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Gravitational Description of the Conformally Invariant Quantum Mechanics of Large Matrices

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signed:

Date: July 6, 2017

Abstract

We study the collective field theory of a free multi-matrix model in the radial sector, which has an emergent $1/r^2$ term, and take the large N limit. We show that it is possible to generate $2-d$ metrics with generic dependence on the collective field Lagrange multiplier (μ) and potential and which are distinguished by the choice of the potential. The Lagrange multiplier is shown to depend on an induced scale parameter after an I.R. regularization and breaks scale invariance. The collective field $sl(2,\mathbb{R})$ algebras of the free Hamiltonian and a related alternative compact operator only close in the absence of μ. We point out that the broken conformal symmetry is contained in the associated metrics which suggests that they are related to a Near-AdS_2 geometry. We also comment on the resemblance of these metrics to black hole solutions.

To my wife
and parents

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1 Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence has proven to be an imperative framework for understanding both gravitational theories and quantum field theories and their equivalence. The correspondence has been realized for a variety of spacetime dimensions, but investigations into the AdS_2/CFT_1 correspondence have taught us that the lower dimensional case is not always the most straightforward in physics. In the gauge theory context, we have matrix models at our disposal, which provide a convenient toy model to investigate string theory. An obvious candidate for a toy CFT_1 would be the theory of a single Hermitain matrix valued field in $d = 1$ dimensions with a conformally invariant potential. However, many extremal black holes in string theory have a near-horizon limit in which $AdS_2 \times X$ arises for some compact space X. This leads to an emergence of AdS_2 black holes in a dimensional reduction with an inherited horizon from the higher dimensional black hole. The CFT dual to such string theories is $\mathcal{N} = 4$ Super Yang Mills theory which has a bosonic sector consisting of 3 complex matrix valued fields. It is therefore natural to enquire whether a corresponding one dimensional conformal field theory emerges through a dimensional reduction from a higher dimensional dual multi-complex matrix model. In contrast to this is the recent idea, in string theory, that spacetime is not an intrinsic feature of fundamental theories of physics but rather an emergent feature. While the same cannot be said about time being an emergent phenomena of some theory, it is immediately plausible, especially in light of the AdS_{d+1}/CFT_d duality, that a spatial dimension may emerge from a quantum mechanical theory of time with a conformal symmetry. Perhaps from this point of view, it would be possible to start with a one dimensional CFT and obtain a two dimensional theory of gravity with an AdS_2 spacetime. It is the purpose of this work to investigate these ideas.

The outline for the dissertation is as follows: section 2 will consist of a review of the AdS/CFT correspondence with particular emphasis on the relevance of multi-matrix models to the study of the correspondence. We provide a brief reminder of both the relationship between the conformal algebra and the Lorentz algebra in $d \geq 3$ dimensions and the causal structure of AdS_{d+1}, for arbitrary d, with particular emphasis on the differences between the global and Poincare coordinate systems [1]. However, the main purpose of section 2 is to illustrate a matching of the $sl(2,\mathbb{R})$ symmetry algebra of conformal quantum mechanics, a well known $d = 1$ quantum mechanical theory with the $SL(2,\mathbb{R})$ conformal symmetry [2], and the isometry algebra of AdS_2. We derive the generators of both algebras and show that they are locally isomorphic to each other and the $so(1,2)$ Lorentz algebra [3]. This matching of symmetries is regarded as evidence for the existence of a possible realization of the AdS_2/CFT_1 correspondence and serves as motivation for the pursuit of such a realization. In section 3 we review various important results pertaining to the existence of AdS_2 geometry in string theory and quantum gravity in the literature [4–9], as well as their consequences for the one dimensional conformal theory that is conjectured to be dual to AdS_2.[1]

In section 4 we study the free matrix valued quantum mechanics and illustrate its equivalent description as a system of non-interacting fermions [10] and its reformulation in terms of the collective field theory [11]. It is here that we illustrate the first two of our three results. Firstly, there is an induced scale parameter that appears in the free Hermitian matrix model of a single matrix and its singlet sector fermionic description and collective field reformulation. The scale parameter arises from the need to regulate the free Hermitian matrix model with a mass term in order to retain well defined observables. Secondly, we study the collective field formulation of the single Hermitian matrix model with a conformally invariant potential and show that the Lagrange multiplier term, required for the eigenvalue density constraint in the collective field theory, depends on this scale parameter and therefore necessarily breaks the conformal invariance. In section 5 we consider the free multi-complex matrix model together with its radial fermionization and collective field theory descriptions [12]. The fermionic theory, which describes a system of fermions in $d+1$ dimensions, and the collective field theory see the emergence of the $1/r^2$ term- that is associated with

[1] We have chosen not to list these results here as they are numerous and we do not wish to break the flow of the introduction.

conformal quantum mechanics- for the free multi-matrix theory with d Hermitian matrices provided that $d \geq 4$. This $1/r^2$ term, however, survives a dimensional reduction in the fermionic description to $d = 1$ dimensions, recovering conformal quantum mechanics [12]. We then study the would-be $sl(2,\mathbb{R})$ algebra for the collective field theory. It turns out that the algebra only closes in the absence of the Lagrange multiplier. We introduce a new set of generators including a new 'Hamiltonian', that has been related to a different choice of AdS_2 time in the pure AdS_2 case [5], which also only closes the $sl(2,\mathbb{R})$ algebra in the absence of the Lagrange multiplier, confirming the result of the previous section- that the Lagrange multiplier breaks conformal invariance. However, this result is extended in this context. While from the perspective of the algebra this is still in the free theory- that is both sets of generators correspond to the free theory- from the perspective of the collective field theory the new 'Hamiltonian' corresponds to a theory with a potential. We then present a third result: we show that in accordance with the emergence of a spatial coordinate, which we take to be a radial coordinate given the radial sector restriction, there is a nontrivial metric associated with the large N background of the collective field which is identified in the quadratic Lagrangian that arises from studying the quantum fluctuations about the large N background. The form of this metric is general and so is the form of the metric's dependence on the large N background. However, the large N background depends on both the Lagrange Multiplier and the potential in the collective field theory. This leads to the conclusion that, in accordance with the distinction between the new and old Hamiltonians mentioned above, the two operators generate distinct spacetime metrics in the collective field theory. The broken conformal symmetry suggests that the emergent geometry may be Near-AdS_2. We also comment on the possible interpretation of these metrics as black holes. Section 6 provides a summary and conclusions. Our original interest in the AdS_2/CFT_1 correspondence was restricted to the collective field theory of a single Hermitian matrix with a conformally invariant potential and its possible relation to AdS_2 black holes. Our interest has since shifted to the multi-matrix case, however, our results are still relevant to the single matrix model. We have therefore resolved to include an overview of the early gauge/gravity duality of Matrix quantum mechanics and 2 dimensional string theory and the possibility of black hole solutions in an appendix [13]. There has also been recent work in the literature related to near-AdS_2 geometries and a breaking of the full conformal group of diffeomorphisms associated with the asymptotic symmetries of AdS_2 as well as the connection between chaos and black holes which can be studied in this context. An overview of some of this work also appears in an appendix.

2 The Anti-de Sitter/Conformal Field theory Correspondence

2.1 A brief overview of the AdS/CFT correspondence

The Maldacena Conjecture[2] is an expectation that $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory on $\mathbb{M}^{1,3}$ is equivalent to Type IIB Superstring theory on $AdS_5 \times S^5$ [14]. This expectation is based on the apparent duality between $\mathcal{N} = 4$ SYM on $\mathbb{M}^{1,3}$ and free Supergravity on $\mathbb{M}^{1,9}$ in the low energy limit of IIB superstring theory. The best understood case is the so-called 'AdS_5/CFT_4 *correspondence*' which can be understood by considering two alternative perspectives of the the same physics describing open and closed strings existing on and in the presence of a coincident stack of N D_3- branes, with the alternative perspectives being distinguished by the strength of the string coupling. The duality has not been entirely tractable in any of its realizations- always characterized by the equivalence between a conformal field theory on a flat spacetime of a given dimension: d and the gravitational theory on the bulk AdS spacetime of dimension: $d+1$ of which the flat d dimensional spacetime is its boundary. The most striking feature of the correspondence is that it relates two frameworks: quantum field theory and string theory, that provide calculational tools to study two physical phenomena: relativistic quantum theory and gravitation, which have remained in contrast since their conception. This may at first thought be expected given that string theory provides a quantum mechanical description of gravity (and of course contains classical theories of general relativity and supergravity) and contains the ingredients of ordinary quantum field theory, albeit through the smearing of the point particle degree of freedom associated with field excitations to a 1- dimensional string and the view of the embedding coordinates as fields on the worldsheet. In fact, it has been well known since the seventies that quantum field theory and string theory should be related through 't Hooft's topological expansion of non-abelian gauge theories, which in the planar limit (Large N) corresponds to classical/tree level string theory [15]. Very recently it has been argued that, in addition to 't Hooft's findings, the large N limit of QCD actually contains the celebrated Veneziano amplitude of tree level string scattering [16]. However, the duality introduces an unexpected feature: it is a strong-to-weak coupling duality which allows us to use our strength at perturbative calculations in one theory to study non-perturbative physics in the corresponding theory. Not only does the correspondence provide a model for studying quantum effects in gravity but it allows us to study strongly coupled physics- something that has eluded theoretical physics for decades. It should also be noted that the duality is a realization of the Holographic principle which may be expected to be a foundational statement for some, at this time, unspecified theory.

Consider Type IIB string theory in $\mathbb{M}^{1,9}$ with a stack of N D_3-branes embedded in the flat space with Neumann boundary conditions in x^μ ($\mu = 0, 1, 2, 3$) and Dirichlet boundary conditions in x^i ($i = 4, .., 9$). This is a theory of closed strings in $\mathbb{M}^{1,9}$ and open strings in $\mathbb{M}^{1,3}$. The effective action for the theory is composed of the open string action, the closed string action and the action for the closed-open string interactions. In the open string perspective one considers $g_s N \ll 1$ and takes the low energy limit: $\alpha' \to 0$. This can be viewed as the limit that the string degrees of freedom become point-like since $\alpha' = l_s^2$. In this case the closed string action is given by 10-d supergravity with the massless string excitations given by a multiplet of $\mathcal{N} = 1$ supergravity. The open string massless excitations are given by a $\mathcal{N} = 4$ multiplet for which there are 6 scalars and their fermionic counterparts as well as a gauge vector. The gauge field is in the D_3-brane while the scalars describe transverse oscillations in the Dirichlet directions. The open string and interaction contributions to the action are derived from the Dirac-Born-Infeld action of the D_3-branes. In the low energy limit, the interaction vanishes and the open and closed strings decouple while the open string action reduces to the bosonic part of $\mathcal{N} = 4$ SYM theory if the couplings g_{YM}^2 and $2\pi g_s$ are identified. Therefore, in this limit, in which the closed and open strings decouple, the theory splits into two parts:

[2]The conjecture is most commonly referred to as the 'AdS/CFT correspondence' however, out of respect for its originator [14], we sometimes refer to it as the 'Maldacena conjecture'. The terms 'AdS/CFT correspondence' and 'gauge/gravity correspondence' are used interchangeably in the literature but we shall consider the latter to be more general and not necessarily referring to either AdS or CFT. We have included a review of the relationship between gauge theories and string theory, in a pre-AdS/CFT gauge/gravity duality, in appendix G- however, the reader need not refer to that appendix at this stage.

the description of the open strings in terms of a non-abelian gauge theory with superconformal symmetry on the flat worldvolume of a stack of N D_3-branes and the closed strings described by a free supergravity theory in $(9+1)$ flat spacetime dimensions.

From the closed string perspective one considers the strong coupling limit $g_sN \to \infty$. Since the D_3-branes are massive and charged they curve spacetime and couple to the four form fields of the string theory. The IIB supergravity solution for the N D_3-branes corresponds to a metric $ds^2 = H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}\delta_{ij}dx^i dx^j$. The 'warp factor' $H(r) = 1 + L^4/r^4$ with $L^4 = 4\pi g_s N\alpha'^2$ which is related to the charge of the N D_3-branes through the five form flux of the field strength of the four form fields of IIB supergravity on the branes. The warp factor is responsible for splitting the background into 2 regions: the so-called 'near-horizon limit/throat' region for which $r \ll L$ and the metric becomes $AdS_5 \times S^5$ and the asymptotic region for which $r \gg L$ which is flat. In either case, L is taken to be much larger than 1 for weakly curved geometry. In the low energy limit ($\alpha' \to 0$) the description splits into two parts. Again, the warp factor plays an important role in an asymptotic ($r \to \infty$) observer's notion of energetic excitations in the two regions. A highly energetic excitation in the throat region is understood by an asymptotic observer to approach zero owing to the warp factor. Therefore, in the low energy limit the two closed string theories are decoupled and there are closed string excitations in asymptotic flat $\mathbb{M}^{1,9}$ supergravity and closed strings described by IIB supergravity in an $AdS_5 \times S^5$ throat region.

These are two alternative perspectives of the same physics. The presence of type IIB supergravity on flat 10-d spacetime in both perspectives suggests that $\mathcal{N}=4$ SYM of $\mathbb{M}^{1,3}$ and type IIB supergravity on $AdS_5 \times S^5$ should be equivalent- which is the basis for the AdS/CFT correspondence. The precise statement of the correspondence relies heavily on the connection between the the free parameters of the two theories: $g^2_{YM} = 2\pi g_s$ (identified in the open string perspective) and $2\lambda_{\text{'t Hooft}} = L^4/\alpha'^2$ (from the closed string perspective).[3] The various limits achieved through the tuning of these parameters allows one to categorize the 'strength' of the statement of the correspondence as described above. In particular the conjecture is based on the low energy limit, however by tuning the couplings appropriately, the conjecture can be made general and the statement of the duality becomes:

$\mathcal{N}=4$ SYM on $\mathbb{M}^{1,3}$ for arbitrary N and λ is equivalent to type IIB string theory on $AdS_5 \times S^5$ with arbitrary g_s and l_s [17], [18].[4,5]

The conjectured correspondence is then roughly cast as:

$$\begin{array}{rcl} \textbf{Q. Gravity} & & \textbf{QFT} \\ \text{Type IIB string theory on } AdS_5 \times S^5 & & \mathcal{N}=4 \text{ Super Yang-Mills in } \mathbb{M}^{1,3} \\ \text{Energy} & \longleftrightarrow & \text{Scaling dimension} \\ \text{Angular Momentum} & & \mathcal{R}-\text{symmetry.} \end{array} \tag{1}$$

[3] The 't Hooft coupling is given by $\lambda_{\text{'t Hooft}} = g^2_{YM}N$.

[4] Obviously one should take $\sqrt{\alpha'} = l_s \neq 0$.

[5] This general statement is based on the identification of free parameters in the two theories. However, it should be noted that for most practical purposes one is certainly interested in using AdS/CFT to learn about strongly coupled field theories. This requires the strong/weak duality. The natural limit from the string theory side is $g_s \to 0$ and $L^2/\alpha' = fixed$ since strings are most understood at weak coupling (i.e. classical string theory); this requires that $N \to \infty$. λ and l_s remain arbitrary in this case. To realize strong coupling for the field theory the effective coupling has to have $\lambda \to \infty$, which is consistent with $l_s/L \to 0$ - the point particle limit in the string theory. The string theory, for the case of strongly coupled gauge theories, is reduced to supergravity. To recognize the correspondence between the point particle limit and supergravity (i.e. the low energy limit of string theory) one recalls that the string tension is $T = 1/2\pi l_s^2$ and the fact that the string spectrum is described by energy relations of the form $M^2 \propto 1/\alpha'$. The general statement of the correspondence is then expected to hold for arbitrary N and λ- when one is not necessarily taking a weakly coupled string limit and/or strongly coupled field theory limit.

This rough illustration can be made more precise with a few clarifying statements. The correspondence relates specific states in the string with specific operators in the quantum field theory. The relation between these states and operators is such that the states in the string theory and the operators in the quantum field theory that have the same quantum numbers are in correspondence with each other. However, before this identification of the corresponding states and operators is possible, one must consider the symmetries of the two theories and relate them by identifying the appropriate generators of these symmetries on each side of the correspondence. In fact, there are many technical details involved in string theory and $\mathcal{N}=4$ SYM that would need to be understood in order to make these statements more definitive. It turns out that in order to develop an expectation that matrix theories be related to string theories we need not consider all the technical details of type IIB and $\mathcal{N}=4$ SYM. Many of the salient features of $\mathcal{N}=4$ SYM and matrix theories in general can be discovered and studied through a toy model known as a matrix model. Only some slight modifications to the results obtained by studying the matrix model are required to retrieve the details of the physical matrix theory. In addition to the matrix model, we can limit our consideration of string theory to the basic idea that a string is an object that exists in some background spacetime and that its motion in spacetime traces out a 2-d manifold in that background. With this simpler approach, one can study the quantum mechanics of a single Hermitian matrix and arrive at the 't Hooft limit. However, since $\mathcal{N}=4$ SYM contains a bosonic part that consists of complex matrix fields, a second step could involve taking a look at a complex matrix model.

Motivated by the original formulation of AdS/CFT, one can elaborate on the matching of the two theories. The symmetries of the two theories are as follows: for the $AdS_5 \times S^5$ of the IIB string theory, AdS_5 has an isometry group of $SO(4,2)$ which is conveniently matched by the conformal group $SO(4,2)$ of $\mathcal{N}=4$ SYM. The S^5 of $AdS_5 \times S^5$ has an $SO(6)$ symmetry and the conserved charge associated with such a symmetry is angular momentum on S^5. On the field theory side, $\mathcal{N}=4$ SYM has six scalar fields ϕ_i $(i=1,...,6)$ that appear in complex matrices e.g. $Z=\phi_1+\phi_2$. It turns out that the corresponding conserved charge on the field theory side is $\mathcal{R}-$ charge. To understand this, we note that energy on the gravity side corresponds to the scaling dimension on the field theory side. For $\mathcal{N}$=4 SYM, the 1/2 BPS part of the action is given by the single complex matrix (Z) kinetic piece $S=\int d^4x tr(\partial_\mu Z \partial^\mu Z^\dagger)+....$[6] It is possible to define suitably normalized trace operators $\mathcal{O}_J(x) \equiv \frac{tr(Z(x))^J}{\sqrt{JN^J}}$ that have scaling dimension $\Delta_{\mathcal{O}_J}=J$ since from the action it is clear that $[Z]=L^{-1} \Rightarrow \Delta_Z=1$. Here, J represents the angular momentum of the state (which will become clear shortly). Correlation functions of such trace operators, referred to as multi-trace operators in the literature, for the one dimensional case, can be inferred from the zero dimensional case by the requirement of conformal invariance- which is consistent with $\mathcal{N}=4$ SYM. The explicit form of, for example, the two point function is found to be $\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{C\delta_{\Delta_1,\Delta_2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$. $\mathcal{R}-$ symmetry transformations rotate the complex matrices Z. As an example one could consider a rotation in the 1-2 plane by θ which would send $\phi_1 \to \phi_1\cos\theta - \phi_2\sin\theta$, $\phi_2 \to \phi_1\sin\theta+\phi_2\cos\theta$ and $\phi_j \to \phi_j$ for $j=3,..,6$. This transformation corresponds to $Z \to e^{i\theta}Z$. A generator of $\mathcal{R}-$ symmetry in the Lie algebra, call it $\hat{\mathcal{R}}$, would generate these transformations in some representation as $e^{i\theta\hat{\mathcal{R}}}Z = e^{i\theta r}Z$ where r would be the eigenvalue of the operator $\hat{\mathcal{R}}$. We can identify this value in $Z \to e^{i\theta}Z$ as the scaling value of Z $(\Delta_Z=1)$. In other words, the $\mathcal{R}-$ charge for a operator $\mathcal{O}_J$ is $\mathcal{R}_{\mathcal{O}_J}=J=$ momentum. So on the gravity side the $SO(6)$ symmetry leads to conservation of angular momentum and on the field theory side the $SO(6)$ symmetry leads to conservation of $\mathcal{R}-$ charge. The mass of such a state in the gravity theory would be $m^2=(Energy)^2-(momentum)^2=J^2-J^2=0$. So this would represent a graviton with angular momentum J on the S^5. In the Maldacena conjecture, states in the gravity theory that are labeled by a given quantum number are in correspondence with operators of the same quantum number in the field theory. Therefore, to calculate the overlap of two states in the gravity theory, we could calculate the correlation function of the corresponding two operators in the field theory. From the t'Hooft limit (with

[6]The $\frac{1}{2}$ BPS part of the action refers to the part for which the trace of operators are invariant under half of the super symmetries of $\mathcal{N}=4$ SYM.

$N \Rightarrow \infty$), it is clear that matrix theories are related to string theories. However, if we consider a graviton of energy and momentum: J then we could represent it by a Fock state $|J\rangle$.[7] For a free theory, we would expect that the overlap of a two particle state with a one particle state would be zero $\langle J_1 J_2 | J_1 + J_2 \rangle = 0$. While this agrees with $\langle \mathcal{O}_J(x_1)\mathcal{O}^\dagger_k(x_2)\rangle = \frac{\delta_{JK}}{|x_1-x_2|^{2J}}$ in the $N \to \infty$ limit, for $J \sim N^h$ with $h \geq 1/2$ this would no longer be the case. We would find a non zero overlap of a 1 and 2 particle state, more characteristic of an interacting theory. In fact, with the definition of the trace operators mentioned above, the correlator $\langle \mathcal{O}_J(\mathcal{O}^\dagger)_J\rangle$, including genus zero and one terms[8], is $\langle \mathcal{O}_J(\mathcal{O}^\dagger)_J\rangle = JN^J + A_1 J^4 N^{J-2} + A_2 J^5 N^{J-2}$ in the zero dimensional case and A_1, A_2 are numerical coefficients. This indicates that if $J \sim N^{1/2}$ or of a higher power then the planar limit is no longer applicable. And since we consider the large N limit, increasing N increases J which is the energy in the gravity theory. Increasing energy means that the gravitational interaction grows with N and one would expect to find that the nature of the objects you are studying in the gravity theory changes from gravitons or strings to more complicated geometric objects. This is illustrated in (2) [19].

order of J in Matrix theory	Gravitational object in string theory	
$\mathcal{O}(1)$	graviton	
$\mathcal{O}(\sqrt{N})$	string	
$\mathcal{O}(N)$	giant graviton (membranes)	
$\mathcal{O}(N^2)$	new spacetimes.	(2)

The first two rows of (2) can be studied with a single trace operator and somewhere in the string theory the planar limit breaks down. What the AdS/CFT correspondence teaches us is that matrix models are related to strings through the 't Hooft (Planar) limit but that at some point in studying strings the planar limit breaks down (when the order of J exceeds powers of N to the half). Investigating matrix theories beyond the planar limit relates matrix theories to gravitational objects of more complicated geometry.[9]

2.2 Expectations for AdS_2 and Conformal Quantum Mechanics

A crucial ingredient of the AdS/CFT correspondence involves the matching of the symmetries of the dual theories. From the CFT side, we provide a short recap of the well known fact that for $d \geq 3$ the conformal group in Minkowski space ($\mathbb{M}^{d-1,1}$), which has Lorentz group $SO(d-1,1)$, is $SO(d,2)$. We then introduce conformal quantum mechanics and derive its symmetry generators and show that they correspond to the group $SL(2,\mathbb{R})$ [2]. However, the algebra can be mapped to the $so(1,2)$ algebra, confirming that the conformal group, $SO(d,2)$, holds for the case of $d=1$. Within the conformal quantum mechanics framework, we also consider an alternative operator to the Hamiltonian which seems more natural to consider the eigenstates of [2]. We then review the conformal structure of anti-de Sitter space, with particular emphasis on the region of the spacetime corresponding to Poincare coordinates, before focusing on the two dimensional case, AdS_2. We derive the AdS_2 isometry group and then proceed to show the matching of the AdS_2 symmetries with those of conformal quantum mechanics [3]. This will make clear our expectation for the existence of an AdS_2/CFT_1 correspondence, at the level of the symmetries, and the illustration of the precise matching of the symmetries will be important for most of the sections to follow.

[7] A two particle state would be of the form $|J_1 J_2\rangle$.

[8] The first term is the genus 0 contribution and the second two terms are genus 1 contributions. The reason for the two distinct powers of J in these two second terms is that the two terms arise from distinct wick contractions.

[9] We thank Robert de Mello Koch for pointing out much of the details relating complex matrix models and string theory that have been discussed in section 2.1 [19].

2.2.1 Isomorphism between the Conformal group in $\mathbb{M}^{d-1,1}(SO(d-1,1))$ and $SO(d,2)$

In $d+1$ spacetime dimensions the conformal algebra is composed of the following generators:

$$\begin{aligned} P_\mu &= -i\partial_\mu, & L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \\ D &= -i(x\cdot\partial), & K_\mu &= -i(2x_\mu(x\cdot\partial) - x^2\partial_\mu). \end{aligned} \tag{3}$$

These generators have the following products

$$\begin{aligned} [D,P_\mu] &= iP_\mu, & [D,K_\mu] &= -iK_\mu \\ [K_\mu,P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}), & [K_\rho,L_{\mu\nu}] &= i(\eta_{\rho,\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\ [P_\rho,L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho,\nu}P_\mu), & [L_{\mu\nu},L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \end{aligned} \tag{4}$$

which close the algebra corresponding to the conformal group. The index $\mu = 0,1,...,d-1$ and the spacetime metric signature is $(+---...-)$. The ordinary Minkowski group has the algebra: $so(d-1,1)$ given by the very last commutator of (4). If we define the generators:

$$\begin{aligned} J_{\mu,\nu} &\equiv L_{\mu\nu} \\ J_{\mu,d} &\equiv \frac{1}{2}(P_\mu - K_\mu) \\ J_{-1,d} &\equiv D \\ J_{\mu,-1} &\equiv \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{5}$$

then it is straightforward to confirm that they close the algebra,

$$[J_{\mu,\nu},J_{\rho,\sigma}] = i(\eta_{\nu\rho}J_{\mu,\sigma} + \eta_{\mu\sigma}J_{\nu,\rho} - \eta_{\mu\rho}J_{\nu,\sigma} - \eta_{\nu\sigma}J_{\mu,\rho}), \tag{6}$$

with a modified Minkowski metric which has entries $\eta_{-1-1} = +1$ and $\eta_{dd} = -1$ appended before the first entry and after the last entry respectively. In other words the new Minkowski metric is $\eta_{\mu\nu} = (\eta_{-1-1},\eta_{00},\eta_{11},...,\eta_{d-1d-1},\eta_{dd}) = (+1,+1,-1,-1,...,-1)$. Therefore the conformal group in $\mathbb{M}^{d-1,1}$ (i.e. $SO(d-1,1)$) is locally[10] isomorphic to $SO(d,2)$.

2.2.2 Conformal Quantum Mechanics

We now introduce the conformal quantum mechanics as presented by de Alfaro, Fubini and Furlan (dAFF) [2]. The AdS/CFT correspondence indicates that the dual to AdS_2 should be a one dimensional theory with a conformal symmetry. In this section, we will restrict our attention to the global conformal group in one dimension which amounts to a scale invariant one dimensional scalar field theory.[11] The requirement of scale invariance means that the coupling, λ say, is dimensionless. The general scale invariant Lagrangian is

[10] By simply matching the algebra for infinitesimal generators we cannot be sure that there are no global differences.

[11] We shall see in the next section (see sub-subsection 3.5.3), where we make reference to [20], that there is evidence to suggest that there is an asymptotic, infinite dimensional, symmetry group of time reparameterizations for AdS_2. This suggests the possibility of a local Virasoro algebra corresponding to one dimensional time diffeomorphisms as the algebra associated with the CFT_1 dual to AdS_2. However, much of our work, to appear in sections 4 and 5, will be more closely related to the $SL(2,\mathbb{R})$ group which is the global sub-group of such an infinite dimensional local symmetry. It is for this reason that we focus on the conformal quantum mechanics of [2] to begin with. Specifically, the conformal killing equation $\bigtriangledown_\mu\xi_\nu + \bigtriangledown_\nu\xi_\mu = 2\omega(x)g_{\mu\nu}$ reduces, in one dimension, to the form $\partial_t\xi = h(t)$ where $h(t)$ is an arbitrary function of time. By Laurent expansion, the generators can be identified as $\xi_n = it^{n+1}\partial_t$. These generators close the Virasoro algebra, $[\xi_n,\xi_m] = i(m-n)\xi_{m+n}$ for which the generators H, D, and K, derived below, close the $sl(2,\mathbb{R})$, global, sub-algebra (see (28)) [20].

$$S = \int d^dx \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda\phi^{2d/d-2}\right]. \tag{7}$$

The power of the coupling term is expressed in energy units and is determined by the requirement that the action be dimensionless which in turn determines the field dimensions to be $\Delta_\phi = \frac{d-2}{2}$. In $d = -1$ (that is 1 length dimension) the theory becomes a relativistic one dimensional quantum field theory which we will refer to as conformal quantum mechanics

$$S = \int dx \left[\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{\phi^2}\right]. \tag{8}$$

We relabel the scalar field $\phi(t) \to q(t)$ so that the action becomes

$$S = \frac{1}{2}\int dt \left(\dot{q}^2 - \frac{\lambda}{q^2}\right) \tag{9}$$

and we consider λ to be a positive coupling constant. To identify the conformal symmetry we note that, under coordinate transformation $t \to t'$ of the form of a linear fractional transformation[12]

$$t \to t' = \frac{\alpha t + \beta}{\gamma t + \rho}, \qquad \alpha, \beta, \gamma, \rho \in \mathbb{R}, \qquad \alpha\rho - \beta\gamma = 1, \tag{10}$$

the action is invariant provided that the 'conformal weight' of the Jacobian is $-\frac{1}{2}$, i.e.

$$q'(t') = \left(\frac{dt}{dt'}\right)^{-\frac{1}{2}} q(t) = \frac{q(t)}{\gamma t + \rho}. \tag{11}$$

Although the Lagrangian is not invariant under such a transformation, the variation in the action is given by an overall time derivative term. The overall derivative term F, together with the usual Noether current, makes up the constant of motion of the theory

$$\partial_t \left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\delta q - F\right) = 0. \tag{12}$$

The transformation of (10) has a matrix

$$\epsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \rho \end{pmatrix} \tag{13}$$

of which the real parameters, α, β, γ, and ρ are the elements. This matrix is an element of the $SL(2,\mathbb{R})$ special linear group. We introduce a unitary representation $U(\epsilon)$ such that

$$q'(t) = U(\epsilon)q(t)U^{-1}(\epsilon) \tag{14}$$

and

$$U^{-1}(\epsilon)q(t)U(\epsilon) = \left(\frac{dt}{dt'}\right)^{\frac{1}{2}} q(t'). \tag{15}$$

[12] This is the well known Mobius transformation (restricted to real parameters) which is the general conformal transformation of $SL(2,\mathbb{R})$, constructed by the composition of the dilatation, translation and inversion transformations.

The evolution of the state is as usual

$$U(\epsilon)|\Psi(t)\rangle = |\Psi(t')\rangle. \tag{16}$$

The transformation equations ((15) and (16)) allow us to determine the variation in the field associated with a given variation in time (associated with conformal transformations: time translation, dilatation and inversion). We turn to these transformations next.

1. Time translation: $t \to t' = t - \epsilon$
   This corresponds to equations (10) and (13) with $\alpha = 1 = \rho, \beta = -\epsilon$ and $\gamma = 0$. For a generator T corresponding to the transformation in the Unitary representation U, it follows from (15) that

$$\begin{aligned} e^{-i\epsilon T_1} q(t) e^{i\epsilon T_1} &= q(t-\epsilon) \qquad (17) \\ \mathbb{I} - i\epsilon[T_1, q] + \mathcal{O}(\epsilon^2) &= q - \epsilon\dot{q} + \mathcal{O}(\epsilon^2) \end{aligned}$$

   and from (16)

$$\begin{aligned} U(\epsilon)|\Psi(t)\rangle &= |\Psi(t-\epsilon)\rangle \qquad (18) \\ \left(\mathbb{I} + i\epsilon T_1 + \mathcal{O}(\epsilon^2)\right)|\Psi(t)\rangle &= |\Psi(t)\rangle - \epsilon|\dot{\Psi}(t)\rangle. \end{aligned}$$

   Therefore we identify

$$\delta q = i[T_1, q(t)] = \dot{q}(t) \tag{19}$$

$$iT_1|\Psi(t)\rangle = -|\dot{\Psi}(t)\rangle. \tag{20}$$

   We associate the generator of time translations with the Hamiltonian $T_1 = H = i\partial_t$.

2. Time dilatations: $t \to t' = t - \epsilon t$
   This corresponds to equations (10) and (13) with $\alpha = e^{-\frac{\epsilon}{2}} = 1/\rho$ and $\beta = 0 = \gamma$. Now

$$\begin{aligned} e^{-i\epsilon T_2} q(t) e^{i\epsilon T_2} &= e^{\epsilon/2} q(t-\epsilon t) \qquad (21) \\ \mathbb{I} - i\epsilon[T_2, q] + \mathcal{O}(\epsilon^2) &= q(t) - \epsilon\left(t\dot{q} - \frac{1}{2}q\right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

   and

$$\begin{aligned} e^{i\epsilon T_2}|\Psi(t)\rangle &= |\Psi(t-\epsilon t)\rangle \qquad (22) \\ \left(\mathbb{I} + i\epsilon T_2\right)|\Psi(t)\rangle &= |\Psi(t)\rangle - \epsilon t|\dot{\Psi}(t)\rangle. \end{aligned}$$

   So

$$\delta q = i[T_2, q(t)] = t\dot{q}(t) - \frac{1}{2}q(t) \tag{23}$$

$$iT_2|\Psi(t)\rangle = -t|\dot{\Psi}(t)\rangle. \tag{24}$$

   The generator T_2 is then identified with the dilatation operator: $T_2 = D = it\partial_t$.

3. Conformal transformation: $t \to t' = (1+\epsilon t)^{-1} t = t - \epsilon t^2$.
   This corresponds to $\alpha = 1 = \rho, \beta = 0$ and $\gamma = \epsilon$. Then

$$\mathbb{I} - i\epsilon[T_3, q] + \mathcal{O}(\epsilon^2) = q(t) - \epsilon\left(t^2\dot{q} - tq\right) + \mathcal{O}(\epsilon^2) \tag{25}$$
$$\delta q = i[T_3, q(t)] = t^2\dot{q} - tq \tag{26}$$
$$iT_3|\Psi(t)\rangle = -t^2|\dot{\Psi}(t)\rangle. \tag{27}$$

   $T_3 = K = it^2\partial_t$ is the generator of conformal transformations.

The explicit forms of T_1, T_2 and T_3 can be used to confirm the following algebra

$$[D, K] = iK, \qquad [H, K] = 2iD, \qquad [H, D] = iH. \tag{28}$$

This is the $sl(2,\mathbb{R})$ algebra of the conformal group in $d = 1$ dimensions. Since H, D and K are constants of motion[13] one can form a new constant of motion, say C, by linear combination of H, D and K:

$$C = lH + mD + nK, \qquad l, n, m \in \mathbb{R}. \tag{29}$$

A convenient choice of such linear combinations is:

$$l_1 = \frac{1}{2}\left(\frac{1}{a}K - aH\right), \qquad l_2 = D, \qquad l_3 = \frac{1}{2}\left(\frac{1}{a}K + aH\right), \tag{30}$$

where the dimensionful constant a is necessary since H and K have different units. These three generators, for a metric signature $(- - +)$, have the algebra:

$$[l_2, l_3] = il_1, \qquad [l_1, l_3] = -il_2, \qquad [l_1, l_2] = -il_3. \tag{31}$$

This is the $so(1,2)$ algebra. The $SL(2,\mathbb{R})$ group is isomorphic to $SO(1,2)$ and we have confirmed that the expected result of $SO(d,2)$, for the conformal group, holds in $d = 1$ dimensions.
The conserved generators H, D and K are expressed in terms of the field operator q by making use of (19),(23) and (26) and Noether's theorem. The variation in the Lagrangian is:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q}\delta q + \frac{\partial\mathcal{L}}{\partial\dot{q}}\delta\dot{q} \tag{32}$$

where $\delta\dot{q}$ is the time derivative of δq. One obtains the following:

$$\delta\mathcal{L} = \frac{d}{dt}\mathcal{L},$$
$$\delta\mathcal{L} = \frac{d}{dt}\left(t\mathcal{L}\right),$$
$$\delta\mathcal{L} = \frac{d}{dt}\left(t^2\mathcal{L} - \frac{q^2}{2}\right), \tag{33}$$

[13]While they have explicit time dependence t (see (34), (35) and (36)) their Heisenberg equations of motion $\frac{d\mathcal{O}}{dt} = i[H,\mathcal{O}] + \partial_t\mathcal{O} = 0$ as can be confirmed using the algebra (28). It is in this sense that they are constants of motion- as expected by the Noether currents (33).

for time translations, dilatations and special conformal transformations respectively. Therefore, the conserved generators (12) are:

$$H = \frac{1}{2}\left(\dot{q}^2 + \frac{\lambda}{q^2}\right), \tag{34}$$

$$D = tH - \frac{1}{4}\left(q\dot{q} + \dot{q}q\right), \tag{35}$$

$$K = t^2 H - \frac{t}{2}\left(q\dot{q} + \dot{q}q\right) + \frac{q^2}{2}, \tag{36}$$

where in the second terms of both (35) and (36) the appropriate symmetrization for bosons was applied to avoid an ordering ambiguity in the quantization of the theory.
Having derived the constants of motion (34), (35) and (36), which close the algebra (28) provided that one demands the usual commutator $[q(t), p(t)] = i$, we now illustrate the existence of an alternative operator to the Hamiltonian, identified by [2], that is apparently more appropriate for the stationary eigenstates of the system. The usual equal time canonical commutation relations can be applied and the generators can be expressed at $t = 0$:

$$\begin{aligned} H|_{t=0} &= \frac{1}{2}\left(p^2 + \frac{\lambda}{q^2}\right), \\ D|_{t=0} &= -\frac{1}{4}\left(qp + pq\right), \\ K|_{t=0} &= \frac{q^2}{2}. \end{aligned} \tag{37}$$

Here it is understood that $p = p(t=0) = \dot{q}(t=0)$ and $q = q(t=0)$.
By direct substitution it is observed that the equations of motion of C are:

$$i[C, q(t)] = f_c\dot{q} - \frac{1}{2}\dot{f}_c q \tag{38}$$

$$C|\Psi(t)\rangle = if_c|\dot{\Psi}(t)\rangle \tag{39}$$

where $f_c = l + mt + nt^2$. By a change of the time coordinate,

$$\begin{aligned} d\tau &= \frac{dt}{f_c}, \\ \phi(\tau) &= \frac{q(t)}{\sqrt{f_c}}, \end{aligned} \tag{40}$$

the equations of motion become:

$$i[C, \phi(\tau)] = \dot{\phi}(\tau), \tag{41}$$

$$C|\Psi(\tau)\rangle = i|\dot{\Psi}(\tau)\rangle. \tag{42}$$

These equations are solved by

$$\phi(\tau) = e^{iC(\tau-\tau_o)}\phi(\tau_o)e^{-iC(\tau-\tau_o)}, \tag{43}$$

$$|\Psi(\tau)\rangle = e^{-iC(\tau-\tau_o)}|\Psi(\tau_o)\rangle. \tag{44}$$

One can define stationary states (with respect to τ) that are eigenstates of C at $\tau = \tau_o$. In order for the solutions to describe time evolution over all times ($-\infty < t < \infty$) as a function of τ, it is required that $\lambda > 0$ and $\Delta = m^2 - 4ln < 0$.[14] Δ is the discriminant of f_c. These conditions are a result of the integral equation

$$\tau = \int_{t_0}^{t} \frac{dx}{f_c(x)} + \tau_0. \tag{45}$$

Only for $\Delta < 0$ will t vary over $(-\infty, \infty)$ without singularities in τ. This case is associated with the operator l_3 which is a generator of compact rotations. Under (40) the Lagrangian becomes:

$$\mathcal{L}' = \frac{1}{2}\left(\dot{\phi}^2 + \frac{\Delta}{4}\phi^2 - \frac{\lambda}{\phi^2}\right) + \frac{1}{2}\frac{d}{d\tau}\left(\left(\frac{m^2}{2} + nt\right)\phi^2\right) \tag{46}$$

and so

$$C(q, \dot{q}) = H_c(\phi, \dot{\phi}) = \frac{1}{2}\left(\dot{\phi}^2 - \frac{\Delta}{4}\phi^2 + \frac{\lambda}{\phi^2}\right). \tag{47}$$

In the Schrodinger picture one has[15]

$$i\partial_\tau \psi(x, \tau) = H_c(x, -i\partial_x)\psi(x, \tau) \tag{48}$$

with a separable stationary state eigenvalue equation

$$C\psi_{c'}(x) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + \frac{\lambda}{x^2} - \frac{\Delta}{4}x^2\right)\psi_{c'}(x) = c'\psi_{c'}(x). \tag{49}$$

This describes single particle quantum mechanics in a potential: $V(x) = \lambda/x^2 - x^2\Delta/4$. From (47) it is clear that the potential has a global minimum which leads to a discrete spectrum of a localized particle that's evolution is well described for all t only when $\lambda > 0$ and $\Delta < 0$. It should be noted that (49) is not the typical Schrodinger time independent equation defined for the Hamiltonian. It is an eigenvalue equation for C, which is an operator that is dependent on Δ (see definition of C - equation (29)) and the variables x and $-i\partial_x$ are Schrodinger picture equivalents of ϕ and $\dot{\phi}$ under their canonical commutation relation at a given time ($t = 0$). $C\psi$ is a solution to the time dependent Schrodinger equation whenever ψ is, which is true for all time; this is a consequence of the fact that C is a constant of motion. By defining $D = tH + D_0$ and $K = t^2H + 2tD_0 + K_0$, consistent with (34), (35) and (36), we have that $C = f_cH + f_c'D_0 + nK_0$ which for a particular choice of time, we can take $t = -\frac{m}{2n}$, is $C = -\frac{\Delta}{4n}H + nK_0$. Demanding that $C\psi_{c'}(x, t) = c'\psi_{c'}(x, t)$ and considering $H\psi(x, t) = i\partial_t\psi(x, t)$ allows one to solve for $\psi_{c'}(x, t)$ in terms of a function $F_{c'}(x)$ that is dependent on the parameter choice (i.e. choice of t in C, which we now consider to be a parameter) such that $(-\frac{d^2}{dx^2} + \frac{\lambda}{x^2} - \frac{\Delta}{4})F_{c'}(x) = c'F_{c'}(x)$. This highlights the distinction between the usual Schrodinger equation $H\psi(x, t) = i\partial_t\psi(x, t)$ and equation (49). Note that this is entirely consistent with the choice of l_3 for C. This choice sets $m = 0$, $n = 1/2a$ and $l = a/2$, which means that the parameter choice $t = -\frac{m}{2n} = 0$. This was the choice taken for t in (37), from which the remainder of the discussion ensued. We refer the reader to appendix A of [2] for further details.

[14] For the three operators (30) one finds that $\Delta = +1 > 0$ for l_1, $\Delta = +1 > 0$ for l_2 and $\Delta = -1 < 0$ for l_3. This identifies l_3 as an appropriate generator that is well defined for all time.

[15] It is possible to obtain a wave function description since the field is a function of time only. The fields become $\phi(\tau = 0) \equiv x$ and $\dot{\phi}(\tau = 0) \equiv -i\frac{d}{dx}$.

In summary, the above work of de Alfaro, Fubini and Furlan [2] shows that the algebra of conformal quantum mechanics is $sl(2,\mathbb{R})$ which can be mapped to the $so(1,2)$ algebra by defining new generators that are linear combinations of the $SL(2,\mathbb{R})$ generators. This establishes the isomorphism, at the level of the algebras, of the conformal group, $SL(2,\mathbb{R})$, and the Lorentz group, $SO(1,2)$. In the case of conformal quantum mechanics, the constant of motion l_3 (30) has a potential that grows as $q \to \pm\infty$ which leads to a discrete spectrum. On the contrary, the Hamiltonian corresponds to the case $\Delta = 0$ which does not have a discrete spectrum. With regard to the normalization properties of the eigenfunctions and the discreteness of the spectrum, l_3 appears to be preferable to the Hamiltonian as the operator for which one should seek to find eigenstates [2].[16] We next turn our attention to anti-de Sitter space in order to develop some ideas about the gravitational side of the correspondence.

2.2.3 AdS_{d+1} geometry and its causal structure

As with any spacetime there are many interesting features to study in detail. For Anti-de Sitter space we shall focus on the causal structure. There are two particular coordinate systems in AdS that are used more frequently than any others: these are the global and Poincare coordinate systems. A prominent feature of AdS is the existence of a boundary to the spacetime. After mapping the global coordinates to the conformal coordinates[17] in order to understand the causal structure of the spacetime, we follow Bayona and Braga [1] who identified the relationship between the conformal coordinates and the Poincare coordinates which established that the AdS boundary in these two coordinate systems is different. This distinction between the boundaries leads to distinct causal structure for AdS depending on the choice of coordinates.

AdS_{d+1} is a $(d+1)-dim$ hyperboloid embedded in a flat $(d+2)-dim$ Minkowski space. The manifold and metric are

$$-R^2 = -X_{-1}^2 - X_0^2 + \sum_{i=1}^{d} X_i^2, \tag{50}$$

and

$$ds^2 = (dX_{-1})^2 + (dX_0)^2 - \sum_{i=1}^{d} (dX_i)^2 \tag{51}$$

respectively.

To parameterize the manifold we choose the global coordinates ϕ, τ and Ω_i such that:

$$\begin{aligned} X_{-1} &= R\cosh\phi\cos\tau, \\ X_0 &= R\cosh\phi\sin\tau, \\ X_i &= R\sinh\phi\Omega_i \end{aligned} \tag{52}$$

[16] In the next section (see subsection 3.2) we shall see that the choice of l_3 over H, form the gravitational perspective, will be associated with a different choice of time coordinate in AdS_2.

[17] The true global coordinates will be identified below along with the map to conformal coordinates which are necessary in order to study the causal structure of the spacetime in a penrose diagram. We will make clear the distinction between the global coordinates and conformal coordinates; however, from the point of view of the penrose diagram, the conformal coordinates supersede the global coordinates. For this reason we shall refer to the two interchangeably as the particular coordinates should be clear from the context.

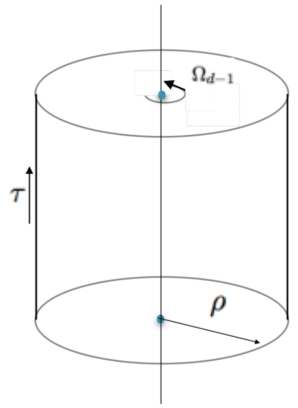

Figure 1: We have illustrated the conformal diagram of AdS_3 which is the manifold $\mathbb{R}^1 \times B^2$ with an S^1 boundary as $\rho \to \pi/2$.

where $\phi \in (0, \infty)$, $-\pi < \tau < \pi$, $-1 \leq \Omega_i \leq 1$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$ are the usual spherical coordinates of S^{d-1}. Making use of the constraint $\Omega_i \Omega_i = 1$ and the implied result $\Omega_i d\Omega_i = 0$ it is straightforward to confirm that the metric in global coordinates is

$$ds^2 = R^2(\cosh^2 \phi d\tau^2 - d\phi^2 - \sinh^2 \phi d\Omega_{d-1}^2). \tag{53}$$

It is apparent from the signature in (53) that one of the two times has been eliminated. The time coordinate τ is such that $-\pi < \tau < \pi$- i.e. time appears to be periodic. However one can achieve the universal cover of AdS_2 by unwrapping the manifold and 'gluing' infinite copies of the spacetime to itself in order to eliminate closed time-like curves. We bring the global AdS coordinate ϕ to a finite range by defining

$$\tan \rho \equiv \sinh \phi \tag{54}$$

with $\rho \in [0, \frac{\pi}{2})$. Then the spacetime coordinate parameterization becomes [1]

$$\begin{aligned} X_{-1} &= R \sec \rho \cos \tau, \\ X_0 &= R \sec \rho \sin \tau, \\ X_i &= R \tan \rho \Omega_i. \end{aligned} \tag{55}$$

with the corresponding metric:

$$ds^2 = \frac{R^2}{\cos^2 \rho}(d\tau^2 - d\rho^2 - \sin^2 \rho d\Omega_{d-1}^2). \tag{56}$$

These are referred to as the conformal coordinates (ρ, τ and Ω_i). Since the coordinate $\rho \in [0, \frac{\pi}{2})$, the metric defines the manifold $\mathbb{R}^1 \times S^d/2$, where $S^d/2$ is specified to indicate that only one hemisphere of the S^d sphere is covered. The manifold is therefore more accurately labeled $\mathbb{R}^1 \times B^d$. Therefore, the conformal diagram of AdS is the volume contained within a cylinder. This is illustrated in figure 1.

The $X_{-1} = X_d$ plane splits the spacetime in two (see figure 2). By introducing the light-cone coordinates

$$u \equiv \frac{1}{R^2}(X_{-1} - X_d) \qquad \text{and} \qquad v \equiv \frac{1}{R^2}(X_{-1} + X_d) \tag{57}$$

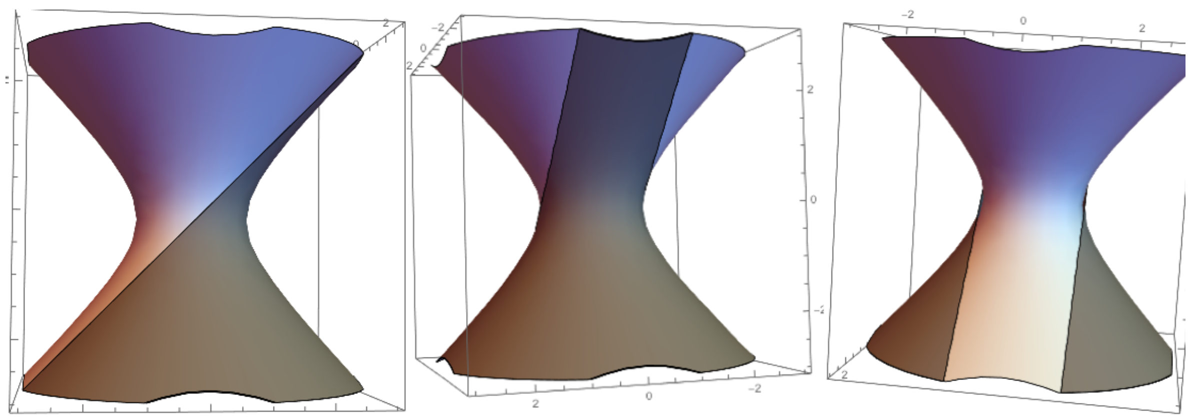

Figure 2: The AdS_2 hyperboloid ($R^2 = X_{-1}^2 + X_0^2 - X_1^2$) is cleaved in two by the cutting plane $X_{-1}^2 = X_1^2$- instead of including the plane we have simply shaded the two regions on either side of the slicing plane.

and the so called Poincare coordinates

$$x^0 \equiv t = \frac{X_0}{Ru}, \qquad x^i \equiv \frac{X_i}{Ru}, \qquad z \equiv \frac{1}{u} = \frac{R^2}{(X_{-1} - X_d)}, \tag{58}$$

where $i = 1, 2, ..., d-1$ we can re-express the hyperboloid as

$$R^2 = R^4 uv + R^2 u^2 (t^2 - x^i \cdot x^i). \tag{59}$$

(59) provides an equation for v. More specifically:

$$\begin{aligned} v &= \frac{1}{R^2}(z - u(t^2 - \vec{x}^2)), \\ \frac{1}{2}[R^2 u + z - u(t^2 - \vec{x}^2)] &= X_{-1}, \\ \frac{1}{2}[-R^2 u + z - t^2 + \vec{x}^2] &= X_d. \end{aligned} \tag{60}$$

Making use of the above equations, (57) and (58), we find that

$$\begin{aligned} X_{-1} &= \frac{1}{2z}[R^2 + z^2 - t^2 + \vec{x}^2], \\ X_d &= \frac{1}{2z}[-R^2 + z^2 - t^2 + \vec{x}^2], \\ X_0 &= \frac{Rt}{z}, \\ X_i &= \frac{Rx^i}{z}, \end{aligned} \tag{61}$$

with no explicit dependence on u or v. Evidently the form of z in (58) indicates that AdS_{d+1} has a boundary at $z = 0$ with two distinct regions corresponding to $z > 0$ and $z < 0$.

The Poincare patch is defined as the half of AdS with $z > 0$ and metric

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2); \qquad x = (x^\mu, z), \mu = 0, 1, ..., d-1. \tag{62}$$

Since the plane partitioning the spacetime ($X_{-1} = X_d$) corresponds to $u = 0$ and $z \to \pm\infty$, the plane is not part of either of the two Poincare patches but must certainly contain some points of the hyperboloid. In fact, in the $z \to \infty$ limit we must concurrently take $t \to \infty$ in order to satisfy (50) and from (55) we see that the partitioning plane corresponds to

$$\cos\tau = \sin\rho\Omega_1. \tag{63}$$

If we consider the case of $\rho \to \frac{\pi}{2}$- taking the embedding coordinate to spatial infinity (the AdS boundary in global coordinates)- then

$$\cos\tau = \Omega_1. \tag{64}$$

This is equivalent to $\cos\tau - \Omega_1 \to 0$ with $\rho \to \frac{\pi}{2}$ but the interesting thing is that this limit can be achieved while $0 < |z| < \infty$.

It is possible to map the Poincare boundary to the global coordinates. Using

$$\begin{aligned} X_{-1} = R\sec\rho\cos\tau &= \frac{1}{2z}[R^2 + z^2 - t^2 + \vec{x}^2], \\ X_0 = r\sec\rho\sin\tau &= \frac{Rt}{z}, \\ X_i = R\tan\rho\Omega_i &= \frac{Rx^i}{z}, \\ X_d = R\tan\rho\Omega_d &= \frac{1}{2z}[-R^2 + z^2 - t^2 + \vec{x}^2] \end{aligned} \tag{65}$$

we obtain

$$\begin{aligned} \sec^2\rho = \frac{X_{-1}^2 + X_0^2}{R^2} &= \frac{1}{R^2(2z^2)}\left([R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2\right) \\ \Rightarrow \sec\rho &= \frac{1}{2R|z|}\sqrt{[R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2}, \end{aligned} \tag{66}$$

$$\begin{aligned} \cos\tau = \frac{X_{-1}}{R\sec\rho} &= \frac{|z|}{z}\frac{\left(R^2 + z^2 - t^2 + \vec{x}^2\right)}{\sqrt{[R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2}} \\ \Rightarrow \cos\tau &= \text{sign(z)}\frac{\left(R^2 + z^2 - t^2 + \vec{x}^2\right)}{\sqrt{[R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2}}, \end{aligned} \tag{67}$$

$$\sin\tau = \frac{X_0}{r\sec\rho} = \text{sign(z)}\frac{2Rt}{\sqrt{[R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2}}. \tag{68}$$

Now $\tan^2\rho = \frac{\vec{x}^2}{\Omega_i\Omega_i z^2}$ and taking the square of (66) we have

$$\begin{aligned} 1 &= \frac{1}{4R^2z^2}\left([R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2\right) - \frac{\vec{x}^2}{\Omega_i\Omega_i z^2} \\ \Rightarrow |\overline{\Omega}| = \sqrt{\Omega_i\Omega_i} &= \frac{2R|\vec{x}|}{\sqrt{[R^2 + z^2 - t^2 + \vec{x}^2]^2 + (2Rt)^2 - (2Rz)^2}} \end{aligned} \tag{69}$$

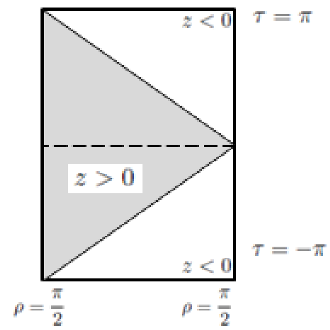

Figure 3: Causal diagram of AdS. The Poincare region is illustrated as the hatched triangle [1].

and lastly, since $\tan\rho = \frac{1}{2R\Omega_d z}\left(-R^2+z^2-t^2+\vec{x}^2\right)$ and making use of the square of (66):

$$\begin{aligned}
\frac{1}{4R^2\Omega_d^2 z^2}\left(-R^2+z^2-t^2+\vec{x}^2\right)^2 &= \frac{1}{4R^2|z|}\left([R^2+z^2-t^2+\vec{x}^2]^2+(2Rt)^2-(2Rz)^2\right)\\
\Rightarrow (\Omega_d)^2 = (\text{sign(z)})^2 &\frac{\left(-R^2+z^2-t^2+\vec{x}^2\right)^2}{[R^2+z^2-t^2+\vec{x}^2]^2+(2Rt)^2-(2Rz)^2}\\
\Rightarrow \Omega_d = \text{sign(z)}&\frac{\left(-R^2+z^2-t^2+\vec{x}^2\right)}{\sqrt{[R^2+z^2-t^2+\vec{x}^2]^2+(2Rt)^2-(2Rz)^2}}. \qquad (70)
\end{aligned}$$

Equations (66)-(70) hold for both $z<0$ and $z>0$ and making use of the coordinate transformations (66)-(70) one can divide the Poincare AdS boundary into regions with well defined points in the global AdS coordinates. This is done in fine detail in [1], where 17 such regions were identified and mapped to the global AdS spacetime. The Poincare patch is illustrated within the global AdS spacetime in figure 3. Evidently the boundary of the Poincare patch contains points in the global AdS bulk ($\rho\in[0,\frac{\pi}{2})$) as well as the global AdS boundary ($\rho=\frac{\pi}{2}$). It should be noted that the global AdS boundary and the Poincare AdS boundary are distinct and can be distinguished in the penrose diagram of figure 3. As discussed above, the global coordinate metric (56) is conformally related to $S^d\times\mathbb{R}$ but with the restriction of the angular coordinate $\rho\in[0,\pi/2)$ which covers only the 'northern hemisphere' of S^d. So, an alternative penrose diagram is given in figure 4. The region of the Poincare boundary for which z is finite and $t\to\infty$ corresponds, in figure 4, to the part of the Poincare patch that coincides with the global AdS boundary- that is, the surface of the cylinder ($\rho\to\frac{\pi}{2}$).[18]

2.2.4 Relating the Isometries of AdS_2, the Conformal Group $SL(2,\mathbb{R})$ and the Group $SO(1,2)$

To understand the symmetries of AdS we shall study the isometry group of AdS_2, the case that is relevant for us. We will then see how the generators of the isometries of AdS_2 are related to the generators of the conformal transformations in $d=1$ dimensions, which we have seen can be mapped to the $SO(1,2)$ Lorentz algebra.

The embedding equation of the manifold (50) and the induced metric (51) indicate that AdS_{d+1} has the isometry group $SO(d,2)$. In fact, AdS_{d+1} is a coset manifold: $SO(d,2)/SO(d,1)$.[19] In the case of AdS_2 the coset manifold has $\frac{3(2)}{2}-\frac{2(1)}{1}=2$ dimensions, as expected. AdS is maximally symmetric since for $AdS_D=AdS_{d+1}$ ($\Rightarrow D=d+1$) we have that $\frac{(d+2)(d+1)}{2}=\frac{D(D+1)}{2}=3$, which is the maximum number of killing vectors that AdS_D can have.

[18] Again a detailed description of the specific regions of Poincare AdS in the global AdS space for a causal diagram such as figure 4 can be found in [1].

[19] The reason for the signature of the sub-manifold is that the invariant subspace can be chosen to be transformations that leave a time direction unit vector invariant.

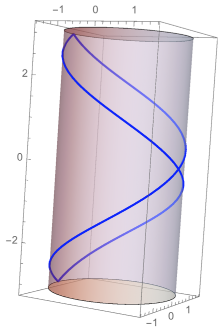

Figure 4: Alternative causal diagram of AdS_3. The parameterization for AdS_3 is $X_0 = R\sec\rho\cos\tau = 1/2z(z^2{+}r^2{+}x^2{-}t^2)$, $X_1 = R\tan\rho\Omega_1 = \frac{Rx}{z}$, $X_2 = R\tan\rho\Omega_2 = \frac{1}{2z}(z^2{-}R^2{+}x^2{-}t^2)$, $X_3 = R\sec\rho\sin\tau = \frac{Rt}{z}$ with $\Omega_1 = \cos\phi$ and $\Omega_2 = \sin\phi$. The coordinates range over $\rho \in [0, \frac{\pi}{2})$, $\phi \in [-\pi, \pi]$ and $\tau \in [-\pi, \pi]$. This particular figure illustrates the limit for which z =finite, $t \to \pm\infty$, $\rho = \frac{\pi}{2}$, $\tau \in [-\pi, \pi]$ which leads to the parametric curves in dark blue that are defined by $\cos\tau = \cos\phi$ with the restriction that $\tau \in [0, \pi]$ and $\phi \in [-\pi, \pi]$ as well as $\tau \in [-\pi, 0]$ and $\phi \in [-\pi, \pi]$. These curves define the global boundary curves for which the global boundary is cut by the Poincare boundary surfaces- that is, the Poincare patch corresponds to the volume contained within the 'wedge' corresponding to these surfaces. See [1] for various limits of the Poincare coordinates and their corresponding location within the global AdS space. Figure plotted with Mathematica.

The Isometry generators are the killing vectors ξ which solve the Lie derivative equation

$$\begin{aligned}&\mathcal{L}_\xi g_{\mu\nu}(x) = 0\\ &\Rightarrow \xi^\alpha(x) D_\alpha g_{\mu\nu}(x) + g_{\beta\nu}(x) D_\mu \xi^\beta(x) + g_{\mu\sigma}(x) D_\nu \xi^\sigma(x) = 0.\end{aligned} \tag{71}$$

The Poincare metric for AdS_2 is $ds^2 = R^2/z^2\left(dt^2 - dz^2\right)$. The non-zero Christoffel symbols are $\Gamma^z_{zz} = \Gamma^t_{tz} = \Gamma^z_{tt} = -1/z$. Then,

$$\begin{aligned}\partial_t \xi^t - \frac{1}{z}\xi^z &= 0,\\ \partial_t \xi^z - \partial_z \xi^t &= 0,\\ \partial_z \xi^z - \frac{1}{z}\xi^z &= 0.\end{aligned} \tag{72}$$

One solution can immediately be identified by the result $D_t g_{\mu\nu}(x) = 0$ which implies that ∂_t is a killing vector.[20] This can be re-expressed as follows: $\xi^\mu_{(1)}\partial_\mu = \xi^t_{(1)}\partial_t + \xi^z_{(1)}\partial_z = \partial_t$ and therefore

$$\xi^\mu_{(1)} = R(1, 0) \tag{73}$$

is a solution.[21] The remaining two solutions are

$$\begin{aligned}\xi^\mu_{(2)} &= (t, z),\\ \xi^\mu_{(3)} &= \frac{1}{R}(t^2 + z^2, 2tz).\end{aligned} \tag{74}$$

[20]We use the notation D_μ for the covariant derivative in this section. Since both D_μ and $\bigtriangledown_\mu$ are both commonly used for the covariant derivative in gauge theories we may use them interchangeably.

[21]The length parameter of AdS: R has been reintroduced for convenience [3]. The same is true of the scaling factor in the second of the equations in (74).

As differential operators

$$
\begin{aligned}
I_{(1)} &\equiv \xi^{\mu}_{(1)}\partial_\mu = R\partial_t, \\
I_{(2)} &\equiv \xi^{\mu}_{(2)}\partial_\mu = t\partial_t + z\partial_z, \\
I_{(3)} &\equiv \xi^{\mu}_{(3)} = \frac{1}{R}((t^2+z^2)\partial_t + 2tz\partial_z).
\end{aligned}
\tag{75}
$$

For (50) ($SO(1,2)$, with signature $\eta = (+,+,-)$), we can define the generators

$$
L_{\mu\nu} \equiv i(X_\nu\partial_\mu - X_\mu\partial_\nu). \tag{76}
$$

There is a slight difference between this definition of the generators of $SO(1,2)$ compared to our previous definition for the $SO(d,2)$ generators, which were defined in terms of the generators $L_{\mu\nu}$ in (3); they differ by a sign. This definition has been chosen for convenience and is just a matter of convention. These can be identified as

Rotation	$L_{0-1} = i(X_{-1}\partial_0 - X_0\partial_{-1})$
Boosts	$L_{01} = i(X_1\partial_0 - X_0\partial_1)$ $L_{1-1} = i(X_{-1}\partial_1 - X_1\partial_{-1})$

and will satisfy the algebra:

$$
[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\mu\rho}L_{\nu\sigma} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\sigma}L_{\nu\rho}). \tag{77}
$$

In particular

$$
\begin{aligned}
[L_{0-1}, L_{10}] &= iL_{1-1}, \\
[L_{0-1}, L_{1-1}] &= iL_{01}, \\
[L_{10}, L_{1-1}] &= -iL_{0-1}.
\end{aligned}
\tag{78}
$$

Expressing the differential generators (75) in terms of Poincare coordinates where $z = 1/u = R^2/(X_{-1} - X_1)$ and $t = zX_0/R = X_0R/(X_{-1} - X_1)$ we find that

$$
\begin{aligned}
L_{0-1} &= i(X_{-1}\frac{\partial}{\partial X^0} - X_0\frac{\partial}{\partial X^{-1}}) = \frac{i}{2}\left(\left[\frac{t^2+z^2}{R} + R\right]\partial_t + \frac{2zt}{R}\partial_z\right), \\
L_{10} &= -i(X_0\frac{\partial}{\partial X^1} + X_1\frac{\partial}{\partial X^0}) = -\frac{i}{2}\left(\left[\frac{z^2+t^2}{R} - R\right]\partial_t + \frac{2zt}{R}\partial_z\right), \\
L_{1-1} &= -i(X_{-1}\frac{\partial}{\partial X^1} + X_1\frac{\partial}{\partial X^{-1}}) = -i\left(z\partial_z + t\partial_t\right).
\end{aligned}
\tag{79}
$$

We now make the following identification at the AdS_2 boundary at $z = 0$ [3]:

$$
\begin{aligned}
\frac{i}{a}I_{(1)}\Big|_{z=0} &= \frac{iR}{a}\partial_t = \frac{1}{a}(L_{0-1} + L_{10})\Big|_{z=0} = H \Leftrightarrow R = a, \\
iaI_{(3)}\Big|_{z=0} &= \frac{ia}{R}t^2\partial_t = a(L_{0-1} - L_{10})\Big|_{z=0} = K \Leftrightarrow a = R, \\
iI_{(2)}\Big|_{z=0} &= it\partial_t = -L_{1-1}\Big|_{z=0} = D,
\end{aligned}
\tag{80}
$$

where a is a dimensional parameter introduced for the correct dimensions of H and K. We see that at the $z = 0$ boundary, the isometry generators of AdS_2 match the Hamiltonian, special conformal generator and dilatation operator provided that the AdS length scale R and the dimensionful parameter a are equal [3]. This identification is paramount to the understanding that the conformal ($SL(2,\mathbb{R})$) symmetry of conformal quantum mechanics, the isometries of AdS_2 and the Lorentz group ($SO(1,2)$) are isomorphic to one another, locally. Indeed the three algebras (28), (78) and

$$[I_{(1)}, I_{(2)}] = I_{(1)}, \qquad [I_{(1)}, I_{(3)}] = 2I_{(2)}, \qquad [I_{(2)}, I_{(3)}] = I_{(3)}, \tag{81}$$

are consistent with one another based on the identification (80). This reproduces the $so(1,2)$ algebra defined in (31) where[22]

$$\begin{aligned} L_{0-1} &= \frac{1}{2}\left(aH + \frac{K}{a}\right) = l_3, \\ L_{10} &= \frac{1}{2}\left(aH - \frac{K}{a}\right) = -l_1, \\ L_{1-1} &= -D = -l_2. \end{aligned} \tag{82}$$

The fact that the $sl(2,\mathbb{R})$, $so(1,2)$ and isometry algebra of AdS_2 can be mapped to each other in this way exhibits a local isomorphism between the groups, at the level of the algebras. The symmetries of conformal quantum mechanics and AdS_2 have therefore been shown to match with one another- a feature that, in the very least, is expected for any example of the AdS/CFT correspondence. In the sections to follow we shall frequently make reference to this matching of symmetries for the case of AdS_2 and conformal quantum mechanics.

[22]The annoying negative signs in (82) can be removed by simply raising indices and recalling that $L_{0-1} = L^{0-1}$, $L_{10} = -L^{10}$ and $L_{1-1} = -L^{1-1}$ due to the metric signature $(++-)$.

3 AdS_2 in String Theory

AdS_2/CFT_1 is seemingly the simplest case of the correspondence, however it has proven to be resistant to all attempts at its realization so far. In fact, it is not yet clear whether the dual CFT should be a global $SL(2,\mathbb{R})$ theory or the chiral half of a 2 dimensional conformal field theory that arises through a dimensional reduction. AdS_2 usually appears in the near-horizon (NH) limit of higher dimensional black holes and the black holes of AdS_2 have unique thermodynamic properties. To further elucidate the possible AdS_2/CFT_1 correspondence we review the appearance of AdS_2 in the contexts of string theory, quantum gravity and conformal field theory. [23,24] This section follows closely the presentation of [4], [5], [6], [7], [8], [9] in subsections 3.1, 3.2, 3.3, 3.4, 3.4 and 3.5 respectively.

3.1 Vacuum States for AdS_2 Black Holes

3.1.1 AdS_2 black holes and vacua

Spradlin and Stominger have investigated the possibility of realizing an AdS_2/CFT_1 correspondence through the identification of $SL(2,\mathbb{R})$ invariant vacua of AdS_2 black holes [4]. These $SL(2,\mathbb{R})$ invariant vacua are used to calculate the conformal boundary correlation functions which, owing to the isomorphism between the isometry group of AdS_2, $SO(1,2)$, and the conformal group, $SL(2,\mathbb{R})$, of conformal quantum mechanics, relate to string theory in AdS_2. The AdS_2 spacetime, which appears as $AdS_2 \times S^2$ in the near-horizon limit of four dimensional extremal Reissner-Nordstrom (RN) black holes, has a 'preferred' Killing vector that generates translations in the asymptotically flat region of the RN geometry and it is the corresponding Killing horizon that defines the black hole horizon of AdS_2.[25] A particular coordinate transformation can eliminate the black hole temperature from the near-horizon metric. This temperature independence does not however hold up to quantum corrections. Central to these conclusions is the identification of the time Killing vectors for the stationary spacetime which is also necessary for the definition of the various vacua associated to the black holes (defined via the condition that positive frequency modes annihilate the vacuum). Therefore, various vacua are defined which include the $SL(2,\mathbb{R})$ ones namely: the global, Poincare and Hartle-Hawking vacua, which are shown to be equivalent, and the non-conformally invariant Schwarzschild vacuum, which is equivalent to the Boulware vacuum. Since these vacua are crucial to the results of Spradlin and Strominger we spell out the coordinate transformations that highlight how they relate to one another [4].

Consider the following magnetically charged Reissner-Nordstrom solution:

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega_{(2)}^2. \tag{83}$$

Excitations above the extreme solution are given by: $E = M - Q/L_p$ (where $L_p^2 = G$ is the Planck length) [6]. Typically: $r_\pm = GM \pm \sqrt{(GM)^2 - (Q^2G)}$ so we re-express the roots in terms of the excitation energy:

$$r_\pm = QL_p + EL_p^2 \pm \sqrt{2QEL_p^3 + E^2L_p^4}. \tag{84}$$

[23] Since the time of writing, more recent developments have occurred regarding AdS_2 and we have collected some of these developments in an appendix (see appendix F).

[24] The impatient reader, or one who is well acquainted with these topics, may wish to skip to subsection 3.6, on a first reading, where we have bulleted these results.

[25] Strictly speaking, one does not expect the familiar $4-d$ RN metric to appear in string theory. Rather, these black holes are qualitatively similar to the types of black holes that do appear in string theory and quantum gravity. One of the similarities is the appearance of AdS_2 in the near horizon limit of these black holes. We shall see examples of the emergence of AdS_2 in the near horizon limit for other black holes in string theory (see 3.4). We also point out some of the differences between the RN solution and charged string theory black holes in a footnote in 3.3.

The relationship between the Hawking temperature and the roots $r_\pm$ is determined by considering the surface gravity of (83):

$$\kappa = Va|_{r=r_+} \tag{85}$$
$$T_H = \frac{\kappa}{2\pi} \tag{86}$$

where a is the magnitude of acceleration at the Killing horizon as measured by an asymptotic observer and the corresponding redshift factor is V. The Killing field ∂_t associated with the Killing horizon r_+ is proportional to the static observer velocity: $k^\mu = Vu^\mu$. Then:

$$k^\mu = (1,0,0,0) \tag{87}$$
$$u^\mu = \left(\frac{r}{\sqrt{(r-r_+)(r-r_-)}},0,0,0\right)$$

$$V = \frac{k^0}{u^0} = \frac{\sqrt{(r-r_+)(r-r_-)}}{r} \tag{88}$$
$$a = \sqrt{a^\mu a_\mu} = \left[\nabla^\mu ln(V)\,\nabla_\mu\, ln(V)\right]^{1/2} \tag{89}$$
$$= \left[\left(\partial_r \frac{\sqrt{(r-r_+)(r-r_-)}}{r}\right)^2\right]^{1/2}$$

and by the entropy relation $S = A/4$ one obtains:

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2}, \qquad S = \frac{\pi r_+^2}{L_p^2}. \tag{90}$$

The energy temperature relation (for the near-extremal case) is $E \sim 2\pi^2 Q^3 T_H^2 L_p$. [4] then take the near-horizon limit ($L_p \to 0$), i.e. the low energy limit, where $E \to 0$ and $r_+ = QL_p$ while keeping Q and T_H fixed. By taking:

$$U \equiv \frac{r - r_+}{L_p^2} \tag{91}$$

$$dU^2 = \frac{dr^2}{L_P^4}, \tag{92}$$

the metric becomes

$$\frac{L_p^4 U\left(U + 4\pi Q^2 T_H\right)}{Q^2 L_p^2} = \frac{(r-r_+)(r-r_-)}{r^2} \tag{93}$$

$$\Rightarrow \frac{ds^2}{Q^2 L_p^2} = -\frac{U\left(U + 4\pi Q^2 T_H\right)}{Q^4}dt^2 + \frac{1}{U\left(U + 4\pi Q^2 T_H\right)}dU^2 + d\Omega_{(2)}^2. \tag{94}$$

The time coordinate in this metric is referred to as the Schwarzschild time and is the preferred choice of time mentioned earlier. It should be noted that the evolution of this time coordinate only covers part of one of the spatial boundaries (the left hand side of the AdS_2 strip) and the uncovered asymptotic boundary is considered to be spatial infinity. The corresponding Killing horizon prevents anything from reaching asymptotic infinity (in the case of the future horizon) and anything from approaching the spacetime from spatial infinity (in the case of the past horizon). Following [4], the dependence of (94) on temperature is removed by the coordinate change:

$$\begin{aligned} \pi T_H \left(t \pm \frac{1}{4\pi T_H} \ln\left(\frac{U}{U + 4\pi Q^2 T_H} \right) \right) &= \tanh^{-1}\left(t' \pm \frac{Q^2}{U'} \right) \\ &\equiv \tanh^{-1} y_\pm \end{aligned} \tag{95}$$

$$\begin{aligned} -\frac{U'^2}{Q^2} dt'^2 + \frac{Q^2}{U'^2} dU'^2 = \frac{4Q^2\pi^2 T_H^2 \text{sech}^2 y_+ \text{sech}^2 y_-}{(\text{tanh} y_+ - \text{tanh} y_-)^2} \times \\ \frac{Q^2}{U(U + 4\pi Q^2 T_H)} \left(\frac{ds^2}{L_p^2} - Q^2 d\Omega_{(2)}^2 \right). \end{aligned} \tag{96}$$

Making use of the identity:

$$\frac{(\text{tanh} y_+ - \text{tanh} y_-)^2}{\text{sech}^2 y_+ \text{sech}^2 y_-} = \sinh^2(y_+ - y_-) \tag{97}$$

the new metric has the form

$$\frac{ds^2}{Q^2 L_p^2} = -\frac{U'^2}{Q^4} dt'^2 + \frac{1}{U'^2} dU'^2 + d\Omega_{(2)}^2. \tag{98}$$

By a further coordinate transformation, $\tau \pm \sigma \pm \frac{\pi}{2} = 2\tan^{-1}\left(t' \pm \frac{Q^2}{U'} \right)$, one obtains AdS_2 with the S^2 compact manifold in the form of the Bertotti-Robinson metric (101) (i.e. a universe in constant magnetic field) [4]. That is, this coordinate transformation gives

$$-d\tau^2 + d\sigma^2 = 4\left(\frac{1}{y_+^2 + 1} \right)\left(\frac{1}{y_-^2 + 1} \right)\left(dt'^2 + \frac{Q^4}{U'^4} dU'^2 \right), \tag{99}$$

which together with the identity,

$$\cos^2\sigma = 4\frac{Q^4}{U'^2}\left(\frac{1}{y_-^2 + 1} \right), \tag{100}$$

leads to

$$\frac{ds^2}{Q^2 L_p^2} = \frac{-d\tau^2 + d\sigma^2}{\cos^2\sigma} + d\Omega_2^2. \tag{101}$$

The τ, σ coordinates are the universal cover (infinite strip) of AdS_2. Therefore, the vacuum states annihilated by positive frequency modes (defined with respect to τ for a scalar field) will be a global vacuum state. One can define equivalent vacuum states in Poincare coordinates (T, y- which cover a patch of AdS_2) via the following coordinate transformation [4]:[26]

[26] Here, y, is not to be confused with the variable $y_\pm$ defined in previous coordinate transformations (see (95)).

$$(T \pm y) = \tan\frac{1}{2}\left(\tau \pm \sigma \pm \frac{\pi}{2}\right) \tag{102}$$

so that the metric takes the form

$$-d\tau^2 + d\sigma^2 = \frac{4}{\left(1+(T+y)^2\right)}\frac{\left(-dT^2+dy^2\right)}{\left(1+(T-y)^2\right)} \tag{103}$$

$$\cos\sigma = \frac{2y}{\sqrt{(1+(T+y)^2)}\sqrt{(1+(T-y)^2)}} \tag{104}$$

$$\Rightarrow \frac{ds^2}{Q^2L_p^2} = \frac{-d\tau^2+d\sigma^2}{\cos^2\sigma} = \frac{-dT^2+dy^2}{y^2}. \tag{105}$$

The Poincare and global coordinate vacuum states are equivalent. This follows from considering the overlap of a positive frequency mode of a scalar field in Poincare coordinates with a positive (or negative) frequency scalar field in global coordinates: $\langle\phi^P_{+\omega}|\phi^G_n\rangle = i\int_0^\infty dy\left[\phi^P_{-\omega}\partial_T\phi^G_n - \phi^G_n\partial_T\phi^P_{+\omega}\right]|_{T=0}$. The Scalar fields have the form: $\phi^P_{+\omega} = (1/\sqrt{\pi\omega})\,e^{-i\omega T}\sin\omega y$ and $\phi^G_n = \left(1/\sqrt{\pi|n|}\right)e^{-in\tau}\sin n(\sigma+\pi/2)$. The overlap leads to the conclusion that the Bogoliubov transformation[27] is block diagonal and expressed in terms of the associated Laguerre polynomials (L^α_n) since $\langle\phi^P_{+\omega}|\phi^G_{+n}\rangle = (-1)^n\sqrt{(n\omega)}e^{-\omega}L^{-1}_n(2\omega)$ and $\langle\phi^P_{+\omega}|\phi^G_{-n}\rangle = 0$. Therefore, [4] show that the two vacua are equivalent:

$$|0\rangle_{global} = |0\rangle_{Poincare}. \tag{106}$$

One can also define a Schwarzschild vacuum (corresponding to time as defined in (94)) by making use of the coordinate transformation $x = \frac{1}{4\pi T_H}\ln\left(\frac{U}{U+4\pi Q^2T_H}\right)$. [28]

Under this transformation:

$$\frac{U(U+4\pi Q^2T_H)}{Q^4}dx^2 = \frac{1}{U(U+4\pi Q^2T_H}dU^2 \tag{107}$$

$$\sinh^2(2\pi T_Hx) = \frac{4\pi^2Q^4T_H^2}{U(U+4\pi Q^2T_H}, \tag{108}$$

and the metric becomes

$$\frac{ds^2}{Q^2L_p^2} = \left[\frac{2\pi T_H}{\sinh(2\pi T_Hx)}\right]^2(-dt^2+dx^2). \tag{109}$$

[27]The Bogoliubov transformation relates one set of creation and annihilation operators to another set. The relation may be such that the annihilation operators of one set are a linear combination of creation and annihilation operators of the other set which is consistent with the detection of particles by the observer whose theory is defined in terms of the second set of operators in the vacuum of the theory corresponding to the first set of operators. In the diagonal case, the two observers will agree on the absence of particles with respect to each other's vacua. The Bogoliubov transformation is what characterizes the disagreement of observers who use different positive frequency modes to define their respective vacua, a phenomena that plagues the semi classical theory of quantum field theory in, fixed, curved backgrounds.

[28]This transformation which leads to a conformally flat form is distinct from the previous metric (94) however this coordinate transformation is time independent and thus positive frequency modes are preserved so that the Schwarzschild vacuum is invariant.

Observers at fixed U coordinates have a proper time proportional to t- Schwarzschild time. The vacuum state for such an observer in the Schwarzschild coordinates (109) is equivalent to the Boulware vacuum:

$$|0\rangle_{Schwarzschild} = |0\rangle_{Boulware}. \tag{110}$$

By taking into account that the Schwarzschild and global coordinates are related by $\tan \frac{1}{2}(\tau \pm \sigma) = \mp e^{\mp 2\pi T_H(t \pm x)}$, one can see that the global time coordinate is invariant under an imaginary translation of Schwarzschild time: $t \rightarrow t + \frac{i}{T_H}$. Therefore, observers at fixed U see particle production in the global vacuum and those traveling along the proper time worldline will experience a thermal bath of particles related to T_H. This defines the Hartle- Hawking vacuum. So one identifies: $|0\rangle_{global} = |0\rangle_{Hartle-Hawking}$. Therefore, the Schwarzschild and Boulware vacuum states are equivalent and distinct from the equivalent Poincare, Hartle-Hawking and global vacuum states [4].

3.1.2 Entropy and the logarithmic violation of decoupling in AdS_2

It was shown in (90) that the black hole entropy was classically temperature independent in the near-horizon limit: $S = \pi Q^2$. After taking into account quantum effects, [4] have found that the entropy of entanglement of the quantum states at asymptotic flat space of the RN black hole and the quantum states of the near-horizon AdS_2 spacetime for finite temperature include a temperature dependent correction term. This indicates that for non-zero temperature, the quantum states of asymptotic flat space fail to decouple from the AdS_2 states [4]. This may have implications for the boundary conformal field theory associated to the AdS_2 bulk in the elusive AdS_2/CFT_1 correspondence. For a quantum field theory, the typical procedure for computing entanglement entropy involves defining a constant time slice surface in the spacetime separating the two regions (A and B) under consideration. Then one traces out the degrees of freedom of B in region B- this provides the entanglement entropy for region A [17]. The result is typically divergent and dependent on a short distance scale (UV cut off). For the specific case of an even dimensional spacetime, there is a logarithmic dependence on the scale of the form $c \ln \Delta$ where c is the central charge and Δ is the cut-off. In the near-horizon limit, it is understood that a throat region develops in the spacetime between the near-horizon AdS_2 region and the asymptotically flat region [6]. One chooses the boundary such that it occurs somewhere in this mouth region. That is, the near-horizon AdS_2 region has U coordinate: $0 < U < U_{\max}$ with quantum states $|\Psi^i_{AdS}\rangle$ and the asymptotically flat region has $U_{\max} < U < \infty$ with states $|\Psi^I_{Flat}\rangle$. By defining $U_{\max} = K\frac{Q}{L_P}$ for some constant $K << 1$ one finds that in the near-horizon limit U_{max} will be located in the throat region. U_{max} must therefore be located deep within the AdS_2 region before taking the limit as $L_p \rightarrow 0 \Rightarrow U_{max} \rightarrow \infty$ but this is suppressed by small K. The density matrix on a given region is obtained by tracing out the states of the other region in the usual way: $\rho_{AdS_2} = Tr_{flat}|\Psi\rangle\langle\Psi|$ where the full state is of course the tensor product: $|\Psi\rangle = \sum_{\alpha,a} c_{a,\alpha}|\Psi^a_{AdS}\rangle|\Psi^\alpha_{flat}\rangle$. S_{ent} was calculated in [21] with:

$$S_{ent} = \frac{c}{6}\rho(\sigma_{max}) - \frac{c}{6}\ln\Delta. \tag{111}$$

$\rho(\sigma_{max})$ is the conformal factor associated to the spacetime at the boundary that defines the two regions under consideration. This computation is carried out for the Hartle-Hawking vacuum for which the conformal factor is $\rho = -\ln\cos\sigma$ which in the near-horizon limit has $\sigma_{max} \sim \frac{2\pi Q^2 T_H}{U_{max}}$. Ignoring temperature independent terms, we find that

$$S_{ent} = \frac{-c}{6}\ln(QT_H). \tag{112}$$

So for finite temperature there is a logarithmic violation of the decoupling of the asymptotic flat states from the near-horizon AdS_2 states [4].

3.1.3 Greens Functions

Greens functions for the Hartle-Hawking and Boulware vacua can be calculated exactly for massive fields in AdS_2 [4]. In general, the vacuum state is necessarily ambiguous for a scalar field in curved space as a result of the Bogoliubov transformation of solution mode functions for the wave equation. The positive frequency modes define the vacuum state of the theory and since a given set of positive frequency mode solutions are not favored over the Bogoliubov transformed set, one singles out a preferred vacuum as indicated by a specific observer (or detector). The Greens function depends on the mode functions and for this reason the Greens function is understood to completely specify the vacuum state.[29] As such, the choice of Greens function, of which there are several alternatives, is of particular importance. The Hadamard Greens function is a particularly attractive choice given its universal divergences which provide for a convenient regularization scheme known as point splitting regularization which is an invaluable technique for defining quantum stress tensors- to which we shall return shortly.[30]

The Hadamard Greens function[31] is given by[32]

$$G^1_H(x,y) = 2Re\int d\omega \phi^*_\omega(x)\phi_{\omega'}(y). \tag{113}$$

In the case of the massive Klein Gordon equation, the fields appearing are those solving the standard KG wave equation with the appropriate time corresponding to the vacuum under consideration. There is a subtlety regarding the normalization of the mode functions with respect to the Klein Gordon inner product in conformal gauge. The basis set $\{\phi_\omega(y)\}$ have oscillatory behavior which prevents their normalization. This can be fixed by requiring that they satisfy $\phi_\omega(y) \to \frac{1}{\sqrt{\pi\omega}}\sin(\omega y - \delta_\omega)$ when y approaches infinity. This leads to a satisfactory delta function normalization $\langle\phi_\omega|\phi_{\omega'}\rangle = 2\omega\int_0^\infty dy\phi^*_\omega\phi_{\omega'} = \delta(\omega-\omega')$.

The wave equation and normalized solutions for the massive (of mass m) positive frequency modes corresponding to the equivalent global and Poincare vacua are:

Global	Poincare
$\left(cos^2\sigma(\partial^2_\sigma - \partial^2_\tau) - m^2\right)\phi = 0,$	$\left(\partial^2_y + \omega^2 - \frac{m^2}{y^2}\right)\psi = 0$
$\phi_i = \Gamma(h)2^{2h-1}\sqrt{\frac{n!}{\pi\Gamma(2h+n)}}cos^h\sigma C^h_i(sin\sigma), i\in\mathbb{Z}^+,$	$\phi_\omega(T,y) = e^{-i\omega T}\sqrt{\frac{y}{2}}J_{h-1/2}(\omega y)$

where $\mathbb{Z}^+$ represents the non-negative integers, the mass has been redefined as $m^2 = h(h-1)$ and C^h_i is a Gegenbauer polynomial. The corresponding Hadamard function for the global vacuum expressed in terms of the hypergeometric function and the distance function $D_G = \frac{cos(\tau_1-\tau_2)-cos(\sigma_1-\sigma_2)}{cos\sigma_1 cos\sigma_2}$, which is $SL(2,\mathbb{R})$ invariant, on global AdS_2 is [4]:

$$G^1_H(\tau_1,\sigma_1;\tau_2,\sigma_2) = \frac{\Gamma(h)^2}{2\pi\Gamma(2h)}Re\left[\frac{2^h}{D^h_G}F(h,h;2h;-\frac{2}{D_G})\right]. \tag{114}$$

As is expected, given the equivalence of the vacua, the Poincare Hadamard function expressed in terms of the Poincare distance function $D_P = \frac{(y_1-y_2)^2-(T_1-T_2)^2}{2y_1y_2}$ is found to be identical. These solutions have the appropriate singular behavior at small separation anticipated by the definition of the Hadamard function. The massless Hadamard functions are obtained from these for $h = 1$ [4].

For the Boulware vacuum, the wave equation

$$\left(\partial^2_x + \omega^2 - \frac{m^2}{sinh^2 x}\right)e^{-i\omega t}\phi = 0,$$

[29] A thorough analysis of the Friedmann universe scalar field theory appears in [22] where many of these ideas are illustrated.

[30] A nice discussion of the Hadamard function appears in the text [23] and the introductory note [24]- both of which include some discussion of the point splitting regularization scheme.

[31] The Hadamard function is related to the familiar Feynman propagator by $G_H = 2ImG_F$ [4].

[32] We have used the standard notation for the unregularized form of the Hadamard function G^1.

is solved by the Legendre functions (P):

$$\phi(x,t) = \sqrt{\frac{\omega}{2}}\frac{\Gamma(i\omega+h)}{\Gamma(i\omega+1)}e^{-i\omega t}sinh^{0.5}xP^{-h+1/2}_{-i\omega-1/2}(coshx). \tag{115}$$

The Boulware Hadamard function is not solvable in terms of elementary functions in the massive case. It is given by

$$G^1_H(t_1,x_1;t_2,x_2)\sqrt{sinhx_1 sinhx_2}\int_0^\infty d\omega\omega\left|\frac{\Gamma(i\omega+h)}{\Gamma(i\omega+1)}\right|^2\cos\omega(t_1-t_2)P^{1/2-h}_{-i\omega-1/2}(\cosh x_1)P^{1/2-h}_{i\omega-1/2}(\cosh x_2). \tag{116}$$

The massless ($h=1$) restriction can be solved. An important point is that the distance function corresponding to the Boulware AdS_2 vacuum is not $SL(2,\mathbb{R})$ invariant which reiterates the earlier statement that the Boulware vacuum does not respect the $SL(2,\mathbb{R})$ symmetry. Although result (116) has implicit temperature dependence (it was considered for $2\pi T_H = 1$) ,when restored, it reduces to the global Hadamard function at $T_H = 0$. This is evidence for the appropriateness of considering T_H to be a measure of the failure of the Boulware vacuum to be $SL(2,\mathbb{R})$ invariant [4].

3.1.4 Stress Tensors

Since various observers detect different particle density distributions, the quantum stress tensor needs to be computed in the different vacua. The computation for the stress tensor for massless fields is simplified by a formula that relates the stress tensors corresponding to two different coordinate systems analogous to the coordinates of the Rindler and Minkowski coordinate systems in flat space. The case for massive scalars is not so straightforward given that there is no analogous equation relating the stress tensor in the two coordinate systems and one is required to perform the point splitting regularization, that was alluded to earlier, in the renormalization of the quantum stress tensor. The equation relating the stress tensor, which is normal ordered with respect to a Minkowski vacuum, to that which is normal ordered in the Rindler vacuum is[33]

$$T_{++}(A^+) = \left(\frac{\partial a^+}{\partial A^+}\right)^2 T_{++}(a^+) + \frac{1}{12\pi}\sqrt{\frac{\partial a^+}{\partial A^+}}\frac{\partial^2}{\partial A^{+2}}\sqrt{\frac{\partial A^+}{\partial a^+}}. \tag{117}$$

For null versions of the Schwarzschild and Poincare coordinate systems related by $2\pi T_H A^\pm = 2\pi T_H(t\pm x) = \ln(T\pm y) = \ln a^\pm$, (117) becomes [4]

$$T_{++}(A^+) = [2\pi T_H a^+]^2 T_{++}(a^+) + \frac{\pi T_H^2}{12}. \tag{118}$$

In the global and Boulware vacua one finds
$\langle T_{++}(A^+)\rangle_G = \frac{\pi T_H^2}{12}, \quad \langle T_{++}(a^+)\rangle_B = -\frac{1}{48a^{+2}}$.

Similarly, for the global null coordinates $\tau^\pm = 1/2(\tau\pm\sigma\pm\pi/2) = \tan^{-1}a^\pm$, the global vacuum leads to

$$\langle T_{++}(\tau^+)\rangle_G = \frac{-1}{12\pi}. \tag{119}$$

[33] $a^\pm$ are the lightcone Minkowski coordinates and the corresponding Rindler coordinates are $A^\pm$. The two coordinates are related by $\pm Ca^\pm = e^{\pm CA^\pm}$, where C is a constant [4].

The point splitting procedure for the massive fields is summarized as follows [4]. One considers a geodesic, that is necessarily non-null, passing through a given spacetime point. A second point, located at a proper distance $\delta > 0$ away from that point on the same geodesic and parameterized by δ, is additionally defined in terms of two functions: $x(\delta)^\mu = (\alpha^+(\delta), \alpha^-(\delta))$. The functions $\alpha^\pm$ are solved by the geodesic equations and take the form of a power series expansion in δ. The expectation value of the point split stress tensor, evaluated in conformal gauge, in a given vacuum are then defined to be

$$\langle T_{++}(x,\delta,\tilde{t^\mu})\rangle_{vac} = M_\delta M_{-\delta}\partial_{\alpha_1^+}\partial_{\alpha_2^+}\frac{1}{2}G^1_{vac}(x_1,x_2)\Big|_{x_1=x(\delta),x_2=x(-\delta)} \tag{120}$$

$$\langle T_{+-}(x,\delta,\tilde{t^\mu})\rangle_{vac} = \frac{-m^2}{2}g_{+-}\frac{1}{2}G^1_{vac}(x_1,x_2)\Big|_{x_1=x(\delta),x_2=x(-\delta)} \tag{121}$$

where $\tilde{t^\mu}$ is the tangent vector to the geodesic under consideration at the point x (i.e. at $\delta = 0$) and $M_\delta \equiv \left(\frac{d\alpha^+(0)}{d\delta}\right)^{-1}\frac{d\alpha^+(\delta)}{d\delta}$ which is required for the appropriate tensor transformation properties of the solution. The benefit of using the Hadamard functions becomes apparent when one considers its short distance behavior $G^1(\alpha_1^+,\alpha_1^-;\alpha_2^+,\alpha_2^-) = [-\frac{1}{2\pi}ln(\alpha_1^+-\alpha_2^+)(\alpha_1^- - \alpha_2^-) + \text{terms non-div as } x_2 \to x_1]$ since the solutions to the point split tensor take the form

$$\langle T_{\mu\nu}(x,\delta,\tilde{t^\mu})\rangle_{reg} = \frac{1}{8\pi}\left[\frac{\Sigma}{\delta^2} - 16\pi k_2(x)\right](g_{\mu\nu} - 2\Sigma\tilde{t_\mu}\tilde{t_\nu}) + \theta_{\mu\nu}(x) + \frac{m^2}{4\pi}g_{\mu\nu}(ln\delta + k_3(x)) + \mathcal{O}(\delta ln\delta) \tag{122}$$

where, $k_1(x), k_2(x)$ and $k_3(x)$ are functions that depend only on x, $\Sigma = \tilde{t^\mu}\tilde{t_\mu} = \pm 1$ and $\theta_{\mu\nu}(x)$ is a traceless tensor and its non-vanishing components are $\theta_{++} = \theta_{--} = k_1$. [4] (122) is the regularized point split stress tensor, however, the remarkable feature of this solution is that in the limit that the points coincide (i.e. $\delta \to 0$) all divergent pieces of the regularized tensor are universal and do not contain any information about the quantum state under consideration.[34] Therefore, one can define a renormalized quantum stress tensor by simply dropping all divergent terms (which depend on the direction $\tilde{t^\mu}$) provided one considers the difference between stress tensors calculated in the different vacua as such terms do not appear in the differences. The general renormalized tensor is therefore

$$\langle T_{\mu\nu}(x)\rangle_{ren} = g_{\mu\nu}\left[\frac{m^2}{4\pi}k_3(x) - 2k_2(x)\right] + \theta_{\mu\nu}(x). \tag{123}$$

In the null Poincare coordinates mentioned above, the stress tensor computed with respect to the global vacuum has $k_1 = 0$, $k_2 = \frac{1+3m^2}{48\pi}$ and $k_3 + \psi(h) + \gamma$.[35] The Renormalized stress tensor therefore has the form

$$\langle T_{\mu\nu}(a^+,a^-)\rangle_{G-ren} = \frac{1}{2\pi}g_{\mu\nu}\left(-\frac{1}{12} - \frac{m^2}{2}(\frac{1}{2} - \psi(h) - \gamma)\right). \tag{124}$$

Note that the form of the result (124) may have been predicted given that the metric is the only $SL(2,\mathbb{R})$ invariant two index tensor and consequently $\langle T_{\mu\nu}(a^+,a^-)\rangle_{G-ren}\ \alpha\ g_{\mu\nu}$. The massless case reduces to the Weyl anomaly

$$\langle T_{\mu\nu}\rangle = -\frac{1}{24\pi}g_{\mu\nu} = \frac{R}{48\pi}g_{\mu\nu} \tag{125}$$

[34] All such information is contained in the functions k_1, k_2 and k_3.

[35] $\psi(z)$ is the Digamma function and γ is the Euler-Mascheroni constant.

with the AdS_2 curvature scalar $R = -2$.

The Boulware vacuum is not expected to have a similar form to the global case since the vacuum fails to satisfy the $SL(2, \mathbb{R})$ symmetry possessed by the global vacuum. It proves convenient to consider the difference

$$\langle T_{\mu\nu} \rangle' = \langle T_{\mu\nu} \rangle_G - \langle T_{\mu\nu} \rangle_B \tag{126}$$

and given that $\langle T_{++} \rangle_G = 0$ one finds that $\langle T_{++} \rangle' = -\langle T_{++} \rangle_B$. The result for arbitrary value of h is then conjectured to take the form:

$$\begin{aligned} \frac{1}{\pi T_H^2} \langle T_{++} \rangle' &= \frac{1}{12} - \frac{m^2}{4\sinh^2 2\pi T_H x} \left(1 - m^2 \int_0^{2\pi T_H x} \frac{dv}{v} \frac{\sinh^2 v}{\sinh^2 2\pi T_H x} F(h+1, 2-h, 3, \frac{\sinh^2 v}{\sinh^2 2\pi T_H x}) \right) \\ \langle T_{+-} \rangle' &= \frac{1}{4\pi} m^2 g_{+-} \left(\psi(h) + \gamma - \int_0^{2\pi T_H x} dv (\coth v - \frac{1}{v} F(h, 1-h, 1, \frac{\sinh^2 v}{\sinh^2 2\pi T_H x})) \right) \end{aligned} \tag{127}$$

where again $m^2 = h(h-1)$ [4].

3.1.5 Boundary correlators

The $SO(1, 2)$ isometry group of the AdS_2 bulk, by the AdS/CFT correspondence, guarantees that the boundary quantum mechanics is conformal and one can make use of the bulk Hadamard Greens functions to define boundary correlation functions for the local operators in the boundary theory. As is familiar in the case of, for example the Poincare patch, the bulk coordinates approach the AdS boundary as $y \to 0$. The remaining coordinate simply specifies the location on the boundary and is taken to parameterize the boundary theory. The boundary theory of AdS_2 is then naturally a quantum mechanics theory being parameterized by time and possessing conformal symmetry. The bulk-to-boundary propagator is defined in terms of the Hadamard Greens function with the appropriate bulk coordinate dependence as the boundary is approached. The approximate form of the bulk-to-boundary propagator is [25] [4]

$$K_{vac}(y, t; t') = \lim_{y' \to 0} [y']^{-h} G_{vac}(y, t; y', t'), \tag{128}$$

where h is the conformal dimension of the corresponding boundary theory operators (and is related to the mass of the bulk scalar field as mentioned before). The subscript 'vac' is to indicate that the propagator necessarily depends on the choice of vacuum. Given some boundary field configuration $\phi_b(t')$, one can make use of the bulk-to-boundary (b-t-b) propagator to extend the field into the bulk where the fact that the b-t-b propagator satisfies the bulk equation of motion in the bulk coordinates, means that the extended field will satisfy the bulk equations of motion.[36] The b-t-b propagator has the limiting boundary behavior of $K_{vac} \to y^{-h+1}\delta(t - t')$ and substituting the extended fields (defined in terms of the limiting b-t-b propagator) into the Klein Gordon action one finds

$$S = \frac{1}{2} \int dt \int dt' \phi_b(t) \phi_b(t') \left(\lim_{y \to 0} y^{-h+1} \partial_y K_{vac}(y, t; t') \right). \tag{129}$$

The derivative in the action arises out of an integration by parts. This action defines a generating function for the boundary conformal operators $\mathcal{O}_h$ sourced by the boundary functions $\phi_b(t')$. The Boundary correlation functions take the form [26] [4]

[36] This definition of the extended field has the usual Greens function integral form in terms of the bulk-to-boundary propagator K.

$$\langle \mathcal{O}_h(t)\mathcal{O}_h(t'))\rangle = \lim_{y,y'\to 0}(y')^{-h}y^{-h+1}\partial_y G_{vac}(y,t;y',t'). \tag{130}$$

The Poincare b-t-b propagator is determined by substitution of the global Hadamard function computed in Poincare coordinates and takes the form

$$K(y,T_1;T_2) = \frac{y^h}{[y^2-(T_1-T_2)^2]^h}. \tag{131}$$

By a coordinate transformation from Poincare time to Schwarzschild time via $2\pi T_H t = lnT$ together with a conformal transformation of the boundary operators, one can compare the two correlation functions

Poincare Schwarzschild

$\langle \mathcal{O}_h(T)\mathcal{O}_h(0)\rangle_G = \frac{1}{T^{2h}}, \quad \langle \mathcal{O}_h(t)\mathcal{O}_h(0)\rangle_G = \left(\frac{T_H}{\sinh \pi T_H t}\right)^{2h},$

from which periodicity of $1/T_H$ in imaginary Schwarzschild time indicates that the state is thermal with the Hawking temperature. The Boulware case involves a technical calculation that is not very informative but leads to the approximate form

$$\langle \mathcal{O}_h(t)\mathcal{O}_h(0)\rangle_B = \left(\frac{T_H}{\sinh \pi T_H t}\right)^{2h}\Bigg|_{singular} \tag{132}$$

which should be restricted to singular contributions in the Maclaurin expansion in t.[37] This is almost (up to non-singular terms) identical to the global case [4].

3.2 AdS$_2$/CFT$_1$ from type 0A strings

The quantum mechanics of a single Hermitian matrix has been argued to provide a holographic dual description of two dimensional type 0 string theories [27].[38] More precisely, using the map to a system of free fermions [10], the holographic dual for the 2 dimensional Type 0 theory is a system of free fermions moving in the potential $V(r) = \frac{-r^2}{4\alpha'} + \frac{q^2}{2r^2}$. For the case of an extremal black hole, the low energy effective action of the Type 0A theory with q units of RR flux and setting $\alpha' = 1$ is [27]

$$S = \int d^2x\sqrt{-g}\left[e^{-2\Phi}\left(8+R+4\bigtriangledown\Phi\cdot\bigtriangledown\Phi - \frac{1}{2}\bigtriangledown T\cdot\bigtriangledown T + T^2\right) - \frac{1}{2}q^2 - q^2T^2 + ...\right]. \tag{133}$$

For the static case, and taking $T=0$, the solution is found to be the linear dilaton background in the asymptotic region and the near horizon limit reduces to the $SL(2,\mathbb{R})$ invariant AdS_2 spacetime in Poincare coordinates

$$\begin{aligned} ds^2 &= \frac{(-dt^2+d\sigma^2)}{4\sigma^2}, \\ \Phi &= \Phi_0, \qquad\qquad \sigma\to\infty, \end{aligned} \tag{134}$$

[37]As an example for $h=2$, $\langle \mathcal{O}_h(t)\mathcal{O}_h(0)\rangle_B \sim \frac{1}{\pi^4 t^4} - \frac{2T_H^2}{3\pi^2 t^2} + \frac{11T_H^4}{45}$ [4].

[38]For a brief review of the modern interpretation of the matrix model description of two dimensional Type 0 strings see sub-subsection G.1.3 of appendix G.

which is of more interest for the AdS_2/CFT_1 conjecture of [5].[39] (134) is regarded as the AdS_2 solution of the Type 0A theory at weak string coupling- the string coupling is $e^{\Phi_0} = \frac{4}{q}$ which is weak at large q. As mentioned above, the matrix model describes a non-interacting fermion theory with common potential

$$V(r) = -\frac{r^2}{4} + \frac{q^2}{2r^2} \tag{135}$$

where r is the radial coordinate and q is proportional to the $D0$ brane flux of the string theory. This potential can be associated with the linear dilaton background when $q \ll r^2$ and the potential is dominated by $-r^2/4$. The $r \ll q$ region however is associated with the AdS_2 solution of the string theory in which case $V(r)$ reduces to

$$V(r) = \frac{q^2}{2r^2}. \tag{136}$$

The Hamiltonian operator that acts on spherically symmetric wave functions (restricted to the singlet sector) is, in second quantized form, given by

$$H = \frac{1}{2}\int d^2x \Psi^\dagger \left(p^2 + \frac{q^2}{r^2}\right)\Psi \tag{137}$$

where $p = -i\partial_r$. The remaining two generators of the $SL(2,\mathbb{R})$ symmetry are:

$$\begin{aligned} K &= \frac{1}{2}\int d^2x \Psi^\dagger r^2 \Psi, \\ D &= \frac{1}{2}\int d^2x \Psi^\dagger \left(rp + pr\right)\Psi. \end{aligned} \tag{138}$$

Motivation is then provided for an $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence by relating the isometries of AdS_2 to the $SL(2,\mathbb{R})$ conformal group of the one dimensional conformal quantum mechanics matrix model [5]. The isometry generators of AdS_2 in Poincare coordinates were computed in (73) and (74). We restate them with a slight change in the notation for convenience [5]:

$$\begin{aligned} X^\mu_{(1)} &= (1,0) & X_{(1)} &= \partial_t \\ X^\mu_{(2)} &= (-2t,-2\sigma) & X_{(2)} &= -2\left(t\partial_t + \sigma\partial_\sigma\right) \\ X^\mu_{(3)} &= \left(t^2+\sigma^2, 2t\sigma\right) & X_{(3)} &= 2t\sigma\partial_\sigma + t^2\partial_t + \sigma^2\partial_\sigma. \end{aligned} \tag{139}$$

If we define the generators[40]

$$H = i\partial_t, \qquad K = i\left(2t\sigma\partial_\sigma + t^2\partial_t + \sigma^2\partial_\sigma\right), \qquad D = -2i\left(\sigma\partial_\sigma + t\partial_t\right), \tag{140}$$

then the commutators of these generators of the $SO(1,2)$ group match the $SL(2,\mathbb{R})$ group thats algebra is closed by (137) and (138)[41]

$$[H,D] = -2iH, \qquad [H,K] = -iD, \qquad [D,K] = -2iK. \tag{141}$$

[39] As we saw in the previous section (see (112)), the asymptotically flat region and the AdS_2 near horizon region do not necessarily decouple completely. So it is not clear if it is entirely appropriate to consider the linear dilaton and AdS_2 regions as distinct however the conjecture of [5] proceeds as if this is the case.

[40] Here, i is introduced so that the Lie group representation is unitary.

[41] There is a slight difference between this algebra and that of (28) that is solely due to the difference in the definition of the $SL(2,\mathbb{R})$ generators in (137) and (138).

The matching of these symmetries is what provides the motivation for this AdS_2/CFT_1 correspondence that conjectures that (137) describes Type 0A string theory on AdS_2 [5].

To focus on the choice of time coordinate in the matrix mode, one performs a change of coordinates to the global coordinates, related to t and σ by [5]

$$\tau \pm w = 2 \arctan (t \pm \sigma). \tag{142}$$

Then

$$\begin{aligned} -dt^2 + d\sigma^2 &= \frac{1}{4}(-\sec^2 \frac{1}{2}(\tau + w) \sec^2 \frac{1}{2}(\tau - w) d\tau^2 + \\ &\sec^2 \frac{1}{2}(\tau + w) \sec^2 \frac{1}{2}(\tau - w)) \end{aligned} \tag{143}$$

$$4\sigma^2 = \sin^2(w) sec^2 \frac{1}{2}(\tau + w) \sec^2 \frac{1}{2}(\tau - w) \tag{144}$$

where use was made of the identity $\tan(x+y) - \tan(x-y) = \sin(2y) \sec(x-y) \sec(x+y)$. The resulting metric is found to be

$$ds^2 = \frac{-d\tau^2 + dw^2}{4 \sin^2 w}. \tag{145}$$

The generators of global time translations, $\partial_\tau = \frac{\partial t}{\partial \tau}\partial_t + \frac{\partial \sigma}{\partial \tau}\partial_\sigma$, are

$$\begin{aligned} \partial_\tau = &\frac{1}{4}\left[\sec^2 \frac{1}{2}(\tau + w) + \sec^2 \frac{1}{2}(\tau - w)\right] \partial_t \\ + &\frac{1}{4}\left[\sec^2 (\tau + w) - \sec^2 \frac{1}{2}(\tau - w)\right] \partial_\sigma. \end{aligned} \tag{146}$$

It is straightforward then to confirm that

$$\frac{1}{2}(H + K) = i\partial_\tau \equiv L_o, \tag{147}$$

since

$$-iL_o = \frac{1}{2}\left(1 + t^2 + \sigma^2\right) \partial_t + \sigma t \partial_\sigma \tag{148}$$

is identical to (147). The new Hamiltonian associated with this generator in the matrix model is $H = \frac{1}{2}(p^2 + \frac{q^2}{r^2} + r^2)$. The correct interpretation of the original Hamiltonian is that it generates time translations in the Poincare time coordinate whereas the new generator L_0 generates time translations in the global time coordinate. This is natural for the gravitational perspective for which there are a vast number of time slices. Therefore, the different 'Hamiltonians', of which we would consider L_0 to be a new Hamiltonian, describe evolution along different time slices. This is a conceptually pleasing notion that would not be apparent in the matrix model perspective where the motivation for choosing L_0 over H is a result of the discrete spectrum of L_0, in contrast to the continuous spectrum of H [5].

By taking the generators to be

$$L_o = \frac{1}{2}(H+K), \qquad L_{\pm 1} = \frac{1}{2}(H - K \mp iD), \tag{149}$$

we see that they satisfy the $so(1,2)$ algebra as expected [5]:

$$[L_o, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_{+1}, L_{-1}] = 2L_o. \tag{150}$$

3.3 Anti-de Sitter Fragmentation

AdS_2 has been shown by Maldacena, Michelson and Strominger (MMS) [6] to fragment into disconnected AdS_2 universes- we turn to their work in this subsection. It is well known that the near-horizon geometry of extremal black holes is AdS_2 with a compact manifold (X): AdS_2xX. It has been shown in particular that for the case of a Reissner-Nordstrom, this near-horizon geometry is AdS_2xS^2 (see (101)). However the most basic approach to taking the near-horizon limit leads to configurations in which the excitation energy of the AdS space is zero [6]. In fact there are various inequivalent ways of taking the near-horizon limit ($M_p \to \infty$ or equivalently $L_p \to 0$ for the Planck mass and length). These inequivalent configurations are a consequence of the impossibility of keeping all three parameters: the black hole temperature, energy and charge, fixed in approaching the horizon. The simplest case of the near-horizon limit (describing only zero energy states) indicates that there are configurations corresponding to multi-black holes that are asymptotically flat at infinity (with some total charge) and in approaching the horizon, the AdS_2 space develops a throat region that branches into multiple AdS_2 regions with composite charges that collectively make up the total charge.
The generic form of the $4-d$ Reissner-Nordstrom black hole is given by (83) with horizon radii given by (84) and temperature and entropy given by (90). In the string theory context, the $2-d$ gravity theory is that of a charged dilaton gravity which is deduced from the $4-d$ charged dilaton model (of string theory) and is closely related to the the $4-d$ Reissner-Nordstrom solution since they are both solutions corresponding to charged black holes [28].[42,43] The Reissner-Nordstrom black hole is considered because of its close similarity to the charged dilaton and its similar qualitative behavior. It was found that the charged dilaton theory exhibits strange features related to its thermodynamics. In contrast with the RN black hole, which has finite entropy at a temperature of zero in the extremal case, the dilaton theory entropy goes to zero for finite temperatures. In both cases, the semi-classical description of the thermodynamics become invalid near extremality [29]. The general interpretation of entropy is a measure of the number of accessible states of a system. It has been shown for the charged dilaton theory, that if that interpretation is to be applied then there is an effective thermodynamic mass gap that separates the ground state of the black hole from the first excited state [29]. Since the Reissner-Nordstrom black hole also has a mass gap in the extremal limit, it is a suitable near horizon AdS_2 black hole analogue for the string examples.

3.3.1 Near-horizon limits

The Reissner-Nordstrom metric (83) is considered in the limit that $L_p \to 0$ in a similar manner to the limit in section (3.1). In this instance, however, [6] take the energy to be zero ($E \sim 2\pi^2 Q^3 T_H^2 L_p \to 0$) and $T_H \to 0$. The horizon coordinate becomes $r_+ = QL_p$. With the coordinate U, as defined in (91), and the charge fixed one obtains the metric:

[42] The action is for the $2-d$ theory is given by $S = \frac{1}{4}\int d^2x\sqrt{-g}\left[e^{-2\phi}\left(R - F^2 + 2(\bigtriangledown\phi)^2\right) + \frac{2}{L_p}\right]$ [6].

[43] There are however, appreciable differences which include the fact that these two black holes become naked singularities at different values of their charge and they become extremal at different values of their charge [28].

$$\frac{ds^2}{L_p^2} = -\frac{U^2}{Q^2}dt^2 + \frac{Q^2}{U^2}dU^2 + Q^2 d\Omega^2_{(2)}. \tag{151}$$

A null coordinate change

$$u^{\pm} = \arctan(t \pm \frac{Q^2}{U}) \tag{152}$$

leads to the following $AdS_2 \times S^2$ metric

$$\frac{ds^2}{L_p^2} = -\frac{4Q^2 du^+ du^-}{\sin^2(u^+ - u^-)} + Q^2 d\Omega^2_{(2)}, \tag{153}$$

which has time-like boundaries located at $u^+ = u^-$ (outside the horizon) and $u^+ = u^- + \pi$ (inside the horizon) [6].

There are four alternative limits [6]:

1. $L_p \to 0$ with E and Q held fixed
   In this case the temperature diverges and it is not clear whether or not the theory is physical.

2. $L_p \to 0$ with T_H and Q held fixed
   In this instance, the near-horizon limit is taken as it was in (95) to eliminate temperature dependence:

$$\frac{ds^2}{L_P^2} = -\frac{U^2}{Q^2}dt^2 + \frac{Q^2}{U^2}dU^2 + Q^2 d\Omega^2_{(2)}. \tag{154}$$

   It is well known that if one considers the quantum case, then there is Hawking radiation and the possible vacuum states are dependent on the choice of Killing time [4]. As a consequence, at the quantum level there is a non-zero stress-energy associated with the black hole. In the $2-d$ charged dilaton gravity

$$S = \frac{1}{4}\int d^2x\sqrt{-g}\left[e^{-2\phi}\left(R - F^2 + 2(\nabla\phi)^2\right) + \frac{2}{L_p}\right] \tag{155}$$

   ($e^{-2\phi}$ is related to the volume of the S^2 factor and $F = dA$ is the electromagnetic 2-form) it becomes evident that it is not possible to have finite excitation energies without singularities at the classical level. In conformal gauge ($g_{+-} = g_{-+} = -\frac{1}{2}e^{2\rho}$, $g_{++} = g_{--} = 0$), the action (155) leads to the constraint equation (for g_{++}): $-2e^{-\phi}\nabla_+\nabla_- e^{-\phi} = T_{++} \geq 0$. If this equation is integrated over the AdS_2 region with $u^- = 0$ and limits $0 \to \pi$ with the integration measure $du^+ e^{\phi-\rho}$, then the inequality becomes

$$e^{-2\rho}\partial_+ e^{-\phi}|_{u^+=0} - e^{-2\rho}\partial_+ e^{-\phi}|_{u^+=\pi} = \frac{1}{2}\int du^+ e^{\phi-2\rho}T_{++} \geq 0. \tag{156}$$

   This solution only has asymptotics of $AdS_2 x S^2$ for non-zero T_{++} as $e^{-\phi}$ in (156) has to diverge at one of the AdS_2 boundaries if $T_{++} \neq 0$. The energy vanishes for the non-singular spacetime, however, it may have significant degeneracy in its ground state which may be related to its entropy (if the entropy is to be interpreted as the number of accessible states). This case has yet to be understood with regard to an appropriate description of the degenerate lowest state.

3. $L_p \to 0$ with E and T_H held fixed (requiring $Q \to \infty$)
   Since Q is required to diverge, this case corresponds to the large N limit. The expressions for the energy near the extremal limit $E \sim 2\pi^2 Q^3 T_H^2 L_P$ and the gap energy (mentioned in the introductory paragraph of section (3.3)) $E_g = 1/(Q^3 L_p)$ remain fixed when $Q \sim L_p^{1/3}$ and $L_p \to 0$. Then the metric becomes

$$\frac{(E_g L_p)^{2/3} ds^2}{Q^2 L_p^2} = -V(V + 4\pi T_H)dt^2 + \frac{1}{V(V+4\pi T_H)} dV^2 + d\Omega^2_{(2)} \tag{157}$$

   where V was defined:

$$V \equiv \frac{r - r_+}{Q^2 L_p^2}. \tag{158}$$

   Since the dilaton term $e^{-\phi}$ is proportional to Q, it diverges in the $L_p \to 0$ limit.[44] (156) then diverges on the left hand side at finite values of the right hand side because the energy is fixed in this limit.

4. E, Q and T_H held fixed for finite but small L_p
   In this limit, one introduces an infrared cut-off in the AdS_2xS^2 theory with a corresponding ultraviolet cut-off in the CFT which would be dual to that theory. By injecting energy into the theory, the dilaton is made to grow (outlined in limit 2).It is possible to choose the cut-off in such a way that the cut off AdS_2 and CFT_1 should match for small energies. The important point is that there should be a duality describing the low energy part of these regulated theories.

3.3.2 Multi-black hole configuration

There are alternative physical (zero energy) solutions to the limit 2 (above). The alternatives are Reissner-Nordstrom multi-black hole solutions [6]

$$ds^2 = -V^{-2}dt^2 + V^2 d\vec{x}\cdot d\vec{x}, \tag{159}$$

$$*F = \frac{1}{L_p} dt \wedge \Lambda dV^{-1}, \tag{160}$$

$$V = 1 + \frac{Q_1 L_p}{|\vec{x} - \vec{x_1}|} + \frac{Q_2 L_p}{|\vec{x} - \vec{x_2}|}. \tag{161}$$

Defining $\vec{U} \equiv \vec{x}/L_p^2$ and similarly for $\vec{U_1}$ and $\vec{U_2}$. In the near-horizon limit

$$V \to \frac{Q_1}{|\vec{U} - \vec{U_1}|} + \frac{Q_2}{|\vec{U} - \vec{U_2}|}, \tag{162}$$

$$*F = dt \wedge dV^{-1}. \tag{163}$$

For (159) in the limit that $\vec{x} \to \infty$, $V \to 1$ so the asymptotic limit is Minkowskian with total charge $Q = Q_1 + Q_2$. As finite $\vec{x}$ is approached, a throat region develops which branches and (162) is attained corresponding to two AdS_2xS^2 black hole regions. In the near-horizon limit, the throat region grows to

[44] In the four dimensional case the black hole solution of the charged dilaton gravity is $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(\frac{1}{1-\frac{2M}{r}}\right)dr^2 + r\left(r - \frac{Q^2 e^{2\phi_o}}{M}\right)d\Omega$ with $e^{-2\phi} = e^{-2\phi_o} - \frac{Q^2}{Mr}$ and $F = Q\sin\theta d\theta \wedge d\phi$ where ϕ_o is the asymptotic constant dilaton value [28].

infinity and the Minkowski region becomes decoupled from the two black holes. The dynamics of these black holes can be described by an action that is deduced from [30] and is given by [6]:

$$S_{12} = \frac{1}{2}\left(Q_1^3 Q_2 - Q_1 Q_2^3\right)\int dt \frac{\left(\partial_t \vec{U_{12}}\right)^2}{|\vec{U_{12}}|^3}. \tag{164}$$

U_{ij} is a collective coordinate describing the separations $\vec{U_i}-\vec{U_j}$. There is a flat geometry associated with this effective action and it possesses a singularity of the conical type. The volume of the moduli space for a multi black hole system is characterized by the separation of the black holes and therefore by the limits of the collective coordinate. In the limit that the collective coordinate goes to zero, the space corresponds to the asymptotically flat part of the geometry whereas the limit in which the black holes are separated to a vast extent corresponds to the conical singularity. The volume of the moduli space becomes infinite in the first limit and becomes small in the second limit. This behavior of the moduli space is important for the proposal for a dual one dimensional conformal quantum mechanics made by Maldacena, Michelson and Strominger- a discussion of this appears in 3.3.4 below [6].

3.3.3 Partitioning of AdS_2 spacetime

An equally important and related feature of AdS_2 black hole spaces is the occurrence of the partitioning of a single AdS_2 universe into multiple AdS_2 universes. While the AdS_2 spaces of interest would arise in the context of string theory, it is instructive to focus on the extremal Reissner Nordstrom solutions discussed in detail so far. The partitioning of the $AdS_2 \times S^2$ spacetime of charge Q into, for example, two $AdS_2 \times S^2$ spacetimes, with charges Q_1 and Q_2 respectively,[45] occurs in two instances. In the 'zerobrane' limit ($Q_1 \ll Q_2$), the spacetime splits from a macroscopic AdS_2 universe into a macroscopic $AdS_2 \times S^2$ universe with charge Q_2 and a microscopic $AdS_2 \times S^2$ universe with charge Q_1. In this limit, the Q_1 black hole has the interpretation of being a $0-brane$ with BPS charge which describes geodesic trajectories in the $AdS_2 \times S^2$. The specific nature of the partitioning of the universe is then understood to occur through the arrival of the charged geodesic at the asymptotic flat boundary region of the original $AdS_2 \times S^2$ space. In the second instance, the partitioning of the universe occurs through the splitting of the throat region in an instanton tunneling procedure [31]. This contrasts the $0-$brane limit as the macroscopic $AdS_2 \times S^2$ universe is cleaved into two macroscopic $AdS_2 \times S^2$ universes consistent with conservation of charge [6].

The $0-$brane approximation leads to the expectation that the CFT_1 dual to AdS_2 in the near-horizon limit of a two black hole space will only be well defined on the AdS_2 boundary on the outside of the horizon. To understand this, one considers the $0-$brane action [6]:

$$S_2 = \frac{1}{4}\int d^2x\sqrt{-g}\left[(e^{-2\phi}R - Fe^{-2\phi}F^2 + 2(\bigtriangledown^2\phi)^2 e^{-2\phi})\right] + \int d^2x\sqrt{-g}\frac{1}{2L_p^2} + \oint dx\sqrt{h}\frac{2^{-2\phi}}{2}K_{tr} + \frac{Q_1}{L_p}\int A - \frac{Q_1}{L_p}\int ds \tag{165}$$

where the field ϕ is related to S^2, K_{tr} is the trace of the extrinsic coordinate and h is the induced metric on the boundary of the AdS_2 space. By averaging $\vec{x_1}$ over the S^2 in (162) and transforming to Eddington-Finkelstein coordinates (for a full extension),[46] one finds the metric

$$\frac{ds^2}{L_p^2} = \frac{-U^2}{Q_2^2 h^2}d\tilde{t}^2 + [2 - \frac{2U^2}{Q_2^2 h^2}]d\tilde{t}dU + (2 - \frac{U^2}{Q_2^2 h^2})dU^2 \tag{166}$$

[45] The total charge of the first $AdS_2 \times S^2$ spacetime is $Q = Q_1 + Q_2$.

[46] The precise coordinate transformation has the form $d\tilde{t} = dt - dU(1 - \frac{Q_2^2 h(U)^2}{U^2})$ [6].

with the factor $h(U) = 1 + \frac{Q_1}{Q_2}[\frac{1}{a}U\Theta(a-U) + \Theta(U-a)]$ which describes a 0−brane gas floating at a fixed distance a from the horizon. This metric has a horizon at $U = 0$ and a curvature singularity at $U = \frac{-aQ_2}{Q_1}$. This singularity corresponds to the inner boundary of the AdS_2 space.[47] The conclusion drawn is that a dual CFT_1 could only be defined on the outer boundary [6].
Another important feature of the 0−brane approximation is its representation as a charged particle that traces out a geodesic in the $AdS_2 \times S^2$ spacetime. This representation is applicable when the gravitational backreaction of the 0−brane on the background geometry is neglected. In this instance, the branching point of the $AdS_2 \times S^2$ throat actually recedes all the way back to the boundary and cleaves the space in two. A microscopic black hole with the charge of the 0− brane and a macroscopic black hole of the remaining charge result. In Euclidean signature, the geodesic worldline of the 0−brane in Poincare coordinates for AdS_2 is[48]

$$S = \int dt \frac{m}{y}\left[\sqrt{1+\dot{y}^2} - 1\right]. \tag{167}$$

There are two general solutions to this action given by the circle $(t-t')^2 + (y-r)^2 = r^2$ as well as the solution $y = y'$. The Lorentzian solutions can be obtained via the standard analytic continuation. The geodesic trajectories can be plotted and one finds, for the case of the Lorentzian strip solution $\cos(\tau - \tau') = \frac{\sin(\sigma_{max}/2-\sigma)}{\sigma_{max}/2}$, periodic evolution that extends to σ_{max} in the spatial direction, but most importantly reaches back to the boundary in finite time. The point being that this 0−brane makes contact with the boundary in finite time indicating that the branch in the throat can reach the boundary and cut the spacetime.[49]

The partitioning of the spacetime can be explicitly described through a topology changing instanton's tunneling solution. There are two different cases under which the tunneling takes place. The first case, which is the non-supersymmetric case, involves a 'bounce' solution for the instanton. In the second case, which is supersymmetric, instanton tunneling takes place between a pair of degenerate ground states. These two cases are distinguished by the brane to tension ratio: q.

In the non-supersymmetric case, one considers an AdS_2 space with a 2−form field strength.[50] This system admits pair production of a 0−brane- anti 0−brane pair. This is found by studying the bounce instanton but excising its trajectory at the 'moment of time symmetry' for which the Euclidean trajectory turns.[51] This can, for convenience, be chosen to be at $\tau = 0$. One then makes use of a test brane approximation where the charge of the brane is not taken into account- i.e. the background flux associated with the constant electric field is accounted for alone. This has the action

$$S = TR^{d-1}V_{d-2}\int d\tau \left(\sinh^{d-2}\rho\sqrt{\cosh^2\rho + \dot{\rho}^2} - q\sinh^{d-1}\rho\right) \tag{168}$$

[47] Recall that the near-horizon limit of an extremal RN black hole has the AdS_2 strip with an inherited horizon from the 4-d black hole. One of the AdS_2 boundaries lies behind the inherited horizon.

[48] While it is not explicitly derived in [6], this action can be obtained from the near-horizon limit of the RN black hole in isotropic coordinates. This metric has the form $ds^2 = -\left(\frac{r}{m}\right)^2 dt^2 + \left(\frac{m}{r}\right)^2 dr^2 + m^2 d\Omega_2^2$. By defining $\phi \equiv \frac{r}{m}$, changing coordinates to $\phi = \frac{m}{y}$ in the gauge field $A = \phi dt$ and dropping the S^2 contribution, which has radius m, one obtains $ds^2 = m^2\left(\frac{-dt^2+dy^2}{y^2}\right)$ [32]. The last two terms in the action (165) gives the desired action (167). The Euclidean version is obtained by wick rotation.

[49] While it is the Euclidean geodesic solution that is responsible for instanton tunneling, as we shall see shortly, the Lorentzian solution is relevant for post tunneling propagation.

[50] The results for the non-supersymmetric case in [6] are in fact valid for arbitrary dimensions.

[51] A detailed discussion of the quantum mechanical double well instanton, the quantum mechanical false vacuum decay instanton and the the Yang-mills instanton solutions can be found in either of the two references [33], [34].

in global coordinates and V_{d-2} is the volume of a $d-2$ sphere. The SUSY BPS bound requires that $q \leq 1$ so for the non-SUSY case we consider $q > 1$. The Euclidean instanton solution is independent of the spacetime and found to be [6]

$$\cosh\rho = \frac{\cosh\rho_{max}}{\cosh\tau} \tag{169}$$

with corresponding Lorentzian solution

$$\cosh\rho = \frac{\cosh\rho_{max}}{\cos\tau}. \tag{170}$$

The solutions match at $\tau = 0$. This matching is necessary to describe the time evolution of the 0−brane after it has tunneled. Figure 5 of [6] shows a plot of the evolution of the 0−brane which illustrates the evolution, after tunneling, to the $\rho \to \infty$ boundary in finite global time $\tau = \pi/2$.

String theory examples for the non-SUSY case ($q > 1$) are considered in [6]. These stem from studying $AdS_3 \times S^3 \times K3$ spacetime with Type IIB string on the $K3$. The possibility for the spacetime to partition (or 'fragment') in this context suggests that it is also possible for other spaces, such as AdS_2 arising in the NH limit of extremal RN black holes, to admit such vacuum instabilities. It is known that these RN solutions decay via a discharging process whereby the black hole emits electrons in a pair creation process near the horizon [35]. In fact the Energy of a spherical brane at a fixed radius

$$E = TR^{d-2}V_{d-2}\left(\sinh^{d-2}\rho\cosh\rho - q\sinh^{d-1}\rho\right) \tag{171}$$

confirms that only the two dimensional case ($d = 2$) is consistent with the saturation of the BPS bound $q = 1$. Since in the limit $E(\rho \to \infty)|_{d=2} = 0$ while $E(\rho \to \infty)|_{d>2} > 0$. The supersymmetric case $q = 1$ for AdS_2 has the solution (which follows from (165))

$$e^{\tau} = \cosh\rho \tag{172}$$

for its geodesic trajectory. This instanton describes, in the early time limit, a single macroscopic AdS_2 of charge $Q = Q_1 + Q_2$ and in the late time limit, it describes a macroscopic AdS_2 space of charge Q_2 and a microscopic 0−brane at the boundary with charge Q_1. Its instanton action solution is

$$S_{Inst} = \pi Q_1 Q_2 = -\frac{\Delta S_{BH}}{2}. \tag{173}$$

ΔS_{BH} represents the change in Bekenstein-Hawking entropy at early and late times. The factor of half appearing (173) is consistent with the tunneling amplitude $A \sim \left(e^{S_{inst}}\right)^2 = e^{-\Delta S_{BH}}$- a remarkable finding, which suggests that the instanton measures the number of black hole microstates for the AdS_2 black hole [6].

The decay of a single AdS_2 universe into two more, as noted, was studied first by Brill [31]. Brill considered the Einstein-Maxwell action with Euclidean metric signature:

$$S = -\int \frac{d^4x}{16\pi}\sqrt{g}[R - F^2] - \oint \frac{d^3x}{8\pi}\sqrt{h}K_{tr} \tag{174}$$

with the solution $ds^2 = \left(\frac{Q}{|\vec{x}|}\right)^2 d\vec{x}^2 + \left(\frac{Q}{|\vec{x}|}\right)^{-2} dv^2$, $*F = -dv \wedge dV^{-1}$ and Laplacian $\bigtriangledown^2 V(x) = 0$.[52] The magnetic charge can be split into two and distributed among two $AdS_2 \times S^2$ black holes with the obvious generalization (see (162)). To interpret the solution in the context of a tunneling instanton, hypersurfaces have to be defined in the throats. But, in contrast to the hypersurfaces of Brill, MMS require that the extrinsic curvature of the hypersurfaces vanish. (These hypersurfaces should correspond to spatial slices of $AdS_2 \times S^2$ with induced metrics consistent with zero extrinsic curvature). This requirement is necessary for a possible analytic continuation between Lorentzian and Euclidean solutions to match. One defines a new coordinate $y = \left[\sum_{i=1}^2 \frac{Q_i}{|\vec{x}-\vec{x}_i|}\right]^2$ and introduces a time coordinate that is appropriate for Euclidean time as the map from the half plane to the radial coordinates on the half plane:

$$e^{\tau - i\sigma} = v + iy, \qquad \sigma \in [0, \pi]. \tag{175}$$

This allows for the realization of hypersurfaces for which there is a single AdS_2 space of charge $Q = Q_1 + Q_2$ in the early time limit and two such universes of charges Q_1 and Q_2 at late times. The two spaces join at the points $\sigma = 0$ and $\sigma = \pi$ on the boundary. For an alternative analysis and conclusion see [36].[53] Brill's instanton solution, which is implicitly assumed to be the same as that of MMS, is found to be

$$A \sim |e^{-\frac{\Delta S_{BH}}{2}}|^2 = e^{-S_{BH}} = e^{\pi Q_1 Q_2}, \tag{176}$$

which matches the result of the microscopic and macroscopic daughter universes of the SUSY case (173). It is also important to note that, while it might be expected that the daughter universes would have different time definitions that prevent the comparison of Hamiltonians describing instantons, the daughter universes stem from the same asymptotically flat configuration and therefore inherit the same preferred time coordinate [6].

3.3.4 Comments on the CFT_1 dual to AdS_2 gravity.

The ground state vacuum AdS_2 trees, that arise in the branching procedures described above, represent various classical backgrounds of the conformal quantum mechanics that is dual to AdS_2 for the system. The quantum mechanical theory is therefore expected to explore, in a continuous manner, the various vacua. In the AdS_2 context, the 0-branes describing the splitting of the space may correspond to instantons in the Higgs branch. This would seem like a natural interpretation when considering, for example, a system of N D_3-branes and k $D_{-1}-$branes embedded in $10 - d$ Minkowski space with the end points of the strings holding the strings between the D_3-branes and the D_{-1} branes. To investigate the ground state solutions for the D_{-1} branes one considers the minimum of the potential. The potential for the D-brane system has two branches: the Higgs branch and the Coulomb branch. The Higgs branch, for which the F-terms and D-terms are constrained by $V = 0$, has the configuration in which the D_{-1}branes reside inside the D_3 branes, where they have the Yang-Mills interpretation of being instantons. The manifold defining the Higgs branch, which has all worldsheet fields in the direction transverse to the D_3 branes set to zero and quotiented with the $U(k)$ gauge symmetry, has the same dimension as the Moduli space of Yang-Mills. This appears to be consistent with one of the main claims of the ADHM construction that states that the the moduli space of Yang-Mills is isomorphic to the Higgs branch manifold [37] [38].[54]
Similarly, one may have an expectation of this sort for the $0-$branes in AdS_2 since the moduli space for these is finite (when taking the black hole separation to be large- see discussion below (164)). This suggests that the trees correspond to the Higgs branch but it cannot be ruled out that the Coulomb branch may

[52] $V = \frac{Q}{|\vec{x}|}$.

[53] [36] have a different result from MMS but perform explicit calculations that pertain to this discussion of the precise matching of the hypersurfaces.

[54] For a more general discussion of the Higgs and Coulomb branches associated with the Yang-Mills instantons see [38].

feature as well. Specifically, the trees could be the corners of the Higgs branch near the point where the coulomb branches meet [6].

3.4 AdS_2 Quantum gravity and String theory and its Central Charge

The AdS_2/CFT_1 correspondence suggests that, dual to a gravitational theory on AdS_2, one should expect a conformal quantum mechanics theory. As discussed, such a theory with the required $SL(2,\mathbb{R})$ symmetry exists [2]. However, another line of thought has developed in the context of quantum gravity that indicates that the isometry of AdS_2 is extended to the full local 2−d conformal group of diffeomorphisms. Additionally, investigations based on the reduction of five dimensional black holes to AdS_2 string theory, which passes through an AdS_3 theory, show that the AdS_2 symmetry comes from a single copy of the Virasoro algebra of diffeomorphisms from the right chiral half, $SL(2,\mathbb{R})_R$, of the AdS_3 theory. We briefly review some of these arguments and results for the AdS_2/CFT_1 correspondence that can be found in references [7] and [8]. For the full details one should consult these papers and references therein.

3.4.1 Extension of the global conformal group to the local group in AdS_2 quantum gravity

In conformal gauge, AdS_2 quantum gravity is described by Liouville theory where the conformal (or Weyl) factor is naturally associated with the Liouville field.[55] Two dimensional classical gravity, and in particular AdS_2, has descriptions in terms of Jackiw Teitelboim (JT) gravity, which in the stringy generalizations includes the dilaton, so possible dilaton couplings could appear as well. The Liouville theory has $2-d$ local conformal symmetry which on the AdS_2 strip has two boundaries. These two boundaries are a signal of open string-like qualities so that the theory has a diffeomorphism symmetry with a single Virasoro algebra. In addition, this Liouville theory is known to have a ground state that is destroyed by the action of the global $SL(2,\mathbb{R})$ subgroup of the Virasoro algebra [40] [41]. That is to say that the vacuum state is $SL(2,\mathbb{R})$ invariant. This invariance of the vacuum is consistent with the conformal factor [7]

$$e^{\phi} = \frac{l}{\sin(u^+ - u^-)}, \tag{177}$$

which in conformal gauge $ds^2 = -e^{2\phi}du^+du^-$, takes the form

$$ds^2 = -\frac{l^2}{\sin^2(u^+ - u^-)}du^+du^-. \tag{178}$$

This is immediately recognized to be AdS_2. The global sub-algebra, $sl(2,\mathbb{R})$, is understood to be the isometry group of AdS_2. Hence the global isometry of AdS_2 is lifted to the full local conformal group in 2−d with a Virasoro algebra. This is a general result for AdS_2 in the context of quantum gravity and string theory and as such is applicable to AdS_2 string theory regardless of whether it appears in the near-horizon limit of black holes that pass through AdS_3 or not [7]. This has an important consequence for understanding the equivalence of the thermodynamic mass gap and the conformal symmetry breaking scale for AdS_2 black holes [42].[56]

[55] The canonical structure of the Liouville theory is studied in [39] which includes a treatment of both the classical and quantum theory.

[56] We summarize the matching of the thermodynamic mass gap and the conformal symmetry breaking scale identified by [42] in an appendix (appendix F). [42] argue that this equivalence should be expected to exist when one identifies the conformal symmetry in the AdS_2/CFT_1 correspondence with the chiral half of a $2-d$ conformal theory. This appears to be natural for the $AdS_3 \to AdS_2$ reduction described below. However, they point out that in the absence of such a reduction it is not obvious how the CFT_1 should arise from the $2-d$ conformal field theory.

3.4.2 AdS_3 Reduction to AdS_2, Mass Gap and Twisted Stress Tensor

AdS_2 string theory can be obtained from a dimensional reduction from string theories in AdS_3. Black string solutions in Type-IIB string theory on the $K3$ surface that are characterized by D-branes admit, in the near-horizon limit, an $AdS_3 \times S_3 \times K3$ geometry. The compactification of these solutions gives an S^1 compactification of AdS_3. The so-called 'very-near-horizon' limit of the S^1 compactification of AdS_3 reduces to AdS_2. Several remarkable features associated with this dimensional reduction lead to interesting results of significance to 2 dimensional black hole physics as well as the AdS_2/CFT_1 correspondence. One such feature is related to the distinction between the 'very-near-horizon', which is accompanied by an infinite mass gap on AdS_2 solutions, and the near-horizon limit, which only provides Near AdS_2 (or near-extremal AdS_2) geometry. A second feature is the extension of the global $SL(2,\mathbb{R})$ symmetry to the full infinite order $(1+1)$ dimensional conformal group- this contrasts the similar case of AdS_3 which has 2 chiral copies of the global $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ transformations which are extended to the full local 2d conformal group [43]. Through the dimensional reduction, the AdS_2 case is identified with the $SL(2,\mathbb{R})_L$ quotient of AdS_3 so that the extension of the AdS_2 Virasoro group is related to the chiral half $SL(2,\mathbb{R})_R$ of the AdS_3 algebra, however the AdS_2 isometry is augmented by a gauge transformation. The connection between the two cases for the global isometries leads to the expectation that it should hold for the full conformal groups. The S^1 compactification of the AdS_3 that arises from the $II-B$ 5 dimensional black hole theory is given by the $2d$ metric [7]

$$ds^2 = -\frac{R^2U^4}{T^2l^6 + l^2U^2R^2}dt^2 + \frac{1}{U^2}l^2dU^2, \tag{179}$$

where R is the S^1 radius and l is the length related to the six dimensional string coupling. To obtain an AdS_2 metric, one takes the very near-horizon limit: $\lambda \to 0$ with $\lambda = U^2kR^2/l^4n$[57]:

$$ds^2 \to \frac{-l^2}{(t^+ - t^-)^2}dt^+dt^-. \tag{180}$$

The energy exceeding the extremal case $E_{above} \sim \lambda^2 n/R$ goes to zero in this limit and one finds that there are no possible excitations of the AdS_2 black hole solutions. This is further evidence for the black hole mass gap barrier for AdS_2 theories. To understand how the AdS_2 case corresponds to half of the chiral $SL(2,\mathbb{R})$ group of AdS_3, it is best to consider the Poincare form of the S^1 compactification of AdS_3 [7]:

$$ds^2 = \frac{l^2}{y^2}(dv^+dv^- + dy^2). \tag{181}$$

The S^1 compactification has $x^5 \sim x^5 + 2\pi$ and this is consistent with $v^+ \sim e^{4\pi T}v^+$, $v^- \sim v^- + 2\pi R$ and $y \sim ye^{2\pi T}$.[58] By performing a conformal transformation $v^\pm \to v^\pm/\lambda'$ and $y \to y/\lambda'$, one can take the very near-horizon limit $\lambda' \to 0$ because the horizon is at $y \to \infty$. In this limit, the transformed radius $\lambda' R \to 0$ and one finds that $v^- \sim v^- + 2\pi R$, mentioned in the relations before, becomes $v^- \sim v^-$. Hence the transformations are restricted to the left chiral half $SL(2,\mathbb{R})$ and so the near-horizon AdS_2 spacetime will be an $SL(2,\mathbb{R})$ quotient of the full AdS_3 conformal symmetry. The AdS_3 and AdS_2 isometry generators were compared in [7] and they were identified up to a gauge transformation in the S^1 compactified coordinate. It was concluded that the $SL(2,\mathbb{R})$ isometries of AdS_3 reduce to the $SL(2,\mathbb{R})$ isometries of AdS_2 with the addition of a gauge transformation. Under the dimensional reduction, the AdS_3 conformal transformations map to a twisted stress tensor for the 2-d conformal group of AdS_2. Hartman and Strominger [8] have shown that this twisted stress tensor leads to a central charge in Maxwell dilaton quantum gravity on

[57] n is related to the D_1 brane momentum density n/R^2.

[58] T is related to the temperature associated with the left chiral modes in the full conformal field theory.

AdS_2.
The metric (181) can in fact be mapped to the global form of (178) by appropriate coordinate definitions. In two dimensions, Einstein gravity for AdS_2 with a constant electric field is described by the action (in the notation of [8])[59]

$$S = \frac{1}{2\pi}\int dt^2\sqrt{-g}\left[\eta(R+\frac{8}{l^2}) - \frac{2f^2}{l^2} + f\epsilon^{\mu\nu}F_{\mu\nu}\right] + S_{matter}. \tag{182}$$

f is an auxiliary field that is included to make sure that terms quadratic in the gauge field are not present and η is the dilaton field (- a Lagrange multiplier that constrains the curvature to be negative). This theory has the AdS_2 metric vacuum solution (180) and in particular, the electric field and gauge potential are given by $F_{+-} = 2E\epsilon_{+-}$ and $A_\pm = El^2/4\sigma$ respectively. By taking conformal gauge $ds^2 = -e^{2\phi}dt^+dt^-$ and Lorentz gauge $\partial_+A_- + \partial_-A_+ = 0$ the action takes a more obvious CFT form [8]

$$S = \frac{1}{2\pi}\int dt^2\left[-4\partial_-\eta\partial_+\phi + 4\partial_-f\partial_+b + \frac{4e^{2\phi}}{4}\eta - \frac{1}{l^2}e^{2\phi}f^2\right] + S_{matter}, \tag{183}$$

where b is a scalar that determines the gauge field in Lorentz gauge $A_\pm = \pm\partial_\pm b$ and satisfies the background solution $b = \frac{1}{2}El^2\ln(\sigma)$. In a similar way in which the residual gauge degrees of freedom appear in bosonic string theory, conformal gauge leads to residual conformal coordinate transformations generated by $(\xi^+(t^+), \xi^-(t^-))$. The $U(1)$ gauge in Lorentz gauge also has associated residual gauge transformations which are generated by $\theta(t^+)+\theta'(t^-)$. Static boundary conditions (at $\sigma = 0$) require that the diffeomorphism generators satisfy $\xi^+(t,0) = \xi^-(t,0)$. In order for the variational principle to be well defined, [8] also have the requirement that $\partial_t b = A_\sigma = 0$ at $\sigma = 0$. This presents a problem as the gauge field ($A \sim \frac{1}{\sigma}$) blows up and the generators ξ fail to preserve the static boundary condition. As a consequence, it is not possible to specify the action of the conformal diffeomorphism generators on the boundary Hilbert space. This can however be rectified by augmenting the coordinate transformation by the gauge transformation $\theta(t^+) + \theta'(t^-)$ where [8]

$$\theta(t^+) = -\frac{1}{4}El^2\partial_+\xi^+ \qquad\qquad \theta'(t^-) = \frac{1}{4}El^2\partial_-\xi^-. \tag{184}$$

The conformal diffeomorphisms are generated by the Dirac bracket of line integrals over the current $\frac{1}{2}(T_{++}\xi^+, T_{--}\xi^-)$. Hartman and Strominger take J to define the current associated with the residual gauge transformations. When these currents are conserved holomorphically ($\partial_+J_- = 0$ and similarly for J_+) then the generators of the gauge transformations will be Dirac brackets of the line integral over the current $\frac{1}{2\pi}(\theta J_+, \theta' J_-)$. The generators for the 'gauge augmented' transformations are the Virasoro charges:

$$L(\xi^+) = \int\frac{dt^+}{2\pi}T'_{++}\xi^+, \qquad\qquad L(\xi^-) = \int\frac{dt^-}{2\pi}T'_{--}\xi^-. \tag{185}$$

These contain the so-called 'twisted' stress tensors

$$T'_{\pm\pm} \equiv T_{\pm\pm} \pm \frac{1}{4}El^2\partial_\pm J_\pm. \tag{186}$$

By computing the twisted stress tensor Dirac bracket, one finds a 'gauge-augmented' anomaly [8]:

[59]The notation dt^2 for the integration measure over the two dimensional spacetime volume reflects the coordinate labels $t^\pm = t \pm \sigma$ for the Poincare wedge of AdS_2.

$$[T'_{--}(t^-), T'_{--}(\tilde{t}^-)]_{DB} = -2\pi\left[2\partial_-\delta(t^- - \tilde{t}^-)T'_{--}(\tilde{(t)}^-) - \delta(t^- - \tilde{t}^-)\partial_- T'_{--}(\tilde{t}')\right] + \frac{c}{2}\partial_-^3\delta(t^- - \tilde{t}^-) \quad (187)$$

with the central charge given by

$$c = \frac{3}{4}E^2 l^4 k \tag{188}$$

where k is the level of the $U(1)$ current. This is an interesting result as one expects that two dimensional quantum gravity should have vanishing total central charge [44]. This is understood to arise from the gauge augmentation to the stress tensor that was required to satisfy the boundary conditions and remove the problem of divergent $U(1)$ gauge field associated with the constant electric field- the original stress tensor necessarily has a vanishing central charge [8].

3.5 State operator correspondence and entanglement in AdS_2/CFT_1

Based on the operator state correspondence for a one dimensional CFT, Sen [9] has identified the origin of string theory states in the bulk of an $AdS_2 \times X$ spacetime (X represents a compact manifold) apart from the Hartle-Hawking vacuum state. Since the Hartle-Hawking state is the maximally entangled tensor product state produced by an identity operator insertion, this is clear evidence of the possibility for non-trivial quantum states to exist in AdS_2 string theory. We summarize his work in this subsection.

3.5.1 AdS_2 geometry and CFT_1 operator state correspondence

The presence of the mass gap in the dimensional reduction to AdS_2 in string theory requires that the AdS_2 black hole has microstates of zero energy. As a consequence, it is reasonable to take the extremal limit of branes in $d > 2$ in coordinates for which the horizons have finite separation. The Lorentzian metric for AdS_2 in such a limit is

$$ds^2 = b^2\left(-(R^2-1)dt^2 + (R^2-1)^{-1}dR^2\right), \tag{189}$$

with b being a constant. The Euclidean version is obtained by the transformation $t \to -i\theta$. After a further transformation of the form $r = \sqrt{\frac{(R-1)}{(R+1)}}$ one has

$$ds^2 = \frac{4b^2}{(1-r^2)^2}(dr^2 + r^2 d\theta^2). \tag{190}$$

By the coordinate transformation $T + \sigma = 2\arctan\tanh\frac{1}{2}\left(t \pm \frac{1}{2}\ln\frac{R-1}{R+1}\right)$ [6] the metric (189) takes the form

$$ds^2 = \frac{b^2}{\sin^2\sigma}(-dT^2 + d\sigma^2), \tag{191}$$

on the AdS_2 strip which has the corresponding Euclidean version ($T \to -i\tau$)

$$ds^2 = \frac{b^2}{\sin^2\sigma}(d\tau^2 + d\theta^2). \tag{192}$$

The Euclidean disk (190) has coordinates that label points on the surface of a cone. Only when θ is 2π periodic does the cone flatten to a disk with the a conical singularity at $r = 0$ avoided. There is an interesting map from the disk to the strip. For example, the coordinate transformation

$$\sigma + i\tau = 2arctan \tanh \frac{1}{2}(\ln r + i\theta) \tag{193}$$

that maps the Lorentzian disk boundary ($r = 1$) region, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, to the right-hand side boundary of the strip $\theta = 0$, and it maps $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ to the left hand boundary of the strip at $\theta = -\pi$. This provides a natural description for the expected dual CFT_1 on S^1 which can be mapped to the two lines $S^0 \times R$ when $r = 1$ in (190). In this case, the operator-state correspondence bijection has the feature of matching local operators which have an action on the single Hilbert space on S^1 to a tensor product of this Hilbert space with a clone on $S^0 \times R$.[60] The insertion point of an operator $\hat{\mathcal{O}}$ on the point $\sigma = -\pi/2$ will correspond to a state [9]

$$|\mathcal{O}\rangle_\otimes \equiv \langle A|\hat{\mathcal{O}}|B\rangle \left(|A\rangle \otimes |B\rangle\right). \tag{194}$$

The obvious 2 point correlation function on the circle is given by $Tr(\mathcal{O}^\dagger \mathcal{O}') =_\otimes \langle \mathcal{O}|\mathcal{O}'\rangle_\otimes$. The Hilbert space will be finite dimensional and have N, degenerate, ground states. The observables are represented by Hermitian matrices of dimension N but it is convenient to use unitary $N \times N$ matrices $\hat{\mathcal{U}}$. Then for $\hat{\mathcal{U}}$, whose action is on one copy of the Hilbert space, there is a correlation function which is described by the 'vacuum' expectation value: $Tr(\mathcal{U}) =_\otimes \langle Id|\mathcal{U}|Id\rangle_\otimes$. The 'vacuum' here refers to the identity that corresponds to the state $|Id\rangle_\otimes = |A\rangle \otimes |A\rangle$.[61] The inner product of two unitary operators $_\otimes\langle \mathcal{U}|\mathcal{U}'\rangle_\otimes$ gives the twist $Tr(\mathcal{U}^{-1}\mathcal{U}')$. The density matrix is computed in the standard way by tracing over the second Hilbert space to obtain the matrix for the first Hilbert space: $[\mathcal{O}^\dagger \mathcal{O}]_{AB}|A\rangle\langle B|$. A consequence of this is that both the Identity state and the state for any unitary operator are maximally entangled having density operators $|A\rangle\langle A|$ [9].

In order to calculate observables in the string theory, it is necessary to evaluate the bulk partition function which involves integrating over independent gauge fields after fixing the electric fields at infinity. This partition function blows up on account of the infinite extent of the AdS_2 disk in Euclidean space. This divergence can be controlled with an infrared cut-off at some finite radius $r = 1 - \delta$. This renders the AdS_2 space a near-AdS_2 ($NAdS_2$) space and results in a finite length S^1 boundary curve of extent, say, l. Since by the AdS_2/CFT_1 correspondence this partition function should be matched by the CFT_1 partition function on the S^1 boundary curve, one can compute quantities of interest in the conformal field theory using the AdS_2 theory. The CFT, regulated, partition function has the form $Z = Tr(e^{-lH}) = Ne^{-lE}$ for the N degenerate ground states. By taking the ground state energy to be zero ($E = 0$), which can be achieved through a modification of the boundary terms in the string action, the partition function is reduced to the ground state degeneracy N (also known as the quantum entropy function) for the black hole microstates. Therefore, the AdS_2 partition function allows one to calculate the 2−d black hole entropy.

Under the restriction to unitary operators $\mathcal{U}$ on the CFT Hilbert space there is a $U(N)$ gauge symmetry that is valid unconditionally- this is a consequence of the states all being ground states. There should be a realization of this exact $U(N)$ gauge symmetry in string theory on $AdS_2 \times X$ which is not apparent at present. Although, there is a known example for which the action of the unitary operator of the CFT on the fields in AdS_2 is realized. In general, the correspondence will be [9]

[60]Note that all operators in the CFT_1 are local given that there are no possible spatial separations because the theory is parameterized by time alone.

[61]The 'vacuum' defined in this way is simply a convention as all states are ground states. Each copy of the Hilbert space is spanned by the basis set $\{|A\rangle\}$.

$$Z_{AdS_2} \text{ with } \mathcal{U} \text{ twisted B.C. on Bulk fields under } \theta \to \theta + 2\pi \quad \longleftrightarrow \quad Tr(\mathcal{U}) \text{ in boundary } CFT. \tag{195}$$

The example of the precise realization of $\mathcal{U}$ in the string theory is the case that $\mathcal{U}$ is a generator of a discrete $\mathbb{Z}_k$ symmetry. This is only sensible on a $\mathbb{Z}_k$ orbifold of $AdS_2 \times X$ since the contractibility of the AdS_2 boundary in Euclidean signature leads to vanishing $Tr(\mathcal{U})$ [9].

3.5.2 Origin of string theory states on $AdS_2 \times X$.

The prescription for constructing string theory states on $AdS_2 \times X$ is outlined as follows.
States on $S^0 \times R$ corresponding to the unitary operators on S^1 have corresponding string theory wave functions ψ_1 and ψ_2 with field dependence. The inner product of these states is responsible for the conformal field theory two point function on the boundary. As discussed, this two point function corresponds to the partition function for the $AdS_2 \times X$ bulk with particular boundary conditions that under $\theta \to \theta + 2\pi$, the fields get twisted according to $\mathcal{U}^{-1}\mathcal{U}'$. The string wave functions are path integrals over a 'hemisphere' of the AdS_2 disk with a cut extending from the boundary toward the disk diameter. This cut represents the action of the unitary operator $\mathcal{U}$. ${}_{\otimes}\langle \mathcal{U}|\mathcal{U}'\rangle_{\otimes}$ is computed by taking the path integral of the AdS_2 disk composed of the two 'hemispheres' connected at the diameter line where the fields appearing in the string wave functions ψ_1 and ψ_2 exist. The path integral must be computed with the boundary twist condition $\mathcal{U}^{-1}\mathcal{U}'$. Note that the cuts may extend all the way to the diameter as can be understood by considering the inner product of $|\mathcal{U}\rangle_{\otimes}$ with itself. Fields on each side of the cut at the boundary are related by the action of the $\mathcal{U}$. The orbifold example indicates that it is possible to construct alternative quantum states to the 'vacuum' states generated by the identity state inner product (for useful illustrations see [9]).

3.5.3 Asymptotic Symmetry of AdS_2

It should be noted that, since all quantum mechanical states in the theory are ground states, the correlation functions of the boundary CFT, which consist of the trace of a string of operators $Tr(\mathcal{U}_1...\mathcal{U}_m)$, will be $SL(2,\mathbb{R})$ invariant. This occurs because the correlation functions depend only on the ordering of the time arguments for the operator insertions (in a cyclic manner under the cyclicity of the trace) and not on their specific time argument values. This admits $SL(2,\mathbb{R})$ invarince on the AdS_2 boundary which changes the time locations of the operator insertions but does not alter their cyclic order. This is consistent with the bulk description, which has $\mathcal{U}_n$ cuts which do not change their order when the boundary is mapped to itself under $SL(2,\mathbb{R})$. Therefore the isometry of AdS_2 is in accord with $SL(2,\mathbb{R})$ invariance of the boundary CFT. Since the $SL(2,\mathbb{R})$ CFT_1 is known to be enhanced to the full one dimensional conformal symmetry of diffeomorphisms, the same symmetry should occur in the bulk. This is guaranteed by the certainty that apart from the global $SL(2,\mathbb{R})$ isometry, it is possible to identify a diffeomorphism group on AdS_2 that asymptotically tends to the one dimensional conformal group of diffeomorphisms that still respects the asymptotic boundary conditions of fields in $AdS_2 \times X$. The bulk theory path integral, corresponding to a correlation function described above, will not alter the order of the cuts that represent the unitary group generators and therefore the bulk correlations functions with the boundary conditions, complete with twists, will have the one dimensional conformal group diffeomorphism invariance [9].

It is in fact reasonable that one might expect the existence of this diffeomorphism group for AdS_2 since the asymptotic symmetries of AdS_2 are a subgroup of the two dimensional diffeomorphisms that leave the asymptotic form of the two dimensional AdS_2 metric invariant. For the Jackiw-Teitelboim model, in which AdS_2 geometry can be enforced, the AdS_2 asymptotic symmetry has been shown to tend to time reparameterizations of the time-like AdS_2 boundary. Furthermore, by expanding the functions that characterize the boundary time reparameterizations in Fourier series, motivated by the periodic nature of time on the S^1 boundary, the generators of the asymptotic symmetry lead to a single Virasoro algebra. This acts on the time-like boundary as an infinite dimensional group of diffeomorphisms in one dimension [20].

3.6 Reflection

Having presented a technical overview of what is known about AdS_2/CFT_1 in the literature, we now endeavor to present a brief summary of what has been learned.

- AdS_2 typically appears in the near horizon limit of black holes that arise in string theory. These black holes have a preferred choice of time and the choice of time coordinate distinguishes between various definitions of the vacuum state. The vacuum defined with regard to the Boulware time is equivalent to the vacuum defined with regard to the Schwarzschild time coordinate. The Poincare time vacuum, the Hartle-Hawking time vacuum and the global time vacuum are equivalent and distinct from the Boulware and Schwarzschild vacua. Greens functions can be computed for the various vacua and these Greens functions have been used to compute boundary correlation functions- a key ingredient for the AdS_2/CFT_1 correspondence. The AdS_2 black holes have a temperature independent entropy at the classical level, but in the quantum case, it has logarithm temperature dependence. A consequence of this is that the asymptotically flat and near horizon AdS_2 regions fail to decouple at finite temperature [4].

- Motivated by the holographic correspondence between Matrix quantum mechanics and Type 0 string theory, Matrix quantum mechanics with a conformal symmetry (dAFF potential), is conjectured to be dual to the Type 0A string theory on AdS_2. The evidence for the correspondence is based on the matching of the AdS_2 isometries, for the Poincare patch of AdS_2, and the $SL(2,\mathbb{R})$ symmetry of the conformal quantum mechanics. Transforming from the conformal Hamiltonian, which generates time translations in terms of the Poincare time coordinate, to a new operator L_0, which generates time translations in the global AdS_2 time coordinate, the free fermion theory operator for which eigenstates are sought, is swapped from a theory with a continuous spectrum, to a theory with a discrete spectrum with some degree of confinement- this is certainly preferable for the free fermion theory. However, one gains insight from considering the gravitational interpretation which says that the different Hamiltonians generate time evolution of different time slices, of which there are many to choose from in a theory of gravity [5].

- There are various approaches to the near horizon limit for string theory black holes- this arises from the fact that the black hole temperature, energy and charge cannot all be kept fixed in the limit. The simplest limit leads the mass gap that restricts AdS_2 excitations to the zero energy ground state. Non-trivial excitations exist, which in passing from the asymptotically flat region to the near horizon AdS_2 throat, sees the throat branching into multiple AdS_2 branches. These are referred to as AdS_2 trees. There are two cases of interest in which these AdS trees can actually cleave into separate AdS_2 universes. In the non-SUSY case, one takes the test brane approximation for which instantons describe brane creation- for AdS_2 this corresponds to $0-$brane- $0-$ anti-brane pair production. The mechanism for the fragmentation of the AdS_2 into multiple universes corresponds to the brane reaching the boundary in finite time, in which case the AdS_2 universe splits into a microscopic AdS_2 universe, described by the brane, and a macroscopic AdS_2 universe. For the $SUSY$ case, the fragmentation of the AdS_2 universe into two macroscopic AdS_2 universes is mediated by Brill instanton tunneling. In the zero brane quantum gravity limit, the CFT_1 dual can only be sensibly defined on the outer AdS_2 boundary. From the string theory perspective of the dual conformal quantum mechanical theory, the AdS_2 trees have the interpretation of being various classical backgrounds for the CFT_1. The discussion of 3.3.4 suggests that the trees correspond to the Higgs branch of the dual field theory [6].

- AdS_2 quantum gravity was argued to be a two dimensional conformal field theory as a result of its description in terms of the Liouville theory in conformal gauge on the strip with two boundaries. Therefore, the isometry group of AdS_2 should be extended to the full two dimensional conformal group and this is a general result for AdS_2 quantum gravity theories and string theory. AdS_2 appears in the 'very near horizon' limit of the S^1 compactification of AdS_3 , which emerges in the near horizon limit of some black holes in string theory. The 'very near horizon' makes manifest the mass gap for AdS_2 black holes which is responsible for the absence of non-zero excitations in the theory. This limit is consistent with the $SL(2,\mathbb{R})_L$ quotient of AdS_3 that leaves the $SL(2,\mathbb{R})_R$ isometry of AdS_3 unbroken. The $SL(2,\mathbb{R})_R$ isometry of AdS_3 reduces to the $SL(2,\mathbb{R})$ isometry of AdS_2 together with a $U(1)$ gauge transformation. The global AdS_3 isometry is known to be extended to a full two dimensional conformal field theory with two copies of the Virasoro algebra- the AdS_2 case is therefore expected to be extended to a single copy of the Virasoro algebra [7]. These diffeomorphisms of the local conformal group are associated with a twisted stress tensor and for the case of Maxwell dilaton gravity, the extra gauge transformation is understood to remedy the problem of a singular $U(1)$ potential at the boundary and leads to a central charge for the AdS_2 theory [8].

- The boundary of the Euclidean AdS_2 disk is S^1, which can be mapped to the strip with two boundaries, $S^0 \times \mathbb{R}$, separated by a horizon in the case of $AdS_2 \times X$ geometries for the near horizon limit of extremal black holes in string theory. This leads to the natural interpretation of the state-operator correspondence for the dual CFT_1 as the map from operators on an N dimensional Hilbert space, of zero energy excitations due to the mass gap, to N^2 states on the tensor product of two copies of the Hilbert space on the two boundaries of $S^0 \times \mathbb{R}$. The identity operator of the CFT_1 corresponds to the maximally entangled, between the two Hilbert spaces, state known as the Hartle-Hawking state; however, there are other non-trivial states in the AdS_2 theory which are distinct from the AdS_2 fragmented spaces of [6]. The CFT_1 can be spanned by a basis of $N \times N$ unitary matrices with correlation functions for the S^1 boundary CFT given by traces of such operators, which have a $U(N)$ symmetry due to the ground state degeneracy of the conformal theory. The bulk partition function, for which AdS_2 is regulated with an infrared cut-off that renders the space to be near-AdS_2, can accommodate the unitary symmetry in the case of a $\mathbb{Z}_k$ orbifold of $AdS_2 \times X$. In this case, the correspondence has the AdS_2 partition function with a $\mathcal{U}$ twisted boundary condition on the bulk AdS_2 fields at the S^1 boundary under $\theta \to \theta + 2\pi$ in correspondence with the correlation function $Tr\mathcal{U}$ for the boundary CFT. The bulk partition function can also be used to compute the quantum entropy function, which is the ground state degeneracy of the black hole. The states on the $S^0 \times \mathbb{R}$ boundary, corresponding to operators on the S^1 boundary, are represented by wave functions in the bulk. These string wave functions are computed via path integrals over a hemisphere of the AdS_2 disk with a cut extending from the boundary representing the action of the unitary operators. The path integral over a pair of these hemispheres, connected along the diameter, corresponds to the CFT two point function and must be computed with the twist boundary condition. The conformal field theory correlation functions obey both the $SL(2,\mathbb{R})$ global group as well as the local diffeomorphisms of time corresponding to the AdS_2 asymptotic symmetries [9].

4 Matrix Quantum Mechanics: Chemical Potential, Conformal Symmetry Breaking and Induced Scale Parameter

Having, in the previous section, reviewed what is known about AdS_2 and the AdS_2/CFT_1 correspondence in string theory, we now remind the reader of our line of thought. We wish to present our method for generating two dimensional metrics from a multi-matrix model in $d = 1$ dimensions with a conformal symmetry. It will turn out that this conformal symmetry is broken by an induced scale parameter for a theory with a potential of the form: $1/x^2$. [62] Therefore, we turn, in this section, to the single Hermitian matrix model in $d = 1$ dimensions as the presence of the induced scale parameter can be identified by studying the free matrix model, which we shall argue requires a mass regularization term, and comparing observables computed within the matrix model to those obtained in the collective field theory formulation of the matrix model.[63] It is of value to present the well known equivalence of the matrix model to a system of fermions [10] and we shall therefore include a treatment of this equivalent description in first and second quantization.

This section consists of two cases: the free theory case and the $1/x^2$, conformal, case. Our approach to the free theory case is as follows: we begin with a treatment of the matrix model, which in $d = 1$ dimensions requires a mass regularization term, and compute observables such as the ground state energy and correlation functions. This is followed by the fermionic description and a second quantized treatment of the system of fermions in which the ground state energy is also computed. The theory is regulated by placing the system of fermions into a box of length L. We then turn to the collective field theory,[64] where once again, we compute correlation functions and the ground state energy. An important point that is learned is that the Lagrange multiplier, that is required to enforce the eigenvalue density constraint, is found to be equivalent to a Fermi energy (or chemical potential). The equivalence of the three descriptions of the model suggests that we compare the results of these observables in the different treatments. As expected, the results are found to agree for the collective field theory and fermionic descriptions, however, the regularization of the matrix model spoils this matching of observables at first glance. It is at this point that the scaling parameter R will be shown to emerge. The new scale factor R is proportional to the length, L, with proportionality constant $\sqrt{N}$. Together these parameters define the standard thermodynamic, double scaling, limit: $L \to \infty$, $1/\sqrt{N} \to 0$ with R fixed. We then consider case 2, the conformal potential, and show that the Fermi energy (Lagrange multiplier) can be solved but is scale dependent. This scale dependence of the Lagrange multiplier is responsible for the breaking of conformal invariance.

4.1 Free Theory

4.1.1 $d = 1$ free matrix valued field theory

In the free field theory of 1 spacetime dimension (i.e. fields as a function of time) we have the following free Hamiltonian:

$$H = -\frac{1}{2} l_s Tr \frac{\partial}{\partial M} \frac{\partial}{\partial M}. \tag{196}$$

[62] For the fermionic and collective field theory descriptions this is the potential that would be expected to be associated with an $SL(2, \mathbb{R})$ conformal symmetry.

[63] Since a more complete and more general treatment of the collective field theory will appear in the next section (section 5) we postpone a discussion of the general collective field method until then. We have included an appendix (appendix B) if the reader wishes to review the method immediately.

[64] In the next chapter we consider the more general case of multi-matrix systems for which a conformally invariant potential emerges in the free theory. For the single Hermitian matrix theory, the conformally invariant potential is inserted by hand. Since the treatment of the conformally invariant potential is similar for the multi- matrix theory, we present it here for the single matrix case as a warm up.

In order to make the connection to quantum mechanics more accurate, we have taken care to introduce a length parameter l_s.[65] In some places we have not exercised such care (for example appendix A); however, it can simply be restored by dimensional analysis. The massless field theory is typically retrieved from the massive case in the limit that the mass goes to zero. The $d = 1$ case is distinct in this regard. General observables, such as free field correlators, are divergent in the $m \to 0$ limit. This suggests that one retain the mass term in order to regularize the theory. This allows one to compute a well defined $2-$ point function in the usual way from which it is possible to compute any $n-$ point function by applying Wick contractions. The regularized theory:

$$L = \frac{1}{2ls} Tr\dot{M}^2 - \frac{m^2}{2ls} TrM^2, \tag{197}$$

has the following partition function

$$\int [M]\, e^{-\frac{i}{2l_s}\int dt TrM(\partial_t^2+m^2)M} = \int [M]\, e^{-\frac{1}{2}\int dt TrM\hat{O}M}, \tag{198}$$

where we have defined the operator $\hat{O} = \frac{i}{l_s}(\partial_t^2 + m^2)$. The propagator is calculated from the two point function

$$\langle M_{ij}(t_1)M_{i'j'}(t_2)\rangle = \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} G(\omega)\delta_{ij'}\delta_{ji'}. \tag{199}$$

The Greens function in frequency space is then clearly $G(\omega) = \frac{il_s}{\omega^2 - m^2}$. To account for the poles we make use of the Feynman time ordering prescription:

$$\langle M_{ij}(t_1)M_{i'j'}(t_2)\rangle = \int \frac{d\omega}{2\pi} \frac{i\delta_{ij'}\delta_{ji'}l_s e^{-i\omega(t_1-t_2)}}{\omega^2 - m^2 + i\epsilon}. \tag{200}$$

Applying Cauchy's integral theorem we arrive at the propagator solution

$$\begin{aligned}\langle M_{ij}(t_1)M_{i'j'}(t_2)\rangle &= \delta_{ij'}\delta_{ji'}\frac{il_s}{2m}\int \frac{d\omega}{2\pi}\left[\frac{e^{-i\omega(t_1-t_2)}}{\omega - m + i\eta} - \frac{e^{-i\omega(t_1-t_2)}}{\omega + m - i\eta}\right] \\ &= \frac{\delta_{ij'}\delta_{ji'}l_s}{2m}\left[\theta(t_1 - t_2)e^{-im(t_1-t_2)} + \theta(t_2 - t_1)e^{im(t_1-t_2)}\right]. \end{aligned} \tag{201}$$

We are interested in considering the collective field theory of the matrix model which is formulated in terms of a Hamiltonian. We therefore consider the equal-time commutator which is easily recovered from (201) in the limit that $t_1 \to t_2$

$$\langle TrM^2(t)\rangle = \frac{N^2 l_s}{2m}. \tag{202}$$

It is evident that there is not a well defined $m \to 0$ limit in this theory.

We determine the energy by considering the velocity two point function (for which we make use of the result (201))

[65] We consider the canonical momentum $P \sim \frac{\partial}{\partial M}$ to have units of energy which leads to the requirement that we introduce ℓ_s with unit -1 (i.e. inverse energy) to ensure that the Hamiltonian has units of energy.

$$
\begin{aligned}
Tr\langle \partial_t M(t)\partial_t M(t)\rangle &= \lim_{t_1\to t_2} Tr\partial_{t_1}\partial_{t_2}\langle M(t_1)M(t_2)\rangle \\
&= \lim_{t_1\to t_2}\frac{N^2 l_s}{2m}\left[\theta(t_1-t_2)e^{-im(t_1-t_2)}m^2+\theta(t_2-t_1)e^{im(t_1-t_2)}m^2\right] \\
&= \frac{N^2 m l_s}{2}.
\end{aligned} \tag{203}
$$

The expectation energy, $\langle\hat{H}\rangle = \frac{1}{2l_s}\left(\frac{N^2 m l_s}{2}\right)+\frac{m^2}{2l_s}\left(\frac{N^2 l_s}{2m}\right)$, has the ground state solution

$$
\langle\hat{H}\rangle = \frac{N^2 m}{2}. \tag{204}
$$

To determine higher order correlation functions for the free theory, one can apply Wick's theorem. As an example, we consider the case of the four point function:

$$
\begin{aligned}
\langle TrM^4(t)\rangle &= \langle M_{i_1i_2}M_{i_2i_3}M_{i_3i_4}M_{i_4i_1}\rangle \\
&= \langle M_{i_1i_2}M_{i_2i_3}M_{i_3i_4}M_{i_4i_1}\rangle + \langle M_{i_1i_2}M_{i_2i_3}M_{i_3i_4}M_{i_4i_1}\rangle \\
&+ \langle M_{i_1i_2}M_{i_2i_3}M_{i_3i_4}M_{i_4i_1}\rangle \\
&= \frac{N^3 l_s^2}{2m^2}+\frac{N l_s^2}{4m^2}.
\end{aligned} \tag{205}
$$

Which is just $\langle TrM^4(t)\rangle = \frac{N^3 l_s^2}{2m^2}$ to leading order in the large N limit.

4.1.2 Fermionic Description

The single Hermitian matrix model of quantum mechanics has a dual description in terms of a set of N non-interacting, non-relativistic fermions [10]. To be specific, it is the Hamiltonian of the continuous eigenvalues of the matrix model (in the large N limit) that resembles N one fermion Hamiltonians with a background potential and the absence of any interactions. For a quantum field theory, one is interested in studying the action and the partition function. It turns out that the generalization to matrix valued fields involves traces of matrices and functions of matrices. In the case of a matrix model of a single Hermitian matrix, the matrix is diagonalizable by a unitary transformation which defines a gauge symmetry. The action can therefore be entirely re-expressed in terms of the eigenvalues of the matrix. This naturally leads one to focus on re-writing the matrix integration measure in terms of the eigenvalues as opposed to the general matrix elements. The definition of the coordinate invariant integration measure requires one to consider the generalization of the invariant length element to the case of matrices. A proper understanding of the transformation properties of the metric ($g' = gJ^2$) associated with the length element identifies the volume element with a true geometrical interpretation as that involving the determinant of the metric. From the manifold associated with the Hermitian matrix we introduce the metric that can be used to define a coordinate invariant integration measure and a coordinate invariant Laplace operator that acts on wave functions in the Hamiltonian .[66] The determinant of the metric is found to be the Vandermonde determinant which is responsible for the eigenvalue dynamics that are characteristic of random matrix theories [45].

The Hermitian matrix M is diagonalized by unitary transformation $M \to UDU^\dagger$ where D is the diagonal matrix of eigenvalues. The infinitesimal element of M is obtained by applying the Leibniz product rule $dM = dUDU^\dagger + UdDU^\dagger + UDdU^\dagger$. It is important to note that the product $UdU^\dagger$ is anti-Hermitian.[67]

[66] The coordinate invariant Laplace operator on a manifold is commonly known as the Laplace-Beltrami operator. It is the generalization from the standard Laplace operator of Euclidean space to curved spaces.

[67] This follows from taking the derivative of the defining equation $UU^\dagger = \mathbb{1}$

It is convenient to define the the matrix infinitesimal $dX \equiv U^\dagger dU$. Then

$$dM = U\left(dD + [dX, D]\right)U^\dagger = dM^\dagger. \tag{206}$$

The right hand side of this equation is obvious because of the Hermiticity of M. The generalization of the invariant length element in this case is

$$\begin{aligned} Tr(dM^2) = dM^\dagger dM = Tr(dD^2) + Tr(dD\,[dX, D]) \\ + Tr([dX, D]\,dD) + Tr([dX, D]\,[dX, D]). \end{aligned} \tag{207}$$

The cyclic property of the trace together with the fact that $[D, dD] = 0$ implies that the two terms $Tr([dX, D]\,dD)$ and $Tr(dD\,[dX, D])$ both vanish. We are left with

$$Tr(dM^2) = \sum_{i=1}^{N} d\lambda_i^2 + \sum_{i,j=1}^{N} [dX, D]_{ij}\,[dX, D]_{ji} \tag{208}$$

which, given that D is diagonal, further simplifies to

$$\begin{aligned} Tr(dM^2) &= \sum_{i=1}^{N} d\lambda_i^2 - \sum_{i,j=1}^{N} dX_{ij}dX_{ji}(\lambda_i - \lambda_j)^2 \\ &= \sum_{i=1}^{N} d\lambda_i^2 + 2\sum_{i<j=1}^{N} \left|dX_{ij}\right|^2(\lambda_i - \lambda_j)^2. \end{aligned} \tag{209}$$

Use was made of the anti-Hermiticity of dX in going from the first to the second line of (209). The invariant length element provides an equation for the metric tensor ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu$) in terms of the coordinates: $x^\mu = \{\lambda_i, X_{ij}, X_{ij}^*\}$.The coordinate invariant integration measure is then given by[68]

$$\begin{aligned} dM &= \mathcal{N}\prod_{i}^{N} dM_{ii}\prod_{i<j} dRe(M_{ij})dIm(M_{ij}) = \sqrt{|g|}\prod_{i} dx^i \\ &= \mathcal{N}\,\triangle^2(\lambda)\prod_{i=1}^{N} d\lambda_i \prod_{i,j(i<j)}^{N} dX_{ij}dX_{ij}^* \end{aligned} \tag{211}$$

$\mathcal{N}$ is a numeric factor. The term $\triangle^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2 = \sqrt{|g|}$ is the so called Vandermonde determinant of eigenvalues. The Laplace operator $\bigtriangledown^2 = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu}$ is easily calculated once the inverse metric $g^{\mu\nu}$ is known. Due to the simple form of the metric (see (210)) its inverse is straightforward to write down and one determines the Laplace operator to be of the form [45]

[68] The metric is explicitly given by:

$$g_{\mu\nu} = \begin{matrix} & \lambda_i & X_{ij(i<j)} & X^*_{ij(i<j)} \\ \lambda_i & 1 & 0 & 0 \\ X^*_{ij(i<j)} & 0 & (\lambda_i - \lambda_j)^2 & 0 \\ X_{ij(i<j)} & 0 & 0 & (\lambda_i - \lambda_j)^2 \end{matrix}. \tag{210}$$

$$\bigtriangledown^2 = \sum_i \frac{1}{\triangle^2(\lambda)} \frac{\partial}{\partial \lambda_i} \triangle^2(\lambda) \frac{\partial}{\partial \lambda_i} + \tag{212}$$
$$2 \sum_{i,j(i<j)} \frac{1}{\triangle^2(\lambda)} \frac{\partial}{\partial X_{ij}} \triangle^2(\lambda) \frac{1}{(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial X^*_{ij}}.$$

Making use of anti-Hermiticity of X and changing the summation index conditions in the second term of (212) leads to the Laplacian

$$\bigtriangledown^2 = \sum_i \frac{1}{\triangle^2(\lambda)} \frac{\partial}{\partial \lambda_i} \triangle^2(\lambda) \frac{\partial}{\partial \lambda_i} - \sum_{i,j(i \neq j)} \frac{1}{(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial X_{ij}} \frac{\partial}{\partial X_{ji}}. \tag{213}$$

If one considers the possibility of concentrating on the singlet sector of the theory then the 'angular' degrees of freedom X associated with the unitary matrices U in the second term of (213) can be ignored. What remains of (213) is familiar from quantum mechanics and by redefining the wave functions on which it acts it is possible to obtain a fermionic description in terms of the eigenvalues. The original wave function $\psi(\lambda_1, ..., \lambda_N)$ on which the first term in (213) acts is a symmetric function of the eigenvalues. However, with the definition $\phi(\lambda_1, ..., \lambda_N) \equiv \triangle(\lambda)\psi(\lambda_1, ..., \lambda_N)$, the original wave function becomes a quotient of a wave function $\phi(\lambda_1, ..., \lambda_N)$ that is anti-symmetric in the eigenvalues and a function $\triangle(\lambda)$ which is also anti-symmetric in the eigenvalues and the N-particle fermionic description emerges. The operator equation:

$$-\frac{1}{2} \bigtriangledown^2 \psi(\lambda) = -\frac{1}{2} \left(\sum_i \frac{1}{\triangle^2(\lambda)} \frac{\partial}{\partial \lambda_i} \triangle^2(\lambda) \frac{\partial}{\partial \lambda_i} \right) \frac{\phi(\lambda_1 ... \lambda_N)}{\triangle(\lambda)} \tag{214}$$
$$= E \frac{\phi(\lambda_1 ... \lambda_N)}{\triangle(\lambda)}$$

can be simplified to arrive at an operator equation acting on the anti-symmetric wave function $\phi(\lambda_1, ..., \lambda_N)$ only.

$$-\frac{1}{2} \left(\sum_i \frac{1}{\triangle(\lambda)} \frac{\partial}{\partial \lambda_i} \triangle^2(\lambda) \frac{\partial}{\partial \lambda_i} \frac{1}{\triangle(\lambda)} \right) \phi(\lambda_1 ... \lambda_N) = E \phi(\lambda_1 ... \lambda_N). \tag{215}$$

By considering the operator of (215) in two parts: $\frac{1}{\triangle(\lambda)} \frac{\partial}{\partial \lambda_i} \triangle(\lambda)$ and $\triangle(\lambda) \frac{\partial}{\partial \lambda_i} \frac{1}{\triangle(\lambda)}$ it is found that

$$-\frac{1}{2} \left(\sum_i \frac{1}{\triangle(\lambda)} \frac{\partial}{\partial \lambda_i} \triangle^2(\lambda) \frac{\partial}{\partial \lambda_i} \frac{1}{\triangle(\lambda)} \right) \phi(\lambda_1 ... \lambda_N) \tag{216}$$
$$= -\frac{1}{2} \sum_i \left(\frac{\partial^2}{\partial \lambda_i^2} - \sum_{j(\neq i)} \sum_{k(\neq i,j)} \frac{1}{(\lambda_i - \lambda_j)} \frac{1}{(\lambda_i - \lambda_k)} \right) \phi(\lambda_1 ... \lambda_N).$$

By including the summation over i, the second term in (216) vanishes identically. Therefore, the kinetic operator acting on an anti-symmetric many body wave function has the form

$$-\frac{1}{2} \bigtriangledown^2 \phi(\lambda_i, ..., \lambda_N) = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} \phi(\lambda_i, ..., \lambda_N) = E \phi(\lambda_i, ..., \lambda_N). \tag{217}$$

The derivation of (217) has been completely general for the singlet sector of the Hermitian matrix model and the inclusion of a background potential is straightforward. The potential is assumed to be diagonalizable by unitary transformation such that $V(M) \to UVU^\dagger = \sum_i V(\lambda_i)$ so that

$$H\phi(\lambda_i, ..., \lambda_N) = \sum_i \left(-\frac{1}{2}\frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i)\right)\phi(\lambda_i, ..., \lambda_N) = E\phi(\lambda_i, ..., \lambda_N) \tag{218}$$

The antisymmetrized many body wave function $\phi(\lambda_i, ..., \lambda_N)$ could be constructed from an orthonormal basis of single particle wave functions in the form of a Slater determinant. We consider the free case for the time being[69], which has the single particle Hamiltonian:

$$\left(-\frac{1}{2}l_s\partial^2_{x_i}\right)\phi_n(x_i) = \epsilon_n\phi_n(x_i). \tag{219}$$

We have replaced the eigenvalue coordinate $\lambda_i \to x_i$ so as to conform to more natural conventions and we have restored l_s. If we put the system into a box then the complete set of wave functions are given by

$$\phi_n(x) = \frac{1}{\sqrt{L}}e^{\frac{2\pi i n x}{L}} \tag{220}$$

with the orthonormality condition $\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\phi_n^*(x)\phi_m(x) = \delta_{nm}$. The quantum number n labeling the state is taken to be $|n| \leq \frac{N}{2}$ with the momentum quantum numbers (still labeling states in terms of the principle quantum number) $k_n = \frac{2\pi n}{L}$ and corresponding energy $\epsilon_n = \frac{l_s}{2}k_n^2$. The energy of N free fermions in $d = 1$ space dimensions is

$$E = \sum_{|n|\leq\frac{N}{2}} \frac{l_s k_n^2}{2} = \frac{4\pi^2 l_s}{L^2}\sum_{n=1}^{N/2} n^2. \tag{221}$$

The sum over the first N/2 integers squared is $\sum_{n=1}^{N/2} n^2 = \frac{N^3}{24} + \frac{N^2}{8} + \frac{N}{12}$ which implies that in the limit that N becomes large $\sum_{n=1}^{N/2} = \frac{1}{3}\frac{N^3}{8}$. Then for N large

$$E = \frac{\pi^2}{6}\frac{N^3 l_s}{L^2} = \frac{N}{3}\epsilon_F \tag{222}$$

and we obtain the Thomas-Fermi energy with the standard thermodynamic limit (ϵ_F =const, $N \to \infty$).

4.1.3 Second quantized approach

In the second quantization operator approach we define the field operators in terms of the single particle eigenfunctions

$$\hat{\psi}(x) = \sum_n \phi_n(x)c_n,$$
$$\hat{\psi}^\dagger(x) = \sum_n \phi_n^*(x)c_n^\dagger. \tag{223}$$

[69] We have included a treatment of the harmonic oscillator potential in appendix A as well as a more general approach to the fermionic problem in terms of orthogonal polynomials.

The density is given by

$$\phi(x) = \langle 0|\hat{\psi}^\dagger(x)\hat{\psi}(x)|0\rangle. \tag{224}$$

Recalling that $c_n|0\rangle \neq 0 \Leftrightarrow |n| \leq \frac{N}{2}$ and $c_n^\dagger|0\rangle \neq 0 \Leftrightarrow |n| > \frac{N}{2}$ and making use of (223) we find that

$$\phi(x) = \sum_{|n|\leq \frac{N}{2}} \phi_n^*(x)\phi_n(x) = \frac{N+1}{L} \simeq \frac{N}{L} \tag{225}$$

at large N. One can, instead of putting the system in a box, consider Dirac normalized wave functions:

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{226}$$

Then the field operators are given by

$$\hat{\psi}(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} c_k \tag{227}$$

and we require the anti-commutation relations $\{c_p, c_k^\dagger\} = \delta(k-p)$. Constructing second quantized operators in the usual way we find that the Hamiltonian is

$$H = \int dx \hat{\psi}^\dagger(x) \left(-\frac{1}{2} l_s \partial_{x_i}^2\right) \hat{\psi}(x) = \int dk \frac{l_s k^2}{2} c_k^\dagger c_k. \tag{228}$$

For fermions in the ground state we impose the constraints $c_k^\dagger|0\rangle \neq 0 \Leftrightarrow |k| > k_{_F}$ and $c_k|0\rangle \neq 0 \Leftrightarrow |k| < k_{_F}$. Then

$$\langle E\rangle = \int dk \frac{l_s k^2}{2} \langle 0|c_k^\dagger c_k|0\rangle = \int_{-\frac{k_F}{2}}^{\frac{k_F}{2}} dk \frac{l_s k^2}{2} \langle 0|\{c_k^\dagger, c_k\}|0\rangle = \delta(k=0) \frac{l_s}{3} \left(\frac{k_{_F}}{2}\right)^3. \tag{229}$$

From the Dirac delta function $\delta(k-k') = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi} e^{ix(k-k')} \to \delta(k=0) = \frac{L}{2\pi}$ so that the ground state energy is

$$\langle E\rangle = \frac{L l_s}{2\pi} \frac{1}{3} \frac{k_{_F}^3}{8}. \tag{230}$$

Recalling the Fermi wave vector from the previous section and the limits of integration on (229) we take $k_F = \frac{2\pi N}{L}$, which upon substitution into (230) confirms the result of (222)

$$\langle E\rangle = \frac{\pi^2}{6} \frac{l_s N^3}{L^2}. \tag{231}$$

4.1.4 Density description ($d = 2$ Field Theory) and the induced scale parameter R

The general set of invariants that are of interest for the collective field theory formulation of the matrix model are the quantities $\phi_k = Tr\left(e^{ikM}\right) = \sum_i e^{ik\lambda_i}$. The λ_i are the eigenvalues of M. By taking the Fourier transform we find the density to be given by $\phi(x) = \sum_j \delta(x - x_j)$. Following appendix B, the splitting and joining operators are found to be $\omega(x;[\phi]) = -2\partial_x\phi(x)\fint dy\frac{\phi(y)}{x-y}$ and $\Omega(x,y;[\phi]) = \partial_x\partial_y[\delta(x-y)\phi(x)]$ respectively. The Lagrange multiplier μ is introduced in order to account for the constraint that $\int dx\phi(x) = N$. The corresponding collective field theory Hamiltonian (see (368)) is[70]

$$H_{coll} = \frac{l_s}{2}\int dx(\partial_x\pi(x))\phi(x)(\partial_x\pi(x)) + \frac{\pi^2 l_s}{3}\int dx\phi^3(x) + \mu\left(N - \int dx\phi(x)\right). \tag{232}$$

The effective potential, which is the leading contribution after taking the large N limit, is

$$V_{\text{eff}}[\phi] = \frac{\pi^2 l_s}{6}\int dx\phi^3(x) + \mu(N - \int dx\phi(x)). \tag{233}$$

In the large N limit we apply the saddle point approximation: $\left.\frac{\delta V_{eff}}{\delta\phi(x)}\right|_{\phi_o(x)} = 0$ which has the solution

$$\phi_o(x) = \frac{1}{\pi}\sqrt{\frac{2\mu}{l_s}}. \tag{234}$$

However, this result is a constant which prevents the recovery of the constraint when the the full domain of integration is considered. We therefore are required to put the system into a finite box and let the integration run over $x\epsilon[-\frac{L}{2},\frac{L}{2}]$. Then

$$N = \int_{\frac{-L}{2}}^{\frac{L}{2}} dx\phi_o(x) = \frac{L\sqrt{2\mu/l_s}}{\pi} \tag{235}$$

and we find that $\phi_o(x) = \frac{N}{L} = \frac{1}{\pi}\sqrt{2\mu/l_s}$ with $\mu = \frac{\pi^2 N^2 l_s}{2L^2}$. The wave vector is defined such that $k_n \equiv \frac{2\pi n}{L}$ for $|n| \leq \frac{N}{2}$. This definition is consistent, in the free theory, with the Fermi wave vector $k_{_F} = \frac{\pi N}{L}$ and the Fermi energy $\epsilon_{_F} = \frac{l_s k_{_F}^2}{2} = \frac{\pi^2 N^2 l_s}{2L^2} = \mu$. The background collective field is then expressed in terms of the Fermi wave vector $\phi_o(x) = \phi_o = \frac{1}{\pi}k_{_F}$ or the Fermi energy $\phi_o = \frac{1}{\pi}\sqrt{2\epsilon_{_F}/l_s}$. From the effective potential we find that the total energy is

$$E = \frac{\pi^2 l_s}{6}\frac{N^3}{L^2} = \frac{N}{3}\epsilon_{_F}. \tag{236}$$

Where again we have obtained the so called Thomas-Fermi energy with $\frac{E}{N} = \frac{1}{3}\epsilon_{_F}$ finite, which is the standard thermodynamic limit. This confirms the results of the first and second quantization of fermions: (222) and (231). Correlators in the density description are calculated in the usual way

$$\langle TrM^2\rangle = \int dx x^2\phi_o(x) = \frac{N}{L}\left(\frac{x^3}{3}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{NL^2}{12} \tag{237}$$

[70]The reader is urged to consult appendix B for details. Our objective in this section is to make use of the collective field theory to compute certain quantities. The treatment of section 5 is general enough that any details skipped here are treated comprehensively there and can be deduced for the single matrix case.

and more generally

$$\langle TrM^{2n}\rangle = \int dx x^{2n}\phi_o = \frac{NL^{2n}}{(2n+1)2^{2n}}. \tag{238}$$

We are now in a position to compare the $d = 1$ theory results with those of the collective field theory for leading order contributions at large N:

$$\begin{array}{cc} \text{d=1 field theory} & \text{density} \\ \langle TrM^2\rangle \sim \frac{N^2 l_s}{m} & \langle TrM^2\rangle \sim NL^2 \\ E \sim N^2 m & E \sim \frac{N^3}{L^2} \\ \langle TrM^4\rangle \sim \frac{N^3 l_s^2}{m^2} & \langle TrM^4\rangle \sim NL^4. \end{array} \tag{239}$$

We find that the results are compatible provided that $L \sim \sqrt{N}$. It is at this point that we identify the induced length (scale) parameter R defined by $L \equiv R\sqrt{N}$. Then comparing the results for the 2 point function we see that

$$\langle TrM^2\rangle = \frac{NL^2}{12} = \frac{N^2R^2}{12} = \frac{N^2 l_s}{2m} \Leftrightarrow R^2 = \frac{6l_s}{m}. \tag{240}$$

The limit $m \to 0$ is consistent with $R \to \infty$. In other words the massless limit corresponds to R large. Strictly speaking, the numerical coefficients in the quantities (239) will not all match identically due to the regulating mass in the matrix model, however, we find the general relation that $R \sim \sqrt{l_s/m}$.

If we apply the standard rescaling[71] of the collective field theory to make explicit the factors of N, that is $x \to \sqrt{N}x$ and $\mu \to N\mu$ with $\phi_o \to \sqrt{N}\phi_o$, then in the free case $\int_{\frac{-L}{2}}^{\frac{L}{2}} dx \frac{1}{\pi}\sqrt{2\mu} = N$ becomes

$$N\int_{\frac{-L}{2\sqrt{N}}}^{\frac{L}{2\sqrt{N}}} dx \frac{1}{\pi}\sqrt{\frac{2\mu}{l_s}} = N. \tag{241}$$

Again, this suggests the definition $L = R\sqrt{N}$. With this definition, the field theory results become

$$\begin{aligned} \langle TrM^{2n}\rangle &= \frac{R^{2n}N^{n+1}}{(2n+1)2^{2n}}, \\ E &= \frac{N^2\pi^2 l_s}{6R^2}. \end{aligned} \tag{242}$$

At this point, we can confirm that by substitution of $L = R\sqrt{N}$ into first (222) and second quantized (231) ground state energy results shows that they agree with the collective field theory result (242) and the matrix model ground state energy (204) (taking $R^2 = \frac{\pi^2 l_s}{3m}$ after matching the energy (242) with (204)).

[71] The standard rescaling makes factors of N explicit in the collective field theory Hamiltonian. This rescaling, in the large N limit, leads to an effective potential which solves the background collective field theory which makes a significant appearance in the next chapter in the form of an emergent spacetime geometry.

4.2 Conformal Invariant Potential in the Radial Sector of Single Matrix Systems and breaking of conformal invariance due to the induced scale parameter R

The $(d = 1)$ matrix model with a conformally invariant potential has the form

$$H = -\frac{1}{2}\ell_s Tr\frac{\partial}{\partial M}\frac{\partial}{\partial M} + \frac{\ell_s N^2\eta^2}{2}Tr\frac{1}{M^2} \tag{243}$$

where the dimensionful constant ℓ_s has been introduced to ensure that the Hamiltonian has the appropriate units of energy. In the density description, we have the effective potential

$$V_{eff}[\phi] = \ell_s\frac{\pi^2}{6}\int dx\phi^3(x) + \int dx\frac{1}{2}\frac{\ell_s N^2\eta^2}{x^2}\phi(x) - \mu(\int dx\phi(x) - N). \tag{244}$$

We consider the coordinate x to be radial which implies that on the real half line a particle in this theory would have one turning point and should be described by a free particle wave function (with the appropriate boundary condition at the turning point). The turning point occurs at $x_o = N\eta\sqrt{\frac{\ell_s}{2\mu}}$. Then the constraint

$$N = \int_{x_o}^{L}\frac{dx}{x}\frac{1}{\pi}\sqrt{\frac{2\mu}{\ell_s}x^2 - N^2\eta^2} \tag{245}$$

can be rewritten, in the standard rescaling: $x = \sqrt{N}\bar{x}$ and $\mu = N\bar{\mu}$, in the form[72]

$$1 = \frac{\eta}{\pi}\int_{\eta\sqrt{\frac{\ell_s}{2\bar{\mu}}}}^{\frac{L}{\sqrt{N}}}\frac{d\bar{x}}{\bar{x}}\sqrt{\frac{2\bar{\mu}}{\eta^2\ell_s}\bar{x}^2 - 1}. \tag{246}$$

We make a change of variables: $z^2 = \frac{2\mu}{\eta^2\ell_s}x^2 - 1$ for which it is seen that $\frac{dx}{x}\sqrt{\frac{2\mu}{\eta^2\ell_s}x^2 - 1} = dz\frac{z^2}{z^2+1}$. And so

$$\begin{aligned} 1 &= \frac{\eta}{\pi}\int_0^{\sqrt{\frac{2\mu L^2}{\eta^2\ell_s N}-1}} dz\frac{z^2}{z^2+1} \\ &= \frac{\eta}{\pi}\int_0^{\sqrt{\frac{2\mu L^2}{\eta^2\ell_s N}-1}} dz[1 - \frac{1}{z^2+1}] \\ &= \frac{\eta}{\pi}\left[\sqrt{\frac{2\mu L^2}{\eta^2\ell_s N} - 1} - \arctan\sqrt{\frac{2\mu L^2}{\eta^2\ell_s N} - 1}\right] \end{aligned} \tag{247}$$

where long division of the integrand was carried out from the first line of (247) to the second line. We make the definition $\epsilon \equiv \sqrt{\frac{2\mu L^2}{\eta^2\ell_s N} - 1}$. Then (247) has the form

$$\frac{\pi}{\eta} = \epsilon - \arctan\epsilon. \tag{248}$$

Given the form of the function $f(x) = x - \arctan x$, it is clear that for a given value of η one will always be able to find the corresponding $\epsilon^*(\eta)$ that solves (248). In [12] it was shown that a free matrix model

[72] From now on we will suppress the bar of $\bar{x}$ and $\bar{\mu}$.

of multiple complex matrices, in the radial sector, was mapped to a system of N non-interacting fermions in a universal $1/x^2$ potential. In the second quantized approach, the higher dimensional Hamiltonian was mapped to a conformal quantum mechanical one with the strength of the potential given by $\eta^2 = \frac{N^2(d-2)^2-1}{4}$.[73] In the equations above, we wrote $N^2\eta^2$ so that the factor of N was explicit and ignored the 1/4 contribution which would be sub-leading. Therefore we note that η^2, for large N after the rescaling, is $\eta^2 \sim (d-2)^2/4 = (m-1)^2$ where $d = 2m$ is the number of Hermitian matrices of the multi-matrix model from which this strength was derived. It is clear that the strength of the potential is given by an integer $\eta = 1, 2, 3, ..$ (consistent with $m = 2, 3, 4, ..$). Taking $L = R\sqrt{N}$ the Lagrange multiplier is given by

$$\bar{\mu} = \frac{\eta^2 \ell_s}{2R^2}\left(1 + \epsilon^{*2}(\eta)\right) \tag{249}$$

where $\bar{\mu}$ is independent of N through the standard rescaling. The first 5 solutions corresponding to a given value of η are provided below

η	d	ϵ
1	4	4.49341
2	6	2.79839
3	8	2.18957
4	10	1.86373
5	12	1.65581

(250)

The fixed quantity $\bar{\mu}$ in (249) has been rescaled and no longer depends on N but certainly depends on the induced length parameter R. μ is therefore scale dependent- a fundamental feature that breaks conformal invariance. However, μ is necessary in the collective field theory in order to enforce the constraint $N = \int dx\phi(x)$. We shall see, in the next section, the link between μ and the breaking of conformal symmetry in the $sl(2, \mathbb{R})$ algebra for the collective field theory of the free multi-matrix theory.
It is worth noting that we have deduced the existence of an induced scale parameter by considering the free theory of a single Hermitian matrix model. This induced scale parameter was shown to be related to the length L through the equation $L = \sqrt{N}R$, where L is the length of the box that we consider for the system of fermions. Placing the system into a box is a common method for any system without a global minimum and a discrete energy spectrum. The reason why we are interested in such systems is that the $1/x^2$, dAFF, potential of conformal quantum mechanics appears naturally in a dimensional reduction in the fermionic description of multi-matrix systems to which we turn in the next section. This is particularly appealing as it is an inherent feature of the radial sector of the free multi-matrix theory as opposed to a particular choice of potential. This reflects a general feature of the radial sector of multi-matrix systems.

[73] See subsection 5.1.2 where we have provided the details of this dimensional reduction.

5 Radial Sector of Free Multi Matrix Systems, Breaking of Conformal Symmetry and The Emergence of 2 Dimensional Metrics

Section 4 showed that within the collective field theory formulation of the free single Hermitian matrix model there is an induced scale parameter R. This extends to the case of a dAFF ($1/x^2$) potential where the Lagrange multiplier enforcing the eigenvalue constraint was shown to be explicitly dependent on R, a feature that resulted in a breaking of the $SL(2,\mathbb{R})$ conformal invariance. In this section, we study the radial sector of a free multi-matrix model in $d = 1$ dimensions. This model is formulated in terms of m complex matrices, which are defined in terms of $d = 2m$ Hermitian matrices [12].

We have broken the section down into two subsections. The first subsection deals with the radial fermionic description of the multi-matrix model. Masuku and Rodrigues [12] have shown that the multi-matrix model of $m(d = 2m)$ complex(Hermitian) matrices is mapped to a system of N non-interacting fermions in $d + 1 = 2m + 1$ dimensions with an emergent dAFF potential provided that $m \geq 2(d \geq 4)$ and that in a second quantization it is mapped, through a dimension reduction, to a second quantized formulation of the $SL(2,\mathbb{R})$ conformal quantum mechanics of [2]. That is, the higher dimensional dAFF potential survives the dimensional reduction. We present these findings in subsection 5.1.

The second subsection provides a thorough treatment of the collective field theory of the free multi-matrix model in $d = 1$ dimensions. Again, following [12] we derive the collective field Hamiltonian which, in agreement with the radial fermionic treatment, contains an emergent dAFF potential for $m \geq 2$ but also includes a cubic interaction. Having presented a derivation of this Hamiltonian, we deduce the corresponding dilatation and special conformal operators and show that the $sl(2,\mathbb{R})$ algebra in the collective field theory is broken by the Lagrange multiplier. As a consequence, the $so(1,2)$ algebra associated with the compact generator L_0,[74] defined in terms of the $sl(2,\mathbb{R})$ generators, is also broken. We then consider the large N background of the generic collective field theory of the multi-matrix model and the quantum fluctuations about this background from which we present the emergence of the two dimensional geometry. The emergent geometry is obtained for both the free collective field Hamiltonian and the compact generator L_0.[75] In the pure AdS_2 case, the Hamiltonian and L_0 would be associated with a difference in the choice of time- i.e. global time versus Poincare time [5], however as noted above, the $sl(2,\mathbb{R})$ algebra is broken. Our treatment of the collective field $sl(2,\mathbb{R})$ algebra will make clear that from the perspective of the $SL(2,\mathbb{R})$ generators these two operators, corresponding to the different time coordinates, are associated with the free multi-matrix theory. However, from the perspective of the collective field theory the two operators correspond to two different Hamiltonians- one associated with the free multi-matrix theory and one with a potential. As a consequence, in the collective field theory the two operators will lead to distinct emergent metrics. This would appear to be in-line with the interpretation of the operators being related by a change in time coordinate but we reiterate that the broken $SL(2,\mathbb{R})$ symmetry suggests that these metrics are more likely associated with a near-AdS_2 geometry.

5.1 Free Radial Fermionic Description

5.1.1 Radial fermionic description and emergence of a $1/r^2$ potential

In the context of investigating the AdS_2/CFT_1 correspondence, much attention has been given to the conformal quantum mechanics proposed by de Alfaro, Fubini and Furlan in [2]. Investigations of the $SL(2,\mathbb{R})$ conformal symmetry has established an isomorphism with $SO(1,2)$, the isometry group of AdS_2 (see subsections (2.2.2) and (3.2)). It is with this matching of symmetries in mind that we turn our

[74] Strictly speaking it is correct to refer to this algebra as the $so(1,2)$ algebra rather than the $sl(2,\mathbb{R})$ algebra, however due to their equivalence we shall sometimes relax this restriction.

[75] Strominger introduced the compact operator L_0 in a second quantized system of fermions in [5]. Masuku and Rodrigues re-expressed this operator in the collective field theory and computed its large N background in [12].

attention to the fermionic description of the multi-matrix model. It has been shown [46]that the Laplacian of the complex (2 Hermitian) matrix model quantum mechanics separates into two parts. One relevant to the radial eigenvalue degrees of freedom and another that transforms angular degrees of freedom.[76] We are able to restrict our attention to radial wave functions by choosing to consider diagonalizable potentials-that is potentials which are equivalent to functions of the radial eigenvalues. However, we focus on the free case. For a multi-complex matrix model we consider the Hermitian positive definite matrix

$$\sum_{A=1}^{m} Z_A^\dagger Z_A \tag{251}$$

constructed from a set of complex matrices, $Z_1 = X_1 + iX_2, Z_2 = X_3 + iX_4, .., Z_m = X_{2m-1} + iX_{2m}$, where we have indicated their definition in terms of the set of Hermitian matrices $\{X_i\}$. Evidently, for a m complex matrix theory there are $d = 2m$ Hermitian matrices. The eigenvalues of (251) are $\rho_i = r_i^2$, which we take to be radial eigenvalues. It is straightforward to generalize the radial contribution to the 2 Hermitian matrix model Laplacian (388) to that corresponding to m complex matrices

$$\bigtriangledown_R^2 = \sum_i \frac{1}{\prod_k r_k^{2m-1} \triangle^2(r^2)} \frac{\partial}{\partial r_i} \left(\prod_k r_k^{2m-1} \triangle^2(r^2) \frac{\partial}{\partial r_i} \right). \tag{252}$$

Under the change of variables $r_i^2 \to \rho_i$

$$\bigtriangledown_R^2 = 4 \sum_i \frac{1}{\rho_i^{m-1} \triangle^2(\rho)} \frac{\partial}{\partial \rho_i} \left(\rho_i^m \triangle^2(\rho) \frac{\partial}{\partial \rho_i} \right). \tag{253}$$

The Schrodinger equation, which acts on symmetric wave functions of the radial eigenvalues $\Phi(\rho_i)$ is $\left(-\frac{1}{2} \bigtriangledown_R^2 + V(\rho_i)\right) \Phi(\rho_i) = E\Phi(\rho_i)$. This is mapped to a system of non-interacting fermions by defining $\Phi(\rho_i) \equiv \Psi(\rho_i)/\triangle$ for completely antisymmetric wave functions $\Psi(\rho_i)$ of the eigenvalues and the generalized Vandermonde determinant $\triangle(\rho) = \prod_{i>j} \rho_i^{\frac{m-1}{2}} \rho_j^{\frac{m-1}{2}} (\rho_i - \rho_j)$. In analogy with the single Hermitian matrix model (see the discussion below (211)), the Vandermonde determinant is associated with the square root of the determinant of the metric in the coordinate invariant integration measure of the partition function when writing the measure in terms of the eigenvalues exclusively. The contribution from the radial eigenvalues, as opposed to the angular degrees of freedom which can be integrated out [47], have been computed in the collective field theory treatment to follow. We simply note that this Vandermonde determinant appears in the collective field theory Jacobian (288) (see also (407) of appendix E). The radial Laplacian now acts on the anti-symmetric wave functions

$$-\frac{1}{2}\left(4 \sum_i \left[\left(\frac{1}{\rho_i^{m-1} \triangle} \frac{\partial}{\partial \rho_i} \triangle \right) \rho_i^m \left(\triangle \frac{\partial}{\partial \rho_i} \frac{1}{\triangle} \right) \right] + v(\rho_i) \right) \Psi(\rho_i) = E\Psi(\rho_i). \tag{254}$$

A detailed proof that

$$\begin{aligned} &4 \sum_i \left[\left(\frac{1}{\rho_i^{m-1} \triangle} \frac{\partial}{\partial \rho_i} \triangle \right) \rho_i^m \left(\triangle \frac{\partial}{\partial \rho_i} \frac{1}{\triangle} \right) \right] \\ &= \left(\sum_i \frac{4}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - \frac{(N^2-1)(m-1)^2}{\rho_i} \right) \end{aligned} \tag{255}$$

[76] We review the case of two Hermitian matrices (a single complex matrix) with an explicit parameterization. It appears in appendix C with a brief discussion of the radial eigenvalue and its significance for the radial fermionic description.

appears in the appendix of [12]. Following this proof we note that (254) contains $\triangle \frac{\partial}{\partial \rho_i} \frac{1}{\triangle} = -\frac{\partial \ln \triangle}{\partial \rho_i} + \frac{\partial}{\partial \rho_i} = \left(-\sum_{j(\neq i)} \frac{1}{\rho_i - \rho_j} - \frac{(N-1)(m-1)}{2\rho_i} + \frac{\partial}{\partial \rho_i} \right)$ and $\frac{1}{\triangle} \frac{\partial}{\partial \rho_i} \triangle = \frac{\partial \ln \triangle}{\partial \rho_i} + \frac{\partial}{\partial \rho_i} = \left(\frac{\partial}{\partial \rho_i} + \frac{(N-1)(m-1)}{2\rho_i} + \sum_{j(\neq i)} \frac{1}{\rho_i - \rho_j} \right)$ which allows us to re-write (254), after some simplification, as

$$4 \sum_i \left[\frac{1}{\rho_i^b} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} - \frac{(ab + a^2)}{\rho_i} - (m - 2a) \sum_{j(\neq i)} \frac{1}{\rho_i - \rho_j} \right. \\ \left. + \sum_{j(\neq i)} \frac{\rho_i}{(\rho_i - \rho_j)^2} - \sum_{k(\neq i)} \sum_{j(\neq i)} \frac{\rho_i}{(\rho_i - \rho_k)(\rho_i - \rho_j)} \right] \Psi(\rho_i) = E\Psi(\rho_i), \tag{256}$$

where $a = (N-1)(m-1)/2$ and $b = m - 1$. The third term on the LHS of (256) is zero and the next two consecutive terms are zero by (393). Therefore

$$\sum_i \left(-\frac{2}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \frac{\partial}{\partial \rho_i} + \frac{(N^2-1)(m-1)^2}{2\rho_i} + V(\rho_i) \right) \Psi(\rho_i) = E\Psi(\rho_i) \tag{257}$$

for which $V(\rho) = -\frac{1}{2}v(\rho_i)$. This may be expressed in terms of r_i by the appropriate changes as

$$\sum_i \left(\frac{-1}{2r_i^{2m-1}} \frac{\partial}{\partial r_i} r_i^{2m-1} \frac{\partial}{\partial r_i} + \frac{(N^2-1)(m-1)^2}{2r_i^2} + V(r_i) \right) \Psi(r_i) = E\Psi(r_i). \tag{258}$$

To summarize the above results from [12], the multi complex matrix quantum mechanics has been mapped to a system of fermions in $d + 1 = 2m + 1$ spacetime dimensions[77] restricted to 's-state' wave functions. The key contribution of this treatment is that for the free theory (multi matrix) there is an emergent $1/r^2$ universal potential experienced by the N non-interacting fermions. The choice of wording 'emergent' reflects the fact that this is occurs only for $m \geq 2$. The strength of this potential depends only on the number of complex matrices of the corresponding matrix model and N^2.

5.1.2 Second quantization and dimensional reduction to conformal quantum mechanics

Following [12] and [5], we consider the second quantization of the conformal quantum mechanics of [2]. We re-state the results of our review of the $SL(2,\mathbb{R})$ generators in subsection 3.2 here with a slight change to the notation which is convenient in what follows where we reserve the coordinate label, r, for the higher dimensional radial fermionic coordinates.[78] The first quantized version of the $SL(2,\mathbb{R})$ generators of (137) and (138) and their algebra (141) are

$$\hat{h} = \tfrac{1}{2}(p^2 + \tfrac{\eta^2}{x^2}), \qquad \hat{k} = \tfrac{x^2}{2}, \qquad \hat{d} = \tfrac{1}{2}(xp + px) \tag{259}$$

and

$$[\hat{d}, \hat{h}] = 2i\hat{h}, \qquad [\hat{d}, \hat{k}] = -2i\hat{k}, \qquad [\hat{h}, \hat{k}] = -i\hat{d}. \tag{260}$$

The corresponding $SO(1,2)$ generators and algebra are given by the first quantized version of (149) and (150) consistent with the notation of (259). Using the map from the matrix theory to the system of non-interacting fermions one can express (as was done in [5] and [12]) the conformal generators corresponding

[77] The dimensional argument here is that it is well known that the radially symmetric Laplacian in N spatial dimensions has a radial part that goes like $\frac{1}{r^{N-1}} \frac{\partial}{\partial r} r^{N-1} \frac{\partial}{\partial r}$.

[78] We wish to point out that the second quantized generators of [5] are for a system of fermions in the plane whereas those of [12] are in one dimensional with a one dimensional radial coordinate understood. To keep this distinction clear, and for reasons that will become clear below, we prefer to use x as opposed to r at this stage.

to a matrix model with the $1/M^2$ potential term in a second quantized formulation. These operators become[79]

$$\hat{H} = \int dx \Psi^\dagger(x)(\frac{p^2}{2} + \frac{\eta^2}{2x^2})\Psi(x),$$
$$\hat{K} = \int dx \Psi^\dagger(x)\frac{x^2}{2}\Psi(x),$$
$$\hat{D} = \int dx \Psi^\dagger(x)\frac{1}{2}(xp + px)\Psi(x). \tag{261}$$

This idea can be applied to the free multi-matrix system. The number of spatial dimensions in the fermionic description is $d = 2m$. Then, mapping the matrix theory to the system of $d+1$ dimensional non-interacting fermions one immediately finds

$$\hat{h} = \frac{1}{2}\frac{1}{r^{d-1}}p_r r^{d-1} p_r + \frac{(N^2-1)(d-2)^2}{8r^2}$$
$$= \frac{p^2}{2} - \frac{i(d-1)}{2r}p + \frac{(N^2-1)(d-2)^2}{8r^2} \tag{262}$$

and

$$\hat{k} = \frac{r^2}{2}. \tag{263}$$

The algebra allows us to deduce the generalization of $\hat{d}$ to higher dimensions. From (262) and (263):

$$[\hat{h}, \hat{k}] = -\frac{1}{2} - r\frac{\partial}{\partial r} - \frac{(d-1)}{2r}r = -i(rp - \frac{id}{2}). \tag{264}$$

This implies the following explicit form of the dilatation operator

$$\hat{d} = rp - \frac{id}{2}. \tag{265}$$

Together (262), (263) and (265) close the $sl(2,\mathbb{R})$ algebra. We can define the second quantized generators for the radial sector by defining the appropriate anti-commutation relation: $\{\Psi(r), \Psi^\dagger(r')\} = \frac{\delta(r-r')}{r^{d-1}}$.[80] The general second quantized generator $\hat{O}$ takes the form $\hat{O} = \int dr r^{d-1}\Psi^\dagger(r)\hat{o}\Psi(r)$, however, one can make a field redefinition $\tilde{\Psi}^\dagger(r) \equiv r^{\frac{d-1}{2}}\Psi^\dagger(r)$ and $\tilde{\Psi}(r) \equiv r^{\frac{d-1}{2}}\Psi(r)$ which maps the higher dimensional second quantized system to that of a one dimensional one (i.e. ordinary quantum mechanics). Under the field re-definition there is an associated first quantized operator re-definition required since one cannot simply pull the Jacobian from the left of the first quantized operator to the right of it if there is a derivative in the operator[81]. The re-definition is such that $J\hat{\Psi}^\dagger\hat{o}\hat{\Psi} \to J^{\frac{1}{2}}\hat{\Psi}^\dagger J^{-\frac{1}{2}}\hat{o}J^{\frac{1}{2}}\hat{\Psi}J^{\frac{1}{2}} \equiv \tilde{\Psi}^\dagger\hat{o}'\tilde{\Psi}$ where $\hat{o}' = J^{\frac{1}{2}}\hat{o}J^{-\frac{1}{2}}$. This similarity-type transformation removes terms linear in the derivative when applied to the Hamiltonian.

[79] The usual anti-commutation relation: $\{\Psi(x), \Psi^\dagger(x')\} = \delta(x - x')$ applies to (261).

[80] In the radial sector we have the inner product definition $\langle\Psi|\Psi\rangle = \int dr r^{d-1}\Psi^*(r)\Psi(r) = 1$. This is consistent with the coordinate basis orthogonality relation $\langle r'|r\rangle = \frac{\delta(r'-r)}{r^{d-1}}$ or by making use of a discrete basis $\sum_n |n\rangle\langle n| = 1$ this becomes $\sum_n \Psi_n^*(r')\Psi_n(r) = \frac{\delta(r'-r)}{r^{d-1}}$. One can confirm that, with the anti-commutation relation $\{c_k, c_{k'}^\dagger\} = \delta_{kk'}$, the relation $\{\Psi(r), \Psi^\dagger(r')\} = \frac{\delta(r-r')}{r^{d-1}}$ holds.

[81] For this reason only the Hamiltonian and the dilatation operators are affected by this transformation

The transformation maps the dilatation operator to the form $rp - \frac{i}{2}$ which is equivalent to $\frac{1}{2}(rp+pr)$. The operators become manifestly Hermitian. Then

$$\hat{H} = \int dr \tilde{\Psi}^\dagger(r)(\frac{p^2}{2} + \frac{N^2(d-2)^2-1}{8r^2})\tilde{\Psi}(r),$$
$$\hat{K} = \int dr \tilde{\Psi}^\dagger(r)\frac{r^2}{2}\tilde{\Psi}(r),$$
$$\hat{D} = \int dr \tilde{\Psi}^\dagger(r)\frac{1}{2}(rp+pr)\tilde{\Psi}(r) \tag{266}$$

close the $sl(2,\mathbb{R})$ algebra. Remarkably, the multi-matrix fermionic description, which contains an emergent $1/r^2$ potential if and only if $d(m) \geq 4(2)$, has been mapped to the second quantized generators of the $SL(2,\mathbb{R})$ group for a quantum mechanical ($d=1$) system. Despite only emerging in the higher dimensional case of $d+1 \geq 5$, the system has undergone a dimensional reduction in which the $1/r^2$ potential has survived all the way down to $d=1$ dimensions [12]. That is, the dimensional reduction of the multi-matrix theory in the radial fermionization has been mapped to a second quantized formulation of ordinary conformal quantum mechanics. In this quantum mechanical system, the conformal potential has well defined strength $\eta^2 = \frac{N^2(d-2)^2-1}{4}$ in terms of d, twice the number of complex matrices in the original formulation before the dimensional reduction. The symmetry of this theory matches that of the AdS_2 isometry group $SO(1,2)$ with generators (140) which motivates the fact that there is a possible emergence of AdS_2 geometry in the matrix model. The wave functions appearing in the second quantized operators (266) can be solved. We have included these wave function solutions in appendix A (see subsection A.2 where they are expressed in terms of the asymptotic form of the Bessel function due to the large N limit).

5.2 Collective Field Theory Description

5.2.1 Collective field description of a multi-matrix system

We now begin with the collective field formulation of a multi-matrix system.[82] The collective field theory [11] provides an alternative description of a many body quantum system in instances for which the interaction potential appearing in the many body Hamiltonian, say $V = V(q_1, q_2, ..., q_m)$ for the m particle degrees of freedom q_i, can be expressed in terms of an infinite combination of its finite set of degrees of freedom: $\phi(x) \equiv f(x; q_1, ..., q_m)$ and the wave functions, $\psi(\{q_i\})$, corresponding to the many body Hamiltonian can be expressed as a functional of $\phi(x)$: $\psi(\{q_i\}) = \Phi[\phi]$. We call $\phi(x)$ the collective field. In changing variables to the collective field there is a non-trivial Jacobian that can be used to define a new wave functional $\Psi[\phi] \equiv J^{1/2}[\phi]\Phi[\phi]$ with an inner product of the standard form

$$(\Psi, \Psi) = \int [d\phi]\Psi^*[\phi]\Psi[\phi]. \tag{267}$$

The change of variables also modifies the kinetic operator in the Hamiltonian which, when expressed in terms of the collective field, has the form (see 368)

$$H_{coll} = \frac{1}{2}\int dx \int dy \pi_x \Omega_{xy} \pi_y + \frac{1}{8}\int dx \int dy [\omega_x + i\int dz \pi_z \Omega_{xz}]\Omega^{-1}_{xy}[\omega_y + i\int dz' \pi_{z'}\Omega_{yz'}]$$
$$- \frac{1}{4}\int dx \frac{\delta \omega_x}{\delta\phi(x)} - \frac{1}{4}\int dx \int dy \frac{\delta^2 \Omega_{xy}}{\delta\phi(x)\delta\phi(y)}. \tag{268}$$

in terms of the functional variables

[82] The general collective field theory has been reviewed in some detail in appendix B.

$$\omega(x;[\phi]) \equiv -\sum_i \frac{\partial^2\phi(x)}{\partial {q_i}^2}$$
$$\Omega(x,y;[\phi]) \equiv \sum_i \frac{\partial\phi(x)}{\partial q_i}\frac{\partial\phi(y)}{\partial q_i}, \tag{269}$$

known as the 'splitting' and 'joining' operators respectively.

We consider the collective field theory of the m complex matrices[83] Z_A, $A = 1,2,3,..,m$ introduced in the radial description. The remainder of 5.2.1 presents a derivation of the multi-matrix collective field theory Hamiltonian which consists of the work of [12] that relied on the previous works of [46, 47]. We have attempted to present the derivation making use of all three references for completeness and for the readers benefit. The requirement that the wave function can be expressed as a functional of the collective field is readily satisfied when one focuses on the invariant subspace of the Hilbert space. We focus on the matrix (251):[84]

$$\sum_A Z^\dagger_{{}_A} Z_{{}_A} \tag{270}$$

which is Hermitian positive definite and can be used to define such a set of invariants for the collective field theory. These invariants are[85]

$$\phi_k \equiv Tr\left(e^{ik\sum_B Z^\dagger_{{}_B} Z_{{}_B}}\right) = \sum_i e^{ikr_i^2},$$
$$\phi(\rho) = \int \frac{dk}{2\pi} e^{ik\rho}\phi_k = \sum_i \delta(\rho - r_i^2). \tag{271}$$

We have defined the eigenvalues of (270) to be r_i^2. By taking the power series expansion of (271) it is found that

$$\frac{\partial\phi_k}{\partial(Z_{{}_A})^\dagger_{ij}} = ik\left(Z_{{}_A} e^{ik\sum_B Z^\dagger_{{}_B} Z_{{}_B}}\right)_{ij},$$
$$\frac{\partial\phi_k}{\partial(Z_{{}_A})_{ij}} = ik\left(e^{ik\sum_B Z^\dagger_{{}_B} Z_{{}_B}} Z^\dagger_{{}_A}\right)_{ji}. \tag{272}$$

The so-called 'joining' operator can be obtained quite easily from (272)

$$\Omega_{kk'} \equiv \frac{\partial\phi_k}{\partial(Z_{{}_A})^\dagger_{ij}}\frac{\partial\phi_{k'}}{\partial(Z_{{}_A})_{ji}} = -kk'Tr\left(\sum_A Z^\dagger_{{}_A} Z^{{}^A} e^{ik\sum_B Z^\dagger_{{}_B} Z_{{}_B}}\right)$$
$$= -kk'\sum_i r_i^2 e^{i(k+k')r_i^2}. \tag{273}$$

Taking the Fourier transform and making use of the density of eigenvalues (271), we find that

[83] Such a case may be of interest when $m = 3$ as this would correspond to the 3 Higgs fields of the bosonic sector of $\mathcal{N} = 4$ super Yang-Mills theory in the context of the AdS/CFT correspondence.

[84] Each term in (270) is Hermitian positive definite and therefore the entire sum is an Hermitian positive definite matrix. As a consequence (270) can be diagonalized by a unitary similarity transformation.

[85] We use ρ instead of x to be suggestive of the radial interpretation of the eigenvalues r_i^2 of the matrices (270). Then $r_i^2 \equiv \rho_i$.

$$\Omega_{\rho\rho'} = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ik\rho} e^{-ik'\rho'} \phi_{kk'} = \partial_\rho \partial_{\rho'} \left[\rho\phi(\rho)\delta(\rho - \rho') \right]. \tag{274}$$

The 'splitting' operator requires some careful attention. By considering the case of, for example, 3 complex matrices one finds that

$$\begin{aligned}
&\sum_A \frac{\partial^2 \phi_k}{\partial (Z_A)^\dagger_{ij} \partial (Z_A)_{ba}} = \sum_A \frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} ik \left(e^{ik \sum_B Z_B^\dagger Z_B} Z_A^\dagger \right)_{ab} \\
&= 3ik\delta_{jb}(e^{ik\sum_B Z_B^\dagger Z_B})_{ai} + ik \left[ik\delta_{ai} \left(\sum_B Z_B^\dagger Z_B \right)_{jb} \right. \\
&+ \frac{(ik)^2}{2}\delta_{ai} \left(Z_1 \left(\sum_B Z_B^\dagger Z_B \right) Z_1^\dagger + Z_2 \left(\sum_B Z_B^\dagger Z_B \right) Z_2^\dagger \right. \\
&\left. \left. + Z_3 \left(\sum_B Z_B^\dagger Z_B \right) Z_3^\dagger \right)_{jb} + \frac{(ik)^2}{2} \left(\sum_B Z_B^\dagger Z_B \right)_{ai} \left(\sum_B Z_B^\dagger Z_B \right)_{jb} + ... \right].
\end{aligned} \tag{275}$$

For $a = i$ and $b = j$ we recognize the second term in (275) to be[86]

$$\begin{aligned}
&(ik)^2 \int_0^1 d\alpha \left(e^{ik\alpha \left(\sum_B Z_B^\dagger Z_B \right)} \right)_{ii} \left(\sum_A Z_A^\dagger Z_A e^{ik(1-\alpha)\left(\sum_B Z_B^\dagger Z_B \right)} \right)_{jj} \\
&\qquad = -k \int_0^k dk' \phi_{k'} Tr \left(\sum_A Z_A^\dagger Z_A e^{i(k-k')\left(\sum_B Z_B^\dagger Z_B \right)} \right).
\end{aligned} \tag{276}$$

We deduce the general result for m complex matrices

$$\begin{aligned}
\omega_k &\equiv \sum_A \frac{\partial^2 \phi_k}{\partial (Z_A)^\dagger_{ij} \partial (Z_A)_{ji}} \\
&= -k \int_0^k dk' \phi_{k'} Tr \left(\sum_A Z_A^\dagger Z_A e^{i(k-k')\left(\sum_B Z_B^\dagger Z_B \right)} \right) + ikmN\phi_k.
\end{aligned} \tag{277}$$

Considering all Trace operations in (277), including those of ϕ_k , we find

$$\omega_k = -k \sum_{i,j} \int_0^k dk' e^{ik' r_i^2} e^{i(k-k')r_j^2} r_j^2 + ikmN \sum_i e^{ikr_i^2}. \tag{278}$$

We separate the first term of (278) into two terms

[86] Again, this may be confirmed by evaluating the integral of (276) for a system of 3 complex matrices. In going from the first line of (276) to the second we make the change of variables $k' = \alpha k$.

$$\omega_k = -k \sum_{i,j(\neq i)} \int_0^k dk' e^{ik' r_i^2} e^{i(k-k') r_j^2} r_j^2 - k \sum_i \int_0^k dk' e^{ikr_i^2} r_i^2$$
$$+ ikmN \sum_i e^{ikr_i^2}$$
$$= ik \sum_{i,j(\neq i)} \left(\frac{r_j^2 e^{ikr_i^2} - r_j^2 e^{ikr_j^2}}{r_i^2 - r_j^2} \right) - k^2 \sum_i r_i^2 e^{ikr_i^2} + ikmN \sum_i e^{ikr_i^2}. \quad (279)$$

It is convenient to re-write the second term of this equation as: $ik \sum_{i,j(\neq i)} \left(\frac{r_j^2 e^{ikr_i^2} + r_j^2 e^{ikr_j^2}}{r_i^2 - r_j^2} \right) - 2ik \sum_{i,j(\neq i)} \frac{r_j^2 e^{ikr_j^2}}{r_i^2 - r_j^2}$. By considering the case of 3×3 complex matrices one can deduce the simplified form of this term to be: $-2ik \sum_{i,j(\neq i)} \frac{r_j^2 e^{ikr_j^2}}{r_i^2 - r_j^2} - ik(N-1) \sum_i e^{ikr_i^2}$. The 'splitting' operator becomes

$$\omega_k = -2ik \sum_{i,j(\neq i)} \frac{r_j^2 e^{ikr_j^2}}{r_i^2 - r_j^2} - k^2 \sum_i r_i^2 e^{ikr_i^2} + ik \sum_i e^{ikr_i^2} + ikN(m-1) \sum_i e^{ikr_i^2}. \quad (280)$$

The density of eigenvalues allows us to re-express functions of the eigenvalues in the following way: $\sum_i f(r_i^2) = \int d\rho f(\rho) \phi(\rho)$. Then by taking the Fourier transform we find that

$$\omega_\rho = -\partial_\rho \left[\rho \phi(\rho) \left(2 \fint dy \frac{\phi(\rho')}{\rho - \rho'} - \frac{\partial_\rho \phi(\rho)}{\rho} + \frac{N(m-1)}{\rho} \right) \right]. \quad (281)$$

The term $\frac{\partial_\rho \phi(\rho)}{\rho}$ in (281) is sub-leading in large N[87] so that

$$\omega_\rho = -\partial_\rho \left[\rho \phi(\rho) \left(2 \fint d\rho' \frac{\phi(\rho')}{\rho - \rho'} + \frac{N(m-1)}{\rho} \right) \right]. \quad (282)$$

The Hermiticity requirement of the collective field theory reads [11] (see(366))

$$-\omega_\rho + i \int d\rho' (\pi_{\rho'} \Omega_{\rho\rho'}) - 2 \int d\rho' \Omega_{\rho\rho'} C_{\rho'} = 0 \quad (283)$$

where π_ρ is the canonical conjugate momentum to $\phi(\rho)$ and $C_\rho = -\frac{1}{2} \frac{\partial \ln J}{\partial \phi(\rho')}$. The second term of (283) is zero. With (274) and (282) and integrating by parts we establish that the Jacobian satisfies

$$\partial_\rho \frac{\partial \ln J}{\partial \phi(\rho)} = 2 \fint d\rho' \frac{\phi(\rho')}{\rho - \rho'} + \frac{N(m-1)}{\rho}. \quad (284)$$

In the 2 Hermitian matrix theory the term $N(m-1)/\rho$ is not present and the Jacobian is proven to be the generalized Vandermonde determinant of the radial eigenvalues (see [46]).[88] The solution $\ln J$ is, by inspection, identified to be

[87] The factors of N are made explicit after a rescaling of the fields- see (314).

[88] In a zero dimensional Hermitian matrix model field theory it is known that the Vandermonde determinant is associated with an inter-eigenvalue repulsive potential [10].

$$\ln J = \int d\rho'' \phi(\rho'') \fint d\rho' \phi(\rho') \ln|\rho - \rho'| + N(m-1) \int d\rho' \phi(\rho') \ln \rho'. \tag{285}$$

The eigenvalue density allows us to rewrite the equation for the Jacobian as

$$\ln J = \sum_{i,j(\neq i)} \ln|\rho_i - \rho_j| + N(m-1) \sum_i \ln \rho_i. \tag{286}$$

Noting that $N(m-1)\sum_i \ln \rho_i = \sum_{j,i} \ln \rho_i^{m-1}$, which we split into three terms $\frac{1}{2}\sum_{i,j(\neq i)} \ln \rho_i^{m-1} + \frac{1}{2}\sum_{j,i(\neq j)} \ln \rho_j^{m-1} + \sum_i \ln \rho_i^{m-1}$, leads to

$$\ln J = \ln \prod_i \rho_i^{m-1} \prod_{i \neq j} \rho_i^{\frac{m-1}{2}} \rho_j^{\frac{m-1}{2}} |\rho_i - \rho_j|. \tag{287}$$

In other words[89]

$$J = \prod_i \rho_i^{m-1} \prod_{i \neq j} \rho_i^{\frac{m-1}{2}} \rho_j^{\frac{m-1}{2}} |\rho_i - \rho_j| = \prod_i \rho_i^{m-1} \prod_{i > j} \rho_i^{m-1} \rho_j^{m-1} (\rho_i - \rho_j)^2. \tag{288}$$

We have included an alternative derivation of the Jacobian, that of [47], which uses the Schwinger-Dyson equations. This appears in appendix E.The Jacobian was important for the Laplace-Beltrami operator where we considered the radial fermionic description (see the discussion above (254)).

The collective field Hamiltonian- in schematic form- (see (368) in appendix B):

$$\begin{aligned} H_{coll} = &\frac{1}{2} \int_\rho \int_{\rho'} \pi_\rho \Omega_{\rho\rho'} \pi_{\rho'} + \frac{1}{8} \int_\rho \int_{\rho'} \left[\omega_\rho + i \int_{\rho''} \pi_{\rho''} \Omega_{\rho\rho''}\right] \Omega^{-1}_{\rho\rho'} \left[\omega_{\rho'} + i \int_{\rho'''} \pi_{\rho'''} \Omega_{\rho'\rho'''}\right] - i\frac{1}{4} \int_\rho \pi_\rho \omega_\rho \\ &+ \frac{1}{4} \int_\rho \int_{\rho'} \pi_\rho \pi_{\rho'} \Omega_{\rho\rho'} \end{aligned} \tag{289}$$

becomes

$$H = \frac{1}{2} \int_\rho \int_{\rho'} \pi_\rho \Omega_{\rho\rho'} \pi_{\rho'} + \frac{1}{8} \int_\rho \int_{\rho'} \omega_\rho \Omega^{-1}_{\rho\rho'} \omega_{\rho'} \tag{290}$$

$$\begin{aligned} &= \frac{1}{2} \int d\rho \int d\rho' \pi_\rho \left(\partial_\rho \partial_{\rho'} [\rho \phi(\rho) \delta(\rho - \rho')]\right) \pi_{\rho'} \\ &+ \frac{1}{8} \int d\rho \int d\rho' \partial_\rho [\rho \phi(\rho) \left(2 \fint d\rho'' \frac{\phi(\rho'')}{\rho - \rho''} + \frac{N(m-1)}{\rho}\right)] \Omega^{-1}_{\rho\rho'} \partial_{\rho'} [\rho' \phi(\rho') \left(\fint d\rho'' \frac{\phi(\rho'')}{\rho' - \rho''} + \frac{N(m-1)}{\rho'}\right)] \end{aligned} \tag{291}$$

We have discarded the term $i\frac{1}{4}\int_\rho \pi_\rho \omega_\rho$ since it will prove to be sub-leading once we make powers of N explicit. The terms $\frac{1}{4}\int \rho \int_{\rho'} \pi_\rho \pi_{\rho'} \Omega_{\rho\rho'}$ and $\int_{\rho'''} \pi_{\rho'''} \Omega_{\rho'\rho'''}$ are easily found to be zero. We integrate both terms by parts twice to obtain

[89] It is evident that for $m = 1$ this result confirms that of the single complex matrix (see [46]).

$$H = \frac{1}{2}\int d\rho \partial_\rho \pi_\rho \rho\phi(\rho)\partial_\rho \pi_\rho$$
$$+\frac{1}{8}\int d\rho \int d\rho' [\rho\phi(\rho)\left(2\fint d\rho'' \frac{\phi(\rho'')}{\rho-\rho''}+\frac{N(m-1)}{\rho}\right)]\left(\partial_\rho\partial_{\rho'}\Omega^{-1}_{\rho\rho'}\right)[\rho'\phi(\rho')\left(\fint d\rho''\frac{\phi(\rho'')}{\rho'-\rho''}+\frac{N(m-1)}{\rho'}\right)]. \tag{292}$$

The Hamiltonian requires the inverse joining operator Ω^{-1}. We avoid a calculation of this inverse in any detail by noting that the equation $\int dp''\Omega^{-1}_{\rho\rho''}\Omega_{\rho''\rho'} = \delta(\rho-\rho')$ can be used to establish the result $\partial_\rho\partial_{\rho'}\Omega^{-1}_{\rho\rho'} = \frac{\delta(\rho-\rho')}{\rho\phi(\rho)}$. Then:

$$H = \frac{1}{2}\int d\rho \partial_\rho \pi_\rho \rho\phi(\rho)\partial_\rho \pi_\rho$$
$$+\frac{1}{8}\int d\rho\rho\phi(\rho)\left(2\fint d\rho''\frac{\phi(\rho'')}{\rho-\rho''}\right)^2+\frac{N(m-1)}{2}\int d\rho\phi(\rho)\fint d\rho''\frac{\phi(\rho'')}{\rho-\rho''}+\frac{N^2(m-1)^2}{8}\int d\rho\frac{\phi(\rho)}{\rho}. \tag{293}$$

The term $N(m-1)/2\int d\rho\phi(\rho)\fint d\rho''\frac{\phi(\rho'')}{\rho-\rho''}$ can be re-expressed as $N(m-1)/4\sum_i\sum_{j(\neq i)}\frac{1}{\rho_i-\rho_j} = 0$.[90] The free theory Hamiltonian therefore takes the form

$$H = \frac{1}{2}\int d\rho\partial_\rho\pi(\rho)\rho\phi(\rho)\partial_\rho\pi(\rho)+\frac{1}{2}\int d\rho\rho\phi(\rho)\rho\left(\fint d\rho'\frac{\phi(\rho')}{\rho-\rho'}\right)^2+\frac{N^2(m-1)^2}{8}\int d\rho\frac{\phi(\rho)}{\rho}. \tag{294}$$

It should also be noted that since we take the interpretation then that the eigenvalues ($x = r^2 = \rho$) are radial and range from $0 \to \infty$ the integration limits also range from $0 \to \infty$. The constraint that $\int_0^\infty d\rho\phi(\rho) = N$ is enforced by the introduction of a Lagrange multiplier μ and is made manifest in the Hamiltonian through the inclusion of a term of the form $\mu\left(N - \int d\rho\phi(\rho)\right)$ in (294). The possibility of a potential term in the original Hamiltonian, i.e. a term of the form $Tr\left(V(M)\right)$ in the matrix Hamiltonian, can be accommodated in (294) as well by introducing a term $\int d\rho V(\rho)\phi(\rho)$. The second term in (294) is easily simplified by making use of the identity $\int_{-\infty}^\infty dr\Phi(r)\left[\fint_{-\infty}^\infty dr'\frac{\Phi(r')}{r-r'}\right]^2 = \frac{\pi^2}{3}\int_{-\infty}^\infty dr\Phi^3(r)$. This is done as follows: we change variables from $\rho \to r$ (noting that $\rho = r^2$) and extend the domain of the collective field to the full real line. We define $\Phi(r) \equiv 2r\phi(r^2)$; since we are considering a radial coordinate in 1 dimension we demand that $\Phi(r) = \Phi(-r)$. Then

$$\int_0^\infty d\rho\rho\phi(\rho)\left(\fint d\rho'\frac{\phi(\rho')}{\rho-\rho'}\right)^2 = \frac{1}{8}\int_{-\infty}^\infty dr\Phi(r)\left[\fint_{-\infty}^\infty dr'\frac{\Phi(r')}{r-r'}\right]^2 = \frac{\pi^2}{24}\int_{-\infty}^\infty dr\Phi^3(r) = \frac{\pi^2}{3}\int_0^\infty d\rho\rho\phi^3(\rho). \tag{295}$$

So we have the Hamiltonian

$$H = \frac{1}{2}\int d\rho\partial_\rho\pi(\rho)\rho\phi(\rho)\partial_\rho\pi(\rho)+\frac{\pi^2}{6}\int_0^\infty d\rho\rho\phi^3(\rho)+\frac{N^2(m-1)^2}{8}\int d\rho\frac{\phi(\rho)}{\rho}+\mu\left(N-\int d\rho\phi(\rho)\right). \tag{296}$$

In radial coordinates one has the collective field $\Phi(r)$. Its canonical conjugate is some function $\tilde{\pi}(r)$ such that

[90] This is possible since the eigenvalue density is $\phi(\rho) = \sum_i \delta(\rho-\rho_i)$.

$$[\phi(\rho),\pi(\rho')] = [\frac{\Phi(r)}{2r},\tilde{\pi}(r')] = i\frac{\delta(r-r')}{2r}$$
$$\Rightarrow [\Phi(r),\tilde{\pi}(r')] = i\delta(r-r') \tag{297}$$

so that $\pi(\rho)$ does not transform (i.e. $\pi(\rho) = \tilde{\pi}(r)$). Then the Hamiltonian in radial coordinates is

$$H = \frac{1}{8}\int_0^\infty dr\,(\partial_r\tilde{\pi}(r))\,\Phi(r)\,(\partial_r\tilde{\pi}(r)) + \frac{\pi^2}{24}\int_0^\infty dr\Phi^3(r) + \frac{N^2(m-1)^2}{8}\int_0^\infty dr\frac{\Phi(r)}{r^2} + \mu\left(N - \int dr\Phi(r)\right). \tag{298}$$

In the Hamiltonian (298) we have chosen not to include a potential term. This makes explicit a significant feature. Indeed (298) is an important result of this section. We note that a $1/r^2$ term has emerged in the collective field theory description from a free multi-complex matrix model and this is true as long as the number of complex(Hermitian) matrices is greater than or equal to 2(4) [12]. This is the conformal potential of which we are now familiar with. We also note that the purely kinetic piece of the Hamiltonian includes a cubic term [12]- later on we shall expand about a background solution to obtain a quadratic Hamiltonian that will be significant for a Holographic description and possible interpretation of emergent gravity. We also wish to point out that the radial sector of the free mutli-matrix theory has a collective field theory reformulation that is identical to the single Hermitian matrix model, restricted to the singlet sector, collective field theory Hamiltonian with a scale invariant potential inserted by hand. The only distinction is the strength of the dAFF potential, which vanishes for the case of a single complex matrix or equivalently for 2 Hermitian matrices or less. In this regard, much of what follows is applicable to the single Hermitian matrix model with a conformally invariant potential. We shall continue to consider specifically the free multi-matrix model though as this is of more interest to us.

5.2.2 Conformal algebra in the density description and breaking of conformal invariance

We now consider the possibility of writing down a density description of the conformal algebra associated with the free multi-matrix model. We shall show that, by knowing the collective field Hamiltonian and the form of the generators of conformal quantum mechanics, we can deduce the forms of the dilatation and special conformal operators in the collective field theory. Consider the collective field Hamiltonian for the one dimensional multi-matrix quantum mechanics, rescaled by $H \to 4H$,

$$H_{coll} = \frac{1}{2}\int dr\partial_r\tilde{\pi}(r)\Phi(r)\partial_r\tilde{\pi}(r) + \frac{\pi^2}{6}\int dr\Phi^3(r) + \frac{\eta^2}{2}\int dr\frac{\Phi^2(r)}{r^2} + L.M.T, \tag{299}$$

where L.M.T. stands for Lagrange multiplier terms. The Lagrange multiplier term of the Hamiltonian (299) will be ignored at present and we will assume that for the time being there are no Lagrange multiplier terms in the dilatation operator and special conformal operator. As a consequence, the results that follow would have to be modified to include such terms in order for the algebra of the full generators to close. We reiterate however, that based on the previous section (section 4.2) it will not be possible to close the $sl(2,\mathbb{R})$ algebra as μ breaks the conformal symmetry. For this reason we refer to the generators presently under consideration as the 'reduced' generators.

We make a simple redefinition of the generators appearing in (28) in terms of the reduced generators ($\tilde{H}, \tilde{K}$ and $\tilde{D}$):

$$H = \tilde{H},$$
$$K = \tilde{K},$$
$$D = -\frac{1}{2}\tilde{D} \tag{300}$$

so that the algebra becomes

$$\begin{aligned}
[\tilde{D},\tilde{H}] &= 2i\tilde{H},\\
[\tilde{D},\tilde{K}] &= -2i\tilde{K},\\
[\tilde{H},\tilde{K}] &= -i\tilde{D}.
\end{aligned} \tag{301}$$

From now on we will drop the tildes and make reference to (301) or(28) to clarify which form of the algebra is being referred to. The collective field and its canonical conjugate satisfy $[\Phi(r),\tilde{\pi}(r')] = i\delta(r-r')$. The form of the charges of conformal quantum mechanics suggests the definition

$$K = \frac{1}{2}\int drr^2\Phi(r) \tag{302}$$

for special conformal transformations. Making use of the expected product in the final commutator of (301) as well as the fundamental commutator for the collective field:[91]

$$\begin{aligned}
[H,K] &= \int dr \int dr' \left(\frac{1}{4}r'^2[\partial_r\tilde{\pi}(r)\Phi(r)\partial_r\tilde{\pi}(r),\Phi(r')] + \frac{\pi^2}{12}r'^2[\Phi^3(r),\Phi(r')] + \frac{\eta^2}{4r^2}r'^2[\Phi(r),\Phi(r')]\right)\\
&= \frac{-i}{4}\int dr \int dr' \left(\partial_r\tilde{\pi}(r)\Phi(r)\left(\partial_r\delta(r-r')\right)r'^2 + \left(\partial_r\delta(r-r')\right)r'^2\Phi(r)\partial_r\tilde{\pi}(r)\right)\\
&= -\frac{i}{2}\int drr\left(\partial_r\tilde{\pi}(r)\Phi(r) + \Phi(r)\partial_r\tilde{\pi}(r)\right)\\
&= -iD
\end{aligned} \tag{303}$$

where in going from the second to the third line we made use of the identity $\partial_r\delta(r-r') = -\partial_{r'}\delta(r-r')$ and integrated by parts. We therefore make the identification:

$$D = \frac{1}{2}\int drr\left(\partial_r\tilde{\pi}(r)\Phi(r) + \Phi(r)\partial_r\tilde{\pi}(r)\right). \tag{304}$$

Next we study the second commutator of (301).

$$\begin{aligned}
[D,K] &= \frac{1}{4}\int dr \int dr' rr'^2\left([\partial_r\Phi(r),\Phi(r')] + [\Phi(r)\partial_r\tilde{\pi}(r),\Phi(r')]\right)\\
&= \frac{i}{2}\int dr \int dr' rr'^2\Phi(r)\left(\partial_{r'}\delta(r-r')\right)\\
&= -i\int drr^2\Phi(r)\\
&= -2iK.
\end{aligned} \tag{305}$$

The final commutator can be analyzed by first noting that

$$\begin{aligned}
[D,\Phi(r)] &= \frac{1}{2}\int dr'r'\left[\left(\partial_{r'}[\tilde{\pi}(r'),\Phi(r)]\right)\Phi(r') + \Phi(r')\partial_{r'}[\tilde{\pi}(r'),\Phi(r)]\right]\\
&= i\partial_r(r\Phi(r))\\
&= i\left(\Delta_\Phi + r\cdot\partial_r\right)\Phi(r)
\end{aligned} \tag{306}$$

[91] It is useful to use the commutator identities of the form: $[ABC,D] = AB[C,D] + A[B,D]C + [A,D]BC$, $[AB,C] = A[B,C] + [A,C]B$, $[AB,CDE] = A[B,C]DE + AC[B,D]E + ACD[B,E] + [A,C]DEB + C[A,D]EB + CD[A,E]B$ etc. in many of the calculations that follow.

as it should for a conformal primary of weight $\Delta_\Phi = +1$.[92] Similarly

$$[D, \tilde{\pi}(r)] = ir\partial_r\tilde{\pi}(r) = i\left(\Delta_{\tilde{\pi}} + r\partial_r\right)\tilde{\pi}(r). \tag{307}$$

Then considering each term in (299) individually:

$$\begin{aligned}
[D, H_{kin}] &= \frac{3i}{2}\int dr\partial_r\tilde{\pi}(r)\Phi(r)\partial_r\tilde{\pi}(r) + \frac{i}{2}\int drr\partial_r\left(\partial_r\tilde{\pi}(r)\Phi(r)\partial_r\tilde{\pi}(r)\right)\\
&= \frac{3i}{2}\int dr\partial_r\tilde{\pi}(r)\Phi(r)\partial_r\tilde{\pi}(r) - \frac{i}{2}\int dr\partial_r\tilde{\pi}(r)\Phi(r)\partial_r\tilde{\pi}(r)\\
&= 2iH_{kin},
\end{aligned} \tag{308}$$

$$\begin{aligned}
[D, \frac{\pi^2}{6}\int dr\Phi^3(r)] &= i\frac{\pi^2}{2}\int dr\Phi^3(r) + \frac{i\pi^2}{6}\int drr\partial_r\Phi^3(r)\\
&= i\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right)\int dr\Phi^3(r)\\
&= 2i\left(\frac{\pi^2}{6}\int dr\Phi^3(r)\right),
\end{aligned} \tag{309}$$

$$\begin{aligned}
[D, \int dr\frac{\eta^2}{2r^2}\Phi(r)] &= \int dr\frac{\eta^2}{2r^2}i\left(\Delta_\Phi + r\partial_r\right)\Phi(r)\\
&= 2i\left(\int dr\frac{\eta^2}{2r^2}\Phi(r)\right).
\end{aligned} \tag{310}$$

So we find that

$$[D, H] = 2iH \tag{311}$$

and the algebra (301) is true up to Lagrange multiplier terms.

In the absence of the constraint, $N = \int dr\Phi(r)$, there is no Lagrange multiplier and the collective field theory Hamiltonian, dilatation operator and special conformal operators close the $sl(2,\mathbb{R})$ algebra. As noted before, the restoration of such a term in the free collective field theory spoils conformal invariance. In the first quantized formulation it is possible to promote the Lagrange multiplier μ to a field and extend the definition of the $SL(2,\mathbb{R})$ generators in a consistent way such that the conformal invariance is unbroken- see appendix E. There is no apparent natural way to do so for the second quantized generators- a fact that strengthens our expectations of the link between μ and the breaking of conformal invariance. We also note that, following [5], we could define the generators of (149)

$$L_0 \equiv \frac{1}{2}(H+K), \qquad L_{\pm1} \equiv \frac{1}{2}(H - K \mp iD), \tag{312}$$

which close the $sl(2,\mathbb{R})$ algebra (150)

[92] It is straightforward to confirm this field dimension from the conformal matrix quantum mechanics with Hamiltonian: $H = -\frac{1}{2}l_s Tr\frac{\partial}{\partial M}\frac{\partial}{\partial M} + \frac{l_s N^2\eta^2}{2}Tr\frac{1}{M^2}$, where l_s is a dimensionful parameter with energy units of -1. The corresponding density description then requires that $\Phi(r)$ have energy dimension $\Delta_\Phi = +1$. Since $N = \int dr\Phi(r) = \int dr\sum_i\delta(r-r_i)$ and $[r] = -1$ it is implied that $[\delta(r-r')] = +1$. Then $[\Phi(r), \tilde{\pi}(r')] = i\delta(r-r') \Rightarrow [\tilde{\pi}(r)] \equiv \Delta_{\tilde{\pi}} = 0$.

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_{+1}, L_{-1}] = 2L_0 \tag{313}$$

without the Lagrange multiplier terms. The motivation for this is twofold. Firstly, L_0 is preferable to H from the point of view as a compact generator with a discrete spectrum [2](see subsection 2.2.2). Secondly, this L_0 is associated, from the gravitational perspective, with an alternative choice of time- in particular the global time of AdS_2 [5] (see subsection 3.2). This algebra closes due to the closing of the algebra, (301), of the reduced generators. In the same way that the $SL(2,\mathbb{R})$ symmetry (301) is broken by the inclusion of the Lagrange multiplier, the Lagrange multiplier breaks the $sl(2,\mathbb{R})$ algebra (313). In the case of the generators H, D and K this confirms the breaking of conformal invariance due to the Lagrange multiplier that was emphasized in (249). However for the generators L_0, $L_{\pm 1}$ the context is somewhat distinct. From the point of view of the algebras above, we have been working in the free theory and L_0 is an alternative constant of motion, in the spirit of conformal quantum mechanics, or different choice of time, in the gravity picture, of the free theory. However, from the collective field theory perspective H corresponds to the free theory (with emergent dAFF term) whereas L_0 represents a new 'Hamiltonian' with the emergent dAFF term as well as a potential term of the form $V(r) = \frac{1}{2}\int drr^2\Phi(r)$. We have pointed this out as this distinction in the collective field theory is important for the large N background to be discussed below.

5.2.3 Large N background and emergence of a 2 dimensional metric

In order to make the following discussion as general as possible we consider the Hamiltonian for the free multi-complex matrix model (298), rescaled as $H \to 4H$ for convenience, with an arbitrary potential term $\int drV(r)\Phi(r)$. We wish to rescale variables in such a way that factors of N are explicit. This will make clear the relevant terms in the Hamiltonian once one takes the large N limit. The appropriate rescaling is the so called standard rescaling:[93]

$$\begin{aligned}
r' &= \sqrt{N}r \\
\Phi'(r') &= \sqrt{N}\Phi(r) \\
\tilde{\pi}'(r') &= \frac{1}{N}\tilde{\pi}(r) \\
\mu' &= N\mu.
\end{aligned} \tag{314}$$

However, we make an important assumption: we assume that the form of the potential $V(r)$ is such that under the standard rescaling $V \to NV$. This leads to the large N Hamiltonian

$$\begin{aligned}
H = {}& \frac{1}{2N^2}\int_0^\infty dr(\partial_r\tilde{\pi}(r))\Phi(r)(\partial_r\tilde{\pi}(r)) + N^2\left[\frac{\pi^2}{6}\int_0^\infty dr\Phi^3(r) + \frac{(m-1)^2}{2}\int_0^\infty dr\frac{\Phi(r)}{r^2}\right. \\
&\left. + \int_0^\infty drV(r)\Phi(r) + \mu\left(1 - \int_0^\infty dr\Phi(r)\right)\right].
\end{aligned} \tag{315}$$

The first term in (315) is quadratic in the canonical momentum, which we consider to be analogous to the kinetic term $\frac{P^2}{2m}$ in an atomic Hamiltonian except that the role of the 'mass' in this case is given by N^2. In the large N limit the 'mass' of the field becomes large and one takes the point of view that the field configuration corresponding to the ground state is that which solves the minimum of the effective potential, which is given by the remainder of the large N Hamiltonian and is certainly of leading order as $N \to \infty$. Again, the analogy with the atomic Hamiltonian is useful as one considers the ground state to

[93] The scaling of $\tilde{\pi}(r)$ is dependent on how $\Phi(r)$ and r scale. If α is a variable then after the rescaling $[\sqrt{N}\Phi(r), \alpha\tilde{\pi}(r')] = \frac{i\delta(r-r')}{\sqrt{N}}$ which implies that $\alpha = 1/N$ and $\tilde{\pi}(r) \to \frac{\tilde{\pi}(r)}{N}$.

correspond to the 'particle' sitting at the bottom of the effective potential well. The uniform field $\Phi_0(r)$, which is the saddle point solution defined by

$$\frac{\partial V_{eff}}{\partial \Phi(r)}\Big|_{\Phi_0(r)} = 0, \tag{316}$$

$$\Phi_0(r) = \frac{1}{\pi}\sqrt{2\mu - \frac{(m-1)^2}{r^2} - 2V(r)}, \tag{317}$$

is termed the 'large N background' and is to be considered the classical solution of the theory in the large N limit. For any quantum field theory, however, the object of principle interest is the quadratic ('free') piece of the action, even in the case of an interacting theory, as it is used to compute observables. For example, even in a perturbative expansion of an interacting theory one computes correlation functions in terms of free field correlation functions. To obtain quantum fluctuations about the uniform classical configuration $\Phi_0(r)$ one expands about the background field by introducing a shift in the field: $\Phi(r) = \Phi_0(r) + \frac{\partial_r \sigma(r)}{N}$. Doing so will allow us to identify the quadratic quantum action.[94] The canonical conjugate momentum to the fluctuation field $\sigma(r)$ is determined as follows. The usual commutation relation $[\Phi(r), \tilde{\pi}(r')] = i\delta(r-r')$ after taking a derivative becomes $-\partial_{r'}[\Phi(r), \tilde{\pi}(r')] = -i\partial_{r'}\delta(r-r') = i\partial_r\delta(r-r')$ so that $[\Phi(r), -\partial_{r'}\tilde{\pi}(r')] = i\partial_r\delta(r-r')$. Now, substituting in $\Phi(r) = \Phi_0(r) + \frac{1}{N}\partial_r\sigma(r)$ gives $\partial_r[\frac{1}{N}\sigma(r), -\partial_{r'}\tilde{\pi}(r')] = i\partial_r\delta(r-r')$. If we define $\nu(r)$ to be the canonical conjugate to $\sigma(r)$ then $\partial_r[\sigma(r), \nu(r')] = i\partial_r\delta(r-r')$ and we deduce that

$$\nu(r) = \frac{-\partial_r\tilde{\pi}(r)}{N} \tag{318}$$

and

$$[\sigma(r), \nu(r')] = i\delta(r-r'). \tag{319}$$

Setting $\Phi(r) = \Phi_0(r) + \frac{\partial_r\sigma(r)}{N}$ in (315), we identify the quadratic Hamiltonian

$$H_2 = \frac{1}{2}\int_0^\infty dr\Phi_o(r)\nu^2(r) + \frac{\pi^2}{2}\int_o^\infty dr\Phi_o(r)\left(\partial_r\sigma(r)\right)^2. \tag{320}$$

With the equation of motion $\nu(r) = \dot{\sigma}(r)/\Phi_o(r)$ the quadratic Lagrangian has the form

$$\mathcal{L}_2 = \frac{1}{2}\int_0^\infty drG^{\mu\nu}(x;\Phi_o(r))\partial_\mu\sigma(r)\partial_\nu\sigma(r) \tag{321}$$

from which we identify the metric

$$G_{\mu\nu} = \begin{pmatrix} \Phi_o(r) & 0 \\ 0 & \frac{-1}{\pi^2\Phi_o(r)} \end{pmatrix}$$
$$ds^2 = \Phi_o(r)dt^2 - \frac{1}{\pi^2\Phi_o(r)}dr^2. \tag{322}$$

[94] The method we have been describing above is the so called background field method and is most lucid in the path integral formulation where one can, by expanding about the background field, obtain an effective action in the partition function expressed in terms of the quantum fluctuations to quadratic order in which the integral takes a Gaussian form.

The large N background has previously been interpreted as a gravitational metric in [48] for a single Hermitian matrix model. To make general covariance explicit in the quadratic Lagrangian we note that $G \equiv \det G_{\mu\nu} = -\pi^2$- so $\sqrt{-G} = \pi$. Therefore, the true coordinate invariant Lagrangian density is $\mathcal{L}_2' = \pi \mathcal{L}_2$ such that

$$\mathcal{L}_2' = \frac{1}{2} \int dr \sqrt{-G} G^{\mu\nu}(r; [\Phi_0]) \partial_\mu \sigma(r) \partial_\nu \sigma(r). \tag{323}$$

We have shown that a two dimensional metric has emerged due to the large N background. We now devote some attention to the background field (317) and the metric appearing in (321). Due to the generic form of the free multi-matrix collective field Hamiltonian (315)- by generic we refer to the cubic interaction and emergent dAFF term- the fluctuations about the collective field theory will lead to the quadratic Lagrangian (323) and metric (322). The dependence of the metric on the background field $\Phi_0(r)$ is therefore general. We therefore attribute the features of the spacetime associated with the metric to the background field. The background field, being determined by the effective potential, has the general form of (317) for which the only unspecified details are the number, m, of complex matrices under consideration and the form of the potential (the Lagrange multiplier is solved from the eigenvalue density constraint once these details have been specified). For a given number of complex matrices, one can then generate different metrics by choice of the potential. This makes clear the need for our stressing, in the discussion of the conformal algebra of 5.2.2, the distinction between the operators H and L_0 in the collective field theory- they lead to distinct metrics. In addition, it is important to note that the general dependence of $\Phi_0(r)$ on μ together with the role played by μ in the breaking of conformal invariance in the collective field theory means that the metric and two dimensional geometry has some memory[95] of the breaking of conformal invariance despite the fact that the quadratic Lagrangian does not appear to have explicit dependence on μ. This indicates that the emergent geometry should perhaps be considered to be some near-AdS_2 ($NAdS_2$) geometry as opposed to pure AdS_2. This is similar to the ideas of [49], although they work in the Almheiri-Polchinski model and still have unbroken $SL(2,\mathbb{R})$ symmetry.

Motivated by these findings, we determine the large N backgrounds for the free multi-matrix Hamiltonian and the compact operator L_0 next.

5.2.4 Black hole geometry for the compact generator L_0.

The collective field Hamiltonian (298) corresponds to a 2 dimensional quantum field theory and, given the emergent dAFF term and metric, the system should correspond to an example of $NAdS_2/NCFT_1$. It is possible to obtain a large N background for the $SO(1,2)$ generator L_0. We now study the Hamiltonian but introduce a special conformal piece, K, such that the new operator $2L_0 \equiv H + K$ is an element of the global sub-group of the Virasoro algebra, which is isomorphic to $SL(2,\mathbb{R})$ [12].[96]

$$\begin{aligned} 2L_0 = & \frac{1}{2} \int_0^\infty dr (\partial_r \tilde{\pi}(r)) \Phi(r) (\partial_r \tilde{\pi}(r)) + \frac{\pi^2}{6} \int_0^\infty dr \Phi^3(r) + \frac{N^2(m-1)^2}{2} \int_0^\infty dr \frac{\Phi(r)}{r^2} \\ & + \frac{\omega^2}{2} \int_0^\infty dr r^2 \Phi(r) - \mu \left(N - \int_0^\infty dr \Phi(r) \right). \end{aligned} \tag{324}$$

After applying the standard rescaling (314) we solve for the background field via

$$\frac{\delta V_{eff}}{\delta \Phi(r)} |_{\Phi_0(r)} = 0. \tag{325}$$

[95] The notion of the geometry 'remembering' or being aware of the broken conformal symmetry is due to [49]. See F.1.3 of appendix F for details.

[96] The definition of the Special conformal operator, according to [12], includes a constant of mass dimensions ω. This is for dimensional consistency. We have also included a Lagrange multiplier to constrain $\int_0^\infty dr \Phi(r) = N$.

The background field solution is [12][97]

$$\Phi_0(r) = \frac{1}{\pi}\left(\frac{\omega}{2}(d-1) - \omega^2 r^2 - \frac{(d-2)^2}{4}\frac{1}{r^2}\right)^{1/2}, \tag{326}$$

and has support on the region $r_- \leq r \leq r_+$ defined by

$$r_\pm^2 = \frac{(d-1)}{4\omega} \pm \sqrt{\frac{(d-1)^2}{16\omega^2} - \frac{(d-2)^2}{4\omega^2}}. \tag{327}$$

It appears natural to identify the background field solution (326) with a two horizon black hole for which the radial coordinate is only well defined between the two horizons enforced by its support [12]. Alternatively, the two boundaries could be suggestive of AdS_2 features directly; given the two boundaries we may take the interpretation that: the inner boundary corresponds to an event horizon and the outer corresponding to the boundary of AdS. Note that the motivation to study L_0 is not limited to the fact that the potential increases as $|r| \to \infty$, which generates a discrete spectrum of states in the fermionic quantum mechanical description. The choice of L_0, as opposed to H, is related to the differences between Poincare time and global time coordinates that was seen in the pure AdS_2 case. If the two dimensional spacetime description is to be understood as a gravitational theory then one is required to take into account the various equivalent descriptions of the gravitational theory based on unique time slices- that is in terms of different 'Hamiltonians' [5]. This perspective was not apparent from the conformal quantum mechanical description of the matrix model. Having two horizons, it is possible that the metric for L_0 could be associated with a Reissner-Nordstrom black hole. Of course our restriction to the radial sector hinders our ability to identify this as a spinning black hole since we no longer have the angular degrees of freedom.

5.2.5 Black hole geometry for the free multi-matrix system

The emergence of the $1/r^2$ potential and its accompanying dimensional reduction to conformal quantum mechanics is a completely general feature of the free multi-matrix model fermionic description [12]. The local two dimensional collective field reformulation has the emergence of a radial coordinate and associated geometry. This clearly signals a holographic description and we have seen that the collective field theory closes the conformal algebra, which is isomorphic to the $so(1,2)$ isometry algebra of AdS_2, provided that the chemical potential μ is removed. Given that the chemical potential breaks conformal invariance, we expect that the collective field theory should be related to some kind of near-AdS_2 ($NAdS_2$) geometry. We consider this aspect of investigations into the AdS_2/CFT_1 correspondence of fundamental interest.[98] Therefore, in lieu of the special conformal modification to the Hamiltonian above, we consider now the the free multi-matrix Hamiltonian with emergent dAFF potential and investigate the type of spacetime that emerges.[99] We confirm, that for the multi-matrix collective field theory, the fixed chemical potential μ is explicitly dependent on the induced scale parameter and breaks conformal invariance. The large N background is found to have limited support on the radial coordinate which suggests that the geometry, while related to AdS_2 in the way mentioned above, describes a black hole. The collective field Hamiltonian is:

[97] Recall that $m = \frac{d}{2}$.

[98] The significance of $NAdS$ geometry will become clear in the following section (see section 6: Summary and conclusions- as well as appendix F).

[99] We have restored the length parameter l_s for dimensional consistency of the matrix model to mirror the analysis of the previous section (section 4).

$$H = \frac{l_s}{2}\int_0^\infty dr\,(\partial_r\tilde{\pi}(r))\,\Phi(r)\,(\partial_r\tilde{\pi}(r)) + \frac{\pi^2 l_s}{6}\int_0^\infty dr\Phi^3(r) + \frac{N^2(m-1)^2 l_s}{2}\int_0^\infty dr\frac{\Phi(r)}{r^2}$$
$$+\mu\left(N - \int dr\Phi(r)\right), \tag{328}$$

which under the standard rescaling (314) becomes

$$H = \frac{l_s}{2N^2}\int_0^\infty dr\,(\partial_r\tilde{\pi}(r))\,\Phi(r)\,(\partial_r\tilde{\pi}(r)) + N^2\left[\frac{\pi^2 l_s}{6}\int_0^\infty dr\Phi^3(r) + \frac{l_s(m-1)^2}{2}\int_0^\infty dr\frac{\Phi(r)}{r^2}\right.$$
$$\left.+\mu\left(1 - \int dr\Phi(r)\right)\right]. \tag{329}$$

The background field solution is

$$\Phi_0(r) = \frac{1}{\pi}\sqrt{\frac{2\mu}{l_s} - \frac{(m-1)^2}{r^2}}. \tag{330}$$

We define $q \equiv (m-1) = (d-2)/2$. It is convenient to control the IR divergences associated with the infinite expanse of the radial direction in the theory by placing the system in a box of length $L \equiv R\sqrt{N}$, where, R is a new length scale in the theory (of section 4). The eigenvalue density constraint is then

$$1 = \int_{q\sqrt{l_s/2\mu}}^{L/\sqrt{N}} dr\frac{1}{\pi}\sqrt{\frac{2\mu}{l_s} - \frac{q^2}{r^2}}. \tag{331}$$

The lower bound $q\sqrt{l_s/2\mu}$ reflects the turning point associated with the limited support of the background collective field $|r| \geq q\sqrt{l_s/2\mu}$.[100] By changing variables $z^2 = \frac{2\mu}{l_s q^2}r^2 - 1$ the integral becomes straightforward:

$$1 = \frac{q}{\pi}\int_0^{\sqrt{\frac{2\mu L^2}{l_s q^2 N}-1}} dz\frac{z^2}{z^2+1} = \frac{q}{\pi}\left[\sqrt{\frac{2\mu L^2}{l_s q^2 N} - 1} - \arctan\sqrt{\frac{2\mu L^2}{l_s q^2 N} - 1}\right]. \tag{332}$$

Defining $\epsilon = \sqrt{\frac{2\mu L^2}{l_s q^2 N} - 1} = \sqrt{\frac{2\mu R^2}{l_s q^2} - 1}$ since $L = \sqrt{N}R$, we find that

$$\frac{\pi}{q} = \epsilon - \arctan\epsilon \tag{333}$$

can be solved numerically for integer values of $q = m-1 = 1, 2, 3, 4....$ The first 5 integer values are listed below:

q	d	$\epsilon^*(q)$
1	4	4.49341
2	6	2.79839
3	8	2.18957
4	10	1.86373
5	12	1.65581.

(334)

[100] The upper bound is $L/\sqrt{N}$ because of the rescaling that took place between (328) and (329).

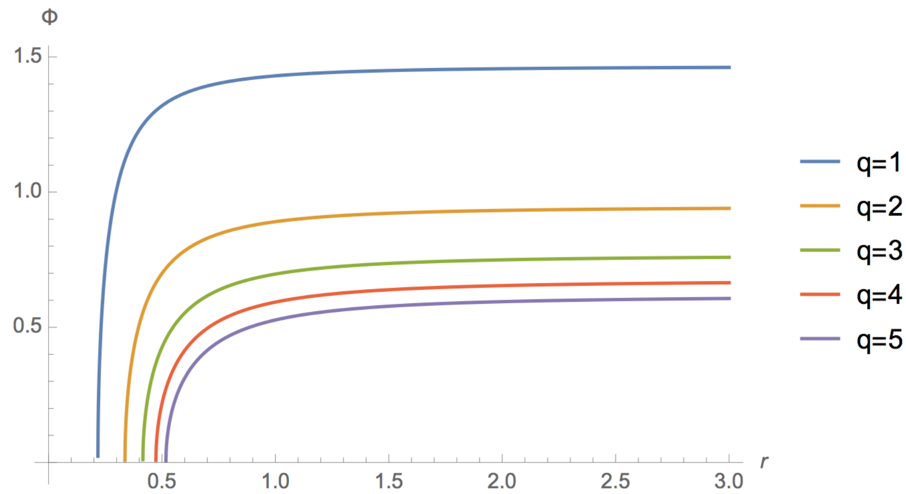

Figure 5: Plot of the background collective field for integer values of q with $R = 1$. (Plots generated in Mathematica)

This allows one to determine the Lagrange multiplier μ appearing in the background collective field $\Phi_0(r)$.

$$\mu = \frac{l_s q^2}{2R^2}(1 + \epsilon^{*2}(q)). \tag{335}$$

Clearly the chemical potential has explicit R dependence and will certainly break conformal invariance. This is confirmation of the single Hermitian matrix result (249). In section 4 the dAFF potential was inserted by hand, but following the map of the multi-matrix theory to conformal quantum mechanics (see 5.1.2) we now expect the theories to be related in the obvious way. Therefore, the background field, parameterized by q and the new scale parameter R, is found to be

$$\Phi_{0(q,R)}(r) = \frac{q}{\pi}\sqrt{\frac{1}{R^2}(1 + \epsilon^{*2}(q)) - \frac{1}{r^2}}. \tag{336}$$

The quadratic Lagrangian and emergent metric are given by

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2}\int dr \left[\frac{1}{l_s\Phi_0(r)}\dot{\sigma}^2(r) - \pi^2 l_s \Phi_0(r)(\partial_r\sigma(r))^2\right], \\ ds^2 &= \Phi_o(r) l_s dt^2 - \frac{1}{\pi^2 l_s \Phi_o(r)} dr^2. \end{aligned} \tag{337}$$

The black hole has a horizon defined by $\Phi_0(r) = 0$:

$$r_H = \frac{R}{(1 + \epsilon^{*2}(q))}. \tag{338}$$

If the emergent geometry defined by the free multi-matrix model is to be identified with a black hole, then the restriction to a radial coordinate exterior to the black hole appears reminiscent of the 'brick wall' black hole of [50]. We have plotted the first 5 solutions to the background field in figure 5.

6 Summary and Conclusions

In order to motivate the AdS_2/CFT_1 correspondence, we illustrated the matching of the $SL(2,\mathbb{R})$ symmetry of conformal quantum mechanics and the $SO(1,2)$ isometry group of AdS_2. This was followed by a review of fairly recent contexts in which AdS_2 has appeared in string theory and quantum gravity. This highlighted several important findings which include: the appearance of AdS_2 in the near horizon geometry of higher dimensional black holes in string theory, the significance of preferred time coordinates in AdS_2 black hole geometries which leads to various vacuum definitions that affect the boundary correlation functions in the holographic CFT [4] and fragmentation of AdS_2 geometries in the ground state configurations of AdS_2 black holes due to the mass gap for such spacetimes [6]. In addition to the fragmented AdS_2 ground state configurations there are other non-trivial AdS_2 ground state excitations identified by considering the state operator correspondence for the CFT_1 [9]. We also saw that there is in fact an ambiguity in the definition of the CFT dual to AdS_2 where it was argued that the dual CFT might be a conformal quantum mechanics [5] or the chiral half single copy of the Virasoro algebra of diffeomorphisms of AdS_3 through which one passes in the dimensional reduction to AdS_2 [7] [8] or essentially the infinite dimensional group of diffeomorphisms in one dimension associated with arbitrary diffeomorphisms of time in the boundary theory for the asymptotic symmetries of AdS_2 [9] [20]. Aquainting ourselves with these known results was useful and may be important for future work. In particular, while it appears that our work is more likely related to a $NAdS_2$ geometry, the mass gap, choices of time and fragmenting processes discussed above may have some relevance on future work related to our apparent black hole solutions. It should also be noted that with regard to the ambiguities in the definition of the dual CFT_1 it is the conformal quantum mechanical $SL(2,\mathbb{R})$ symmetry that is important for the collective field theory and the work of [5] has certainly had a bearing on our work.

We have shown that matrix model theories in $d = 1$ dimensions have a fermionic and a collective field theory description, both of which are formulated in terms of the eigenvalue degrees of freedom. A general feature of the collective field theory formulation, in the singlet sector, is the emergence of the radial eigenvalue coordinate and a two dimensional metric which depends on the large N background. The authors consider the most interesting case of such theories to be the free multi-matrix theory which has an emergent $1/r^2$ potential term for a system of fermions in $d+1 = 2m+1$ dimensions that is present if and only if the number of complex(Hermitian) matrices, $m(d)$, is greater than or equal to 2(4). In the second quantization a redefinition of the fields, in terms of a Jacobian, mapped the system of $d+1 = 2m+1$ dimensional fermions to a $d = 1$ system- a dimensional reduction in which the $1/r^2$ potential survives [12].

We emphasize three new results: firstly, there is an induced scale parameter, R, in the free matrix model and associated fermionic and collective field theories that arises from the need to regulate the massless matrix model limit. This scale is related to the IR regularization of the free fermion and collective field theories which consist of a set of fermions in a box of length L. The precise relation is $L = \sqrt{N}R$ which defines the standard thermodynamic limit $L \to \infty$, $N \to \infty$ with R fixed. Secondly, for the free multi-matrix collective field Hamiltonian this induced scale features explicitly in the fixed chemical potential μ that is required in the collective field theory as a constraint on the eigenvalue density. A consequence of the R dependence of μ is that μ necessarily breaks conformal invariance. This was originally discovered in the single matrix model, where the $1/r^2$ term was inserted by hand, and then confirmed in the multi-matrix model where the $1/r^2$ term emerges in a natural way. The breaking of conformal invariance due to the presence of μ for the free multi-matrix model was shown to appear in a more general way in the collective field realization of the conformal generators, which only close the $sl(2,\mathbb{R})$ algebra in the absence of μ and therefore only the reduced generators realize the $SO(1,2)$ symmetry of AdS_2. We have also pointed out that, from the perspective of the collective field $sl(2,\mathbb{R})$ algebra, the symmetry was broken for the free multi-matrix model for both the generators for the original Hamiltonian and for the generators of the 'new Hamiltonian', related by a different choice in time which was motivated by the pure AdS_2 global and Poincare times [5]. However, from the perspective of the collective field large N background, the new Hamiltonian, which would be associated with the global time in the AdS_2 case, has a non-trivial potential.

Therefore, the two Hamiltonians generate distinct metrics.

This is our third result: we have a method for generating two dimensional metrics from the collective field theory Hamiltonian. The metrics have a general form in terms of the large N background, being distinguished only by the precise form of the collective field theory potential. While the quadratic collective field Lagrangian, obtained by studying the quantum fluctuations about the large N background, has no explicit μ dependence, the emergent metric $G_{\mu\nu}(r, [\Phi_0(r)])$ depends on μ (and for the 'Poincare time' Hamiltonian- on R)- a feature that means that the emergent geometry has some 'memory' of the breaking of conformal invariance. This suggests that the appearance of μ in the collective field theory might be related to a $NAdS_2$ geometry [49]. The breaking of conformal invariance by μ and the expectation that the collective field theory is related to a $NAdS_2$ geometry is important for future investigation of the AdS_2/CFT_1 correspondence for multi-matrix models.[101] AdS_2 has asymptotic symmetries consisting of arbitrary time reparameterizations of the boundary [20]. Since the symmetries are asymptotic to AdS_2 they are spontaneously broken to the $SL(2,\mathbb{R})$ group in pure AdS_2 and are explicitly broken for any deviation of the form $AdS_2 \to NAdS_2$ [49]. Strictly speaking, this differs from the conformal symmetry breaking of our work as, in the mechanism of symmetry breaking in [49], the $SL(2,\mathbb{R})$ symmetry remains, even in the case of $NAdS_2$ geometry. For the collective field theory it is the $SL(2,\mathbb{R})$ symmetry that is explicitly broken. It would be interesting to see in what way, if any, one might be able to recover the $SL(2,\mathbb{R})$ symmetry of the collective field which would achieve a pure AdS_2- i.e. AdS_2 without the asymptotic symmetries.

Near AdS_2 gravity of this kind- that is, pure AdS_2 without the asymptotic symmetries- is well described by the so-called Jackiw-Teitelboim (JT) gravity [51] which is contained in the more general Almheiri-Polchinski (AP) model [52]. This model is universal, given that it describes the near-horizon $NAdS_2$ geometry of near-extremal black holes in generic dilaton gravity theories. The AP model captures the leading order gravitational corrections to correlation functions due to the backreaction of matter that are responsible for the breaking of conformal symmetry. The leading order gravitational effect induces a conformal symmetry breaking scale that, by the universality of the AP model, is equivalent to the thermodynamic mass gap associated with near-extremal black holes in $NAdS_2$ [42]. The gravitational features of the JT model are captured by an emergent Schwarzian boundary action that is invariant under bulk time $SL(2,\mathbb{R})$ transformations. This action exhibits the conformal symmetry breaking gravitational corrections in the four point function [49] and recovers the exponential Lyapunov growth of out-of-time-order correlators at early time and the accompanying late time exponential Ruelle decay of these quantities- features indicative of the chaotic behavior of black holes [49] [53]. The conformal symmetry breaking scale, thermodynamic mass gap and out-of-time-order correlation functions have fairly general forms[102] and may in principle be able to be computed in the collective field theory. That remains beyond the scope of this work.

We also wish to point out that we have not studied the 'black hole' solutions suggested in section 5 in detail. It would be of value to compute quantities such as the temperature, entropy and total energy of the solutions and to determine if they might correspond to extremal or near extremal black holes and if there is a mass gap. It is not yet clear how similar the metric of the free multi-matrix Hamiltonian is to the 'brick wall' solution of [50]. We have introduced an IR cut-off by placing the system of fermions into a box which had a direct effect on the form of the large N background, Φ. This regulator, L (or after the rescaling $L/\sqrt{N} = R$), is to be implemented on the domain of the radial coordinate r. This appears to be in accord with the IR regulator of 'brick wall' black holes, for which the black hole is placed in a box. However, the 'brick wall' UV cut-off, roughly a Planckian distance from the horizon, is imposed. In the case of the free multi-matrix collective field Hamiltonian we have not imposed the 'UV' turning point- it is inherited in the metric due to the emergent $1/r^2$ term. This has a significant consequence- an in falling observer will certainly be aware of the 'wall' at r_H (see (338)) which is in conflict with the expectation

[101] We have included a somewhat comprehensive review of $NAdS_2$ geometry in appendix F which also includes details concerning the relationship between chaos and black holes. The reader should consult that appendix if unfamiliar with some of the topics we refer to regarding possible future work.

[102] See appendix F for details.

that any in falling observer shall not be able to detect the horizon when passing through it. This may bring into question how physical the black hole solution is; however reconciling the existence of a smooth horizon and a consistent quantum mechanical description (i.e. unitary evaporation) is a delicate subject in light of the arguments of [54] which suggest that black holes either have a violation of unitarity or a detectable horizon- the 'firewall'.

While the metric generated by L_0 has the appearance of a charged black hole one would like to be able to lift the restriction to the radial sector and introduce angular degrees of freedom in the multi-matrix model- in this way it may be possible to determine whether the black hole is more likely to be identified with a charged black hole or a spinning black hole.

A black hole solution for the collective field theory has been hypothesized in the past [13]. This involved a deformed matrix model. That proposal consisted of the collective field theory that corresponded to a matrix model that was distinct from the usual $c = 1$ matrix model of two dimensional string theory with the distinction being that the Lagrange multiplier, μ, was absent (being set to zero) and there was an extra $1/x^2$, dAFF, term with explicit dependence on the black hole mass.[103] The absence of the Lagrange multiplier in the deformed matrix model is attractive, especially since the deformed matrix model fermionic Hamiltonian is an element of a set of three generators that close the $sl(2, \mathbb{R})$ algebra (see (499) of appendix G). One of these generators is recognized as the generator, L_0, in first quantized form that has been discussed above.[104] This deformed model may provide an alternative matrix model with which to investigate the possibility of an emergent AdS_2 geometry.

We would like to revisit some of these ideas in future work.

[103]We have provided a review of the collective field theory description of the $c = 1$ matrix model of two dimensional string theory together with the proposed deformed matrix model of [13]. This appears together with the discussion of the matrix model-Type 0 string theory duality in appendix G.

[104]The first quantized operator is $\frac{1}{2}(p^2 + \frac{\eta^2}{r^2} + r^2)$ which appears in $L_0 = \frac{1}{2}\int dr \Psi^\dagger(p^2 + \frac{\eta^2}{r^2} + r^2)\Psi$ [5].

A Appendix A

A.1 Harmonic oscillator potential for the fermionic description

If we introduce the single particle energy eigenvalues $\{\epsilon_i\}$ with an harmonic oscillator potential the eigenfunction problem

$$\sum_i \left(-\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} + V(\lambda_i)\right)\phi(\lambda_i) = \epsilon_i\phi(\lambda_i), \qquad V(\lambda_i) = \frac{1}{2}\omega^2\lambda_i^2, \tag{339}$$

can be mapped to the familiar quantum mechanical harmonic oscillator problem, which is solved in any standard book on quantum mechanics [55]. We define $z \equiv \sqrt{\omega}\lambda_i$ and apply this coordinate redefinition in (340):

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} + \frac{1}{2}\omega^2\lambda_i^2\right)\phi(\lambda_i) = \epsilon_i\phi(\lambda_i). \tag{340}$$

The problem is then mapped to the form

$$\frac{d^2\phi}{dz^2} = \left(z^2 - k^2\right)\phi \tag{341}$$

where $k \equiv \sqrt{\frac{2\epsilon}{\omega}}$. The single particle eigenfunctions of (341) are expressed in terms of the Hermite polynomials:[105]

$$\phi_n(\lambda_i) = \left(\frac{\omega}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(z)e^{-z^2/2}, \qquad n = 0, 1, 2, ... \tag{342}$$

A.1.1 Second quantization

The many body problem of quantum mechanics is best approached from the non-relativistic second quantized field description. We adopt the notation $\phi \to \psi$ for convention in what follows, however it is to be understood that from now on ψ no longer has the same meaning that it had in (214) but now refers to the single particle states of the left hand side of (342) or a general Hamiltonian (the context should make this clear). The fermions are spinless.

It is possible to construct the density operator corresponding to the harmonic oscillator problem from the field operators. The field operators are defined to be $\hat{\psi}(x) \equiv \sum_n \psi_n(x)c_n$ and $\hat{\psi}^\dagger(x) \equiv \sum_n \psi_n^\dagger(x)c_n^\dagger$. In this case the wave functions are the Hermite polynomial solutions (342) and the sum is over n, the principle quantum number labeling energy eigenstates. We define the normal ordered density

$$\hat{\phi}(x) \equiv: \hat{\psi}^\dagger(x)\hat{\psi}(x) := \hat{\psi}^\dagger(x)\hat{\psi}(x). \tag{343}$$

The expectation value

$$\phi(x) \equiv \langle\hat{\phi}(x)\rangle = \langle 0|\hat{\psi}^\dagger(x)\hat{\psi}(x)|0\rangle = \sum_{n,m=0}^{N-1}\psi_n^\dagger(x)\psi_m(x)\langle 0|c_n^\dagger c_m|0\rangle. \tag{344}$$

For the system of fermions we can consider the operators $c_n^\dagger$ to be decomposed into two parts:

[105] The harmonic oscillator problem has eigenvalue/function solutions, which by convention, are defined for eigenstates with quantum number $n = 0, 1, 2, ...$ For a system of N fermions in an harmonic oscillator potential we take $n = 0, 1, 2, ..., N-1$ and this range will appear in many sums to follow.

$$c_n = \begin{cases} b_n^\dagger, & \text{if } n < N-1 \\ a_n, & \text{if } n > N-1 \end{cases} \tag{345}$$

$b_n^\dagger$ is a hole creation operator and a_n is the fermion annihilation operator as follows in the obvious way from (345). With this separation the vacuum expectation value (344) becomes:

$$\begin{aligned}\phi(x) &= \sum_{n,m=0}^{N-1} \psi_n^\dagger(x)\psi_m(x)\langle 0|c_n^\dagger c_m|0\rangle = \sum_{n,m=0}^{N-1} \psi_n^\dagger(x)\psi_m(x)\langle 0|\{c_n^\dagger c_m\}|0\rangle \\ &= \sum_{n=0}^{N-1} \psi_n^\dagger(x)\psi_n(x).\end{aligned} \tag{346}$$

The second last equality in (346) follows from the fact that the summation is bounded by N and therefore, $c_n|0\rangle = b_n^\dagger|0\rangle$. Given that in this instance $c_n^\dagger|0\rangle = b_n|0\rangle = 0$ it is completely valid to replace $\langle 0|c_n^\dagger c_m|0\rangle$ by $\langle 0|\{c_n^\dagger c_m\}|0\rangle$. We make use of the anti-commutation relation $\{b_n, b_m^\dagger\} = \delta_{nm}$ to obtain the final equality in (346).

The traced operators of a matrix model are of interest given the map from matrix quantum mechanics to a system of non-interacting fermions. The density of eigenvalues in that case was particularly useful since a trace valued operator could be re-expressed in terms of the density and a function of the eigenvalues (the functional form being identical to the that of the traced operators). Explicitly: $Tr\,(f(M)) = \int dx f(x)\phi(x)$. The density of (346) shares this property. For instance $\int dx\phi(x) = \sum_{n=0}^{N-1}\int dx\psi_n^\dagger(x)\psi_n(x) = N$ since the Hermite polynomials are orthogonal and provided that they are appropriately normalized. Then the two point correlation function that follows is:

$$\begin{aligned}\langle Tr\left(M^2\right)\rangle &= \int dx x^2 \sum_{n=0}^{N-1} \psi_n^\dagger(x)\psi_n(x) = \sum_{n=0}^{N-1}\int dx \int dy \langle n|y\rangle\langle y|\hat{X}^2|x\rangle\langle x|n\rangle \\ &= \sum_{n=0}^{N-1}\langle n|\hat{X}^2|n\rangle.\end{aligned} \tag{347}$$

The harmonic oscillator is the classic example for introducing the notion of elementary excitations as opposed to a ladder of energy levels of first quantization. However, the form of (347) allows one to go in the opposite direction and consider the ladder operators of the algebra associated with the harmonic oscillator of first quantized quantum mechanics since $|n\rangle$ is an eigenstate of these operators. The explicit form of these (bosonic) operators is identified by completing the square of the Hamiltonian: $H = \frac{1}{2}\frac{p^2}{2} + \frac{1}{2}m^2x^2$[106]

$$H = m\left(\sqrt{\frac{m}{2}}x - i\sqrt{\frac{1}{2m}}p\right)\left(\sqrt{\frac{m}{2}}x + i\sqrt{\frac{1}{2m}}p\right) + \frac{m}{2} \equiv m(a^\dagger a + \frac{1}{2}) \tag{348}$$

with $a^\dagger \equiv \sqrt{\frac{m}{2}}x - i\sqrt{\frac{1}{2m}}p$ and $a \equiv \sqrt{\frac{m}{2}}x + i\sqrt{\frac{1}{2m}}p$. Therefore $x = \frac{1}{\sqrt{2m}}(a^\dagger + a)$. These oscillator ladder operators satisfy the commutation relation $\left[a, a^\dagger\right] = 1$. Using these operators it is straightforward to confirm that $\langle n|x^2|n\rangle = \frac{1}{2m}(1+2n)$ and as a result one finds that (347) implies that

$$\langle Tr\left(M^2\right)\rangle = \frac{N^2}{2m} \tag{349}$$

[106]My convention for distinguishing coordinates from operators is to label them x and $\hat{X}$ respectively. However, I will neglect this notational distinction whenever the context makes clear which is being referred to. For example, I have elected to refrain from putting a hat on the momentum operator in the Hamiltonian.

where the sum over the natural numbers (to N-1) $\sum_{n=1}^{N-1} n = \frac{N(N-1)}{2}$ has been used to simplify the result.

A.1.2 Orthogonal Polynomials

The approach of the previous sub-subsection (A.1.1) is convenient although it is not very general as, apart from the special cases of the harmonic oscillator or a free theory, the ladder/creation-annihilation operators elude any attempts to write down their explicit form. The theory of classical orthogonal polynomials provides a far more general approach [56] and we shall use the harmonic oscillator as an example to illustrate this. For the case of the harmonic oscillator we focus on the Hermite polynomials. The Hermite polynomials have the following generating function:

$$g(s,z) = e^{-s^2+2sz} = e^{z^2-(s-z)^2} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(z). \tag{350}$$

It is useful to establish some preliminary results before considering the wave functions of the harmonic oscillator (342) in any kind of detail. We note that by taking the derivative of the generating functional with respect to z one establishes that $\frac{dH_n(z)}{dz} = 2nH_{n-1}$. Alternatively, taking the derivative with respect to s leads to the recursion relation: $H_{n+1} = 2zH_n - 2nH_{n-1}$. Now, taking the derivative, with respect to z, of the first expression and considering the second expression with the replacement $n \to n-1$ we find that the Hermite polynomials satisfy:

$$\frac{d^2H_n(z)}{dz^2} - 2z\frac{dH_n(z)}{dz} + 2nH_n(z) = 0. \tag{351}$$

The form of the generating function (350) implies that $\left.\frac{\partial^n g(s,z)}{\partial s^n}\right|_{s=0} = H_n(z)$. One can easily verify that by substituting the second form of the generating function (350) into this expression it is found that:

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{352}$$

We concentrate on the wave functions in what follows. The generating function implies that

$$\int_{-\infty}^{\infty} dz e^{-z^2} g(s,z)g(t,z) = \sum_{n,m}^{\infty} \frac{s^n t^m}{n!m!} \int_{-\infty}^{\infty} dz H_n(z)H_m(z)e^{-z^2}. \tag{353}$$

It is particularly convenient to use (in (353)) the very first expression for the generating function in (350) to prove that the wave functions of (342) are orthogonal. That is $\int_{-\infty}^{\infty} dz H_n(z)H_m(z)e^{-z^2} = 0$ for $n \neq m$. One also finds that for $n = m$ the result $\int_{-\infty}^{\infty} dz H_n^2(z)e^{-z^2} = \sqrt{\pi}2^n n!$ leads to the Normalization factor: $N_n = \sqrt{\frac{1}{2^n n!}} \left(\frac{m}{\pi}\right)^{1/4}$ which appears in (342). We are now in a position to to solve the expectation value $\langle n|x^2|n\rangle$. Using the coordinate basis completeness relation:

$$\langle n|x^2|n\rangle = \int dx \psi_n^*(x) x^2 \psi_n(x) = \frac{N_n^2}{m^{3/2}} \int dz H_n^2(z) z^2 e^{-z^2}, \tag{354}$$

where the second equality of (354) follows form the change of variables $z = \sqrt{m}x$. We identify the factor $(H_n(z)z)^2$ in the integrand. The recursion relation: $H_{n+1} = 2zH_n - 2nH_{n-1}$ can be re-expressed as

$zH_n(z) = nH_{n-1}(z) + \frac{1}{2}H_{n+1}(z)$. Squaring this expression and noting that the cross term is orthogonal under the integration, we find:

$$\langle n|x^2|n\rangle = \frac{N_n^2}{m^{3/2}} n^2 \int dz H_{n-1}^2(z) z^2 e^{-z^2} + \frac{N_n^2}{m^{3/2}}\left(\frac{1}{4}\right)\int dz H_{n+1}^2(z) z^2 e^{-z^2}$$
$$= \frac{(2n+1)}{2m}. \tag{355}$$

Then:

$$\sum_{n=0}^{N-1}\langle n|x^2|n\rangle = \frac{N^2}{2m} \tag{356}$$

which confirms the result obtained by the operator method (349). We have made use of operator (and orthogonal polynomial) techniques in our discussion of second quantization, however, the usual path integral approach can be used instead. We chose to use the operator approach as it is better suited to the non-relativistic oscillator theory.

A.2 General scale invariant potential wave functions

Since matrix quantum mechanics has a free many-body fermionic description its Schrodinger wave functions are of interest from both a single particle perspective as well as a second quantized description. We provide a calculation of these wave functions for the case of a scale invariant potential. It is shown that, when viewed as the reduction of the multi-matrix theory to the radial sector in one dimension [12], the solution wave functions are Bessel functions of integral order. The scale invariant Schrodinger equation is

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{\eta^2}{2x^2}\right)\psi(x) = E\psi(x). \tag{357}$$

Defining $k^2 \equiv 2E$ and making a change of variables: $z \equiv kx$ allows one to redefine the wave functions in the following way: $\psi(x) = \sqrt{kz}\phi(kx) = \sqrt{z}\phi(z)$. This maps the differential equation to the Bessel wave equation:

$$z^2\phi''(z) + z\phi'(z) + \left[z^2 - (\eta^2 + \frac{1}{4})\right]\phi(z) = 0. \tag{358}$$

In this instance, the solution $\phi(z)$ is a Bessel function $\phi(z) = J_{\sqrt{\eta^2+1/4}}(z)$ for which the full solution is given by $\psi(x) = \sqrt{z}J_{\sqrt{\eta^2+1/4}}(z)$. In the dimensional reduction to quantum mechanics (266) the identified conformal interaction strength is given by $\eta^2 = \frac{1}{4}(N^2(d-2)^2 - 1)$ [12]. So, $\psi(x) = \sqrt{z}J_{\frac{1}{2}N(d-2)}(z)$. The form of this result tempts one to express the Bessel function in terms of spherical Bessel functions of the first kind. One typically has $j_n = \sqrt{\pi}2zJ_{n+1/2}(z)$, $n \in \mathbb{Z}$, with j_n being the spherical Bessel function of the first kind and $J_{n+1/2}$ the Bessel function of the first kind. So, the typical case corresponds to a Bessel function, J, that is half integral order. This requirement can be investigated through the possibility

$$\frac{1}{2}N(d-2) \stackrel{?}{=} n + \frac{1}{2}, \qquad n \in \mathbb{Z}. \tag{359}$$

Noting that $d = 2m$ and that $N, m \in \mathbb{Z}$ we see that $\frac{N(2m-2)-1}{2} = \frac{e_1 - e_2 - 1}{2}$ where we have defined the even integers $2Nm \equiv e_1$ and $2N \equiv e_2$. Since the difference between two even integers $e_3 \equiv e_1 - e_2$ is itself even, the numerator $e_3 - 1$ must be odd. Then, $\frac{N(2m-2)-1}{2} = \frac{e_3-1}{2} \equiv \frac{o_1}{2}$, where $o_1 = e_3 - 1$ is an odd integer and n is half integral. Therefore $\frac{N(d-2)}{2} \in \mathbb{Z}$. We therefore use the Bessel functions of integral order. We are interested in taking the large N limit ($N \to \infty$). Large order asymptotic forms of the Bessel functions exist for order $\nu \to \infty$ with ν passing through positive reals and for a fixed value of z. The general solution is a superposition of the Bessel and Neumann functions [57]:

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^{\nu}, \qquad Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu}. \tag{360}$$

B Appendix B

B.1 General Formalism: Quantum Collective Field Method

This appendix indicates how a general quantum many body system can be reformulated in terms of the so called collective field theory of Jevicki and Sakita [11].
The collective field method of Jevicki and Sakita allows one to re-express a quantum many body Hamiltonian, that depends on say, m, degrees of freedom, as a functional operator in terms of the collective field. This is possible if the many body interaction potential can be expressed in terms of an infinite combination of its degrees of freedom and so can be expressed in the form $F(x, \{q_i\}_{i=1}^m) = \phi(x)$- where $\phi(x)$ is the collective field and x is continuous. Provided that the many body wave function can be expressed as a functional of the collective field then, by beginning with the inner product of the wave functions, one can determine the Jacobian associated with the collective field. Making use of the Jacobian one can define new wave functionals, rescaled by the Jacobian, and the inner product in terms of the new wave functionals has a simpler and more natural form.

In order to re-express the kinetic piece of the Hamiltonian as a functional operator of the collective field one makes use of the chain rule, generalized for infinite degrees of freedom, to re-write

$$-\sum_i \frac{1}{2}\frac{\partial^2}{\partial q_i^2}\psi(\{q_i\}) \to \frac{i}{2}\int dx\omega(x;[\phi])\pi(x) + \frac{1}{2}\int dx\int dy\Omega(x,y;[\phi])\pi(x)\pi(y), \tag{361}$$

where the functionals $\omega(x;[\phi])$ and $\Omega(x,y;[\phi])$ are defined

$$\omega(x;[\phi]) \equiv -\sum_i \frac{\partial^2\phi(x)}{\partial q_i^2}, \qquad \Omega(x,y;[\phi]) \equiv \sum_i \frac{\partial\phi(x)}{\partial q_i}\frac{\partial\phi(y)}{\partial q_i}, \tag{362}$$

and $\pi(x) \equiv -i\delta/\delta\phi(x)$ is the canonical conjugate momentum to $\phi(x)$. The Jacobian, which can be identified as discussed above in the wave function inner product through the change of variables to the collective field, can alternatively be determined in a far easier way by performing a similarity transformation on the kinetic part of the Hamiltonian in its new functional form:

$$H[\phi] \to J^{1/2}[\phi]H[\phi]J^{-1/2}[\phi]. \tag{363}$$

This has no effect on $\phi(x)$ and therefore no effect on $\omega(x;[\phi])$ and $\Omega(x,y;[\phi])$ however, the canonical conjugate momentum transforms as

$$\pi(x) \to \pi(x) + \frac{i}{2}\frac{\delta \ln J[\phi]}{\delta\phi(x)}. \tag{364}$$

After transforming the kinetic part of the Hamiltonian and simplifying one obtains

$$\tilde{H} = \frac{i}{2}\int dx[\omega_x + i\int dy(\pi_y\Omega_{xy}) + \int dy\Omega_{xy}\rho_y]\pi_x + \frac{1}{2}\int dx\int dy\pi_x\Omega_{xy}\pi_y - \frac{1}{8}\int dx\int dy\rho_x\Omega_x y\rho_y$$
$$-\frac{1}{4}\int dx\omega_x\rho_x + \frac{i}{4}\int dx\int dy\Omega_x y(\pi_x\rho_y), \tag{365}$$

where we have used the condensed notation $\omega(x;[\phi]) \equiv \omega_x$, $\Omega(x,y;[\phi]) \equiv \Omega_{xy}$ and $\frac{\delta \ln J}{\delta\phi(x)} \equiv \rho_x$. Since Hermitian operators are of interest, one should demand that $\tilde{H}$ is Hermitian, which requires

$$\omega_x + i\int dy\pi_y\Omega_{xy} + \int dy\Omega_{xy}\rho_y = 0. \tag{366}$$

This is referred to as the Hermiticity requirement and it allows for one to solve for ρ_y in order to determine the Jacobian. By multiplying both sides of (366) by $\int dx\Omega^{-1}_{zx}$, ρ_z is solved:

$$\rho_z \equiv \frac{\delta \ln J}{\delta\phi(z)} = -\int dx\Omega^{-1}_{zx}\omega_x - i\int dy\int dx\Omega^{-1}_{zx}(\pi_y\Omega_{xy}). \tag{367}$$

After substitution of ρ_x and implementing the Hermiticity requirement the kinetic part of the collective field Hamiltonian becomes

$$\begin{aligned} H_{coll} = \frac{1}{2}\int dx\int dy\pi_x\Omega_{xy}\pi_y + \frac{1}{8}\int dx\int dy[\omega_x + i\int dz\pi_z\Omega_{xz}]\Omega^{-1}_{xy}[\omega_y + i\int dz'\pi_{z'}\Omega_{yz'}] \\ -\frac{1}{4}\int dx\frac{\delta\omega_x}{\delta\phi(x)} - \frac{1}{4}\int dx\int dy\frac{\delta^2\Omega_{xy}}{\delta\phi(x)\delta\phi(y)}. \end{aligned} \tag{368}$$

This re-expression of the kinetic part of the Hamiltonian in terms of ϕ, provided by Jevicki and Sakita, is completely general and can be used once one has determined $\omega(x;[\phi])$ and $\Omega(x,y;[\phi])$ for some system under consideration.

C Appendix C

C.1 Polar matrix coordinates and radial fermionization of the 2 Hermitian matrix model

Given a pair of $N \times N$ Hermitian matrices: $\{X_1, X_2\}$ one can define a complex matrix Z [46]. This is achieved in the natural way, not unlike the way in which one defines a complex number in terms of 2 reals. The pair of Hermitian matrices are then related to the complex matrix through the definitions $X_1 \equiv \frac{1}{2}\left(Z + Z^\dagger\right)$ and $X_2 \equiv \frac{-i}{2}\left(Z - Z^\dagger\right)$ such that

$$Z \equiv X_1 + iX_2 \tag{369}$$

where it is clear that, since $X_1^\dagger = X_1$ and $X_2^\dagger = X_2$, Z is constructed from the sum of an Hermitian and an anti-Hermitian matrix (iX_2). The complex matrix can also be constructed from a product of a Hermitain and a unitary matrix

$$Z = RU \tag{370}$$

where $R^\dagger = R$ and $U^\dagger U = \mathbb{1}$. R and U, being normal matrices, are each diagonalizable by unique unitary similarity transformations. It is clear (or at least conceivable at this stage) that in (370) the matrix R is associated with radial degrees or freedom and that U is associated with angular degrees of freedom. With this definition, it is convenient to introduce two parameterizations of the complex matrices and their Hermitian conjugates in order to obtain the Laplacian associated with the Hamiltonian of a 2 Hermitian matrix model quantum mechanics. The reason why this is convenient is that, for a given parameterization, it is possible to define Lie-algebra valued differential, anti-Hermitian, matrices which lead to a metric that is expressed entirely in terms of the eigenvalues associated with the Hermitian matrix R once one restricts to the radial sector. In what follows, we consider the two(one) Hermitian(complex) matrix model of Masuku and Rodrigues [46] in order to show how the metric is obtained for the complex matrix coordinate and leads to the corresponding Laplace-Beltrami operator. The radial piece of this operator, which is the part that is relevant for us, has been shown to agree in both parameterizations [46] and it is for this reason that we simply review one of these parameterizations.[107] After restricting our interest to the radial sector of the theory we obtain the dual non-interacting system of fermions of ordinary quantum mechanics- a process referred to as radial fermionization.

C.2 2 Hermitian matrix model

The Hamiltonian of the two Hermitian matrix model is [46]

$$\hat{H} = -\frac{1}{2}\left(\frac{\partial}{\partial(X_1)_{ij}}\frac{\partial}{\partial(X_1)_{ij}} + \frac{\partial}{\partial(X_2)_{ij}}\frac{\partial}{\partial(X_2)_{ji}}\right) + V(X_1, X_2) \tag{371}$$

with summation over repeated indices. Consistent with (369) and (370) we complexify with the introduction of the complex matrix coordinate Z and its Hermitian conjugate:

$$\begin{aligned} Z &= RU \\ Z^\dagger &= U^\dagger R. \end{aligned} \tag{372}$$

[107] In the main text we wish to study the more general case of many complex matrices in the radial sector. The distinction of parameterizations in [46] is therefore too specific and not relevant, at this stage, for the radial collective field Hamiltonian. The generalization of the radial Laplace Beltrami operator to more complex matrices is straightforward and appears in (252) in the main text.

By labeling the eigenvalues of R by r_i [46] express the radial matrix in terms of the diagonal eigenvalue matrix and the corresponding unitary matrix V as $R = V^\dagger r V$. This allows for the definition of the two mentioned parameterizations. [46] define parameterization I in terms of the matrix coordinates (r, V, U) as

$$\begin{aligned} Z &= V^\dagger r V U \\ Z^\dagger &= U^\dagger V^\dagger r V. \end{aligned} \tag{373}$$

The Hermiticity of the matrix product $V^\dagger r V$ is convenient for introducing parameterization II, which we shall focus on from now on, in terms of (r, V, W) where $W \equiv VU$. Then

$$\begin{aligned} Z &= V^\dagger r W \\ Z^\dagger &= W^\dagger r V. \end{aligned} \tag{374}$$

The complex matrix coordinate metric is determined by introducing the Lie-algebra valued differential, anti-Hermitian, matrices: $dX \equiv V dU U^\dagger V^\dagger$, $dS \equiv dV V^\dagger$ and $dT \equiv dW W^\dagger$ and computing the infinitesimal arc length squared. It should be noted that, like dS, dT and dX, the product $U^\dagger U = UU^\dagger = \mathbb{1}$ leads to an anti-Hermitian differential since $dU^\dagger U + U^\dagger dU = dU U^\dagger + U dU^\dagger = 0$. Making use of these anti-Hermitian differential product matrices it is straightforward to show that

$$\begin{aligned} dZ &= V^\dagger (dr + r dT - dS r) W, \\ dZ^\dagger &= W^\dagger (dr + r dS - dT r) V. \end{aligned} \tag{375}$$

The infinitesimal arc length squared of the complex matrix is the trace of the product of the two equations in (375). The Trace operation is cyclic and the commutator $[dr, r] = [r, dr] = 0$ as a consequence of the fact that r and dr are real and diagonal. This leads to the following result

$$Tr(dZ^\dagger dZ) = Tr\left((dr)^2 - r^2 (dS)^2 - r^2 (dT)^2 + 2 r dS r dT\right), \tag{376}$$

of the arc length of parameterization II in (376). By defining the coordinates Y^+ and Y^- such that $dY^+ \equiv \frac{1}{\sqrt{2}}(dT + dS)$ and $dY^- \equiv \frac{1}{\sqrt{2}}(dT - dS)$, it becomes a straightforward exercise to confirm that the arc length in terms of the new coordinates Y^+ and Y^- is

$$Tr(dZ^\dagger dZ) = Tr\left(dr^2 + \frac{1}{2}[r, dY^+][r, dY^+] - \frac{1}{2}\{r, dY^-\}\{r, dY^-\}\right). \tag{377}$$

Making use of the anti-Hermiticity of dY^+ and dY^- one arrives at the result

$$\begin{aligned} Tr(dZ^\dagger dZ) = &\sum_i dr_i^2 + \sum_{i,j(i<j)} (r_i - r_j)^2 dY_{ij}^+ dY_{ij}^{*+} \\ &+ 2 \sum_i r_i^2 dY_{ii}^- dY_{ii}^{*-} + \sum_{i,j(i<j)} (r_i + r_j)^2 dY_{ij}^- dY_{ij}^{*-}. \end{aligned} \tag{378}$$

The metric tensor is extracted from the equation $ds^2 = g_{\mu\nu} dx^\mu dx^{*\nu}$ where $x^\mu = \{r_i, Y_{ii}^-, Y_{ij(i<j)}^+, Y_{ij(i<j)}^{*+}, Y_{ij(i<j)}^-, Y_{ij(i<j)}^{*-}\}$. The second and fourth terms of (378) are split such that

$$Tr(dZ^\dagger dZ) = \sum_i dr_i^2 + \frac{1}{2}\sum_{i,j(i<j)} (r_i - r_j)^2 dY_{ij}^{+} dY_{ij}^{*+} \tag{379}$$
$$+ \frac{1}{2}\sum_{i,j(i<j)} (r_i - r_j)^2 dY_{ij}^{*+} dY_{ij}^{+} + 2\sum_i r_i^2 dY_{ii}^{-} dY_{ii}^{*-}$$
$$+ \frac{1}{2}\sum_{i,j(i<j)} (r_i + r_j)^2 dY_{ij}^{-} dY_{ij}^{*-} + \frac{1}{2}\sum_{i,j(i<j)} (r_i + r_j)^2 dY_{ij}^{*-} dY_{ij}^{-}.$$

The metric tensor in this case is

$$g_{\mu\nu} = \begin{array}{c} \\ r_i \\ Y_{ii}^- \\ Y_{ij(i<j)}^+ \\ Y_{ij(i<j)}^{*+} \\ Y_{ij(i<j)}^- \\ Y_{ij(i<j)}^{*-} \end{array} \begin{array}{c} \begin{array}{cccccc} r_i^* & Y_{ii}^{*-} & Y_{ij(i<j)}^{*+} & Y_{ij(i<j)}^{+} & Y_{ij(i<j)}^{*-} & Y_{ij(i<j)}^{-} \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2r_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(r_i-r_j)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(r_i-r_j)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(r_i+r_j)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(r_i+r_j)^2 \end{array}\right) \end{array}. \tag{380}$$

The inverse metric can simply be read off of (380). The determinant is clearly

$$g \equiv \det g_{\mu\nu} = \prod_i 2r_i^2 \left(\triangle^2(r^2)\right)^2 \tag{381}$$

where $\triangle^2(r^2)$ is given by $\Delta^2(r^2) \equiv \Pi_{i<j}\frac{1}{4}(r_i^2 - r_j^2)^2$. The Laplacian associated with parameterization II is

$$\bigtriangledown_{II}^2 = \sum_i \frac{1}{\prod_k r_k(\triangle^2(r^2))}\frac{\partial}{\partial r_i}\left(\prod_k r_k(\triangle^2(r^2)\frac{\partial}{\partial r_i}\right) + \sum_i \frac{1}{2r_i}\frac{\partial}{\partial Y_{ii}^-}\frac{\partial}{\partial Y_{ii}^-} \tag{382}$$
$$+ \sum_{i,j(\neq i)} \frac{2}{(r_i + r_j)^2}\frac{\partial}{\partial Y_{ij}^-}\frac{\partial}{\partial Y_{ij}^{*-}} + \sum_{i,j(\neq i)} \frac{2}{(r_i - r_j)^2}\frac{\partial}{\partial Y_{ij}^+}\frac{\partial}{\partial Y_{ij}^{*+}}.$$

C.2.1 Invariant states

It is possible to map the Laplacian (382) to a form that makes the distinction between the angular and radial parts of the operators more apparent. For this, one considers invariant states on which (382) act. These states are constructed as the trace of a string of the complex matrices and their Hermitian conjugates: $Tr\left(...Z^{n_p}Z^{\dagger m_p}...Z^{n_q}Z^{\dagger m_q}...\right)$. Evidently for $n_i > m_i, \forall i$ the invariant state depends only on the eigenvalues of R and on $Q \equiv VUV^\dagger = WV^\dagger$. If $n_p > m_p$ and $n_q < m_q$ for some p and q then the invariant state depends on r, Q and $Q^\dagger$. Lastly, if $n_i < m_i, \forall i$ then the invariant states depend only on r and $Q^\dagger$ [46]. Now

$$dQ = dTQ - QdS = \frac{1}{\sqrt{2}}(dY^+Q - QdY^+) + \frac{1}{\sqrt{2}}(dY^- + QdY^-). \tag{383}$$

It is possible to extract from (383) the operators $\frac{\partial}{\partial Y_{ij}^+}$ and $\frac{\partial}{\partial Y_{ij}^-}$ expressed in terms of Q.[108] This is done by considering

[108] Equation (383) has the apparent form of the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. We can therefore read off of $dQ_{ab} = \frac{1}{\sqrt{2}}(dY_{ak}^+Q_{kb} - Q_{ak}dY_{kb}^+) + \frac{1}{\sqrt{2}}(dY_{ak}^-Q_{kb} + Q_{ak}dY_{kb}^-)$ the form of $\frac{\partial Q_{ab}}{\partial Y_{ij}^+}$ and $\frac{\partial Q_{ab}}{\partial Y_{ij}^-}$ that appear in (384).

$$\frac{\partial}{\partial Y_{ij}^{+}} = \frac{\partial Q_{ab}}{\partial Y_{ij}^{+}}\frac{\partial}{\partial Q_{ab}} = \frac{1}{\sqrt{2}}(Q_{jb}\frac{\partial}{\partial Q_{ib}} - Q_{ai}\frac{\partial}{\partial Q_{aj}})$$
$$\frac{\partial}{\partial Y_{ij}^{-}} = \frac{\partial Q_{ab}}{\partial Y_{ij}^{-}}\frac{\partial}{\partial Q_{ab}} = \frac{1}{\sqrt{2}}(Q_{jb}\frac{\partial}{\partial Q_{ib}} + Q_{ai}\frac{\partial}{\partial Q_{aj}}). \tag{384}$$

The generators of left and right rotations are defined: $E_{ji}^{L} \equiv Q_{jb}\frac{\partial}{\partial Q_{ib}}$ and $E_{ji}^{R} \equiv Q_{ai}\frac{\partial}{\partial Q_{aj}}$ in terms of which (384) becomes

$$\frac{\partial}{\partial Y_{ij}^{+}} = \frac{1}{\sqrt{2}}(E_{ji}^{L} - E_{ji}^{R}), \qquad \frac{\partial}{\partial Y_{ij}^{-}} = \frac{1}{\sqrt{2}}(E_{ji}^{L} + E_{ji}^{R}). \tag{385}$$

By recalling that from the definitions of dY^{+} and dY^{-}, it is seen that they are both anti-Hermitian which implies that both partial derivatives in (385) are anti-Hermitian. This, together with the results of (385), can be used to convert (382) to the form

$$\triangledown_{II}^{2} = \sum_{i}\frac{1}{\prod_{k} r_{k} \triangle^{2}(r^{2})}\frac{\partial}{\partial r_{i}}\left(\prod_{k} r_{k} \triangle^{2}(r^{2})\frac{\partial}{\partial r_{i}}\right) - \sum_{i}\frac{1}{4r_{i}}(E_{ii}^{L} + E_{ii}^{R})^{2}$$
$$- \sum_{i,j(\neq i)}\frac{2(r_{i}^{2}+r_{j}^{2})}{(r_{i}-r_{j})^{2}}(E_{ij}^{L}E_{ij}^{L} + E_{ji}^{R}E_{ij}^{R}) + \sum_{i,j(\neq i)}\frac{4r_{i}r_{j}}{(r_{i}^{2}-r_{j}^{2})^{2}}(E_{ji}^{L}E_{ij}^{R} + E_{ji}^{R}E_{ij}^{L}). \tag{386}$$

Since dY_{ii}^{+} does not appear in (377) we find that $0 = \frac{\partial}{\partial Y_{ii}^{+}} = \frac{1}{\sqrt{2}}(E_{ii}^{L} - E_{ii}^{R})$. Therefore, $E_{ii}^{L} = E_{ii}^{R}$ and

$$\triangledown_{II}^{2} = \sum_{i}\frac{1}{\prod_{k} r_{k} \triangle^{2}(r^{2})}\frac{\partial}{\partial r_{i}}\left(\prod_{k} r_{k} \triangle^{2}(r^{2})\frac{\partial}{\partial r_{i}}\right) - \sum_{i}\frac{1}{2r_{i}}E_{ii}^{L}E_{ii}^{L}$$
$$- \sum_{i,j(\neq i)}\left[\frac{2(r_{i}^{2}+r_{j}^{2})}{(r_{i}-r_{j})^{2}}(E_{ij}^{L}E_{ij}^{L} + E_{ji}^{R}E_{ij}^{R}) - \sum_{i,j(\neq i)}\frac{4r_{i}r_{j}}{(r_{i}^{2}-r_{j}^{2})^{2}}(E_{ji}^{L}E_{ij}^{R} + E_{ji}^{R}E_{ij}^{L})\right]. \tag{387}$$

For further details one can consult [46] directly. The main purpose of this section has been to illustrate the defnition of and distinction between the radial and angular degrees of freedom of the complex matrix. As noted above, the primary interset is the radial piece of the operator (387).

C.3 Radial fermionization (fermionic description)

In the Hamiltonian formulation, as has been discussed already, the Laplacian is modified by a Jacobian. Since we restrict our attention to the radial sector of the theory, all terms in the Laplacian which include the generators of left and right rotations are irrelevant (see (387)). The wave functions on which the Hamiltonian (371) acts are symmetric under rotations (U(N) transformations), but one can re-express these symmetric wave functions as the quotient of an anti-symmetric wave function and the Vandermonde determinant. Acting with the Hamiltonian on the $1/\triangle^{2}(r^{2})$ part of the wave function leads to another Hamiltonian operator acting on the completely antisymmetric wave functions. This will be shown to be the case for a two Hermitian matrix model in what follows from [46].

We consider a potential of the form $Tr[v(ZZ^{\dagger})]$ where $v(ZZ^{\dagger})$ is a polynomial function. [46] redefine the radial egienvalues by $\rho_{i} \equiv r_{i}^{2}$. In the radial sector (corresponding to s-states) the kinetic part of the Hamiltonian is

$$-\frac{1}{2}\bigtriangledown^2 = -\frac{1}{2}\sum_i \frac{1}{\prod_k r_k \triangle^2 (r^2)} \frac{\partial}{\partial r_i}\left(\prod_k r_k \triangle^2 (r^2) \frac{\partial}{\partial r_i}\right) \tag{388}$$
$$= -\frac{2}{\triangle^2(\rho)} \sum_i \frac{\partial}{\partial \rho_i}\left(\rho_i \triangle^2 (\rho)\right) \frac{\partial}{\partial \rho_i}.$$

The $U(N)\times U(N)$ symmetric wave functions Φ on which (388) act can be redefined as $\Phi \equiv \Psi/\triangle$ where Ψ are completely anti-symmetric wave functions and $\triangle$ is simply the square root of $\Delta^2(r^2) \equiv \Pi_{i<j}\frac{1}{4}(r_i^2 - r_j^2)^2$. The Schrodinger equation becomes

$$\left[-\frac{2}{\triangle^2(\rho)} \sum_i \frac{\partial}{\partial \rho_i}\left(\rho_i \triangle^2 (\rho)\right) \frac{\partial}{\partial \rho_i}\right] \frac{\Psi}{\triangle} = E\frac{\Psi}{\triangle} \tag{389}$$

or

$$\left[-\sum_i \frac{2}{\triangle(\rho)} \frac{\partial}{\partial \rho_i}\left(\rho_i \triangle^2 (\rho)\right) \frac{\partial}{\partial \rho_i}\frac{1}{\triangle}\right] \Psi = E\Psi. \tag{390}$$

By regrouping terms in the left hand side of (390) one obtains

$$-2\sum_i \left(\frac{1}{\triangle}\frac{\partial}{\partial \rho_i}\triangle\right) \rho_i \left(\triangle \frac{\partial}{\partial \rho_i}\frac{1}{\triangle}\right)\Psi = E\Psi, \tag{391}$$
$$-2\sum_i \left(\frac{\partial \ln \triangle}{\partial \rho_i} + \frac{\partial}{\partial \rho_i}\right) \rho_i \left(-\frac{\partial \ln \triangle}{\partial \rho_i} + \frac{\partial}{\partial \rho_i}\right)\Psi = E\Psi.$$

We note that $\frac{\partial \ln \triangle}{\partial \rho_i} = \sum_{k,j(k<j)} \frac{1}{|\rho_j - \rho_k|}(\delta_{ik} - \delta_{ij}) = \sum_{j(j>i)} \frac{1}{(\rho_i - \rho_j)} + \sum_{k(k<i)} \frac{1}{(\rho_i - \rho_k)}$ which leads to

$$-2\sum_i \left(\sum_{j(\neq i)} \frac{1}{\rho_i - \rho_j} + \frac{\partial}{\partial \rho_i}\right) \rho_i \left(-\sum_{k(\neq i)} \frac{1}{\rho_i - \rho_k} + \frac{\partial}{\partial \rho_i}\right)\Psi = E\Psi,$$
$$-2\sum_i \left(\frac{\partial}{\partial_{\rho_i}}\rho_i \frac{\partial}{\partial_{\rho_i}} - \sum_{j(\neq i),k(\neq i)} \frac{\rho_i}{(\rho_i - \rho_j)(\rho_i - \rho_k)} - \sum_{j(\neq i)} \frac{1}{\rho_i - \rho_j} + \sum_{j(\neq i)} \frac{\rho_i}{(\rho_i - \rho_j)^2}\right)\Psi = E\Psi. \tag{392}$$

The third term of (392) is zero and the remaining non-derivative terms sum to zero since

$$\sum_{j(\neq i)} \frac{\rho_i}{(\rho_i - \rho_j)^2} - \sum_{j(\neq i),k(\neq i)} \frac{\rho_i}{(\rho_i - \rho_j)(\rho_i - \rho_k)} = -\sum_{i(\neq j,k),j(\neq i,k),k(\neq i,j)} \frac{\rho_i}{(\rho_i - \rho_j)(\rho_i - \rho_k)} \tag{393}$$

and the right hand side of (393) is identically zero. This is easily verified by inspection for i, j, k running over $1, 2, 3$. The Schrodinger equation is [46]

$$\left(-2\sum_i \frac{\partial}{\partial \rho_i}\rho_i \frac{\partial}{\partial \rho_i} + v(\rho_i)\right)\Psi = E\Psi. \tag{394}$$

D Appendix D

D.1 Jacobian From Schwinger-Dyson Equations

The so called Schwinger-Dyson equations are derived by considering a general property of integration. Masuku and Rodrigues [47] apply this to the generating functional for the multi-complex matrix theory (evaluated for zero source)[109]:

$$\int \prod_A \prod_{ij} dZ^\dagger_{Aij} dZ_{Aij} \frac{\partial}{\partial (Z_A)_{ji}} \left(\frac{\partial \phi_k}{\partial (Z^\dagger_A)_{ij}} F[\phi] e^{-S} \right) = 0. \tag{395}$$

$F[\phi]$ is an arbitrary product of invariant functions. This may be re-expressed as

$$\left\langle \left(\frac{\partial^2 \phi_k}{\partial (Z_A)^\dagger_{ij} \partial (Z_A)_{ji}} \right) F[\phi] \right\rangle + \left\langle \frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial F[\phi]}{\partial (Z_A)_{ji}} \right\rangle - \left\langle F[\phi] \frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ji}} \frac{\partial S}{\partial (Z_A)_{ji}} \right\rangle = 0 \tag{396}$$

where the notation reflects the usual definition for time ordered correlation functions (arbitrary) of a quantum field theory. In particular, [47] make the definition of the Jacobian clear by stating: $\langle F[\phi] \rangle = \int \prod_A \prod_{ij} dZ^\dagger_{Aij} dZ_{Aij} F[\phi] e^{-S} \equiv \int [d\phi] J[\phi] F[\phi] e^{-S}$ and noting that up to an overall constant: $\int [d\phi] \sim \int \prod_i d\rho_i$. The following identity is also true

$$\int [d\phi] \int dk' \frac{\partial}{\partial \phi_{k'}} \left(\left[\frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \phi_{k'}}{\partial (Z_A)_{ji}} \right] J[\phi] F[\phi] e^{-S} \right) = 0. \tag{397}$$

Carrying out the differentiation, and applying the chain-rule: $\int dk' \frac{\partial \phi_{k'}}{\partial (Z_A)_{ji}} \frac{\partial F[\phi]}{\partial \phi_{k'}} = \frac{\partial F[\phi]}{\partial (Z_A)_{ji}}$ and $\int dk' \frac{\partial \phi_{k'}}{\partial (Z_A)_{ij}} \frac{\partial S}{\partial (Z_A)_{ij}} = \frac{\partial S}{\partial (Z_A)_{ji}}$, one obtains

$$\int dk' \left\langle \left(\frac{\partial}{\partial \phi_{k'}} \left[\frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \phi_{k'}}{\partial (Z_A)_{ji}} \right] \right) F[\phi] \right\rangle + \int dk' \left\langle \left[\frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \phi_{k'}}{\partial (Z_A)_{ji}} \right] F[\phi] \frac{\partial \ln J}{\partial \phi_{k'}} \right\rangle + \left\langle \frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial F[\phi]}{\partial (Z_A)_{ji}} \right\rangle - \left\langle F[\phi] \frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial S}{\partial (Z_A)_{ji}} \right\rangle = 0. \tag{398}$$

By comparing (396) and (398), which are equivalent given that $F[\phi]$ is an arbitrary invariant function, [47] deduce

[109] It should be noted that in [47] a gaussian ensample of m complex matrices is considered. This leads to a density of eigenvalues which differs from the expected Wigner distribution. For the case of 1 complex matrix a restriced Wigner distribution is obtained but for $m \geq 2$ the distribution is no longer of the Wigner type. For the present task of obtaining the Jacobian associated with the change of variables from matrix degrees of freedom to radial eigenvalues it is not necessary to specify the form of the Action functional i.e. at this stage it is not necessarily gaussian. The treatment does however require that the potential is diagonalizable by unitary transformation and can therefore be expressed in terms of the radial eigenvalues.

$$\int dk' \frac{\partial}{\partial \phi_{k'}} \left[\frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \phi_{k'}}{\partial (Z_A)_{ji}} \right] + \int dk' \left[\frac{\partial \phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial \ln J}{\partial \phi_{k'}} = \frac{\partial^2 \phi_k}{\partial (Z_A)^\dagger_{ij} \partial (Z_A)_{ji}}. \tag{399}$$

The joining and splitting operators (of (277) and (273)) are apparent in (399).[110] Then

$$\int dk' \Omega_{kk'} \frac{\partial \ln J}{\partial \phi_{k'}} + \int dk' \frac{\partial}{\partial \phi_{k'}} \Omega_{kk'} = \omega_k. \tag{400}$$

Each term of (400) is a function of k. We multiply (400) by $\frac{1}{2\pi} e^{-ik\rho}$, integrate over k, making use of the definition of the Fourier transform and relabel the dummy integration variable $k' \to \rho'$. Then

$$\int \frac{dk}{2\pi} e^{-ik\rho} \left(\int dk' \Omega(k,k';[\phi]) \frac{\partial \ln J}{\partial \phi(k')} \right) + \int \frac{dk}{2\pi} e^{-ik\rho} \left(\frac{\partial \Omega(k,k';[\phi])}{\partial \phi(k')} \right) = \int \frac{dk}{2\pi} e^{-ik\rho} \omega(k;[\phi]) \tag{401}$$

becomes

$$\int d\rho' \Omega_{\rho\rho'} \frac{\partial \ln J}{\partial \phi(\rho')} + \int d\rho' \frac{\partial \Omega_{\rho\rho'}}{\partial \phi(\rho')} = \omega_\rho. \tag{402}$$

Due to the form of (274) it is clear that the second term of (402) is zero. Making use of the explicit forms of ω_ρ and $\Omega_{\rho\rho'}$ and integrating by parts leads to the equation satisfied by the Jacobian:

$$\partial_\rho \frac{\partial \ln J}{\partial \phi(\rho)} = 2 \fint d\rho' \frac{\phi(\rho')}{\rho - \rho'} + \frac{N(m-1)}{\rho}, \tag{403}$$

which is the same result obtained through the collective field method (284). The solution $\ln J$ is, by inspection, identified to be

$$\ln J = \int d\rho'' \phi(\rho'') \fint d\rho' \phi(\rho') \ln|\rho - \rho'| + N(m-1) \int d\rho' \phi(\rho') \ln \rho'. \tag{404}$$

The eigenvalue density allows us to rewrite the equation for the Jacobian as

$$\ln J = \sum_{i,j(\neq i)} \ln|\rho_i - \rho_j| + N(m-1) \sum_i \ln \rho_i. \tag{405}$$

Noting that $N(m-1)\sum_i \ln \rho_i = \sum_{j,i} \ln \rho_i^{m-1}$, which can be split into three terms $\frac{1}{2}\sum_{i,j(\neq i)} \ln \rho_i^{m-1} + \frac{1}{2}\sum_{j,i(\neq j)} \ln \rho_j^{m-1} + \sum_i \ln \rho_i^{m-1}$, leads to

$$\ln J = \ln \prod_i \rho_i^{m-1} \prod_{i \neq j} \rho_i^{\frac{m-1}{2}} \rho_j^{\frac{m-1}{2}} |\rho_i - \rho_j|. \tag{406}$$

[110] Note that the summation of the indices A, i and j in (399) is implicit through the Einstein summation convention for which summation over repeated indices is implied.

In other words [47][111]

$$J = \prod_i \rho_i^{m-1} \prod_{i \neq j} \rho_i^{\frac{m-1}{2}} \rho_j^{\frac{m-1}{2}} |\rho_i - \rho_j| = \prod_i \rho_i^{m-1} \prod_{i>j} \rho_i^{m-1} \rho_j^{m-1} (\rho_i - \rho_j)^2. \tag{407}$$

[111]It is evident that for $m = 1$ this result confirms that of the single complex matrix (see (381)).

E Appendix E

E.1 Minimum extension of the $SL(2,\mathbb{R})$ generators

In this appendix we illustrate the closure of the first quantized $sl(2,\mathbb{R})$ algebra without dropping the Lagrange multiplier μ. In order to do so we assume the existence of Lagrange multiplier terms in the special conformal and dilatation operators which are independent of the canonical conjugate coordinates x and p. Given that we know the explicit form of the Lagrange multiplier term in the Hamiltonian we can make use of the $sl(2,\mathbb{R})$ algebra to deduce the Lagrange multiplier terms in the two remaining generators. We shall show that this can be achieved by promoting the Lagrange multiplier μ to a 'field'.

The generators of conformal quantum mechanics in first quantization are

$$\begin{aligned}\hat{h} &= \frac{1}{2}p^2 + \frac{\eta^2}{2r^2} + \mu \equiv h + h_\mu,\\ \hat{k} &= \frac{1}{2}r^2 + k_u,\\ \hat{d} &= \frac{1}{2}(rp + pr) + d_\mu.\end{aligned} \tag{408}$$

Since we have assumed k_μ and d_μ to be independent of x and p, only commutators involving Lagrange multiplier terms of the form $[\mathcal{O}_\mu, \mathcal{O}'_\mu]$ are not necessarily zero. We deduce the explicit form of the generators k_μ and d_μ by studying

$$\begin{aligned}[][d_\mu, h_\mu] &= 2ih_\mu,\\ [d_\mu, k_\mu] &= -2ik_\mu,\\ [h_\mu, k_\mu] &= -id_\mu,\end{aligned} \tag{409}$$

since we already know the form of the algebra (301) in the absence of the Lagrange multiplier terms. We refer to this first quantized algebra as the minimum extension. So:

$$[d_\mu, \mu] = d_\mu\mu - \mu d_\mu = 2i\mu. \tag{410}$$

If we assume that d_μ is linear in derivatives with respect to μ i.e. $d_\mu = A(\mu)\partial_\mu$, where $A(\mu)$ is an arbitrary function of μ, then:

$$\begin{aligned}[][d_\mu, \mu] &= d_\mu\mu - \mu d_\mu\\ &= (A(\mu)\partial_\mu\mu) = 2i\mu\\ \Rightarrow A(\mu) &= 2i\mu.\end{aligned} \tag{411}$$

Then by defining the canonical conjugate momentum $P_\mu \equiv -i\partial_\mu$ we find that

$$d_\mu = 2i\mu\partial_\mu = -2\mu P_\mu. \tag{412}$$

Similarly, if we take k_μ to be quadratic in derivatives with respect to μ i.e. $k_\mu = B(\mu)\partial_\mu^2$ then

$$\begin{aligned}[][\mu, k_\mu] &= \mu k_\mu - \mu k_\mu - 2B(\mu)\partial_\mu = 2\mu\partial_\mu\\ \Rightarrow B(\mu) &= -\mu\end{aligned} \tag{413}$$

but written in terms of the momentum

$$k_\mu = \mu(-i\partial_\mu)(-i\partial_\mu) = \mu P_\mu^2. \tag{414}$$

Checking the last commutator confirms that the subalgebra closes:

$$\begin{aligned}[d_\mu, k_\mu] &= -2i[\mu\partial_\mu, \mu\partial_\mu^2] \\ &= -2i(-\mu\partial_\mu^2) \\ &= -2ik_\mu \end{aligned} \tag{415}$$

and we have

$$\begin{aligned}[\hat{h}, \hat{k}] &= -i\hat{d}, \\ [\hat{d}, \hat{k}] &= -2i\hat{k}, \\ [\hat{d}, \hat{h}] &= 2i\hat{h}. \end{aligned} \tag{416}$$

F Appendix F

F.1 Near AdS_2 Geometry

In this section we summarize important results form the papers [51][112], [42], [49][113], [61] and [53] which are covered in subsections F.1.1, F.1.2, F.1.3, F.1.3 and F.1.4 respectively.

F.1.1 Jackiw-Teitelboim Gravity

Einstein gravity cannot be sensibly constructed in two dimensions since the Einstein tensor ($G_{\mu\nu}$) automatically vanishes. It is in fact entirely un-geometrical since the usual Einstein-Hilbert action is topological- being the well known Gauss-Bonnet/Euler term. Jackiw and Teitelboim [51, 58, 59] have however postulated a sensible geometric model. The model suggested is a constant curvature model for which the scalar curvature is defined in terms of a cosmological constant. The corresponding gravitational action is not completely geometric though as it is defined in terms of a scalar field ϕ which appears as a Lagrange multiplier which is responsible for imposing AdS_2 geometry. The precise form of this constraint is

$$R = -2\Lambda, \qquad\qquad (\Lambda > 0). \tag{417}$$

This equation can be derived from the variation of a flat space conformal field theory (Liouville action) after considering that in 2-d all pseudo-Riemannian manifolds are conformally flat. The scale factor (Liouville field), say ϕ in $g_{\mu\nu} = e^{\phi}\eta_{\mu\nu}$, determines the curvature scalar $R = -e^{-\phi}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi$. Variation with respect to the Liouville field in the Liouville action reproduces the AdS_2 constraint (417). However, a more natural action, from both the classical and quantum perspectives, is the Jackiw-Teitelboim (JT) action:

$$S_{JT} = \int d^2x\sqrt{-g}\phi(R + 2\Lambda). \tag{418}$$

Classically, this action follows from a dimensional reduction of the 3-d Einstein-Hilbert action and, quantum mechanically, the constant curvature constraint (417) arises due to anomalies. It should be noted that the Liouville description is of interest from the point of view that it can be quantized and is found to be $SO(1,2)$ invariant- which is the isometry group of AdS_2. The dilaton[114] equation of motion obviously returns the constraint (417) and the metric equation of motion is[115]

$$(\bigtriangledown_\mu \bigtriangledown_\nu - g_{\mu\nu} \bigtriangledown^2 + g_{\mu\nu}\Lambda)\phi = 0. \tag{419}$$

The trace of this equation with the metric gives $(\bigtriangledown^2 - 2\Lambda)\phi = 0$. This equation is solved for $\Lambda\phi$ which, when substituted into (419) leads to

$$(\frac{1}{2}g_{\mu\nu} \bigtriangledown^2 - \bigtriangledown_\mu \bigtriangledown_\nu)\phi = 0. \tag{420}$$

Together (419) and (420) in conformal gauge reduce to the simple form

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - 2\Lambda e^{\phi})\phi = 0. \tag{421}$$

[112]The model was presented by both Jackiw and Teitelboim in the same volume of [58] and Teitelboim's own related work can be found in [59]. We focus on the paper by Jackiw [51].

[113] [49] has extensions of the results found in [60] relating to the Sachdev-Ye-Kitaev (SYK) model.

[114]The Lagrange multiplier scalar field is labeled as a dilaton due to the appearance of the corresponding action in string theory contexts where the Lagrange multiplier is the string dilaton field.

[115]By the variations: $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta R = R_{\mu\nu}\delta g^{\mu\nu} - (\bigtriangledown_\mu \bigtriangledown_\nu - g_{\mu\nu}\Box)\delta g^{\mu\nu}$, which follow by varying the metric.

A matter term can be included in the action. By choosing ϕ dependence in the form $\int d^2x\sqrt{-g}\phi\mathcal{L}_{matter}$, one learns that curvature/gravity is sourced by the matter Lagrangian $R = 16\pi G\mathcal{L}_{matter}$. This is not surprising given that the stress-energy tensor $T_{\mu\nu}$ cannot source gravity in the usual sense since the Einstein tensor vanishes in 2-d.[116]

The JT model [51] has resurfaced recently with an important role in the $AdS_2/CFT_1(NAdS_2/NCFT_1)$ context where it provides the leading order gravitational corrections that are responsible for breaking the full conformal symmetry associated with AdS_2.

F.1.2 Universal A.P. model and equivalence of thermodynamic mass gap and conformal symmetry breaking scale

As noted at the end of F.1.1, leading order gravitational effects are responsible for breaking the full conformal symmetry associated with AdS_2. These effects are present in the IR limit of higher dimensional extremal black holes which are known to be dimensionally reduced to AdS_2 (appearing in $AdS_2 \times X$.) These effects are introduced by the backreaction due to matter and appear in the boundary correlation functions dual to the bulk scalar fields. This was discovered by Almheiri and Polchinski (AP) [52]. The AP model provides a description of certain attributes of extremal black holes (and near-extremal black holes for non-zero temperature). [117]For the zero-temperature case the AP model action can be expressed as

$$S_{AP} = \frac{1}{16\pi G}\int d^2x\sqrt{-g}(\Phi^2 R + C(\Phi^2 - \Phi_0^2)) + S_{matter} \tag{422}$$

and has the Poincare patch solutions

$$\begin{aligned} ds^2 &= \frac{2}{C}\frac{1}{z^2}(-dt^2 + dz^2), \\ \Phi^2 &= \Phi_0^2 + \frac{a}{z}. \end{aligned} \tag{423}$$

Here C and Φ_0 are positive constants and a is an integration constant that parameterizes a set of solutions. The connected boundary 4-point correlation functions obtained in [52] were found to have the form

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle \sim \frac{G}{at^3}. \tag{424}$$

This contrasts the expected conformal scaling behavior which would require scaling of the form $\sim \frac{1}{t^4}$ which is obtained for the disconnected contribution.[118] The presence of the the dimensionless two dimensional Newton constant G in (424) is to be understood as arising due to gravitational backreaction induced by the bulk matter and as a consequence one identifies the parameter a as being responsible for regulating the backreaction. Indeed, for $a \to \infty$ the backreaction is no longer important and the dilaton (423) blows up; this is the UV limit. However, in the IR limit ($a \to 0$) the backreaction causes the 4-point function to blow up and the dilaton approaches a constant value. The important feature due to the backreaction is its significance in the IR and the associated energy scale $E \sim \frac{G}{a}$ below which the IR theory breaks the conformal symmetry of the full AdS_2 theory- as is apparent in the 4-point function (424).

The correct interpretation of the AP model is that it describes the near horizon $NAdS_2$ geometry of near-extremal black holes for various dilaton gravity theories [42] [61]. This is seen by considering the action[119]

[116] In the applications of interest to us however the matter Lagrangian will be independent of the dilaton (see subsection (F.1.3)).

[117] The AP model [61] is summarized in [42], which is included in our presentation of the work of [42].

[118] The dual field theory operators have conformal dimension -1 in length units which explains the expectations discussed.

[119] F in (425) is the field strength for the case of Maxwell-dilaton models.

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}(\Phi^2 R + \lambda(\bigtriangledown\Phi)^2 - U(\Phi) - f(\Phi)F^2). \tag{425}$$

The static (r dependent only) solutions, after scaling $g_{\mu\nu} \to g_{\mu\nu}\Phi^{-\lambda/2}$ and setting $\lambda = f(\Phi) = 0$ with the gauge choice $ds^2 = -e^{2\omega}dt^2 + e^{-2\omega}dr^2$, have the equations of motion

$$\begin{aligned} 0 &= 2\omega'(\Phi^2)' + (\phi^2)'' + e^{-2\omega}U(\Phi),\\ 0 &= (e^{2\omega})'' + e^{-2\omega}\partial_{\Phi^2}U(\Phi),\\ (\Phi^2)' &= -\frac{\alpha}{2}. \end{aligned} \tag{426}$$

The primes represent differentiation with respect to r and α is a parameter. The final equation in (426) implies that $\Phi^2 = \Phi_H^2 - \frac{\alpha r}{2}$. Φ_H^2 is an integration constant that will correspond to the value of the dilaton on the horizon. If one Taylor expands about the solution Φ_H^2 the solution is $(e^{2\omega})' = \frac{2}{\alpha}U(\Phi_H) - \frac{\alpha r}{2}\partial_{\Phi^2}U(\Phi_H) + \frac{\alpha^2 r^2}{8}\partial^2_{\Phi^2}U(\Phi_H) + ... \equiv (\star)$. The near horizon limit ($r \to 0$ and retaining up to first order in r) obtains the AP model equations of motion. This determines the universality of the AP model since the generic dilaton gravity theory (425) with near-extremal black holes has near horizon geometry described by the AP model. Integrating the Taylor expanded solution $(\star)$ and defining $e^{2\omega} \equiv g(r)$ gives

$$g(r) = C + (\star)r, \tag{427}$$

where C is a constant. In terms of $g(r)$, the form of the metric is

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)}. \tag{428}$$

From thermodynamic considerations alone, one can determine the thermodynamic mass gap associated with near-extremal black holes. The mass above extremality for such black holes has the generic form: $\Delta M = M - M_{ext} = M_g^{-1}T^2$. Below this mass gap, the mass/energy goes to zero at a faster rate than the rate at which the temperature goes to zero. In this case the Hawking process of black hole evaporation can no longer occur. The Hawking temperature can be computed from the surface gravity or via the Gibbons-Perry approach [62]. In the latter approach, we note that the metric is expected to have roots $g(r_H) = 0$.[120] This implies $g(r) = g'(r_H)(r - r_H) + \frac{1}{2}g''(r_H)(r - r_H)^2 + ...$, which, in the near horizon limit ($r \to r_H$), becomes $g(r) \simeq g'(r_H)(r - r_H)$. We analytically continue to Euclidean time $-i\tau = t$ and perform a diffeomorphism [17]

$$R = 2\sqrt{\frac{r - r_H}{g'(r_H)}}, \qquad \theta = \frac{1}{2}|g'(r_H)|\tau \tag{429}$$

to obtain

$$\begin{aligned} ds^2 \to\, & g'(r_H)(r - r_H)d\tau^2 + \left[g'(r_H)(r - r_H)\right]^{-1}dr^2\\ =& R^2 d\theta^2 + dR^2. \end{aligned} \tag{430}$$

[120]The roots $g(r_H) = 0$ can be made consistent with (427) if one takes $C \to 0$.

This is the familiar polar coordinate metric on the cone which is only geodesically complete after removing the conical singularity which requires periodicity $\theta \sim \theta + 2\pi \Rightarrow \tau \sim \tau + \frac{4\pi}{|g'(r_H)|}$. But finite temperature field theory requires Euclidean time periodicity $\tau \sim \tau + \beta$. Therefore the Hawking temperature is

$$T_H = \frac{1}{4\pi}|g'(r_H)|. \tag{431}$$

For $r_H = 0$ this becomes $T_H = \frac{1}{2\pi\alpha}|U(\Phi_H)|$ with the solution $U(\Phi_0) = 0$ at zero temperature. Expanding about the zero temperature solution $\Phi_H^2 = \Phi_0^2 + \delta\Phi^2$: $U(\Phi_H) = \partial_{\Phi^2}U(\Phi_0)\delta\Phi^2 + \mathcal{O}((\delta\Phi^2)^2)$. Therefore $\delta\Phi^2 = \frac{2\pi T_H \alpha}{|\partial_{\Phi^2}U(\Phi_0)|}$. For $AdS_2 \times S^2$, $\Phi_H^2 = \Phi_0^2 + \delta\Phi^2$ is the compact S^2 area and Φ_0^2 is the extremal black hole horizon area with near extremal corrections $\delta\Phi^2$. The Wald entropy formula $Area/4G$ becomes

$$S_W = \frac{1}{4G}\left(\Phi_0^2 + \frac{2\pi T_H \alpha}{|\partial_{\Phi^2}U(\Phi_0)|}\right). \tag{432}$$

By the equation $dS = \frac{dE}{T}$, this can be integrated to find

$$\Delta M = \frac{\pi\alpha}{4G|\partial_{\Phi^2}U(\Phi_0)|}T^2 + \mathcal{O}(T). \tag{433}$$

This result was determined simply by the near-horizon features ($U(\Phi_0)$) of a near-extremal black hole which, as discussed, is universally contained in the AP model description. Reading off from (433), the mass gap is

$$M_g = \frac{4G}{\pi\alpha}|\partial_{\Phi^2}U(\Phi_0)|. \tag{434}$$

The near-extremal, near-horizon, action is found to be

$$S = \int dtdz\left[\frac{\alpha}{4[-\partial_{\Phi^2}U(\Phi_0)]}\frac{1}{z}(\partial_z g)^2 + \mathcal{O}(\frac{1}{z})\right] \tag{435}$$

in Poincare coordinates. g is a bulk linearized graviton perturbation which Almheiri and Kang [42] have shown to exhibit the IR feature of conformal symmetry breaking. The conformal symmetry breaking scale is read off of correlation functions (the 4-point function) and from (435) is

$$E_{break} \sim \frac{G|\partial_{\Phi^2}U(\Phi_0)|}{\alpha} \tag{436}$$

which is the same as the thermodynamic mass gap. The equality of E_{break} and M_g is, by the universality of the AP model, a generic feature of all dilaton gravity theories that have AP IR descriptions [42].

F.1.3 Emergence of $SL(2,\mathbb{R})$ invariant boundary Schwarzian action

F.1.3 consists of a review of the emergence of an $SL(2,\mathbb{R})$ invariant boundary Schwarzian action from the JT model which was identified by Maldacena *et al.* [49] and [61].

The thermodynamic mass gap discussed in F.1.2 is responsible for the IR limit pure AdS_2 restriction to the ground state. This restriction can be avoided by considering the $NAdS_2$ gravity which exhibits the leading order gravitational effects due to matter backreaction that are intimately connected with the breaking of conformal symmetry. It is in this context that the JT gravity is relevant (appearing from the broader AP model). This 2-d gravity has some 'memory' of the breaking of conformal symmetry as opposed to the pure AdS_2 case [49].

Consistent with the AP approach, the geometry is expected to appear in a dimensional reduction, in the IR, from a gravity theory that posses a UV description. In order to have a well defined boundary field theory, it is also necessary to consider the UV/IR (scale/radius) duality- a UV cut off in the boundary theory coincides with an IR cut off in the bulk. In other words, a large radius cut off on the AdS_2 disk is required [63]. [121] The IR cut off can be implemented via a trajectory $(t(u), z(u))$ parameterized by the boundary time u which has a fixed proper length $g_{uu} = \epsilon^{-2} = \frac{(dt/du)^2+(dz/du)^2}{z^2}$ that becomes $z = \epsilon dt/du = \epsilon t'$ as the boundary is approached $(z \to 0)$ [49]. The functions $t(u)$ parameterize a class of equivalent (up to $SL(2,\mathbb{R})$ transformations) AdS_2 cut off spaces. $SL(2,\mathbb{R})$ transformations on $t(u)$ lead to the same AdS_2 cut off spaces. This suggests that the $SL(2,\mathbb{R})$ symmetry on the regularized AdS_2 has a gauge symmetry interpretation.

Cadoni and Mignemi [20] have computed, from the JT model, the generators of the asymptotic symmetries of AdS_2 (- which they have also shown to satisfy the full conformal group Virasoro algebra). The leading order (from the $z \to 0$ boundary) form of the generators are

$$\chi^t = \epsilon(u), \qquad \chi^z = -z\epsilon'(u). \tag{437}$$

These generators are reparameterizations of the fields $t(u)$ that send boundary curves into new boundary curves thats geometry of course leaves the topological term of the action invariant and therefore corresponds to the same extremal entropy. The AdS_2 bulk breaks the full conformal group to $SL(2,\mathbb{R})$ since the full reparameterization symmetry is asymptotic and, as a consequence, the boundary fields $t(u)$ have an interpretation as pseudo-Goldstone zero modes [49].

The full action for the theory is

$$S = S_{top} + S_{JT} + S_m, \tag{438}$$

with topological, JT and matter terms. The JT action is

$$S_{JT} = -\frac{1}{16\pi G}\int d^2x\sqrt{g}(R+\Lambda) - \frac{1}{8\pi G}\int K\phi_B \tag{439}$$

with the cosmological constant term $\Lambda = 2$, the boundary dilaton value ϕ_B and the trace of the extrinsic curvature K. The boundary term (Gibbons-Hawking) is included for a well defined variational principle; it is required to remove terms associated with the boundary of AdS_2.

As discussed for the JT gravity, the metric equation of motion (ignoring matter) is

$$T_{\mu\nu} = \frac{1}{8\pi G}\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\nabla^2 + g_{\mu\nu}\right)\phi = 0. \tag{440}$$

The tracelessness condition from (440) in the euclidean Poincare coordinates becomes

$$-z^2(\partial_t^2 + \partial_z^2)\phi + 2\phi = T^\mu_\mu = 0 \tag{441}$$

[121] These ideas for the relationship between the UV and IR can be attributed to the work of [64].

and is shown to be satisfied by the by anzats

$$\phi = \frac{1}{z}(a + bt + c(t^2 + z^2)). \tag{442}$$

In other words, the dilaton equation of motion is solved exactly in terms of the constants a, b and c. Evidently, the dilaton diverges at the boundary ($z \to 0$) and for this reason one introduces a dilaton coupling at $z = 0$ which specifies the strength of the divergence [49]:

$$\phi_B(u) \equiv \frac{\phi_R(u)}{\epsilon} \tag{443}$$

in which the renormalized dilaton ϕ_R remains finite for $\epsilon \to 0$. The two equations $z = \epsilon t'$ and $\epsilon = \frac{\phi_R}{\phi_B}$ together with the solution to the metric equation of motion (442) solve the dilaton boundary condition ϕ_B appearing the Gibbons-Hawking term with:

$$\phi_R(u) = \frac{1}{t'(u)}(a + bt(u) + ct^2(u)). \tag{444}$$

This solution has been determined entirely by bulk considerations and a boundary condition for the dilaton. However, if one imposes the dilaton equation of motion $R = -2\Lambda$, i.e. the Lagrange multiplier is no longer necessitated, then the JT action is reduced solely to the boundary term in (439). The induced metric on the boundary $g_{uu} = \epsilon^{-2}$ and the dilaton boundary coupling give

$$S = \frac{-1}{8\pi G}\int \frac{du}{\epsilon}\frac{\phi_R(u)}{\epsilon} \tag{445}$$

and by taking $z = \epsilon t'(u)$ the trace of the extrinsic curvature becomes

$$K = \frac{t'^2 + \epsilon^2(t't''' - 3/2t''^2)}{t'^2} = 1 + \epsilon^2\{t, u\}. \tag{446}$$

$\{t, u\} = \frac{t'''}{t'} - \frac{3}{2}\frac{t''^2}{t'^2}$ is the well known Schwarzian derivative. The JT action takes the form:

$$S_{JT} = \frac{-1}{8\pi G}\int du \left[\phi_R(u)\{t, u\} + \frac{\phi_R(u)}{\epsilon^2}\right]. \tag{447}$$

The second term in (447) is of no interest so it is neglected. Therefore, we have a boundary theory action with a spatially (time in 1-d) dependent coupling for the pseudo-Goldstone zero modes $t(u)$ which, due to the Schwarzian form, exhibits $SL(2,\mathbb{R})$ symmetry (in t(u)).[122] The variation of this action with respect to t is

$$\delta S = \int du\delta t \left[-3\phi_R\frac{t''^2}{t'^3} + \phi_R\frac{t'''}{t'2} - \left(\frac{\phi_R}{t'}\right)'' - 3\left(\frac{\phi_R t''}{t'^2}\right)'\right]' = 0, \tag{448}$$

which after some work is found to be

[122]Note that $\{t, u\} = \{\tau, u\}$ under $\tau = \frac{at+b}{ct+d}$ with $ad - bc = 1$.

$$
\begin{aligned}
0 &= \left[\frac{1}{t'^3}\left(\phi_R t''^2 - \phi_R t''' t' - \phi_R' t'' t' - \phi_R'' t'^2\right)\right]' \\
&= \left[-\frac{1}{t'}\left(\frac{(t'\phi_R)'}{t'}\right)'\right]'.
\end{aligned} \tag{449}
$$

Integrating this equation of motion gives

$$
\phi_R = \frac{1/2\tilde{c}t^2(u) + bt(u) + a}{t'(u)} \tag{450}
$$

where a, b and $\tilde{c}$ are integration constants. Simply redefining $c \equiv 1/2\tilde{c}$ reproduces the bulk solution

$$
\phi_R(u) = \frac{1}{t'(u)}(a + bt(u) + ct^2(u)). \tag{451}
$$

Therefore, from the work of [49], one concludes that the boundary Schwarzian action captures the bulk gravity information associated with the dilaton field and should therefore contain many gravitational features associated with the $NAdS_2$ spacetime.

This result is in agreement with those of [61] where it was shown that the boundary dynamical time variable in the AP model leads to an equation of motion for the regulated AdS_2 boundary of the form[123]

$$
\frac{a}{16\pi G}\frac{d^2}{du^2}\log z + (p_+ - p_-)z = 0 \tag{452}
$$

for the boundary trajectory $(t(u), z(u))$ and $z = \frac{dt}{du}$. This equation of motion includes an interaction term between the boundary and the stress energy $p_+ = \int_u^\infty dx T_{++}(x)$, $p_- = \int_{-\infty}^u dx T_{--}(x)$. The corresponding boundary stress energy (which for 1-d is simply the energy) is given by

$$
\langle T^B_{uu}\rangle = \frac{-1}{16\pi G}\{t, u\}. \tag{453}
$$

In the case that matter dynamics are ignored (i.e. $p_+ - p_- = \lambda = const$), a field redefinition $\phi = \log z$ maps the boundary equation of motion to

$$
\frac{a}{16\pi G}\partial_u^2\phi + \lambda e^\phi = 0 \tag{454}
$$

which has the appearance of a 1-d Liouville analogue. (454) can be derived from an action $S = \int du \left[\frac{a}{32\pi G}(\partial_u\phi)^2 - \lambda e^\phi + \lambda\partial ut\right]$. By noting that the t equation of motion implies that $\lambda = const$ and making use of the ϕ equation of motion: $\lambda = -\frac{1}{t'}\frac{a}{16\pi G}\partial_u^2\phi$, the Lagrangian density corresponding to this action is mapped to

$$
\mathcal{L} = \frac{a}{32\pi G}(\partial_u\phi)^2 - \frac{a}{16\pi G}\partial_u^2\phi. \tag{455}
$$

The field redefinition discussed above then sets

[123] a appearing here is the parameter responsible for regulating the backreaction in the AP model.

$$\mathcal{L} = \frac{a}{32\pi G}(\partial_u \phi)^2 - \frac{a}{16\pi G}\partial_u^2 \phi = -\frac{a}{16\pi G}\{t, u\}. \tag{456}$$

This Lagrangian [61] is consistent with the boundary Schwarzian action that summarizes bulk gravitational features (447) [49]:

$$S = -\frac{a}{16\pi G}\int du\{t, u\}. \tag{457}$$

The corresponding Hamiltonian for this theory is [124]

$$H = \frac{8\pi G}{a}\pi_\phi^2 + e^\phi \pi_t \tag{458}$$

which has equations of motion $-e^\phi \pi_t = \partial_u \pi_\phi$, $\partial_u \pi_t = 0$, $\frac{16\pi G}{a}\pi_\phi = \partial_u \phi$ and $\partial_u t = e^\phi = z$. By coupling the boundary theory to the bulk matter stress energy $p_0 = p_+ - p_-$ the Hamiltonian is modified:

$$H = \frac{8\pi G}{a}\pi_\phi^2 + e^\phi(\pi_t + p_0). \tag{459}$$

This has the same equations of motion as the AP model apart from the equation $\partial_u \phi_\phi = -e^\phi(\pi_t + p_0)$. Together the equations of motion imply that $\frac{a}{16\pi G}\partial_u^2 \phi = e^\phi(\pi_t + p_0)$. For $\pi_t = 0$ this reproduces the boundary equation of motion (452). Substitution of the equations of motion back into the Hamiltonian returns the AP model energy (453) [61]:

$$H = -\frac{a}{16\pi G}\{t, u\}. \tag{460}$$

F.1.4 Black hole chaos and out-of-time order correlation functions

Chaos is identified through the exponential divergence of initially neighboring phase space trajectories. This is observed in the chaotic behavior of trajectories of an initially ordered state of a system that undergoes thermalization over some time scale. This leads to a physically natural description of the thermodynamic macrostate of a system as the ergodic sampling of phase space points in the possible microstates that are deemed accessible by a given macrostate. This connection between thermodynamics, and in particular the second law in thermalization, and chaos is well known, as is the thermodynamic properties of black holes. A key recent finding then is the connection between chaos and black holes. Two features of chaotic behavior of black holes should be emphasized: 1) Chaos in black holes is conveniently diagnosed through the exponential growth of out-of-time order (OTO) correlation functions- specifically the four point function. 2) In the context of black holes, the connection between the thermodynamics of black holes and chaos is understood to arise from an exponential growth in the delay of emitted/scattered quanta due to backreaction at the horizon- i.e. in the near horizon region [53]. Entropy, ergodicity and chaos are intimately related and depend on the sensitivity of the dynamics of an observable to its initial conditions. This is typically stated in the form

[124] The Hamiltonian formalism is implemented in [61] as it is convenient for correlation function computations that identify chaotic behavior. Note that [61] work in Lorentzian signature as opposed to the Euclidean results discussed above from [49] who showed that, even though in the Lorentzian picture negative energy modes associated with ghost fields appear, by treating the $SL(2,\mathbb{R})$ symmetry of the Schwarzian action as a gauge symmetry it is found that the conserved charges set the amount that the negative energy modes can be excited. In other words, one cannot arbitrarily introduce negative energy excitations.

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\}. \tag{461}$$

In quantum mechanics this is modified to the quantum commutator $\frac{-i}{\hbar}[q(t), p(0)]$. This clarifies why the completely unfamiliar object, the OTO correlation function, is significant for diagnosing chaotic behavior. However, this can be understood in a practical way as follows: in quantum mechanics, chaos should be understood through perturbations. A small perturbation at a given time will lead to an exponentially large effect at a later time (or earlier time in the time reverse). But a basic assumption, that requires little motivation, is that the precise form of the disturbing perturbation should not be significant in this 'butterfly effect'. This is reflected in a simple generic quantum system such as an 'Ising-like' spin chain [65]. The Hamiltonian of the spin chain is

$$H = -\sum_i \left(Z_i Z_{i+1} + gX_i + hZ_i\right). \tag{462}$$

Z_i, X_i and Y_i represent the Pauli matrix spin operators at lattice site i. The time evolution of an arbitrary local operator W is $W(t) = e^{iHt}W(0)e^{-iHt}$. W may be a simple operator but for a chaotic system (e.g. $g = -1.5$ and $h = 0.5$) the operator grows in time and consequently its effects grow [65]. This is seen through the expansion $W(t) = W(0) - it[H, W(0)] - \frac{t^2}{2}[H, [H, W(0)]] + ...$ Provided the measurement/operator W has an effect- i.e. has a non-vanishing commutator with Z_i or X_i- the size of the operator grows in time. The precise form of the growth is characterized by the sum of the squares of the time dependent coefficients of terms with a given number of operators in a product of local operators [65]. For example: if there are n terms that consist of a product of k operators then:

$$f_k(t) \equiv \sum_{i=1}^{n} c_{i,k}^2(t) \tag{463}$$

is the quantity that characterizes the growth. This function decreases with time except for $k \sim \mathcal{O}(L)$ where L is the length of the spin chain system defined by (462). In which case, the chaotic effect is manifest in growth of the size of the operator followed by a saturation in $f_k(t)$.[125]

To characterize chaos, one can then simply consider a thermal state at two times $t = 0$ and $t = t'$. Chaos causes a $t = 0$ perturbation $V(0)$ to thermalize at $t = t'$. The time reverse from the thermal state at $t = t'$ back to $t = 0$ will reproduce this perturbed initial state. Alternatively, if the system is perturbed by $V(0)$ and later by $W(t')$ then the time reverse does not recover the initial perturbed state with $V(0)$ but generates a thermal state. In this thought experiment, it is evident that a local operator $W(t')$ has a global macroscopic effect, which defines the quantum butterfly effect [66]. The OTO commutator squared thermal correlation function:

$$F(t) = \langle -[V(0), W(t)]^2 \rangle \tag{464}$$

for generic quantum operators V, W, which can be taken to be Pauli matrices in the spin chain example, have the behavior illustrated in figure (6).

The main point is that a chaotic quantum system exhibits growth in the effect of perturbations with time evolution. Consistent with natural expectations, the butterfly effect is not limited to very specific perturbations but in fact occurs for general operators. The OTO correlator exhibits exponential growth in time, which is associated with Lyapunov behavior, followed by exponential decay associated with Ruelle

[125] The initial growth of the operators that do not have chaotic dynamic evolution decrease after their initial growth.

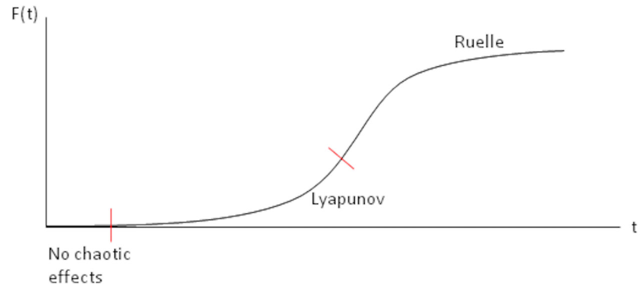

Figure 6: Sketch of the behavior of a generic quantum chaotic spin chain system. At early times there is no chaos. This is followed by exponential growth with Lyapunov exponent. At late times there is an exponential decay.

behavior [53].
The rate of growth of chaos in thermal quantum systems has been computed and conjectured to be bounded by exponential increase with Lyapunov exponent [67]

$$\lambda_L \leq 2\pi T. \tag{465}$$

The bound is saturated for boundary CFT's in AdS/CFT. This has been discovered through the study of eternal $AdS-$Schwarzschild black holes that have a thermofield double for the two exterior black hole boundaries [67] [68]. In this context, the operators $V(0)$ and $W(t)$ are represented by wave functions in the bulk. The OTO correlator is then computed in the bulk through an apparent scattering process where the 'scattering amplitude' for the wave packets scattering near the horizon is the OTO:

$$\langle\beta|VW_tVW_t|\beta\rangle \sim \langle VV\rangle\langle WW\rangle(1+\mathcal{O}(Ge^{2\pi Tt})). \tag{466}$$

Clearly, the Lyapunov exponent bound is saturated [67]. The extension to more physical black holes, which involve collapse and subsequent evaporation, was provided by Polchinski [53] by making use of 't Hooft's S-matrix ansatz [69] [70]. This ansatz states that

Processes involving free particles/states in asymptotically flat spacetime at extremely early and late times should be described by an S-matrix. This should include processes for which a black hole forms and subsequently evaporates. The form of the S-matrix is $S = \langle in|out\rangle$. Then, for unitary processes, the effect of an extra ingoing particle $|in+\delta in\rangle$ will lead to S-matrix elements $\langle in+\delta in|out+\delta out\rangle$ where information is imparted from the extra ingoing particle to outgoing particles.[126]

't Hooft and Dray's computations [71] showed that the positions of outgoing particles are dragged back by the shock wave of the extra particle such that the outgoing particles positions depend on the momentum of the extra ingoing particle. The main contribution of Polchinski was to compute an identity relating the S-matrix to another S-matrix with an extra particle. The S-matrix is factorized

$$S = S_{in}S_HS_{out}. \tag{467}$$

[126]For black hole creation processes, the original in state $|in\rangle$ could consist of a spherically infalling massless matter shell. This generates a Schwarzschild spacetime. The effect of the extra ingoing particle is to generate a backreaction on the, exterior to the the horizon, spacetime that manifests itself in a shock wave effecting outgoing Hawking quanta.

The asymptotic in and outgoing pieces are related to greybody factors that describe the scattering of wave modes off of the effective Schrodinger potential.[127] The modes near the horizon, H, are expressed as $H = \frac{1}{T_I} I - \frac{R_I}{T_I} r$, where I is the asymptotic incident wave packet, H is the transmitted wave packet and r is the reflected wave packet. Outgoing modes have the same name but are primed. T_I, R_I are the transmission and reflection greybody amplitudes. The NH S-matrix, which is the piece of interest, can then be expressed without the greybody factors and in terms of the operators that create ingoing and outgoing horizon modes H and H' respectively:

$$S_H = \langle 0| a_{H'_1 A'_1} ... a_{H'_n A'_n} a^\dagger_{H_1 A_1} ... a^\dagger_{H_m A_m} |0\rangle. \tag{468}$$

The A_i labels specify internal degrees of freedom. 't Hooft and Dray's shock wave analysis due to an extra ingoing particle leads to a back-shift of subsequent outgoing particles in the direction of the horizon

$$\tilde{H}'(u, \theta) = H'(u - \Delta u(\theta), \theta). \tag{469}$$

The tilde on the left hand side represents the outward propagating near horizon mode after the shock wave in the backreacted spacetime. u is the Kruskal coordinate for infalling null matter related to the Schwarzschild time and tortoise coordinate through $u = -e^{2\pi T(r*-t)}$.[128] The relation between Schwarzschild time t and Kruskal coordinate u is

$$\Delta t(\theta) = \Delta u(\theta) \frac{dt}{du} = \Delta u(\theta) e^{2\pi T(t-r*)}. \tag{470}$$

(470) exhibits a clear relation to chaotic behavior [53]. The exponential growth in the delay as observed by an asymptotic Schwarzschild observer is evidence of the chaotic behavior of the black hole near the horizon. This is related, by taking the reciprocal, to the exponential red-shift as observed by asymptotic observers when a fixed frequency pulse is sent out by an object thrown toward the horizon.[129] As one might have expected, due to the backreaction generated by a single extra ingoing wave packet, the backreaction effect on the outgoing waves depends on the angular (transverse to the radial direction) separation of the outgoing particle form the ingoing one. The dependence of the separation in the transverse direction is logarithmic [70].

To study the S-matrix with an extra ingoing particle one makes use of the operator identity [53]

$$a_{\tilde{H}' A'} a^\dagger_{HA} = a^\dagger_{\tilde{H} A} a_{H', A'} \tag{471}$$

which simply reflects the fact that the operators to the left experience the shock wave (or time delay) caused by the operators to the right (at earlier time). The corresponding occupation numbers of the operators $a_{H'A'}$, from before the extra particle passes, are transferred to operator $a_{\tilde{H}'A'}$ after it has passed. The extra particle S-matrix is

$$\tilde{S}_H = \langle 0| a_{\tilde{H}'_1 A'_1} ... a_{\tilde{H}'_n A'_n} a_{HA} a^\dagger_{H_1 A_1} ... a^\dagger_{H_m A_m} |0\rangle = \langle 0| a_{\tilde{H}'_1 A'_1} ... a_{\tilde{H}'_{j-1} A'_{j-1}} a^\dagger_{\tilde{H}' A'} a_{H'_j A'_j} ... a_{H'_n A'_n} a^\dagger_{H_1 A_1} ... a^\dagger_{H_m A_m} |0\rangle. \tag{472}$$

[127] A discussion of such factors can be found in [72].

[128] The tortoise coordinate has the usual definition $\frac{dr*}{dr} = (1 - \frac{2GM}{r})^{-1}$ that describes the black hole exterior.

[129] Polchinski makes a note of associating the red-shifting to Lyapunov behavior and the Ruelle behavior of exponential decay at late times with quasinormal modes generated by horizon perturbations. This is clearly illustrated in figure 2 of [53]. Also seen in that figure, is the very early time absence of chaotic effects before the exponential growth.

The extra particle operator $a^\dagger_{HA}$ has been commuted back $n-j$ times- this reflects a subtlety that indicates that the operator identity (471) can only be trusted up to a scrambling time. Not all of the particles in (472) are relevant for the S-matrix since some of the outgoing particles are overtaken by the outward jump of the horizon generated by the extra particle.[130] Polchinski traces over these states ($\equiv |X\rangle$ before the extra particle and $|\tilde{X}\rangle$ after) and takes the mod-squared

$$\sum_{\tilde{X}} |\langle \tilde{X}|a_{\tilde{H}'_j A'_j}...a_{\tilde{H}'_n A'_n} a^\dagger_{HA} a^\dagger_{H_1 A_1}...a^\dagger_{H_m A_m}|0\rangle|^2 = \sum_X |\langle X|a_{H'_j A'_j}...a_{H'_n A'_n} a^\dagger_{H_1 A_1}...a^\dagger_{H_m A_m}|0\rangle|^2. \tag{473}$$

This identity relates the S-matrix with the extra particle to that without it; it makes clear how the full S-matrix observer would see the effect of chaos in the black hole- as each later time mode being more delayed due the the shock wave for scattering near the horizon. This links the thermal properties to chaos as the time delay (with Lyapunov behavior) depends only on the temperature, which is entirely determined by the horizon through the surface gravity. The NH scattering process is argued by Polchinski to be a re-statement of the firewall paradox with an observable energy flux at the horizon which carries information away from the black hole and can have information imparted to it by the vaporized infalling observer [53].

The AP-JT description of $NAdS_2$ provides and accurate account of the backreaction and Lyapunov behavior of the NH black hole geometry. To take into account of the backreaction, one introduces matter into the system. The inclusion of matter does not modify the AdS_2 geometry as it is still fixed by the dilaton equation of motion, which the matter Lagrangian is independent of. However, the metric equation of motion is modified by the presence of the matter stress tensor. The Schwarzian description for massless fields has the boundary trajectory solution satisfying

$$\frac{8\pi G}{\bar{\phi}_r} \frac{\{t,u\}'}{t'} = -t' T_{tz}. \tag{474}$$

$t(u)$ is the solution to this equation [49]. In order to compute the OTO correlation functions Maldacena *et al.* compute an effective action for the matter fields which are coupled to the boundary zero modes t. By performing a perturbative expansion of the Schwarzian action after making a change of variables $t = \tan\frac{\tau}{2}$ and setting $\tau = u + \epsilon(u)$[131]- the expansion is truncated at second order in ϵ. This allows them to obtain an expression for the OTO 2 point function of the zero modes $\langle \epsilon(u)\epsilon(0)\rangle$, which together with the effective action leads to the OTO four point function, analytically continued to Lorentzian signature, of the form [49]

$$\langle V(a)W(b+\tilde{u})V(0)W(\tilde{u})\rangle \sim \frac{8\pi G}{T\bar{\phi}_r} e^{2\pi T\tilde{u}} \tag{475}$$

$1/2\pi T \leq \tilde{u} \leq 1/2\pi T \log\frac{\bar{\phi}_r T}{8\pi G}$ and $a, b \sim 1/T$. This clearly has the exponential time growth that saturates the Lyapunov exponent bound $\lambda_L = 2\pi T$.[132]

The matter fields do not interact in the bulk AdS_2 but they do generate a dilaton profile, which when relating the bulk time to the boundary time manifests as an interaction (see (443)). This matches the results of the eternal black hole bulk scattering described above. An interesting new result of [49] is that

[130] This interpretation is due to Polchinski- the backward shock wave experienced by outgoing particles is conveniently understood instead as the outward growth of the horizon after the extra particle falls through. After the horizon jumps out the outgoing particles are again closer to the horizon and therefore take longer to reach the asymptotic flat space.

[131] Note that u no longer represents the Kruskal coordinate, but the boundary time. $\tilde{u}$ is the Lorentzian version.

[132] The bound is saturated for the AP model in the approach of [61] as well, where the exponential growth in the delay of an outgoing signal due to the backreaction of matter for a black hole was shown to be $2\pi T$ in a way that makes the connection to Polchinski's black hole S-matrix treatment (see 470 as well as the discussion and footnote following that equation) slightly more apparent.

by computing the four point function to higher orders in G using the Schwarzian action one can observe the late time exponential Ruelle decay of quasinormal modes. This involves a similar bulk scattering process to that of the eternal black holes [68]. The operators V and W are again quanta that scatter near the horizon. One replaces the AdS_2 metric by a set of 2 shock waves with shifts X^+ and X^- on the future and past horizons. For the boundary calculation, one associates the shifts with specific zero modes $t(u)$ in the Schwarzian action. Since the four point function is OTO, one is required to consider folded time contours. The computation is technical but the result[133] is

$$\frac{\langle V_1 W_3 V_2 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \frac{\Gamma(2\Delta)^{-1} \int_0^\infty ds e^{-s/z} \frac{s^{2\Delta-1}}{(1+s)^{2\Delta}}}{z^{2\Delta}}. \tag{476}$$

z is a complicated function of the Lorentzian boundary time but for specific values it takes the form $z = \frac{e^{\tilde{u}} G\pi}{\bar{\phi}_r}$. Figure 4 of [49] shows the early and late time behavior of (476) and clearly illustrates that, while the expected initial Lyapunov growth occurs, at late times the exponential Ruelle decay of quasinormal modes takes over in the boundary action approach for the near horizon $NAdS_2$ geometry [49].

[133]The subscript numbers in (476) label the folded time contours to which the operators are assigned under the specific time ordering.

G Appendix G

There is a pre-AdS/CFT gauge theory duality with gravity based on the $c = 1$ matrix model and its equivalent two dimensional string theory description. We refer to this as the 'early model of gauge/gravity correspondence' and we have outlined the basis for the correspondence based on the matrix model lattice and spacetime triangulations in (G.1).

G.1 Early Models of Gauge/String Correspondence

The study of matrix models has led to some interesting progress in the fields of string theory and strongly coupled field theories. A central feature relating the two cases, principally motivated by the development of non-perturbative effects in string theory which led to the Maldacena conjecture, is the emergence of spacetime [73], which is closely linked to the ambiguities, arising out of dualities such as the Toroidal duality (T), in geometry and topology in string theory.[134] It is also reasonable to expect that, being a gauge symmetry, general covariance will be an emergent symmetry of a fundamental theory. What can be considered as an early example of the gauge/gravity correspondence is the holographic duality between matrix quantum mechanics and string theory in one or two dimensions. There are two distinct ways in which the spatial direction emerges from the matrix model: 1) the emergence of a two dimensional string worldsheet, which is generally covariant, from the continuum limit of the discretized worldsheet theory. 2) the emergence of the Liouville direction in the string worldsheet embedding (target) spacetime. In the first case, one considers the discretization of the string worldsheet by a dual matrix quantum mechanics. The discretized 'triangulation' of the worldsheet has a dual 'triangular' lattice provided by the ribbon graphs of the matrix model.[135] The two lattices are dual in the sense that the faces, edges and vertices of the two lattices have the following correspondence:

$$\begin{array}{ccc} \textbf{Matrix} & & \textbf{string} \\ V & & F \\ E & \longleftrightarrow & E \\ F & & V \end{array} \tag{477}$$

Therefore, the sum over topologies and integration over metrics in the string partition function is replaced with a sum over random triangulations provided by the dual matrix model Feynman perturbation series. The string partition function is identified with the free energy (logarithm of the partition function) of the matrix model. The large N limit is the 't Hooft topological expansion which suppresses non-planar contributions, leaving only the perturbative expansion in the matrix model coupling on the sphere.[136] However, at large order in the matrix coupling the spherical topology partition function has a critical coupling value controlled by the so-called string exponent [74]. This critical behavior is identical for all genus contributions. The average area of the triangles in the worldsheet triangulation diverges at the same

[134] For a string theory compactified toroidally, for example on a circle of radius R, there is an equivalence with the same theory compactified on a circle of radius $R' = \alpha'/R$- under which winding modes and momentum modes are exchanged. Therefore, features of the background geometry on scales smaller than l_s cannot be detected. In fact, examples exist for which there is a T-duality between compactifications on a circle of radius $R = 2\sqrt{\alpha'}$ and a $\mathbb{Z}_2$ quotient of a circle of radius $R' = \sqrt{\alpha'}$- that is a line segment. It appears that T-duality introduces a fundamental ambiguity in the background geometry in string theory and, given the example of the circle-line symmetry, the topology of the background. If this is the case, then T-duality is motivating factor against the existence of intrinsic spacetime in physical theories [73].

[135] The discretization is not required to be achieved via a specific polygonization as long as the two lattices are dual.

[136] We make a distinction between the matrix model partition function $Z(g)$ and the set of partition functions $\{Z_h(g)\}_h$. h labels the number of handles for the surfaces in the topological expansion. The matrix partition function, $Z(g)$, is a sum over the topologies for a given value of the matrix coupling, g, and each topology is weighted by the appropriate power of N. There is therefore a clear difference between the perturbation expansion in the topology and one in the matrix model coupling, g [74].

critical point. So the coupling partition function diverges at the critical point for each topology leading to a dominant contribution from graphs (discretizations) with infinite number of vertices. By scaling the area of each triangle to zero, one can define a continuous limit of the worldsheet. In order to counteract the suppression of non-planar contributions to the full partition function, one can tune the coupling to the critical point. This is the so called double scaling limit which realizes the integration over topologies and metrics for the gravity theory (the double scaling limit is discussed in a comprehensive manner in both [74] and [75]). The action for the string theory consists of the worldsheet metric, a worldsheet field corresponding to the time at which each vertex occurs and a cosmological constant which is included simply because there is no reason to expect that the action should be Weyl invariant [76]. This action defines a non-critical string in one dimension with a non-trivial metric that describes quantum gravity on the worldsheet. The second case arises when one studies the non-critical string action in conformal gauge where the Weyl factor appears as an extra spatial dimension and the metric is reduced to a fiducial metric. This gives the interpretation of the action as that of a critical string theory in two dimensions with a coupling to the fiducial gravity [76]. This two dimensional action is the same as the Liouville action and the Weyl factor is really the Liouville field. The connection between the two is best understood to arrive from the string theory with background field solutions for which the metric is flat, the Kalb-Ramond field is absent and the dilaton is linear, once it has been restricted to two dimensions, and augmented with the inclusion of a tachyon field in order to make perturbation theory well defined. In two dimensions the only propagating degree of freedom is the tachyon, whose mass is lifted to zero by the presence of the Liouville field. The theory can be understood to be a theory of a single massless scalar field in two dimensions with a non-trivial spatially dependent interaction which happens to be exponential.

G.1.1 Collective field, $c = 1$ matrix model and 2 dimensional string theory

The connection between the $c = 1$ matrix model and two dimensional string has a most lucid account in terms of the collective field formulation [11] of Das and Jevicki (DJ) [77]. They have shown that the emergent spatial direction associated with the continuous eigenvalue coordinate in the collective field formulation of matrix quantum mechanics, mapped to the so-called time of flight coordinate, leads to an interpretation of the collective field as the massless tachyon of two dimensional string theory and the identification of the Liouville coordinate with the time of flight coordinate. This identification is intimately linked to the appearance of two features of the collective field theory: the massless Klein Gordon field and a spatially dependent coupling [77]. These two features are shared by the low energy effective action of the tachyon in the linear dilaton vacuum.[137] Polchinski gave a classical derivation of the DJ model which clarified the connection between the collective field and the massless tachyon through the Fermi liquid branch profile in the free fermion picture [78]. In the remainder of G.1.1 and in G.1.2 we follow the paper [13] which neatly reviewed the above identification of [77] and made a proposal for a deformed matrix model associated with a black hole.

The appearance of the linear massless bosonic field in the collective field theory (identified with the tachyon) is in fact universal. The only explicit dependence on the potential is contained in the precise form of the space dependent coupling which depends on the time of flight coordinate. The DJ Hamiltonian is

$$H_{DJ} = \int_0^\infty dq \left[\frac{1}{2} \left(\pi_\xi^2 + (\partial_q \xi)^2 \right) + \frac{\sqrt{\pi}}{12} \left(\frac{dx}{dq} \right)^{-2} \left[(\pi_\xi - \partial_q \xi)^3 - (\pi_\xi - \partial_q \xi)^3 \right] \right] \tag{478}$$

where q is the time of flight coordinate and the coupling has the form

$$\left(\frac{dx}{dq} \right)^{-2} = \frac{1}{2\mu \sinh^2 q}. \tag{479}$$

[137] This is understood from the beta function equations required by the conformal gauge symmetry of string theory.

Asymptotically ($q \to \infty$) the coupling tends to $\left(\frac{dx}{dq}\right)^{-2} \to \frac{2e^{-2q}}{\mu}$. μ is the Lagrange multiplier enforcing the eigenvalue constraint that appears in the collective field theory and plays the role of the chemical potential in the fermionic picture whereas in the string picture it plays the role of a cosmological constant. By comparison with the low energy effective tachyon action in the linear dilaton vacuum, after making the field redefinition $\tilde{T}(t,\phi) = e^{\sqrt{2}\phi}T(t,\phi)$, it is found that:[138]

$$S_{eff} = \frac{1}{2}\int dtd\phi \left[\frac{1}{2}\tilde{T}(-\partial_t^2 - \partial_\phi^2)\tilde{T} - \frac{e^{-\sqrt{2}\phi}}{3!}\tilde{T}^3\right], \tag{480}$$

where one identifies the string coupling in the two theories[139]

$$g_s \sim \mu^{-1}e^{-2q}, \qquad\qquad g_s \sim e^{-\sqrt{2}\phi}. \tag{481}$$

The chemical potential/Fermi energy in the matrix model is related to the presence of tachyon condensation in the string theory $T = \mu e^{-\sqrt{2}\phi}$.[140] Then asymptotically there is the correspondence between the time of flight coordinate and the Liouville coordinate, the collective field and the tachyon, and between quantum mechanical time in the matrix model (t_{qm}) and time in the string theory:

$$\begin{aligned} q &\longleftrightarrow \frac{\phi}{\sqrt{2}} \\ t_{qm} &\longleftrightarrow \frac{t}{\sqrt{2}} \\ \xi(t,q) &\longleftrightarrow \tilde{T}(t,\phi). \end{aligned} \tag{482}$$

The typical procedure for then computing the spectrum of the theory involves ignoring the nonlinear terms- that is taking the cosmological constant to be zero for which the Virasoro operator then takes the form of a free massless field. The spectrum consists of the massless tachyon as well as an infinite set of discrete states with an associated w_∞ algebra. These states are characterized by imaginary energy and momenta for the Liouville coordinate. The origin of the discrete states is considered to be the excitations of strings in higher dimensions so that they appear residually.

An important point to be noted is that the identification made between the collective field and the tachyon relied on matching asymptotically ($q \to \infty$) where the tachyon condensation ($\mu e^{-\sqrt{2}\phi}$) is no longer taken into account. If the tachyon condensation is taken into account then the Virasoro constraint has the form[141]

$$L_0 T = [\frac{1}{2}(\partial_t^2 - \partial_\phi^2) - \sqrt{2}\partial_\phi + \mu e^{-\sqrt{2}\phi}]T = T. \tag{483}$$

In this case the collective field is related to the tachyon via a non-local field re-definition [13]. The asymptotic form of this field re-definition, in terms of the Fourier transformed collective field, is

[138] The cubic interaction was chosen by [13] for ease of comparison with the collective field- in general the form of the tachyon potential is not well known.

[139] This identification is based on the linear dilaton vacuum which has the background field configuration: $G_{\mu\nu} = \eta_{\mu\nu}$, $\Phi = 2\sqrt{2}\phi$ and $T(x) = 0$. Here, $G_{\mu\nu}$ is the embedding spacetime metric, Φ is the dilaton which appears in the string coupling as $g_s = e^{-\Phi/2}$ and $T(x)$ is the tachyon. For the linear dilaton vacuum background fields mentioned, the string coupling becomes $g_s = e^{-\Phi/2} = e^{-\sqrt{2}\phi}$ [13].

[140] This is the static (time independent) linearized tachyon solution.

[141] This is a statement of the mass shell constraint of a bosonic string theory for which $(L_0 - 1)|T\rangle = 0$ for a tachyonic field in the linear dilaton background (the inclusion of the tachyon background results in the Liouville theory discussed above in connection with the matrix model). A general discussion of this can be found in [79].

$$\tilde{T}(t,\phi) \sim \int dp\tilde{\xi}(p)\gamma(p)e^{-ipt_{qm}}\left[\Gamma(ip)\mu e^{-ip/2+ip\phi/\sqrt{2}} + \Gamma(-ip)\mu e^{ip/2-ip\phi/\sqrt{2}}\right]. \tag{484}$$

$\gamma(p)$ is an arbitrary function that depends on the normalization. The arbitrary function in the field renormalization of the collective field appearing in (484) $\gamma(p)\Gamma(\pm ip)\mu^{\mp ip/2}$ is required, for the unitarity of the $S-$ matrix to be $\gamma(p) = \Gamma(\pm ip)^{-1}$. In which case the scattering matrix is decomposed into two pieces, a product of external leg factors and the collective field theory amplitude:

$$S = \Pi_{j=1}^{N}(-)\mu^{-ip_j}\frac{\Gamma(-ip_j)}{\Gamma(+ip_j)}A_{collective}(p_1,...,p_N). \tag{485}$$

The reflection coefficient is read-off to be $R(p) = -\mu^{ip}\Gamma(-ip)/\Gamma(+ip)$ for each external leg. These external leg factors amount to a phase which is not physical in Minkowski space. However, when one continues to Euclidean momenta ($p \to -i|k|$) the numerator has poles at $|k| \in \mathbb{Z}^+$ which have the interpretation of being the momenta at which resonances occur between single particle tachyon states and the tachyon background. The problem of determining the S-matrix is reduced somewhat to the problem of computing the collective field amplitude $A_{collective}$ [13] [80].

G.1.2 Proposal for the deformed matrix model black hole solution

The linear dilaton vacuum solution modified to account for tachyon perturbation (i.e. tachyon condensate) has a description in terms of a $c = 1$ conformal field theory coupled to the Liouville Field of the form [80][142]

$$\mathcal{L} = \int \frac{d^2z}{8\pi}\left[\partial X\overline{\partial}X + \partial\phi\overline{\partial}\phi - 2\sqrt{2}R\phi + \mu e^{-\sqrt{2}\phi}\right]. \tag{486}$$

It turns out that the $SL(2,\mathbb{R})/U(1)$[143] nonlinear sigma model describing the Wess-Zumino-Witten (WZW) two dimensional black hole, of mass M, with action [81][144]

$$S_{WZW} = \int \frac{d^2zk}{8\pi}Tr(g^{-1}\partial gg^{-1}\overline{\partial}g) - ik\Gamma_{WZW} + gauge \tag{487}$$

with $k = 9/4$, can in a particular parameterization of the $SL(2,\mathbb{R})$ elements g, be expressed as [80]

$$S_{effective} = \int \frac{d^2z}{8\pi}\left[(\partial\tilde{X})^2 + (\partial\tilde{\phi})^2 - 2\sqrt{2}R\tilde{\phi} + M|\frac{1}{2\sqrt{2}}\partial\tilde{\phi} + \frac{i\sqrt{k}}{\sqrt{2}}\partial\tilde{X}|^2 e^{-2\sqrt{2}\tilde{\phi}}\right]. \tag{488}$$

This is immediately recognized as the linear dilaton theory above except that the tachyon (cosmological constant) term is replaced by a black hole mass perturbation. Martinec and Shatashvili [82] have actually shown that the gauged WZW model in momentum space is reduced to the CFT of a scalar coupled to the Liouville coordinate with a scaling variable given by the cosmological constant term (μ), which in position space corresponds to the the black hole mass (M). This suggests that, since the linear dilaton theory is related to the continuum matrix model (collective field theory), there should be a matrix model description of the black hole theory. In other words, the matrix model should also be able to be described by an alternative background (classical) solution that has non-trivial target space geometry. This was the

[142] X is a $c = 1$ matter field and the Liouville field has $c = 25$ which gives $c_{total} = 26$ consistent with a bosonic string theory.
[143] For Minkowski metric signature one has an $SL(2,\mathbb{R})/O(1,1)$ sigma model.
[144] This theory has no tachyon condensation and has the dilaton $\Phi = \log(-uv + M)$ and metric $ds^2 = \frac{-2kdudv}{2(M-uv)}$.

motivation behind the work of Jevicki and Yoneya [13].

The zero mode of the Virasoro condition represents the linear tachyon in the black hole background. This zero mode is given by $L_0 = -\triangle_{cas} + 1/4(u\partial_u - v\partial_v)^2$ which becomes

$$L_0 T = \frac{1}{(k-2)}\left[(1-uv)\partial_v\partial_u - \frac{1}{2}(u\partial_u + v\partial_v) - \frac{1}{2k}(u\partial_u - v\partial_v)^2\right]T. \tag{489}$$

$\triangle_{cas}$ is the $SL(2,\mathbb{R})$ Casimir operator. When in the continuous representation of the group, the eigenvalues are $\triangle_{cas} = -1/4 - \lambda^2$ and $-i\partial_t = 2i\omega$.[145] After re-writing the Virasoro constraint for the linear tachyon solution in a covariant form, the metric and dilaton are found from the Laplacian to have the forms

$$\begin{aligned} ds^2 &= \frac{k-2}{2}(dr^2 - \beta(r)^2 d\tilde{t}^2) \\ \Phi &= \log(\sinh\frac{r}{\beta(r)}) + a \end{aligned} \tag{490}$$

where a is a parameter, $\beta(r) = 2\sqrt{\coth^2\frac{r}{2} - 2/k}$ and $u = \sinh r/2e^{\tilde{t}}$, $v = -\sinh r/2e^{-\tilde{t}}$.[146] The parameter a is related to the ADM black hole mass $M = \sqrt{2/(k-2)}e^a$. The metric (490) has no curvature singularity but since the dilaton, expressed in the coordinates u, v and rescaled to $u \to 1/\sqrt{M}u, v \to 1/\sqrt{M}v$ with $M \equiv e^a$, becomes $\Phi = \log(4\sqrt{-uv(M-uv)(\frac{-(M-uv)}{uv} - 2/k)}) + a$, there is a divergence in the string coupling. The approximate form of the string coupling scales as

$$g_s \sim e^{-a/2} = \frac{1}{\sqrt{M}} \tag{491}$$

in contrast to the case $g_s \sim 1/\mu$ related to the the trivial background gravity. There is also a way to asymptotically identify the black hole solution with the linear dilaton case. In the $r \to \infty$ limit where $u \to e^{r/2+\tilde{t}}, v \to e^{r/2-\tilde{t}}$ the Virasoro constraint and the dilaton tend to

$$\begin{aligned} L_o &\to \frac{1}{4k}\partial_{\tilde{t}}^2 + \frac{1}{(4k-8)}(\partial_r^2 - \partial_r) \\ \Phi &\to r + a - \log 4 \end{aligned} \tag{492}$$

and so asymptotically, the coordinates of the exterior of the black hole and the linear dilaton coordinates have the correspondence (recall $k = \frac{9}{4}$)

$$\begin{aligned} \tilde{t} &\longleftrightarrow \frac{\sqrt{3}}{2}t \\ r &\longleftrightarrow 2\sqrt{2}\phi. \end{aligned} \tag{493}$$

And for the energy and momentum

[145] There is a mass shell condition at $k = 9/4$ which is $\lambda^2 = 9\omega^2$.

[146] Note that the coordinates $(r, \tilde{t})$ describe the spacetime points outside the horizon of the black hole. $\tilde{t}$ can be referred to as the black hole exterior time coordinate- at this stage this coordinate is distinct from the quantum mechanical time of the matrix model and the time coordinate in the linear dilaton theory. However, we shall indicate a relationship between this black hole time and the linear dilaton theory time below in (493).

$$
\begin{array}{ccc}
ip_\phi & \longleftrightarrow & \dfrac{ip_q}{\sqrt{2}} - \sqrt{2} \\
ip_t & \longleftrightarrow & \dfrac{ip_{t_{qm}}}{\sqrt{2}}.
\end{array} \tag{494}
$$

This indicates that the black hole and linear dilaton states are in $1-1$ correspondence.
Consistent with the Virasoro condition, the asymptotic reflection and absorption coefficients for scattering tachyons off of the black hole geometry are

$$
R(\lambda) = \frac{\alpha(\omega,\lambda)}{\alpha(\omega,-\lambda)} \tag{495}
$$

$$
A(\lambda) = \frac{\beta(\omega,\lambda)}{\alpha(\omega,-\lambda)} \tag{496}
$$

which are defined in terms of the Beta function and the Gamma function as $\alpha(\omega,\lambda) = (\Gamma(\rho_+)\Gamma(\rho_-^*\rho_+))/(\Gamma(\rho_-^*))$ and $\beta(\omega,\lambda) = B(\rho_+,\rho_-)$ with $\rho_\pm = -i(\lambda \pm \omega) + 1/2$. These coefficients are subject to the constraint

$$
|R|^2 + \frac{\omega}{\lambda}|A|^2 = 1. \tag{497}
$$

The multiplicative factor of the absorption coefficient accounts for the difference between the massless particle at the horizon and at asymptotic infinity.
Jevicki and Yoneya consider the perturbations corresponding to the black hole mass and the cosmological constant as distinct- in other words they are to be understood as two alternative augmentations of the matrix model that, based on the arguments above, correspond to the same two dimensional string theory. As a consequence of this distinction the black hole theory should have no cosmological constant terms and therefore the Fermi energy will be absent in the dual matrix model. The proposal for the deformed matrix model is then stated as follows [13]

S-matrix elements of the black hole theory should have a decomposed form in terms of the collective field theory amplitudes and external leg factors that arise out of the asymptotic non-local field redefinition of the collective field. The collective field theory will correspond to a matrix model that differs from the usual $c = 1$ model by a deformation with vanishing Fermi energy. The deformation will be related to a finite black hole mass term since one is not taking the extremal limit.

An important feature of the deformed matrix model is, that by considering the poles of the reflection amplitudes that have resonances at $i\sqrt{2}p_t = -2, -4, -6, ..$ and the energy for an incoming tachyon that scatters to $N-1$ tachyons $i\sqrt{2}p_t = -(2r+N-2)$[147] one is forced to conclude, that the on shall scattering amplitudes should vanish unless N =even. This is not possible for the ordinary $c = 1$ matrix model but must be satisfied by the deformed model. Another important feature that must appear in the matrix model is the $1/\sqrt{M}$ behavior of the string coupling.
The solution to the Virasoro condition (492) leads to the non-local field re-definition with asymptotic Fourier transformed ($y \to \infty$) form [13]

$$
T(v,u) \to \int_\infty^\infty dp\tilde{\xi}(p)\gamma(p)e^{-2i\omega t}\left((-y)^{-1/2+i\lambda(p)}\alpha(\omega(p),\lambda(p)) + (-y)^{-1/2-i\lambda(p)}\alpha(\omega(p),-\lambda(p))\right). \tag{498}
$$

[147] r is a measure of the number of insertions of the black hole mass.

In contrast to the ordinary case, the arbitrary weight functions do not have poles at real values of the momentum.
A suitable ansatz $h(p,x) = \frac{(p^2-x^2)}{2} + M\delta(p,x)$ for the deformed matrix model can be simplified by noting that the the coupling is determined from $\left(\frac{dx}{dq}\right)^{-2}$ and therefore one expects the scaling behavior $\delta(p,x) \to 1/c^2\delta(p,x)$ for some constant c when $(x,p) \to (cx, cp)$. This is consistent with the form $\delta(p,x) = g(p/x)/2x^2$ with the function $g(p/x)$ not yet specified. To determine the unspecified function g, one notes that the w_∞ algebra associated with theory is related to the $sl(2,\mathbb{R})$ algebra of [2] with the quadratic Casimir $L_1^2 + L_2^2 - L_3^2 = \frac{3\hbar}{16}$. By choosing the function $g = 1$ this algebra still closes with the explicit elements

$$\begin{aligned} L_1 &= \frac{1}{4}(p^2 - x^2 + \frac{M}{x^2}) \\ L_2 &= -\frac{1}{4}(px + xp) \\ L_3 &= \frac{1}{4}(p^2 + x^2 + \frac{M}{x^2}) \end{aligned} \tag{499}$$

and quadratic Casimir

$$L_1^2 + L_2^2 - L_3^2 = \frac{3\hbar}{16} - \frac{M}{2}. \tag{500}$$

Therefore, the deformed matrix model has the Hamiltonian

$$H = \frac{(p^2 - x^2)}{2} + \frac{1}{2}\frac{M}{x^2}. \tag{501}$$

The corresponding doubling scaling limit for the theory takes a new form as well. If the matrix model potential has the form $V = Tr(-\frac{m^2}{2} + \frac{M'}{2m^2})$, with m representing the matrix and M' a scale parameter related to the black hole mass, then the appropriate double scaling limit as determined by the free energy on the sphere for the limit $M' \to 0$ requires that $M = N^2M'$ is fixed in the $M' \to 0$ and large N limits. The standard collective field rescaling then maps the potential to the form in (501).
By studying the tree level scattering in Polchinski's approach [78][148] the functional relation between incoming and outgoing tachyon waves are found to obey the relationship

$$\psi_\pm(z) = \psi_\mp(z \mp \frac{1}{2}\log(1 + \frac{\psi_\pm^2(z)}{M})). \tag{502}$$

The remarkable feature of this result is that it satisfies the requirement of only being defined for even wave functions. This has a polynomial in momentum form in a power series solution [13].[149]
The S-matrix elements can be computed by determining the amplitudes A (see (503)) using the methods of [48] to which one should refer for details. The scattering operators have the the form

$$S_{collective} = \sum_{p,q=0}^{\infty} \frac{1}{p!q!}\Pi\int_0^\infty \Pi d\lambda_i A(\lambda_1,..,\lambda_p;\lambda_{p+1},...,\lambda_{p+q})\Pi_j^p\alpha_-(\lambda_j)\Pi_{i=p+1}^{p+q}\alpha_+(-\lambda_i) \tag{503}$$

for the collective field and

[148] Polchinksi's approach in [78] was for the $c = 1$ model. For the details of the black hole theory consult [13].
[149] For the $c = 1$ matrix model a solution of this kind was computed in [83].

$$S_{\tilde{T}} = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} \Pi \int_0^{\infty} \Pi d\lambda_i A(\lambda_1, .., \lambda_p; \lambda_{p+1}, ..., \lambda_{p+q}) \Pi_j^p \left[|R(\lambda_j)| \alpha_-^{\mathcal{I}} + \sqrt{\omega_j/\lambda_j} |A(\lambda_j)| \alpha_-^{Hor} \right] \Pi_{i=p+1}^{p+q} \alpha_+^{\mathcal{I}}(-\lambda_i) \tag{504}$$

for the tachyon [13].

G.1.3 Modern interpretation of the matrix model description of two dimensional strings

A more modern view of the matrix model-two dimensional string theory arises from the inclusion of D-branes and their consequences for the open/closed string duality- much of the details below were learned by the authors from the reviews [84] and [85]. The original matrix model description of the closed string worldsheet is now recognized to correspond to the open string tachyon condensate on D_0-branes. These are not the only D-branes in the theory which are closely related to the boundary states of the Liouville theory. Strongly coupled (non-perturbative) physics is known to be described by the dynamics of D-branes- this knowledge leads to the identification of the two boundary states of interest in the Liouville theory. We refer to these two D-branes in the Type-0 string theory as: the FZZT D_1-brane and the ZZ D_0-brane.
The FZZT branes are spacelike and reside at fixed time, where as usual for D=2, the Liouville coordinate is taken to be the spatial coordinate. These branes correspond to the macroscopic loop operators, which in the matrix description, cuts holes in the worldsheet. The Dirichlet boundary condition is in the time coordinate and the Neumann condition in the Liouville coordinate. They have the obvious interpretation of absorbing (emitting) closed strings inserted by the tachyon operator but can also be understood to probe the D_0-particle trajectories (to which we refer to in what follows) by closed strings. A clear computation of such a probe calculation appears in [84] with accompanying informative illustrations. Since the D-brane is fixed in time it certainly does not have dynamics.
The more interesting brane of the theory is the D_0-brane (D-particle) or ZZ brane. A stack of these branes has a natural correspondence with the matrix model eigenvalues which are trapped at a given spatial coordinate, which as indicated by DJ, is related to the Liouville coordinate.[150,151] Having time 'evolution', the D_0-branes are dynamical and have Dirichlet conditions in the Liouville coordinate. To appreciate their significance it is instructive to recall the worldsheet description of the string theory in which the dynamical metric $\sim e^{\phi} g_{\mu\nu}$ (for which the Weyl factor ϕ becomes the Liouville coordinate in the spacetime picture). The factor ϕ shifts when a scaling transformation takes place and for this reason conformal invariance is preserved only if the Dirichlet condition is applied at $\phi \to \infty$. That is, the D_0-brane is located deep within the Liouville wall and its dynamics describe strongly coupled physics. In the context of the Poincare disk (or half plane of Euclidean AdS_2), the appropriate conformally invariant boundary condition is that of the so-called Rolling tachyon which has boundary interaction contribution to the action [27]:

$$\delta S = \oint \lambda d\sigma \cos(x) \tag{505}$$

where x is the time coordinate.[152] Therefore one has a description of an open string tachyon which is concerned with the decay of the unstable D_0 branes at infinite Liouville coordinate.[153]

[150]This is made convincing by considering the Fermi liquid with bosonic collective perturbations propagating in the Liouville (eigenvalue) direction though transverse (fixed Liouville coordinate) fluctuations.

[151]The stack of such branes is necessary for identification of the matrix eigenvalues with N D_0-branes in the the two dimensional string theory. This introduces a matrix Chan-Paton description for open string boundary locations/conditions and associated $U(N)$ gauge theory. Open string operators become matrix valued fields and there is a boundary term in the action, in addition to (505), which involves a gauge field that is responsible for projecting onto the singlet subspace of the theory [84].

[152]See [84] for details.

[153]There is a vast amount of literature on tachyon condensation. References to the original literature can be found in the review [85].

A proposal has been made for a correspondence between the the matrix model and a fermionic string theory [27]. The proposal states that the symmetric matrix model provides a description for NSR strings in two dimensions and that an $\mathcal{N}=1$ supersymmetric Liouville theory, coupled to matter, describes the closed string background. A particular GSO projection gives this string theory as a two dimensional fermionic Type-0 theory. Supersymmetry is restricted to the worldsheet as the precise form of the GSO projection does not lead to spacetime fermions. The truncation to various sectors leads to two distinct Type-0 theories that are distinguished, as per the conventions that distinguish IIA and IIB, by the $R-R$ sectors. For the Type-0 A theory, the left and right chiral truncation is different for the $R-R$ sector whereas for the Type-0 B theory they are the same [27]:

$$\begin{aligned} A :&(NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R-) \oplus (R-, R+) \\ B :&(NS-, NS-) \oplus (NS+, NS+) \oplus (R+, R+) \oplus (R-, R-). \end{aligned} \tag{506}$$

Since spacetime bosons arise from the doubly bosonic or doubly fermionic sectors, from the worldsheet point of view, and spacetime fermions from the (NS, R) or (R, NS) sectors, it is clear that the theory has no spacetime fermions. Note as well that the appearance of the $NS-$ sector leads to tachyonic states in the closed string theory.
The spacetime effective action for the theory has the linear dilaton background closed string solution augmented with a non-zero tachyon $G_{\mu\nu} = \eta_{\mu\nu}$, $\Phi = \phi$ and $T = \mu e^{\phi}$. This particular background has the super Liouville worldsheet description. Compactification of the Euclidean time makes the Type-0 A and B theories T dual, consistent with the more familiar IIA and IIB theories.
For the Type-0 B theory, as discussed, the ZZ boundary has an associated open string tachyon whose mass turns out to be $m^2 = -1/2\alpha'$.[154] The matrix model eigenvalues describe the dynamics of the large number of D_0 branes. The open string tachyon effective theory requires a $\mathbb{Z}_2$ symmetry which maps the matrix to its negative and this has the implication of requiring that both sides of the inverted oscillator potential be filled in a symmetric way to the Fermi level. Then the duality, which is slightly different to the ordinary bosonic string case, matches matrix quantum mechanics in a double well potential, that represents the unstable D_0 branes with Dirichlet conditions in the strongly coupled $\phi \to \infty$ region, with closed Type 0 B string theory.[155] The Type 0 A theory is obtained from the $\mathbb{Z}_2$ quotient of the Type 0 B theory. This theory has $(N+q)$ D-branes and N anti-D-branes which leads to a Quiver gauge theory description and it describes the a background with q stable D_0 branes. Having charge, the D_0 branes have q units of background $R-R$ flux [27].

[154]The closed string tachyon still has its mass sent to zero by the presence of the Liouville field.

[155]Filling the other side of the quadratic maximum was originally conjectured to account for non-perturbative instabilities in the bosonic case for which eigenvalues could tunnel through the maximum however, the D-brane picture makes this convincingly justified.

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