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Anomalous dimensions of heavy operators from magnon energies

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Abstract: We study spin chains with boundaries that are dual to open strings suspended between systems of giant gravitons and dual giant gravitons. Motivated by a geometrical interpretation of the central charges of $su(2|2)$, we propose a simple and minimal all loop expression that interpolates between the anomalous dimensions computed in the gauge theory and energies computed in the dual string theory. The discussion makes use of a description in terms of magnons, generalizing results for a single maximal giant graviton. The symmetries of the problem determine the structure of the magnon boundary reflection/scattering matrix up to a phase. We compute a reflection/scattering matrix element at weak coupling and verify that it is consistent with the answer determined by symmetry. We find the reflection/scattering matrix does not satisfy the boundary Yang-Baxter equation so that the boundary condition on the open spin chain spoils integrability. We also explain the interpretation of the double coset ansatz in the magnon language.

Keywords: 1/N Expansion, AdS-CFT Correspondence, Gauge-gravity correspondence

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1 Introduction

In this article we will connect two distinct results that have been achieved in the context of gauge/gravity duality. The first result, which is motivated by the Penrose limit in the $AdS_5 \times S^5$ geometry [1], is the natural language for the computation of anomalous dimensions of single trace operators in the planar limit provided by integrable spin chains (see [2] for a thorough review). For the spin chain models we study, using only the symmetries of the system, one can determine the exact large N anomalous dimensions and the two magnon scattering matrix. Using integrability one can go further and determine the complete scattering matrix of spin chain magnons [3, 4]. The second results which we will

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use are the powerful methods exploiting group representation theory, which allow one to study correlators of operators whose classical dimension is of order N. In this case, the large N limit is not captured by summing the planar diagrams. Our results allow a rather complete understanding of the anomalous dimensions of gauge theory operators that are dual to giant graviton branes with open strings suspended between them. These results generalize the analysis of [5] to systems that include non-maximal giant gravitons and dual giant gravitons. The boundary magnons of an open string attached to a maximal giant graviton are fixed in place — they can not hop between sites of the open string. In the case of non maximal giant gravitons and dual giant gravitons there are non-trivial interactions between the open string and the brane, allowing the boundary magnons to move away from the string endpoints.

The operators we focus on are built mainly out of one complex U(N) adjoint scalar Z, and a much smaller number M of impurities given by a second complex scalar field Y, which are the "magnons" that hop on the lattice of the Zs. The dilatation operator action on these operators matches the Hamiltonian of a spin chain model comprising of a set of defects that scatter from each other. The spin chain models enjoy an SU(2|2)^2 symmetry. The symmetries of the system determine the energies of impurities, as well as the two impurity scattering matrix [3, 4]. The SU(2|2) algebra includes two sets of bosonic generators ($R^a{}_b$ and $L^\alpha{}_\beta$) that each generate an SU(2) group. The action of the generators is summarized in the relations

$$[R^a{}_b, T^c] = \delta^c_b T^a - \frac{1}{2}\delta^a_b T^c\,, \qquad [L^\alpha{}_\beta, T^\gamma] = \delta^\gamma_\beta T^\alpha - \frac{1}{2}\delta^\alpha_\beta T^\gamma \tag{1.1}$$

where T is any tensor transforming as advertised by its index. The algebra also includes two sets of super charges Q^α_a and S^b_β. These close the algebra

$$\{Q^\alpha_a, S^b_\beta\} = \delta^b_a L^\alpha_\beta + \delta^\alpha_\beta R^b_a + \delta^b_a \delta^\alpha_\beta C\,, \tag{1.2}$$

where C is a central charge, and

$$\{Q^\alpha_a, Q^\beta_b\} = 0\,, \qquad \{S^a_\alpha, S^b_\beta\} = 0. \tag{1.3}$$

We will realize this algebra on states that include magnons. When the magnons are well separated, each magnon transforms in a definite representation of $su(2|2)$ and the full state transforms in the tensor product of these individual representations. Acting on the ith magnon we can have a centrally extended representation [3, 4]

$$\{Q^\alpha_a, S^b_\beta\} = \delta^b_a L^\alpha_\beta + \delta^\alpha_\beta R^b_a + \delta^b_a \delta^\alpha_\beta C_i\,, \tag{1.4}$$

$$\{Q^\alpha_a, Q^\beta_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k_i}{2}\,, \quad \{S^a_\alpha, S^b_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab}\frac{k^*_i}{2}\,. \tag{1.5}$$

The total multimagnon state must be in a representation for which the central charges k_i, k^*_i vanish. Thus the multi magnon state transforms under the representation with

$$C = \sum_i C_i\,, \qquad \sum_i k_i = 0 = \sum_i k^*_i\,. \tag{1.6}$$

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A key ingredient to make use of the $su(2|2)$ symmetry entails determining the central charges k_i, k_i^* and hence the representations of the individual magnons. There is a natural geometric description of the system, first obtained by an inspired argument in [6] and later put on a firm footing in [7], which gives an elegant and simple description of these central charges. The two dimensional spin chain model that is relevant for planar anomalous dimensions is dual to the worldsheet theory of the string moving in the dual $AdS_5 \times S^5$ geometry. This string is a small deformation of a $\frac{1}{2}$ BPS state. A convenient description of the $\frac{1}{2}$-BPS sector (first anticipated in [8]) is in terms of the LLM coordinates introduced in [9], which are specifically constructed to describe $\frac{1}{2}-$BPS states built mainly out of Zs. In the LLM coordinates, there is a preferred LLM plane on which states that are built mainly from Zs orbit with a radius $r = 1$ (in convenient units). Consider a closed string state dual to a single trace gauge theory operator built mainly from Zs, but also containing a few magnons M. The closed string solution looks like a polygon with vertices on the unit circle. The sides of the polygon are the magnons. The specific advantage of these coordinates is that they make the analysis of the symmetries particularly simple and allow a perfect match to the $SU(2|2)^2$ superalgebra of the gauge theory described above. Matching the gauge theory and gravity descriptions in this way implies a transparent geometrical understanding of the k_i and k_i^*, as we now explain. The commutator of two supersymmetries in the dual gravity theory contains NS-B_2 gauge field transformations. As a consequence of this gauge transformation, strings stretched in the LLM plane acquire a phase which is the origin of the central charges k_i and k_i^*. It follows that we can immediately read off the central charges for any particular magnon from the sketch of the closed string worldsheet on the LLM plane: the straight line segment corresponds to a complex number which is the central charge [7].

The gauge theory operators that correspond to closed strings have a bare dimension that grows, at most, as $\sqrt{N}$. We are interested in operators whose bare dimension grows as N when the large N limit is taken. These operators include systems of giant graviton branes. The key difference as far as the sketch of the state on the LLM plane is concerned, is that the giant gravitons can orbit on circles of radius $r < 1$ while dual giant gravitons orbit on circles of radius $r > 1$. The magnons populating open strings which are attached to the giant gravitons can be divided into boundary magnons (which sit closest to the ends of the open string) and bulk magnons. The boundary magnons will stretch from a giant graviton located at $r \neq 1$ to the unit circle, while bulk magnons stretch between points on the unit circle. We will also consider the case below that the entire open string is given by a single magnon, in which case it will stretch between two points with $r \neq 1$.

The computation of correlators of the corresponding operators in the field theory is highly non-trivial. Indeed, as a consequence of the fact that we now have order N fields in our operators, the number of ribbon graphs that can be drawn is huge. These enormous combinatoric factors easily overpower the usual $\frac{1}{N^2}$ suppression of non-planar diagrams so that both planar and non-planar diagrams must be summed to capture even the leading large N limit of the correlator [10]. This problem can be overcome by employing group representation theory techniques. The article [11] showed that it is possible to compute the correlation functions of operators built from any number of Zs exactly, by using the

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Schur polynomials as a basis for the local operators of the theory. In [12] these results were elegantly explained by pointing out that the organization of operators in terms of Schur polynomials is an organization in terms of projection operators. Completeness and orthogonality of the basis follows from the completeness and orthogonality of the underlying projectors. With these insights [11, 12], many new directions opened up. A basis for the local operators which organizes the theory using the quantum numbers of the global symmetries was given in [13, 14]. Another basis, employing projectors related to the Brauer algebra was put forward in [15] and developed in a number of interesting works [16–22]. For the systems we are interested in, the most convenient basis to use is provided by the restricted Schur polynomials. Inspired by the Gauss Law which will arise in the world volume description of the giant graviton branes, the authors of [23] suggested operators in the gauge theory that are dual to excited giant graviton brane states. This inspired idea was pursued both in the case that the open strings are described by an open string word [24–26] and in the case of minimal open strings, with each open string represented by a single magnon [27, 28]. The operators introduced in [24, 27] are the restricted Schur polynomials. Further, significant progress was made in understanding the spectrum of anomalous dimensions of these operators in the studies [25, 26, 29–34]. Extensions which consider orthogonal and symplectic gauge groups and other new ideas, have also been achieved [35–40].

In this paper we will connect the string theory description and the gauge theory description of the operators corresponding to systems of excited giant graviton branes. Our study gives a concrete description of the central charges k_i and some of the consequences of the $su(2|2)$ symmetry. We will see that the restricted Schur polynomials provide a natural description of the quantum brane states. For the open strings we find a description in terms of open spin chains with boundaries and we explain precisely what the boundary interactions are. The double coset ansatz of the gauge theory, which solves the problem of minimal open strings consisting entirely of a single magnon, also has an immediate and natural interpretation in the same framework.

There are closely related results which employ a different approach to the questions considered in this article. A collective coordinate approach to study giant gravitons with their excitations has been pursued in [41–45]. This technique employs a complex collective coordinate for the giant graviton state, which has a geometric interpretation in terms of the fermion droplet (LLM) description of half BPS states [8, 9]. The motivation for this collective coordinate starts from the observation that within semiclassical gravity, we think of the D-branes as being localized in the dual spacetime geometry. It might seem however, that since in the field theory the operators we write down have a precise $\mathcal{R}$-charge and a fixed energy, they are dual to a delocalized state. Indeed, since gauge/gravity duality is a quantum equivalence it is subject to the uncertainty principle of quantum mechanics. The $\mathcal{R}$-charge of an operator is the angular momentum of the dual states in the gravity theory, so that by the uncertainty principle, the dual giant graviton-branes must be fully delocalized in the conjugate angle in the geometry. The collective coordinate parametrizes coherent states, which do not have a definite $\mathcal{R}$-charge and so may permit a geometric interpretation of the position of the D-brane as the value of the collective coordinate. With the correct

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choice for the coherent states, mixing between different states of a definite $\mathcal{R}$-charge would be taken into account and so when diagonalizing the dilatation operator (for example) the mixing between states with different choices of the values of the collective coordinate might be suppressed. This computation would be, potentially, much simpler than a direct computation utilizing operators with a definite $\mathcal{R}$-charge. Of course, by diagonalizing the dilatation operator for operators dual to giant graviton brane plus open string states, one would expect to recover the collective coordinates, but this may only be possible after a complicated mixing problem in degenerate perturbation theory is solved. Some of the details that have emerged from our study do not support this semiclassical reasoning. Specifically, we find that the brane states are given by restricted Schur polynomials and these do not receive any corrections when the perturbation theory problem is solved, so that there does not seem to be any need to solve a mixing problem which constructs localized states from delocalized ones. Our large N eigenstates do have a definite $\mathcal{R}$-charge. The nontrivial perturbation theory problem involves mixing between operators corresponding to the same giant graviton branes, but with different open string words attached. Thus, it is an open string state mixing problem, solved with a discrete Fourier transform, as it was for the closed string. However, there is general agreement between the approaches: the Fourier transform solves a collective coordinate problem which diagonalizes momentum, rather than position.

For an interesting recent study of anomalous dimensions, at finite N, using a very different approach, see [46].

This article is organized as follows: in section 2 we recall the relevant facts about the restricted Schur polynomials. The action of the dilatation operator on these restricted Schur polynomials is studied in section 3 and the eigenstates of the dilatation operator are constructed in section 4. Section 5 provides the dual string theory interpretation of these eigenstates and perfect agreement between the energies of the string theory states and the corresponding eigenvalues of the dilatation operator is demonstrated. In sections 6 and 7 we consider the problem of magnon scattering, both in the bulk and off the boundary magnons. We have checked that the magnon scattering matrix we compute is consistent with scattering results obtained in the weak coupling limit of the theory. One important conclusion is that the spin chain is not integrable. In section 8 we review the double coset ansatz and describe the dual string theory interpretation of these results. Our conclusions and some discussion is given in section 9. The appendices collect some technical details.

2 Giants with open strings attached

In this section we will review the gauge theory description of the operators dual to giant graviton branes with open string excitations. In this description, each open string is described by a word with order $\sqrt{N}$ letters. Most of the letters are the Z field. There are however $M \sim O(1)$ impurities which are the magnons of the spin chain. For simplicity we will usually take all of the impurities to be a second complex matrix Y. This idea was first applied in [47] to reproduce the spectrum of small fluctuations of giant gravitons [48]. The description was then further developed in [49–53]. The articles [51–53] in particular

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developed this description to the point where interesting dynamical questions[1] could be asked and answered. The open string words are then inserted into a sea of Zs which make up the giant graviton brane(s). Concretely, the operators we consider are

$$O\left(R, R_1^k, R_2^k; \{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k\right)$$
$$= \frac{1}{n!} \sum_{\sigma \in S_{n+k}} \chi_{R,R_1^k,R_2^k}(\sigma) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (W_k)^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots (W_2)^{i_{n+k-1}}_{i_{\sigma(n+k-1)}} (W_1)^{i_{n+k}}_{i_{\sigma(n+k)}} \tag{2.1}$$

where the open string words are

$$(W_I)^i_j = (Y Z^{n_1} Y Z^{n_2-n_1} Y \cdots Y Z^{n_{M_I}-n_{M_I-1}} Y)^i_j \,. \tag{2.2}$$

We have used the notation $\{n_i\}_I$ in (2.1) to describe the integers $\{n_1, n_2, \cdots, n_{M_I}\}$ which appear in the Ith open string word. This is a lattice notation, which lists the number of Zs appearing to the left of each of the Ys, starting from the second Y: the Zs form a lattice and the n_i give a position in this lattice. This notation is particularly convenient when we discuss the action of the dilatation operator. We will also find an occupation notation useful. The occupation notation lists the number of Zs between consecutive Ys, and is indicated by placing the n_i in brackets. Thus, for example $O(R, R_1^1, R_2^1, \{n_1, n_2, n_3\}) = O(R, R_1^1, R_2^1, \{(n_1), (n_2 - n_1), (n_3 - n_2)\})$. R is a Young diagram with $n + k$ boxes. A bound state of p_s giant gravitons and p_a dual giant gravitons is described by a Young diagram R with p_a rows, each containing order N boxes and p_s columns, each containing order N boxes. $\chi_{R,R_1^k,R_2^k}(\sigma)$ is a restricted character [24] given by

$$\chi_{R,R_1^k,R_2^k}(\sigma) = \mathrm{Tr}_{R_1^k,R_2^k}\left(\Gamma_R(\sigma)\right). \tag{2.3}$$

R^k is a Young diagram with n boxes, that is, it is a representation of S_n. The irreducible representation R of S_{n+k} is reducible if we restrict to the S_n subgroup. R^k is one of the representations that arise upon restricting. In general, any such representation will be subduced more than once. Above we have used the subscripts 1 and 2 to indicate this. We have in mind a Gelfand-Tsetlin like labeling to provide a systematic way to describe the possible R^k we might consider. In this labeling, we use the transformation of the representation under the chain of subgroups $S_{n+k} \supset S_{n+k-1} \supset S_{n+k-2} \supset \cdots \supset S_n$. This is achieved by labeling boxes in R. Dropping the boxes with labels $\leq i$, we obtain the representation of S_{n+k-i} to which R^k belongs. We have to spell out how this chain of subgroups are embedded in S_{n+k}. Think of S_q as the group which permutes objects labeled $1, 2, 3, \cdots, q$. Here we have $q = n + k$ and the objects we have in mind are the Z fields or the open string words. We associate an integer to an object by looking at the upper indices in (2.1); as an example, the open string described by W_2 is object number $n + k - 1$. To go from S_{n+k-i} to $S_{n+k-i-1}$, we keep only the permutations that fix $n+k-i$. We can put the states in R_1^k and R_2^k into a 1-to-1 correspondence. The trace $\mathrm{Tr}_{R_1^k,R_2^k}$ sums the column index over R_1^k and the row index over R_2^k. If we associate the row and column indices with the endpoints of the open string, we can associate the endpoints of the open string I with

[1] For example, one could consider the force exerted by the string on the giant.

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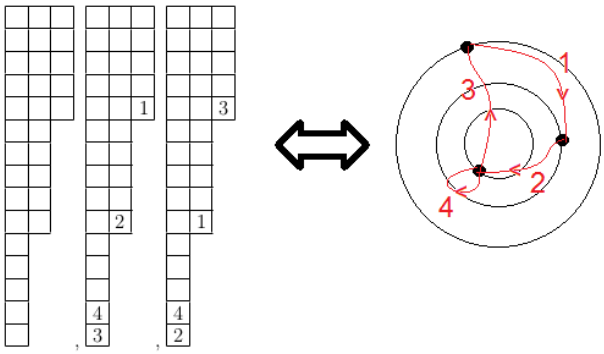

Figure 1. A cartoon illustrating the R, R_1^k, R_2^k labeling for an example with $k = 4$ open strings and 3 giant gravitons. The shape of the strings stretching between the giants is not realistic — only the locations of the end points of the open strings is accurate. The giant gravitons are orbiting on the circles shown; the radius shown for each orbit is accurate. They wrap an S^3 which is transverse to the plane on which they orbit. The smaller the radius of the giant's orbit, the larger the S^3 it wraps. The size of the S^3 that the giant wraps is given by its momentum, which is equal to the number of boxes in the column which corresponds to the giant. The numbers appearing in the boxes of R_1^4 tell us where the open strings start and the numbers appearing in the boxes of R_2^4 where they end.

the box labeled I in R_1^k and R_2^k. The numbers appearing in the boxes of R_1^k literally tell us where the k open strings start and the numbers in R_2^k where the k open strings end. See figure 1 for an example of this labeling. Each Y in an open string word is a magnon. We will take the number of magnons $M_I = O(1)\ \forall I$. The $Z^{ij}_{i_{\sigma(j)}}$ with $1 \leq j \leq n$ belong to the system of giants and the Z's appearing in W_I belong to the Ith open string. It is clear that $n \sim O(N)$.

Each giant graviton is associated with a long column and each dual giant graviton with a long row in the Young diagrams labeling the restricted Schur polynomial. Our notation for the Young diagrams is to list row lengths. Thus a Young diagram that has two columns, one of length n_1 and the second of length n_2 with $n_2 < n_1$ is denoted $(2^{n_2}, 1^{n_1-n_2})$, while a Young diagram with two rows, one of length n_1 and one of length n_2 $(n_1 > n_2)$ is denoted (n_1, n_2).

We want to use the results of [24–26] to study correlation functions of these operators. The correlators are obtained by summing all contractions between the Zs belonging to the giants, and by grouping the open string words in pairs and summing only the planar diagrams between the fields in each pair of the open string words. To justify the planar approximation for the open string words we take $n_i \geq 0$ and $\sum_{i=1}^{L} n_i \leq O(\sqrt{N})$. For a nice careful discussion of related issues, see [54].

We can put these operators into correspondence with normalized states

$$O(R, R_1^k, R_2^k; \{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k) \leftrightarrow |R, R_1^k, R_2^k; \{n_i\}_1, \{n_i\}_2, \cdots, \{n_i\}_k\rangle \tag{2.4}$$

by using the usual state-operator correspondence available for any conformal field theory. In what follows we will mainly use the state language.

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3 Action of the dilatation operator

The one loop dilatation operator, in the SU(2) sector, is [55, 56]

$$D = -\frac{g_{\rm YM}^2}{8\pi^2}{\rm Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right]\right). \tag{3.1}$$

Our goal in this section is to review the action of this dilatation operator on the restricted Schur polynomials, which was constructed in general in [25, 26]. When we act with D on $O(R,R_1^k,R_2^k;\{n_i\}_1,\{n_i\}_2,\cdots,\{n_i\}_k)$ the derivative with respect to Y will act on a Y belonging to a specific open string word. Thus, in the large N limit we can decompose the action of D into a sum of terms, with each individual term being the action on a specific open string. If we act on a magnon belonging to the bulk of the open string word, then the only contribution comes by acting with the derivative respect to Z on a field that is immediately adjacent to the magnon. We act only on the adjacent Z fields because to capture the large N limit we should use the planar approximation for the open string word contractions. To illustrate the action on a bulk magnon, consider the operator corresponding to a single giant graviton with a single open string attached. The giant has momentum n so that R is a single column with $n+1$ boxes: $R = 1^{n+1}$. Further, $R_1^1 = R_2^1 = 1^n$. The open string has three magnons and hence we can describe the corresponding state as $|1^{n+1},1^n,1^n;\{n_1,n_2\}\rangle$. The action on the bulk magnon at large N is

$$\begin{aligned} D_{\text{bulk magnon}}|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle = \frac{g_{\rm YM}^2 N}{8\pi^2}\Big[&2|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle \\ &- |1^{n+1},1^n,1^n;\{(n_1-1),(n_2+1)\}\rangle - |1^{n+1},1^n,1^n;\{(n_1+1),(n_2-1)\}\rangle\Big]. \end{aligned} \tag{3.2}$$

If we act on a magnon which occupies either the first or last position of the open string word, we realize one of the four possibilities listed below.

1. The derivative with respect to Z acts on the Z adjacent to the Y, belonging to the open string and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the same order. None of the labels of the state change. This term has a coefficient of 1 [25, 26].

2. The derivative with respect to Z acts on the Z adjacent to the Y, belonging to the open string word and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the opposite order. In this case, a Z has moved out of the open string word and into its own slot in the restricted Schur polynomial — a hop off interaction in the terminology of [25]. In the process the Young diagrams labeling the excited giant graviton grows by a single box. If the string is attached to a giant graviton, the column the endpoint of the relevant open string belongs to inherits the extra box. If the string is attached to a dual giant graviton, the row the endpoint of the relevant open string belongs to inherits the extra box. The coefficient of this term is given by minus one times the square root of the factor associated with the open string box divided by N [25, 26]. We remind the reader that a box in row i and column j is assigned the factor $N-i+j$.

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3. The derivative with respect to Z acts on a Z belonging to the giant and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the opposite order. In this case, a Z has moved from its own slot in the restricted Schur polynomial and onto the open string word — a hop on interaction in the terminology of [25]. In the process the Young diagrams labeling the giant graviton shrinks by a single box. The details of which column/row shrinks is exactly parallel to the discussion in point 2 above. The coefficient of this term is given by minus one times the square root of the factor associated with the open string box divided by N [25, 26].

4. The derivative with respect to Z acts on a Z belonging to the giant and the coefficient of the product of derivatives with respect to Y and Z replaces these fields in the same order. This is a kissing interaction in the terminology of [25]. None of the labels of the state change. The coefficient of this term is given by the factor associated with the open string box divided by N [25, 26].

For the example we are considering the dilatation operator has the following large N action on the magnons closest to the string endpoints

$$D_{\text{first magnon}}|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle = \frac{g_{\rm YM}^2 N}{8\pi^2}\Bigg[\left(1+1-\frac{n}{N}\right)|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle$$
$$-\sqrt{1-\frac{n}{N}}\left(|1^{n+2},1^{n+1},1^{n+1};\{(n_1-1),(n_2)\}\rangle+|1^n,1^{n-1},1^{n-1};\{(n_1+1),(n_2)\}\rangle\right)\Bigg] \quad (3.3)$$

and

$$D_{\text{last magnon}}|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle = \frac{g_{\rm YM}^2 N}{8\pi^2}\Bigg[\left(1+1-\frac{n}{N}\right)|1^{n+1},1^n,1^n;\{(n_1),(n_2)\}\rangle$$
$$-\sqrt{1-\frac{n}{N}}\left(|1^{n+2},1^{n+1},1^{n+1};\{(n_1),(n_2-1)\}\rangle+|1^n,1^{n-1},1^{n-1};\{(n_1),(n_2+1)\}\rangle\right)\Bigg]. \quad (3.4)$$

There are a few points worth noting: the complete action of the dilatation operator can be read from the Young diagram labels of the operator. The factors of the boxes in the Young diagram for the endpoints of a given open string determine the action of the dilatation operator on that open string. When the labels $R_1^k \neq R_2^k$, the string end points are on different giant gravitons and the two endpoints are associated with different boxes in the Young diagram so that the action of the dilatation operator on the two boundary magnons is distinct. To determine these endpoint interactions we must go beyond the planar approximation. Notice that for a maximal giant graviton we have $n = N$. In this case, most of the boundary magnon terms in the Hamiltonian vanish and the boundary magnons are locked in place at the string endpoints. The giant graviton brane is simply supplying a Dirichlet boundary condition for the open string. For non-maximal giants, all of the boundary magnon terms are non-zero and, for example, Z fields that belong to the open string can wander into slots describing the giant. Alternatively, since the split between open string and brane is probably not very sharp, we might think that the magnons can wander from the string endpoints into the bulk of the open string. The coefficient of these hopping terms is modified by the presence of the giant graviton, so that the boundary magnons do not behave in the same way as the bulk magnons do.

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As a final example, consider a dual giant graviton which carries momentum n. In this case, R is a single row of n boxes and we have

$$D_{\text{first magnon}}|n+1,n,n;\{(n_1),(n_2)\}\rangle = \frac{g_{\rm YM}^2 N}{8\pi^2}\Big[\left(1+1+\frac{n}{N}\right)|n+1,n,n;\{(n_1),(n_2)\}\rangle$$
$$-\sqrt{1+\frac{n}{N}}\left(|n+2,n+1,n+1;\{(n_1-1),(n_2)\}\rangle+|n,n-1,n-1;\{(n_1+1),(n_2)\}\rangle\right)\Big]. \quad (3.5)$$

In the appendix B we discuss the action of the dilatation operator at two loops.

4 Large N diagonalization: asymptotic states

We are now ready to construct eigenstates of the dilatation operator. We will not construct exact large N eigenstates. Rather, we focus on states for which all magnons are well separated. From these states we can still obtain the anomalous dimensions. In section 6 we will describe how one might use these asymptotic states to construct exact eigenstates, following [3, 4]. In the absence of integrability however, this can not be carried to completion and our states are best thought of as very good approximate eigenstates.

The Zs in the open string word define a lattice on which the Ys hop. Our construction entails taking a Fourier transform on this lattice. The boundary interactions allow Zs to move onto and out of the lattice, so the lattice size is not fixed. It is not clear what the Fourier transform is, if the size of the lattice varies. The goal of this section is to deal with these complications. With each application of the one-loop dilatation operator, a single Z can enter or leave the open string word. At γ loops at most γ Zs can enter or leave. At any finite loop order (γ) the change in length $\Delta L = \gamma$ of the lattice is finite while the total length L of the lattice is $\sqrt{N}$. Thus, at large N the ratio $\frac{\Delta L}{L} \to 0$ and we can treat the lattice length as fixed. To implement this idea, we introduce the phases

$$q_a = e^{\frac{i2\pi k_a}{J}} \quad (4.1)$$

with $k_a = 0, 1, \ldots, J-1$, as well as a cut off function whose form is shown in figure 2. The eigenstate with two magnons is then given by

$$|\psi(q_1)\rangle = \sum_{m_2=0}^{n+J}\sum_{m_1=0}^{m_2} f(m_2) q_1^{m_1-m_2}|1^{n+J+m_1-m_2+1},1^{n+J+m_1-m_2},1^{n+J+m_1-m_2};\{m_2-m_1\}\rangle$$
$$+\sum_{m_1=0}^{J+m_2}\sum_{m_2=0}^{n} f(m_1)f(J-m_1+m_2)$$
$$\times\, q_1^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle\,. \quad (4.2)$$

For a detailed discussion of the construction, we refer the reader to appendix A. At large N it is now simple to show that

$$D|\psi(q_1)\rangle = 2\times\frac{Ng_{\rm YM}^2}{8\pi^2}\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_1+q_1^{-1})\right)|\psi(q_1)\rangle$$
$$= 2g^2\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_1+q_1^{-1})\right)|\psi(q_1)\rangle\,. \quad (4.3)$$

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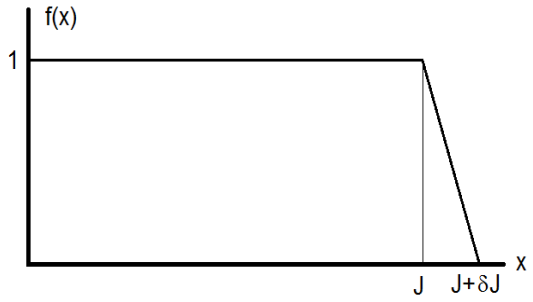

Figure 2. The cutoff function used in constructing large N eigenstates.

Since both magnons are boundary magnons, the above formula shows that boundary magnons carry momentum and it characterizes their anomalous dimension. The analysis for the dual giant graviton of momentum n leads to

$$\begin{aligned} D|\psi(q_1)\rangle &= 2 \times \frac{N g_{\rm YM}^2}{8\pi^2} \left(1 + \left[1 + \frac{n}{N}\right] - \sqrt{1 + \frac{n}{N}}(q_1 + q_1^{-1})\right) |\psi(q_1)\rangle \\ &= 2g^2 \left(1 + \left[1 + \frac{n}{N}\right] - \sqrt{1 + \frac{n}{N}}(q_1 + q_1^{-1})\right) |\psi(q_1)\rangle \,. \end{aligned} \tag{4.4}$$

For the generalizations to states with more magnons and further details, the reader should consult appendix A. This completes our discussion of the large N asymptotic eigenstates. We will now consider the dual string theory description of these states.

5 String theory description

The string theory description of the gauge theory operators is most easily developed using the limit introduced by Maldacena and Hofman [7], in which the spectrum on both sides of the correspondence simplifies. The limit considers operators of large $\mathcal{R}$ charge J and scaling dimension Δ holding $\Delta - J$ and the 't Hooft coupling λ fixed. Both sides of the correspondence enjoy an SU(2|2) × SU(2|2) supersymmetry with novel central extensions as realized by Beisert in [3, 4]. Once the central charge of the spin-chain/worldsheet excitations have been determined, their spectrum and constraints on their two body scattering are determined. A powerful conclusion argued for in [7] using the physical picture developed in [6] is that there is a natural geometric interpretation for these central charges in the classical string theory. This geometric interpretation also proved useful in the analysis of maximal giant gravitons in [5]. In this section we will argue that it is also applicable to the case of non-maximal giant and dual giant gravitons.

Giant gravitons carry a dipole moment under the RR five form flux F_5. When they move through the spacetime, the Lorentz force like coupling to F_5 causes them to expand in directions transverse to the direction in which they move [57]. The giant graviton orbits on a circle inside the S^5 and wraps an S^3 transverse to this circle but also contained in the S^5. Using the complex coordinates $x = x^5 + ix^6$, $y = x^3 + ix^4$ and $z = x^1 + ix^2$ the S^5 is described by

$$|z|^2 + |x|^2 + |y|^2 = 1 \tag{5.1}$$

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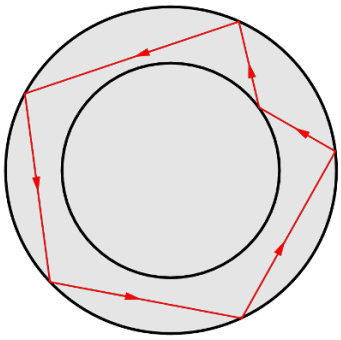

Figure 3. The giant is orbiting on the smaller circle shown. Each red segment is a magnon. The arrows in the figure simply indicate the orientation of the central charge k_i of the ith magnon. The LLM disk is shaded in this and subsequent figures. This is done to distinguish the rim of the LLM disk from the orbits of the giant gravitons.

in units with the radius of the S^5 equal to 1. The giant is orbiting in the $1-2$ plane on the circle $|z| = r$. The size to which the giant expands is determined by canceling the force causing them to expand, due to the coupling to the F_5 flux, against the D3 brane tension, which causes them to shrink. Since the coupling to the F_5 flux depends on their velocity, the size of the giant graviton is determined by its angular momentum n as [58–60]

$$|x|^2 + |y|^2 = \frac{n}{N}\,. \tag{5.2}$$

Using (5.1) we see that the giant graviton orbits on a circle of radius [58]

$$r = \sqrt{1 - \frac{n}{N}} < 1\,. \tag{5.3}$$

Consider now the worldsheet geometry for an open string attached to a giant graviton. Following [7], we will describe this worldsheet solution using LLM coordinates [9]. The worldsheet for this solution, in these coordinates, is shown in figure 3. The figure shows an open string with 6 magnons. Each magnon corresponds to a directed line segment in the figure. The first and last magnons connect to the giant which is orbiting on the smaller circle shown. Between the magnons we have a collection of $O(\sqrt{N})$ Zs. These are pushed by a centrifugal force to the circle $|z| = 1$ giving the string worldsheet the shape shown in figure 3.

In the limit that the magnons are well separated, each magnon transforms in a definite $\mathrm{SU}(2|2)^2$ representation. The open string itself transforms as the tensor product of the individual magnon representations. The representation of each individual magnon is specified by giving the values of the central charges k_i, k_i^* appearing in (1.5). Regarding the plane shown in figure 3 as the complex plane, k is given by the complex number determined by the vector describing the directed segment corresponding to the magnon. In particular, the magnitude of k is given by the length of the line corresponding to the magnon. The energy of the magnon, which transforms in a short representation, is determined by supersymmetry to be [3, 4]

$$E = \sqrt{1 + 2\lambda|k|^2} = 1 + \lambda|k|^2 - \frac{1}{2}\lambda^2|k|^4 + \dots \tag{5.4}$$

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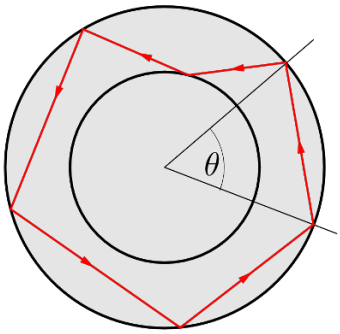

Figure 4. A bulk magnon subtending an angle θ has a length of $2\sin\frac{\theta}{2}$.

For a magnon which subtends an angle θ as shown in figure 4, we find [7]

$$E = 1 + 4\lambda \sin^2 \frac{\theta}{2} + O(\lambda^2) = 1 + \lambda(2 - e^{i\theta} - e^{-i\theta}) + O(\lambda^2)\,. \tag{5.5}$$

This is in perfect agreement with the field theory answer (A.12) if we set $\lambda = g^2$ and

$$q = e^{i\frac{2\pi k}{J}} = e^{i\theta} \quad \Rightarrow \quad \theta = \frac{2\pi k}{J}\,. \tag{5.6}$$

Thus the angle that is subtended by the magnon is equal to its momentum, which is the well known result obtained in [7]. Consider now the boundary magnon, as shown in figure 5. The circle on which the giant orbits has a radius given by

$$r = \sqrt{1 - \frac{n}{N}}\,. \tag{5.7}$$

The large circle has a radius of 1 in the units we are using. Thus, the length of the boundary magnon is given by the length of the diagonal of the isosceles trapezium shown in figure 5. Consequently

$$\begin{aligned} E &= 1 + \lambda\left((1-r)^2 + 4r\sin^2\frac{\theta}{2}\right) + O(\lambda^2) \\ &= 1 + \lambda\left(1 + r^2 - r(e^{i\theta} + e^{-i\theta})\right) + O(\lambda^2)\,. \end{aligned} \tag{5.8}$$

This is again in complete agreement with (A.12) after we set $\theta = \frac{2\pi k}{J}$ and recall that $r = \sqrt{1 - \frac{n}{N}}$. This is a convincing check of the boundary terms in the dilatation operator and of our large N asymptotic eigenstates. In the description of maximal giant gravitons, the boundary magnon always stretches from the center of the disk to a point on the circumference of the circle $|z| = 1$. Consequently, for the maximal giant the boundary magnon subtends an angle of zero and it never has a non-zero momentum. For submaximal giants we see that the boundary magnons do in general carry non-zero momentum. This is completely expected: in the case of a maximal giant graviton, the boundary magnons are locked in the first and last position of the open string lattice. As we move away from the maximal giant graviton, the coefficients of the boundary terms which allow the boundary

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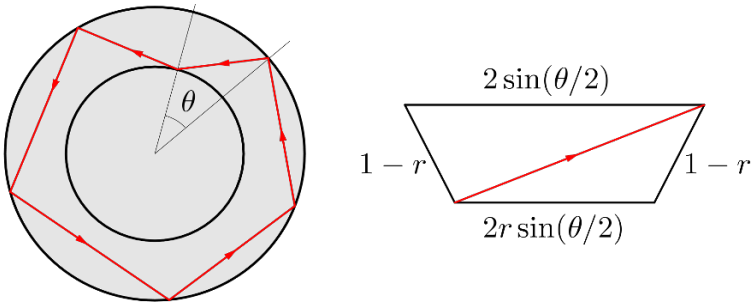

Figure 5. A boundary magnon subtending an angle θ has a length of $\sqrt{(1-r)^2 + 4r\sin^2\frac{\theta}{2}}$.

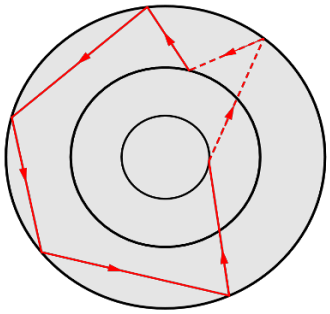

Figure 6. A two strings attached to two giant gravitons state. To distinguish the two strings, one of them has been indicated with dashed lines. Both giants are submaximal and so are moving on circles with a radius $|z| < 1$. One of the strings has only two boundary magnons. The second string has two boundary magnons and three bulk magnons. Notice that each open string has a non-vanishing central charge. It is only for the full state that the central charge vanishes. See [45] for closely related observations.

magnons to hop in the lattice, increase from zero, allowing the boundary magnons to move and hence, to carry a non-zero momentum. In the appendix B we have checked that the two loop answer in the field theory agrees with the $O(\lambda^2)$ term of (5.4).

Notice that the vector sum of the directed line segments vanishes. This is nothing but the statement that our operator vanishes unless $q_M^{-1} = q_1 q_2 \cdots q_{M-1}$. This condition ensures that although each magnon transforms in a representation of $su(2|2)^2$ with non-zero central charges, the complete state enjoys an $su(2|2)^2$ symmetry that has no central extension. It is for this reason that the central charges must sum to zero and hence that the vector sum of the red segments must vanish. This is achieved in an interesting way for certain multi-string states: each open string can transform under an $su(2|2)^2$ that has a non-zero central charge and it is only for the full state of all open strings plus giants that the central charge vanishes. An example of this for a two string state is given in figure 6.

To conclude this section, we will consider an example involving a dual giant graviton. In this case, the giant graviton orbits on a circle [59, 60]

$$r = \sqrt{1 + \frac{n}{N}} > 1\,. \tag{5.9}$$

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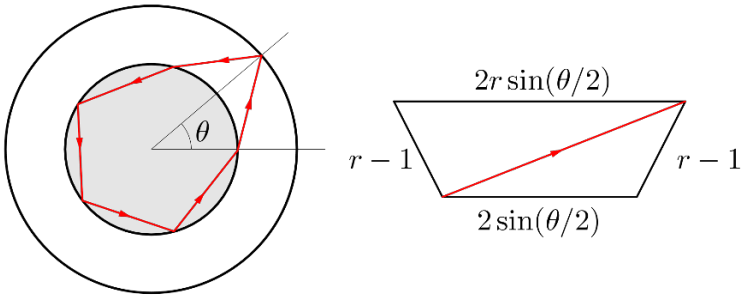

Figure 7. A boundary magnon subtending an angle θ has a length of $\sqrt{(r-1)^2 + 4r\sin^2\frac{\theta}{2}}$.

The length of the line segment corresponding to the boundary magnon is again given by the length of the diagonal of an isosceles trapezium, as shown in figure 7. Consequently

$$\begin{aligned} E &= 1 + \lambda\left((r-1)^2 + 4r\sin^2\frac{\theta}{2}\right) + O(\lambda^2) \\ &= 1 + \lambda\left(1 + r^2 - r(e^{i\theta} + e^{-i\theta})\right) + O(\lambda^2) \end{aligned} \tag{5.10}$$

which is in perfect agreement with (4.4) after we set $\theta = \frac{2\pi k}{J}$ and $r = \sqrt{1 + \frac{n}{N}}$.

6 From asymptotic states to exact eigenstates

The states we have written down above are asymptotic states in the sense that we have implicitly assumed that all of the magnons are well separated. In this case the excitations can be treated individually and the symmetry algebra acts as a tensor product representation. However, the magnons can come close together and even swap positions. When they swap positions, we get different asymptotic states that must be combined to obtain the exact eigenstate. The asymptotic states must be combined in a way that is compatible with the algebra, as explained in [3]. This requirement ultimately implies a unique way to complete the asymptotic states to obtain the exact eigenstate.

When two bulk magnons swap positions, the corresponding asymptotic states are combined using the two particle S-matrix. The relevant two particle S-matrix has been determined in [3, 4]. It is also possible for a bulk magnon to reflect/scatter off a boundary magnon. For maximal giant gravitons [5], the reflection from the boundary preserves the fact that the boundary magnon has zero momentum and it reverses the sign of the momentum of the bulk magnon. In this section we would like to investigate the scattering of a bulk magnon off a boundary magnon for a non-maximal giant graviton.

We must require that the total central charge k of the state vanishes. Thus, after the scattering the directed line segments must still sum to zero. Further the central charge C of the state must remain unchanged. Taken together, these conditions uniquely fix the momentum of both bulk and boundary magnon after the scattering.

In figure 8 the process of scattering a bulk magnon off the boundary magnon is shown. After the scattering the magnons that have a different momentum, corresponding to line

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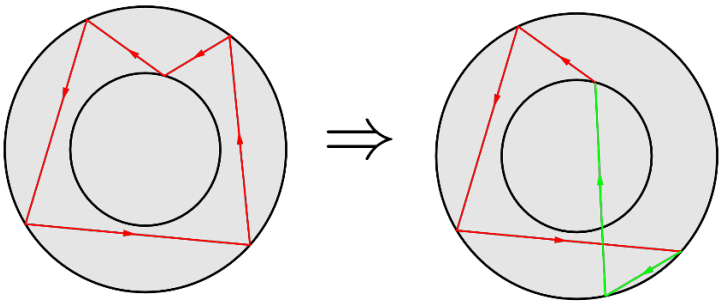

Figure 8. A bulk magnon scatters with a boundary magnon. In the process the direction of the momentum of the bulk magnon is reversed.

segments that have changed and these are shown in green. In this case the giant graviton is close enough to a maximal giant that the momentum of the boundary magnon is reversed, so this is a reflection-like scattering. Before and after the scattering the line segments line up to form a closed circuit, so that the central charge k of the state before and after scattering is zero. To analyze the constraint arising from fixing the central charge C, we parameterize the problem as shown in figure 9. There is a single parameter θ which is fixed by requiring

$$\begin{aligned}&\sqrt{1+8\lambda\sin^2\frac{\varphi_2}{2}}+\sqrt{1+8\lambda\left([1+r]^2+4r\sin^2\frac{\varphi_1}{2}\right)}\\&\quad=\sqrt{1+8\lambda\sin^2\frac{\theta}{2}}+\sqrt{1+8\lambda\left([1+r]^2+4r\sin^2\left(\frac{\varphi_1+\varphi_2+\theta}{2}\right)\right)}\end{aligned}\tag{6.1}$$

which is the condition that the state has the correct central charge C. In the above formula we have

$$r=\sqrt{1-\frac{b_0}{N}}\,.\tag{6.2}$$

The equation (6.1) has two solutions, one of which is negative $\theta=-\varphi_2$ and describes the state before the scattering. We need to choose the solution for which $\theta\neq-\varphi_2$. Notice that for $b_0=N$ this condition implies that $\theta=\varphi_2$ which is indeed the correct answer [5]. In this case, the bulk magnon reflects off the boundary with a reverse in the direction of its momentum but no change in its magnitude. The momentum of the bulk magnon remains zero. When $b_0=0$ the momenta of the two magnons is exchanged which is again the correct answer [3, 4]. When $0<b_0<N$ we find the solution to (6.1) for the momentum of the bulk magnon interpolates between reflection like scattering (when the momentum of the magnon is reversed) and magnon like scattering (when the momenta of the two magnons are exchanged). In this case though, in general, the magnitude of the momenta of the bulk and the boundary magnons are not preserved by the scattering — the scattering is inelastic.

The fact that the scattering between boundary and bulk magnons is not elastic has far reaching consequences. First, the system will not be integrable. In the case of purely elastic scattering for all magnon scatterings, the number of asymptotic states that must be combined to construct the exact energy eigenstate is roughly $(M-1)!$ for M magnons.

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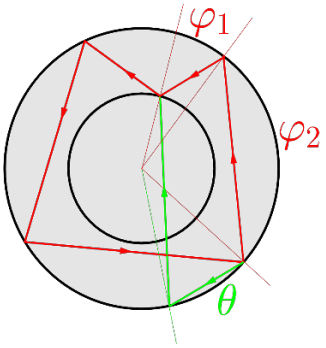

Figure 9. A bulk magnon scatters with a boundary magnon. In the process the direction of the momentum of the bulk magnon is reversed. Before the scattering the boundary magnon subtends an angle φ_1 and the bulk magnon subtends an angle φ_2. After the scattering the boundary magnon subtends an angle $\varphi_1 + \varphi_2 + \theta$ and the bulk magnon subtends an angle $-\theta$.

This is the number of ways of arranging the magnons (distinguished by their momentum) up to cyclicity. There are M magnon momenta appearing and these momenta are the same for all the asymptotic states. The exact eigenstates can then be constructed using a coordinate space Bethe ansatz. For the case of inelastic scattering, the momenta appearing depend on the specific asymptotic state one considers and there are many more than $(M-1)!$ asymptotic states that must be combined to construct the exact eigenstate. In this case constructing the exact eigenstates from the asymptotic states appears to be a formidable problem.

7 *S*-matrix and boundary reflection matrix

We have a good understanding of the symmetries of the theory and the representations under which the states transform. Following Beisert [3, 4], this is all that is needed to obtain the magnon scattering matrix. In this section we will carry out this analysis.

Each magnon transforms under a centrally extended representation of the SU(2|2) algebra

$$\{Q_a^\alpha, Q_b^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\frac{k_i}{2}\,, \qquad \{S_\alpha^a, S_\beta^b\} = \epsilon^{ab}\epsilon_{\alpha\beta}\frac{k_i^*}{2}\,, \tag{7.1}$$

$$\{S_\alpha^a, Q_b^\beta\} = \delta_b^a L_\alpha^\beta + \delta_\alpha^\beta R_b^a + \delta_b^a\delta_\alpha^\beta C_i\,. \tag{7.2}$$

There are also the usual commutators for the bosonic $su(2)$ generators. There are three central charges k_i, k_i^*, C_i for each SU(2|2) factor. Following [5] we set the central charges of the two copies to be equal. It is useful to review how the bosonic part of the $\mathrm{SU}(2|2)^2$ symmetry acts in the gauge theory. $\mathcal{N}=4$ super Yang-Mills theory has 6 hermitian adjoint scalars ϕ^i that transform as a vector of SO(6). We have combined them into the complex fields as follows

$$\begin{aligned} X &= \phi^1 + i\phi^2\,, & \bar{X} &= \phi^1 - i\phi^2\,,\\ Y &= \phi^3 + i\phi^4\,, & \bar{Y} &= \phi^3 - i\phi^4\,,\\ Z &= \phi^5 + i\phi^6\,, & \bar{Z} &= \phi^5 - i\phi^6\,. \end{aligned} \tag{7.3}$$

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The bosonic subgroup of $SU(2|2)^2$ is $SU(2) \times SU(2) = SO(4)$ that rotates $\phi^1, \phi^2, \phi^3, \phi^4$ as a vector. In terms of complex fields, Y, X and $\bar{Y}, \bar{X}$ transform under different SU(2|2) groups. $Z, \bar{Z}$ do not transform. To specify the representation that each magnon transforms in, following [3, 4] we specify parameters a_k, b_k, c_k, d_k for each magnon, where

$$Q_a^\alpha|\phi^b\rangle = a_k \delta_a^b |\psi^\alpha\rangle\,, \qquad Q_a^\alpha|\psi^\beta\rangle = b_k \epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b\rangle\,, \tag{7.4}$$

$$S_\alpha^a|\phi^b\rangle = c_k \epsilon_{\alpha\beta}\epsilon^{ab}|\psi^\beta\rangle\,, \qquad S_\alpha^a|\psi^\beta\rangle = d_k \delta_\alpha^\beta|\phi^a\rangle\,, \tag{7.5}$$

for the kth magnon. We are using the non-local notation of [4]. Using the representation introduced above

$$Q_1^1 Q_2^2|\phi^2\rangle = a_k Q_1^1|\psi^2\rangle = b_k a_k \epsilon^{12}\epsilon_{12}|\phi^2\rangle\,, \qquad Q_2^2 Q_1^1|\phi^2\rangle = 0\,, \tag{7.6}$$

so that $k_k = 2\,a_k\,b_k$. An identical argument using the S_α^a supercharges gives $k_k^* = 2\,c_k\,d_k$. Consider next a state with a total of K magnons. If we are to obtain a representation without central extension, we must require that the central charges vanish

$$\begin{aligned} \frac{k}{2} &= \sum_{k=1}^K \frac{k_k}{2} = \sum_{k=1}^K a_k b_k = 0\,, \\ \frac{k^*}{2} &= \sum_{k=1}^K \frac{k_k^*}{2} = \sum_{k=1}^K c_k d_k = 0\,. \end{aligned} \tag{7.7}$$

To obtain a formula for the central charge C consider

$$Q_a^\alpha S_\beta^b|\phi^c\rangle = c_k Q_a^\alpha \epsilon^{bc}\epsilon_{\beta\gamma}|\psi^\gamma\rangle = c_k b_k \epsilon^{bc}\epsilon_{\beta\gamma}\epsilon^{\alpha\gamma}\epsilon_{ad}|\phi^d\rangle\,. \tag{7.8}$$

Now set $a = b$ and $\alpha = \beta$ and sum over both indices to obtain

$$Q_a^\alpha S_\alpha^a|\phi^c\rangle = 2 b_k c_k |\phi^c\rangle\,. \tag{7.9}$$

Very similar manipulations show that

$$S_\alpha^a Q_a^\alpha|\phi^c\rangle = 2 a_k d_k |\phi^c\rangle \tag{7.10}$$

so that we learn the value of the central charge C_k

$$\{Q_a^\alpha, S_\alpha^a\}|\phi^c\rangle = 4C|\phi^c\rangle = 2(a_k d_k + b_k c_k)|\phi^c\rangle\,, \quad \Rightarrow \quad C_k = \frac{1}{2}(a_k d_k + b_k c_k)\,. \tag{7.11}$$

Using

$$\{S_2^1, Q_1^1\} = L_2^1 \qquad L_2^1|\psi^2\rangle = |\psi^1\rangle \tag{7.12}$$

we easily find

$$\{S_2^1, Q_1^1\}|\psi^2\rangle = (a_k d_k - b_k c_k)|\psi^1\rangle \qquad \Rightarrow \qquad a_k d_k - b_k c_k = 1\,. \tag{7.13}$$

This is also the condition to get an atypical representation of $su(2|2)$ [4].

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Following [3], a useful parametrization for the parameters of the representation is given by

$$a_k = \sqrt{g}\eta_k\,, \qquad b_k = \frac{\sqrt{g}}{\eta_k} f_k \left(1 - \frac{x_k^+}{x_k^-}\right)\,, \tag{7.14}$$

$$c_k = \frac{\sqrt{g} i \eta_k}{f_k x_k^+}\,, \qquad d_k = \frac{\sqrt{g} x_k^+}{i\eta_k}\left(1 - \frac{x_k^-}{x_k^+}\right)\,. \tag{7.15}$$

The parameters $x_k^\pm$ are set by the momentum p_k of the magnon

$$e^{i\frac{2\pi p_k}{J}} = \frac{x_k^+}{x_k^-}\,. \tag{7.16}$$

The parameter f_k is a pure phase, given by the product $\prod_j e^{ip_j}$, where j runs over all magnons to the left of the magnon considered. To ensure unitarity $|\eta_k|^2 = i(x_k^- - x_k^+)$. The condition $a_k d_k - b_k c_k = 1$ to get an atypical representation implies that

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}\,. \tag{7.17}$$

This equation will be very useful in verifying some of the S-matrix formulas given below. A useful parametrization for the parameters specifying the representation for a boundary magnon is given by

$$a_k = \sqrt{g}\eta_k\,, \qquad b_k = \frac{\sqrt{g}}{\eta_k} f_k \left(1 - r\frac{x_k^+}{x_k^-}\right)\,, \tag{7.18}$$

$$c_k = \frac{\sqrt{g} i \eta_k}{f_k x_k^+}\,, \qquad d_k = \frac{\sqrt{g} x_k^+}{i\eta_k}\left(1 - r\frac{x_k^-}{x_k^+}\right)\,, \tag{7.19}$$

where $r = \sqrt{1 - \frac{n}{N}}$ is the radius of the path on which the giant graviton of momentum n orbits[2] and the parameters $x_k^\pm$ are again set by the momentum carried by the boundary magnon according to (7.16). For the boundary magnon, f_k is again a phase as described above and now $|\eta_k|^2 = i(rx_k^- - x_k^+)$. For a maximal giant graviton $r = 0$ and the boundary magnon carries no momentum and $|\eta_k|^2 = -ix_k^+$. For the boundary magnon, the condition $a_k d_k - b_k c_k = 1$ to get an atypical representation implies that

$$x_k^+ + \frac{1}{x_k^+} - rx_k^- - \frac{r}{x_k^-} = \frac{i}{g} \tag{7.20}$$

This equation will again be useful below. Equation (7.20) interpolates between (7.17) for $r = 1$, which is the correct condition for a bulk magnon and the condition obtained for $r = 0$

$$x_k^+ + \frac{1}{x_k^+} = \frac{i}{g} \tag{7.21}$$

which was used in [5] for the boundary magnon attached to a maximal giant graviton.

[2]For an open string attached to a dual giant graviton, we would have $r = \sqrt{1 + \frac{n}{N}}$ where n is the momentum of the dual giant graviton.

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Following [3, 4] one can check that the above parametrization obeys (7.7). Finally,

$$\begin{aligned} a_k b_k c_k d_k &= g^2(e^{-ip_k}-1)(e^{ip_k}-1) = 4g^2\sin^2\frac{p_k}{2} \\ &= \frac{1}{4}\Big[(a_k d_k + b_k c_k)^2 - (a_k d_k - b_k c_k)^2\Big] = \frac{1}{4}\Big[(2C_k)^2 - 1\Big] \end{aligned} \tag{7.22}$$

so that

$$C_k = \pm\sqrt{\frac{1}{4} + 4g^2\sin^2\frac{p_k}{2}}\,. \tag{7.23}$$

The components of an energy eigenstate in different asymptotic regions are related by the bulk-bulk and boundary-bulk magnon scattering matrices S and R. S and R must commute with the $su(2|2)$ group. The labels of the representations of individual magnons can change under the scattering but they must do so in a way that preserves the central charges of the total state. In the picture of the energy eigenstates provided by the LLM plane, the central charges are given by the directed line segments (which are vectors and hence can also be viewed as complex numbers), one for each magnon. The fact that these line segments close into polygons is the statement that the central charges k and k^* of our total state vanishes. The sum of the lengths squared of these line segments determines the central charge C. By scattering these segments can rearrange themselves as long as the sums $\sum_i \sqrt{1+2\lambda l_i^2}$ with l_i the length of segment i is preserved and so long as they still form a closed polygon.

Implementing the consequences of invariance under $\mathrm{SU}(2|2)^2$ is exactly parallel to the analysis of [3–5]. For the S-matrix describing the scattering of two bulk magnons, the reader is referred to [3, 4]. When considering the equations for the reflection/scattering matrix describing the reflection/scattering of a bulk magnon from a boundary magnon, we need to pay attention to the fact that the central charges of the representation are no longer swapped between the two magnons. Rather, the central charges after the reflection are determined by solving (6.1). Denote the central charge of the boundary magnon before the reflection by p_B. Denote the central charge of the bulk magnon before the reflection by p_b. Denote the central charge of the boundary magnon after the reflection by k_B. Denote the central charge of the bulk magnon after the reflection by k_b. Denote the reflection/scattering matrix by $\mathcal{R}$. Since the S-matrix has to commute with the bosonic $su(2)$ generators Schur's Lemma implies that it must be proportional to the identity in each given irreducible representation of $su(2)$. This immediately implies that

$$\mathcal{R}|\phi^a_{p_B}\phi^b_{p_b}\rangle = A^R_{12}|\phi^{\{a}_{k_B}\phi^{b\}}_{k_b}\rangle + B^R_{12}|\phi^{[a}_{k_B}\phi^{b]}_{k_b}\rangle + \frac{1}{2}C^R_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\alpha_{k_B}\psi^\beta_{k_b}\rangle \tag{7.24}$$

$$\mathcal{R}|\psi^\alpha_{p_B}\psi^\beta_{p_b}\rangle = D^R_{12}|\psi^{\{\alpha}_{k_B}\psi^{\beta\}}_{k_b}\rangle + E^R_{12}|\psi^{[\alpha}_{k_B}\psi^{\beta]}_{k_b}\rangle + \frac{1}{2}F^R_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\phi^a_{k_B}\phi^b_{k_b}\rangle \tag{7.25}$$

$$\begin{aligned} \mathcal{R}|\phi^a_{p_B}\psi^\beta_{p_b}\rangle &= G^R_{12}|\psi^\beta_{k_B}\phi^a_{k_b}\rangle + H^R_{12}|\phi^a_{k_B}\psi^\beta_{k_b}\rangle \\ \mathcal{R}|\psi^\alpha_{p_B}\phi^b_{p_b}\rangle &= K^R_{12}|\psi^\alpha_{k_B}\phi^b_{k_b}\rangle + L^R_{12}|\phi^b_{k_B}\psi^\alpha_{k_b}\rangle\,. \end{aligned} \tag{7.26}$$

The analysis now proceeds as in [3, 4]. Demanding the S-matrix commutes with the

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supercharges implies

$$
\begin{aligned}
A_{12}^R &= S_0^{12}\frac{\eta_1\eta_2 x_1'^+ x_1^+(x_1^- - x_2^+)\left((x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-)x_2^+ + (x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)x_2'^+\right)}{\eta_1'\eta_2' x_2'^+ x_2^+(x_1^- - x_1^+)(x_1^+ - x_1'^+)(x_1^+(rx_2^+ - x_2^-) + x_2^-(rx_2^- - x_2^+))}\\
B_{12}^R &= A_{12}^R\left[1 + \frac{2x_2'^-(x_1'^- - x_1'^+)}{x_1'^+(x_1^- - x_2^+)(x_1'^- x_2'^- - rx_1'^+ x_2'^+)}\frac{B_1}{B_2}\right]\\
B_1 &= x_2^- x_1'^+\Big[(x_1^- - x_1^+)(2x_1^- - x_1'^-)(x_2^+ x_1'^+ - x_1^+ x_2^+) - x_1'^+ x_1^-(x_2^+ - rx_2^-)(x_1^- - x_2^+)\Big]\frac{rx_2'^+ - x_2'^-}{rx_2'^- - x_2'^+}\\
&\quad + \Big[x_1^+ x_1'^+(x_1^- - x_2^+)(x_2^- - rx_2^+) + (x_1^- - x_1^+)x_2^- x_2^+(x_1'^+ - x_1^+)\Big]x_1'^- x_2'^-\\
B_2 &= (rx_2^- - x_2^+)\left[x_1^+ x_2'^- x_1'^- \frac{rx_2^+ - x_2^-}{rx_2^- - x_2^+} - x_1'^+ x_1^- x_2^- \frac{rx_2'^+ - x_2'^-}{rx_2'^- - x_2'^+}\right]\\
C_{12}^R &= S_{12}^0\frac{2\eta_2\eta_1 C_1}{fx_2^+(x_1^+ - x_1'^+)(x_1^+(rx_2^+ - x_2^-) + x_2^-(rx_2^- - x_2^+))(x_1'^- x_2'^- - rx_1'^+ x_2'^+)}\\
C_1 &= x_1'^+\frac{x_1^- - x_2^+}{x_1^- - x_1^+}\Big(x_1'^+ x_1^- x_2^-(x_2^+ - rx_2^-)(rx_2'^+ - x_2'^-) + x_1^+ x_1'^- x_2'^-(x_2^- - rx_2^+)(x_2'^+ - rx_2'^-)\Big)\\
&\quad + x_2^- x_2^+(x_1^+ - x_1'^+)\Big(x_1^-(rx_1'^+ x_2'^+ + x_1'^- x_2'^- - 2x_1'^+ x_2'^-) + x_1'^- x_2'^-(rx_2'^- - x_1'^- + x_1'^+ - x_2'^+)\Big)\\
D_{12}^R &= -S_{12}^0\\
E_{12}^R &= -S_{12}^0\left[1 - 2x_1^+ x_2'^- \frac{\frac{x_1'^-}{x_1^-}(x_1'^- - x_1'^+ + x_2'^+ - rx_2'^-) - (x_1'^- - x_1'^+) - \frac{x_1'^+ x_2^-}{x_1^+ x_2'^-}\frac{x_2^+ - rx_2^-}{x_2^- - rx_2^+}(x_2'^- - rx_2'^+)}{\left[x_1^+ + x_2^-\frac{x_2^+ - rx_2^-}{x_2^- - rx_2^+}\right]\left[rx_1^{+\prime}x_2^{+\prime} - x_1^{-\prime}x_2^{-\prime}\right]}\right]\\
F_{12}^R &= S_{12}^0\frac{2x_1^+ x_1'^+ f(x_1^{-\prime} - x_1^{+\prime})(x_2'^- - rx_2'^+)(x_2^- - rx_2^+)}{\eta_1'\eta_2' x_1^- x_1^{-\prime}\left[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\right]\left[x_1^{-\prime}x_2^{-\prime} - rx_1^{+\prime}x_2^{+\prime}\right]}\\
&\quad \times\left[x_1^- - x_1'^- + \frac{rx_2^- - x_2^+}{x_2^- - rx_2^+}\frac{x_2^- x_1^-}{x_1^+} + \frac{x_2'^+ - rx_2'^-}{x_2'^- - rx_2'^+}\frac{x_1'^- x_2'^-}{x_1'^+}\right]\\
G_{12}^R &= S_{12}^0\frac{\eta_1 x_1^+\Big[x_2^+(rx_2^- - x_2^+)(rx_2'^+ - x_2'^-) + x_2'^+(rx_2^+ - x_2^-)(x_2'^+ - rx_2'^-)\Big]}{\eta_2' x_2'^+(x_1^- - x_1^+)\Big[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\Big]}\\
H_{12}^R &= S_{12}^0\frac{\eta_1(x_1^{-\prime} - x_1^{+\prime})\Big[x_1^- x_2^-(rx_2^- - x_2^+) + x_1^+ x_1^{-\prime}(rx_2^+ - x_2^-)\Big]}{\eta_1' x_1^{-\prime}(x_1^- - x_1^+)\Big[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\Big]}\\
K_{12}^R &= S_{12}^0\frac{\eta_2 x_2^-\Big[x_1^- x_1^{+\prime}(rx_2^{+\prime} - x_2^{-\prime}) + x_1^{-\prime}x_2^{-\prime}(rx_2^{-\prime} - x_2^{+\prime})\Big]}{\eta_2' x_1^{-\prime}x_2^{-\prime}\Big[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\Big]}\\
L_{12}^R &= S_{12}^0\frac{\eta_2 x_2^-(x_1^- - x_1^{-\prime})(x_1^{-\prime} - x_1^{+\prime})}{\eta_1' x_1^{-\prime}\Big[x_1^+(x_2^- - rx_2^+) + x_2^-(x_2^+ - rx_2^-)\Big]}
\end{aligned}
\tag{7.27}
$$

where

$$
\frac{x_1^+}{x_1^-} = e^{ip_b} \qquad \frac{x_2^+}{x_2^-} = e^{ip_B}\,, \tag{7.28}
$$

$$
\frac{x_{1'}^+}{x_{1'}^-} = e^{ik_b} \qquad \frac{x_{2'}^+}{x_{2'}^-} = e^{ik_B}\,. \tag{7.29}
$$

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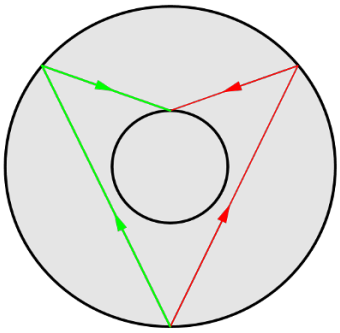

Figure 10. A bulk magnon scatters with a boundary magnon. The sum of the momenta of the two magnons is π. Here we only show two of the magnons; we indicate them in red before the scattering and in green after the scattering. In the process the direction of the momentum of both magnons is reversed.

Thus, the S-matrix is determined up to an overall phase. Here we have simply chosen $D_{12} = -S^0_{12}$ which specifies the overall phase. This overall phase is constrained by crossing symmetry [61]. It is simple to verify that this R matrix is unitary for any value of r and any momenta, and further that it reproduces the bulk S matrix for $r = 1$ and the reflection matrix for scattering from a maximal giant graviton for $r = 0$. In performing this check we compared to the expressions in [62]. To provide a further check of these expressions, we have considered the case that the boundary and the bulk magnons have momenta that sum to π, as shown in figure 10. In this situation it is very simple to compute the final momenta of the two magnons — the final momenta are minus the initial momenta. In appendix E we have computed the value of $\frac{1}{2}\left(1 + \frac{B^R_{12}}{A^R_{12}}\right)$ at one loop. We find this agrees perfectly with the answer obtained from (7.27). To perform this check, one needs to express $x^\pm$ in terms of p by solving $x^+ = x^- e^{ip}$ and (7.20) for the boundary magnon or (7.17) for the bulk magnon. Doing this we find

$$x^- = e^{-i\frac{p}{2}}\left(\frac{1}{2g\sin\frac{p}{2}} + 2g\sin\frac{p}{2}\right) + O(g^2), \tag{7.30}$$

for a bulk magnon and

$$x^- = -\frac{i}{g(r - e^{ip})} + ige^{-ip}(r - e^{ip})\frac{re^{ip} - 1}{r + e^{ip}} + O(g^2) \tag{7.31}$$

for a boundary magnon. Inserting these expansions into (7.27) and keeping only the leading order (which is g^0) at small g, we reproduce (E.13) for any allowed value of r.

It is a simple matter to verify that the boundary Yang-Baxter equation is not satisfied by this reflection matrix, indicating that the system is not integrable. This conclusion follows immediately upon verifying that changing the order in which the bulk magnons scatter with the boundary magnon leads to final states in which the magnons have different momenta. Consequently, the integrability is lost precisely because the scattering of the boundary and bulk magnons, for boundary magnons attached to a non-maximal giant graviton, is inelastic.

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8 Links to the double coset Ansatz and open spring theory

There is an interesting limiting case that we can consider, obtained by taking each open string word to simply be a single Y, i.e. each open string is a single magnon. In this case one must use the correlators computed in [27, 28] as opposed to the correlators computed in [24]. The case with distinguishable open strings is much simpler since when the correlators are computed, only contractions between corresponding open strings contribute; when the open strings are identical, it is possible to contract any two of them. In this case one must consider operators that treat these "open strings" symmetrically, leading to the operators constructed in [27]. In a specific limit, the action of the dilatation operator factors into an action on the Zs and an action on the Ys [31, 32]. The action on the Ys can be diagonalized by Fourier transforming to a double coset which describes how the magnons are attached to the giant gravitons [32, 33]. For an operator labeled by a Young diagram R with p long rows or columns, the action on the Zs then reduces to the motion of p particles along the real line with their coordinates given by the lengths of the Young diagram R, interacting through quadratic pair-wise interaction potentials [34]. For interesting related work see [63]. Our goal in this section is to explain the string theory interpretation of these results.

The conclusion of [32, 33] is that eigenstates of the dilatation operator given by operators corresponding to Young diagrams R that have p long rows or columns can be labeled by a graph with p vertices and directed edges. The number of directed edges matches the number of magnons Y used to construct the operator. These graphs have a natural interpretation in terms of the Gauss Law expected from the worldvolume theory of the giant graviton branes [23]. Since the giant graviton has a compact world volume, the Gauss Law implies the total charge on the giant's world volume vanishes. Each string end point is charged, so this is a constraint on the possible open string configurations: the number of strings emanating from the giant must equal the number of strings terminating on the giant. Thus, the graphs labeling the operators are simply enumerating the states consistent with the Gauss Law. To stress this connection we use the language "Gauss graphs" for the labels, we refer to the vertices of the graph as branes since each one is a giant graviton brane and we identify the directed edges as strings since each is a magnon. The action of the dilatation operator is nicely summarized by the Gauss graph labeling the operator. Count the number n_{ij} of strings (of either orientation) stretching between branes i and j in the Gauss graph. The action of the dilatation operator on the Gauss graph operator is then given by

$$DO_{R,r}(\sigma) = -\frac{g_{\rm YM}^2}{8\pi^2}\sum_{i<j} n_{ij}(\sigma)\Delta_{ij}O_{R,r}(\sigma)\,. \tag{8.1}$$

The operator Δ_{ij} is defined in appendix D. For a proof of this, see [32, 33]. To obtain anomalous dimensions one needs to solve an eigenproblem on the R, r labels, which has been accomplished in [34] in complete generality.

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For three open strings stretched between three giant gravitons we have to solve the following eigenvalue problem

$$\begin{aligned}
&\frac{g_{\rm YM}^2}{8\pi^2}\Big[(2N-c_1-c_2+3)O(c_1,c_2,c_3)-\sqrt{(N-c_1+1)(N-c_2+1)}O(c_1+1,c_2-1,c_3)\\
&\quad-\sqrt{(N-c_1)(N-c_2+2)}O(c_1-1,c_2+1,c_3)\Big]\\
&\quad+\frac{g_{\rm YM}^2}{8\pi^2}\Big[(2N-c_2-c_3+5)O(c_1,c_2,c_3)-\sqrt{(N-c_2+1)(N-c_3+3)}O(c_1,c_2-1,c_3+1)\\
&\quad-\sqrt{(N-c_2+2)(N-c_3+2)}O(c_1,c_2+1,c_3-1)\Big]\\
&\quad+\frac{g_{\rm YM}^2}{8\pi^2}\Big[(2N-c_1-c_3+4)O(c_1,c_2,c_3)-\sqrt{(N-c_3+2)(N-c_1+1)}O(c_1+1,c_2,c_3-1)\\
&\quad-\sqrt{(N-c_3+3)(N-c_1)}O(c_1-1,c_2,c_3+1)\Big]\\
&\qquad=\gamma O(c_1,c_2,c_3)
\end{aligned}\tag{8.2}$$

where c_1, c_2 and c_3 are the lengths of the columns = momenta of the three giant gravitons and γ is the anomalous dimension. At large N, approximating for example $O(c_1,c_2,c_3)=O(c_1+1,c_2,c_3-1)$ which amounts to ignoring back reaction on the giant gravitons, we have

$$\begin{aligned}
&\frac{g_{\rm YM}^2N}{8\pi^2}\left[\sqrt{1-\frac{c_1}{N}}-\sqrt{1-\frac{c_2}{N}}\right]^2O(c_1,c_2,c_3)+\frac{g_{\rm YM}^2N}{8\pi^2}\left[\sqrt{1-\frac{c_2}{N}}-\sqrt{1-\frac{c_3}{N}}\right]^2O(c_1,c_2,c_3)\\
&\quad+\frac{g_{\rm YM}^2N}{8\pi^2}\left[\sqrt{1-\frac{c_3}{N}}-\sqrt{1-\frac{c_1}{N}}\right]^2O(c_1,c_2,c_3)=\gamma O(c_1,c_2,c_3)\,.
\end{aligned}\tag{8.3}$$

The Gauss graph associated with this operator has a string stretching between the brane of momentum c_1 and the brane of momentum c_3, a string stretching between the brane of momentum c_1 and the brane of momentum c_2 and a string stretching between the brane of momentum c_2 and the brane of momentum c_3.

On the string theory side, since our magnons don't carry any momentum, we have three giants moving in the plane with magnons stretched radially between them. Identifying the central charges, we find they are radial vectors with length equal to the distance between the giants. With these central charges we can write down the energy

$$E=\sqrt{1+2\lambda(r_1-r_2)^2}+\sqrt{1+2\lambda(r_1-r_3)^2}+\sqrt{1+2\lambda(r_3-r_2)^2}\,.\tag{8.4}$$

Using the usual translation between the momentum of the giant graviton and the radius of the circle it moves on

$$r_i=\sqrt{1-\frac{c_i}{N}}\qquad i=1,2,3\tag{8.5}$$

we find that the order λ term in the expansion of (8.4) precisely matches the gauge theory result (8.3).

If we don't ignore back reaction on the giant graviton, we find that (8.2) leads to a harmonic oscillator eigenvalue problem. In this case, we are keeping track of the Zs slipping past a magnon, from one giant onto the next. In this way, one of the giants will grow and one will shrink thereby changing the radius of their orbits and hence the length

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of the magnon stretched between them. In this process we would expect the energy to vary continuously, which is exactly what we see at large N. A specific harmonic oscillator state (see [34] for details) corresponds to two giant gravitons executing a periodic motion. In one period, the giants first come towards each other and then move away from each other again. Exciting these oscillators to any finite level, we find an energy that is of order the 't Hooft coupling divided by N. These very small energies translate into motions with a huge period.

There is an important point worth noting. The harmonic oscillator problem that arises from (8.2) is obtained by expanding (8.2) assuming that $c_1 - c_2$ is order $\sqrt{N}$ and c_1, c_2 are of order N. The oscillator Hamiltonian then arises as a consequence of (and depends sensitively on) the order 1 shifts in the coefficients of the terms in (8.2). Thus to really trust the oscillator Hamiltonian we find we must be sure that (8.2) is accurate enough that we can expand it and the order 1 term we obtain is accurate. This is indeed the case, as we discuss in appendix D.

9 Conclusions

In this study we have used the descriptions of the action of the dilatation operator derived using an approach which relies heavily on group representation theory techniques, to study the anomalous dimensions of operators with a bare dimension that grows as N, as the large N limit is taken. For these operators, even just to capture the leading large N limit, we are forced to sum much more than just the planar diagrams and this is precisely what the representation theoretic approach manages to do. We have demonstrated an exact agreement with results coming from the dual gravity description, which is convincing evidence in support of this approach. It gives definite correct results in a systematic large N expansion, demonstrating that the representation theoretic methods provide a useful language and calculational framework with which to tackle the kinds of large N but non-planar limits we have studied in this article. Of course, we have mainly investigated the leading large N limit and the computation of $\frac{1}{N}$ corrections is an interesting problem that we hope to return to in the future.

The progress that was made in understanding the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory is impressive (see [2] for a comprehensive review). Of course, much of the progress is thanks to integrability. There are however results that do not rely on integrability, only on the symmetries of the theory. In our study we clearly have a genuine extension of methods (giant magnons, the SU(2|2) scattering matrix) that worked in the planar limit, into the large N but non-planar setting. Further, even though integrability does not persist, it is present when the radius r of the circle on which the graviton moves is $r = 0$ (maximal giant graviton) or $r = 1$ (point-like giant graviton). If we perturb about these two values of r, we are departing from integrability in a controlled way and hence we might still be able to exploit integrability. For more general values of r, we have managed to find asymptotic eigenstates in which the magnons are well separated and we expect these to be very good approximate eigenstates. Indeed, anomalous dimensions computed using these asymptotic eigenstates exactly agree with the dual string theory energies. Without

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the power of integrability it does not seem to be easy to patch together asymptotic states to obtain exact eigenstates.

We have a clearer understanding of the non-planar integrability discovered in [29–34]. The magnons in these systems remain separated and hence free, so they are actually non-interacting. One of the giants would need to lose all of its momentum before any two magnons would scatter. It is satisfying that the gauge theory methods based on group representation theory are powerful enough to detect this integrability directly in the field theory. The results we have found here give the all loops prediction for the anomalous dimensions of these operators. In the limit when we consider a very large number of fields there would seem to be many more circumstances in which one could construct operators that are ultimately dual to free systems. This is an interesting avenue that deserves careful study, since these simple free systems may provide convenient starting points, to which interactions may be added systematically.

A possible instability associated to open strings attached to giants has been pointed out in [51]. In this case it seems that the spectrum of the spin chain becomes continuous, the ground state is no longer BPS and supersymmetry is broken. The transition that removes the BPS state is simply that the gap from the ground state to the continuum closes. Of course, the spectrum of energies is discrete but this is only evident at subleading orders in $1/N$ when one accounts for the back reaction of the giant graviton-branes. The question of whether these BPS states with given quantum numbers exist or not has been linked to a walls of stability type description [64] in [45]. It would be interesting to see if these issues can be understood using the methods of this article.

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A Large N eigenstates

In section 4 we explained that at any finite loop order (γ) the change in length $\Delta L = \gamma$ of the open string word lattice is finite while the total length L of the lattice is $\sqrt{N}$. This implies that at large N the ratio $\frac{\Delta L}{L} \to 0$ and we can treat the lattice length as fixed. This observation is most easily used by first introducing "simple states" that have a definite number of Zs, in the lattice associated to each open string. This is accomplished by relaxing the identification of the open string word with the lattice. The dilatation operator's action now allows magnons to move off the open string, mixing simple states with states that are not simple. However, by modifying these simple states we can build states that are closed under the action of the dilatation operator. Our simple states are

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defined by taking a "Fourier transform" of the states (2.4). The simplest system to consider is that of a single giant, with a single string attached, excited by only two magnons (i.e. only boundary magnons — no bulk magnons). The string word is composed using J Z fields and the complete operator using $J+n$ Zs. Introduce the phases

$$q_a = e^{\frac{i2\pi k_a}{J}} \tag{A.1}$$

with $k_a = 0, 1, \ldots, J-1$. As a consequence of the fact that the lattice is a discrete structure, momenta are quantized with the momentum spacing set by the inverse of the total lattice size. This explains the choice of phases in (A.1). The simple states we consider are thus given by

$$\begin{aligned} |q_1, q_2\rangle = &\sum_{m_1=0}^{J-1} \sum_{m_2=0}^{m_1} q_1^{m_1} q_2^{m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle \\ &+ \sum_{m_2=0}^{J-1} \sum_{m_1=0}^{m_2} q_1^{m_1} q_2^{m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2-m_1\}\rangle \,. \end{aligned} \tag{A.2}$$

This Fourier transform is a transform on the lattice describing the open string worldsheet. The two magnons sit at positions m_1 and m_2 on this lattice. If $m_2 > m_1$, there are $m_2 - m_1$ Zs between the magnons. If $m_1 > m_2$, there are $J + m_2 - m_1$ Zs between the magnons. The Zs before the first magnon of the string and after the last magnon of the string, are mixed up with the Zs of the giant — they do not sit on the open string word. All of the terms in (A.2) are states with different positions for the two magnons, but each is a giant that contains precisely n Zs with an open string attached, and the open string contains precisely J Zs. We can't distinguish where the string begins and where the giant ends: the open string and giant morph smoothly into each other. This is in contrast to the case of a maximal giant graviton, where the magnons mark the endpoints of the open string.[3] If this interpretation is consistent we must recover the expected inner product on the lattice and we do: consider a giant with momentum n. An open string with a lattice of J sites is attached to the giant. The string is excited by M magnons, at positions $n_1, \ldots, n_{M-1}$ and n_M, with $n_{j+1} > n_j$. The corresponding normalized states, denoted by $|n; J; n_1, n_2, \cdots, n_k\rangle$ will obey[4]

$$\langle n; J; n_1, m_2, \cdots, m_M | n, J, n_1, n_2, \cdots, n_M\rangle = \delta_{m_2 n_2} \cdots \delta_{m_M n_M} \quad n_{k+1} > n_k, m_{k+1} > m_k \,. \tag{A.3}$$

This is the statement that, up to the ambiguity of where the open string starts, the magnons must occupy the same sites for a non-zero overlap. It is clear that ($G(x) \equiv 1^{x+1}, 1^x, 1^x$ and again, $n_{j+1} > n_j, m_{j+1} > m_j$)

$$\langle G(n+J+m_1-m_2); \{m_2, \cdots, m_M\} | G(n+J+n_1-n_2); \{n_2, \cdots, n_M\}\rangle = \delta_{m_2 n_2} \cdots \delta_{m_k n_k}$$

[3] For the maximal giant graviton, the boundary magnons are not able to hop and so sit forever at the end of the open string. For a non-maximal giant graviton the boundary magnons can hop. Even if they are initially placed at the string endpoint, they will soon explore the bulk of the string.

[4] As a consequence of the fact that it is not possible to distinguish where the open string begins and where the giant ends, there is no delta function setting the positions of the first magnons to be equal to each other — we have put this constraint in by hand in (A.3).

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reproducing the lattice inner product. The simple states are an orthogonal set of states. To check this, compute the coefficient c_a of the state $|1^{n+a+1}, 1^{n+a}, 1^{n+a}; \{J-a\}\rangle$. Looking at the two terms in (A.2) we find the following two contributions

$$\begin{aligned} c_a &= \sum_{m_1=a}^{J-1} q_1^{m_1} q_2^{m_1-a} + \sum_{m_1=0}^{a-1} q_1^{m_1} q_2^{m_1-a} \\ &= \begin{cases} J q_2^{-a} & \text{if} \quad k_1+k_2=0 \\ 0 & \text{if} \quad k_1+k_2 \neq 0 \end{cases} . \end{aligned} \tag{A.4}$$

Thus, $q_1 = q_2^{-1}$ to get a non-zero result. We will see that this zero lattice momentum constraint maps into the constraint that the $su(2|2)$ central charges of the complete magnon state must vanish. Our simple states are then given by setting $q_2 = q_1^{-1}$ and are labeled by a single parameter q_1; denote the simple states using a subscript s as $|q_1\rangle_s$.

The asymptotic large N eigenstates are a small modification of these simple states. When we apply the dilatation operator to the simple states nothing prevents the boundary magnons from "hopping past the endpoints of the open string", so the simple states are not closed under the action of the dilatation operator. We need to relax the sharp cut off on the magnon movement, by allowing the sums that appear in (A.2) above to be unrestricted. We accomplish this by introducing a "cut off" function, shown in figure 2. In terms of this cut off function $f(\cdot)$ our eigenstates are

$$\begin{aligned} |\psi(q_1)\rangle = &\sum_{m_2=0}^{n+J} \sum_{m_1=0}^{m_2} f(m_2) q_1^{m_1-m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2-m_1\}\rangle \\ &+ \sum_{m_1=0}^{J+m_2} \sum_{m_2=0}^{n} f(m_1) f(J-m_1+m_2) \\ &\times q_1^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle . \end{aligned} \tag{A.5}$$

The dilatation operator can not arrange that the number of Zs between two magnons becomes negative. Thus, any bounds on sums in the definition of our simple states enforcing this are respected. On the other hand, the dilatation operator allows boundary magnons to hop arbitrarily far beyond the open string endpoint. Bounds in the sums for simple states enforcing this are not respected. Replace these bounds enforced as the upper limit of a sum, by bounds enforced by the cut off function. From figure 2 we see that the cut off function is defined using a parameter δJ. We require that $\frac{\delta J}{J} \to 0$ as $N \to \infty$, so that at large N the difference between these eigenstates and the simple states $|q_1\rangle_s$ vanishes, as demonstrated in appendix C. We also want to ensure that

$$f(i) = f(i+1) + \epsilon \qquad \forall i \tag{A.6}$$

with $\epsilon \to 0$ as $N \to \infty$. (A.6) is needed to ensure that we do indeed obtain an eigenstate. It is straight forward to choose a function $f(x)$ with the required properties. We could for example choose δJ to be of order $N^{\frac{1}{4}}$. Our large N answers are not sensitive to the details of the cut off function $f(x)$. When $1/N$ corrections to the eigenstates are computed $f(x)$

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may be more constrained and we may need to reconsider the precise form of the cut off function and how we implement the bounds.

It is now straight forward to verify that, at large N, we have

$$\begin{aligned} D|\psi(q_1)\rangle &= 2 \times \frac{N g_{\rm YM}^2}{8\pi^2} \left(1 + \left[1 - \frac{n}{N}\right] - \sqrt{1 - \frac{n}{N}}(q_1 + q_1^{-1})\right) |\psi(q_1)\rangle \\ &= 2g^2 \left(1 + \left[1 - \frac{n}{N}\right] - \sqrt{1 - \frac{n}{N}}(q_1 + q_1^{-1})\right) |\psi(q_1)\rangle \,. \end{aligned} \tag{A.7}$$

For the dual giant graviton of momentum n we find

$$\begin{aligned} D|\psi(q_1)\rangle &= 2 \times \frac{N g_{\rm YM}^2}{8\pi^2} \left(1 + \left[1 + \frac{n}{N}\right] - \sqrt{1 + \frac{n}{N}}(q_1 + q_1^{-1})\right) |\psi(q_1)\rangle \\ &= 2g^2 \left(1 + \left[1 + \frac{n}{N}\right] - \sqrt{1 + \frac{n}{N}}(q_1 + q_1^{-1})\right) |\psi(q_1)\rangle \,. \end{aligned} \tag{A.8}$$

The generalization to include more magnons is straight forward. We will simply consider increasingly complicated examples and for each simply quote the final results. The discussion is most easily carried out using the occupation notation. For example, the simple states corresponding to three magnons are

$$\begin{aligned} |q_1, q_2, q_3\rangle = &\sum_{n_3=0}^{J-1} \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} q_1^{n_1} q_2^{n_2} q_3^{n_3} |G(n + J + n_1 - n_3); \{(n_2 - n_1), (n_3 - n_2)\}\rangle \\ &+ \sum_{n_1=0}^{J-1} \sum_{n_3=0}^{n_1} \sum_{n_2=0}^{n_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} |G(n + n_1 - n_3); \{(J + n_2 - n_1), (n_3 - n_2)\}\rangle \\ &+ \sum_{n_2=0}^{J-1} \sum_{n_1=0}^{n_2} \sum_{n_3=0}^{n_1} q_1^{n_1} q_2^{n_2} q_3^{n_3} |G(n + n_1 - n_3); \{(n_2 - n_1), (J + n_3 - n_2)\}\rangle \end{aligned} \tag{A.9}$$

where we have again lumped together the Young diagram labels $G(x) = R, R_1^1, R_2^1 = 1^{x+1}, 1^x, 1^x$. The coefficient of the ket $|G(n + J - a - b); \{(a), (b)\}\rangle$ is given by the sum

$$\sum_{n_1=0}^{J-1} (q_1 q_2 q_3)^{n_1} q_2^a q_3^{a+b} \tag{A.10}$$

which vanishes if $k_1 + k_2 + k_3 \neq 0$. Consequently we can set $q_3 = q_1^{-1} q_2^{-1}$. Including the cut off function, our energy eigenstates are given by

$$\begin{aligned} |\psi(q_1, q_2)\rangle = &\sum_{n_3=0}^{\infty} \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} q_1^{n_1 - n_3} q_2^{n_2 - n_3} f(n_3) |G(n + J + n_1 - n_3); \{(n_2 - n_1), (n_3 - n_2)\}\rangle \\ &+ \sum_{n_1=0}^{J+n_2} \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{n_3} q_1^{n_1 - n_3} q_2^{n_2 - n_3} f(n_1) f(J + n_3 - n_1) |G(n + n_1 - n_3); \{(J + n_2 - n_1), (n_3 - n_2)\}\rangle \\ &+ \sum_{n_2=0}^{J+n_3} \sum_{n_1=0}^{n_2} \sum_{n_3=0}^{\infty} q_1^{n_1 - n_3} q_2^{n_2 - n_3} f(n_2) f(J + n_3 - n_1) |G(n + n_1 - n_3); \{(n_2 - n_1), (J + n_3 - n_2)\}\rangle. \end{aligned}$$

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It is a simple matter to see that

$$D|\psi(q_1,q_2)\rangle = (E_1+E_2+E_3)|\psi(q_1,q_2)\rangle \tag{A.11}$$

where

$$\begin{aligned}
E_1 &= g^2\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_1+q_1^{-1})\right)\\
E_2 &= g^2\left(2-q_2-q_2^{-1}\right)\\
E_3 &= g^2\left(1+\left[1-\frac{n}{N}\right]-\sqrt{1-\frac{n}{N}}(q_3+q_3^{-1})\right).
\end{aligned} \tag{A.12}$$

Now consider the extension to states containing many magnons: for an M magnon state, consider all M cyclic orderings of the "magnon positions"

$$\begin{aligned}
&n_1 \le n_2 \le n_3 \le \cdots \le n_{M-2} \le n_{M-1} \le n_M \le J-1\\
&n_M \le n_1 \le n_2 \le n_3 \le \cdots \le n_{M-2} \le n_{M-1} \le J-1\\
&n_{M-1} \le n_M \le n_1 \le n_2 \le n_3 \le \cdots \le n_{M-2} \le J-1\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&n_2 \le n_3 \le \cdots \le n_{M-2} \le n_{M-1} \le n_M \le n_1 \le J-1\,.
\end{aligned} \tag{A.13}$$

Construct the differences $\{n_2-n_1, n_3-n_2, n_4-n_3, \cdots, n_M-n_{M-1}, n_1-n_M\}$. Every difference except for one is positive. Add J to the difference that is negative, i.e. the resulting differences are $\{\Delta_2,\Delta_3,\Delta_4,\cdots,\Delta_M,\Delta_1\}$ with

$$\Delta_i = \begin{cases} n_i - n_{i-1} & \text{if} \quad n_i \ge n_{i-1} \\ \\ J + n_i - n_{i-1} & \text{if} \quad n_i \le n_{i-1} \end{cases}. \tag{A.14}$$

For each ordering in (A.13) we have a term in the simple state. This term is obtained by summing over all values of $\{n_1,n_2,\cdots,n_L\}$ consistent with the ordering considered, of the following summand

$$q_1^{n_1}q_2^{n_2}\cdots q_L^{n_M}|1^{n+\Delta_1+1},1^{n+\Delta_1},1^{n+\Delta_1};\{(\Delta_2),(\Delta_3),\cdots,(\Delta_M)\}\rangle\,. \tag{A.15}$$

Repeating the argument we outlined above, this term vanishes unless $q_M^{-1} = q_1q_2\cdots q_{M-1}$ so that the summand can be replaced by

$$q_1^{n_1-n_M}q_2^{n_2-n_M}\cdots q_{M-1}^{n_{M-1}-n_M}|1^{n+\Delta_1+1},1^{n+\Delta_1},1^{n+\Delta_1};\{(\Delta_2),(\Delta_3),\cdots,(\Delta_M)\}\rangle\,. \tag{A.16}$$

Finally, consider the extension to many string states and an arbitrary system of giant graviton branes. Each open string word is constructed as explained above. We add extra columns (one for each giant graviton) and rows (one for each dual giant graviton) to R. The labels R_1^k and R_2^k specify how the open strings are connected to the giant and dual giant gravitons. When describing twisted string states, the strings describe a closed loop, "punctuated by" the giant gravitons on which they end. As an example, consider a two

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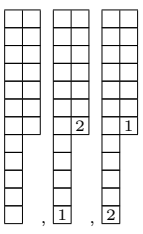
giant graviton state, with a pair of strings stretching between the giant gravitons. The two strings carry a total momentum of J. Notice that we are using the two strings to define a single lattice of J sites. One might have thought that the two strings would each define an independent lattice. To understand why we use the two strings to define a single lattice, recall that we are identifying the zero lattice momentum constraint with the constraint that the $su(2|2)$ central charges of the complete magnon state must vanish. There is a single $su(2|2)$ constraint on the two string state, not one constraint for each string. We interpret this as implying there is a single zero lattice momentum constraint for the two strings, and hence there is a single lattice for the two strings. This provides a straight forward way to satisfy the $su(2|2)$ central charge constraints. The first giant graviton has a momentum of b_0 and the second a momentum of b_1. The first string is excited by M magnons with locations $\{n_1, n_2, \cdots, n_{M-1}, n_M\}$ and the second by $\tilde{M}$ magnons with locations $\{\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_{\tilde{M}-1}, \tilde{n}_{\tilde{M}}\}$ where we have switched to the lattice notation. We need to consider the $M + \tilde{M}$ orderings of the $\{n_i\}$ and $\{\tilde{n}_i\}$. Given a specific pair of orderings, we can again form the differences

$$\begin{aligned}
\Delta_1 &= \begin{cases} n_1 - \tilde{n}_M & \text{if} \quad n_1 \geq \tilde{n}_M \\ J + n_1 - \tilde{n}_M & \text{if} \quad n_1 \leq \tilde{n}_M \end{cases} \\
\Delta_i &= \begin{cases} n_i - n_{i-1} & \text{if} \quad n_i \geq n_{i-1} \\ & \qquad\qquad i = 2, 3, \cdots, M \\ J + n_i - n_{i-1} & \text{if} \quad n_i \leq n_{i-1} \end{cases} \\
\Delta_{M+1} &= \begin{cases} \tilde{n}_1 - n_M & \text{if} \quad n_M \leq \tilde{n}_1 \\ J + \tilde{n}_1 - n_M & \text{if} \quad n_M \geq \tilde{n}_1 \end{cases} \\
\Delta_{M+i} &= \begin{cases} \tilde{n}_i - \tilde{n}_{i-1} & \text{if} \quad \tilde{n}_i \geq \tilde{n}_{i-1} \\ & \qquad\qquad i = 2, 3, \cdots, \tilde{M} \ . \\ J + \tilde{n}_i - \tilde{n}_{i-1} & \text{if} \quad \tilde{n}_i \leq \tilde{n}_{i-1} \end{cases}
\end{aligned} \tag{A.17}$$

For each ordering we again have a term in the simple state, obtained by summing over all values of $\{n_1, n_2, \cdots, n_M, \tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_{\tilde{M}}\}$ consistent with the ordering considered, of the following summand

$$\begin{aligned}
q_1^{n_1} \cdots q_M^{n_M} \tilde{q}_1^{\tilde{n}_1} \cdots \tilde{q}_{\tilde{M}}^{\tilde{n}_{\tilde{M}}} | G(\Delta_1, \Delta_{M+1}); \{(\Delta_2), (\Delta_3), \cdots, (\Delta_M)\}, \\
\{(\Delta_{M+2}), (\Delta_{M+3}), \cdots, (\Delta_{M+\tilde{M}})\}\rangle
\end{aligned} \tag{A.18}$$

where

$G(x, y) \equiv$  . (A.19)

In the first Young diagram above there are $b_1 + y + 1$ rows with 2 boxes in each row and $b_0 + x - b_1 - y - 1$ rows with 1 box in each row. Repeating the argument we outlined

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above, this term vanishes unless $\tilde{q}_{\tilde{M}}^{-1} = q_1 \cdots q_M \tilde{q}_1 \cdots \tilde{q}_{\tilde{M}-1}$ so that the summand can be replaced by

$$q_1^{n_1-\tilde{n}_{\tilde{M}}} q_2^{n_2-\tilde{n}_{\tilde{M}}} \cdots \tilde{q}_{\tilde{M}-1}^{\tilde{n}_{\tilde{M}-1}-\tilde{n}_{\tilde{M}}} |G(\Delta_1, \Delta_{M+1}); \{(\Delta_2), (\Delta_3), \cdots, (\Delta_M)\}, \{(\Delta_{M+2}), (\Delta_{M+3}), \cdots, (\Delta_{M+\tilde{M}})\}\rangle \,. \tag{A.20}$$

B Two loop computation of boundary magnon energy

The dilatation operator, in the su(2) sector, can be expanded as [55, 56]

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{\rm YM}^2}{16\pi^2}\right)^k D_{2k} = \sum_{k=0}^{\infty} g^{2k} D_{2k} \,, \tag{B.1}$$

where the tree level, one loop and two loop contributions are

$$D_0 = \mathrm{Tr}\left(Z\frac{\partial}{\partial Z}\right) + \mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right) , \tag{B.2}$$

$$D_2 = -2 : \mathrm{Tr}\left([Z,Y]\left[\frac{\partial}{\partial Z}, \frac{\partial}{\partial Y}\right]\right) : , \tag{B.3}$$

$$D_4 = D_4^{(a)} + D_4^{(b)} + D_4^{(c)} , \tag{B.4}$$

$$\begin{aligned} D_4^{(a)} &= -2 : \mathrm{Tr}\left(\left[[Y,Z], \frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], Z\right]\right) : \\ D_4^{(b)} &= -2 : \mathrm{Tr}\left(\left[[Y,Z], \frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], Y\right]\right) : \\ D_4^{(c)} &= -2 : \mathrm{Tr}\left([[Y,Z], T^a]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], T^a\right]\right) : \,. \end{aligned} \tag{B.5}$$

The boundary magnon energy we computed above came from D_2. By computing the contribution from D_4 we can compare to the second term in the expansion of the string energies. Since we are using the planar approximation when contracting fields in the open string words, in the limit of well separated magnons, the action of D_4 can again be written as a sum of terms, one for each magnon. Thus, if we compute the action of D_4 on a state $|1^{n+1}, 1^n, 1^n; \{n_1, n_2\}\}\rangle$ with a single string and a single bulk magnon, its a trivial step to obtain the action of D_4 on the most general state.

A convenient way to summarize the result is to quote the action of D_4 on a state for which the magnons have momenta q_1, q_2, q_3. Of course, we will have to choose the q_i so that the total central charge vanishes as explained in the article above. Thus we could replace $q_3 \to (q_1 q_2)^{-1}$ in the formulas below. We will write the answer for a general giant graviton system with strings attached. For the boundary terms, each boundary magnon corresponds to an end point of the string and each end point is associated with a specific box in the Young diagram. Denote the factor of the box corresponding to the first magnon by c_F and the factor of the box associated to the last magnon by c_L. A straight forward

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but somewhat lengthy computation, using the methods developed in [25, 26] gives

$$\begin{aligned}(D_4)_{\text{first magnon}}|\psi(q_1,q_2,q_3)\rangle &= -\frac{g^4}{2}\left[\left(1+\frac{c_F}{N}\right)^2 - 2\left(1+\frac{c_F}{N}\right)\sqrt{\frac{c_F}{N}}(q_1+q_1^{-1})\right.\\ &\qquad \left.+\frac{c_F}{N}(q_1^2+2+q_1^{-2})\right]|\psi(q_1,q_2,q_3)\rangle\\ &= -\frac{g^4}{2}\left[1+\frac{c_F}{N}-\sqrt{\frac{c_F}{N}}(q_1+q_1^{-1})\right]^2|\psi(q_1,q_2,q_3)\rangle\\ &= -\frac{1}{2}\left[g^2\left(1+\frac{c_F}{N}-\sqrt{\frac{c_F}{N}}(q_1+q_1^{-1})\right)\right]^2|\psi(q_1,q_2,q_3)\rangle \quad \text{(B.6)}\end{aligned}$$

in perfect agreement with (5.4). The term $D_4^{(b)}$ does not make a contribution to the action on distant magnons, since we sum only the planar open string word contractions. The remaining terms $D_4^{(a)}, D_4^{(c)}$ both make a contribution to the action on distant magnons. For completeness note that

$$(D_4)_{\text{bulk magnon}}|\psi(q_1,q_2,q_3)\rangle = -\frac{1}{2}\left[2g^2\left(2-(q_2+q_2^{-1})\right)\right]^2|\psi(q_1,q_2,q_3)\rangle\,. \quad \text{(B.7)}$$

C The difference between simple states and eigenstates vanishes at large N

In this section we want to quantify the claim made in section 4 that the difference between our simple states and our exact eigenstates vanishes in the large N limit. We will do this by computing the difference between the simple states and eigenstates and observing this difference has a norm that goes to zero in the large N limit.

For simplicity, we will consider a two magnon state. The generalization to many magnon states is straight forward. Our simple states have the form

$$\begin{aligned}|q\rangle = \mathcal{N}\left(\sum_{m_1=0}^{J-1}\sum_{m_2=0}^{m_1} q^{m_1-m_2}|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle \quad \text{(C.1)}\right.\\ \left.+\sum_{m_2=0}^{J-1}\sum_{m_1=0}^{m_2} q^{m_1-m_2}|1^{n+J+m_1-m_2+1},1^{n+J+m_1-m_2},1^{n+J+m_1-m_2};\{m_2-m_1\}\rangle\right).\end{aligned}$$

Requiring that $\langle q|q\rangle = 1$ we find

$$\mathcal{N} = \frac{1}{J\sqrt{J+1}}\,. \quad \text{(C.2)}$$

With this normalization we find that the simple states are orthogonal

$$\langle q_a|q_b\rangle = \delta_{k_a k_b} + O\left(\frac{1}{J}\right) \qquad \text{where} \qquad q_a = e^{i\frac{2\pi k_a}{J}}, \quad q_b = e^{i\frac{2\pi k_b}{J}}\,. \quad \text{(C.3)}$$

This is perfectly consistent with the fact that in the planar limit the lattice states, given by $|1^{n+m_1-m_2+1},1^{n+m_1-m_2},1^{n+m_1-m_2};\{J-m_1+m_2\}\rangle$ are orthogonal and our simple states are simply a Fourier transform of these.

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Our eigenstates have the form (we will see in a few moments that the normalization in the next equation below is the same as the normalization in (C.2))

$$|\psi(q)\rangle = \mathcal{N}\Bigg(\sum_{m_2=0}^{\infty} \sum_{m_1=0}^{m_2} f(m_2) q^{m_1-m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2-m_1\}\rangle$$
$$+ \sum_{m_1=0}^{J+m_2} \sum_{m_2=0}^{\infty} f(m_1) f(J-m_1+m_2) q^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle \Bigg)$$
$$\equiv |q\rangle + |\delta q\rangle \tag{C.4}$$

where

$$|\delta q\rangle = \mathcal{N}\Bigg(\sum_{m_2=J}^{n+J+1} \sum_{m_1=0}^{m_2} f(m_2) q^{m_1-m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2-m_1\}\rangle$$
$$+ \sum_{m_1=J}^{J+m_2} \sum_{m_2=0}^{n+m_1} f(J-m_1+m_2) f(m_1) q^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle$$
$$+ \sum_{m_1=0}^{J-1} \sum_{m_2=m_1+1}^{n+m_1} f(J-m_1+m_2) q^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle \Bigg)$$
$$= \mathcal{N}\Bigg(\sum_{m_2=J}^{J+\delta J} \sum_{m_1=0}^{m_2} f(m_2) q^{m_1-m_2} |1^{n+J+m_1-m_2+1}, 1^{n+J+m_1-m_2}, 1^{n+J+m_1-m_2}; \{m_2-m_1\}\rangle$$
$$+ \sum_{m_1=J}^{l_-} \sum_{m_2=0}^{J+\delta J} f(J-m_1+m_2) f(m_1) q^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle$$
$$+ \sum_{m_1=0}^{J-1} \sum_{m_2=m_1+1}^{m_1+\delta J} f(J-m_1+m_2) q^{m_1-m_2} |1^{n+m_1-m_2+1}, 1^{n+m_1-m_2}, 1^{n+m_1-m_2}; \{J-m_1+m_2\}\rangle \Bigg)$$

and l_- is the smallest of $J+m_2$ and $J+\delta J$. It is rather simple to see that $|\delta q\rangle$ is given by a sum of $O(J)$ terms and that each term has a coefficient of order δJ. Consequently, up to an overall constant factor $c_{\delta q}$ which is independent of J, we can bound the norm of $|\delta q\rangle$ as

$$\langle \delta q | \delta q \rangle \leq c_{\delta q} J (\delta J)^2 \mathcal{N}^2 = c_{\delta q} \frac{(\delta J)^2}{J(J+1)} \tag{C.5}$$

which goes to zero in the large J limit, proving our assertion that the difference between the simple states and the large N eigenstates vanishes in the large N limit.

D Review of dilatation operator action

The studies [29, 30] have computed the dilatation operator action without invoking the distant corners approximation. The only approximation made in these studies is that correlators of operators with p long rows/columns with operators that have p long rows/columns and some short rows/columns, vanishes in the large N limit. These results are useful since they provide data against which the distant corners approximation could be compared. Further, we have demonstrated that the action of the dilatation operator reduces to a set of decoupled harmonic oscillators in [31–34]. However, to obtain this result we needed to

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expand one of the factors in the dilatation operator to subleading order. The agreement of the resulting spectrum[5] is strong evidence that the distant corners approximation is valid. It is worth discussing these details and explaining why we do indeed obtain the correct large N limit. This point is not made explicitly in [31–34].

In terms of operators belonging to the SU(2) sector and normalized to have a unit two point function, the action of the one loop dilatation operator

$$DO_{R,(r,s)}(Z,Y) = \sum_{T,(t,u)} N_{R,(r,s);T,(t,u)} O_{T,(t,u)}(Z,Y)$$

is given by

$$N_{R,(r,s);T,(t,u)} = -g_{\rm YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \,\mathrm{hooks}_T \,\mathrm{hooks}_r \,\mathrm{hooks}_s}{f_R \,\mathrm{hooks}_R \,\mathrm{hooks}_t \,\mathrm{hooks}_u}} \times \\ \times \,\mathrm{Tr}\left(\Big[\Gamma_R((n,n+1)), P_{R\to(r,s)}\Big] I_{R'\,T'} \Big[\Gamma_T((n,n+1)), P_{T\to(t,u)}\Big] I_{T'\,R'} \right).$$

The above formula is exact. After using the distant corners approximation to simplify the trace and prefactor, this becomes

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{\rm YM}^2 \sum_{u\nu_1\nu_2} \sum_{i<j} \delta_{\vec{m},\vec{n}} M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} \Delta_{ij} O_{R,(r,u)\nu_1\nu_2} \,. \tag{D.1}$$

Notice that we have a factorized action: the Δ_{ij} (explained below) acts only on the Young diagrams R, r and

$$M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} = \frac{m}{\sqrt{d_s d_u}} \Big(\langle \vec{m}, s, \mu_2 \,;\, a | E^{(1)}_{ii} | \vec{m}, u, \nu_2 \,;\, b \rangle \langle \vec{m}, u, \nu_1 \,;\, b | E^{(1)}_{jj} | \vec{m}, s, \mu_1 \,;\, a \rangle \\ + \langle \vec{m}, s, \mu_2 \,;\, a | E^{(1)}_{jj} | \vec{m}, u, \nu_2 \,;\, b \rangle \langle \vec{m}, u, \nu_1 \,;\, b | E^{(1)}_{ii} | \vec{m}, s, \mu_1 \,;\, a \rangle \Big) \tag{D.2}$$

where a and b are summed, acts only on the s, μ_1, μ_2 labels of the restricted Schur polynomial. a labels states in the irreducible representation s and b labels states in the irreducible representation t. To spell out the action of operator Δ_{ij} it is useful to split it up into three terms

$$\Delta_{ij} = \Delta^+_{ij} + \Delta^0_{ij} + \Delta^-_{ij} \,. \tag{D.3}$$

Denote the row lengths of r by r_i and the row lengths of R by R_i. Introduce the Young diagram r^+_{ij} obtained from r by removing a box from row j and adding it to row i. Similarly r^-_{ij} is obtained by removing a box from row i and adding it to row j. In terms of these Young diagrams we have

$$\Delta^0_{ij} O_{R,(r,s)\mu_1\mu_2} = -(2N + R_i + R_j - i - j) O_{R,(r,s)\mu_1\mu_2} \,, \tag{D.4}$$

$$\Delta^+_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + R_i - i)(N + R_j - j + 1)}\, O_{R^+_{ij},(r^+_{ij},s)\mu_1\mu_2} \,, \tag{D.5}$$

$$\Delta^-_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + R_i - i + 1)(N + R_j - j)}\, O_{R^-_{ij},(r^-_{ij},s)\mu_1\mu_2} \,. \tag{D.6}$$

[5] One can also compare the states that have a definite scaling dimension. The states obtained in the distant corners approximation are in perfect agreement with the states obtained in [29, 30] by a numerical diagonalization of the dilatation operator.

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As a matrix Δ_{ij} has matrix elements

$$\Delta_{ij}^{R,r;T,t} = \sqrt{(N+R_i-i)(N+R_j-j+1)}\delta_{T,R_{ij}^+}\delta_{t,r_{ij}^+} \\ + \sqrt{(N+R_i-i+1)(N+R_j-j)}\delta_{T,R_{ij}^+}\delta_{t,r_{ij}^+} - (2N+R_i+R_j-i-j)\delta_{T,R}\delta_{t,r}\,. \tag{D.7}$$

In terms of these matrix elements we can write (D.1) as

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{\rm YM}^2 \sum_{T,(t,u)\nu_1\nu_2} \sum_{i<j} \delta_{\vec{m},\vec{n}} M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}\, \Delta_{ij}^{R,r;T,t} O_{T,(t,u)\nu_1\nu_2}\,. \tag{D.8}$$

Although the distant corners approximation has been used to extract the large N value of $M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}$, the action of $\Delta_{ij}^{R,r;T,t}$ is computed exactly. In particular, the coefficients appearing in (D.7) are simply the factors associated with the boxes that are added or removed by $\Delta_{ij}^{R,r;T,t}$, and hence in developing a systematic large N expansion for $\Delta_{ij}^{R,r;T,t}$ we can trust the shifts of numbers of order N by numbers of order 1.

The limit in which the dilatation operator reduces to sets of decoupled oscillators corresponds to the limit in which the difference between the row (or column) lengths of Young diagram R are fixed to be $O(\sqrt{N})$ while the row lengths themselves are order N. The continuum variables are then

$$x_i = \frac{R_{i+1}-R_i}{\sqrt{R_1}} \qquad i = 1, 2, \cdots, p-1 \tag{D.9}$$

when R has p rows (or columns) and the shortest row (or column) is R_1. In this case, the leading and subleading (order N and order $\sqrt{N}$) contribution to $\Delta_{ij}O_{R,(r,s)\mu_1\mu_2}$ vanish, leaving a contribution of order 1. This contribution is sensitive to the exact form of the coefficients appearing in (D.7), and it is with these shifts that we reproduce the numerical results of [29, 30].

E One loop computation of bulk/boundary magnon scattering

In this appendix we will compute the scattering of a bulk and boundary magnon, to one loop, using the asymptotic Bethe ansatz. See [65] where studies of this type were first suggested and [66] for related systems. We can introduce a wave function $\psi(l_1, l_2, \cdots)$ as follows

$$O = \sum_{l_1,l_2,\cdots} \psi(l_1,l_2,\cdots) O(R, R_1^k, R_2^k; \{l_1, l_2, \cdots\})\,. \tag{E.1}$$

We assume that the boundary magnon (at l_1) and the next magnon along the open string (at l_2) are very well separated from the remaining magnons. These magnons are both assumed to be Y impurities. To obtain the scattering we want, we only need to focus on these two magnons. The time independent Schrödinger equation following from our one loop dilatation operator is

$$E\psi(l_1,l_2) = \left(3+\frac{c}{N}\right)\psi(l_1,l_2) - \sqrt{\frac{c}{N}}\left(\psi(l_1-1,l_2)+\psi(l_1+1,l_2)\right) \\ -\left(\psi(l_1,l_2-1)+\psi(l_1,l_2+1)\right) \tag{E.2}$$

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where c is the factor of the box that the endpoint associated to the magnon at l_1 belongs to. The equation (E.2) is valid whenever the two magnons are not adjacent in the open string word, i.e. when $l_2 > l_1 + 1$.[6] In the situation that the magnons are adjacent, we find

$$E\psi(l_1, l_1+1) = \left(1+\frac{c}{N}\right)\psi(l_1, l_1+1) - \sqrt{\frac{c}{N}}\psi(l_1-1, l_2) - \psi(l_1, l_1+2)\,. \tag{E.3}$$

We make the following Bethe ansatz for the wave function

$$\psi(l_1, l_2) = e^{ip_1 l_1 + ip_2 l_2} + R_{12}\, e^{ip'_1 l_1 + p'_2 l_2}\,. \tag{E.4}$$

It is straight forward to see that this ansatz obeys (E.2) as long as

$$E = 3 + \frac{c}{N} - \sqrt{\frac{c}{N}}(e^{ip_1} + e^{-ip_1}) - (e^{ip_2} + e^{-ip_2}) \tag{E.5}$$

and

$$\sqrt{\frac{c}{N}}(e^{ip_1} + e^{-ip_1}) + e^{ip_2} + e^{-ip_2} = \sqrt{\frac{c}{N}}(e^{ip'_1} + e^{-ip'_1}) + e^{ip'_2} + e^{-ip'_2}\,. \tag{E.6}$$

Note that (E.5) is indeed the correct one loop anomalous dimension and (E.6) can be obtained by equating the $O(\lambda)$ terms on both sides of (6.1), as it should be. From (E.3) we can solve for the reflection coefficient R. The result is

$$R_{12} = -\frac{2e^{ip_2} - \sqrt{\frac{c}{N}}e^{ip_1+ip_2} - 1}{2e^{ip'_2} - \sqrt{\frac{c}{N}}e^{ip'_1+ip'_2} - 1} \tag{E.7}$$

Two simple checks of this result are

1. We see that $R_{12}R_{21} = 1$.

2. If we set $c = N$ we recover the S-matrix of [65].

We will now move beyond the $su(2)$ sector by considering a state with a single Y impurity and a single X impurity. The operator with a Y impurity at l_1 and an X impurity at l_2 is denoted $O(R, R_1^k, R_2^k; \{l_1, l_2, \cdots\})_{YX}$ and the operator with an X impurity at l_1 and a Y impurity at l_2 is denoted $O(R, R_1^k, R_2^k; \{l_1, l_2, \cdots\})_{XY}$. We now introduce a pair of wave functions as follows

$$\begin{aligned} O = \sum_{l_1, l_2, \cdots} \Big[& \psi_{YX}(l_1, l_2, \cdots) O(R, R_1^k, R_2^k; \{l_1, l_2, \cdots\})_{YX} \\ & + \psi_{XY}(l_1, l_2, \cdots) O(R, R_1^k, R_2^k; \{l_1, l_2, \cdots\})_{XY} \Big]\,. \end{aligned} \tag{E.8}$$

From the one loop dilatation operator we find the time independent Schrödinger equation (E.2) for each wave function, when the impurities are not adjacent. When the impu-

[6] Notice that we are associating a lattice site to every field in the spin chain and not just to the Zs.

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rities are adjacent, we find the following two time independent Schrödinger equations

$$\begin{aligned} E\psi_{YX}(l_1,l_1+1) &= \left(2+\frac{c}{N}\right)\psi_{YX}(l_1,l_1+1) - \sqrt{\frac{c}{N}}\psi_{YX}(l_1-1,l_1+1) \\ &\quad - \psi_{XY}(l_1,l_1+1) - \psi_{YX}(l_1,l_1+2) \qquad \text{(E.9)} \\ E\psi_{XY}(l_1,l_1+1) &= \left(2+\frac{c}{N}\right)\psi_{XY}(l_1,l_1+1) - \sqrt{\frac{c}{N}}\psi_{XY}(l_1-1,l_1+1) \\ &\quad - \psi_{YX}(l_1,l_1+1) - \psi_{XY}(l_1,l_1+2)\,. \qquad \text{(E.10)} \end{aligned}$$

Making the following Bethe ansatz for the wave function

$$\begin{aligned} \psi_{YX}(l_1,l_2) &= e^{ip_1l_1+ip_2l_2} + Ae^{ip_1'l_1+ip_2'l_2} \\ \psi_{XY}(l_1,l_2) &= Be^{ip_1'l_1+ip_2'l_2} \end{aligned} \qquad \text{(E.11)}$$

we find that the two equations of the form (E.2) imply that both $\psi_{XY}(l_1,l_2)$ and $\psi_{YX}(l_1,l_2)$ have the same energy, which is given in (E.5). The equations (E.9) and (E.10) imply that

$$\begin{aligned} A &= \frac{e^{ip_2'}+e^{ip_2}-1-\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}}{1+\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}-2e^{ip_2'}}\,, \\ B &= \frac{e^{ip_2}-e^{ip_2'}}{1+\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}-2e^{ip_2'}}\,. \end{aligned} \qquad \text{(E.12)}$$

It is straight forward but a bit tedious to check that $|A|^2+|B|^2=1$ which is a consequence of unitarity. To perform this check it is necessary to use the conservation of momentum $p_1+p_2=p_1'+p_2'$, as well as the constraint (E.6). We now finally obtain

$$\frac{A}{R_{12}} = \frac{e^{ip_2'}+e^{ip_2}-1-\sqrt{\frac{c}{N}}e^{ip_1'+ip_2'}}{2e^{ip_2}-\sqrt{\frac{c}{N}}e^{ip_1+ip_2}-1}\,. \qquad \text{(E.13)}$$

This should be equal to

$$\frac{1}{2}\left(1+\frac{B_{12}}{A_{12}}\right) \qquad \text{(E.14)}$$

where A_{12} and B_{12} are the S-matrix elements computed in section 7, describing the scattering between a bulk and a boundary magnon. This allows us to perform a non-trivial check of the S-matrix elements we computed.

F No integrability

The (boundary) Yang-Baxter equation makes use of the boundary magnon (B) and two bulk magnons (1 and 2). For our purposes, it is enough to track only scattering between bulk and boundary magnons. The Yang-Baxter equation requires equality between the scattering[7] which takes $B+1\to B'+1'$ and then $B'+2\to\tilde{B}'+\tilde{2}$ and the scattering which

[7]There are some bulk magnon scatterings that we are ignoring as they don't affect our argument.

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takes $B+2 \to B'+2'$ and then $B'+1 \to \tilde{B}'+\tilde{1}$. For the first scattering, given the initial momenta p_1, p_2, p_B, we need to solve

$$\sqrt{1+8\lambda \sin^2 \frac{p_1}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{p_B}{2}\right)} = \sqrt{1+8\lambda \sin^2 \frac{k_1}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{q}{2}\right)} \tag{F.1}$$

$$\sqrt{1+8\lambda \sin^2 \frac{p_2}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{q}{2}\right)} = \sqrt{1+8\lambda \sin^2 \frac{k_2}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{k_B}{2}\right)} \tag{F.2}$$

for the final momenta k_1, k_2, k_B. For the second scattering we need to solve

$$\sqrt{1+8\lambda \sin^2 \frac{p_2}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{p_B}{2}\right)} = \sqrt{1+8\lambda \sin^2 \frac{l_2}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{s}{2}\right)} \tag{F.3}$$

$$\sqrt{1+8\lambda \sin^2 \frac{p_1}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{s}{2}\right)} = \sqrt{1+8\lambda \sin^2 \frac{l_1}{2}} + \sqrt{1+8\lambda \left((1+r)^2 + 4r \sin^2 \frac{l_B}{2}\right)} \tag{F.4}$$

for the final momenta l_1, l_2, l_B. It is simple to check that, in general, $k_1 \neq l_1$, $k_2 \neq l_2$ and $k_B \neq l_B$, so the two scatterings can't possibly be equal.

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