3)$.
(iv) If (Q2) holds, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, then $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_4, I_4, D_4, V_4, W_4, Z_4)$.

Proof. By Theorems 5.1 and 5.2, it clearly follows that the solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ of system (2.3) is defined for all $s \in \mathbb{R}$ and satisfies

$$(H^*(-\infty), I^*(-\infty), D^*(-\infty), V^*(-\infty), W^*(-\infty), Z^*(-\infty)) = (H_0, 0, 0, 0, 0, 0)$$

and

$$0 < H^*(s) < H_0, \quad I^*(s) > 0, \quad D^*(s) > 0, \quad V^*(s) > 0, \quad W^*(s) > 0 \quad \text{and} \quad Z^*(s) > 0, \text{ for all } s \in \mathbb{R}.$$

Now, we discuss the asymptotic behavior of $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ when $s \to \infty$. For item (i), define the following Lyapunov function

$$\mathcal{M}_1 = c\left(x_1 - \int_{H_1}^{x_1} \frac{\ell(H_1)}{\ell(t)}dt\right) + \left(\frac{\ell(H_1)}{\ell(x_1)} - 1\right) d_1x_2 + c\left(x_3 - I_1\int_{I_1}^{x_3}\frac{1}{t}dt\right) + \left(\frac{I_1}{x_3} - 1\right) d_2x_4$$
$$+\frac{\beta_1 f(H_1,V_1)}{kI_1}\left[c\left(x_5 - D_1\int_{D_1}^{x_5}\frac{1}{t}dt\right) + \left(\frac{D_1}{x_5} - 1\right) d_3x_6\right] + \frac{\beta_1 f(H_1,V_1)}{\alpha D_1}\left[c\left(x_7 - V_1\int_{V_1}^{x_7}\frac{1}{t}dt\right)\right.$$
$$\left.+\left(\frac{V_1}{x_7} - 1\right) d_4x_8\right] + \frac{r\beta_1 f(H_1,V_1)}{b\alpha D_1}\left(cx_9 - d_5x_{10}\right) + \frac{p}{a}\left(cx_{11} - d_6x_{12}\right).$$

The derivative of $\mathcal{M}_1$ along the trajectories of system (5.1), denoted by $\frac{d\mathcal{M}_1}{ds}$, gives

$$\frac{d\mathcal{M}_1}{ds} \le dH_1\left(1 - \frac{\ell(H_1)}{\ell(x_1)}\right)\left(1 - \frac{x_1}{H_1}\right)$$
$$+\beta_1 f(H_1,V_1)\left[5 - \frac{\ell(H_1)}{\ell(x_1)} - \frac{x_3D_1}{x_5I_1} - \frac{V_1x_5}{D_1x_7} - \frac{f(x_1,x_7)I_1}{f(H_1,V_1)x_3} - \frac{f(H_1,V_1)\ell(x_1)x_7}{f(x_1,x_7)\ell(H_1)V_1}\right]$$
$$+\beta_1 f(H_1,V_1)\left(1 - \frac{f(H_1,V_1)\ell(x_1)}{f(x_1,x_7)\ell(H_1)}\right)\left(\frac{f(x_1,x_7)\ell(H_1)}{f(H_1,V_1)\ell(x_1)} - \frac{x_7}{V_1}\right)$$
$$+\beta_2 \mathrm{g}(H_1,I_1)\left[3 - \frac{\ell(H_1)}{\ell(x_1)} - \frac{\mathrm{g}(x_1,x_3)I_1}{\mathrm{g}(H_1,I_1)x_3} - \frac{\mathrm{g}(H_1,I_1)\ell(x_1)x_3}{\mathrm{g}(x_1,x_3)\ell(H_1)I_1}\right]$$
$$+\beta_2 \mathrm{g}(H_1,I_1)\left(1 - \frac{\mathrm{g}(H_1,I_1)\ell(x_1)}{\mathrm{g}(x_1,x_3)\ell(H_1)}\right)\left(\frac{\mathrm{g}(x_1,x_3)\ell(H_1)}{\mathrm{g}(H_1,I_1)\ell(x_1)} - \frac{x_3}{I_1}\right)$$
$$+\frac{r\sigma\beta_1 f(H_1,V_1)}{b\alpha D_1}(\mathcal{R}_1 - 1)x_9 + \frac{pq}{a}(\mathcal{R}_2 - 1)x_{11}.$$

Thus, the conditions of the theorem ensure that $\frac{d\mathcal{M}_1}{ds} \le 0$ and $\frac{d\mathcal{M}_1}{ds} = 0$ if and only if $x_1 = H_1$, $x_3 = I_1$, $x_5 = D_1$, $x_7 = V_1$, $x_9 = W_1 = 0$ and $x_{11} = Z_1 = 0$. Thus by [19, Theorem 5.3.1], it follows that $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_1, I_1, D_1V_1, 0, 0)$. Hence, item (i) is proved.

For item (ii), define the following Lyapunov function

$$\mathcal{M}_2 = c\left(x_1 - \int_{H_2}^{x_1} \frac{\ell(H_2)}{\ell(t)}dt\right) + \left(\frac{\ell(H_2)}{\ell(x_1)} - 1\right) d_1x_2 + c\left(x_3 - I_2\int_{I_2}^{x_3}\frac{1}{t}dt\right) + \left(\frac{I_2}{x_3} - 1\right) d_2x_4$$
$$+\frac{\beta_1 f(H_2,V_2)}{kI_2}\left[c\left(x_5 - D_2\int_{D_2}^{x_5}\frac{1}{t}dt\right) + \left(\frac{D_2}{x_5} - 1\right) d_3x_6\right] + \frac{\beta_1 f(H_2,V_2)}{\alpha D_2}\left[c\left(x_7 - V_2\int_{V_2}^{x_7}\frac{1}{t}dt\right)\right.$$
$$\left.+\left(\frac{V_2}{x_7} - 1\right) d_4x_8\right] + \frac{r\beta_1 f(H_2,V_2)}{b\alpha D_2}\left[c\left(x_9 - W_2\int_{W_2}^{x_9}\frac{1}{t}dt\right) + \left(\frac{W_2}{x_9} - 1\right) d_5x_{10}\right] + \frac{p}{a}\left(cx_{11} - d_6x_{12}\right).$$

The derivative of $\mathcal{M}_2$ along the trajectories of system (5.1), denoted by $\frac{d\mathcal{M}_2}{ds}$, yields

$$\frac{d\mathcal{M}_2}{ds} \le dH_2\left(1 - \frac{\ell(H_2)}{\ell(x_1)}\right)\left(1 - \frac{x_1}{H_2}\right)$$
$$+\beta_1 f(H_2,V_2)\left[5 - \frac{\ell(H_2)}{\ell(x_1)} - \frac{x_3D_2}{x_5I_2} - \frac{V_2x_5}{D_2x_7} - \frac{f(x_1,x_7)I_2}{f(H_2,V_2)x_3} - \frac{f(H_2,V_2)\ell(x_1)x_7}{f(x_1,x_7)\ell(H_2)V_2}\right]$$
$$+\beta_1 f(H_2,V_2)\left(1 - \frac{f(H_2,V_2)\ell(x_1)}{f(x_1,x_7)\ell(H_2)}\right)\left(\frac{f(x_1,x_7)\ell(H_2)}{f(H_2,V_2)\ell(x_1)} - \frac{x_7}{V_2}\right)$$
$$+\beta_2 \mathrm{g}(H_2,I_2)\left[3 - \frac{\ell(H_2)}{\ell(x_1)} - \frac{\mathrm{g}(x_1,x_3)I_2}{\mathrm{g}(H_2,I_2)x_3} - \frac{\mathrm{g}(H_2,I_2)\ell(x_1)x_3}{\mathrm{g}(x_1,x_3)\ell(H_2)I_2}\right]$$
$$+\beta_2 \mathrm{g}(H_2,I_2)\left(1 - \frac{\mathrm{g}(H_2,I_2)\ell(x_1)}{\mathrm{g}(x_1,x_3)\ell(H_2)}\right)\left(\frac{\mathrm{g}(x_1,x_3)\ell(H_2)}{\mathrm{g}(H_2,I_2)\ell(x_1)} - \frac{x_3}{I_2}\right) + \frac{pq}{a}(\mathcal{R}_3 - 1)x_{11}.$$

Hence, under the conditions of the theorem, $\frac{d\mathcal{M}_2}{ds} \le 0$ and $\frac{d\mathcal{M}_2}{ds} = 0$ if and only if $x_1 = H_2$, $x_3 = I_2$, $x_5 = D_2$, $x_7 = V_2$, $x_9 = W_2$ and $x_{11} = Z_2 = 0$. Thus by [19, Theorem 5.3.1], it follows that

$(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_2, I_2, D_2V_2, W_2, 0)$. This completes the proof of item (ii).

For item (iii), define the following Lyapunov function

$$\begin{aligned}\mathcal{M}_3 &= c\left(x_1 - \int_{H_3}^{x_1} \frac{\ell(H_2)}{\ell(t)}dt\right) + \left(\frac{\ell(H_3)}{\ell(x_1)} - 1\right) d_1x_2 + c\left(x_3 - I_3\int_{I_3}^{x_3} \frac{1}{t}dt\right) + \left(\frac{I_3}{x_3} - 1\right) d_2x_4 \\ &+ \frac{\beta_1 f(H_3, V_3)}{kI_3}\left[c\left(x_5 - D_3\int_{D_3}^{x_5} \frac{1}{t}dt\right) + \left(\frac{D_3}{x_5} - 1\right) d_3x_6\right] + \frac{\beta_1 f(H_3, V_3)}{\alpha D_3}\left[c\left(x_7 - V_3\int_{V_3}^{x_7} \frac{1}{t}dt\right)\right. \\ &\left. + \left(\frac{V_3}{x_7} - 1\right) d_4x_8\right] + \frac{r\beta_1 f(H_3, V_3)}{b\alpha D_3}(cx_9 - d_5x_{10}) + \frac{p}{a}\left[c\left(x_{11} - Z_3\int_{Z_3}^{x_{11}} \frac{1}{t}dt\right) + \left(\frac{Z_3}{x_{11}} - 1\right) d_6x_{12}\right].\end{aligned}$$

The derivative of $\mathcal{M}_3$ along the trajectories of system (5.1), denoted by $\frac{d\mathcal{M}_3}{ds}$ yields

$$\begin{aligned}\frac{d\mathcal{M}_3}{ds} \leq\ & dH_3\left(1 - \frac{\ell(H_3)}{\ell(x_1)}\right)\left(1 - \frac{x_1}{H_3}\right) \\ &+ \beta_1 f(H_3, V_3)\left[5 - \frac{\ell(H_3)}{\ell(x_1)} - \frac{x_3D_3}{x_5I_3} - \frac{V_3x_5}{D_3x_7} - \frac{f(x_1, x_7)I_3}{f(H_3, V_3)x_3} - \frac{f(H_3, V_3)\ell(x_1)x_7}{f(x_1, x_7)\ell(H_3)V_3}\right] \\ &+ \beta_1 f(H_3, V_3)\left(1 - \frac{f(H_3, V_3)\ell(x_1)}{f(x_1, x_7)\ell(H_3)}\right)\left(\frac{f(x_1, x_7)\ell(H_3)}{f(H_3, V_3)\ell(x_1)} - \frac{x_7}{V_3}\right) \\ &+ \beta_2 \mathrm{g}(H_3, I_3)\left[3 - \frac{\ell(H_3)}{\ell(x_1)} - \frac{\mathrm{g}(x_1, x_3)I_3}{\mathrm{g}(H_3, I_3)x_3} - \frac{\mathrm{g}(H_3, I_3)\ell(x_1)x_3}{\mathrm{g}(x_1, x_3)\ell(H_3)I_3}\right] \\ &+ \beta_2 \mathrm{g}(H_3, I_3)\left(1 - \frac{\mathrm{g}(H_3, I_3)\ell(x_1)}{\mathrm{g}(x_1, x_3)\ell(H_3)}\right)\left(\frac{\mathrm{g}(x_1, x_3)\ell(H_3)}{\mathrm{g}(H_3, I_3)\ell(x_1)} - \frac{x_3}{I_3}\right) + \frac{r\sigma\beta_1 f(H_3, V_3)}{b\alpha D_3}(\mathcal{R}_4 - 1)x_9.\end{aligned}$$

Therefore, the conditions of the theorem ensure that $\frac{d\mathcal{M}_3}{ds} \leq 0$ and $\frac{d\mathcal{M}_3}{ds} = 0$ if and only if $x_1 = H_3$, $x_3 = I_3$, $x_5 = D_3$, $x_7 = V_3$, $x_9 = W_3 = 0$ and $x_{11} = Z_1$. Thus by [19, Theorem 5.3.1], it follows that $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_3, I_3, D_3V_3, 0, Z_3)$. Whence item (iii) is demonstrated.

For item (iv), define the following Lyapunov function

$$\begin{aligned}\mathcal{M}_4 = c&\left(x_1 - \int_{H_4}^{x_1} \frac{\ell(H_4)}{\ell(t)}dt\right) + \left(\frac{\ell(H_4)}{\ell(x_1)} - 1\right) d_1x_2 + c\left(x_3 - I_4\int_{I_4}^{x_3} \frac{1}{t}dt\right) + \left(\frac{I_4}{x_3} - 1\right) d_2x_4 \\ &+ \frac{\beta_1 f(H_4, V_4)}{kI_4}\left[c\left(x_5 - D_4\int_{D_4}^{x_5} \frac{1}{t}dt\right) + \left(\frac{D_4}{x_5} - 1\right) d_3x_6\right] \\ &+ \frac{\beta_1 f(H_4, V_4)}{\alpha D_4}\left[c\left(x_7 - V_4\int_{V_4}^{x_7} \frac{1}{t}dt\right) + \left(\frac{V_4}{x_7} - 1\right) d_4x_8\right] \\ &+ \frac{r\beta_1 f(H_4, V_4)}{b\alpha D_4}\left[c\left(x_9 - W_4\int_{W_4}^{x_9} \frac{1}{t}dt\right) + \left(\frac{W_4}{x_9} - 1\right) d_5x_{10}\right] \\ &+ \frac{p}{a}\left[c\left(x_{11} - Z_4\int_{Z_4}^{x_{11}} \frac{1}{t}dt\right) + \left(\frac{Z_4}{x_{11}} - 1\right) d_6x_{12}\right].\end{aligned}$$

The derivative of $\mathcal{M}_4$ along the trajectories of system (5.1), denoted by $\frac{d\mathcal{M}_4}{ds}$ yields

$$\begin{aligned}\frac{d\mathcal{M}_4}{ds} \leq & \, dH_4\left(1-\frac{\ell(H_4)}{\ell(x_1)}\right)\left(1-\frac{x_1}{H_4}\right)\\ & +\beta_1 f(H_4,V_4)\left[5-\frac{\ell(H_4)}{\ell(x_1)}-\frac{x_3D_4}{x_5I_4}-\frac{V_4x_5}{D_4x_7}-\frac{f(x_1,x_7)I_4}{f(H_4,V_4)x_3}-\frac{f(H_4,V_4)\ell(x_1)x_7}{f(x_1,x_7)\ell(H_4)V_4}\right]\\ & +\beta_1 f(H_4,V_4)\left(1-\frac{f(H_4,V_4)\ell(x_1)}{f(x_1,x_7)\ell(H_4)}\right)\left(\frac{f(x_1,x_7)\ell(H_4)}{f(H_4,V_4)\ell(x_1)}-\frac{x_7}{V_4}\right)\\ & +\beta_2 \mathrm{g}(H_4,I_4)\left[3-\frac{\ell(H_4)}{\ell(x_1)}-\frac{\mathrm{g}(x_1,x_3)I_4}{\mathrm{g}(H_4,I_4)x_3}-\frac{\mathrm{g}(H_4,I_4)\ell(x_1)x_3}{\mathrm{g}(x_1,x_3)\ell(H_4)I_4}\right]\\ & +\beta_2 \mathrm{g}(H_4,I_4)\left(1-\frac{\mathrm{g}(H_4,I_4)\ell(x_1)}{\mathrm{g}(x_1,x_3)\ell(H_4)}\right)\left(\frac{\mathrm{g}(x_1,x_3)\ell(H_4)}{\mathrm{g}(H_4,I_4)\ell(x_1)}-\frac{x_3}{I_4}\right).\end{aligned}$$

Hence, under the conditions of the theorem, $\frac{d\mathcal{M}_4}{ds} \leq 0$ and $\frac{d\mathcal{M}_4}{ds} = 0$ if and only if $x_1 = H_4$, $x_3 = I_4$, $x_5 = D_4$, $x_7 = V_4$, $x_9 = W_4$ and $x_{11} = Z_4$. Thus by [19, Theorem 5.3.1], it follows that $(H^*(\infty), I^*(\infty), D^*(\infty), V^*(\infty), W^*(\infty), Z^*(\infty)) = (H_4, I_4, D_4V_4, W_4, Z_4)$. This completes the proof of item (iv). □

Remark 5.1. We see that, in Theorem 5.3, (Q2) is an additional condition introduced to establish the existence of traveling waves solutions of system (2.3) defined for $s \in \mathbb{R}$ connecting equilibria E_0 and E_1, equilibria E_0 and E_2, equilibria E_0 and E_3, and equilibria E_0 and E_4, respectively.

Now, we focus on the traveling wave solution $(H(s), I(s), D(s), V(s), W(s), Z(s))$ of system (2.3) in Cases 4, 5 and 7, given in (2.7), (2.8) and (2.10), respectively.

By the previous Lyapunov functions and LaSalle's invariance principle [19, Theorem 5.3.1], and the linearization methods, it is found that for system (2.3),

- when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$ and $\mathcal{R}_3 \leq 1$, the unique positive equilibrium $E_2 = (H_2, I_2, D_2, V_2, W_2, 0)$ is globally asymptotically stable and equilibrium $E_1 = (H_1, I_1, D_1, V_1, 0, 0)$ is unstable.
- when $\mathcal{R}_0 > 1$, $\mathcal{R}_2 > 1$ and $\mathcal{R}_4 \leq 1$, the unique positive equilibrium $E_3 = (H_3, I_3, D_3, V_3, 0, Z_2)$ is globally asymptotically stable and equilibrium $E_1 = (H_1, I_1, D_1, V_1, 0, 0)$ is unstable.
- when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, the unique positive equilibrium $E_4 = (H_4, I_4, D_4, V_4, W_4, Z_4)$ is globally asymptotically stable and equilibrium $E_1 = (H_1, I_1, D_1, V_1, 0, 0)$ is unstable.

  Therefore, one can easily guess that:
- There exists a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ of system (2.3) defined for $s \in \mathbb{R}$ connecting E_1 and E_2 when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_3 \leq 1$ and $c > \hat{c}$.
- There exists a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ of system (2.3) defined for $s \in \mathbb{R}$ connecting E_1 and E_3 when $\mathcal{R}_0 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_4 \leq 1$ and $c > \hat{c}$.
- There exists a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ of system (2.3) defined for $s \in \mathbb{R}$ connecting E_1 and E_4, when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c > \hat{c}$, for some minimum waves speed $\hat{c}$ defined in (5.6) after a computational process.

Linearizing system (2.3) at equilibrium $E_1 = (H_1, I_1, D_1, V_1, 0, 0)$, we have

$$\begin{cases} d_1H_{ss} - cH_s - dH - \beta_1 f_H(H_1,V_1)H - \beta_2 \mathrm{g}_H(H_1,V_1)H - \beta_1 f_V(H_1,V_1)V - \beta_2\mathrm{g}_I(H_1,V_1)I = 0,\\ d_2I_{ss} - cI_s + \beta_1 f_H(H_1,V_1)H + \beta_2\mathrm{g}_H(H_1,V_1)H + \beta_1 f_V(H_1,V_1)V + \beta_2\mathrm{g}_I(H_1,V_1)I - \delta I - pI_1Z = 0,\\ d_3D_{ss} - cD_s + kI - (\alpha+\delta)D = 0,\\ d_4V_{ss} - cV_s + \alpha D - \mu V - rV_1W = 0,\\ d_5W_{ss} - cW_s + bV_1W - \sigma W = 0,\\ d_6Z_{ss} - cZ_s + aI_1Z - qZ = 0. \end{cases} \tag{5.4}$$

Substituting $(H(s), I(s), D(s), V(s), W(s), Z(s)) = (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5, \tilde{k}_6)e^{\tilde{\gamma}s}$ into (5.4) gives

$$\tilde{A}(\tilde{\gamma})(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5, \tilde{k}_6)^T = c\gamma(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5, \tilde{k}_6)^T, \tag{5.5}$$

where

$$\tilde{A}(\tilde{\gamma}) = \begin{pmatrix} h_1(\tilde{\gamma}) & -\beta_2 g_I & 0 & -\beta_1 f_V & 0 & 0 \\ \beta_1 f_H + \beta_2 g_H & h_2(\tilde{\gamma}) & 0 & \beta_1 f_V & 0 & -pI_1 \\ 0 & k & h_3(\tilde{\gamma}) & 0 & 0 & 0 \\ 0 & 0 & \alpha & h_4(\tilde{\gamma}) & -rV_1 & 0 \\ 0 & 0 & 0 & 0 & h_5(\tilde{\gamma}) & 0 \\ 0 & 0 & 0 & 0 & 0 & h_6(\tilde{\gamma}) \end{pmatrix},$$

with $h_1(\tilde{\gamma}) = d_1\tilde{\gamma}^2 - \beta_1 f_H - \beta_2 g_H - d$, $h_2(\tilde{\gamma}) = d_2\tilde{\gamma}^2 + \beta_2 g_I - \delta$, $h_3(\tilde{\gamma}) = d_3\tilde{\gamma}^2 - \alpha - \delta$, $h_4(\tilde{\gamma}) = d_4\tilde{\gamma}^2 - \mu$, $h_5(\tilde{\gamma}) = d_5\tilde{\gamma}^2 + bV_1 - \sigma$, $h_6(\tilde{\gamma}) = d_6\tilde{\gamma}^2 + aI_1 - q$. The characteristic polynomial of $\tilde{A}(\tilde{\gamma})$ is

$$\begin{aligned} \tilde{P}_{\tilde{\gamma}}(\tilde{\lambda}) &= \det(\tilde{A}(\tilde{\gamma}) - \tilde{\lambda}\mathcal{I}) \\ &= (h_5(\tilde{\gamma}) - \tilde{\lambda})(h_6(\tilde{\gamma}) - \tilde{\lambda}) \Big[\tilde{\lambda}^4 - (h_1 + h_2 + h_3 + h_4)\tilde{\lambda}^3 + (h_1h_2 + h_1h_3 + h_1h_4 + h_2h_3 + h_2h_4 + h_3h_4 \\ &\quad + \beta_2 g_I(\beta_1 f_H + \beta_2 g_H))\tilde{\lambda}^2 - (h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4 - (h_3 + h_4)\beta_2 g_I(\beta_1 f_H + \beta_2 g_H) \\ &\quad - k\alpha\beta_1 f_V)\tilde{\lambda} + h_1h_2h_3h_4 + h_3h_4\beta_2 g_I(\beta_1 f_H + \beta_2 g_H) + k\alpha\beta_1 f_V(h_1 + \beta_1 f_H + \beta_2 g_H)\Big]. \end{aligned}$$

Let $\tilde{\lambda}_1(\tilde{\gamma})$, $\tilde{\lambda}_2(\tilde{\gamma})$, $\tilde{\lambda}_3(\tilde{\gamma})$, $\tilde{\lambda}_4(\tilde{\gamma})$, $\tilde{\lambda}_5(\tilde{\gamma})$ and $\tilde{\lambda}_5(\tilde{\gamma})$ denote the roots of equation $\tilde{P}_{\tilde{\gamma}}(\tilde{\lambda}) = 0$, where $\tilde{\lambda}_1(\tilde{\gamma}) = h_5(\tilde{\gamma})$ and $\tilde{\lambda}_2(\tilde{\gamma}) = h_6(\tilde{\gamma})$. We define

$$\tilde{\lambda}_{\max}(\tilde{\gamma}) = \max\left\{\tilde{\lambda}_1(\tilde{\gamma}), \tilde{\lambda}_2(\tilde{\gamma}), Re\left\{\tilde{\lambda}_3(\tilde{\gamma})\right\}, Re\left\{\tilde{\lambda}_4(\tilde{\gamma})\right\}, Re\left\{\tilde{\lambda}_5(\tilde{\gamma})\right\}, Re\left\{\tilde{\lambda}_6(\tilde{\gamma})\right\}\right\}.$$

It is clear that $h_5(0) = bV_1 - \sigma > 0$ if and only if $\mathcal{R}_1 > 1$ and $h_6(0) = aI_1 - q > 0$ if and only if $\mathcal{R}_2 > 1$. Thus, $\tilde{\lambda}_{\max}(\tilde{\gamma})$ is positive for $\tilde{\gamma} \in [0, \infty)$. The minimum waves speed $\hat{c}$ is defined as

$$\hat{c} = \inf_{\tilde{\gamma}>0} \frac{\tilde{\lambda}_{\max}(\tilde{\gamma})}{\tilde{\gamma}}. \tag{5.6}$$

Note that the minimum wave speeds c^* and $\hat{c}$ are computed from the infection-free equilibrium E_0 and immune-free infection equilibrium E_1, respectively. Therefore, by analogy with the role of the wave speed c^* in proving the existence of a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibrium E_0 and the infection equilibria E_i, $i = 1, 2, 3, 4$, we make the conjectures below, especially that numerical simulations (Figs. 8, 9, 10) suggest so.

Conjecture 5.1. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_3 \leq 1$ and $c > \hat{c}$. Then system (2.3) has a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibria E_1 and E_2.*

Conjecture 5.2. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_4 \leq 1$ and $c > \hat{c}$. Then system (2.3) has a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibria E_1 and E_3.*

Conjecture 5.3. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c > \hat{c}$. Then system (2.3) has a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibria E_1 and E_4.*

Next, we consider the traveling wave solution $(H(s), I(s), D(s), V(s), W(s), Z(s))$ of system (2.3) in Case 8, given in (2.11).

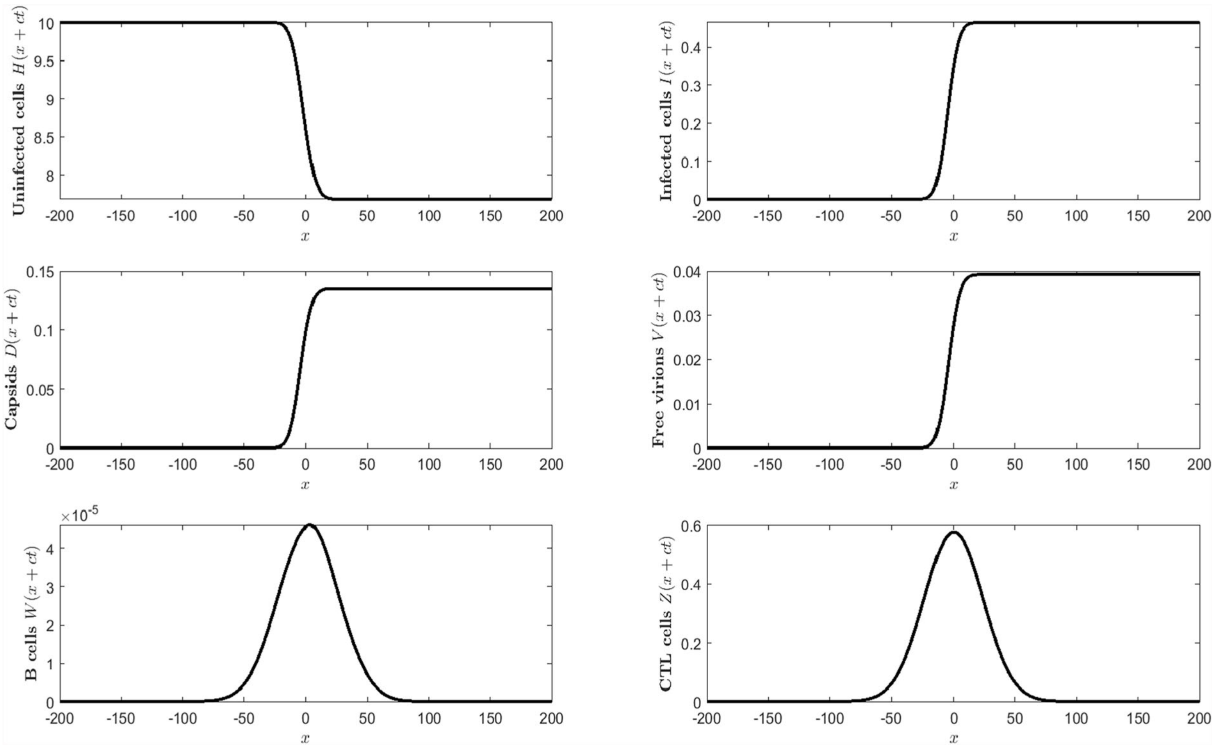

FIG. 2. Profiles of the traveling waves solution of (1.1) connecting E_0 and E_1 at the final time $T = 10$.

Again, by the previous Lyapunov methods, LaSalle's invariance principle [19, Theorem 5.3.1] and the linearization methods, it is found that for system (2.3), when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, the unique positive equilibrium $E_4 = (H_4, I_4, D_4, V_4, W_4, Z_4)$ is globally asymptotically stable and equilibrium $E_2 = (H_2, I_2, D_2, V_2, W_2, 0)$ is unstable. Therefore, we can also guess that there exists a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ of system (2.3), defined for $s \in \mathbb{R}$, and connecting E_2 and E_4 when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c > \check{c}$, for some minimum wave speed $\check{c}$ defined in (5.7).

Linearizing system (2.3) at equilibrium $E_2 = (H_2, I_2, D_2, V_2, W_2, 0)$ and after computation we obtain the following characteristic polynomial

$$\begin{aligned}
\bar{P}_{\bar{\gamma}}(\bar{\lambda}) &= (h_6(\bar{\gamma}) - \bar{\lambda})\left[-\bar{\lambda}^5 + (h_1 + h_2 + h_3 + h_4 + h_5)\bar{\lambda}^4 - (h_1h_2 + h_1h_3 + h_1h_4 + h_2h_3 + h_2h_4 + h_3h_4\right.\\
&+h_1h_5 + h_2h_5 + h_3h_5 + h_4h_5 + \beta_2 g_I(\beta_1 f_H + \beta_2 g_H) - rbV_2W_2)\bar{\lambda}^3 + (h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4\\
&+h_2h_3h_4 + h_1h_2h_5 + h_1h_3h_5 + h_1h_4h_5 + h_2h_3h_5 + h_2h_4h_5 + h_3h_4h_5 - (h_3 + h_4 - h_5)\beta_2 g_I(\beta_1 f_H + \beta_2 g_H)\\
&-k\alpha\beta_1 f_V + rbV_2W_2(h_1 + h_2 + h_3))\bar{\lambda}^2 - (h_1h_2h_3h_4 + h_1h_2h_3h_5 + h_1h_2h_4h_5 + h_1h_3h_4h_5 + h_2h_3h_4h_5\\
&-(h_3h_4 - h_3h_5 - h_4h_5 + rbV_2W_2)\beta_2 g_I(\beta_1 f_H + \beta_2 g_H) + k\alpha\beta_1 f_V(h_1 - h_5 + \beta_1 f_H + \beta_2 g_H)\\
&-rbV_2W_2(h_1h_2 + h_1h_3 + h_2h_3))\bar{\lambda} + h_1h_2h_3h_4h_5 + h_1h_2h_3 rbV_2W_2\\
&+(h_3h_4h_5 + h_3 rbV_2W_2)\beta_2 g_I(\beta_1 f_H + \beta_2 g_H)\\
&\left.+h_5 k\alpha\beta_1 f_V(h_1 + \beta_1 f_H + \beta_2 g_H)\right],
\end{aligned}$$

where $h_1(\bar{\gamma}) = d_1\bar{\gamma}^2 - \beta_1 f_H - \beta_2 g_H - d$, $h_2(\bar{\gamma}) = d_2\bar{\gamma}^2 + \beta_2 g_I - \delta$, $h_3(\bar{\gamma}) = d_3\bar{\gamma}^2 - \alpha - \delta$, $h_4(\bar{\gamma}) = d_4\bar{\gamma}^2 - \mu - r_2W_2$, $h_5(\bar{\gamma}) = d_5\bar{\gamma}^2 + bV_2 - \sigma$, $h_6(\bar{\gamma}) = d_6\bar{\gamma}^2 + aI_2 - q$.

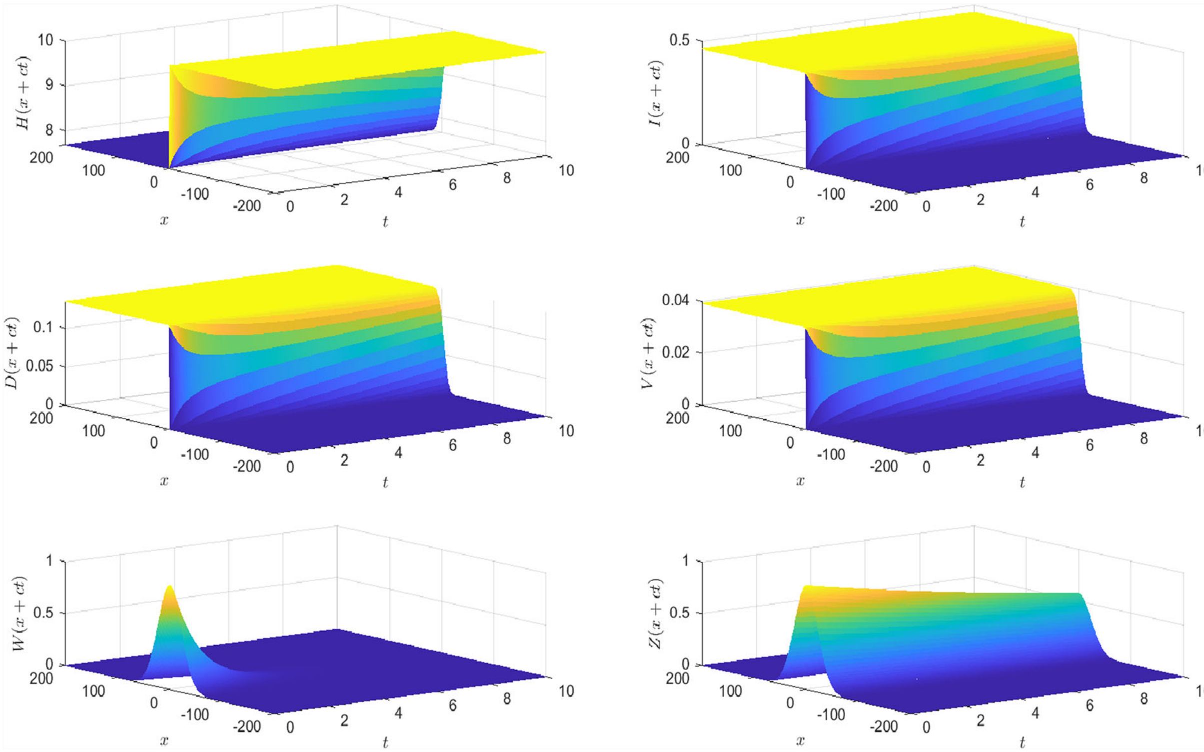

FIG. 3. Numerical simulation illustrating the existence of a traveling waves solution to system (1.1) connecting E_0 and E_1 with minimal speed c^*.

Let $\bar{\lambda}_1(\bar{\gamma})$, $\bar{\lambda}_2(\bar{\gamma})$, $\bar{\lambda}_3(\bar{\gamma})$, $\bar{\lambda}_4(\bar{\gamma})$, $\bar{\lambda}_5(\bar{\gamma})$ and $\bar{\lambda}_5(\bar{\gamma})$ denote the roots of equation $\bar{P}_{\bar{\gamma}}(\bar{\lambda}) = 0$, where $\bar{\lambda}_1(\bar{\gamma}) = h_6(\bar{\gamma})$.

$$\bar{\lambda}_{\max}(\bar{\gamma}) = \max\left\{\bar{\lambda}_1(\bar{\gamma}), Re\left\{\bar{\lambda}_2(\bar{\gamma})\right\}, Re\left\{\bar{\lambda}_3(\bar{\gamma})\right\}, Re\left\{\bar{\lambda}_4(\bar{\gamma})\right\}, Re\left\{\bar{\lambda}_5(\bar{\gamma})\right\}, Re\left\{\bar{\lambda}_6(\bar{\gamma})\right\}\right\}.$$

It is clear that $h_6(0) = aI_2 - q > 0$ if and only if $\mathcal{R}_3 > 1$. Thus, $\bar{\lambda}_{\max}(\bar{\gamma})$ is strictly positive for $\bar{\gamma} \in [0, \infty)$. The minimum waves speed $\check{c}$ is defined in this case as

$$\check{c} = \inf_{\bar{\gamma}>0} \frac{\bar{\lambda}_{\max}(\bar{\gamma})}{\bar{\gamma}}. \tag{5.7}$$

Hence, we further have the following conjecture.

Conjecture 5.4. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c > \check{c}$. Then system (2.3) has a traveling wave solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ defined for $s \in \mathbb{R}$ connecting equilibria E_2 and E_4.*

Finally, we consider the traveling waves solution $(H(s), I(s), D(s), V(s), W(s), Z(s))$ of system (2.3) in Case 9, given in (2.12).

Again, by the previous Lyapunov methods and LaSalle's invariance principle [19, Theorem 5.3.1], together with the linearization methods, it is found that for system (2.3), when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, the unique positive equilibrium $E_4 = (H_4, I_4, D_4, V_4, W_4, Z_4)$ is globally asymptotically stable and the equilibrium point $E_3 = (H_3, I_3, D_3, V_3, 0, Z_2)$ is unstable. Therefore, we can also guess that there exists a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ of

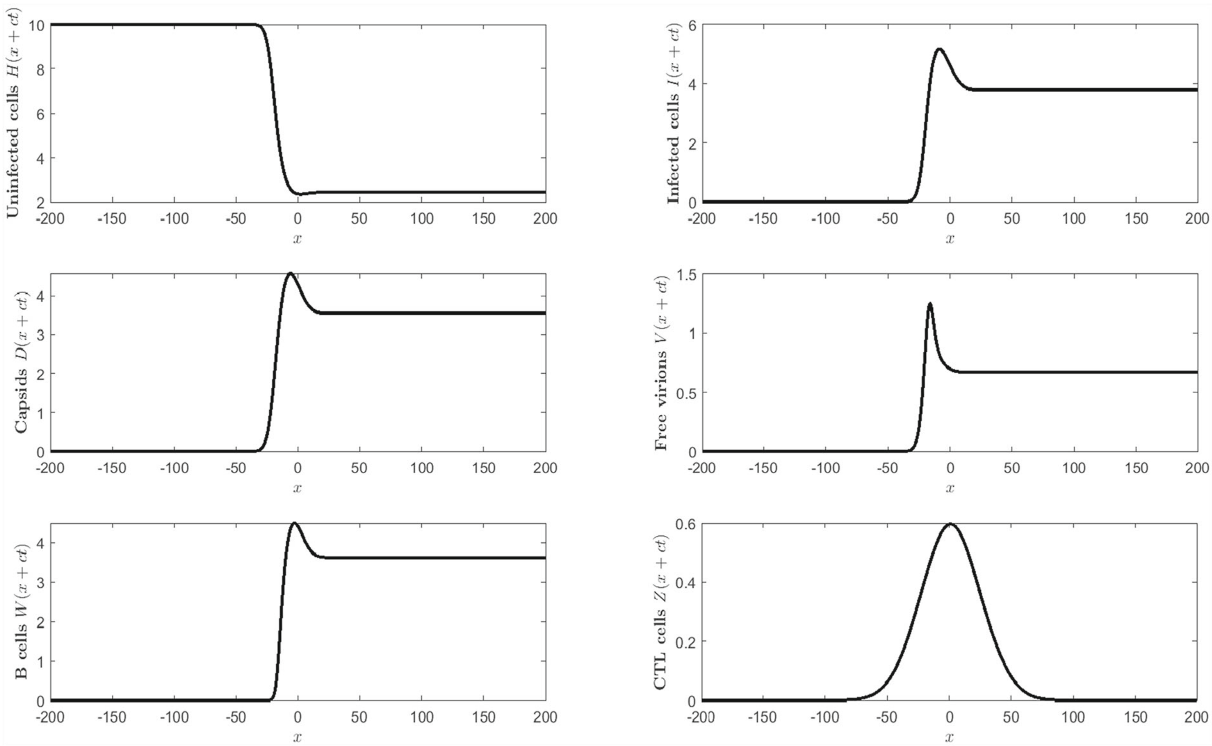

FIG. 4. Profiles of the traveling waves solution of (1.1) connecting E_0 and E_2 at the final time $T = 10$.

system (2.3), defined for $s \in \mathbb{R}$, and connecting E_3 and E_4 when $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c > \hat{\hat{c}}$, for some minimum wave speed $\hat{\hat{c}}$ defined in (5.8).

Linearizing system (2.3) at equilibrium $E_3 = (H_3, I_3, D_3, V_3, 0, Z_2)$ and after some computation, we obtain the following characteristic polynomial

$$\begin{aligned}\bar{\bar{P}}_{\bar{\bar{\gamma}}}(\bar{\bar{\lambda}}) = (h_5(\bar{\bar{\gamma}}) - \bar{\bar{\lambda}}) \Big[& -\bar{\bar{\lambda}}^5 + (h_1+h_2+h_3+h_4+h_6)\bar{\bar{\lambda}}^4 - (h_1h_2+h_1h_3+h_1h_4+h_2h_3+h_2h_4+h_3h_4 \\ & +h_1h_6+h_2h_6+h_3h_6+h_4h_6+\beta_2 g_I(\beta_1 f_H+\beta_2 g_H)+apI_3Z_3)\bar{\bar{\lambda}}^3 + (h_1h_2h_3+h_1h_2h_4+h_1h_3h_4 \\ & +h_2h_3h_4+h_1h_2h_6+h_1h_3h_6+h_1h_4h_6+h_2h_3h_6+h_2h_4h_6+h_3h_4h_6-(h_3+h_4)\beta_2 g_I(\beta_1 f_H+\beta_2 g_H) \\ & -k\alpha\beta_1 f_V - apI_3Z_3(h_1+h_3+h_4))\bar{\bar{\lambda}}^2 - (h_1h_2h_3h_4+h_1h_2h_3h_6+h_1h_2h_4h_6+h_1h_3h_4h_6+h_2h_3h_4h_6 \\ & -(h_3h_6+h_4h_6)\beta_2 g_I(\beta_1 f_H+\beta_2 g_H)+k\alpha\beta_1 f_V(h_1-h_6+\beta_1 f_H+\beta_2 g_H)+apI_3Z_3(h_1h_3+h_1h_3+h_3h_4))\bar{\bar{\lambda}} \\ & +h_1h_2h_3h_4h_6-h_1h_3h_4apI_3Z_3+h_3h_4h_6\beta_2 g_I(\beta_1 f_H+\beta_2 g_H)+h_5k\alpha\beta_1 f_V(h_1+\beta_1 f_H+\beta_2 g_H)\Big],\end{aligned}$$

where $h_1(\bar{\bar{\gamma}}) = d_1\bar{\bar{\gamma}}^2 - \beta_1 f_H - \beta_2 g_H - d$, $h_2(\bar{\bar{\gamma}}) = d_2\bar{\bar{\gamma}}^2 + \beta_2 g_I - \delta - pZ_3$, $h_3(\bar{\bar{\gamma}}) = d_3\bar{\bar{\gamma}}^2 - \alpha - \delta$, $h_4(\bar{\bar{\gamma}}) = d_4\bar{\bar{\gamma}}^2 - \mu$, $h_5(\bar{\bar{\gamma}}) = d_5\bar{\bar{\gamma}}^2 + bV_3 - \sigma$, $h_6(\bar{\bar{\gamma}}) = d_6\bar{\bar{\gamma}}^2 + aI_3 - q$.

Let $\bar{\bar{\lambda}}_1(\bar{\bar{\gamma}})$, $\bar{\bar{\lambda}}_2(\bar{\bar{\gamma}})$, $\bar{\bar{\lambda}}_3(\bar{\bar{\gamma}})$, $\bar{\bar{\lambda}}_4(\bar{\bar{\gamma}})$, $\bar{\bar{\lambda}}_5(\bar{\bar{\gamma}})$ and $\bar{\bar{\lambda}}_6(\bar{\bar{\gamma}})$ denote the roots of equation $\bar{\bar{P}}_{\bar{\bar{\gamma}}}(\bar{\bar{\lambda}}) = 0$, where $\bar{\bar{\lambda}}_1(\bar{\bar{\gamma}}) = h_5(\bar{\bar{\gamma}})$. We define

$$\bar{\bar{\lambda}}_{\max}(\bar{\bar{\gamma}}) = \max\left\{\bar{\bar{\lambda}}_1(\bar{\bar{\gamma}}), Re\left\{\bar{\bar{\lambda}}_2(\bar{\bar{\gamma}})\right\}, Re\left\{\bar{\bar{\lambda}}_3(\bar{\bar{\gamma}})\right\}, Re\left\{\bar{\bar{\lambda}}_4(\bar{\bar{\gamma}})\right\}, Re\left\{\bar{\bar{\lambda}}_5(\bar{\bar{\gamma}})\right\}, Re\left\{\bar{\bar{\lambda}}_6(\bar{\bar{\gamma}})\right\}\right\}.$$

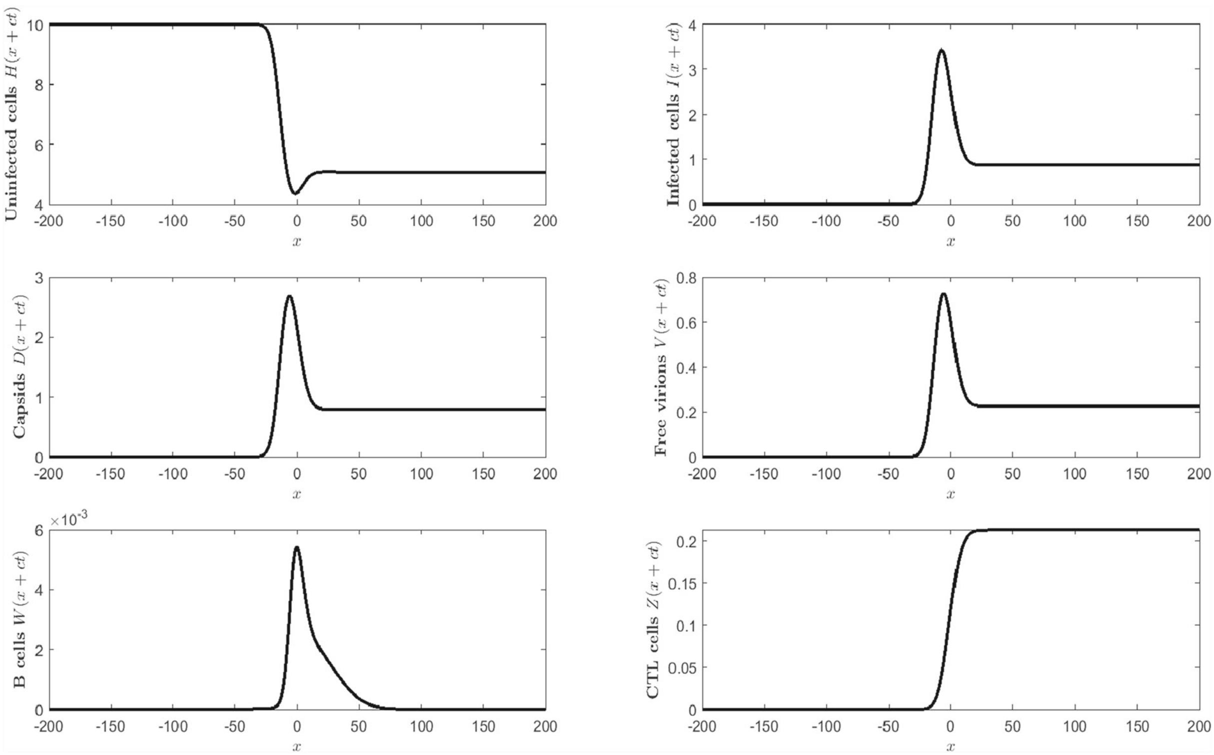

FIG. 5. Profiles of the traveling waves solution of (1.1) connecting E_0 and E_3 at the final time $T = 10$.

It is clear that $h_5(0) = bV_3 - \sigma > 0$ if and only if $\mathcal{R}_4 > 1$. Thus, $\bar{\bar{\lambda}}_{\max}(\bar{\bar{\gamma}})$ is strictly positive for $\bar{\bar{\gamma}} \in [0, \infty)$. The minimum waves speed $\hat{\hat{c}}$ is defined in this case by

$$\hat{\hat{c}} = \inf_{\bar{\bar{\gamma}}>0} \frac{\bar{\bar{\lambda}}_{\max}(\bar{\bar{\gamma}})}{\bar{\bar{\gamma}}}. \tag{5.8}$$

Hence, we further have the following conjecture.

Conjecture 5.5. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c > \hat{\hat{c}}$. Then system (2.3) has a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibria E_3 and E_4.*

Now, regarding the nonexistence of the traveling waves solutions of system (2.3), as it is easy to prove that when $\mathcal{R}_0 \leq 1$, system (2.3) only has an infection-free equilibrium E_0 which is globally asymptotically stable, it follows that there are no traveling waves solution of system (2.3), defined for $s \in \mathbb{R}$, and connecting equilibria E_0 and E_1, equilibria E_0 and E_2, equilibria E_0 and E_3 and equilibria E_0 and E_4, respectively. However, this is evident and the question is: what happens when $c < c^*$?

Remark 5.2. In [48,50], the authors established the existence of traveling wave solution for the critical wave speed $c = c^*$. Here, in the same vein and for model (2.3), we propose the following conjectures:

Conjecture 5.6. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 \leq 1$, $\mathcal{R}_2 \leq 1$ and $c = c^*$. Then system (2.3) has a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ defined for $s \in \mathbb{R}$, connecting equilibria E_0 and E_1.*

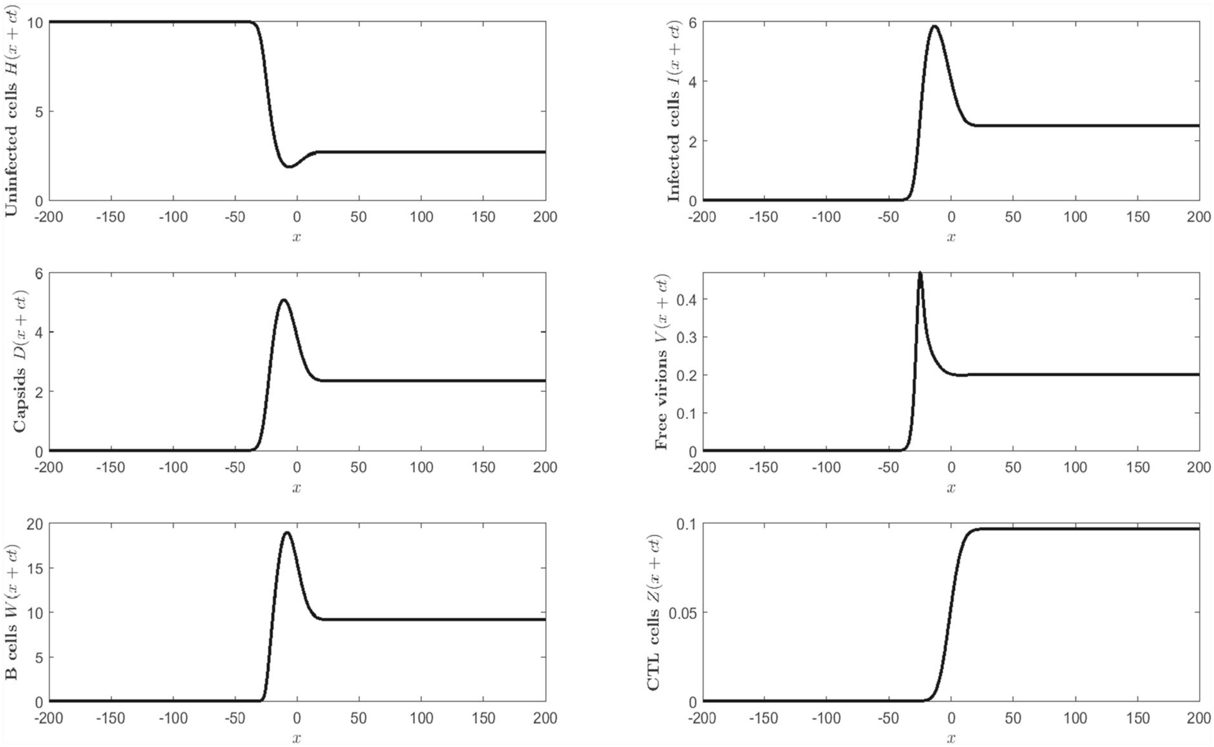

FIG. 6. Profiles of the traveling waves solution of (1.1) connecting E_0 and E_4 at the final time $T = 10$.

Conjecture 5.7. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_3 \leq 1$ and $c = c^*$. Then system* (2.3) *has a traveling waves solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibria E_0 and E_2.*

Conjecture 5.8. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_2 > 1$, $\mathcal{R}_4 \leq 1$ and $c = c^*$. Then system* (2.3) *has a traveling wave solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$, defined for $s \in \mathbb{R}$, connecting equilibria E_0 and E_3.*

Conjecture 5.9. *Assume that (Q2) holds, $\mathcal{R}_0 > 1$, $\mathcal{R}_1 > 1$, $\mathcal{R}_3 \leq 1$, $\mathcal{R}_3 > 1$, $\mathcal{R}_4 > 1$ and $c = c^*$. Then system* (2.3) *has a traveling wave solution $(H^*(s), I^*(s), D^*(s), V^*(s), W^*(s), Z^*(s))$ defined for $s \in \mathbb{R}$ connecting equilibria E_0 and E_4.*

6. Application and numerical simulations

In the preceding sections, we have proved the existence of traveling waves solutions for system (1.1) satisfying appropriate conditions (2.4)–(2.12). In this section, we provide some numerical simulations to illustrate these theoretical results obtained.

Case 1: connecting infection-free equilibrium E_0 and immune-free infection equilibrium E_1.

Case 2: connecting infection-free equilibrium E_0 and infection equilibrium with only antibody immune defense E_2.

Case 3: connecting infection-free equilibrium E_0 and infection equilibrium with only CTL immune response E_3.

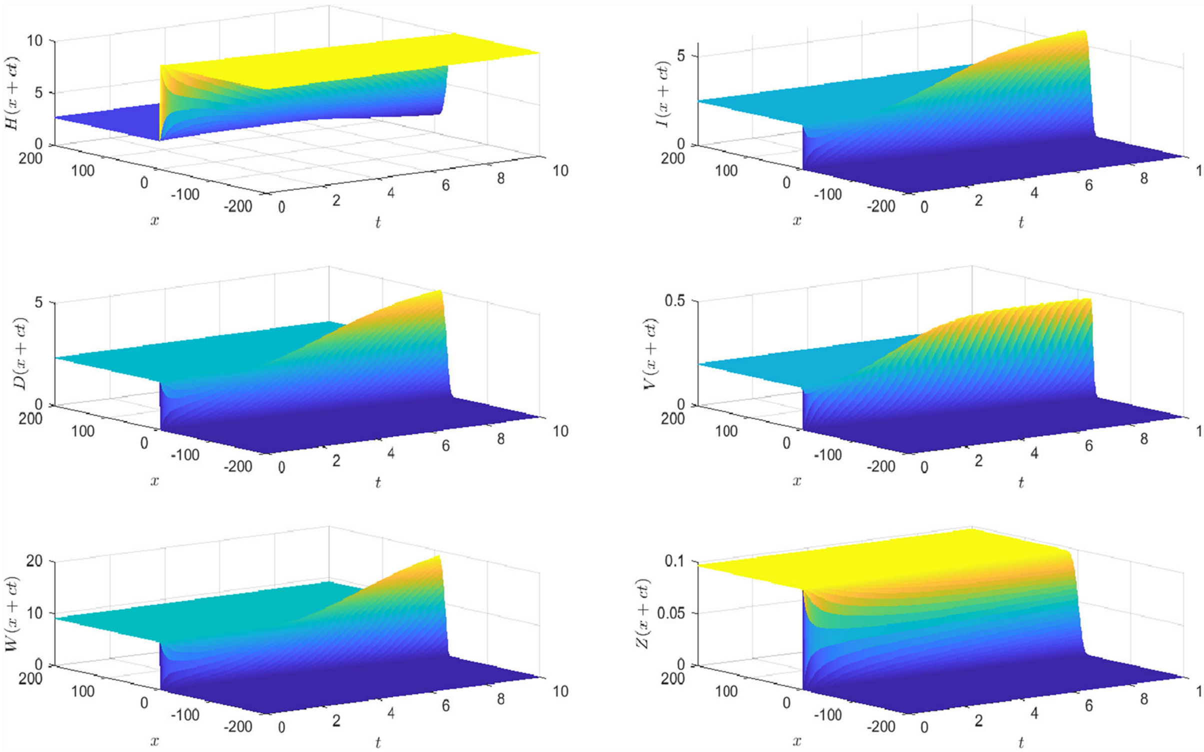

Fig. 7. Numerical simulation illustrating the existence of a traveling waves solution to system (1.1) connecting E_0 and E_4 with minimal speed c^*

Case 4: connecting infection-free equilibrium E_0 and CTL-antibody-present infection equilibrium E_4.

Case 5: connecting immune-free infection equilibrium E_1 and infection equilibrium with only antibody immune defense E_2.

Case 6: connecting immune-free infection equilibrium E_1 and infection equilibrium with only CTL immune response E_3.

Case 7: connecting immune-free infection equilibrium E_1 and CTL-antibody-present infection equilibrium E_4.

Case 8: connecting infection equilibrium with only antibody immune defense E_2 and CTL-antibody-present infection equilibrium E_4.

Case 9: connecting and infection equilibrium with only CTL immune response E_3 and CTL-antibody-present infection equilibrium E_4.

We consider the following particular masse action functions responses $f(H,V) = HV$ and $g(H,I) = HI$. Clearly, it can be seen that for the specific form of functional response chosen, assumption (Q2) is satisfied, with $\ell(H) = H$. We choose the diffusion coefficients $d_1 = d_2 = 0.01$ and $d_i = 0.75$, $i = 3, \cdots, 6$, for the nine cases. For simulations, we further intercept $[-200, 200]$ from spatial domain and $[0, 10]$ from time domain. Furthermore, we take the homogeneous Neumann boundary condition and the below piecewise functions as initial conditions for system (1.1):

$$G(x,t) = \begin{cases} G_i, & -200 \le x \le 0, \\ G_{i+1}, & 0 < x \le 200, \end{cases} \; t = 0, \; i = 0, \cdots, 3,$$

where G represents H, I, D, V, W and Z, respectively.

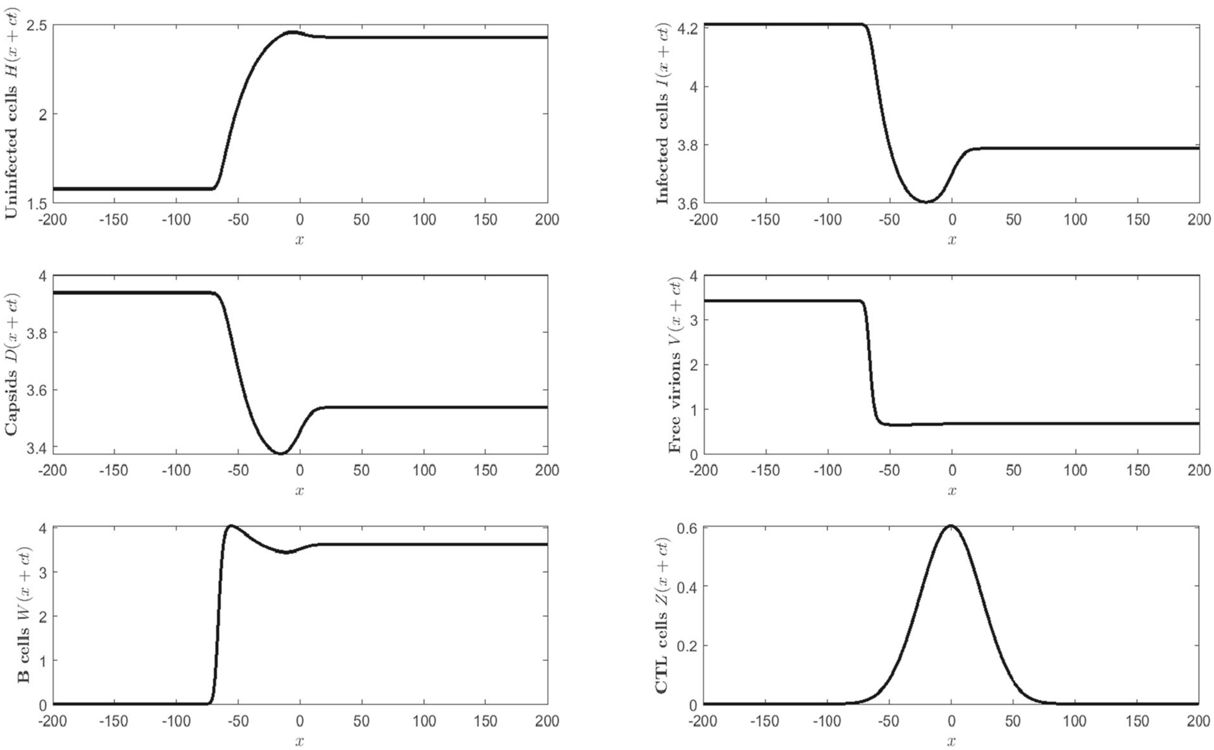

Fig. 8. Profiles of the traveling waves solution of (1.1) connecting E_1 and E_2 at the final time $T = 10$.

Case 1: Here, we take a set of parameters for system (1.1) as Case I in Table 1. As a result, we obtain the reproductive number for virus infection $\mathcal{R}_0 = 1.3016 > 1$, the antibody immune response reproduction number $\mathcal{R}_1 = 0.0589 < 1$, the CTL immune defense reproduction number $\mathcal{R}_2 = 0.1854 < 1$, the infection-free equilibrium $E_0 = (10, 0, 0, 0, 0, 0)$ and the immune-free infection equilibrium $E_1 = (7.6828, 0.4634, 0.1353, 0.0392, 0, 0)$. By Theorem 5.3, it follows that there exists a traveling waves solution for system (1.1) with speed c^* connecting the infection-free equilibrium E_0 and the immune-free infection equilibrium E_1. It can be observed in Fig. 2 and Fig. 3 that this is in fact the case. These figures clearly indicate that there exists a unique trajectory (or heteroclinic orbit) leaving E_1 and entering the lower domain containing E_0. This means biologically that in the absence of adaptive immune response, we can expect a considerable decrease in the population size of infected cells, virus DNA-containing capsids and viruses.

Case 2: Here, we take the parameters as in Case III in Table 1. The calculations show that $\mathcal{R}_0 = 6.3458 > 1$, $\mathcal{R}_1 = 5.1372 > 1$, $\mathcal{R}_3 = 0.7571 < 1$; the infection-free equilibrium is $E_0 = (10, 0, 0, 0, 0, 0)$, the immune-free infection equilibrium is $E_1 = (1.5758, 4.2121, 3.9365, 3.4248, 0, 0)$ and the infection equilibrium with only antibody immune defense is $E_2 = (2.4293, 3.7853, 3.5377, 0.6667, 3.6167, 0)$. Thus, by Theorem 5.3, it follows theoretically that there exists a traveling waves solution for system (1.1) with speed c^* connecting the infection-free equilibrium E_0 and the infection equilibrium with only antibody immune defense E_2. This is illustrated in Fig. 4.

Case 3: Here, we take again the parameter values as Case III in Table 1 except $\mu = 3$ and $a = 0.03$. Then by calculation, we have $\mathcal{R}_0 = 4.4486 > 1$, $\mathcal{R}_2 = 2.3256 > 1$, $\mathcal{R}_4 = 0.6776 < 1$; the infection-free equilibrium is $E_0 = (10, 0, 0, 0, 0, 0)$, the immune-free infection equilibrium is $E_1 = (2.2479, 3.8761, 3.6225, 1.0505, 0, 0)$ and the infection equilibrium with only CTL immune response is

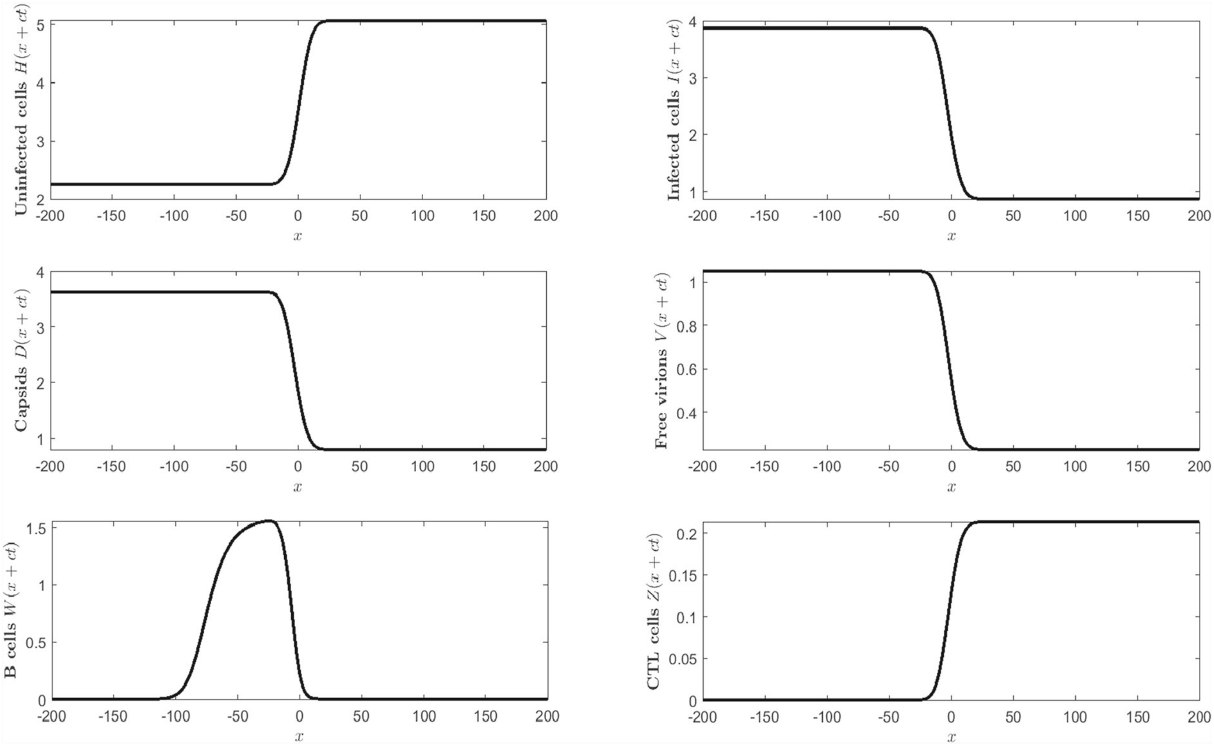

FIG. 9. Profiles of the traveling waves solution of (1.1) connecting E_1 and E_3 at the final time $T = 10$.

$E_3 = (2.5215, 1.6667, 1.5576, 0.4517, 0, 0.2618)$. Therefore, by Theorem 5.3, there exists a traveling waves solution for system (1.1), with speed c^*, connecting the infection-free equilibrium E_0 and the infection equilibrium with only CTL immune response E_3. This is observed in Fig. 5.

Case 4: Here again, we take the parameter values as in Case III in Table 1 except $\beta_1 = 0.1$, $\beta_2 = 0.1$ and $a = 0.02$. Then by calculation, one has $\mathcal{R}_0 = 9.0654 > 1$, $\mathcal{R}_1 = 18.0848 > 1$, $\mathcal{R}_2 = 1.7794 > 1$, $\mathcal{R}_3 = 1.6188 > 1$, $\mathcal{R}_4 = 10.1636 > 1$; The infection-free equilibrium is $E_0 = (10, 0, 0, 0, 0, 0)$, the immune-free infection equilibrium is $E_1 = (1.1031, 4.4485, 4.1574, 3.6170, 0, 0)$, the infection equilibrium with only antibody immune defense is $E_2 = (1.9058, 4.0471, 3.7823, 0.2, 15.4531, 0)$, the infection equilibrium with only CTL immune response is $E_3 = (1.4125, 2.5, 2.3364, 2.0327, 0, 0.1511)$ and the CTL-antibody-present infection equilibrium is $E_4(2.7027, 2.5, 2.3364, 0.2, 9.1636, 0.0967)$. Hence, by Theorem 5.3, there exists a traveling waves solution for system (1.1), with speed c^*, connecting the infection-free equilibrium E_0 and the CTL-antibody-present infection equilibrium E_4. The graphical illustration is displayed in Fig. 6 and Fig. 7.

Case 5: Here, we take the parameter values as in Case 2. Then the numerical results reveal that there exists a traveling wave solution for system (1.1) with speed $\hat{c}$ connecting the immune-free equilibrium E_1 and the infection equilibrium with only antibody immune response E_2. This is illustrated in Fig. 8.

Case 6: Here, one takes the parameter values as in Case 3. Then the numerical results confirm that there exists a traveling wave solution for system (1.1) with speed $\hat{c}$ connecting the immune-free equilibrium E_1 and the infection equilibrium with only CTL immune response E_3, as shown on Fig. 9.

Now, we consider the parameters as in Case 4. Then the numerical simulations confirm the existence of a traveling waves solution for system (1.1), with speed $\hat{c}$, connecting E_1 and E_4, as illustrated on Fig. 10. Figures 11 and 12 illustrate the existence of a traveling waves solution for system (1.1), with speed

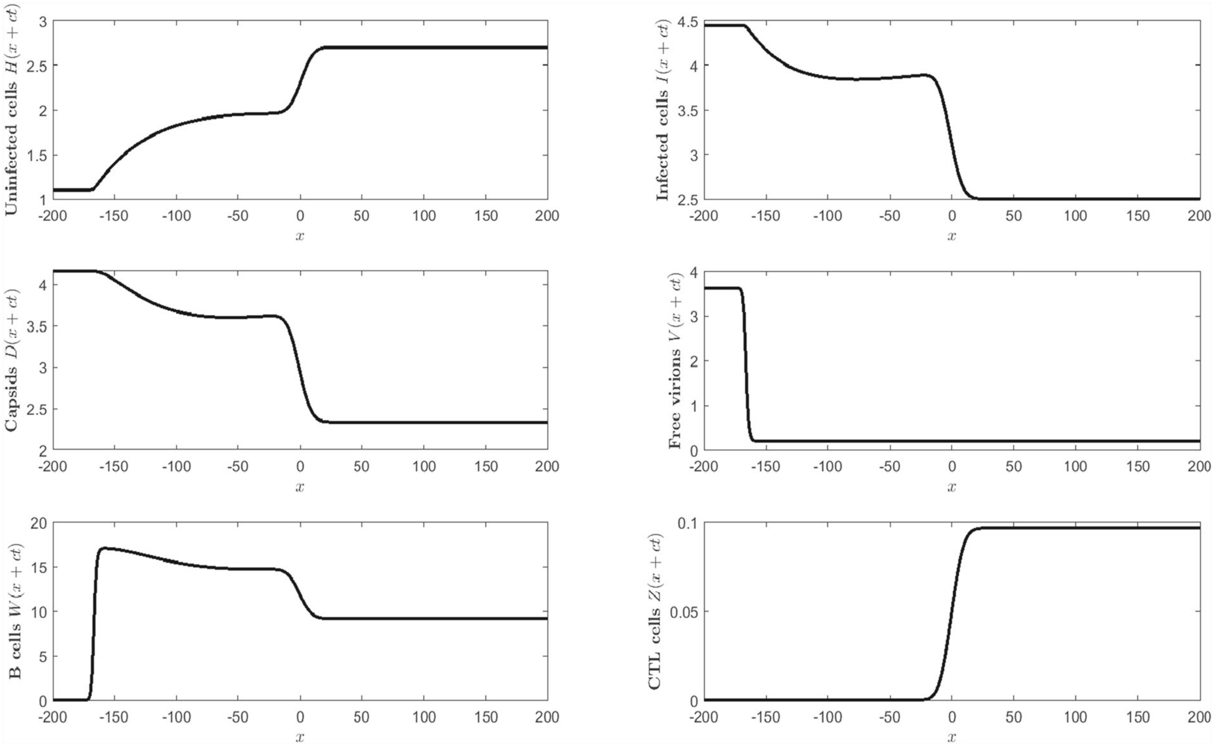

FIG. 10. Profiles of the traveling waves solution of (1.1) connecting E_1 and E_4 at the final time $T = 10$.

$\breve{c}$, connecting E_2 and E_4. The existence of a traveling waves solution for system (1.1), with speed $\hat{\breve{c}}$, connecting E_3 and E_4, is illustrated in Fig. 13 and Fig. 14.

Lemma 2.3 shows that the minimal wave speed c^* is the minimum value of the function $c = h(\gamma)$ given by (2.18) and achieved at the unique number γ^*. To illustrate this, on Fig. 15 (a), we plot the function $c = h(\gamma)$ where it is shown that the minimal wave speed $c^* = 0.2118$ is achieved at $\gamma^* = 0.99$. Also, Fig. 15 (b) is the graph of the function $\tilde{\lambda}_{\max}(\tilde{\gamma})/\tilde{\gamma}$ in (5.6) together with the minimal wave speed of $\hat{c} = 0.4459$ achieved at $\hat{\tilde{\gamma}} = 0.3$. Figure 15 (c) depicts the function $\bar{\lambda}_{\max}(\bar{\gamma})/\bar{\gamma}$ in (5.7) together with the minimal wave speed of $\breve{c} = 0.4629$ achieved at $\breve{\bar{\gamma}} = 0.31$. Figure 15 (d) depicts the function $\bar{\bar{\lambda}}_{\max}(\bar{\bar{\gamma}})/\bar{\bar{\gamma}}$ in (5.8) together with the minimal wave speed of $\hat{\breve{c}} = 5.2432$ achieved at $\hat{\breve{\bar{\bar{\gamma}}}} = 3.52$.

The profiles on Figs. 4, 5, 6, 8, 10 and 13 show that the traveling waves solutions of system (1.1) at the finite time T=10, connecting E_0 and E_2, E_0 and E_3, E_0 and E_4, E_1 and E_2, E_1 and E_4, E_3 and E_4, are not monotone.

7. Conclusion

Investigation on virus infections remains a hot topical issue [31,43,45]. In the present paper, based on [31,43], we developed a virus infection model with spatial diffusion, adaptive immunity, both virus-to-cell infection and cell-to-cell transmission. The virus-to-cell and cell-to-cell incidence rates are modeled by general nonlinear functions. The model obtained gives an insight into the intra-host models of infectious diseases such as HBV, HIV, HCV, HTLV-1, MLV, and so on.

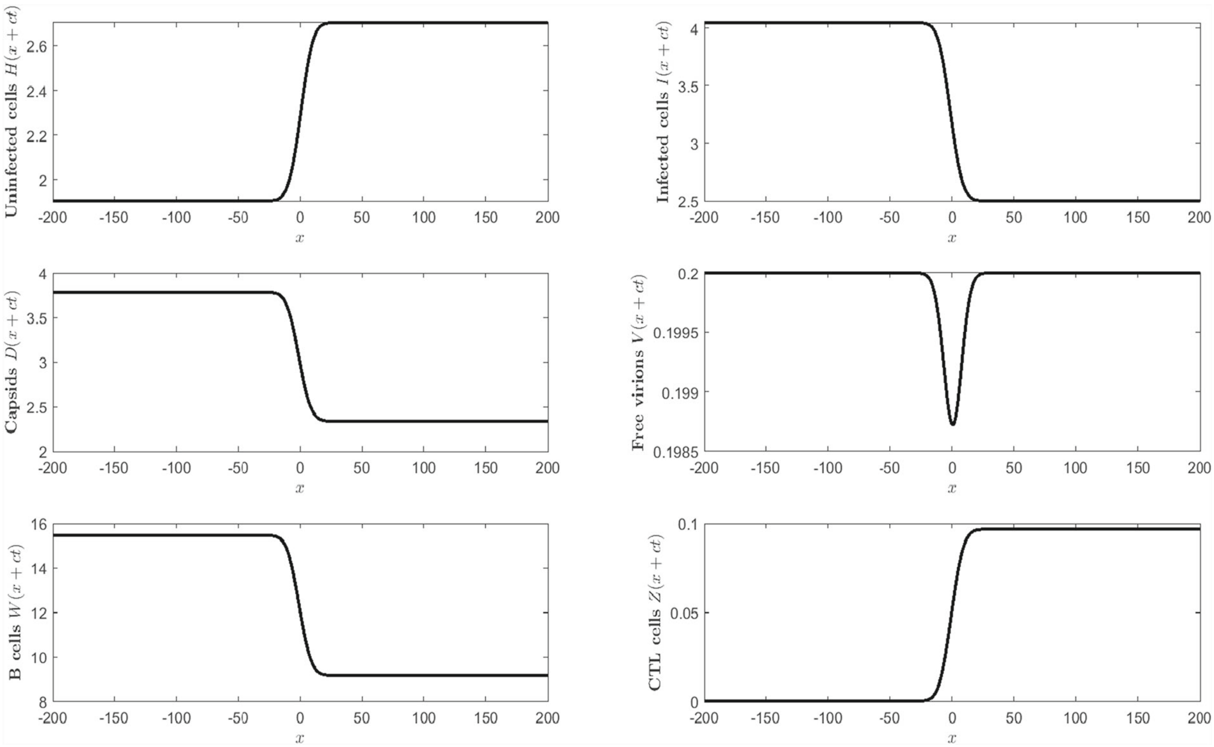

Fig. 11. Profiles of the traveling waves solution of (1.1) connecting E_2 and E_4 at the final time $T = 10$.

Our findings are as follows. We started the analysis of the model by the calculation of the basic reproduction numbers for virus infection $\mathcal{R}_0$, antibody immune response $\mathcal{R}_1$, CTL immune response $\mathcal{R}_2$, CTL immune competition $\mathcal{R}_3$, antibody immune competition $\mathcal{R}_4$, and the critical wave speed c^*. These thresholds made it possible to directly determine the existence of traveling waves solutions connecting the infection-free equilibrium E_0 and the immune-free infection equilibrium E_1, the infection equilibrium with only antibody immune defense E_2, the infection equilibrium with only CTL immune response E_3, and the CTL-antibody-present infection equilibrium E_4. This means that there exists a unique heteroclinic orbit leaving E_1, E_2, E_3 or E_4, and entering the lower domain containing E_0. We continued the analysis by establishing the upper and lower solutions with the aid of an auxiliary system. Moreover, by building the mapping L, given by (4.7), on an appropriate closed and convex set, and applying the Schauder's fixed point theorem combined with Lyapunov methods, we obtained sufficient conditions that guarantee the existence of nontrival and nonnegative traveling waves connecting the infection-free equilibrium to each of the other four equilibria (see Theorem 5.3 and Remark 5.1). Numerical simulations have been carried out to shed more light and illustrate these very relevant theoretical results.

Our planned future work includes:

- The study of the existence of traveling waves connecting the infected equilibria such as E_1 and E_2, E_1 and E_3, E_1 and E_4. We made conjectures about these challenging connections that are proved numerically.
- Extension of the study to the tough n-dimensional space variables setting.

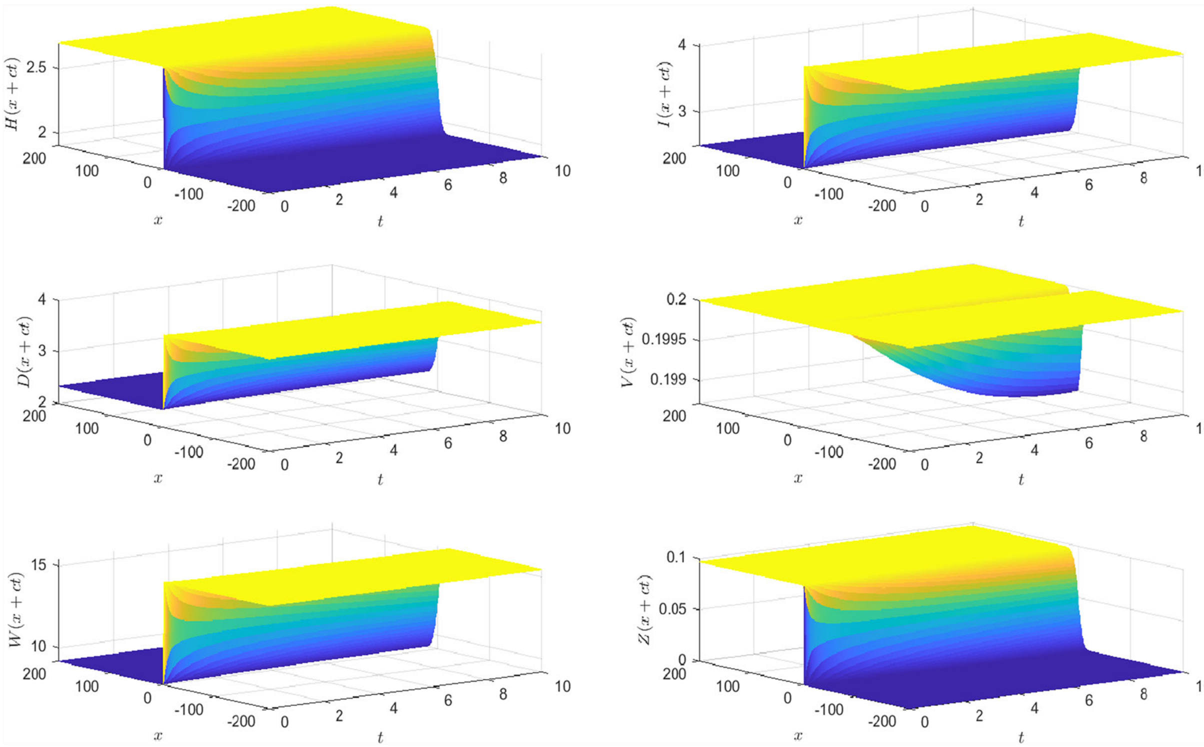

FIG. 12. Numerical simulation illustrating the existence of a traveling waves solution to system (1.1) connecting E_2 and E_4 with minimal speed $\check{c}$.

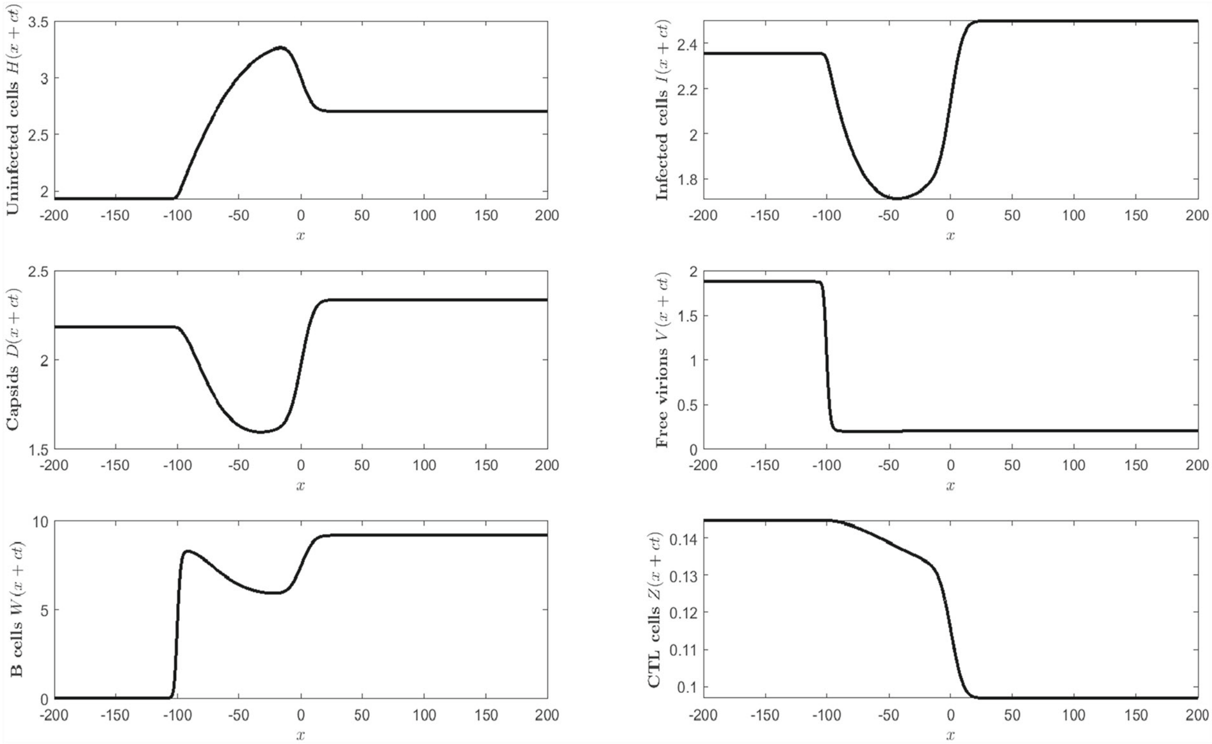

FIG. 13. Profiles of the traveling waves solution of (1.1) connecting E_3 and E_4 at the final time $T = 10$.

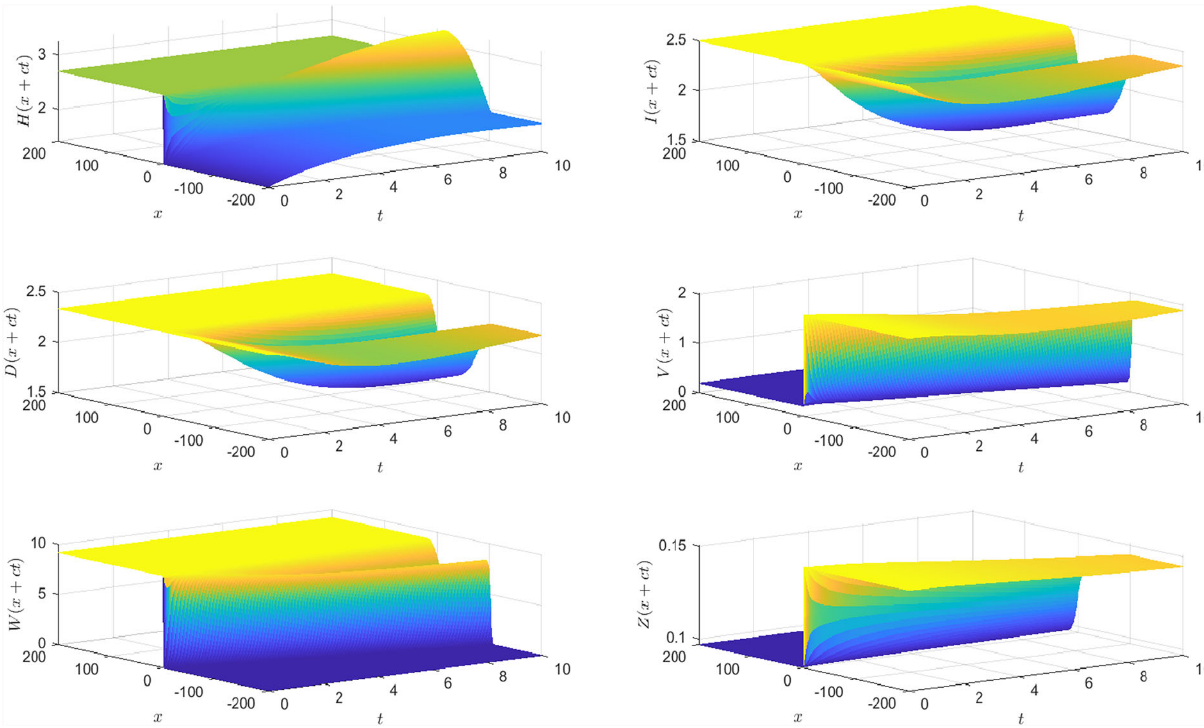

FIG. 14. Numerical simulation illustrating the existence of a traveling waves solution to system (1.1) connecting E_3 and E_4 with minimal speed $\hat{\hat{c}}$.

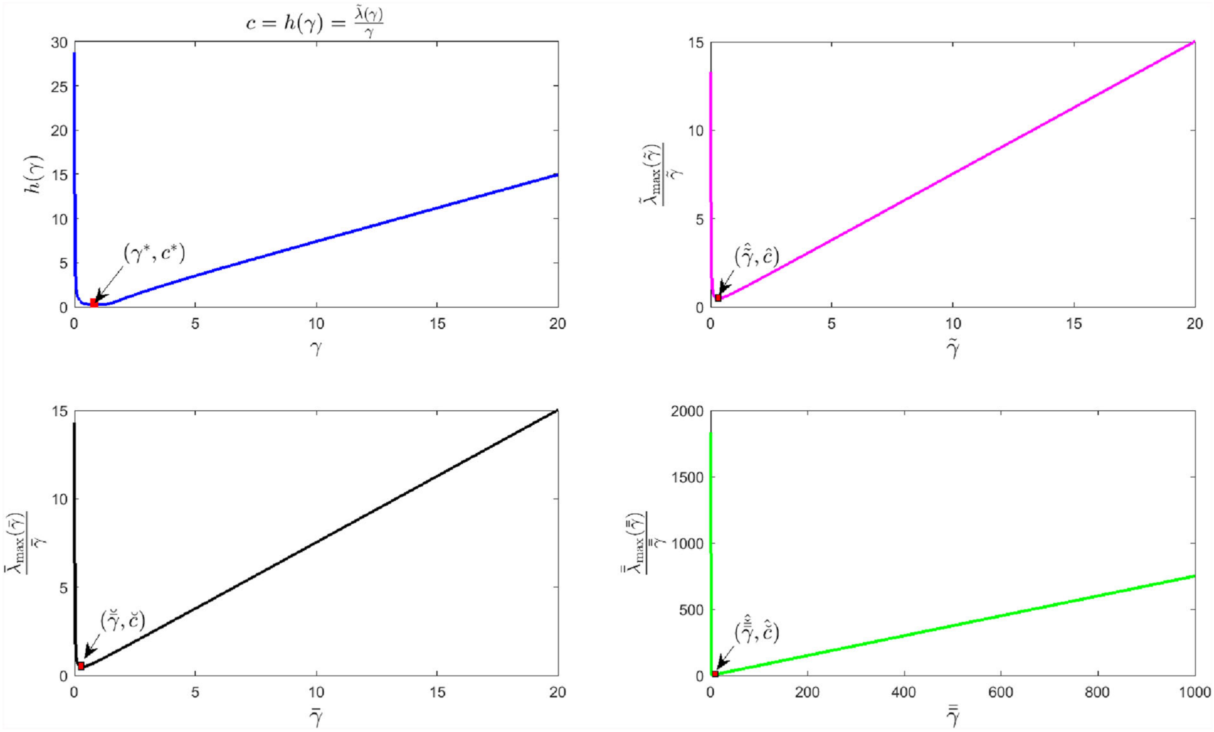

FIG. 15. In (a): graph of the function $h(\gamma)$ in (2.18) illustrating the existence and uniqueness of the minimal wave speed c^*. In (b): graph of the function $\tilde{\lambda}_{\max}(\tilde{\gamma})/\tilde{\gamma}$ illustrating the existence and uniqueness of the minimal wave speed $\hat{c}$. In (c): graph of the function $\bar{\lambda}_{\max}(\bar{\gamma})/\bar{\gamma}$ illustrating the existence and uniqueness of the minimal wave speed $\breve{c}$. In (d): graph of the function $\bar{\bar{\lambda}}_{\max}(\bar{\bar{\gamma}})/\bar{\bar{\gamma}}$ illustrating the existence and uniqueness of the minimal wave speed $\hat{\bar{c}}$.

Acknowledgements

The authors express their sincere thankfulness to the Editor and the referees for their diligent reviews and insightful suggestions which significantly improved the quality of the paper. J.M.-S. Lubuma and S. Foko acknowledge the support of the University of the Witwatersrand under the Science Faculty Start-up Funds for Research and the Postdoctoral Programme. C. Tadmon acknowledges support from the Institute of Mathematics, University of Mainz, Germany, where the paper was revised and finalized during a research stay sponsored by the Alexander von Humboldt Foundation.

Author contributions All authors contributed equally to the following aspects: conceptualization, methodology, software development, validation, formal analysis, writing—original draft, writing—review & editing, and visualization.

Funding Open access funding provided by University of the Witwatersrand. J.M.-S. Lubuma and S. Foko acknowledge the support of the University of the Witwatersrand under the Science Faculty Start-up Funds for Research and the Postdoctoral Programme. C. Tadmon acknowledges support from the Institute of Mathematics, University of Mainz, Germany, where the paper was revised and finalized during a research stay sponsored by the Alexander von Humboldt Foundation.

Data Availability Statement The codes used to perform the simulations presented in this work are available upon request from the authors.

Declarations

Conflict of interest The authors declare that they have no known personal relationships or competing financial interests that could have appeared to influence the work made in this paper.

Ethical approval This manuscript adheres to the ethical standards set by this Journal. We confirm that: The manuscript is original, has not been published elsewhere, and is not under consideration by another journal. All research results are presented clearly and honestly without fabrication, falsification, or inappropriate data manipulation. Proper acknowledgments are given for all referenced works. Relevant code is available to verify the validity of the results if requested. We are committed to maintain the integrity of the scientific record and will address any significant errors discovered in our published article promptly.

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(Received: March 2, 2025; revised: May 16, 2025; accepted: May 21, 2025)