Abstract

In this thesis, our primary interest lies in the investigation of the location of the zeros and the asymptotic zero distribution of hypergeometric polynomials.

The location of the zeros and the asymptotic zero distribution of general hypergeometric polynomials are linked with those of the classical orthogonal polynomials in some cases, notably ${}_2F_1$ and ${}_1F_1$ hypergeometric polynomials which have been extensively studied. In the case of ${}_3F_2$ polynomials, less is known about their properties, including the location of their zeros, because there is, in general, no direct link with orthogonal polynomials. Our introduction in Chapter 1 outlines known results in this area and we also review recent papers dealing with the location of the zeros of ${}_2F_1$ and ${}_1F_1$ hypergeometric polynomials.

In Chapter 2, we consider two classes of ${}_3F_2$ hypergeometric polynomials, each of which has a representation in terms of ${}_2F_1$ polynomials. Our first result proves that the class of polynomials ${}_3F_2(-n, a, b; a-1, d; x)$, $a, b, d \in \mathbb{R}$, $n \in \mathbb{N}$ is quasi-orthogonal of order 1 on an interval that varies with the values of the real parameters b and d. We deduce the location of $(n-1)$ of its zeros and discuss the apparent role played by the parameter a with regard to the location of the one remaining zero of this class of polynomials. We also prove results on the location of the zeros of the classes ${}_3F_2(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x)$, $b \in \mathbb{R}$, $n \in \mathbb{N}$ and ${}_3F_2\left(-n, b, \frac{b-n}{2}+1; b-n, \frac{b-n+1}{2}; x\right)$, $n \in \mathbb{N}$, $b \in \mathbb{R}$ by using the orthogonality and quasi-orthogonality of factors involved in its represen-

tation. We use Mathematica to plot the zeros of these ${}_3F_2$ hypergeometric polynomials for different values of n as well as for different ranges of the parameters. The numerical data is consistent with the results we have proved.

The Euler integral representation of the ${}_2F_1$ Gauss hypergeometric function is well known and plays a prominent role in the derivation of transformation identities and in the evaluation of ${}_2F_1(a,b;c;1)$, among other applications (cf. [1], p.65). The general ${}_{p+k}F_{q+k}$ hypergeometric function has an integral representation (cf. [37], Theorem 38) where the integrand involves ${}_pF_q$. In Chapter 3, we give a simple and direct proof of an Euler integral representation for a special class of ${}_{q+1}F_q$ functions for $q \geq 2$. The values of certain ${}_3F_2$ and ${}_4F_3$ functions at $x = 1$, some of which can be derived using other methods, are deduced from our integral formula.

In Chapter 4, we prove that the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ asymptotically approach the section of the lemniscate $\left\{z : |z(1-z)^2| = \frac{4}{27}; \mathrm{Re}(z) > \frac{1}{3}\right\}$ as $n \to \infty$. In recent papers (cf. [31], [32], [34], [35]), Martínez-Finkelshtein and Kuijlaars and their co-authors have used Riemann-Hilbert methods to derive the asymptotic distribution of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$ when the limits $A = \lim_{n\to\infty} \dfrac{\alpha_n}{n}$ and $B = \lim_{n\to\infty} \dfrac{\beta_n}{n}$ exist and lie in the interior of certain specified regions in the AB-plane. Our result corresponds to one of the transitional or boundary cases for Jacobi polynomials in the Kuijlaars Martínez-Finkelshtein classification.