Classical Symmetry Reductions of Steady Nonlinear One-Dimensional Heat Transfer Models

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Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

We study the nonlinear models arising in heat transfer in extended surfaces (fins) and in solid slab (hot body). Here thermal conductivity, internal generation and heat transfer coefficient are temperature dependent. As such the models are rendered nonlinear. We employ Lie point symmetry techniques to analyze these models. Firstly we employ Lie point symmetry methods and determine the exact solutions for heat transfer in fins of spherical geometry. These solutions are compared with the solutions of heat transfer in fins of rectangular and radial geometries. Secondly, we consider models describing heat transfer in a hot body, for example, a plane wall. We then employ the preliminary group classification methods to determine the cases of the arbitrary function for which the principal Lie algebra is extended by one. Furthermore we analyze the exact solutions.

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Chapter 1

Introduction

1.1 Literature review

Heat transfer is a discipline of thermal engineering that is concerned with the use, conversion and exchange of thermal energy and heat between physical systems. It occurs when a hot object is placed in cold surroundings. The object loses the internal energy, while the surroundings gain internal energy. This change in internal energy is accompanied by a change in temperature or a change in phase. According to the first law of thermodynamics, heat transfer changes the internal energy of the systems involved. The three important modes of heat transfer include heat conduction, thermal radiation and heat convection. The modes often occur simultaneously though they have different characteristics. The discipline of heat transfer seeks to do what thermodynamics is inherently unable to do, namely, to quantify the rate at which heat transfer occurs in terms of the degree of thermal non-equilibrium. This is done through the rate equations for the three modes.

Whenever there exists a temperature difference in a medium or between media, heat transfer must occur. When temperature gradient exists in stationary medium, which may be solid or a fluid, we use the term conduction to refer to heat transfer that will occur across the medium. In fluids, conduction is due to the collusion of the molecules during their random motion. In contrast, in solids, conduction is due to the vibrations of molecules in a lattice and energy transport by free electrons. It is observed that the rate of heat conduction through a wall with constant thickness is proportional to the temperature d-ifference between the surfaces and the normal to the heat flow direction and is inversely proportional to the thickness of the wall. It is possible to quantify heat transfer processes in terms of appropriate rate equations and for heat conduction, the rate equation is known as Fourier's law.

The second mode of heat transfer, convection, comprises of two mechanism-s: energy transfer due to random molecular motion (diffusion) and energy transferred by the bulk, or macroscopic, motion of fluid. Convection may be classified according to the nature of the flow. We speak of forced convection and free (or natural) convection. Forced convection occurs if the fluid is forced to flow over the surface by external means such as a fan, pump or the wind. Convection is called natural or free convection if the fluid is set in motion by temperature differences between the wall surface and the surrounding fluid. Radiation which is the last mode of heat transfer is the energy emitted by matter in the form of electromagnetic waves as a result of the changes in the electron configurations of the atoms or molecules. Unlike conduction and convection, the transfer of energy by radiation does not require the presence of an intervening medium.

Rational and motivation

In this study, we focus on heat transfer through fins of various geometries and heat conduction with the internal heat generations. Firstly, we investigate the heat transfer through fins by employing the symmetry techniques to analyze the problem in order to determine the cases of thermal conductivity and heat transfer coefficient terms. Using symmetry analysis and reduction, exact solutions will be constructed. Secondly, we determine the effect of internal heat generation on temperature distribution. We consider a plane wall with uniform heat generation with asymmetrical boundary conditions.

A fin is a surface that extends from a hot object to increase the rate of heat transfer to the surrounding fluid. They are commonly applied for heat management in electrical appliances such as computer power supplies and substation transformers. Fins are much longer than they are thick, which makes it accurate to assume that the temperature varies only in the lengthwise direction. That is, at any point x along the length of the fin the temperature is essentially uniform across the cross section of the fin, which results in a one-dimensional heat transfer problem. They are also used in situations in which cooling is attained via free (or natural) convection for which the heat transfer coefficients h are small, in heat exchanging devices such as radiators in cars and heat exchangers in power plants. The first mathematical treatment of fins started years ago with the appearance of pioneer work by Harper and Brown [1]. They investigated the heat transfer in air-cooled engines equipped with rectangular or "wedge-like" fins. Many problems describing heat transfer in fins have been well documented [2,3] and attention has been paid to models describing longitudinal trapezoid fins [3], rotating radial fins of rectangular profiles and other profiles [4], radial fins of rectangular profile [4], radial rectangular fins [5] and longitudinal fins of rectangular profile [6]. Many fin problem models assume a constant thermal conductivity and heat transfer coefficient. In fact it is stated in [4] that exact solutions exist only when these parameters are constant. In [6], the authors have shown that solutions may be obtained even when the heat transfer coefficient and thermal conductivity are dependent on temperature.

We also consider a situation in which the thermal behavior of a body is affected by heat generated or absorbed internally. Some examples include heating associated with a flow of electrical current and electrical resistance, exothermic chemical reactions, absorption of radiation in microwave ovens and emission of radiation flame. A common thermal energy generation process involves the conversion from electrical to thermal energy in a current-carrying medium (Ohmic or resistance heating). The implications of energy generation involves a volumetric source of thermal energy due to conversion from another form of energy in a conducting medium. The source may be uniformly distributed or it may be non-uniformly distributed, as in the absorption of radiation passing through a semi-transparent medium. Internal heat generation affects the temperature distribution in the medium and causes the heat rate to vary with location, thereby preventing inclusion of the medium in the thermal circuit. In this research conduction with internal heat generation will be considered for different geometries, including the rectangular, the solid cylinder and the sphere.

In most heat conduction problems for heat transfer in fins, it is assumed that (a) heat transfer is at steady state as such one dimensional ordinary differential equations are solved (b) no heat generation and thermal properties such as heat transfer coefficient and thermal conductivity are given by constants.

It is assumed these thermal properties depend on temperature.

1.2 Steady heat transfer models

We consider the one dimensional energy balance model

$$\frac{1}{r^{\alpha}}\frac{d}{dr}\left[r^{\alpha}k(\theta)\frac{d\theta}{dr}\right] \pm \beta G(\theta) = 0. \tag{1.1}$$

Here θ is the dimensionless temperature, r is the dimensionless space variable, k is the dimensionless thermal conductivity, G is the dimensionless heat transfer coefficient, α describes the geometry of the body and β is a constant. The fin is represented by $-\beta$ with $\beta > 0$ and hot body by $+\beta$, $\forall\ \beta > 0$. We shall discuss the relevant boundary conditions for each of these phenomena in the following chapters.

1.3 Aims and objectives of the dissertation

The main objective of this dissertation is to contribute to the fundamental understanding of heat transfer through extended surfaces and other hot bodies such as plane wall. We employ the Lie point symmetry techniques and preliminary group classification methods in our study.

1.4 Outline of the dissertation

The outline of this dissertation is as follows

- In chapter 2, the definitions and basic operations of the proposed methods of solutions are presented. The application of symmetry techniques will be discussed.

- In chapter 3, we focus on heat transfer through different fin models
- In chapter 4, we discuss and employ preliminary group classification to understand the heat transfer through a hot body.

1.5 Concluding remarks

In this chapter, we have provided the model representing heat transfer processes. Depending on the sign of the term involving $G(\theta)$, this model may represent either heat transfer in a fin or in a hot body. A detailed introduction to the heat transfer theory is also provided.

Chapter 2

Lie point symmetries of differential equations

2.1 A brief historical background

In this chapter we give a brief introduction to the Lie group theory and its applications. This will include the algorithm to determine the Lie point symmetries of ordinary differential equations. Symmetry methods for solving differential equations, unify many ad hoc methods (such as the substitution $y = vx$ for solving first-order homogenous equations) for constructing explicit exact solutions for differential equations and provide powerful new ways to find solutions. It reduces the systems of differential by equations finding equivalent systems of differential equations of simpler forms. Towards the end of the nineteenth century Sophus Lie [31] introduced the notion of the Lie group in order to study solutions of ordinary differential equations. He invented the theory of Lie groups while studying the symmetries of differential equations. This theory has applications in many areas of mathematics such as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, etc. The applications of Lie groups to differential systems were mainly established by Lie and Emmy Noether [32], and then advocated by Elie Cartan [34].

A symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Lie's fundamental discovery was that the complicated nonlinear conditions of invariance of the system under the group transformations could, in the case of a continuous group, be replaced by linear conditions reflecting a form of infinitesimal invariance of the system under the generator of the group. One can safely say that Lie group analysis is the only universal and effective method for solving nonlinear equations analytically. The applications of Lie group theory to differential equations remained dormant until Ovsiannikov [33]revived it in the late 1950s. Thereafter, the Lie group theory has been applied in many problems (described by linear or nonlinear equations) modeling some physical or abstract phenomena.

2.2 Calculation of Lie point symmetries

In this section we provide a brief theory of Lie point symmetry techniques. Here we restrict discussion to Lie point symmetries admitted by ordinary differential equations (ODEs). In short, a symmetry of a differential equation is an invertible transformation of the dependent and independent variables that does not change the form of the original differential equation. Detailed theory and applications of Lie symmetry groups may be found in the texts such as those of [17, 18, 19, 20,21].

2.2.1 One-parameter group of transformations

A symmetry group of a *single* or *system* of equations is a group of transformation which maps any solution to another solutions of the equation or system of equations. We define and illustrate the notion of transformation groups depending on a real parameter, where every one-parameter group of this type is completely determined by the infinitesimal transformation or the corresponding tangent vector. In this section we introduce the definition of a *group* and a $one-parameter\ group\ of\ transformations$. We are interested in applying the one-parameter group of transformations to ODEs, with the aim of constructing invariant solutions.

Definition 2.1. A group G is a set of elements with a law of composition ϕ between elements satisfying the following:

(i) $Closure\ property$. For any elements a and b of G, $\phi(a, b)$ is an element of G.

(ii) $Associative\ property$. If $a, b, c \in G$ and also $\phi(a, b)$ and $\phi(b, c)$ are in G, then
$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$.

(iii) $Identity\ element$. For all a in G, $\phi(e, a) = a = \phi(a, e)$.

(iv) $Inverse\ element$. For each a in G, $\phi(a, a^{-1}) = e = \phi(a^{-1}, a)$.

Remark. A group is *Abelian* if $\phi(a, b) = \phi(b, a)$ holds for all elements a and b in G.

Example 2.1. Some examples of groups

(i) A set of all integers with $\phi(a, b) = a + b$. Here $e = 0$ and $a^{-1} = -a$.

(ii) A set of all positive reals with $\phi(a,b) = a \cdot b$. Here e=1 and $a^{-1} = \frac{1}{a}$.

Definition 2.2. A set G of transformations

$$T_a : \bar{r^i} = f^i(r,\theta,a),\ \bar{\theta}^{\alpha} = g(r,\theta,a), i = 1,...,n; \alpha = 1,...,m, \tag{2.1}$$

with a being a real parameter which continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f^i, g^{α} are differentiable functions, is a $continuous\ one-parameter\ (local)\ Lie\ group\ of\ transformations$ in $\mathbb{R}^{n+m}$ and θ provided the following properties are satisfied, namely:

(i) For T_a, T_b in G where a,b in $\mathcal{D}' \subset \mathcal{D}$ then $T_b\,T_a = T_c \in G$, $c = \phi(a,b)$ in $\mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0\,T_a = T_a\,T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a\,T_{a^{-1}} = T_{a^{-1}}\,T_a = T_0$ (Inverse)

(iv) ϵ is a continuous parameter, that is, $a \in \mathcal{D}'$, where $\mathcal{D}$ is an interval in $\mathbb{R}$.

(v) $\phi(a,b)$ is an analytical function of a and b.

The group property (i) can be written as

$$\bar{\bar{r}} \equiv f(\bar{r},\bar{\theta},b) = f(r,\theta,\phi(a,b)), \quad \bar{\bar{\theta}} \equiv g(\bar{r},\bar{\theta},b) = g(r,\theta,\phi(a,b)). \tag{2.2}$$

Example 2.2. The transformation

$$T_a : \bar{r} = (1+a)r, \quad \bar{\theta} = (1+a)\theta, \quad a \in \mathbb{R}^+,$$

forms a one-parameter group.

The composition of the above transformations is

$$\bar{\bar{r}} = (1+b)\bar{r} = (1+b)(1+a)r = (1+a+b+ab)r.$$

Similarly

$$\bar{\bar{\theta}} = (1 + a + b + ab)\theta.$$

Therefore

$$\bar{\bar{r}} = (1 + c)r, \quad \bar{\bar{\theta}} = (1 + c)\theta$$

where $c = a + b + ab$.

Definition 2.3. A group parameter a is *canonical* if the group composition law is additive, that is, $\phi(a, b) = a + b$.

Theorem 2.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\phi(s, b)}{\partial b}\right|_{b=0}. \tag{2.3}$$

We now give the definition of a symmetry group for ODE by considering the second order ordinary differential equation given by

$$\theta'' = F(r, \theta, \theta'). \tag{2.4}$$

The transformation of the form (2.1) form a symmetry group of equation $F(r, \theta, \theta', ...\theta^{(n)}) = 0$ if the equation is form invariant (has the same form), that is $F(\bar{r}, \bar{\theta}, \bar{\theta}', ...\bar{\theta}^{(n)}) = 0$

2.2.2 Extended infinitesimal transformations (Prolongations)

According to Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*

$$\bar{r} \approx r + a\xi(r, \theta), \quad \bar{\theta} \approx \theta + a\eta(r, \theta) \tag{2.5}$$

obtained from (2.1) by expanding the functions f and g in a Taylor series about $a = 0$ and also taking into account the initial conditions

$$f\mid_{a=0}= r, \;\; g\mid_{a=0}= \theta.$$

Thus, we have

$$\xi(r,\theta) = \frac{\partial f}{\partial a}\mid_{a=0}, \;\; \eta(r,\theta) = \frac{\partial g}{\partial a}\mid_{a=0}. \tag{2.6}$$

The vector (ξ, η) with components (2.6) is the tangent vector at the point (r, θ) to the curve described by the transformed point $(\bar{r}, \bar{\theta})$ and is termed *tangent vector field* of the group G.

One dependent and one independent variable

In this study of invariance properties of the kth order ODE with the independent variable r and dependent variable θ that is $\theta = \theta(r)$, one seeks the admitted one-parameter Lie group of transformation of the form

$$\bar{r} = r + \epsilon\xi(r,\theta) + O(\epsilon^2),$$

$$\bar{\theta} = \theta + \epsilon\eta(r,\theta) + O(\epsilon^2),$$

$$\bar{\theta}_1 = \theta_1 + \epsilon\zeta(r,\theta,\theta_1) + O(\epsilon^2),$$

which corresponds to the (kth extended) infinitesimal generator

$$X^k = \xi(r,\theta)\frac{\partial}{\partial r} + \eta(r,\theta)\frac{\partial}{\partial \theta} + \zeta_1(r,\theta,\theta_1)\frac{\partial}{\partial \theta_1} + ... + \zeta_k(r,\theta,\theta_1,...,\theta_k)\frac{\partial}{\partial \theta_k},$$

k= 1, 2, 3...

Note that

θ_1=θ', $\theta_2 = \theta''$, $\theta_3 = \theta'''$ etc, where prime indicates derivative with respect to r.

Definition 2.4. The total derivative is defined by

$$D_r = \frac{\partial}{\partial r} + \theta_1 \frac{\partial}{\partial \theta} + \theta_2 \frac{\partial}{\partial \theta_1} + ... + \theta_{n+1} \frac{\partial}{\partial \theta_n} + ...$$

Explicit formulas for the extended infinitesimals ζ_k result from theorem 2.8.

Theorem 2.2

$$\zeta_k(r, \theta, \theta_1, ..., \theta_k) = D_r(\zeta_k - 1) - \theta_k D_r(\xi(r, \theta)), \quad k = 1, 2, ...$$

where

$$\zeta_0 = \eta(r, \theta)$$

Consider now a second-order ODE

$$E(r, \theta, \theta', \theta'') = 0 \tag{2.7}$$

where r is an independent variable and θ is the dependent variable. Let

$$X = \xi(r, \theta) \frac{\partial}{\partial r} + \eta(r, \theta) \frac{\partial}{\partial \theta}, \tag{2.8}$$

be the infinitesimal generator of the one-parameter group G of transformations (2.1). The second prolongation of the group G is denoted by $G^{[2]}$ and the symbol of $G^{[2]}$ is given by

$$X^{[2]} = X + \zeta_1(r, \theta, \theta') \frac{\partial}{\partial \theta'} + \zeta_2(r, \theta, \theta', \theta'') \frac{\partial}{\partial \theta''}, \tag{2.9}$$

where

$$\begin{aligned}
\zeta_r &= D_r(\eta) - \theta' D_r(\xi) = \eta_r + (\eta_\theta - \xi_r)\theta' - \xi_\theta \theta'^2, \\
\zeta_{rr} &= D_r(\zeta_1) - \theta'' D_r(\xi) = \eta_{rr} + (2\eta_{r\theta} - \xi_{rr})\theta' + (\eta_{\theta\theta} - 2\xi_{r\theta})\theta'^2 - \xi_{\theta\theta}\theta'^3 \\
&\quad + (\eta_\theta - 2\xi_r - 3\xi_\theta \theta')\theta''.
\end{aligned}$$

Example 2.3 As an example, we consider a nonlinear ODE

$$\frac{1}{r^3} \frac{d}{dr} \left[r^3 \theta^m \frac{d\theta}{dr} \right] - M^2 = 0. \tag{2.10}$$

The invariance criterion for symmetry determination is given by

$$X^{[2]}(\theta'' + \frac{3}{r}\theta' + m\theta^{-1}(\theta')^2 - M^2\theta^{-m}) \mid_{\theta''=-\frac{3}{r}\theta'-m\theta^{-1}(\theta')^2+M^2\theta^{-m}} = 0, \quad (2.11)$$

where

$$X^{[2]} = \xi(r,\theta)\frac{\partial}{\partial r} + \eta(r,\theta)\frac{\partial}{\partial \theta} + \zeta(r,\theta,\theta')\frac{\partial}{\partial \theta'} + \zeta(r,\theta,\theta',\theta'')\frac{\partial}{\partial \theta''} \quad (2.12)$$

is the second prolongation of the symmetry generator (2.8). The resulting determining equations are given by

$$m\xi_\theta - \theta\xi_{\theta\theta} = 0, \quad (2.13)$$

$$-mr\eta + \theta(mr\eta_\theta + \theta(6\xi_\theta + r\eta_{\theta\theta} - 2r\xi_{r\theta})) = 0, \quad (2.14)$$

$$mM^2r\eta + \theta(M^2r\eta_\theta + 3\theta^m\eta_r - 2M^2r\xi_r + r\theta^m\eta_{rr}) = 0, \quad (2.15)$$

$$-3\theta^{1+m}\xi - r(3M^2r\theta\xi_\theta + \theta^m(-2mr\eta_r - 3\theta\xi_r - 2r\theta\eta_{r\theta} + r\theta\xi_{rr})) = 0. \quad (2.16)$$

Solving Eqs. (2.13) - (2.16) yield the following Lie point symmetries

$$\begin{aligned}
X_1 &= \frac{\theta^{-m}}{r^2}\frac{\partial}{\partial\theta}, \\
X_2 &= \frac{4}{M^2r}\frac{\partial}{\partial r} + \theta^{-m}\frac{\partial}{\partial\theta}, \\
X_3 &= (1+m)(r^3+r)\frac{\partial}{\partial r} + 2\theta(r^2-1)\frac{\partial}{\partial\theta}, \\
X_4 &= (1+m)(r^3-r)\frac{\partial}{\partial r} + 2\theta(r^2+1)\frac{\partial}{\partial\theta}, \\
X_5 &= 8(1+m)r\frac{\partial}{\partial r} + (-8\theta + 3(1+m)M^2r^2\theta^{-m})\frac{\partial}{\partial\theta}, \\
X_6 &= 4(1+m)r\frac{\partial}{\partial r} + (-8\theta + 3(1+m)M^2r^2\theta^{-m})\frac{\partial}{\partial\theta}, \\
X_7 &= (-4(1+m)M^2r^2 + 32\theta^{1+m})\frac{\partial}{\partial r} + (-(1+m)M^4r^3\theta^{-3} + M^2r^4\theta)\frac{\partial}{\partial\theta}, \\
X_8 &= ((1+m)^2M^2r^3 + 8(1+m)r\theta^{1+m})\frac{\partial}{\partial r} + (6(1+m)M^2r^2\theta - 16\theta^{2+m})\frac{\partial}{\partial\theta}.
\end{aligned}$$

Clearly equation (2.10) admits eight dimensional Lie algebra. Note that if an equation admits maximal eight symmetries it is equivalent to $y'' = 0$ and is linearizable [22].

2.3 Concluding remarks

In this chapter, we have outlined the Lie symmetry techniques. In particular, a discussion on the calculations of Lie point symmetries is provided. A connection between the one-parameter group of transformations and Lie algebras is highlighted. An example on calculations of Lie point symmetries is also given.

Chapter 3

Comparison of exact solutions for heat transfer in fins of different geometries

K.J. Moleofane and R.J. Moitsheki, Comparison of exact solutions for heat transfer in extended surfaces of different geometries, *Abstract and Applied Analysis*, Volume 2014, Article ID 417098, 2014, 7 pages.

3.1 Introduction

Heat transfer rate from a hot body to the surrounding may be increased by surfaces which extend into that surrounding. These extended surfaces are referred to as fins. Extended surfaces are found in many engineering appliances. Thus, mathematical modeling of the heat transfer through this surfaces and the solution of these models are of continued interest. The heat transfer in fins is governed by boundary value problems (BVPs) which are rendered highly nonlinear by the dependency of thermal properties on temperature. In this study, both the heat transfer coefficient and thermal conductivity are given as a power law function of temperature.

The interest in solutions of fin problems continues unabated. Many symmetry analysts [26, 27, 28, 29, 30] analyzed the fin equation when heat transfer coefficient is given as function of space variable. Such a function was classified by direct methods (see e.g. [26]) and the extended analysis was done in [28]. Only general solution were provided in this case. It was well accepted that exact solutions of steady fin problems exist only when thermal conductivity and heat transfer are given as constants [6]. However Moitsheki *et al.* [11] have shown that solutions may exist even when those thermal properties are temperature dependent. In recent years Moitsheki *et al.* [11, 12, 13, 14] constructed exact solutions for the convective heat transfer in fins of different profiles. Furthermore, Ndlovu and Moitsheki [15] provided the approximate analytical solutions to steady state heat transfer in fins of different profiles which could not be solved exactly. In their studies an excellent comparison between exact and approximate solutions were established. One may also refer to the work by Moradi [16].

In this study, we consider heat conduction problem in fins of different geometries and in particular the spherical fin which has never been studied before. We compare the exact solutions of heat transfer in rectangular, radial and spherical fins. We further compare the fin efficiencies and effectiveness of these fins, and determine the effects of thermal parameters in a spherical fin. This chapter is arranged as follows; in section 2, we present the description of the models considered. In section 3 we briefly discuss the Lie point symmetry methods. Following linearization, the exact solutions are provided in section 4. In section 5, we analyze the problem when linearization fails. Conclusions are provided in section 6.

3.2 Mathematical description of a fin problem

We consider a fin of an arbitrary geometry with the length (or radius) R and a cross-sectional area A_c. The perimeter of the fin is given by P. The fin is attached to a fixed prime surface of temperature T_b and extends to an ambient fluid of temperature T_a, as shown in Figures (3.1), (3.2), (3.3). The energy balance equation is given by

$$\frac{A_c}{R^\alpha}\frac{d}{dR}\left[R^\alpha K(T)\frac{dT}{dR}\right] = PH(T)(T - T_a) \tag{3.1}$$

and the relevant boundary conditions are

$$T(R_0) = T_b, \qquad \left.\frac{dT}{dR}\right|_{R=R_1} = 0 \tag{3.2}$$

where T is temperature, $K(T)$ is the thermal conductivity and $H(T)$ is the heat transfer coefficient and α is a constant representing different geometries for different values.

The first boundary conditions in (3.2) represents a constant temperature at the base of the fin and the second boundary condition implies that the fin is insulated at the tip.

Introducing the nondimensional variables and numbers,

$$\theta = \frac{T - T_a}{T_b - T_a},\ r = \frac{R - R_1}{R_0 - R_1},\ H = h_b\left(\frac{T - T_a}{T_b - T_a}\right)^n,\ K = k_a\left(\frac{T - T_a}{T_b - T_a}\right)^m,$$

$$M^2 = \frac{Ph_bL^2}{k_aA_c},$$

then Eq. (3.1) and the boundary conditions (3.2) become

$$\frac{1}{r^\alpha}\frac{d}{dr}\left[r^\alpha\theta^m\frac{d\theta}{dr}\right] - M^2\theta^{n+1} = 0 \quad 0 \le r \le 1 \tag{3.3}$$

$$\theta(1) = 1, \quad \theta'(0) = 0. \tag{3.4}$$

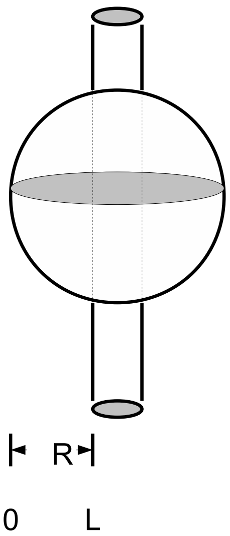

Figure 3.1: Graphical representation of a spherical fin.

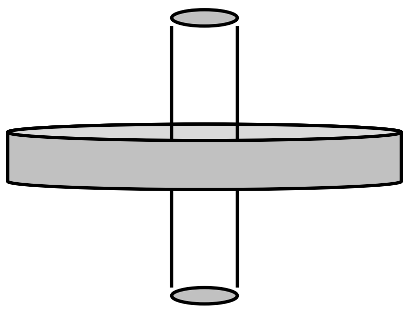

Figure 3.2: Graphical representation of a radial fin.

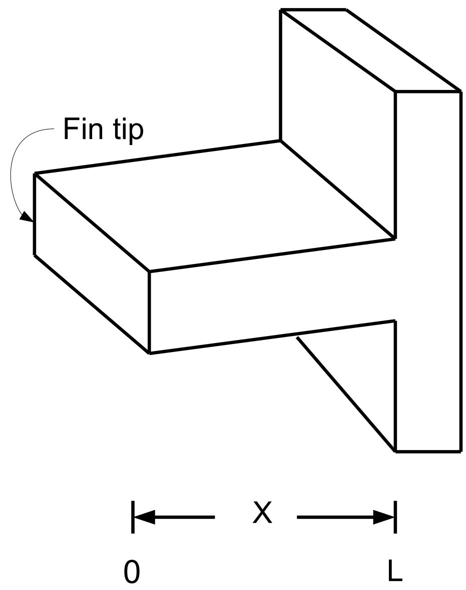

Figure 3.3: Graphical representation of a rectangular fin.

Here θ is the dimensionless temperature, r is the dimensionless space variable, and M is the thermogeometric fin parameter.

Two main cases may be analyzed, namely $m = n$ and $m \neq n$. One may construct exact solution when $m = n$ since the problem is linearizable. When $m \neq n$, symmetry method is employed.

3.3 Linearization and exact solutions

It has been proven in [13] that equations such as Eq.(3.3) is linearizable provided $m = n$. Thus assuming $m = n$ and letting $y = \theta^{n+1}$, Eq.(3.3) is transformed to linear, variable coefficient second order ordinary differential equation given by

$$\frac{d^2y}{dr^2} + \frac{\alpha}{r}\frac{dy}{dr} - (n+1)M^2 y = 0. \tag{3.5}$$

Subject to $y(1) = 1$ and $\frac{dy}{dr}(0) = 0$.

Several subcases arise namely $\alpha = 0, 1, 2$ and arbitrary, given $n < 1$ and $n > 1$.

Subcase (i) $\alpha = 0, \quad n < -1$ and $n > -1$.

This case has been solved in [11]. In this case the solutions are given by

$$\theta = \left\{ \frac{\cosh\left(\sqrt{n+1}\, Mr\right)}{\cosh\left(\sqrt{n+1}\, M\right)} \right\}^{\frac{1}{n+1}} \qquad -1 < n < \infty \tag{3.6}$$

and

$$\theta = \left\{ \frac{\sinh\left(\sqrt{n+1}\, Mr\right)}{\sinh\left(\sqrt{n+1}\, M\right)} \right\}^{\frac{1}{n+1}} \qquad -1 < n < 0. \tag{3.7}$$

The solution for $n < -1$ are given in terms of sine and cosine functions.

Subcase (ii) $\alpha = 1, \quad n < -1$ and $n > -1$.

This case has been solved in [12]. In this case the exact solutions are given

$$\theta = \left\{ \frac{I_0\left(\sqrt{n+1}\, Mr\right)}{I_0\left(\sqrt{n+1}\, M\right)} \right\}^{\frac{1}{n+1}} \qquad -1 < n < \infty. \tag{3.8}$$

where I_0 is a modified Bessel function of the first kind of order zero [35]. The solutions for $n < -1$ are given in terms of Bessel functions.

Subcase (iii) $\alpha = 2, \quad n < -1$ and $n > -1$.

In this case we obtain the exact solutions

$$\theta = \left\{ \frac{1}{r} \left[\frac{\sinh\left(M\sqrt{n+1}\, r\right)}{\sinh\left(M\sqrt{n+1}\right)} \right] \right\}^{\frac{1}{n+1}} \qquad n > -1, \tag{3.9}$$

and

$$\theta = \left\{ \frac{1}{r} \left[\frac{\sin\left(M\sqrt{n+1}\, r\right)}{\sin\left(M\sqrt{n+1}\right)} \right] \right\}^{\frac{1}{n+1}} \qquad n < -1, \tag{3.10}$$

The solutions (3.6), (3.8) and (3.9) are depicted in Fig. 3.4. In Fig. 3.5, the dimensionless temperature, $\theta(r)$, given by Eq.(3.9) is plotted along the dimensionless spatial direction for varying values of M and for $n = 2$, while in Fig. 3.6, $\theta(r)$, given by (3.9) is plotted against r for varying values of n and for $M = 2$.

Subcase (iv) $\alpha =$ arbitrary, $\quad n < -1$ and $n > -1$.

Given an arbitrary α, we obtain the general solutions

$$\theta = \left\{ r^{\frac{1}{2}-\frac{\alpha}{2}} \left[c_1 J_{\frac{\alpha}{2}-\frac{1}{2}} \left(M\sqrt{n+1}\, r \right) + c_2 Y_{\frac{\alpha}{2}-\frac{1}{2}} \left(M\sqrt{n+1}\, r \right) \right] \right\}^{\frac{1}{n+1}} \quad n < -1, \tag{3.11}$$

and

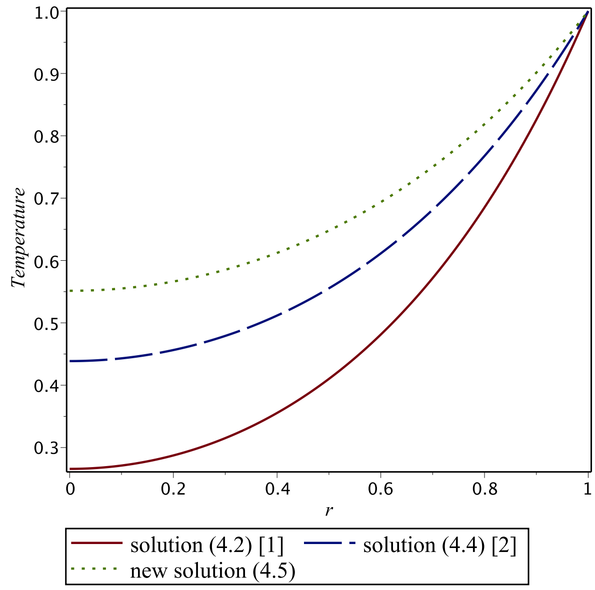

Figure 3.4: Fin temperature $\theta(r)$ given by (3.6), (3.8) and (3.9) plotted against r for $M = 2$ and $n = 0$.

$$\theta = \left\{ r^{\frac{1}{2}-\frac{\alpha}{2}} \left[c_1 J_{\frac{\alpha}{2}-\frac{1}{2}} \left(iM\sqrt{n+1}\, r \right) + c_2 Y_{\frac{\alpha}{2}-\frac{1}{2}} \left(iM\sqrt{n+1}\, r \right) \right] \right\}^{\frac{1}{n+1}} \quad n > -1, \tag{3.12}$$

Note that one may obtain exact solutions which satisfy the boundary conditions only when $\alpha = 1$ but this will coincide with (3.8).

One may also construct exact solution when $m \neq n = -1$. In this case the solutions satisfying the boundary condition is given by

$$\theta = (m+1) \left[\frac{M^2 r^2}{2(\alpha+1)} + \frac{1}{m+1} - \frac{M^2}{2(\alpha+1)} \right]. \tag{3.13}$$

The solution (3.13) is depicted in Fig. 3.7.

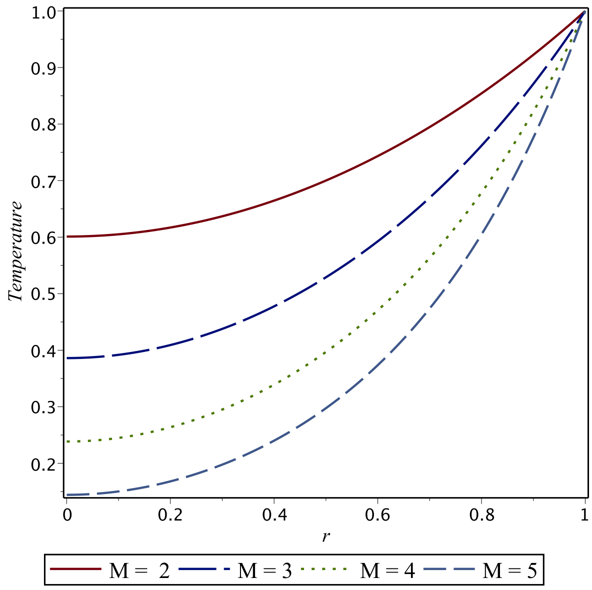

Figure 3.5: Fin temperature $\theta(r)$ given by (3.9) plotted against r for varying values of M and $n = 2$.

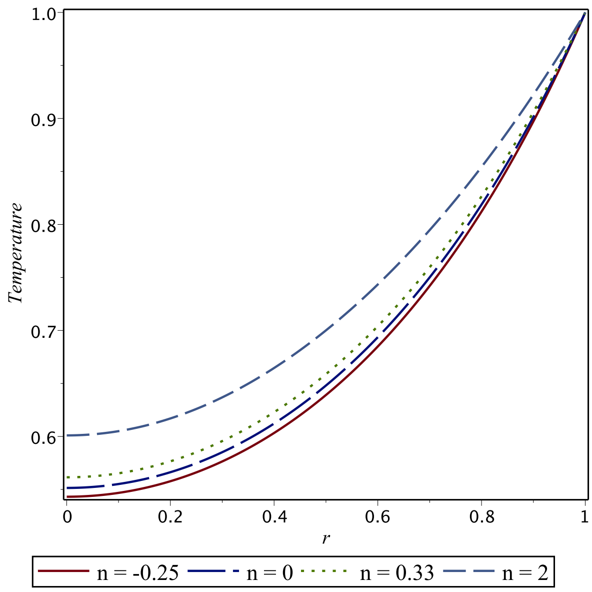

Figure 3.6: Fin temperature $\theta(r)$ given by (3.9) plotted against r for varying values of n and $M = 2$.

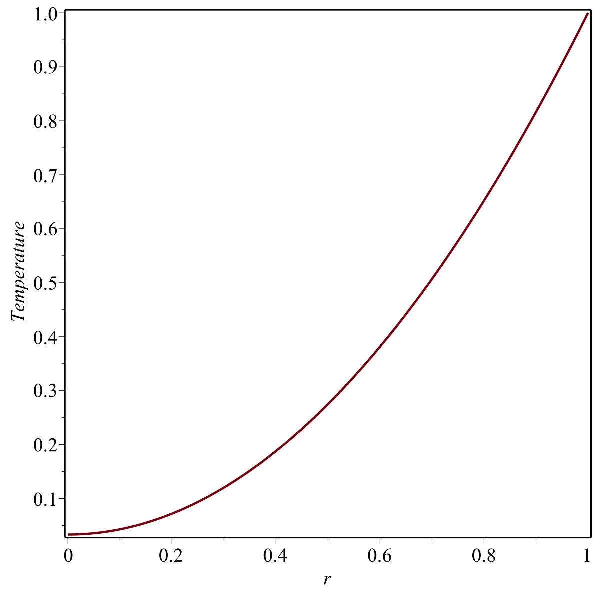

Figure 3.7: Fin temperature $\theta(r)$ given by (3.13) plotted against r for $M = 2$ and $n = 4$.

Table 3.1: Comparison of the temperature profiles in fins with rectangular, radial and spherical profiles. Here $M = 4$ and $n = 1$

r	$\alpha = 0$ (rectangular)	$\alpha = 1$ (radial)	$\alpha = 2$ (spherical)
0	0.125136	0.185069	0.238556
0.1	0.134790	0.192403	0.244902
0.2	0.160849	0.213754	0.363795
0.3	0.199607	0.248095	0.295078
0.4	0.250490	0.295331	0.339080
0.5	0.315255	0.356686	0.396937
0.6	0.397056	0.434579	0.470685
0.7	0.500174	0.532528	0.563297
0.8	0.630101	0.655185	0.678749
0.9	0.793788	0.808503	0.822159
1	1	1	1

3.4 Symmetry reductions

In this section we consider the case $m \neq n$, with $\alpha = 2$ and α being arbitrary. In this case Eq. (3.3) is not linearizable and as such we employ the Lie point symmetry analysis.

3.4.1 Case $\forall\ \alpha \neq 0$.

In this case Eq. (3.3) admits one dimensional Lie algebra spanned by the base vector

$$X = \frac{m}{2(m+1)}\left(-2\theta\frac{\partial}{\partial\theta} + (n-m)r\frac{\partial}{\partial r}\right).$$

Note that an obvious extra symmetry when $\alpha = 0$ is a translation in r. We employ the method of differential invariants to reduce the order of Eq. (3.3) by one. The first prolongation of the generator X is given by

$$X^{[1]} = \frac{m}{2(m+1)}\left(-2\theta\frac{\partial}{\partial\theta} + (n-m)r\frac{\partial}{\partial r}\right) - \left[\frac{2m + m(n-m)}{2(m+1)}\right]\theta'\frac{\partial}{\partial\theta'}$$

and the corresponding characteristic equation is given by

$$\frac{dr}{m(n-m)r} = -\frac{d\theta}{2m\theta} = -\frac{d\theta'}{[2m + m(n-m)]\theta'}. \tag{3.14}$$

Solving the above characteristics gives the invariants

$$I_1 = \theta r^{\frac{2}{n-m}}, \quad I_2 = \theta' r^{\frac{2+n-m}{n-m}}. \tag{3.15}$$

Now we let $I_1 = t$, $I_2 = u$ and writing $u = u(t)$. From the definition of t, and by using chain rule

$$D_r(u) = D_r(t)D_t(u) \tag{3.16}$$

we obtain

$$\frac{2+n-m}{n-m}\theta' + r\theta'' = \left[\frac{2\theta}{(n-m)r} + \theta'\right]u'. \tag{3.17}$$

Substituting $u = u(t)$ into equation (3.3) we have

$$u'\left(\frac{2t^2}{n-m} + tu\right) + \alpha tu + mu^2 - M^2 t^{n-m+2} = 0. \tag{3.18}$$

We notice that the above equation may not be solved exactly.

Subcase $\alpha = \dfrac{3m+n+4}{m+n+2}$.

This choice of α is not physical since there is no relationship between the geometry of the fin and the exponents of the thermal conductivity and heat transfer coefficient. However, this case is mathematically interesting since Eq. (3.3) admits two Lie point symmetries which implies [22] that the ODE in question is integrable or reducible to the one with cubic degree in first derivative. The admitted Lie algebra is spanned by the base vectors

$$X_1 = \frac{1}{m-n}\left(2\theta\frac{\partial}{\partial\theta} + (m-n)r\frac{\partial}{\partial r}\right),$$

$$X_2 = \exp\left(\frac{n+1}{m+n+2}\right) r^2 \left[2(m+n+2)\theta\frac{\partial}{\partial\theta} + (n+2)(2m+n+2)r\frac{\partial}{\partial r}\right].$$

We omit further analysis since the initial assumption is not physically realistic.

Subcase $\alpha = 2 \;\; n = -1.$

In this subcase Eq. (3.3) is integrable and admits an eight dimensional Lie algebra spanned by the base vectors

$$
\begin{aligned}
X_1 &= -\frac{r^2\theta^{-m}}{3}(M^2 r\frac{\partial}{\partial\theta} + 3\theta^m\frac{\partial}{\partial r}),\\
X_2 &= -\frac{2r\theta^{-m}}{3}(M^2 r\frac{\partial}{\partial\theta} + 3\theta^m\frac{\partial}{\partial r}),\\
X_3 &= \frac{m}{2(m+1)}\left\{\left[(m+1)M^2r^2 - 2\theta^{m+1}\right]\frac{\partial}{\partial\theta} + 2(m+1)r\theta^m\frac{\partial}{\partial r}\right\},\\
X_4 &= \frac{r^2\theta^{-m}}{18(m+1)}\left\{\left(mM^4r + M^2r^3 - 6M^2r\theta^{m+1}\right)\frac{\partial}{\partial\theta} + \left[3M^2(m+1)\theta^m - 18\theta^{2m+1}\right]\frac{\partial}{\partial r}\right\},\\
X_5 &= \frac{\theta^{-m}}{12(m+1)^2}\left\{[12\theta^{2(m+1)} - 8(m+1)M^2r^2\theta^{m+1} + M^4(m+1)^2r^4]\right\}\frac{\partial}{\partial\theta} +\\
&\quad [2(m+1)^2M^2r^3\theta^m - 12(m+1)r\theta^{2m+1}]\frac{\partial}{\partial r}\Big\},\\
X_6 &= \frac{m[(m+1)M^2r^2 - 6\theta^{m+1}]}{6(m+1)\theta^m}\frac{\partial}{\partial\theta},\\
X_7 &= -\frac{\theta^{-m}}{r}\frac{\partial}{\partial\theta},\\
X_8 &= -\theta^{-m}\frac{\partial}{\partial\theta}.
\end{aligned}
$$

Equation (3.3) is equivalent to the simple motion equation $y'' = 0$ [22]. We adopt the method of canonical coordinate to demonstrate this claim. We introduce the method of finding the solutions using X_7 and X_8 from the above dimensional Lie algebra vectors. The two symmetries lead to the canonical variables

$$t = r, \quad u = -\frac{r\theta^{1+m}}{1+m} \tag{3.19}$$

and the corresponding canonical forms of X_7 and X_8 are

$$X_1 = \partial_u, \quad X_2 = t\partial_u. \tag{3.20}$$

Writing $u = u(t)$ transforms equation (3.3) to

$$u''(t) + tM^2 = 0. \tag{3.21}$$

Integrating the latter equation and writing it into its original variables we obtain

$$\theta(r) = \left\{ \left[\frac{1}{6} M^2 r^2 - \frac{c_1}{r} - c_2 \right] (1+m) \right\}^{\frac{1}{1+m}}. \tag{3.22}$$

Imposing the boundary conditions, we obtain

$$\theta = r^{\frac{2}{m+1}}, \quad \forall m < 1.$$

Subcase $\alpha = 2 \ \ m = n.$

In this subcase the admitted eight dimensional symmetries include:

$$\begin{aligned}
X_1 &= \theta \frac{\partial}{\partial \theta}, \\
X_2 &= (1+n) r \frac{\partial}{\partial r} - \theta \frac{\partial}{\partial \theta}, \\
X_3 &= \frac{e^{-\sqrt{M^2(1+n)} r} \theta^{-n}}{r} \frac{\partial}{\partial \theta}, \\
X_4 &= \frac{e^{\sqrt{M^2(1+n)} r} \theta^{-n}}{\sqrt{M^2(1+n)} r} \frac{\partial}{\partial \theta}, \\
X_5 &= (1+n) M^2 \frac{\partial}{\partial r} + (M^3 \sqrt{1+n} - \frac{M^2}{r}) \theta \frac{\partial}{\partial \theta}, \\
X_6 &= -(1+n) M^2 \frac{\partial}{\partial r} + (M^3 \sqrt{1+n} + \frac{M^2}{r}) \theta \frac{\partial}{\partial \theta}, \\
X_7 &= (1+n) \frac{\partial}{\partial r} - (1 + \sqrt{M^2(1+n)} r) \theta \frac{\partial}{\partial \theta}, \\
X_8 &= (1+n) \frac{\partial}{\partial r} - (1 - \sqrt{M^2(1+n)} r) \theta \frac{\partial}{\partial \theta}.
\end{aligned}$$

3.5 Fin efficiency and heat flux

The fin efficiency is defined as the ratio of the actual heat transfer from the fin surface to the surrounding fluid while the whole fin is kept at the same temperature. On the other hand, heat flux is the total amount of heat flowing per unit area per unit time. The fin efficiency and the heat flux in dimensionless

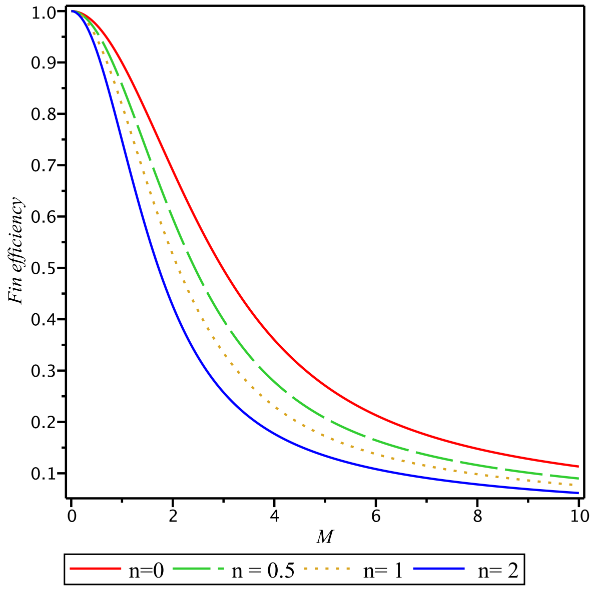

Figure 3.8: Fin efficiency plotted against M for varying values of n.

variables are given by

$$\eta = \int_0^1 \theta^{n+1} dr \tag{3.23}$$

and

$$q = \frac{1}{Bi}\frac{k(\theta)}{h(\theta)}\frac{d\theta}{dr} \tag{3.24}$$

respectively. Here the dimensionless parameter $Bi = \frac{h_b L}{k_a}$ is the Biot number. Given solution (3.9), we obtain

$$\eta = \frac{\ln(M\sqrt{n+1}) - \ln(-M\sqrt{n+1}) + Ei(1, M\sqrt{n+1}) - Ei(1, -M\sqrt{n+1})}{2\sinh(M\sqrt{n+1})} \tag{3.25}$$

where $Ei(a, z)$ is the Exponential Integral [24]. Fin efficiency (3.25) is depicted in Fig. 3.5.

Given solution (3.9) the heat flux becomes

$$q = \frac{1}{Bi}\frac{rM\sqrt{n+1}\cosh(M\sqrt{n+1}r) - \sinh(M\sqrt{n+1}r)}{(n+1)r\sinh(M\sqrt{n+1}r)} \tag{3.26}$$

Heat flux (3.26) is depicted in Fig. 3.6 and Fig. 3.7.

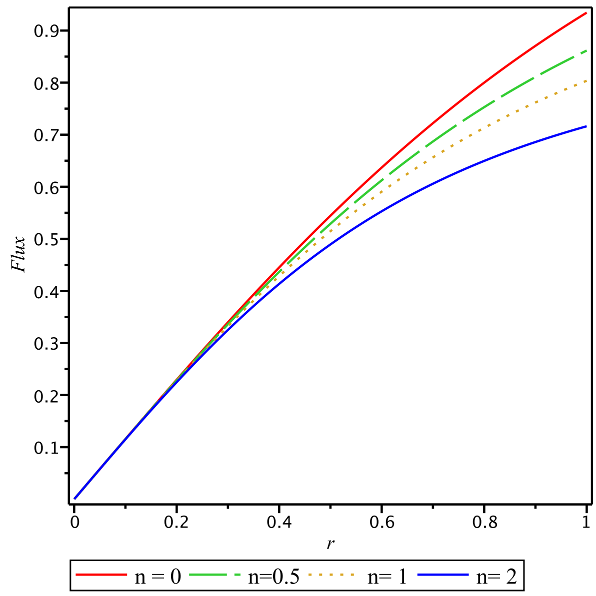

Figure 3.9: Heat flux plotted against r for varying values of n.

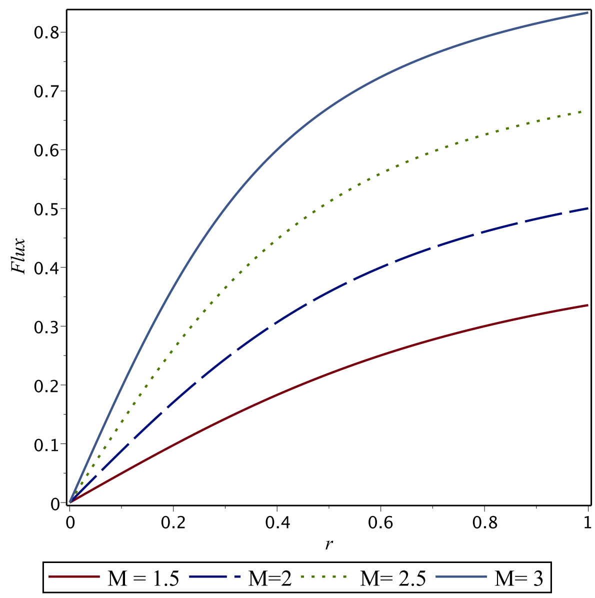

Figure 3.10: Heat flux for varying values of M.

3.6 Discussions

In Fig. 3.4, the new solution is given by the dotted graph and it can be seen that the values of the temperature for spherical fins is the highest across the spatial r direction. In the table, the θ values for $\alpha = 2$ are the highest as well. Now we focus only on spherical fins and observe in Fig. 3.8 that temperature decreases with increase in the values of M. Recall that the thermo-geometric fin parameter is directly proportional to the aspect ratio of the fin. Thus longer fins (M larger) releases heat much more efficiently than shorter ones. In Fig. 3.9, temperature increases with an increase in the values of n. Finally, in Fig. 3.10 depicts the heat transfer where $m \neq n = 1$.

3.7 Concluding remarks

In this chapter we focused on the comparison of temperature distribution (or heat transfer) in fins of different geometries. Lie point symmetry analysis resulted in a number of admitted symmetries for different cases of the parameters appearing in the governing equation. We have applied the method of differential invariants to reduce the order of the equation by one, and on the other case we have used method of canonical variables to construct some exact solutions.The spherical fin is not as effective in transferring heat as the radial or rectangular fins.

Chapter 4

Preliminary group classification of nonlinear model describing heat transfer in a hot body

Some results in this chapter have been submitted to ISI journal for possible publication.

4.1 Introduction

In this chapter we investigate the heat conduction in a hot body (wall or solid slab). We assume the geometry of the body to be either rectangular , radial or spherical. However, more emphasis is in a rectangular body such as a wall. We further assume that the temperature distribution is symmetrical along the center as shown in Fig. 4.1. Furthermore we assume that the thermal behavior of the wall (hot body) is affected by internally generated or absorbed thermal energy (sink or source). Here the heat source is placed at the center of the plane.

4.2 Mathematical description of heat transfer models

We consider a situation in which the thermal behavior of a body is affected by heat generated or absorbed internally. Some examples include heating associated with a flow of electrical current and electrical resistance, exothermic chemical reactions, absorption of radiation in microwave ovens and emission of radiation flame. We denote the temperature dependent internal heat generation term by $\dot{Q}_v'''(T) = 0$ and α describes the geometry of the body, for example $\alpha = 0$ implies the rectangular geometry, $\alpha = 1$ the solid cylinder and $\alpha = 2$ represents the sphere. Applying the energy conservation principle and the Fourier's law we obtain the balanced heat conduction with internal heat generation model given by [23]

$$\frac{1}{R^\alpha}\frac{d}{dR}\left[R^\alpha K(T)\frac{dT}{dR}\right] + \dot{Q}_v'''(T) = 0. \tag{4.1}$$

The prescribed boundary conditions are given by

$$R = 0: \quad \frac{dT}{dR} = 0, \quad and \quad R = R_s : T = T_s. \tag{4.2}$$

Introducing the dimensionless variables

$$r = \frac{R}{R_s}, \quad \theta = \frac{T}{T_s}, \quad k(\theta) = \frac{K(T)}{k_a}, \omega(\theta) = \frac{\dot{Q}_v'''(T)}{N_g}, \quad N_g = \frac{R_1^2\dot{Q}_L'''}{k_a T_s}, \tag{4.3}$$

we obtain the dimensionless model

$$\frac{1}{r^\alpha}\frac{d}{dr}\left[r^\alpha k(\theta)\frac{d\theta}{dr}\right] + N_g\omega(\theta) = 0, \quad 0 \leq r \leq 1, \tag{4.4}$$

and the boundary conditions become

$$r = 0: \quad \frac{d\theta}{dr} = 0, \quad and \quad r = 1: \quad \theta = 1. \tag{4.5}$$

4.3 Equivalence transformations

We determine the equivalence transformations [10] of Eq.(4.4). We recall that an equivalence transformation

$$\bar{r} = \bar{r}(r,\theta), \quad \bar{\theta} = \bar{\theta}(r,\theta)$$

is a non-degradable change of the variables r and θ such that the form of Eq. (4.4) remains invariant generally with different arbitrary function appearing in the equation. To determine the equivalence transformation, one seeks the equivalence algebra generated by the vector field

$$X = \xi(r,\theta)\frac{\partial}{\partial r} + \eta(r,\theta)\frac{\partial}{\partial \theta} + \mu(r,\theta,W)\frac{\partial}{\partial W}. \tag{4.6}$$

The second prolongation is given by

$$\tilde{X}^{[2]} = X + \zeta_r\frac{\partial}{\partial \theta'} + \zeta_{rr}\frac{\partial}{\partial \theta''} + \omega_r\frac{\partial}{\partial W_r}. \tag{4.7}$$

where ζ_r and ζ_{rr} are given by the usual prolongation and ω_r is

$$\omega_r = \tilde{D}_r(\mu) - W_r\tilde{D}_r(\xi) - W_u\tilde{D}_r(\eta) \tag{4.8}$$

with $\tilde{D}_r$ being the total derivative operator defined by

$$\tilde{D}_r = \frac{\partial}{\partial r} + W_r\frac{\partial}{\partial W} + ... \tag{4.9}$$

The invariance surface conditions is given by

$$\tilde{X}^{[2]}\left(\frac{1}{r^\alpha}\frac{d}{dr}[r^\alpha\theta^m\frac{d\theta}{dr}] + N_gW(\theta)\right) |_{(4.4)}= 0, \quad \tilde{X}^{[2]}(W_r = 0) \,|_{W_r=0}= 0. \tag{4.10}$$

Using the second condition in (4.10) results in

$$\mu_r - W_\theta\eta_r = 0$$

which implies that

$$\mu_r = 0, \quad \eta_r = 0$$

The first condition yields the following equation

$$[-\frac{a}{r^2}\theta^m\theta'\xi+m\theta^{m-1}\theta''\eta+\frac{a}{r}m\theta^{m-1}\theta'\eta+(m^2-m)\theta^{m-2}(\theta')^2\eta+N_g\mu+\frac{a}{r}\theta^m\zeta_r+\theta^m\zeta_{rr}]\mid_{(4.4)}=0 \tag{4.11}$$

which gives the following determining equations

$$m\xi_\theta-\theta\xi_{\theta\theta}=0, \tag{4.12}$$
$$-m\theta^{m-2}\eta+\frac{2a}{r}\theta^m\xi_\theta+m\theta^{m-1}\eta_\theta+\theta^m\eta_{\theta\theta}-2\theta^m\xi_{r\theta}=0, \tag{4.13}$$
$$-a\xi+ar\xi_r+2mr^2\theta^{-1}\eta_r+2r^2\eta_{r\theta}-r^2+3r^2\theta^{-m}N_gW(\theta)\xi_\theta=0, \tag{4.14}$$
$$-mN_gW(\theta)\theta^{-1}\eta+N_g\mu+\frac{a}{r}\theta^m\eta_r+\theta^m\eta_{rr}-N_gW(\theta)\eta_\theta+2N_gW(\theta)\xi_r=0. \tag{4.15}$$

Solving equation (4.12) we obtain

$$\xi(r,\theta)=\frac{C(r)}{1+m}\theta^{1+m}+D(r).$$

Substituting the value of ξ in equations (4.13) and (4.14) yield

$$\xi=rc_1+r^ac_2,$$
$$\eta=\theta c_3+\theta^{-m}c_4.$$

Now substitution of ξ and η into equation (4.15) gives

$$\mu=\Big((1+m)c_3-2(c_1+ar^{a-1}c_2)\Big)W.$$

Therefore the admitted operators are

$$\begin{aligned}
\tilde{X}_1 &= r\frac{\partial}{\partial r}-2W\frac{\partial}{\partial W},\\
\tilde{X}_2 &= r\frac{\partial}{\partial r}-2aW\frac{\partial}{\partial W},\\
\tilde{X}_3 &= \theta\frac{\partial}{\partial\theta}+(1+m)W\frac{\partial}{\partial W},\\
\tilde{X}_4 &= \theta^{-m}\frac{\partial}{\partial\theta}.
\end{aligned}$$

The one-parameter group of equivalence transformations corresponding to each operator is

$$\begin{aligned}
\tilde{X}_1 &= \bar{r} = re^{\alpha_1}, \quad \bar{W} = We^{-2\alpha_1}, \quad \bar{\theta} = \theta, \\
\tilde{X}_2 &= \bar{r} = re^{\alpha_2}, \quad \bar{W} = We^{-2a\alpha_1}, \quad \bar{\theta} = \theta, \\
\tilde{X}_3 &= \bar{r} = r, \quad \bar{W} = We^{(1+m)\alpha_3}, \quad \bar{\theta} = \theta e^{\alpha_3}, \\
\tilde{X}_4 &= \bar{r} = r, \quad \bar{W} = W.
\end{aligned}$$

and the composition of transformations gives

$$\bar{r} = (1 + e^{\alpha_1+\alpha_2})r,$$

Since W is dependent on θ, we consider the projections above on the space of (θ, W) to be

$$\begin{aligned}
Z_1 &= pr(\tilde{X}_1), \\
Z_2 &= pr(\tilde{X}_2), \\
Z_3 &= pr(\tilde{X}_3), \\
Z_4 &= pr(\tilde{X}_4).
\end{aligned}$$

Therefore

$$\begin{aligned}
Z_1 &= -2W\frac{\partial}{\partial W}, \\
Z_2 &= -2aW\frac{\partial}{\partial W}, \\
Z_3 &= \theta\frac{\partial}{\partial \theta} + (1+m)W\frac{\partial}{\partial W}, \\
Z_4 &= \theta^{-m}\frac{\partial}{\partial \theta}.
\end{aligned}$$

Note that $Z_i : i = 1, 2, 3, 4$ span the Lie algebra denoted by L_4.

The following propositions contains the essence of the method of preliminary group classification.

Proposition 4.1

(see, examples, [25]).*Let* L_r *be an* r*-dimensional subalgebra of the algebra* L_4*. Denote by* $Z_i, i = 1, ..., r$ *a basis of* L_r *and by* W_i *the elements of the algebra* L_4 *such that* Z_i *is the projections of* W_i *on* (θ, k, g)*. If equations*

$$k = f(\theta), \quad W = \psi(\theta) \tag{4.16}$$

are invariant with respect to the algebra L_r *then the equation*

$$\frac{d}{dr}\left(f(\theta)\frac{d\theta}{dr}\right) - M^2\psi(\theta) = 0 \tag{4.17}$$

admits the operator

$$Z_i = projection \quad of \quad W_i \quad on \ (r, \theta) \tag{4.18}$$

Proposition 4.2

(see, examples, [25]). *Let* (4.17) *and the equation*

$$\frac{d}{dr}\left(f(\bar{\theta})\frac{d\bar{\theta}}{dr}\right) - M^2\psi(\bar{\theta}) = 0 \tag{4.19}$$

be constructed according to Proposition 4.1 via subalgebras L_r *and* $\bar{L_r}$*, respectively. If* L_r *and* $\bar{L_r}$*, are similar subalgebras in* L_4 *then* (4.17) *and* (4.19) *are equivalent with respect to the equivalence group* G_4 *generated by* L_r*. These propositions imply that the problem of preliminary group classification of* (4.4) *is reduced to the algebraic problem of constructing nonsimilar subalgebras of* L_4 *or optimal system of subalgebras* [20]*. We explore method in* [25] *to construct the one-dimensional optimal systems.*

4.4 Principal Lie algebra

In this section we determine the Lie point symmetries, given an arbitrary function $\omega(\theta)$. The Lie point symmetries (or vector fields) admitted when the

Table 4.1: Commutators of L_4

	Z_1	Z_2	Z_3	Z_4
Z_1	0	0	0	0
Z_2	0	0	0	0
Z_3	0	0	0	$-(1+m)Z_4$
Z_4	0	0	$(1+m)Z_4$	0

function appearing in the equation are arbitrary span the principal Lie algebra (L_P). Following the well known procedure (see example 2.9), it turns out that equation (4.4) admits no Lie point symmetries, that is, the principal Lie algebra is null.

4.5 Adjoint group for algebra L_4

We now wish to construct the adjoint group of the algebra L_4. Let us denote the elements of adjoint group L_4 by the letter A. The generators of the adjoint group L_4 are

$$A = \sum [Z_\alpha, Z_\beta] \frac{\partial}{\partial Z_\beta}, \quad \alpha = 1, 2, ..., 4 \tag{4.20}$$

Using Table (4.1) we obtain the following generators

$$\begin{aligned} A_1 &= 0, \\ A_2 &= 0, \\ A_3 &= -(1+m)Z_4 \frac{\partial}{\partial Z_4}, \\ A_4 &= (1+m)Z_4 \frac{\partial}{\partial Z_3}. \end{aligned}$$

The infinitesimal operator of A_1 and A_2 generates the following one-parameter group of linear transformations

$$Z_1' = Z_1, \quad Z_2' = Z_2, \quad Z_3' = -(1+m)a_4 Z_4, \quad Z_4' = (1+m)a_3 Z_4,$$

which are represented by the matrices

$$M_1(a_1) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$M_2(a_2) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$M_3(a_3) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (1+m)a_3 Z_4 \end{vmatrix}$$

Following the same procedure we obtain

$$M_4(a_4) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (1+m)a_4 Z_4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

The product of the matrices give us

$$M = M_1(a_1)...M_4(a_4)= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (1+m)a_3Z_4 \\ 0 & 0 & 0 & (1+m)a_4Z_4 \end{vmatrix}$$

where $a_3, a_4 \,\epsilon\, \mathbb{R}^+$.

Let $e = (e^1, e^2, e^3, e^4)$, $\bar{e} = (\bar{e}^1, \bar{e}^2, \bar{e}^3, \bar{e}^4)$ and $\bar{e} = Me$,

Then the components of $\bar{e}$ are:

$$\bar{e^1} = e^1, \tag{4.21}$$

$$\bar{e^2} = e^2, \tag{4.22}$$

$$\bar{e^3} = e^3, \tag{4.23}$$

$$\bar{e^4} = (1+m)[a_3e^3 + a_4e^4]. \tag{4.24}$$

These transformations give rise to the adjoint group elements of the algebra L_4.

4.6 Construction of the one-dimensional optimal system of subalgebra of L_4

In this section we construct the optimal system of one-dimensional subalgebra of L_4.

CASE 1: $e^1 \neq 0$, $e^2 \neq 0$, $e^3 \neq 0$ In this case we get

$$e^4 = 0 \tag{4.25}$$

and by putting $a_3 = 1$ in equations (4.21)-(4.24)

$$a_4 = \frac{-e^4}{e^3} \tag{4.26}$$

and we obtain,

$$\bar{e} = (e^1, e^2, e^3, 0) \tag{4.27}$$

These vectors give rise to the following nonequivalent generators

$$Z_1 + Z_2 + Z_3 \tag{4.28}$$

Similarly the analysis of the other cases yields the following nonequivalent generators:

CASE 2: $e^1 \neq 0$, $e^2 \neq 0$, $e^3 = 0$

$$Z_1 + Z_2, Z_1 + Z_2 + (1+m)Z_4. \tag{4.29}$$

CASE 3: $e^1 \neq 0$, $e^2 = 0$, $e^3 = 0$

$$Z_1 + Z_3. \tag{4.30}$$

CASE 4: $e^1 = 0$, $e^2 \neq 0$, $e^3 \neq 0$

$$Z_2 + Z_4 \tag{4.31}$$

CASE 5: $e^1 = 0$, $e^2 = 0$, $e^3 = 0$

$$(1+m)Z_4 \tag{4.32}$$

CASE 6: $e^1 \neq 0$, $e^2 = 0$, $e^3 \neq 0$,$e^4 = 0$

$$Z_1, Z_1 + (2+m)Z_3. \tag{4.33}$$

CASE 7: $e^1 \neq 0$, $e^2 \neq 0$, $e^3 = 0$

$$Z_1 + Z_2, Z_1 + Z_2 + (1+m)Z_4 \tag{4.34}$$

CASE 8: $e^1 \neq 0$, $e^2 = 0$, $e^3 = 0$

$$Z_1 + Z_2 + (1+m)Z_4 \tag{4.35}$$

Altogether from (4.26) - (4.33) we have the following optimal system of one-dimensional subalgebras of L_4:

$$\begin{aligned}
Z^{(1)} &= Z_1 + Z_2 + Z_3, \\
Z^{(2)} &= Z_1 + Z_2, \\
Z^{(3)} &= Z_1 + Z_2 + (1+m)Z_4, \\
Z^{(4)} &= Z_1 + Z_3, \\
Z^{(5)} &= Z_2 + Z_4, \\
Z^{(6)} &= (1+m)Z_4, \\
Z^{(7)} &= Z_1, \\
Z^{(8)} &= Z_1 + (2+m)Z_3, \\
Z^{(9)} &= Z_3, \\
Z^{(10)} &= Z_4.
\end{aligned}$$

4.7 Equations admitting an extension of the principal Lie algebra

In this section we are going to employ Propositions 1 and 2 to the optimal system obtained in the previous section to obtain all nonequivalent equations (4.1) admitting an ex+tension by one of the principal Lie algebra L_p.

To illustrate the method we choose the following examples from our optimal system:

Table 4.2: Forms of $W(\theta)$ for which L_p is extended by one element.

Z	W	Equation	Additional operator
$Z^{(1)}$	θ^{m+2a-1}	$\frac{1}{r^a}\frac{d}{dr}[r^a\theta^m\frac{d\theta}{dr}] + N_g\theta^{m-1+2a} = 0$	$X_1 = 2r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial\theta}$
$Z^{(4)}$	θ^{m-1}	$\frac{1}{r^a}\frac{d}{dr}[r^a\theta^m\frac{d\theta}{dr}] + N_g\theta^{m-1} = 0$	$X_1 = 2r\frac{\partial}{\partial r}$
$Z^{(6)}$	γ	$\frac{1}{r^a}\frac{d}{dr}[r^a\theta^m\frac{d\theta}{dr}] + N_g\gamma = 0$	$X_1 = (1+m)\theta^{-m}\frac{\partial}{\partial\theta}$
$Z^{(9)}$	θ^{1+m}	$\frac{1}{r^a}\frac{d}{dr}[r^a\theta^m\frac{d\theta}{dr}] + N_g\theta^{1+m} = 0$	$X_1 = \theta\frac{\partial}{\partial\theta}$
$Z^{(10)}$	γ	$\frac{1}{r^a}\frac{d}{dr}[r^a\theta^m\frac{d\theta}{dr}] + N_g\gamma = 0$	$X_1 = \theta^{-m}\frac{\partial}{\partial\theta}$

(a) Consider: $Z^{(1)}$:

$$\begin{aligned} Z^{(1)} &= Z_1 + Z_2 + Z_3 \\ &= \theta\frac{\partial}{\partial\theta} + (m-1-2a)W\frac{\partial}{\partial W}. \end{aligned}$$

Invariants are found from the subsidiary equations:

$$\frac{d\theta}{\theta} = \frac{dW}{(m-1-2a)W}.$$

We obtain

$$W = \theta^{m-2a-1} \tag{4.36}$$

Note that the operators $Z^{(2)}$, $Z^{(3)}$, $Z^{(5)}$, $Z^{(7)}$, and $Z^{(8)}$ do not lead to any form of the arbitrary function for which the principal Lie algebra is extended. The necessary condition for existence of invariant solution is not satisfied.

4.8 Extra Lie point symmetries

Consider the form $W = \theta^{m+1}$ generated by $Z^{(9)}$. In this case equation (4.4) becomes

$$\frac{1}{r^\alpha}\left[r^\alpha\theta^m\frac{d\theta}{dr}\right] + N_g\theta^{m+1} = 0. \tag{4.37}$$

We focus on three cases, namely $\alpha = 0, 1, 2$. Notice that equation (4.37) is linearizable since the term involving the internal heat generation is a differential consequence of the thermal conductivity (see also [13]).

Subcase $\alpha = 0$

Given $\alpha = 0$, then equation (4.33) using mathematica admits eight Lie point symmetries given by

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial r}, \\
X_2 &= \theta\frac{\partial}{\partial \theta}, \\
X_3 &= \left(e^{r\sqrt{-(1+m)N_g}}\theta^{-m}\right)\frac{\partial}{\partial \theta}, \\
X_4 &= \left(e^{-r\sqrt{-(1+m)N_g}}\theta^{-m}\right)\frac{\partial}{\partial \theta}, \\
X_5 &= \frac{-\cos[2\sqrt{1+m}r\sqrt{N_g}]}{N_g}\frac{\partial}{\partial r} + \frac{\theta\sin[2\sqrt{1+m}r\sqrt{N_g}]}{\sqrt{1+m}N_g}\frac{\partial}{\partial \theta}, \\
X_6 &= \frac{-\sin[2\sqrt{1+m}r\sqrt{N_g}]}{N_g}\frac{\partial}{\partial r} - \frac{\theta\cos[2\sqrt{1+m}r\sqrt{N_g}]}{\sqrt{1+m}N_g}\frac{\partial}{\partial \theta}, \\
X_7 &= \frac{e^{r\sqrt{-(1+m)N_g}}\theta^{1+m}}{1+m}\frac{\partial}{\partial r} + \frac{e^{r\sqrt{-(1+m)N_g}}\theta^{2+m}\sqrt{-(1+m)N_g}}{(1+m)^2}\frac{\partial}{\partial \theta}, \\
X_8 &= \frac{e^{-r\sqrt{-(1+m)N_g}}\theta^{1+m}}{1+m}\frac{\partial}{\partial r} - \frac{e^{-r\sqrt{-(1+m)N_g}}\theta^{2+m}\sqrt{-(1+m)N_g}}{(1+m)^2}\frac{\partial}{\partial \theta}.
\end{aligned}
$$

Subcase $\alpha = 1$

In this case the Lie algebra is spanned by the generators

$$
X_1 = \frac{\partial}{\partial \theta}, X_2 = 2F[r]\frac{\partial}{\partial r} + \left(\frac{-\theta F[r]}{(1+m)r} + \frac{\theta F'[r]}{1+m}\right)\frac{\partial}{\partial \theta}.
$$

Subcase $\alpha = 2$, $m = n$.

In this case equation (4.37) admits the following Lie point symmetries

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial r} - \frac{\theta}{(1+m)r}\frac{\partial}{\partial \theta},\\
X_2 &= \theta\frac{\partial}{\partial \theta},\\
X_3 &= \frac{\left(e^{r\sqrt{-(1+m)Ng}}\theta^{-m}\right)}{(1+m)^2}\frac{\partial}{\partial \theta},\\
X_4 &= \frac{\left(e^{r\sqrt{-(1+m)Ng}}\theta^{-m}\right)}{r\sqrt{-(1+m)N_g}}\frac{\partial}{\partial \theta},\\
X_5 &= \frac{\sin[2\sqrt{1+m}r\sqrt{N_g}]}{\sqrt{(1+m)N_g}}\frac{\partial}{\partial r} + \left(\frac{\theta\cos[2\sqrt{1+m}r\sqrt{N_g}]}{1+m} - \frac{\theta\sin[2\sqrt{1+m}r\sqrt{N_g}]}{(1+m)^{\frac{3}{2}}r\sqrt{N_g}}\right)\frac{\partial}{\partial \theta},\\
X_6 &= \frac{-\cos[2\sqrt{1+m}r\sqrt{N_g}]}{\sqrt{(1+m)N_g}}\frac{\partial}{\partial r} + \left(\frac{\theta\sin[2\sqrt{1+m}r\sqrt{N_g}]}{1+m} + \frac{\theta\cos[2\sqrt{1+m}r\sqrt{N_g}]}{(1+m)^{\frac{3}{2}}r\sqrt{N_g}}\right)\frac{\partial}{\partial \theta},\\
X_7 &= \frac{e^{-r\sqrt{-(1+m)Ng}}r\theta^{1+m}}{1+m}\frac{\partial}{\partial r} - \frac{e^{-r\sqrt{-(1+m)Ng}}\theta^{2+m}}{(1+m)^2}\left(1+r\sqrt{-(1+m)N_g}\right)\frac{\partial}{\partial \theta},\\
X_8 &= \frac{e^{r\sqrt{-(1+m)Ng}}r\theta^{1+m}}{(1+m)\sqrt{-(1+m)N_g}}\frac{\partial}{\partial r} + \left(\frac{e^{r\sqrt{-(1+m)Ng}}r\theta^{2+m}}{(1+m)^2} - \frac{e^{r\sqrt{-(1+m)Ng}}\theta^{1+m}}{(2+m)^2\sqrt{-(1+m)N_g}}\right)\frac{\partial}{\partial \theta}.
\end{aligned}
$$

4.9 Linearization and exact solutions

In this case equation (4.37) is linearized by the point transformation $y = \theta^{m+1}$ to

$$\frac{d^2y}{dr^2} + \frac{\alpha}{r}\frac{dy}{dr} + (m+1)N_g y = 0. \tag{4.38}$$

The boundary conditions (4.5) becomes

$$y'(0) = 0, \quad y(1) = 1.$$

We determine the exact solutions of equation (4.37) using three different cases of α, namely $\alpha = 0, 1, 2$.

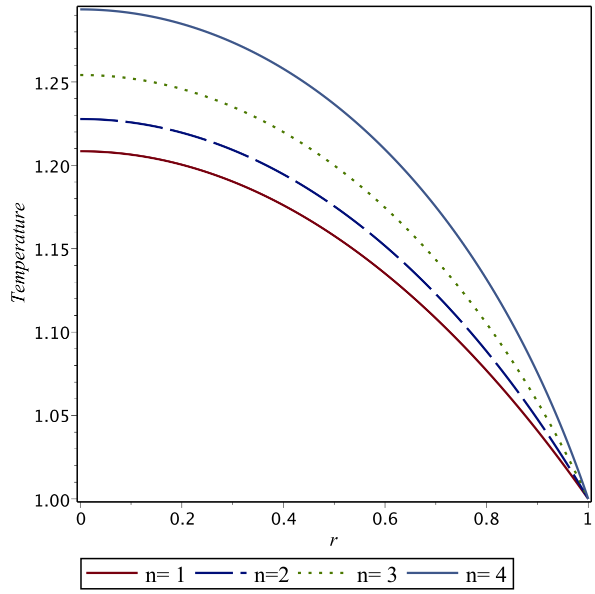

Figure 4.1: Temperature profile for varying value of n.

Subcase $\alpha = 0$

When $\alpha = 0$ the exact solutions in terms of original term satisfying the boundary conditions (4.5) is

$$\theta = \left\{ \frac{cos(\sqrt{(m+1)Ngr})}{cos(\sqrt{(m+1)Ng})} \right\}^{\frac{1}{m+1}} \qquad -1 < m < \infty \tag{4.39}$$

and

$$\theta = \left\{ \frac{sin(\sqrt{(m+1)Ngr})}{sin(\sqrt{(m+1)Ng})} \right\}^{\frac{1}{m+1}} \qquad -1 < m < 0 \tag{4.40}$$

Now the exact solution when $m < -1$ we have

$$\theta = \left\{ \frac{cosh(\sqrt{(m+1)Ngr})}{cosh(\sqrt{(m+1)Ng})} \right\}^{\frac{1}{m+1}} \tag{4.41}$$

The solutions (4.39) and (4.40) are depicted in Fig. 4.1 and Fig. 4.2 respectively.

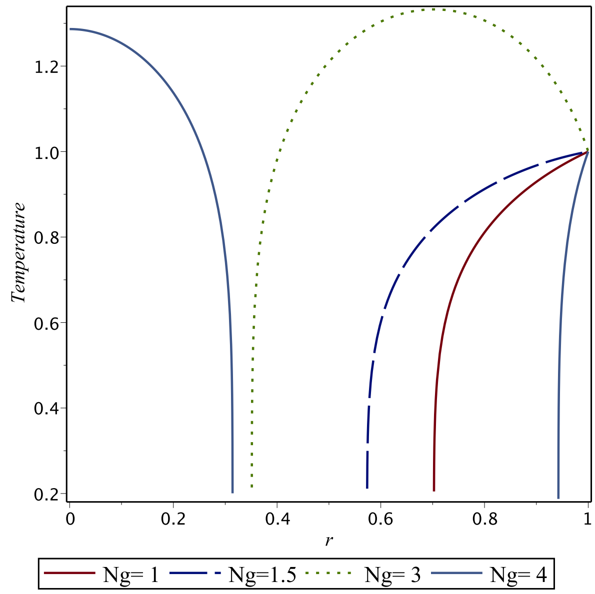

Figure 4.2: Temperature profile for varying values of Ng.

Subcase $\alpha = 1$

The exact solution obtained in terms of the original boundary conditions is

$$\theta(r) = \left\{ \frac{J_0(\sqrt{(1+m)Ng}r}{J_0(\sqrt{(1+m)Ng}} \right\}^{\frac{1}{1+m}}. \tag{4.42}$$

Subcase $\alpha = 2$

In this case we get

$$\theta(r) = \left\{ \frac{\sin(\sqrt{(1+m)Ng}r}{\sin(\sqrt{(1+m)Ng}} \right\}^{\frac{1}{1+m}}. \tag{4.43}$$

4.10 Discussion and concluding remarks

We observe in Fig. 4.1 that temperature increases with the increased values of n. Furthermore in Fig. 4.2, temperature increases with increasing values of N_g. We also observe that there is a threshold value of N_g for which the temperature generated internally reaches a maximum. The analysis of this threshold values is not yet carried out. On the other hand, a deeper understanding of the application of principal Lie algebra and equivalence transformations and the construction of optimal systems of subalgebras using the methods of preliminary group classification has been gained. We have considered a model describing heat transfer in a hot body. Symmetric boundary conditions are imposed. Note that both thermal conductivity and heat generation coefficient are temperature dependent. We have singled out a case where the internal heat generation is a differential consequence of the thermal conductivity. This has resulted in some exciting exact solutions which have been analyzed.

Chapter 5

Conclusions

In this dissertation we focused on reductions of steady nonlinear one-dimensional heat transfer models. We focused on heat transfer in fins and in a hot body. In chapter one, we gave a brief background on heat transfer in fins with different geometries. Chapter 2, a historical background on Lie point symmetries and a full discussion on the calculations of Lie point symmetries with examples were provided. We compared exact solutions for heat transfer in fins with different geometries (rectangular, radial and spherical) in chapter 3. We also compared the fin efficiencies given these geometries. We learned that the temperature values for these three fin geometries increases as r increases. It was also observed that heat transfer is much slower in spherical fins than radial and rectangular fins, which was confirmed in Fig. 3.1 and Table 3.1. We have depicted that the increase in the values of n yielded a decrease in fin performance. In chapter 4 an understanding of the application of principal Lie algebra, equivalence transformations and the construction of optimal system using the preliminary group classification has been gained. We considered models arising in heat transfer in a hot body such as heat transfer in a plane wall. We assumed the internal heat generation and thermal conductivity are temperature dependent. Some geometries are open, for instance, we do not know why there is a jump in graph when N_g is increased.

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