An Essay on Branching Time Logics

A Comprehensive Investigation into Axiomatisations and Decidability of the Logics of Different Branching Time Structures.

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I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

on this 30 th day of May 2024 at Johannesburg.

Abstract

In this thesis we investigate the Priorian logics of a variety of classes of trees. These classes of trees are divided in to irreflexive and reflexive trees, and each of these has a number of subclasses, for example, dense irreflexive trees, discrete reflexive trees, irreflexive trees with branches isomorphic to the natural numbers, etc. We find finite axiomatisations for the logics of these different classes of trees and show that each logic is sound and strongly / weakly complete with respect to the respective class of trees. The methods use to show completeness vary from adapting some known constructions for specific purposes, including unravelling and bulldozing, building a network step-by-step, filtering through a finite set of formulas, as well as using some new processes, namely refining the filtration and unfolding. Once the logics have been shown to be sound and complete with respect to the different classes of trees, we also show that most of these logics are decidable, using methods that include the finite model property, mosaics and conservative extensions. Lastly, we give a glimpse into the available research on other languages used to study branching time structures, including the Peircean and Ockhamist languages, and languages that include additional modal operators like "since" and "until".

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Chapter 1

Introduction

Time has no doubt played a significant role in humanity's attempts of understanding the world for millennia. In Philosophy, it influences our thinking on topics like free will vs determinism, the nature of perception and reality, and the nature and possibility of motion. In more concrete applications, for example computer science, it has played a role in the development of artificial intelligence, and in linguistics when arguing about tenses. For this reason, mathematicians, computer scientists and philosophers have been interested in a more formal approach to time in an attempt to make reasoning about time more precise. Such explorations are mainly conducted within the discipline of Temporal Logic. The term temporal logic, or tense logic, has different meanings, and uses different terminology and concepts, depending on the field within which it is studied. For the purposes of this study, we will view Temporal Logic as a branch of modal logic, an approach that dates back about sixty years ago to the work of Arthur Prior.

Prior's interest in the study of time was primarily motivated by philosophical considerations, which lead to his invention of Tense logics. He believed that a proper logical approach could help to clarify and solve the problems involving these philosophical considerations. Two of his key contributions are metric tense logic and hybrid temporal logic. He also introduced the following temporal operators:

P: 'It has at some time in the past been the case that . . . '

F: 'It will at some time in the future be the case that . . . '

H: 'It has always been the case that . . . '

G: 'It will always be the case that . . . '

Prior also introduced two versions of branching time temporal logic that extend the language containing the above-mentioned temporal operators. These additional temporal logics reflect the views of William of Ockham and C.S. Peirce, respectively (see for example [50] for a detailed history of tense logic). His work sparked a large interest in these issues which lead to a diverse field with many applications in various fields.

One such field of application is computer science (see [20] for a comprehensive summary). Temporal logics are syntactically simple and elegant, have natural semantics in interpreted transition systems, are expressive for properties of computations and have good computational behaviour. Because of this, as stated in [20], temporal logics are frequently applied in computer and information systems. For example, it is used in scheduling of the execution of programs by an operating system; concurrent and reactive systems, specifically, synchronisation of concurrent processes; real-time processes and systems; hardware verification; temporal databases, to name a few. These mostly relate to specification and verification of properties of transition systems. Formally, transition systems can be modelled by directed graphs consisting of states and transitions between them. They are used to model sequential and concurrent processes. There can be different types of transitions which we indicate by assigning different labels to them, for example, initial, terminal, deadlock, safe or unsafe, etc. In [20] it is described as follows:

"One can describe state properties by formulae of a suitable state-description language; on propositional level these are simply atomic propositions. The set of such propositions that are true at a given state is encoded in the label of that state, and a transition system where every state is assigned such label will be called an interpreted transition system. In terms of the semantics of modal logics, transition systems are simply Kripke frames, and the labelling of states corresponds to a valuation of the atomic propositions in such frames, so interpreted transition systems are just Kripke models." Therefore, temporal logics provide a very appropriate logical framework for formal specification and verification of programs and properties of transition systems, as they are naturally interpreted on Kripke frames.

Some more specific applications resulting from this suitability of temporal logics to computer systems include the medical field and AI solutions. For example, formalising clinical practise guidelines [59], modelling patient flows suitable for enhanced reasoning and correct representation [16], developing artificial intelligence and medical natural language processing [75], etc. Various further applications in AI can be found in [69].

Conversely, the applications of temporal logics for specification and verification of computer systems have led to the study of their expressiveness and computational complexity and the development of efficient algorithmic methods for solving their basic logical decision problems.

Even within the boundaries of temporal logic as a branch of modal logic, there is still a wide variety of systems. For example, we could view time as either a instant-based system or an interval system. Furthermore, do we view time as linear or branching, continuous or discrete, reflexive or irreflexive; does time have a beginning or end, etc.

The two most basic types of formal models of time are instant-based and interval-based models. In the instant-based models, the basic entities are instants or points with a binary relation, which is usually a partial ordering on the set of instants. Discrete models of instant-based time are frequently used in computer science, while dense and continuous models are more popular for philosophical and natural sciences applications, for example dynamical systems theory [70]. Interval based models, on the other hand, are useful for reasoning about events with duration rather than instants, as the primitive entities. Interval-based models are normally applied to linear time but have more scope for different relations between intervals, for example temporal precedence, inclusion, and overlap (see for example [2] for more relations in interval-based models). Although either model can be reduced to the other, there are solid reasons to justify a place in the formalisation of time for both (see for example [38] and [37]).

This thesis investigates the instant-based temporal logics of classes of tree-like structures and establishes a range of new complete finite axiomatisation and decidability results for these logics. Having a sound and complete axiomatisation for the logic of a class of trees makes it possible to move from the semantics perspective of validity in a frame for a particular class of trees that, by definition, may require infinitely many situations to be checked, to derivability in the formal syntactic system for this class. The purpose of a deductive system is therefore, to be able to distil the fundamental principles that hold in a certain context and from which all that can be said about that context can be derived. The archetypal example of this is the 5 postulates/axioms of Euclidean Geometry. Euclid used only these postulates and logical deduction to build the theory of Euclidean Geometry. Hence, once these fundamental principles of a system are known, the entire system is known. With a sound and complete deductive system, we can replace the more intangible idea of validity with a finite deduction.

Therefore, with a sound and complete axiomatisation, finding all truths about a structure of a class of structures becomes equivalent to finding all theorems derivable from the axioms of the system. We cannot hope that we will be able to find an axiomatisation for all classes of structures, in fact, in 1931 Gödel (see for example [31]) famously showed that the first order theory in the language of Peano arithmetic is incomplete. In a related but different phenomenon, in modal logic, was explored by Blok in [52]. He showed that, in modal logic, incompleteness is the rule rather than the exception. Finding a sound and complete deductive system for a class of frames, shows that the system for this class is complete, in the sense that all validities can be derived in the formal system. However, with a complete axiomatisation, we still do not necessarily have a decision procedure to find all the theorems of the system. Therefore, even though the system may be determined, decidability of the system must still be determined. What is required is an algorithm which determines, for any given formula, whether it is a theorem of the system or not. For example, the finite model property, in conjunction with a finite axiomatisation, gives such a procedure. Via the soundness and completeness of a system, such an algorithm determining theoremhood is also an algorithm for determining validity.

Tree structures (defined in Section 2.2) have been of particular interest in the study of the logic of time as they arise naturally in the process of answering questions pertaining to a timeline that has a determined past and different possible futures. In these structures, the past is linear, as it has already occurred, but the future is undetermined, represented by the different branches of the tree. For this reason, tree-like models of time have been extensively studied in temporal logic. However, there are other applications of tree structures as well, including decision trees and computation trees which gives these structures a wide range of applications across the sciences. Particular subclasses of trees are also useful for particular applications, for example, many applications require the trees to be discrete or finite, while others consider continuous timelines. We consider two main classes of trees, irreflexive trees and reflexive trees. Within these we will distinguish different subclasses, for example dense trees, discrete trees, and finite trees. We summarise the different classes we consider in this thesis in Section 2.2.1.

Historically, there have been three main alternative temporal languages used to describe and reason about trees: Priorian, Peircean and Ockhamist [38] and [37]. Historically, the Priorian language developed naturally from modal logic for unidirectional time, and temporal logic developed as a bidirectional extension thereof. The Priorian language has a future and past operator but, as it is the standard language used for temporal logic, however, it lacks the power to express certain natural and useful properties of trees. It cannot distinguish between different branches of a tree, which makes statements about the future limited. For example, we can only talk about something happening somewhere in the future, or something happening always in the future. The Peircean language is more expressive and expands the Priorian language to give us the ability to talk about something happening somewhere on all branches through an instant of time. The Ockhamist language, which is an extension of the Peircean language adds additional ability in what we can express, for example, we can talk about something always happening on only some branches through an instant. The Ockhamist language is the most expressive and the Peircean and Priorian languages can be seen as fragments of the full Ockhamist language.

Axiomatisations and decidability results in this thesis will be for the Priorian temporal language and its future fragment introduced in Section 2.3.1, but the semantics of some alternative languages with different modalities will be introduced in Chapter 7 when surveying axiomatisation results in the literature. Of course, linear orders are special cases of trees, and as summarised in Section 2.9, there are many axiomatisations for the temporal logics of linear orders, for example Segerberg [60] axiomatised the logic of the integers, natural numbers, rational numbers and real numbers, to name a few. Burgess [13] also axiomatised some of the logics of these linear orders and others. In [19] more alternative axiomatisations to the ones mentioned before are given for some of these linear orders. There has been some investigation of branching time structures, for example [11], and in [61] Segerberg presents an axiomatisation for the class of irreflexive finite trees, and an axiomatisation for the class of reflexive finite trees is given in [15], following work done by Grzegorczyk. Axiomatisations for the logics of different classes of trees in the Priorian languages have not been extensively studied in the literature. This gap is largely filled by this thesis, where we build on some known results and use different known and new construction methods in completeness proof for the axiomatisations of the logics for different classes of trees are sound and complete with respect to the respective classes. In addition to this, we also show that most of these logics are decidable.

In Chapter 2 we start with the different relations relevant to this thesis in Section 2.1, and then, in Sections 2.2 and 2.3, we define trees and the syntax and semantics of the languages we will use. For the purpose of this thesis, we will need some basic background on constructions and methods that will be used to show completeness of the logics of different classes of trees. We introduce this in Sections 2.5, 2.4, 2.6, 2.7 and 2.8. We also consider some axiomatisation results for linear frames in Section 2.9. Lastly, in Section 2.10, we investigate whether there are sets of formulas that define the classes of trees when restricted to irreflexive (respectively, reflexive) trees.

In Chapter 3, we start our investigation of axiomatisations with the future fragment of the Priorian temporal language that has one modal operator **F** that corresponds semantically to the 'diamond' of modal logic. The benefit of having only a future operator, is that we are not concerned about 'looking back'. This simplifies the construction of models and in most cases, we show completeness of the logics of the classes of trees by using a generated submodel of the canonical model of the logic and 'unravelling' the model to get a tree (see e.g. [61] and [60]). Classes of trees considered include the class of all irreflexive/reflexive trees, discrete irreflexive/reflexive trees, locally finite irreflexive/reflexive trees and the unbounded versions of these classes of trees. Since our language only has future operators, we employ some constructions to build left unbounded models to show strong completeness for the unbounded classes of irreflexive/reflexive trees. Some specific methods will be applied to build a model for a consistent set of formulas of logics of the particular classes of trees, but the simplicity of these processes makes these results rather straight forward. An example of this is the class of dense irreflexive/reflexive trees, where we use the method of building a step-by-step network, and finite irreflexive/reflexive trees, where we use the method of selective filtrations. As special cases of these we also find complete finite axiomatisations for the class of irreflexive/reflexive trees with branches isomorphic to the natural numbers, integers, and rational numbers.

In Chapter 4 we find complete axiomatisations for the Priorian logics of the classes of all irreflexive/reflexive trees and dense irreflexive/reflexive trees. The semantics of the Priorian temporal language corresponds to the basic temporal language with both a future operator and past operator, and hence methods used for completeness results for these logics come in useful when looking for completeness results for branching time logics. For example, [46] introduced completeness via canonicity. The step-by-step method to show completeness for dense trees was mentioned in [49], [11] and [12]. For the classes of all irreflexive/reflexive trees, we show completeness of our axiomatisations of the logics for these classes by bulldozing a generated submodel of the canonical model. We also use a step-by-step method to build a dense network (see for example [44]) for the class of dense irreflexive/reflexive trees to show that our axiomatisations of these logics are sound and strongly complete with respect to these classes of trees. Using the appropriate axioms, we can also find completeness results for the class of unbounded irreflexive/reflexive trees, dense unbounded irreflexive/reflexive trees, and the class of irreflexive/reflexive trees with branches isomorphic to the rational numbers.

In Chapter 5 we consider axiomatisations in the Priorian temporal language for some classes of discrete irreflexive/reflexive trees. Axiomatisations for different linear orders in the Priorian language have also been studied since [51] and in [32], [60], [53], [10] and elsewhere. Some of these axiomatic systems will be modified to fit the class of branching time structures in, for example, Sections 5.8.1 and 5.11. Some methods used in these sections include filtrations and bulldozing, which were used in [61] and [60], and networks as used in [44], to name but a few. Versions of these methods will be used in the results of Sections 4.2, 5.1, 4.5, 5.6. For example, we use networks and selective filtrations for the classes of locally finite irreflexive/reflexive trees, well-founded and conversely well-founded irreflexive/reflexive trees, as well as finite irreflexive/reflexive trees. In Section 5.1, we were not able to find suitable axioms for the class of discrete irreflexive trees and therefore used the concept of anti-rules to add more rules to the chosen logic to show completeness. We introduce a new method designed specifically for branching time structures, namely, unfolding in Section 5.2 so show completeness of the logic for the class of discrete reflexive trees. We also use these and other methods to find axiomatisations of the unbounded versions of these classes, as well as the classes of irreflexive/reflexive trees with branches isomorphic to the natural numbers, and to the integers. All the logics for the classes in Chapter 5 are shown to be sound and weakly complete with respect to the relevant classes of trees. This brings the question whether it is possible to use different methods to show strong completeness instead. However, in Propositions 5.1.12 and 5.1.13, we show that the logics for these classes are not compact and hence we cannot hope for strong complete axiomatisations.

In Chapter 6 we investigate decidability of the logics of the different classes of trees studied in the previous chapters. The methods we will use here include the finite model property using Harrop's Theorem [41], mosaics introduced by [72] and [71] amongst others, and conservative extensions. Most of the logics from the previous chapters have the finite model property, although not necessarily with respect to classes of trees, however, the well-founded and conversely well-founded classes, and the classes of dense trees in the future fragment of the Priorian language do not. We used mosaics to show decidability of the logics of the classes of well-founded irreflexive/reflexive trees.

We end the thesis in Chapter 7 with a summary of the completeness results from this thesis, and survey the literature on axiomatisations of the logics of trees involving other languages. We also discuss some open questions for future investigations.

Chapter 2

Modal and Temporal Logics of Trees - Preliminaries

Many concepts from logic and other disciplines will be used throughout the thesis and are defined in this chapter. The definitions and important results are from [4] and [15] unless otherwise stated. Even though some concepts can be defined in a more general context, we will give the definitions that are relevant to the context in which they will be used. This means that the concepts will be defined in the field of modal logic.

We start this chapter with defining the different orderings we will be investigating in this thesis in Section 2.1. These orderings relate to different applications, for example, when considering discrete or dense time, etc. Next, in Section 2.2, we look at the precise definition of a tree as we will use it in this thesis, as well as the different classes of trees, based on the orderings on their branches. In Section 2.3, we define frames and models, the syntax and semantics of the Priorian languages (which will be the main focus language in this thesis) and some related concepts. Then, in Sections 2.5, 2.6, 2.7 and 2.8 we define the basic concepts we need to build the results in the following chapters. In Section 2.9 we look at some results for linear frames with the aim of learning from those results and adapting them for trees. Lastly, in Section 2.10, we look at whether there is a set of formulas that defines the different classes of trees when restricted to trees. This may give us an idea of candidate axioms we can use to axiomatise these different classes.

2.1 Properties of Relations

Although the language used in this thesis can be interpreted on any model, our focus is classes of tree-like models, as natural models of branching time. In this section we define the classes of trees that we will study, but first we must recall some basic definitions of ordering relations and their properties.

Let (W, R) be a set with a binary relation, i.e., a frame. Let $w, v \in W$. We recall some useful terminology. If $v \in R[w]$ or, equivalently, $w \in R^{-1}[v]$, then v is a **successor** of w and w is a **predecessor** of v. If, moreover, $w \neq v$, then v is a **proper successor** of w and w is a **proper predecessor** of v. We will sometimes write $R^{\neq}wv$ to indicate this. We say that v is an **immediate successor** of w (and w is an **immediate predecessor** of v) if $R^{\neq}wv$ and there is no $u \in W$ such that $R^{\neq}wu$ and $R^{\neq}uv$. We call two elements of W **comparable** if either they are equal, or one is a successor of the other. Recall that R is:

- **irreflexive** if for all $x \in W$, it is not the case that Rxx.
- **reflexive** if for all $x \in W$, it is the case that Rxx.

- **transitive** if for all x, y and z in W, if Rxy and Ryz, then Rxz.
- **asymmetric** if for all $x, y \in W$, if Rxy, then not Ryx.
- **anti-symmetric** if for all $x, y \in W$ with $x \neq y$, if Rxy, then not Ryx.
- a **pre-order** if it is transitive and reflexive.
- a **partial order** if it is an anti-symmetric pre-order.
- a **strict partial order** if it is transitive and irreflexive (and hence also asymmetric).
- a **linear order** if it is a partial order in which every two elements are comparable, i.e., for all $x, y \in W$, either Rxy or Ryx.
- a **strict linear order** if it is a strict partial order in which every two elements are comparable, i.e., for all $x, y \in W$ with $x \neq y$, either Rxy or Ryx

The following concepts can be defined for binary relations in general, but for the purpose of this study, we will define them on transitive binary relations only. R is called

- **left linear** if Rxy and Rzy imply either $x = z$, Rxz or Rzx.
- **connected** if, for any x and y with $x \neq y$, we have that either (1) Rxy or (2) Ryx or (3) there is a z such that Rzx and Rzy.
- **dense** if, for all $u, v \in W$ such that $R^{\neq}uv$, there exists $w \in W$ such that $R^{\neq}uw$ and $R^{\neq}wv$.[1]
- **forwards discrete** if every instant that has a successor has an immediate successor.
- **backwards discrete** if every instant that has a predecessor has an immediate predecessor.
- **discrete** if it is both forwards and backwards discrete.
- **locally finite** if there is a finite number of distinct instants between any two distinct instants. Equivalently, if there are no infinitely ascending sequences $z_1Rz_2Rz_3...$ or descending sequences $...z_3Rz_2Rz_1$ of distinct instants between any two given instants.
- **well-founded** when there are no infinitely descending sequences $...z_3Rz_2Rz_1$ in T, where $z_i \neq zi + 1$ for all i.
- **conversely well-founded** when there are no infinitely ascending sequences $z_1Rz_2Rz_3...$ in T, where $z_i \neq zi + 1$ for all i.
- **left unbounded** if, for all $w\ inW$ there exists a $v \in W$ such that $R^{\neq}wv$.
- **right unbounded** if, for all $w\ inW$ there exists a $v \in W$ such that $R^{\neq}vw$.
- **unbounded** if it is both left and right unbounded.[2]

[1]As density is employed in the modal logic literature, the instant between instants required for density typically does not need to be distinct. In order to build a frame that has dense branches in the sense of our understanding of linear orderings, we require there to be a distinct instant between any two distinct instants.

[2]As the terms are employed in the modal logic literature, the successors required for (left/right) unboundedness typically do not need to be proper. However, our usage is in line with our thinking of frames as orderings.

2.2 Trees

A **tree** $T = \langle T, \prec \rangle$ (also referred to as a branching time structure) is defined as a non-empty set T of instants t with a (strict) partial ordering $\prec$ on T that is left linear and connected.

A **branch** h is a maximal (with respect to set inclusion) subset of T that is linearly ordered by $\prec$. The set of all branches in T will be denoted by $H(T)$, and the set of all branches passing through the instant t will be denoted by $H_t(T)$.

We say that w is a **root** if it has no predecessors. Also, v is a **leaf** if it has no successors.

2.2.1 Classes of Trees

In this thesis we will investigate the different classes of irreflexive trees and reflexive trees, as different orderings are used in different applications, for example, when studying discrete time. The following lists include all the classes we will look at in this thesis. We start with the irreflexive trees:

- C_{basic}: The class of irreflexive trees which is the class of connected branching time structures that are left linear, with a strict partial ordering. All other classes of irreflexive trees will be a subset of this class.
- $C_{fdiscrete}$: The class of forwards discrete irreflexive trees where every branch is irreflexive and forward discrete.
- $C_{bdiscrete}$: The class of backwards discrete irreflexive trees where every branch is irreflexive and backwards discrete.
- $C_{discrete}$: The class of discrete irreflexive trees, i.e. trees that are both forwards and backwards discrete.
- C_{locfin}: The class of locally finite irreflexive trees, i.e., the class of irreflexive trees where every branch is irreflexive and locally finite.
- C_{dense}: The class of dense irreflexive trees, i.e., the class of irreflexive trees where every branch is irreflexive and dense.
- $C_{wellfnd}$: The class of well-founded irreflexive trees, i.e., the class of irreflexive trees where every branch is irreflexive and well-founded.
- $C_{cwellfnd}$: The class of conversely well-founded irreflexive trees, i.e., the class of irreflexive trees where every branch is irreflexive and conversely well-founded.
- C_{finite}: The class of finite irreflexive trees, i.e., the class of irreflexive trees with only finitely many instants.
- C_{lunbnd}: The class of left unbounded irreflexive trees.
- C_{runbnd}: The class of right unbounded irreflexive trees.
- C_{unbnd}: The class of unbounded irreflexive trees.
- $C_{\mathbb{N}}$: The class of irreflexive trees with branches isomorphic to $(\mathbb{N}, <)$.
- $C_{\mathbb{Z}}$: The class of irreflexive trees with branches isomorphic to $(\mathbb{Z}, <)$.
- $C_{\mathbb{Q}}$: The class of irreflexive trees with branches isomorphic to $(\mathbb{Q}, <)$.

- $C_{\mathbb{R}}$: The class of irreflexive trees with branches isomorphic to $(\mathbb{R}, <)$.

Next, we list the reflexive trees that are similarly defined as above, but with a reflexive relation:

- C_{basic^r}: The class of reflexive trees.
- $C_{fdiscrete^r}$: The class of forwards discrete reflexive trees.
- $C_{bdiscrete^r}$: The class of backwards discrete reflexive trees.
- $C_{discrete^r}$: The class of discrete reflexive trees.
- C_{locfin^r}: The class of locally finite reflexive trees.
- C_{dense^r}: The class of dense reflexive trees.
- $C_{wellfnd^r}$: The class of well-founded reflexive trees.
- $C_{cwellfnd^r}$: The class of conversely well-founded reflexive trees.
- C_{finite^r}: The class of finite reflexive trees.
- C_{lunbnd^r}: The class of left unbounded reflexive trees.
- C_{runbnd^r}: The class of right unbounded reflexive trees.
- C_{unbnd^r}: The class of unbounded reflexive trees.
- $C^r_{\mathbb{N}}$: The class of reflexive trees with branches isomorphic to $(\mathbb{N}, \leq)$.
- $C^r_{\mathbb{Z}}$: The class of reflexive trees with branches isomorphic $(\mathbb{Z}, \leq)$.
- $C^r_{\mathbb{Q}}$: The class of reflexive trees with branches isomorphic to $(\mathbb{Q}, \leq)$.
- $C^r_{\mathbb{R}}$: The class of reflexive trees with branches isomorphic to $(\mathbb{R}, \leq)$.

2.3 Syntax and Semantics of Modal and Temporal Logics

Historically there are three main modal languages that have been used to study trees: The Priorian temporal language, the Peircean language and the Ockhamist language. All three of these languages have a future only fragment, and apart from the future and past operators, some versions also include other operators, for example "since" and "until" (see Chapter 7 for a further discussion on these languages). In this thesis, we will focus on the Priorian temporal language and its future fragment. The Peircean and Ockhamist languages can be seen as extensions of the Priorian language.

The future fragment of the Priorian temporal language corresponds with the basic modal language developed in 1918 by Lewis in "A Survey of Symbolic Logic" (see [47] and e.g. [3]) and more recently by authors like S Kripke in the early 1960s, and A Prior in the 1970s and beyond.

Historically, the basic modal language uses $\Diamond$ ("diamond") and $\Box$ ("box") operators. Semantically, $\Diamond$ corresponds with the future operator **F** of the future fragment of the Priorian temporal language and $\Box$ corresponds to its dual operator **G**. This is in line with the idea of seeing the standard modal language as a restriction of the Priorian temporal language where we only consider the future operators.

The Priorian temporal language was introduced by A Prior in [51]. While the future fragment of the Priorian temporal language only looks into the future, the Priorian Temporal language also looks into the past.

We can now look at some concepts needed to define the semantics of the languages we will use to study tree structures.

2.3.1 Fames and Models

If *Prop* is a set of propositional letters and $p \in Prop$, then the set of formulae of the Priorian temporal language, $\mathcal{L}_{Prior}$ can be recursively defined by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \mathbf{P}\varphi \mid \mathbf{F}\varphi \tag{2.1}$$

The dual of $\mathbf{F}$ is $\neg\mathbf{F}\neg$ and is denoted by $\mathbf{G}$ and the dual of $\mathbf{P}$ is $\neg\mathbf{P}\neg$ and is denoted by $\mathbf{H}$. The language can also be defined in terms of $\mathbf{G}$ and $\mathbf{H}$, with $\mathbf{F}$ and $\mathbf{P}$ as secondary modal operators. In this study, both these sets will be used interchangeable for the purposes of proof by induction, etc. We will use $\mathcal{L}_{Prior}$ for both the language and the set of formulas in the language.

Before we take a closer look at the syntax and semantics of the Priorian languages, we need to define frames and models.

Definition 2.3.1. Frame: A **frame** for the Priorian temporal language is a pair $\mathcal{F} = (W, R)$ such that

1. W is a non-empty set.
2. R is a binary relation on W.

Since a natural interpretation of tree (defined in Section 2.2) is in terms of time, **instants** will be used to refer to the elements of W).

Definition 2.3.2. Subframe: A **subframe** $\mathcal{F}' = (W', R')$ of $\mathcal{F} = (W, R)$ is a frame such that $W' \subseteq W$ and R' is the restriction of R to W'.

Definition 2.3.3. Model: A **model** for the Priorian temporal language is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame for the Priorian temporal language, and V is a function, called a **valuation**, assigning to each propositional letter p in Φ a subset $V(p)$ of W.

For every model $\mathcal{M} = (W, R, V)$ and formula φ we define the **extension of** φ **in** $\mathcal{M}$ as $[\![\varphi]\!] = \{w \in W \mid \mathcal{M}, w \Vdash \varphi\}$. We lift V to a function $V : \mathcal{L}_{Prior} \to \mathcal{P}(W)$ (where $\mathcal{P}(W)$ is the powerset of W) by $V(\varphi) = [\![\varphi]\!]_M$.

Definition 2.3.4. Submodel: A **submodel** $\mathcal{M}' = (W', R', V')$ of $\mathcal{M} = (W, R, V)$ is a model such that $W' \subseteq W$, R' is the restriction of R to W', and V' is the restriction of V to W'.

Next, we define the semantics of the Priorian temporal languages.

2.3.2 Semantics of the Priorian Languages

The truth of an arbitrary formula φ of $\mathcal{L}_{Prior}$ at an instant t in a model $\mathcal{M}$ is defined inductively as follows where $p \in Prop$:

- $\mathcal{M}, t \Vdash p$ iff $t \in V(p)$
- $\mathcal{M}, t \nVdash \bot$

- $\mathcal{M}, t \Vdash \neg\varphi$ iff $\mathcal{M}, t \nVdash \varphi$
- $\mathcal{M}, t \Vdash \varphi \vee \psi$ iff $\mathcal{M}, t \Vdash \varphi$ or $\mathcal{M}, t \Vdash \varphi$
- $\mathcal{M}, t \Vdash \mathbf{F}\varphi$ iff $(\exists t' \in W)(Rtt'$ and $\mathcal{M}, t' \Vdash \varphi)$
- $\mathcal{M}, t \Vdash \mathbf{P}\varphi$ iff $(\exists t' \in W)(Rt't$ and $\mathcal{M}, t' \Vdash \varphi)$

The dual operators are defined as:

- $\mathcal{M}, t \Vdash \mathbf{G}\varphi$ iff $(\forall t' \in W$ with $Rtt')(\mathcal{M}, t' \Vdash \varphi)$
- $\mathcal{M}, t \Vdash \mathbf{H}\varphi$ iff $(\forall t' \in W$ with $Rt't)(\mathcal{M}, t' \Vdash \varphi)$

The restriction that only includes the future operators is denoted by $\mathcal{L}^m_{Prior}$ and will be referred to as the future fragment of the Priorian temporal language or modal language, and is defined as follows:

Let $\mathfrak{M}$ be the class of models $(\mathcal{M}, t)$ where $\mathcal{M}$ is a model (W, R, V) and $t \in W$. The truth relation $\Vdash$ is a relation between $\mathcal{M}$ and $\mathcal{L}^m_{Prior}$. The truth of an arbitrary formula φ of $\mathcal{L}^m_{Prior}$ at an instant t in a model $\mathcal{M}$ is defined inductively as follows where $p \in Prop$:

- $\mathcal{M}, t \Vdash p$ iff $t \in V(p)$
- $\mathcal{M}, t \nVdash \bot$
- $\mathcal{M}, t \Vdash \neg\varphi$ iff $\mathcal{M}, t \nVdash \varphi$
- $\mathcal{M}, t \Vdash \varphi \vee \psi$ iff $\mathcal{M}, t \Vdash \varphi$ or $\mathcal{M}, t \Vdash \varphi$
- $\mathcal{M}, t \Vdash \mathbf{F}\varphi$ iff $(\exists t' \in W)(Rtt'$ and $\mathcal{M}, t' \Vdash \varphi)$

The dual operator is defined as:

- $\mathcal{M}, t \Vdash \mathbf{G}\varphi$ iff $(\forall t' \in W$ with $Rtt')(\mathcal{M}, t' \Vdash \varphi)$

The following definitions will be used throughout this thesis.

Definition 2.3.5. Satisfiable: A formula φ is **satisfiable in a model** $\mathcal{M}$ if there is some instant t such that $\mathcal{M}, t \Vdash \varphi$. φ is **satisfiable in a frame** $\mathcal{F}$ if is it satisfiable in some model $\mathcal{M}$ based on $\mathcal{F}$. φ is **satisfiable in a class** C of frames if it is satisfiable in some frame $\mathcal{F}$ in C.

A **set of formulas** Γ **is satisfiable in a model** $\mathcal{M}$ if there is some instant t such that $\mathcal{M}, t \Vdash \varphi$ for each φ in Γ.

Definition 2.3.6. Finite satisfiability: A set of formulas Γ is **finitely satisfiable in a class of frames** F if every finite subset of Γ is satisfiable in F.

Definition 2.3.7. Validity: Let $\mathcal{F}$ be a frame. A formula φ is **valid** on $\mathcal{F}$ if for every model $\mathcal{M} = (W, R, V)$ based on $\mathcal{F}$ and for every instant $t \in W$ we have $\mathcal{M}, t \Vdash \varphi$.

Definition 2.3.8. ***p*-Variant Valuations**: We say that two valuations V and V' are ***p*-variant**, $V \sim_p V'$, if they agree except possibly on where p is true.

Definition 2.3.9. Congruent Instants: let $\mathcal{M} = (W, R, V)$ be a model. We say that two instants w and v in W are congruent (notation: $w \cong v$) if they satisfy the same formulas in the language $\mathcal{L}$. We can parameterise this definition with a set of formulas Φ and then talk about two instants w and v being Φ-congruent if they satisfy the same formulas from Φ, which we denote by $w \cong_\Phi v$. We define the **Φ-type of w in $\mathcal{M}$** to be the set $\Phi_\mathcal{M}(w) = \{\varphi \in \Phi \mid \mathcal{M}, w \Vdash \varphi\}$ of all formulas in Φ satisfied at w in $\mathcal{M}$, i.e. we have that $w \cong_\Phi v$ exactly when w and v are of the same **Φ-type** in $\mathcal{M}$.

Definition 2.3.10. Frame for a logic: The term "frame for a logic" will be used to indicate a frame that validates all the axioms of the logic.

Since many of the frames and models will be trees, we will specifically indicate when the frame or model is indeed a tree. It will be clear from the context when a specific Kripke frame or model is a tree.

Definition 2.3.11. Set of Predecessors / Successors: Let $\mathcal{F} = (W, R)$ be a frame and $X \subseteq W$, then $R[X] = \{v \in W \mid Rwv \text{ for some } w \in X\}$ (the **set of successors of** X) and $R^{-1}[X] = \{w \in W \mid Rwv \text{ for some } v \in X\}$ (the **set of predecessors of** X). We will abuse notation and write $R[x]$ and $R^{-1}[x]$ instead of $R[\{x\}]$ and $R^{-1}[\{x\}]$, respectively.

Definition 2.3.12. Semantic entailment/consequence: A formula φ is a local consequence of a set of formulas Γ over a frame class F, notation $\Gamma \Vdash_\mathcal{F} \varphi$, if for every model $\mathcal{M} = (W, R, V)$ based on a frame $\mathcal{F} \in \mathsf{F}$ and any instant $w \in W$, if $\mathcal{M}, w \Vdash \Gamma$ then $\mathcal{M}, w \Vdash \varphi$.

Definition 2.3.13. Closure of a formula φ: Let φ be a formula in a language $\mathcal{L}$. Then $Cl(\varphi)$ is the set of all subformulas of φ closed under single negations, i.e. $Cl(\varphi) = \{\psi : \psi \in \mathcal{S}\} \cup \{\neg\psi : \psi \in \mathcal{S} \text{ and } \psi \text{ not of the form } \neg\gamma\}$, where $\mathcal{S}$ is the set of subformulas of φ.

We will now define some basic concepts that will be used in this study.

2.4 Truth Preserving Morphisms and Operation on Models and Frames

In this thesis, we will often start with the canonical model of a logic and then transform this model into a tree. For this purpose, it is useful to have some tools that preserve satisfaction in models so that the resulting tree is still a model for the given logic. In this section, we define two such tools, namely, bounded morphisms and generated submodels using the terminology in [4].

2.4.1 Bounded Morphisms

Definition 2.4.1. Bounded Morphism: Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models for the modal or temporal language. A mapping $f : \mathcal{M} \mapsto \mathcal{M}'$ is a **bounded morphism** if it satisfied the following conditions:

- w and $f(w)$ satisfy the same proposition letters.
- f is a homomorphism with respect to the relation R (that is, if Rwv then $R'f(w)f(v)$).
- If $R'f(w)v'$ then there exists v such that Rwv and $f(v) = v'$ (the back condition).
- In the case of the temporal language, the following additional condition must be met to complete the back condition: If $R'v'f(w)$ then there exists v such that Rvw and $f(v) = v'$.

2.4.2 Generated Submodels

Definition 2.4.2. Generated Submodel: We will distinguish between generated submodels in the modal language and the temporal language:

- **Unidirectionally Generated Submodel**: Let $\mathcal{M} = (W, R, V)$ be a model and let U be a subset of W. Then we say that $\mathcal{M}'$ is the **submodel unidirectionally generated by** U (denoted by $\mathcal{M}' \rightarrowtail \mathcal{M}$) if $\mathcal{M}'$ is the smallest submodel of $\mathcal{M}$ containing U and the following closure condition is met:
  - if $u \in W'$ and Ruv then $v \in W'$
- **Bidirectionally Generated Submodel**: Let $\mathcal{M} = (W, R, V)$ be a model and let U be a non-empty subset of W. Then we say that $\mathcal{M}'$ is the **submodel bidirectionally generated by** U (denoted by $\mathcal{M}' \rightarrowtail\!\!\rightarrow \mathcal{M}$) if $\mathcal{M}'$ is the smallest submodel of $\mathcal{M}$ containing U and the following closure conditions are met:
  - if $u \in W'$ and Ruv then $v \in W'$
  - if $u \in W'$ and Rvu then $v \in W'$

When it is clear from the context, we will omit unidirectional / bidirectional and just refer to generated submodels.

When U consists of a single point w we say that $\mathcal{M}'$ is a **point-generated submodel** of $\mathcal{M}$ generated by w. In this thesis we will only work with point-generated submodels and hence will sometimes omit 'point' and just refer to generated submodels.

Note that validity is preserved under generated submodels.

The next lemma tells us more about the instants in a generated submodel.

Lemma 2.4.3. *Let $\mathcal{M} = (W, R, V)$ be the submodel of $\mathcal{M}' = (W', R', V')$ generated by U. Then W consists of exactly those points reachable from elements of U in finitely many steps.*

Proof. Let $\mathcal{M} = (W, R, V)$ be the submodel of $\mathcal{M}' = (W', R', V')$ generated by U. Firstly, by the definition of a generated submodel, all instants that can be reached from elements of U in finitely many steps will be in W.

Next we show that the set of instants that can be reached from elements of U in finitely many steps forms a generated submodel containing U. Let W be the set of instants that can be reached from elements of U in finitely many steps. Firstly, since all instants in U can be reached from instants in U in zero steps, we have that $U \subseteq W$.

Next we show that the conditions for a generated submodel are met: If $u \in U$ and $R'uv$, then v can be reached in finitely many steps from u. Hence $v \in W$. In the bidirectional case we also have that if $u \in U$ and $R'vu$, then v can be reached in finitely many steps from u. Hence $v \in W$. Also for any $w \in W$ with $w \notin U$ and for any $v \in W'$ with $R'wv$ we have that w can be reached in finitely many steps from an instant in U and hence v can also be reached in finitely many steps from an instant in U. Hence $v \in W$. Therefore W consists of exactly those instants in W' that can be reached in finitely many steps from instants in U. □

In this thesis we will use point-generated submodels extensively and will rely on the fact that the generated submodels of transitive and left linear frames are connected. We prove this in the propositions below (one for the modal language and one for the temporal language).

Proposition 2.4.4. *) A unidirectional point-generated submodel of a transitive model is connected.*

Proof. Let $\mathcal{M} = (W, R, V)$ be the generated submodel of a transitive model, generated by w. Then, for all $u, v \in W$ with $u \neq w$ and $v \neq w$ we have that Rwu and Rwv by transitivity, which satisfies the third property of connectedness. Also if $u = w$ then Rwv by transitivity, which gives connectedness. Similarly for when $v = w$. Hence $\mathcal{M}$ is connected. □

Let $\mathcal{M} = (W, R, V)$ be a model. For the proof of the next proposition we define a **path** between two instants $u, v \in W$ to be a sequence $\sigma = u_0, u_1, ..., u_m$, $u_i \in W$, and $u_0 = u$ and $u_m = v$, where we have for each pair (u_i, u_j) either Ru_iu_j or Ru_ju_i. For simplicity we will require there to be no repeated instants in σ. Simply put, a path from u to v is a sequence of instants that starts at u and ends at v, and each instant of the sequence is either a successor or a predecessor of the previous instant. To further simplify, we will factor the path from u to v through transitivity, i.e. if there is a subsequence containing only successors/predecessors we can reduce this to only the first and last instant in the subsequences. In effect we will then end up with a sequence of up and down steps where each up (successor) step is followed by a down (predecessor) step. That is, given a path $\sigma = u, u_1, ..., v$, if for any sub-path $u_j, u_{j+1}, ..., u_{j+n}$ of σ we have either $u_jRu_{j+1}...Ru_{j+n}$ or $u_{j+n}Ru_{j+(n-1)}...Ru_j$, then replace this subsequence with u_j, u_{j+n} A path that has been simplified in this way will be called a **z-path**.

For the proof of the proposition below, we will need the following definition: The concatenation of two paths σ and ρ is the path $\sigma\rho$ formed by joining σ and ρ. Now if σ and ρ were both z-paths, and σ ended on a successor (predecessor) step and ρ began on a successor (predecessor) step, then the resulting path will not be a z-path. We will call the process of turning such a concatenation into a z-path **z-concatenation**.

Proposition 2.4.5. *) The bidirectional point-generated submodel of a transitive, left linear model is connected.*

Proof. Let $\mathcal{M} = (W, R, V)$ be the bidirectionally generated submodel of a transitive, left linear model, generated by w. Let u and v be distinct instants in W. We need to show that either Ruv or Rvu, or they have a common predecessor.

If either Ruv or Rvu then we are done.

Next, suppose we do not have Ruv or Rvu. Then we are looking for a $t \in W$ such that Rtu and Rtv. Notice that, by construction of the submodel, we have a z-path connecting u to w and a z-path connecting w to v (this is also true in the case where either u or v is equal to w, in which case it is the trivial path). Let σ be the z-concatenation of the path starting at u and ending at w and the z-path from w to v. We will use induction on the length of σ to show that u and v have a common predecessor. Hence, we will prove the statement "The endpoints of a z-path of length greater than or equal to 2 have a common predecessor."

First, if σ is of length 2 then we must have Rwu and Rwv, since the case when Ruw and Rvw contradicts left linearity. Therefore, u and v have a common predecessor w.

Next, let ρ be a z-path from u to v of length $k + 1$ and let σ be the subpath of ρ from u to u_k of length k. Suppose that the statement holds for all z-paths of length k or less, where $k \geq 2$. Then by the induction hypothesis u and u_k have a common predecessor s such that Rsu and Rsu_k. Then ρ is the z-path of length $k + 1$ that extends σ such that the last step in the path is either Ru_kv or Rvu_k. If Ru_kv, then we have Rsv by transitivity, and hence u and v have a common predecessor s. Therefore, suppose Rvu_k. If Rsv we are done, therefore suppose we do not have Rsv. Then, since Rsu_k we have a z-path of length 2 from v to s and hence, by the induction hypothesis there exists a t such that Rtv and Rts. But since Rsu, it follows from transitivity that Rtu. Hence t is the required common predecessor.

Hence $\mathcal{M}$ is connected as required. □

An important fact that will be use extensively in this thesis is that modal satisfaction is invariant under generated submodels (see e.g., Proposition 2.6. in [4]).

2.5 Modal and Temporal Logics

In this section we set the background for the work done in this thesis. We define most of the basic concepts and state a few results from literature, and offer short proofs for some results that will come in useful for our purposes. Unless otherwise stated, the definitions below are from [4] and [15]. We start with the definition of a normal logic, as most of this thesis will be to find logics that are sound and complete with respect to a particular class of trees. We also list the axioms that we will use for this purpose, and define some additional concepts that we will use continuously. We then define the canonical model of a logic. In most of our constructions to build a suitable model for the logic in question, we will start with the canonical model of the logic and then transform the canonical model to a tree. We also define decidability and some related concepts that will be used in Chapter 6 to show whether the logics we found in the completeness chapters are decidable.

Definition 2.5.1. Normal Modal Logic: A **normal modal logic** Λ is a set of formulas in $\mathcal{L}^m_{Prior}$ that contains all propositional tautologies, $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$, and $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$, and that is closed under **modus ponens**, **uniform substitution** and **generalisation** defined below.

- **Modus ponens**: if φ and $\varphi \to \psi$, then ψ.
- **Uniform substitution**: if φ, then ψ, where ψ is obtained from φ by uniformly replacing proposition letters in φ by arbitrary formulas.
- **Generalisation**: if φ, then $\mathbf{G}\varphi$.

The smallest normal modal logic is **K**.

Definition 2.5.2. Normal Temporal Logic: A **normal temporal logic** Λ is a set of formulas in $\mathcal{L}_{Prior}$ that contains all propositional tautologies, as well as

- **K** axioms for **G** and **H**: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ and $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$
- Dual axioms: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ and $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$
- Converse axioms of the past-future relations: $p \to \mathbf{GP}p$ and $p \to \mathbf{HF}p$

The logic is also closed under the following rules:

- **Modus ponens**: if φ and $\varphi \to \psi$, then ψ.
- **Uniform substitution**: if φ, then ψ, where φ is obtained from ψ by uniformly replacing proposition letters in φ by arbitrary formulas.
- **Generalisation**: if φ, then $\mathbf{G}\varphi$ and if φ, then $\mathbf{H}\varphi$.

The smallest normal temporal logic is denoted by $\mathbf{K_t}$.

We will refer to normal modal logics and normal temporal logics as modal logics or temporal logics for short.

In the process of finding axiomatisations for the different classes of trees, we will use some axioms known to define certain properties and add additional axioms in the appropriate sections to define properties we need. We give a list of these axioms in Table 2.1 but will also define them in the appropriate sections.

All the axioms are canonical with the exception of **S**, **Q**, $\mathbf{L_l}$, $\mathbf{L_r}$, **Grz** and $\mathbf{Grz_l}$. Note that, although **D**, $\mathbf{U_l}$ and $\mathbf{U_r}$ define density, left unboundedness and right unboundedness of Kripke frames, they are not strong enough to

Abbr	Axiom	Property of Kripke Frames defined by Axiom
$\mathbf{K_G}$	$\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$	Valid on all frames
$\mathbf{K_H}$:	$\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$	Valid on all frames
$\mathbf{Dual_F}$	$\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$	Valid on all frames
$\mathbf{Dual_P}$	$\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$	Valid on all frames
Conv	$p \to \mathbf{GP}p$ and $p \to \mathbf{HF}p$	Valid on all frames
4	$\mathbf{FF}p \to \mathbf{F}p$	$\forall xyz(Rxy \wedge Ryz \to Rxz)$ (Transitivity)
$\mathbf{.3_l}$	$(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$	$\forall xyz(Rxz \wedge Ryz \to Rxy \vee Ryx \vee x = y)$ (Left linear)
D	$\mathbf{F}p \to \mathbf{FF}p$	$\forall xy(Rxy \to \exists x(Rxz \wedge Rzy))$ (Density of Kripke frames - Note that difference with the notion of density as used in Section 2.1, where the intermediate point is required to be different from both the preceding and succeeding points.)
$\mathbf{U_l}$	$\mathbf{P}\top$	$\forall x\exists yRyx$ (Left seriality or left unboundedness of Kripke frames - Note the difference with left unboundedness as defined in Section 2.1.)
$\mathbf{U_r}$	$\mathbf{F}\top$	$\forall x\exists yRxy$ (Right seriality or right unboundedness of Kripke frames - Note the difference with right unboundedness as defined in Section 2.1.)
S	$\neg p \wedge \mathbf{H}\neg p \wedge \mathbf{F}p \to \mathbf{F}(p \wedge \mathbf{H}\neg p)$	No infinitely descending sequences between two instants (irreflexive frames)
Q	$\mathbf{P}p \vee \mathbf{GH}\neg p \vee \mathbf{F}(p \wedge \mathbf{H}(p \vee \mathbf{H}\neg p))$	No infinitely descending sequences of distinct instants between two instants (reflexive frames)
$\mathbf{L_l}$	$\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$	Well-foundedness, transitivity, irreflexivity
$\mathbf{L_r}$	$\mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$	Converse well-foundedness, transitivity, irreflexivity
T	$p \to \mathbf{F}p$	Reflexivity
Grz	$\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$	No infinitely ascending sequences of distinct instants, transitivity, reflexivity
$\mathbf{Grz_l}$	$\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$	No infinitely descending sequences of distinct instants, transitivity, reflexivity

Table 2.1: List of Axioms

define these properties in trees as defined in Section 2.1. These axioms are valid on all reflexive frames, regardless of the property of the frame and hence cannot give us the needed properties in these cases. We will therefore refer to left seriality and right seriality, as oppose to unboundedness, to distinguish between the Kripke frame property and the order property of trees. The use of the term density will be clear from the context.

Definition 2.5.3. Derivability in a Logic: A **derivation** in a logic Λ is a finite sequence of formulas where each formula is either a tautology, an axiom of the logic or the result of applying the rules of the logic to previous steps. A formula φ appearing as the last element in a derivation is called a **theorem** of the logic, which we will write it as $\vdash_\Lambda \varphi$ and say that φ is **derivable** in Λ.

If $\Gamma \cup \varphi$ is a set of formulas, then φ is deducible in Λ from Γ if $\vdash_\Lambda \varphi$ or there are formulas $\varphi_1, ..., \varphi_n \in \Gamma$ such that $\vdash_\Lambda (\varphi_1 \wedge ... \wedge \varphi_n) \to \varphi$. If this is the case we write $\Gamma \vdash_\Lambda \varphi$, if not, $\Gamma \nvdash_\Lambda \varphi$.

Definition 2.5.4. Consistent Set of Formulas: Let Λ be a logic. A set of formulas Γ is **Λ-consistent** if $\Gamma \nvdash_\Lambda \perp$, and Λ-inconsistent otherwise. A formula φ is Λ-consistent if $\{\varphi\}$ is Λ-consistent; otherwise φ is Λ-inconsistent.

Definition 2.5.5. Axiomatisation: The normal temporal (resp. normal modal) logic Λ generated by a set of formulas Γ is the smallest normal temporal, (resp. normal modal) logic containing Γ.

More loosely, any set of axioms and rules from which all and only the theorems of a given logic can be derived, will be called an **axiomatisation** of that logic.

Next, we discuss the canonical model of a logic and some related results and concepts. In most of our constructions to build a suitable model for the logic in question, we will start with the canonical model of the logic and then transform the canonical model to a tree. We also define decidability and some related concepts that will be used in Chapter 6 to show whether the logics we found in the completeness chapters are decidable.

Definition 2.5.6. Maximal Consistent Set: Let Λ be a normal logic. A set of formulas Γ is **maximal Λ-consistent**, abbreviated as mcs, if Γ is Λ-consistent, and any set of formulas properly containing Γ is Λ-inconsistent.

Definition 2.5.7. Canonical Model for Modal Logic: Let Λ be a modal logic. The **canonical model**, $\mathcal{M}^\Lambda$ for a modal logic Λ (in the modal language) is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$ where:

- W^Λ is the set of all maximal Λ-consistent sets.
- R^Λ is the binary relation on W^Λ defined as follows: $R^\Lambda wu$ if for all formulas φ, if $\varphi \in u$ then $\mathbf{F}\varphi \in w$. R^Λ is called the **canonical relation**.
- V^Λ is the valuation defined by $V^\Lambda(p) = \left\{w \in W^\Lambda \mid p \in w\right\}$. V^Λ is called the **canonical (or natural) valuation**.

Definition 2.5.8. Canonical Model for Temporal Logic: The **canonical model for a temporal logic** Λ is the structure $\mathcal{M}^\Lambda = (W^\Lambda, R^\Lambda, V^\Lambda)$ where:

- W^Λ is the set of all Λ-mcs's.
- R^Λ is the binary relation on W^Λ defined by $R^\Lambda wu$ if for all formulas φ, $\varphi \in u$ implies $\mathbf{F}\varphi \in w$, and for all formulas φ, $\varphi \in w$ implies $\mathbf{P}\varphi \in u$.
- V^Λ is the valuation defined by $V^\Lambda(p) = \left\{w \in W^\Lambda \mid p \in w\right\}$

Definition 2.5.9. Canonical for a property: Let φ be a formula, and P be a property of frames. If the canonical frame for any logic Λ containing φ has property P, and φ is valid on any class of frames with property P, then φ is **canonical for** P.

Lemma 2.5.10 (Existence Lemma)**.** *Let $\mathcal{M} = (W, R, V)$ be the canonical model for a logic Λ in the future fragment of the Priorian language. Suppose $w \in W$ and $\mathbf{F}\psi \in w$. Then there is a $u \in W$ such that $\psi \in u$ and Rwu. For a logic in the Priorian temporal language, the following condition is added: Suppose $\mathbf{P}\psi \in w$. Then there is a u such that $\psi \in u$ and Ruw. (See e.g., Lemma 4.20 in [4]. The proof for $\mathbf{P}\psi$ is symmetric.)*

Lemma 2.5.11 (Truth Lemma)**.** *Let $\mathcal{M} = (W, R, V)$ be the canonical model for a logic Λ in the future fragment of the Priorian language and let $w \in W$. Then $\mathcal{M}, w \Vdash \varphi$ iff $\varphi \in w$. (See e.g., Lemma 4.21 in [4].)*

In the completeness chapters we will show that the chosen logic is sound and complete with respect to a certain class of trees. We define these concepts next.

Definition 2.5.12. Soundness: Let F be a class of frames and let Λ_{F} be the set of all formulas that are valid on F (Note that Λ_{F} is a logic.). A modal/temporal logic Λ is sound with respect to F if $\Lambda \subseteq \Lambda_{\mathsf{F}}$. (Equivalently: Λ is sound with respect to F if for all formulas φ, and all structures $\mathcal{F} \in \mathsf{F}$, we have that if $\vdash_{\Lambda} \varphi$ then $\mathcal{F} \Vdash \varphi$). If Λ is **sound** with respect to F, we say that F is a class of frames for Λ.

Definition 2.5.13. Completeness: Let (F) be a class of frames and let Γ be a set of formulas. A logic Λ is **strongly complete** with respect to (F) if $\Gamma \vdash_{\Lambda} \varphi$ whenever $\Gamma \Vdash_{(F)} \varphi$.

A logic Λ is **weakly complete** with respect to a class of frames (F) if, for any formula φ, it is the case that $\vdash_{\Lambda} \varphi$ whenever $\Vdash_{(F)} \varphi$.

Definition 2.5.14. Compactness: The logic of a class C of frames is compact if every set Γ of formulas which is finitely satisfiable in C, is satisfiable in C.

Note that if a logic Λ is strongly complete with respect to a class of frames, then entailment (see Definition 2.3.12) over this class of frames is compact. We can see this when noting that strong completeness requires that any Λ-consistent set of formulas must be satisfied in any model for the logic at a particular instant. Furthermore, any finite subset of a Λ-consistent set of formulas is also Λ-consistent. Hence, compactness follows. Conversely, if a logic is not compact, it cannot be strongly complete with respect to any class of frames, and hence, it is not canonical.

Definition 2.5.15. Subformula Closed: A set of formulas Φ is a **subformula closed** set of formulas if for every $\varphi \in \Phi$ we have that every subformula of φ is also in Φ.

Definition 2.5.16. Decidability: A logic is decidable if there is an algorithm to determine whether any formula is a theorem of the logic or not.

2.6 Tree-like Models

When considering how to transform a general model into a tree, there are some processes or tools that can be used, for example, in the modal language, a model can be unravelled into a tree, and in the temporal language, we can consider the quotient frame and work on the cluster level, when we have a left linear, transitive and connected model. We define these concepts next and will use them throughout the thesis.

Definition 2.6.1. Unravelling: (Note that unravelling only works in the modal language. In Chapter 4 we will define a similar process for the temporal language.) Let $\mathcal{M}$ be a rooted model and let w be the root of $\mathcal{M}$. Define the model $\mathcal{M}' = (W', R', V')$ as follows: W is the set of all finite sequences $w, u_1, ..., u_n$, $n \geq 0$, such that there exists a path $wRu_1...Ru_n$ in $\mathcal{M}$. Define $(w, u_1, ..., u_n)R'(w, u_1, ..., u_{n+1})$ to hold if Ru_nu_{n+1} holds in $\mathcal{M}$. That is, R' relates two sequences iff the second is an extension of the first with an instant from $\mathcal{M}$ that is a successor of the last element of the first sequence. Finally, V' is defined by $(w, u_1, ..., u_n) \in V'(p)$ iff $u_n \in V(p)$. $\mathcal{M}'$ is called an **unravelling** of $\mathcal{M}$ from w.

Furthermore, the mapping $f : (w, u_1, ..., u_n) \mapsto u_n$ defines a surjective bounded morphism from $\mathcal{M}'$ to $\mathcal{M}$, thus $\mathcal{M}'$ and $\mathcal{M}$ satisfy the same modal formulas at corresponding instants (see e.g. p 62 in [4]). Hence, it follows that any satisfiable modal formula is satisfiable in a tree. Also note that unravelling can also be done in a model that is not rooted by selecting any instant in that model and unravelling from there.

Definition 2.6.2. Cluster: Let $\mathcal{F} = (W, R)$ be a transitive frame and let $\sim$ be the relation on W where for all $v, w \in W$ $v \sim w$ iff $v = w$ or (Rvw and Rwv). Then $\sim$ is an equivalence relation and the equivalence classes induced by this relation are called **clusters**.

There are three types of clusters, and using terminology from [15] they are as follows:

- a **degenerate cluster** consists of a single irreflexive point
- a **simple cluster** consists of a single reflexive point
- a **proper cluster** consists of at least two (reflexive) points

Furthermore, the **quotient frame** with respect to $\sim$ of $\mathcal{F}$ is $\mathcal{F}_\sim = (W_\sim, R_\sim)$ and is defined by $W_\sim = \{S(x) \mid x \in W\}$ where $S(x)$ is the cluster containing x and $S(x)R_\sim S(y)$ if and only if xRy. When the quotient frame is a tree (see Section 2.2) then we say that $\mathcal{F}$ is a **tree of clusters**.

To show that $R_\sim$ is well-defined let $x, x' \in S(x)$ and $y, y' \in S(y)$ and suppose $S(x)R_\sim S(y)$. Then Rxy by definition of $R_\sim$. But since $x, x' \in S(x)$ it follows that either $x' = x$, in which case $Rx'y$, or that $Rx'x$ and hence by transitivity $Rx'y$. Also, since $y, y' \in S(y)$ it follows that Ryy' and again by transitivity $Rx'y'$. Hence, by definition of $R_\sim$, we have $S(x')R_\sim S(y')$.

$\mathcal{F}_\sim$ is also called the **skeleton** of $\mathcal{F}$.

We say that quotient frame has a **root cluster** C if for no cluster E do we have $R_\sim EC$. Lastly, a quotient frame has a **unique root cluster** C if C is a root cluster and, for no cluster E with $E \neq C$, is it the case that $R_\sim EC$.

2.7 Finite Models and Frames

In this section we define the process of filtering a model through a finite set of formulas to get a finite model that can more easily be manipulated into a tree. It is also a useful process to consider when showing completeness of the classes of finite trees. We also define the finite model property that will help us show that the logics we consider are decidable.

Definition 2.7.1. Filtration: Let $\mathcal{M} = (W, R, V)$ be a model for a modal logic and let $Cl(\alpha)$ be a subformula closed set of formulas closed under single negations. Define an equivalence relation on the states of $\mathcal{M}$ as follows: $w \sim_{Cl(\alpha)} v$ iff for all $\varphi \in Cl(\alpha)$, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}, v \Vdash \varphi$. We denote the equivalence class containing w by $[w]$. Let $W_{Cl(\alpha)} = \{[w] \mid w \in W\}$.

Suppose $\mathcal{M}^f_{Cl(\alpha)} = (W^f, R^f, V^f)$ is any model such that:

- $W^f = W_{Cl(\alpha)}$.
- If Rwv then $R^f[w][v]$.
- If $R^f[w][v]$, then $\mathcal{M}, v \Vdash \varphi$ implies $\mathcal{M}, w \Vdash \mathbf{F}\varphi$ for all $\mathbf{F}\varphi \in Cl(\alpha)$.
- $V^f(p) = \{[w] \mid \mathcal{M}, w \Vdash p\}$, for all proposition letters p in $Cl(\alpha)$.

Then $\mathcal{M}^f_{Cl(\alpha)}$ is called a **filtration** of $\mathcal{M}$ through $Cl(\alpha)$. If $\mathcal{M}^f_{Cl(\alpha)}$ is a filtration of $\mathcal{M}$ through $Cl(\alpha)$, then for all formulas $\varphi \in Cl(\alpha)$ and all instants $w \in W$ we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \varphi$ (see e.g. Theorem 2.39 in [4]).

The relation R^t, defined as $R^t\,[w]\,[v]$ iff for all φ, if $\mathbf{F}\varphi \in Cl(\alpha)$ and $\mathcal{M}, v \Vdash \varphi \vee \mathbf{F}\varphi$ then $\mathcal{M}, w \Vdash \mathbf{F}\varphi$, is transitive and satisfies the conditions imposed on R^f, if R is transitive. This is called the **transitive filtration for modal logics**.

Definition 2.7.2. Bidirectional Filtration: Let $\mathcal{M} = (W, R, V)$ be a model for a temporal logic, let α be a formula of the logic and let $Cl(\alpha)$ be a subformula closed set of formulas closed under single negations. Define an equivalence relation on the states of $\mathcal{M}$ as follows: $w \sim_{Cl(\alpha)} v$ iff for all $\varphi \in Cl(\alpha)$, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}, v \Vdash \varphi$. We denote the equivalence class containing w by $[w]$. Let $W_{Cl(\alpha)} = \{[w] \mid w \in W\}$.

Suppose $\mathcal{M}^f_{Cl(\alpha)} = (W^f, R^f, V^f)$ is any model such that:

- $W^f = W_{Cl(\alpha)}$
- If Rwv then $R^f\,[w]\,[v]$
- If $R^f\,[w]\,[v]$ then for all $\mathbf{G}\varphi \in Cl(\alpha)$, if $\mathcal{M}, w \Vdash \mathbf{G}\varphi$ then $\mathcal{M}, v \Vdash \varphi$
- If $R^f\,[w]\,[v]$ then for all $\mathbf{H}\varphi \in Cl(\alpha)$, if $\mathcal{M}, v \Vdash \mathbf{H}\varphi$ then $\mathcal{M}, w \Vdash \varphi$
- $V^f(p) = \{[w] \mid \mathcal{M}, w \Vdash p\}$, for all proposition letters p in $Cl(\alpha)$.

Then $\mathcal{M}^f_{Cl(\alpha)}$ is called a **bidirectional filtration** of $\mathcal{M}$ through $Cl(\alpha)$.

Furthermore:

Proposition 2.7.3. *If $\mathcal{M}^f_{Cl(\alpha)}$ is a bidirectional filtration of $\mathcal{M}$ through $Cl(\alpha)$, then for all formulas $\varphi \in Cl(\alpha)$ and all instants $w \in W$ we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \varphi$.*

Proof. Let $\mathcal{M}$, $\mathcal{M}^f_{Cl(\alpha)}$ be defined as above and let $\varphi \in Cl(\alpha)$. We use induction on φ:

Suppose φ contains no connectives or modalities. Then either φ is a propositional letter or $\varphi = \bot$. If $\varphi = \bot$ then $\mathcal{M}, w \not\models \bot$ iff $w \notin V(\bot)$ iff $[w] \notin V'(\bot)$ by definition of V'. And hence $\mathcal{M}^f_{Cl(\alpha)}, [w] \nVdash \bot$ If $\varphi = p$ where p is a propositional letter then $\mathcal{M}, w \models p$ iff $w \in V(p)$ iff $[w] \in V'(p)$ by definition of V' and hence $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash p$.

Assume that for all $[w] \in W'$, $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \psi$ iff $\mathcal{M}, w \Vdash \psi$ and $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \gamma$ iff $\mathcal{M}, w \Vdash \gamma$ for some formulas ψ and γ in $Cl(\alpha)$.

Checking for $\neg\psi$, $\psi \vee \gamma$, $\mathbf{G}\psi$ and $\mathbf{H}p\psi$:

If φ is $\neg\psi$, then $\mathcal{M}, w \Vdash \neg\psi$ iff $\mathcal{M}, w \nVdash \psi$ iff $w \notin V(\psi)$ iff $[w] \notin V'(\psi)$ by the induction hypothesis and as $Cl(\alpha)$ is closed under subformulas, iff $[w] \in V'(\neg\psi)$ and hence $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \neg\psi$.

If φ is $\psi \vee \gamma$, then $\mathcal{M}, w \Vdash \psi \vee \gamma$ iff $\mathcal{M}, w \Vdash \psi$ or $\mathcal{M}, w \Vdash \psi$ iff $w \in V(\psi)$ or $w \in V(\gamma)$ by the induction hypothesis and as $Cl(\alpha)$ is closed under subformulas, iff $[w] \in V'(\psi \vee \gamma)$ and hence $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \psi \vee \gamma$.

If φ is $\mathbf{G}\psi$:

$\Rightarrow$: $\mathcal{M}, w \Vdash \mathbf{G}\psi$ and suppose $R^f[w][v]$. Then by definition of R^f, for all $\mathbf{G}\gamma \in Cl(\alpha)$ with $\mathbf{G}\gamma \in w$ we have $\mathbf{G}\gamma, \gamma \in v$. Hence $\mathcal{M}, v \Vdash \mathbf{G}\psi$ and $\mathcal{M}, v \Vdash \psi$ and by the induction hypothesis $\mathcal{M}^f_{Cl(\alpha)}, [v] \Vdash \psi$. Hence $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \mathbf{G}\psi$.

$\Leftarrow$: Conversely, suppose $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \mathbf{G}\psi$. Suppose $v \in W$ with Rwv, then $R^f[w][v]$ and hence $\mathcal{M}', [v] \Vdash \psi$. Since $\psi \in Cl(\alpha)$ it follows from the induction hypothesis that $\mathcal{M}, v \Vdash \psi$. Hence,$\mathcal{M}, w \Vdash \mathbf{G}\psi$.

The case for when φ is $\mathbf{H}\psi$ follows symmetrically.

Therefore, $\varphi \in Cl(\alpha)$, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^f_{Cl(\alpha)}, [w] \Vdash \varphi$. □

We will also use the finite model property in Chapter 6, which we define next.

Definition 2.7.4. Finite Model Property: A logic Λ has the **finite model property** if there is a class of finite frames $\mathcal{C}$ such that $\Lambda = \{\varphi \mid \text{for all } \mathcal{F} \in \mathcal{C}, \mathcal{F} \Vdash \varphi\}$.

2.8 Networks

In this section we define the concept of networks, which are used to build a tree, step-by-step, from the canonical model, that will be used to help transform the canonical model of a logic to a tree.

Definition 2.8.1. Network for a Λ-consistent formula or a Λ-consistent set of formulas of a logic Λ: A **network** $\mathcal{N} = (N, \ll, \kappa)$ for a Λ-consistent formula / set of formulas, where Λ is a temporal logic, is a set of nodes N with a binary relation $\ll$ ($s\underline{\ll}t$ in the reflexive case) on N and a labelling function κ that maps nodes to mcs's in the canonical model $\mathcal{M} = (W, R, V)$ for Λ (or to instants/mcs's in a submodel of the canonical model for Λ), such that the Λ-consistent formula /set of formulas is the label for some node in N. We take $s\underline{\ll}t$ to mean $s \ll t$ or $s = t$.

Every network gives rise to an underlying frame and model in the following way. For the network $\mathcal{N} = (N, \ll, \kappa)$, $\mathcal{F}_{\mathcal{N}} = (N, \ll)$ is the **underlying frame**. The underlying valuation $V_{\mathcal{N}}$ on $\mathcal{F}_{\mathcal{N}}$ is defined by $V_{\mathcal{N}}(p) = \{s \in N \mid p \in \kappa(s)\}$. $\mathcal{M}_{\mathcal{N}} = (\mathcal{F}_{\mathcal{N}}, V_{\mathcal{N}})$ is the **underlying model**.

A network $\mathcal{N}_1 = (N_1, \ll_1, \kappa_1)$ **extends** a network $\mathcal{N}_0 = (N_0, \ll_0, \kappa_0)$ if the underlying frame $\mathcal{F}_{\mathcal{N}_1}$ is a subframe of the underlying frame (i.e. $\mathcal{F}_{\mathcal{N}_0}$ and κ_0 agrees with κ_1 on $\mathcal{N}_0$).

In the following definitions, we will define coherency in both irreflexive and reflexive networks respectively.

Definition 2.8.2. Strict Coherent Network: For the purposes of branching time structures, we define a **strict coherent network** for a logic Λ to be a tree structure (2.2). A network $\mathcal{N} = (N, \ll, \kappa)$ strict coherent if:

1. $\ll$ is a strict partial ordering.
2. $\ll$ is left linear, i.e. for all s, t and u if $s \ll t$ and $u \ll t$ then either $s\underline{\ll}u$ or $u\underline{\ll}s$.
3. N is connected, i.e. for all $s, t \in N$, there is a $u \in N$ such that $u\underline{\ll}s$ and $u\underline{\ll}t$.
4. $\kappa(s)R\kappa(t)$ for all $s, t \in N$ such that $s \ll t$ where R is the canonical relation.

Definition 2.8.3. Coherent Network: For the purposes of branching time structures, we define a **coherent network** for a logic Λ to be a tree structure (2.2). A network $\mathcal{N} = (N, \ll, \kappa)$ coherent if:

1. $\ll$ is a reflexive partial ordering.
2. $\ll$ is left linear, i.e. for all s, t and u if $s \ll t$ and $u \ll t$ then either $s\underline{\ll}u$ or $u\underline{\ll}s$.
3. N is connected, i.e. for all $s, t \in N$, there is a $u \in N$ such that $u\underline{\ll}s$ and $u\underline{\ll}t$.
4. $\kappa(s)R\kappa(t)$ for all $s, t \in N$ such that $s \ll t$ where R is the canonical relation.

Coherent and strict coherent networks give rise to underlying frames and models that are reflexive/irreflexive trees (2.2). The fourth condition is equivalent to "if $s \ll t$ then $\mathbf{F}\varphi \in \kappa(s)$ for all $\varphi \in \kappa(t)$ and $\mathbf{P}\varphi \in \kappa(t)$ for all $\varphi \in \kappa(s)$".

Definition 2.8.4. Saturated Network: A network for a temporal logic is **saturated** if

1. If $\mathbf{F}\varphi \in \kappa(s)$ for some $s \in N$ then there is a $t \in N$ such that $s \ll t$ and $\varphi \in \kappa(t)$.

2. If $\mathbf{P}\varphi \in \kappa(s)$ for some $s \in N$ then there is a $t \in N$ such that $t \ll s$ and $\varphi \in \kappa(t)$.

A definition of a saturated network for a modal logic can be obtained by simply omitting the second clause in the definition of saturation.

A network that is both coherent/strict coherent and saturated is called a **perfect/strict perfect network**. For all perfect networks and their underlying models we have the following result:

Lemma 2.8.5 (Truth Lemma)**.** *Let $\mathcal{N} = (N, \ll, \kappa)$ be a perfect network. Then for all formulas φ in $\mathcal{L}^m_{Prior}$ (in the modal case) or $\mathcal{L}_{Prior}$ (in the temporal case) and nodes $s \in N$, $\mathcal{M}_\mathcal{N}, s \Vdash \varphi$ iff $\varphi \in \kappa(s)$ where $\mathcal{M}_\mathcal{N} = (W_\mathcal{N}, R_\mathcal{N}, V_\mathcal{N})$ is the underlying model of $\mathcal{N}$.*

Proof. Proof by induction on φ:

Suppose φ contains no connectives or modalities. Then either φ is a propositional letter or $\varphi = \bot$. If $\varphi = \bot$ then $\mathcal{M}_\mathcal{N}, s \nVdash \bot$ iff $s \notin V_\mathcal{N}(\bot)$ iff $\bot \notin \kappa(s)$. If $\varphi = p$ where p is a propositional letter then $\mathcal{M}_\mathcal{N}, s \Vdash p$ iff $s \in V_\mathcal{N}(p)$ iff $p \in \kappa(s)$.

Assume that for all $s \in N$, $\mathcal{M}_\mathcal{N}, s \Vdash \psi$ iff $\varphi \in \kappa(s)$ and $\mathcal{M}_\mathcal{N}, s \Vdash \gamma$ iff $\psi \in \kappa(s)$ for some formulas ψ and γ.

Checking for $\neg\psi$, $\psi \vee \gamma$, $\mathbf{F}\psi$ (and $\mathbf{P}\psi$ in the temporal case):

If φ is $\neg\psi$, then $\mathcal{M}_\mathcal{N}, s \Vdash \neg\psi$ iff $\mathcal{M}_\mathcal{N}, s \nVdash \psi$ iff $s \notin V_\mathcal{N}(\psi)$ iff $\psi \notin \kappa(s)$ by the induction hypothesis, iff $\neg\psi \in \kappa(s)$ as $\kappa(s)$ is an mcs.

If φ is $\psi \vee \gamma$, then $\mathcal{M}_\mathcal{N}, s \Vdash \psi \vee \gamma$ iff $\mathcal{M}_\mathcal{N}, s \Vdash \psi$ or $\mathcal{M}_\mathcal{N}, s \Vdash \psi$ iff $\psi \in \kappa(s)$ or $\gamma \in \kappa(s)$ by the induction hypothesis, iff $\psi \vee \gamma \in \kappa(s)$.

If φ is $\mathbf{F}\psi$:

$\Rightarrow$: $\mathcal{M}_\mathcal{N}, s \Vdash \mathbf{F}\psi$ iff $\exists t$ with $s \ll t$ and $\mathcal{M}_\mathcal{N}, t \Vdash \psi$ iff $\exists t$ with $s \ll t$ and $\varphi \in \kappa(t)$ by the induction hypothesis. Then by the fourth property of Definition 2.8.2 we have $\mathbf{F}\psi \in \kappa(s)$.

$\Leftarrow$: Let $\mathbf{F}\psi \in \kappa(s)$. Then, by the modal saturation (see Definition 2.8.4) there is a $t \in N$ such that $s \ll t$ and $\psi \in \kappa(t)$. Then $\mathcal{M}_\mathcal{N}, t \Vdash \psi$ by the induction hypothesis. Hence $\mathcal{M}_\mathcal{N}, s \Vdash \mathbf{F}\psi$.

(The case for when φ is $\mathbf{P}\psi$ in the temporal case follows similarly.)

Therefore $\mathcal{M}_\mathcal{N}, s \Vdash \varphi$ iff $\varphi \in \kappa(s)$ for all φ in $\mathcal{L}^m_{Prior}$ (respectively $\mathcal{L}_{Prior}$).

□

We are now ready to look into results in the literature that might give us some direction on how to find axiomatisations for the different classes of trees we are interested in.

2.9 Axiomatisations of the Logics of Linear Frames

Although there are few results in the literature on axiomatisations in the Priorian languages of logics of specific classes of trees, several authors have published results on linear frames. Most comprehensive, are the studies by Segerberg in [60], as well as Burgess in [10] and [13]. Other studies include [51], [32] and [53]. We give a short summary of some of these results that in this section, focussing on the axiomatisations that inspired the axiomatisations of different classes of trees in this thesis.

Staying with the notation a defined in Table 2.1 and used in this thesis, **K** is the smallest normal modal logic, **4** the axiom for transitivity and **.3$_\mathbf{r}$** the axiom for right linearity. We know that **K4.3$_\mathbf{r}$** is sound and strongly complete with respect to the class of all linear orders. In [60] Segerberg uses this fact to build axiomatisation of the logics of some subclasses of linear orders. In Table 2.2 we summarise the axiomatisations of the logics of linear frames in the Priorian modal language. Note that we only mention the additional axioms added to **K4.3$_\mathbf{r}$** needed in the axiomatisations.

Class of Trees	**Axiomatisation**
Irreflexive linear orders and well-founded linear orders	No additional axioms
Reflexive linear orders, well-founded reflexive linear orders, reflexive linear orders isomorphic to the rational numbers, reflexive linear orders isomorphic to the real numbers	$\mathbf{T}$
Unbounded irreflexive linear orders	$\mathbf{U_r}$
Reflexive linear orders	$\mathbf{T}$
Unbounded reflexive linear orders	$\mathbf{T}$, $\mathbf{U_r}$
Dense irreflexive linear orders	$\mathbf{D}$
Dense, unbounded irreflexive linear orders, irreflexive linear orders isomorphic to the rational numbers, irreflexive linear orders isomorphic to the real numbers	$\mathbf{D}$
Irreflexive linear orders isomorphic to the integers, Irreflexive linear orders isomorphic to the natural numbers	$\mathbf{U_r}$, $\mathbf{G}(\mathbf{G}p \to p) \to (\mathbf{FG}p \to \mathbf{G}p)$
Finite irreflexive linear orders, conversely well-founded irreflexive linear orders	$\mathbf{L_r}$
Reflexive linear orders isomorphic to the integers, reflexive linear orders isomorphic to the natural numbers	$\mathbf{U_r}$, $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to (\mathbf{FG}p \to p)$
Conversely well-founded reflexive linear orders	$\mathbf{Grz}$

Table 2.2: Axioms added to $\mathbf{K4.3_r}$ of Modal Logics of Linear Frames in [60]

In the Table 2.3 is a summary of axiomatisations of the logics of linear frames in the Priorian temporal language as found by [60]. All axiomatisations include those of $\mathbf{K_t}$, the smallest normal temporal logic, $\mathbf{4}$, $.\mathbf{3_l}$ and $.\mathbf{3_r}$, the axioms for transitivity, and right and left linearity, as well as those mentioned in the table. We use $\mathbf{A}\varphi$ as an abbreviation for $\mathbf{H}\varphi \wedge \varphi \wedge \mathbf{G}\varphi$ and $\mathbf{E}\varphi$ as an abbreviation for $\mathbf{P}\varphi \vee \varphi \vee \mathbf{F}\varphi$.

In Table 2.4, we summarise the results for axiomatisations of the logics of linear orders found by Burgess in [13]. Burgess starts with $\mathbf{K_t}$, the smallest normal temporal logic, as well as the axiom for transitivity, $\mathbf{4}$, and the following: $\mathbf{F}p \wedge \mathbf{F}p \to \mathbf{F}(p \wedge \mathbf{F}p) \vee \mathbf{F}(p \wedge q) \vee \mathbf{F}(\mathbf{F}p \wedge q)$ and the dual $\mathbf{P}p \wedge \mathbf{P}p \to \mathbf{P}(p \wedge \mathbf{P}p) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(\mathbf{P}p \wedge q)$. He shows that this logic is sound and complete with respect to the class of all linear orders and uses this as a basis for some subclasses as listed in the Table 2.4.

Both [60] and [13] used filtrations to show completeness of the logics with respect to discrete linear orders and linear orders isomorphic to the real numbers. However, [19] used mcs's to build a model in order to show strong completeness for these logics. We summarise these results in Table 2.5.

These results come in useful when searching for axiomatisations for the logics of the classes of trees as each branch can be seen as a linear ordering. In some cases, only minor adjustments have to be made to translate these results into axiomatisations for the logics of classes of trees. In other cases, new methods will have to be developed to find these axiomatisations.

Another useful investigation for the work in this thesis, is to see if we can define the classes of trees with a set of axioms when we restrict the frames to be trees. This is done in the next section and will also help us to find axiomatisations.

Class of Trees	Axiomatisation
Irreflexive linear orders	No additional axioms
Reflexive linear orders, reflexive linear orders isomorphic to the rational numbers	$\mathbf{T}$
Irreflexive linear orders isomorphic to the rational numbers	$\mathbf{D}$, $\mathbf{U_l}$, $\mathbf{U_r}$
Irreflexive linear orders isomorphic to the real numbers	$\mathbf{D}$, $\mathbf{U_l}$, $\mathbf{U_r}$, $\mathbf{C_l}$, $(\mathbf{F}p \leftrightarrow \mathbf{H}p) \rightarrow (\mathbf{E}p \rightarrow \mathbf{A}p)$
Reflexive linear orders isomorphic to the real numbers	$\mathbf{T}$, $\mathbf{E}p \wedge \mathbf{E}\neg p \wedge \mathbf{A}(\mathbf{H}p \vee \mathbf{G}\neg p) \rightarrow \mathbf{E}(\mathbf{H}p \wedge (q \rightarrow \mathbf{G}(p \rightarrow q)) \wedge (\neg q \rightarrow \mathbf{G}(p \rightarrow \neg q))) \vee \mathbf{E}(\mathbf{G}\neg p \wedge (r \rightarrow (\neg p \rightarrow r)) \vee (\neg r \rightarrow \mathbf{H}(\neg p \rightarrow \neg r)))$
Finite reflexive linear orders	$\mathbf{T}$, $\mathbf{Grz}$, $\mathbf{Grz_l}$
Finite irreflexive linear orders	$\mathbf{L_l}$, $\mathbf{L_r}$
Reflexive linear orders isomorphic to the integers	$\mathbf{T}$, $\mathbf{G}(\mathbf{G}(p \rightarrow \mathbf{G}p) \rightarrow p) \rightarrow (\mathbf{FG}p \rightarrow p)$, $\mathbf{H}(\mathbf{H}(p \rightarrow \mathbf{H}p) \rightarrow p) \rightarrow (\mathbf{PH}p \rightarrow p)$
Irreflexive linear orders isomorphic to the integers	$\mathbf{U_r}$, $\mathbf{U_l}$, $\mathbf{G}(\mathbf{G}p \rightarrow p) \rightarrow (\mathbf{FG}p \rightarrow \mathbf{G}p)$, $\mathbf{H}(\mathbf{H}p \rightarrow p) \rightarrow (\mathbf{PH}p \rightarrow \mathbf{G}p)$
Reflexive linear orders isomorphic to the natural numbers	$\mathbf{T}$, $\mathbf{G}(\mathbf{G}(p \rightarrow \mathbf{G}p) \rightarrow p) \rightarrow (\mathbf{FG}p \rightarrow p)$, $\mathbf{H}(\mathbf{H}(p \rightarrow \mathbf{H}p) \rightarrow p) \rightarrow (\mathbf{PH}p \rightarrow p)$, $\mathbf{P}(\mathbf{P}p \rightarrow \mathbf{H}p)$, $\mathbf{F}(\mathbf{F}p \rightarrow \mathbf{G}p)$
Irreflexive linear orders isomorphic to the natural numbers	$\mathbf{U_r}$, $\mathbf{EH}\bot$, $\mathbf{G}(\mathbf{G}(p \rightarrow \mathbf{G}p) \rightarrow p) \rightarrow (\mathbf{FG}p \rightarrow p)$, $\mathbf{H}(\mathbf{H}(p \rightarrow \mathbf{H}p) \rightarrow p) \rightarrow (\mathbf{PH}p \rightarrow p)$

Table 2.3: Axioms added to $\mathbf{K_t4.3_l.3_r}$ of Temporal Logics of Linear Frames in [60]

Class of Trees	Axiomatisation
Irreflexive linear orders	No additional axioms
Unbounded linear orders	$\mathbf{G}p \rightarrow \mathbf{F}p$, $\mathbf{H}p \rightarrow \mathbf{P}p$
Bounded linear orders	$\mathbf{G}\bot \rightarrow \mathbf{FG}\bot$, $\mathbf{H}\bot \rightarrow \mathbf{PH}\bot$
Dense linear orders	$\mathbf{D}$
Linear orders isomorphic to the rational numbers	$\mathbf{D}$, $\mathbf{G}p \rightarrow \mathbf{F}p$, $\mathbf{H}p \rightarrow \mathbf{P}p$
Linear orders where each instant has an immediate predecessor and successor	$p \wedge \mathbf{H}p \rightarrow \mathbf{FH}p$, $p \wedge \mathbf{G}p \rightarrow \mathbf{PG}p$
Discrete linear orders	$p \wedge \mathbf{H}p \rightarrow \mathbf{G}\bot \vee \mathbf{FH}p$, $p \wedge \mathbf{G}p \rightarrow \mathbf{H}\bot \vee \mathbf{PG}p$
Complete (no gaps) linear orders	$\mathbf{F}p \wedge \mathbf{FG}\neg p \rightarrow \mathbf{F}(\mathbf{HF}p \wedge \mathbf{G}\neg p)$, $\mathbf{P}p \wedge \mathbf{PH}\neg p \rightarrow \mathbf{P}(\mathbf{GP}p \wedge \mathbf{H}\neg p)$
Linear orders isomorphic to the real numbers	$\mathbf{D}$, $\mathbf{G}p \rightarrow \mathbf{F}p$, $\mathbf{H}p \rightarrow \mathbf{P}p$, $\mathbf{F}p \wedge \mathbf{FG}\neg p \rightarrow \mathbf{F}(\mathbf{HF}p \wedge \mathbf{G}\neg p)$, $\mathbf{P}p \wedge \mathbf{PH}\neg p \rightarrow \mathbf{P}(\mathbf{GP}p \wedge \mathbf{H}\neg p)$
Well-founded linear orders	$\mathbf{L_l}$
Linear orders isomorphic to the natural numbers	$\mathbf{L_l}$, $p \wedge \mathbf{G}p \rightarrow \mathbf{H}\bot \vee \mathbf{PG}p$

Table 2.4: Axioms added to $\mathbf{K_t}, \mathbf{4}$, $\mathbf{F}p \wedge \mathbf{F}p \rightarrow \mathbf{F}(p \wedge \mathbf{F}p) \vee \mathbf{F}(p \wedge q) \vee \mathbf{F}(\mathbf{F}p \wedge q)$, $\mathbf{F}p \wedge \mathbf{P}p \rightarrow \mathbf{P}(p \wedge \mathbf{P}p) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(\mathbf{P}p \wedge q)$ of Temporal Logics of Linear Frames in [13]

Class of Trees	Axiomatisation
Irreflexive linear orders	$\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$
Unbounded irreflexive linear orders	$\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$
Linear orders isomorphic to the rational numbers	$\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $\mathbf{GG}p \to \mathbf{G}p$
Linear orders isomorphic to the real numbers	$\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $\mathbf{GG}p \to \mathbf{G}p$, $(\mathbf{G}p \to \mathbf{PG}p) \to (\mathbf{G}p \to \mathbf{H}p)$
Unbounded discrete irreflexive linear orders	$\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $(p \wedge \mathbf{G}p) \to \mathbf{PG}p$, $(p \wedge \mathbf{H}p) \to \mathbf{FH}p$
Linear orders isomorphic to the integers	$\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $(\mathbf{G}p \to \mathbf{PG}p) \to (\mathbf{G}p \to \mathbf{H}p)$ (weak completeness)

Table 2.5: Axioms added to $\mathbf{K_t4}$ of Temporal Logics of Linear Frames in [19]

2.10 Definability of Classes of Trees

Before we look into axiomatisations for the different classes of trees, we look into whether there are sets of formulas that define these classes when restricted to irreflexive (respectively, reflexive) trees. This may give us an idea of possible candidate axioms we can use to axiomatise the logics of the different classes of trees. Since we are restricting the frames to either irreflexive or reflexive trees, some properties are already inherent in the frame, like connectedness, transitivity, left linearity and irreflexivity/reflexivity. Hence, the empty set will define the class of all irreflexive/reflexive trees relative to the class of irreflexive/reflexive trees.

Considering the other classes of trees, we see that conversely well-foundedness, right unboundedness and density are properties that can be defined by the respective axioms $\mathbf{L_r}$ (irreflexive trees), **Grz** (reflexive trees), $\mathbf{U_r}$ and **D** (only irreflexive trees). In the temporal case, we can also define left-unboundedness with $\mathbf{U_l}$, and well-foundedness with $\mathbf{L_l}$ (irreflexive trees) and $\mathbf{Grz_l}$ (reflexive trees). See e.g., [4] for confirmation that $\mathbf{L_r}$, $\mathbf{U_l}$, $\mathbf{U_r}$ and **D** define the respective properties, and [15] for confirmation that **Grz** and $\mathbf{Grz_l}$ define conversely well-foundedness and well-foundedness. The facts that **S** and **Q** define forward discreteness are shown in Sections 5.1 and 5.2 respectively.

Therefore, the following sets define the respective classes of trees in the Priorian modal language, relative to the class of irreflexive/reflexive trees:

- $\emptyset$ defines the class of all irreflexive/reflexive trees
- $\mathbf{L_r}$ defines the class of conversely well-founded irreflexive trees
- **Grz** defines the class of conversely well-founded reflexive trees
- $\mathbf{U_r}$ defines the class of right unbounded irreflexive trees
- **D** defines the class of dense irreflexive trees
- $\{\mathbf{U_r}, \mathbf{D}\}$ defines the class of dense, right unbounded, irreflexive trees

Likewise, the following sets define the respective classes of trees in the Priorian temporal language, relative to the class of irreflexive/reflexive trees:

- $\emptyset$ defines the class of all irreflexive/reflexive trees
- $\mathbf{L_r}$ defines the class of conversely well-founded irreflexive trees

- $\mathbf{L_l}$ defines the class of well-founded irreflexive trees
- **Grz** defines the class of conversely well-founded reflexive trees
- $\mathbf{Grz_l}$ defines the class of well-founded reflexive trees
- $\mathbf{U_r}$ defines the class of right unbounded irreflexive trees
- $\mathbf{U_l}$ defines the class of left unbounded irreflexive trees
- $\{\mathbf{U_l}, \mathbf{U_r}\}$ defines the class of unbounded irreflexive trees
- **D** defines the class of dense irreflexive trees
- $\{\mathbf{U_r}, \mathbf{D}\}$ defines the class of dense, right unbounded, irreflexive trees
- $\{\mathbf{U_l}, \mathbf{D}\}$ defines the class of dense, left unbounded, irreflexive trees
- $\{\mathbf{U_l}, \mathbf{U_r}, \mathbf{D}\}$ defines the class of dense, unbounded, irreflexive trees
- **S** defines the class of forward discrete irreflexive trees
- **Q** defines the class of forward discrete reflexive trees

However, other classes are not as easy to define. This begs the question whether we can use the Goldblatt-Thomason Theorem (see [67] and [34]) which states that a first-order definable class of frames is modally definable if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflections of ultrafilter extensions, to either show that a property of a frame is definable or is not. However, we see that two of these constructions, namely ultrafilter extension and disjoint unions, fail to preserve trees. In the next proposition we use bounded morphisms to show that certain properties, and hence classes of trees, cannot be defined when restricted to these classes.

Proposition 2.10.1. *The following classes of trees are not definable in the modal language:*

- *dense reflexive trees*
- *left unbounded irreflexive/reflexive trees*
- *right unbounded reflexive trees*
- *unbounded reflexive trees*
- *discrete reflexive trees*
- *reflexive trees with branches isomorphic to the natural numbers*
- *reflexive trees with branches isomorphic to the integers*
- *reflexive trees with branches isomorphic to the rational numbers*
- *reflexive trees with branches isomorphic to the real numbers*

Proof. First, we define bounded morphisms from a frame that has the property in question onto a frame that does not.

Dense reflexive trees: Let $\mathcal{F}$ be the linear ordering of two consecutive copies of the rational numbers and let $\mathcal{F}'$ be two linearly ordered reflexive instants. Let $f : \mathcal{F} \to \mathcal{F}'$ be the function that maps the first copy of the rational numbers, with their usual non-strict ordering, to the first instant and the second copy to the second instant.

Right unbounded reflexive trees: Let $\mathcal{F}$ be a copy of the natural numbers, with their usual non-strict ordering, and let $\mathcal{F}'$ be a single reflexive instant. Let $f : \mathcal{F} \to \mathcal{F}'$ be the function that maps all natural numbers to the reflexive instant.

Left unbounded reflexive trees: Let $\mathcal{F}$ be a copy of the negative integers, with their usual non-strict ordering, and let $\mathcal{F}'$ be a single reflexive instant. Let $f : \mathcal{F} \to \mathcal{F}'$ be the function that maps all negative integers to the reflexive instant.

Unbounded reflexive trees: Let $\mathcal{F}$ be a copy of the integers, with their usual non-strict ordering, and let $\mathcal{F}'$ be a single reflexive instant. Let $f : \mathcal{F} \to \mathcal{F}'$ be the function that maps all integers to the reflexive instant.

Discrete reflexive trees: Let $\mathcal{F}$ be two consecutive copies of the integers, with with their usual non-strict ordering, and let $\mathcal{F}'$ be a copy of the integers with a lower bound, w. Let $f : \mathcal{F} \to \mathcal{F}'$ be the function that maps the first copy of the integers to w and the second copy of the integers in $\mathcal{F}$ isomorphically to the copy of the integers in $\mathcal{F}'$

Reflexive trees with branches isomorphic to the natural numbers, integers, rational numbers and the real numbers: In all these cases, we let $\mathcal{F}$ be be either the natural numbers, integers, rational numbers and the real numbers, respectively, and we let $\mathcal{F}'$ be a single reflexive instant. Let $f : \mathcal{F} \to \mathcal{F}'$ be the function that maps the natural numbers, integers, rational numbers and the real numbers, respectively, to the reflexive instant.

The undefinability of last property will be proved using a generated subframe:

Left unbounded irreflexive trees: Let $\mathcal{F}$ be any rooted tree (with an irreflexive ordering) and let $\mathcal{F}'$ be the tree $\mathcal{F}$ with a copy of the negative integers, with the usual strict ordering, added preceding the root of $\mathcal{F}$. Then $\mathcal{F}$ is a point generated subframe of $\mathcal{F}'$ generated by the root. □

The same bounded morphisms given above will also show undefinability of the classes in the following proposition.

Proposition 2.10.2. *The following classes of trees are not definable in the temporal language:*

- *dense reflexive trees*
- *left unbounded reflexive*
- *right unbounded reflexive trees*
- *unbounded reflexive trees*
- *discrete reflexive trees*

Therefore, many of the classes of trees have been shown to either be definable or not definable. However, for some classes of trees this proves to be more complicated. We have no known results for irreflexive trees with branches isomorphic to the natural numbers, integers, rational numbers and real numbers. Also, no known result for finite irreflexive/reflexive trees, locally finite irreflexive/reflexive trees, backwards discrete irreflexive trees, rooted irreflexive/reflexive trees or well-founded irreflexive/reflexive trees. Hence, to investigate this further we perhaps need to ask if there is a Goldblatt-Thomason equivalent when restricted to trees. Therefore, either finding a set of formulas that would define more classes of trees or showing that a class is not definable, would be an interesting investigation for further study.

We are now ready to investigate axiomatisations of different classes of trees in the Priorian languages.

Chapter 3

Complete Axiomatisations of the Modal Logics of Trees

In this chapter we aim to find complete finite axiomatisations for the logics of the different classes of irreflexive trees and reflexive trees in the future fragment of the Priorian temporal language $\mathcal{L}^m_{Prior}$ (see Section 2.3). All formulas in this chapter will be from this language. Some axiomatisations for the logics of different classes of trees have been established in the literature, for example in [26], [61] and [58], using mostly bulldozing and unravelling, as well as in [48] and [63]. Logics of trees of bounded branching were axiomatised in [26], and more specifically, the axiomatisation of the logic of the single complete binary tree can be found in [33] and [64]. Furthermore, the logic of at most binary branching tree, where each irreflexive point has exactly on reflexive successor is given in [62].

Looking at the semantic clauses, it follows that any formula of the form $\mathbf{F}\varphi$ in the future fragment of the Priorian language corresponds to the modal formula $\Diamond\varphi$ in standard modal logic. For this reasons, results on axiomatisations of modal logics often applies directly to this fragment of the Priorian language.

Building a models for classes of trees can be done using known methods like unravelling and networks (see e.g. [4]). By making slight changes to the axioms of the logic we start with, we can ensure that the resulting model is the required tree. Since we will be working with transitive trees, the basic logic **K4**, which defines the class of transitive frames, will be the basis for the logics in this sections, with extra axioms added as necessary.

3.1 Irreflexive Trees

We start with the section with the class of all irreflexive trees, and other subclasses with respect to which **K4** is also complete.

3.1.1 All, Discrete, Locally Finite, and Rooted Irreflexive Trees

In this section we show that **K4** is strongly complete for the future fragment of the Priorean logic of the class of all irreflexive trees. We do this by building a model for every set of formulas consistent with **K4** based on a the class of all irreflexive trees. The resulting model also has some other properties from which we can deduce additional results.

It is well known that **K4** defined the class of all transitive frames, and based on Section 2.10, the class of irreflexive trees for the logic is the class of all irreflexive trees. Table 3.1 summarises the axioms together with the frame classes relevant to it.

Logic and Axioms	Class of Kripke frames	Class of Irreflexive Trees	Standard Frames [1]
K4: **K** as defined in Definition 2.5.1 **4** : $\mathbf{FF}p \to \mathbf{F}p$	Transitive frames	Irreflexive trees	Irreflexive trees

Table 3.1: Axioms, Kripke frames, Tree Frames and Standard Frames for **K4**

To build a tree model for **K4** satisfying a given **K4**-consistent set Γ, we begin with the canonical model $\mathcal{M}^{\mathbf{K4}} = (W, R, V)$ for **K4** and we let $\mathcal{M}$ be the submodel generated by some **K4** maximal consistent set $w \in W$ containing Γ.

Using the process of unravelling (see Definition 2.6.1) of the canonical model $\mathcal{M}^{\mathbf{K4}}$ from an instant $w \in W$ described in Chapter 4 of [4] to get the model $(\overrightarrow{W}, \overrightarrow{R}, \overrightarrow{V})$ and taking the transitive closure of $\overrightarrow{R}$, denoted by R^*, we get a model $\mathcal{M}^* = (\overrightarrow{W}, R^*, \overrightarrow{V})$. Then $\mathcal{M}^*$ is a model whose underlying frame has the desired properties of a tree, as the unravelling gives a model that is irreflexive, antisymmetric, left linear and connected, and transitivity is given by taking the transitive closure.

Furthermore, $\mathcal{M}^{\mathbf{K4}}$ is a bounded morphic image of $\mathcal{M}^*$ and hence these models satisfy the same formulas at corresponding instants. Hence, we have the following result (see Definition 2.4.1).

Theorem 3.1.1. **K4** *is sound and strongly complete with respect to the class of irreflexive trees.*

Proof. Soundness follows from the fact that axiom **4** defines transitivity and hence is valid on the class of irreflexive trees. The other axioms of this logic are valid on all frames.

For completeness, let $\mathcal{M}^{\mathbf{K4}} = (W, R, V)$ be the canonical model for **K4**. Let Γ be a **K4** consistent set and let w be an mcs in W such that $\Gamma \subseteq w$. Let $\mathcal{M}^* = (\overrightarrow{W}, R^*, \overrightarrow{V})$ be the transitive closure of the unravelling from w of the submodel generated by w. Then $\mathcal{M}^*, w \Vdash \Gamma$.

Hence **K4** is sound and strongly complete with respect to the class of irreflexive trees. □

Note that the unravelling process results in a model that has certain properties which gives us the following completeness results.

Theorem 3.1.2. **K4** *is sound and strongly complete with respect to the class of rooted irreflexive trees.*

Proof. Since we are unravelling the canonical model $\mathcal{M}^{\mathbf{K4}}$ of **K4** from a specific instant $w \in W$ and we are only adding successors of w, it follows that s_0, the sequence consisting of only one instant w, is the root of the unravelled model $\mathcal{M}^*$. □

Theorem 3.1.3. **K4** *is sound and strongly complete with respect to the class of discrete irreflexive trees.*

Proof. $R^* s_1 s_2$ iff s_1 is an initial segment of s_2. Therefore $s_2 = s_1 \circ \langle v_1, ..., v_n \rangle$. But then $s_1 \circ \langle v_1 \rangle$ is an immediate successor of s_1 and $R^*(s_1 \circ \langle v_1 \rangle) s_2$. □

Theorem 3.1.4. **K4** *is sound and strongly complete with respect to the class of locally finite irreflexive trees.*

Proof. Since unravelling of the canonical model $\mathcal{M}^{\mathbf{K4}}$ of **K4** cannot create infinitely many instants between any two given instants, it follows that the unravelled model $\mathcal{M}^*$ is locally finite. Indeed, if $R^* s_1 s_2$ then $s_2 = s_1 \circ \langle v_1, ..., v_m \rangle$ for some sequence of instants $v_1, ..., v_m$, which means that there is exactly $m - 1$ instants between s_1 and s_1. □

Corollary 3.1.5. **K4** *is sound and strongly complete with respect to the class of well-founded irreflexive trees.*

Proof. Since **K4** is sound and strongly complete with respect to the class of rooted and locally finite trees, the result follows. □

[1] Here and what follows, we will use the terminology“Standard Frames” to refer to the intended class of frames for the logic.

Logic and Axioms	**Class of Kripke frames**	**Class of Irreflexive Trees**	**Standard Frames**
$\mathbf{K4U_r}$: **K** as defined in Definition 2.5.1 $\mathbf{4}$: $\mathbf{FF}p \to \mathbf{F}p$ $\mathbf{U_r}$: $\mathbf{F}\top$	Right-unbound transitive frames	Right-unbounded irreflexive trees	Unbounded irreflexive trees

Table 3.2: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{K4U_r}$

3.1.2 Unbounded Irreflexive Trees and Irreflexive Trees with Branches Isomorphic to the Natural Numbers/Integers

As seen in Section 3.1.1, **K4** is sound and strongly complete with respect to the class of rooted, locally finite trees and hence also with respect to the class of well-founded trees. Adding the axiom for right seriality $\mathbf{U_r}$: $\mathbf{G}p \to \mathbf{F}p$, we can look at a summary of the axioms together with the frame classes relevant to it in Table 3.2.

In this section we show that $\mathbf{K4U_r}$ is strongly complete with respect to the class of right unbounded trees and unbounded trees. We do this by building a model for every set of formulas consistent with $\mathbf{K4U_r}$ based on these trees, as we did in Section 3.1.1, but with the right seriality axiom $\mathbf{U_r}$ added. This gives us the following results.

Theorem 3.1.6. $\mathbf{K4U_r}$ *is sound and strongly complete with respect to the class of right unbounded irreflexive trees.*

Proof. Let Γ be a $\mathbf{K4U_r}$-consistent set and let w be an instant in the canonical model containing Γ. Let $\mathcal{M}^*$ be the transitive closure of the unravelling of the canonical model with w as root. Then, as the canonical model of $\mathbf{K4U_r}$ is right unbounded as a Kripke model, $\mathcal{M}^*$ has no leaves and is therefore right unbounded as an order and $\mathcal{M}^*, w \Vdash \Gamma$, as required. □

Theorem 3.1.7. $\mathbf{K4U_r}$ *is sound and strongly complete with respect to the class of well-founded, locally finite right unbounded irreflexive trees.*

Proof. The model $\mathcal{M}^*$ built in the proof of Theorem 3.1.6 above is also well-founded and locally finite. □

Theorem 3.1.8. $\mathbf{K4U_r}$ *is sound and strongly complete with respect to the class of irreflexive trees with branches isomorphic to the natural numbers.*

Proof. Consider $\mathcal{M}^*$ built in proof of Theorem 3.1.6 above. This model is well-founded with root s_0, right unbounded, and locally finite. Hence, since each successive sequence on the branch only adds one more instant to the previous sequence, each branch $h = \{s_0, s_1, ...\}$ has a strict, locally finite linear ordering. Let $f : h \to \mathbb{N}$ be defined as follows: $f(s_0) = 0$, $f(s_i) = i$ for all $i \in \omega$. Then f is an isomorphism from the branch h to the natural numbers and hence the result follows. □

For left unbounded trees, the following construction will be useful. It involves adding a tail to a model to make it left-unbounded. We will use this method to obtain several completeness results, and for this purposes, we will give a more general definition of the set we use when adding the tail, and then be more specific in the different cases, for example, by specifying the rational numbers as the tail when we do dense trees, etc.

Definition 3.1.9. Suppose $\mathcal{M} = (W, R, V)$ is a tree with s_0 the root of the tree. Now, let $W' = \{w_i \mid i \in I\}$ with $W' \cap W = \emptyset$ be a set of instants where I is a countably infinite set. Let R' be a strict left unbounded linear ordering on the elements of W'. Now, let $W^l = W \cup W'$ and define a relation R^l on W^l as follows: $R^l = R \cup R' \cup (W' \times W)$. Let V^l be a valuation on W^l that agrees with V on W and which is arbitrary on W'.

Logic and Axioms	Class of Kripke frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{KL_r}$: **K** as defined in Definition 2.5.1 $\mathbf{L_r} : \mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$	Irreflexive, transitive conversely well-founded frames	Conversely well-founded irreflexive trees	Conversely well-founded irreflexive trees

Table 3.3: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{KL_r}$

Then $\mathcal{M}^l = (W^l, R^l, V^l)$ is a tree. Moreover, $\mathcal{M}$ is a generated submodel of $\mathcal{M}^l$ and hence they satisfy the same formulas at corresponding instants of W. Furthermore, $\mathcal{M}^l$ is left unbounded, as we have added an infinite tail to $\mathcal{M}$. This process will be referred to as **adding a tail**.

Depending on the set I, we can use this process to get different completeness results, for example, if we want a completeness result for dense trees, we can let I be the rational numbers, and R' the usual ordering of the rational numbers, which will give a dense tail.

Theorem 3.1.10. **K4** *is sound and strongly complete with respect to the class of left unbounded irreflexive trees.*

Proof. Let Γ be a **K4**-consistent set of formulas and let w be an instant in the canonical model where Γ is satisfied. Let $\mathcal{M}^*$ be the unravelling of the canonical model with w as root and let $\mathcal{M}^l$ be the model obtained by adding a tail as described above were I is the set of integers and R' is the usual ordering of the integers. Then, since $\mathcal{M}^l$ is a generated submodel of $\mathcal{M}^*$, it follows that $\mathcal{M}^l, w \Vdash \Gamma$. Hence, the result follows. □

Theorem 3.1.11. $\mathbf{K4U_r}$ *is sound and strongly complete with respect to the class of unbounded irreflexive trees.*

Proof. Note that canonical model for $\mathbf{K4U_r}$ is right unbounded as a Kripke frame. Let Γ be a $\mathbf{K4U_r}$-consistent set of formulas and let w be an instant in the canonical model where Γ is satisfied. Let $\mathcal{M}^l$ be the model obtained by adding a tail as described above were I is the set of integers and R' is the usual ordering of the integers. Let $\mathcal{M}^*$ be the unravelling of this model with w as root. This model will be an unbounded tree, as it will have no leaves or roots, and hence the result follows. □

Theorem 3.1.12. $\mathbf{K4U_r}$ *is sound and strongly complete with respect to the class of irreflexive trees with branches isomorphic to the integers.*

Proof. Following the same argument as in the case of the class of left unbounded trees in the proof of Theorem 3.1.11 above, but using the set of negative integers $\mathbb{Z}^-$ as the countably infinite set I in the process of adding a tail, the model $\mathcal{M}^l$ is locally finite and unbounded, and hence, each branch is isomorphic to the integers. □

3.1.3 Finite Irreflexive Trees

Adding the Löb formula $\mathbf{L_r} : \mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$ to **K** to get $\mathbf{KL_r}$ (the logic that defines the class of conversely well-founded, transitive frames - see Table 3.3), it was shown in [61] that $\mathbf{KL_r}$ is sound and weakly complete with respect to the class of finite irreflexive trees. The process starts with a generated submodel of the canonical model for $\mathbf{KL_r}$ which is then filtered through a set $Cl(\alpha)$ that consists of a non-theorem α and all its subformulas, using the transitive filtration. Lastly, the clusters of the filtration are 'untangled' to yield a finite tree.

It has also been shown (see e.g. Theorem 4.43 in [4] and Lemma 5.1.13) that $\mathbf{KL_r}$ is not strongly complete with respect to any class of frames. The best we can therefore hope for is weak completeness. The method we will use in this thesis is different from that used by Segerberg in [61] described above. We will use a selective filtration to build a finite model for this logic by choosing only irreflexive instants to ensure that the resulting model is a tree. This method is described in Section 5.5. of [15] and we will use this method again in Chapter 5 Section 5.12.

What follows are the main results of the construction in [15]:

Beginning with $\mathbf{KL_r}$ and a non-theorem α, we will build a finite model for this logic on which $\neg\alpha$ is satisfied, by constructing a model from the canonical model $\mathcal{M} = (W, R, V)$ through the process of selective filtration. The following lemma will be needed for the construction. This lemma was adapted from Theorem 5.45 in [15], and the proof was also adapted for our purposes.

Lemma 3.1.13. *Let $\mathcal{M} = (W, R, V)$ be a model for Λ, where Λ is a logic containing $\mathbf{L_r}$, and suppose that for some instant $w \in W$, $\mathcal{M}, w \nVdash \mathbf{G}\varphi$. Then there exists an irreflexive instant $v \in W$ such that Rwv and, $\mathcal{M}, v \nVdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{G}\varphi$.*

Proof. Let $\mathcal{M} = (W, R, V)$ be a model for Λ, and suppose that for some instant $w \in W$, $\mathcal{M}, w \nVdash \mathbf{G}\varphi$. Since all substitution instances of $\mathbf{L_r}$ are true in $\mathcal{M}$, we have $\mathcal{M}, w \Vdash \mathbf{G}(\mathbf{G}\varphi \to \varphi) \to \mathbf{G}\varphi$ and hence $\mathcal{M}, w \nVdash \mathbf{G}(\mathbf{G}\varphi \to \varphi)$. Therefore, there is an instant $v \in W$ such that Rwv and, $\mathcal{M}, v \nVdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{G}\varphi$. Clearly v is irreflexive, since $\mathcal{M}, v \Vdash \mathbf{G}\varphi$. □

The lemmas above give us the tools to construct a finite, transitive, irreflexive model. Unravelling this model will give us left linearity which will give us the required finite tree. First we define a set of formulas Φ that is **closed under single negations** as a set of formulas that contains all negations of formulas in the set that are not already a negation, i.e. for all $\varphi \in \Phi$ and φ not of the form $\neg\psi$, it is the case that $\neg\varphi \in \Phi$.

Let α be a non-theorem of $\mathbf{KL_r}$. In the construction of the finite tree we will ensure that, for all formulas φ in $Cl(\alpha)$, and for all $\mathbf{KL_r}$-mcs's w selected in the constructed model, φ is true at w in the canonical model if and only if it is true at w in the constructed. Therefore, even though $\mathbf{L_l}$ is not canonical (as the canonical frame contains reflexive points), we can build a finite and transitive frame from the canonical model without any reflexive points and on which the relevant modal formulas are satisfied.

The following lemma was adapted from Theorems 5.46 and 5.47 in [15], and the proofs were also adapted for our purposes.

Lemma 3.1.14. $\mathbf{KL_r}$ *is sound and weakly complete with respect to the class of transitive, irreflexive, finite frames.*

Proof. Soundness follows, since the axioms are valid on the class of transitive, irreflexive, finite frames.

Let α be a formula that is not a theorem of $\mathbf{KL_r}$. We will filter the canonical model $\mathcal{M} = (W, R, V)$ for $\mathbf{KL_r}$ through $Cl(\alpha)$ to build a finite model on which α is refuted. Since α is not a theorem of $\mathbf{KL_r}$, there exists an instant $w \in W$ such that $\mathcal{M}, w \nVdash \alpha$. If w is reflexive, then $\mathcal{M}, w \nVdash \mathbf{G}\alpha$. Then by Lemmas 3.1.13, there exists $u, v \in W$ with Rwv such that $\mathcal{M}, v \Vdash \mathbf{G}\alpha$ and $\mathcal{M}, v \nVdash \alpha$. Therefore, we can always find an irreflexive instant in the canonical model where α is false.

Let w_0 be any irreflexive instant such that $\mathcal{M}, w_0 \nVdash \alpha$. We will use w_0 as the starting point for building the required model for $\mathbf{KL_r}$ that will refute α by systematically adding more irreflexive instants to the model until all modal formulas in $Cl(\alpha)$ are satisfied at instant iff they are members of those instants, i.e. until we can prove a Truth Lemma with respect to $Cl(\alpha)$. Let $\Phi_w = \{\mathbf{G}\psi \in Cl(\alpha) \mid \mathbf{G}\psi \notin w\}$ for all $w \in W$. These sets will remain fixed throughout the construction process. The systematic selection process is defined inductively as follows:

Set $W_0 = \{w_0\}$. Then, clearly, $\mathcal{F}_0 = (\{w_0\}, \emptyset)$ is a finite irreflexive frame.

Next, suppose W_n has already been constructed.

If, for all $w \in W_n$, either $\Phi_w = \emptyset$ or, for all $\mathbf{G}\psi \in \Phi_w$, there is a $v \in W_n$ such that Rwv with $\neg\psi \in v$, then let $W^* = \bigcup_{s=0}^{n} W_s$ and $\mathcal{M}_n = (W^*, R_n, V_n)$ where R_n is the restriction of R to W^* and V_n is the restriction of V to W^*, i.e. $V_n(p) = \{w \in W^* \mid w \in V(p)\}$. Then $\mathcal{M}_n$ is the required model.

Otherwise for each w such that $\Phi_w \neq \emptyset$:

If $\mathbf{G}\varphi \in \Phi_w$ and there is no $v \in W_n$ with Rwv such that $\neg\varphi \in v$, use Lemma 3.1.13 to select an instant w' with Rww' such that $\varphi \notin w'$ and $\mathbf{G}\varphi \in w'$.

Then, let W_{n+1} be the set of all w' selected above and repeat the process until a set W_k is reached for which, for all $w \in W_k$, either $\Phi_w = \emptyset$ or, for all $\mathbf{G}\psi \in \Phi_w$, there is a $v \in W_n$ such that Rwv with $\neg\psi \in v$.

This process will terminate after a finite number of steps as $|\Phi_{w'}| < |\Phi_w|$ for all $w \in W_k$ (since the canonical frame is transitive), and each step adds only finitely many new instants, and hence the resulting model is finite.

Then let $W^{**} = \bigcup_{s=0}^{k} W_s$ and $\mathcal{M}^{**} = (W^{**}, R_k, V_k)$ where R_k is the restriction of R to W^{**} and V_k the restriction of V to W^{**}.

We now prove a version of the Truth Lemma restricted to formulas in $Cl(\alpha)$ using induction:

The base case and boolean cases follow easily.

Suppose φ is $\mathbf{G}\psi$, and $\mathcal{M}, v \Vdash \psi$ iff $\mathcal{M}^{**}, v \Vdash \psi$ for all $v \in W^{**}$ (Induction hypothesis).

$\Rightarrow$: Suppose $\mathcal{M}, w \Vdash \mathbf{G}\psi$. Then, for all v with Rwv it is the case that $\mathcal{M}, v \Vdash \psi$. By the induction hypothesis, and, since Rwv and R_k is the restriction of R to W^{**}, it follows that $\mathcal{M}^{**}, v \Vdash \psi$ for all v such that $R_k wv$ (by construction of $\mathcal{M}^{**}$). Hence, $\mathcal{M}^{**}, w \Vdash \mathbf{G}\psi$.

$\Leftarrow$: Suppose $\mathcal{M}, w \nVdash \mathbf{G}\psi$. Then, by Lemma 3.1.13, there exists a $v' \in W$ with Rwv' such that $\mathcal{M}, v' \nVdash \psi$ and $\mathcal{M}, v' \Vdash \mathbf{G}\psi$. Let v be the v' guaranteed by Lemma 3.1.13 that was selected in the construction of $\mathcal{M}^{**}$. Then $v \in W^{**}$, and hence by the induction hypothesis we have $\mathcal{M}^{**}, v \nVdash \psi$ and $R_k wv$ by construction. Therefore, $\mathcal{M}^{**}, w \nVdash \mathbf{G}\psi$.

Notice that $\mathcal{M}^{**}$ is transitive as we constructed R_k by taking the restriction of R to W^{**} and R is transitive. It is also connected, irreflexive and finite by construction. Furthermore, since $w_0 \in W^{**}$ it also refutes α. Therefore $\mathbf{KL_r}$ is weakly complete with respect to the class of transitive irreflexive finite frames. □

Next, unravel $\mathcal{M}^{**}$ from w_0 (as in the proof of Lemma 3.1.14). Note that we also have to take the transitive closure of the relation of the unravelling. This procedure gives a left linear model. We just have to make sure that after unravelling we will have a finite model. Since $\mathcal{M}^{**}$ contains only irreflexive instants and is transitive and finite, this process will also be finite, as we argue in the proof of the lemma below.

Lemma 3.1.15. *Let $\mathcal{M}^r$ be the unravelling of $\mathcal{M}^{**}$ from w_0. Then $\mathcal{M}^r$ is finite.*

Proof. Let $\mathcal{M}^r$ be the unravelling of $\mathcal{M}^{**}$ from w_0. Since the instants of $\mathcal{M}^r$ are paths in $\mathcal{M}^{**}$, $\mathcal{M}^{**}$ has only finitely many instant, and $\mathcal{M}^{**}$ is transitive and irreflexive, it follows that no instant can repeat in a path in $\mathcal{M}^{**}$. Hence, $\mathcal{M}^r$ is finite. □

Hence, we have the following completeness result.

Theorem 3.1.16. $\mathbf{KL_r}$ *is sound and weakly complete with respect to the class of finite irreflexive trees.*

Proof. Soundness follows, since the axioms are valid on the class of transitive irreflexive finite frames.

By Lemma 3.1.14 any non-theorem α of $\mathbf{KL_r}$ can be refuted in an instant w_0 in a transitive irreflexive finite model $\mathcal{M}^{**}$, which can be unravelled by Lemma 3.1.15 into a finite irreflexive tree.

□

Since $\mathcal{M}^r$ is a finite, irreflexive tree and hence also conversely well-founded, we have the following consequence.

Corollary 3.1.17. $\mathbf{KL_r}$ *is sound and weakly complete with respect to the class of conversely well-founded irreflexive trees.*

Logic and Axioms	Class of Kripke frames	Class of Irreflexive Trees	Standard Frames
K4D: **K** as defined in Definition 2.5.1 **4** : $\mathbf{FF}p \to \mathbf{F}p$ **D** : $\mathbf{F}p \to \mathbf{FF}p$	Dense, transitive frames	Dense irreflexive trees	Dense irreflexive trees

Table 3.4: Axioms, Kripke frames, Tree Frames and Standard Frames for **K4D**

3.1.4 Dense Irreflexive Trees

Consider the logic **K4D** where **D** is the canonical axiom $\mathbf{F}p \to \mathbf{FF}p$. Table 3.4 summarises the axioms together with the frame classes relevant to it.

In this section we show that **K4D** is strongly complete with respect to the class of dense irreflexive trees. We do this by building a model for every set of formulas consistent with **K4D** based on a dense irreflexive tree, by building a network for a **K4D**-consistent set of formulas. This has already been done in e.g. Chapter 4 in [4].

Borrowing concepts and terminology from [4], [49], [12] and [11], we will build dense irreflexive network for **K4D** (see Section 2.8). All networks referred to in the remainder of this section will be networks for a **K4D**-consistent set of formulas.

To make sure we build a network of the right type, we will require the network to be strict coherent and saturated (see Definitions 2.8.2 and 2.8.4 in Section 2.8). Since we are interested in building a dense model, we need to ensure that the network is dense. Furthermore, since we are working in the future fragment of the Priorian temporal language, we do not have to consider the condition involving **P** formulas. Therefore, we will remove the condition involving **P** formulas, and add the following extra condition:

- Condition for a dense network: $\ll$ is dense, i.e. for all $s, t \in N$ with $s \ll t$ and $s \neq t$, there exists a $u \in N$ different from s and t such that $s \ll u \ll t$.

A saturated network satisfying the above condition will be called a **d-saturated network**. If a network is both strict coherent and d-saturated, it is called a **d-perfect network**. Notice that d-perfect networks will give rise to frames and models that are dense trees.

It follows that if we can construct a d-perfect network for a **K4D**-consistent set of formulas, we will have the required model for **K4D**.

We call the violations of the different conditions of a d-saturated network of a network $\mathcal{N} = (N, \ll, \kappa)$ **defects**, and formalise them as follows: The pair (s, t) is an S_d-defect if the density condition is not satisfied, i.e., $s \ll t$, and for no $u \in N$ do we have $s \ll u \ll t$. The pair $(s, \mathbf{F}\psi)$ is an $S_{\mathbf{F}}$ defect if $\mathbf{F}\psi \in \kappa(s)$ for $s \in N$ but there is no $t \in N$ such that $s \ll t$ and $\psi \in \kappa(t)$.

The following lemma shows that it is possible to repair any defect in a finite, strict coherent network in such a way that the resulting network is also finite and strict coherent. The proof is similar to the proof given in [4] for countable linear orders. We will use $s \underline{\ll} t$ for $s \ll t$ or $s = t$.

Lemma 3.1.18 (Repair Lemma)**.** *For any defect of a finite, strict coherent network $\mathcal{N}$ for a* **K4D***-consistent set of formulas Γ there is a finite strict coherent network $\mathcal{N}'$ for Γ that extends $\mathcal{N}$ and lacks the defect.*

Proof. Let $\mathcal{N} = (N, \ll, \kappa)$ be a finite, strict coherent network for a **K4D**-consistent set of formulas. Let $\mathcal{M} = (W, R, V)$ be the canonical model for **K4D**. The following describes the methods of repairing defects.

S_d defect:

Suppose there are nodes $s, t \in N$ with $s \ll t$ for which there is no intermediate node, i.e. (s, t) is an S_d defect. Since $\mathcal{N}$ is strict coherent, $\kappa(s)R\kappa(t)$ and by canonicity of the density axiom there exists a **K4D**-mcs Γ such that $\kappa(s)R\Gamma R\kappa(t)$. Therefore, add a new node u to N between s and t with $\kappa(u) = \Gamma$, i.e., define $\mathcal{N}' = (N, \ll', \kappa')$ by

$N' = \{u\} \cup N$

$\ll' = \ll \cup \{(x, u) \mid x \underline{\ll} s\} \cup \{(u, x) \mid t \underline{\ll} x\}$

$\kappa' = \kappa \cup \{(u, \Gamma)\}$

Then $\mathcal{N}'$ does not have the defect (s, t) and $\ll'$ is still a strict partial order, left linear and connected. To show strict coherency of the network, we still need to show that property 4 of Definition 2.8.2 is satisfied. Let $x, y \in N'$ such that $x \ll' y$. If neither x nor y is u then $\kappa'(x)R\kappa'(y)$ by the coherency of $\mathcal{N}$. Next, we will consider the case when $x = u$ and note that the case for $y = u$ follows similarly. Now, if $y = t$ then $\kappa'(u)R\kappa'(y)$ by the choice of Γ. So, suppose $y \neq t$. Then we must have that $t \ll y$ and hence by coherency of $\mathcal{N}$ we have $\kappa(t)R\kappa(y)$. But we also have $\Gamma R\kappa(t)$ and hence by transitivity of R we have $\Gamma R\kappa(y)$. But then by definition of κ' we have $\kappa'(u)R\kappa'(y)$.

$S_{\mathbf{F}}$ defects:

Suppose there is a node $s \in N$ with $\mathbf{F}\psi \in \kappa(s)$ but there is no $t \in N$ such that $s \ll t$ and $\psi \in \kappa(t)$. Let u' be a node in N with $s \underline{\ll} u'$ such that $(u', \mathbf{F}\psi)$ is a $S_{\mathbf{F}}$ defect and for all $u \in N$ with $u' \ll u$, $(u, \mathbf{F}\psi)$ is not a $S_{\mathbf{F}}$-defect. Since $\mathcal{N}$ is finite, such a u' exists and is possibly s. Add a new node s' as an immediate successor of u' on a new branch of $\mathcal{N}$ and let Γ' be a **K4D**-mcs containing ψ in $\mathcal{M}$ such that $\kappa(u')R\Gamma'$ (guaranteed by the Existence Lemma for normal logics).

Define $\mathcal{N}' = (N, \ll', \kappa')$ by

$N' = \{s'\} \cup N$

$\ll' = \ll \cup \{(x, s') \mid x \underline{\ll} u'\}$

$\kappa' = \kappa \cup \{(s', \Gamma')\}$

Note that $\mathcal{N}'$ does not contain the $S_{\mathbf{F}}$ defect $(s, \mathbf{F}\psi)$. It is easy to see that it is still a connected, left linear strict partial. We just need to show that property 4 of Definition 2.8.2 is satisfied to prove strict coherency.

Let $x, y \in N'$ such that $x \ll' y$. Then x and y are distinct nodes by irreflexivity and if neither x nor y is s' then $\kappa'(x)R\kappa'(y)$ by coherency of $\mathcal{N}$. Therefore, we just need to check the cases when either x or y is the new node s'. The case when $x = s'$ is not possible as s' has no successors. Therefore, suppose $y = s'$. By construction $\kappa(s') = \Gamma'$ with $\kappa(u')R\Gamma'$ and by coherency $\kappa(x)R\kappa(u')$. Then $\kappa(x)R'\kappa(y)$ follows by the transitivity of the canonical relation.

□

Notice that, once a $S_{\mathbf{F}}$-defect is repaired, the witness added to repair it remains in all future extensions so that defect cannot arise again, and once a S_d-defect has been repaired, the intermediate point remains between the two points so that particular defect cannot arise again. This leads to the following lemma.

Lemma 3.1.19. *Once a defect has been repaired in a strict coherent network for a* **K4D***-consistent set of formulas, no strict coherent extension of that network will have that defect again.*

Therefore, starting with a finite strict coherent network for a **K4D**-consistent set of formulas Γ, the following lemma (already proved for the linear case in e.g. in Chapter 4 of [4]) shows it can be turned into a d-perfect network using this process of repairing defects.

Lemma 3.1.20. *Any finite, strict coherent network $\mathcal{N} = (N, \ll, \kappa)$ for a* **K4D***-consistent set of formulas Γ_0 can be extended to a d-perfect network for Γ_0.*

Proof. Let $S = \{s_i \mid i \in \omega\}$ be the supply of nodes that will be used to build the network. Next, let

$$\mathcal{D} = [(N \cup S) \times (N \cup S)] \cup [(N \cup S) \times \{\mathbf{F}\varphi \mid \varphi \in \mathcal{L}^m_{Prior}\}] \cup [(N \cup S) \times \{\mathbf{P}\varphi \mid \varphi \in \mathcal{L}^m_{Prior}\}],$$

i.e., $\mathcal{D}$ is the set of all possible defects of $\mathcal{N}$ and of any network extending it with nodes from S. Note that $\mathcal{D}$ is countable and let $D_0, D_1, D_2, \ldots$ be an arbitrary but fixed enumeration of its elements.

We will show that we can build a d-perfect network from any finite strict coherent network for a **K4D**-consistent set Γ. Given a **K4D** mcs Γ_0 containing Γ, let $\mathcal{N}_0$ be the network $(\{s_0\}, \varnothing, (s_0, \Gamma_0))$. Then, trivially, $\mathcal{N}_0$ is a finite, strict coherent network for Γ but it may not be d-saturated.

Next, we will use an inductive definition to build a chain of networks. Suppose $\mathcal{N}_n$ is a finite, strict coherent network for $n \geq 0$. Let D be the actual defect of $\mathcal{N}_n$ that is minimal in the enumeration of potential defects. Since $\mathcal{N}_n$ is finite, it has at least a density defect. Next, form $\mathcal{N}_{n+1}$ by repairing the defect D. In this process an extension of $\mathcal{N}_n$ is created without the defect, and any new extension will also not have the defect D (Lemma 3.1.19). Notice that repairing defects creates more potential defects and therefore, in subsequent steps, the repair process will need to check the list of potential defects from the beginning. In this way, all relevant defects will eventually be repaired. Let I be a countable linearly ordered set. Define $\mathcal{N} = (N, \ll, \kappa)$ as follows:

$$N = \bigcup_{n \in I} N_n \qquad \ll = \bigcup_{n \in I} \ll_n \qquad \kappa = \bigcup_{n \in I} \kappa_n \tag{3.1}$$

Then $\mathcal{N}$ is a d-perfect network, for suppose not and let D be the actual defect of $\mathcal{N}$ minimal in the enumeration of potential defects on $\mathcal{N}$, say $D = D_k$. Note that each $\mathcal{N}_i$ is an approximation of $\mathcal{N}$ in the sense that every $\mathcal{N}_i$ removes a defect from $\mathcal{N}_{i-1}$. By construction of $\mathcal{N}$ there must be an $\mathcal{N}_i$ that approximates $\mathcal{N}$ but that still has the defect D (Indeed, if D is a future defect $(s, \mathbf{F}\psi)$, it would be present from the step in the construction in which node s is added until it is repaired, and if D is a density defect (s, t) it would be present from the step in the construction where last of the two nodes s and t are added until it is repaired.). As D is the k-th potential defect of $\mathcal{N}$, which is now an actual defect of $\mathcal{N}_i$, no defect later in the enumeration will be repaired before D is repaired and, there can be at most k potential defects earlier in the enumeration than D and they will be repaired in at most $k + i$ steps after which D will be repaired in the next step. Hence, $\mathcal{N}$ is a d-perfect network, i.e. a dense irreflexive tree.

□

Hence, we have the following completeness result.

Theorem 3.1.21. **K4D** *is sound and strongly complete with respect to the class of dense irreflexive trees.*

Proof. Soundness follows from the fact that the axioms are valid on the class of dense trees.

Given a **K4D** consistent set Γ, extend it to a **K4D**-mcs Γ_0. Let $\mathcal{N}_0$ be the network $(\{s_0\}, \varnothing, (s_0, \Gamma_0))$. Then, trivially, $\mathcal{N}_0$ is a finite, strict coherent network for Γ_0. By Lemma 3.1.20, this can be extended to a d-perfect network $\mathcal{N} = (N, \ll, \kappa)$.

Now, since $\mathcal{N}$ is a d-perfect network, it follows that the underlying model $\mathcal{M}_{\mathcal{N}}$ is a dense tree and satisfies Γ at s_0 (by Lemma 2.8.5).

□

3.1.5 Unbounded Dense Irreflexive Trees

Let $\mathbf{K4DU_r}$ the extension of **K4D** by adding the right seriality axiom $\mathbf{U_r}$: $\mathbf{F}\top$. Table 3.5 summarises the axioms together with the frame classes relevant to it.

In this section we show that $\mathbf{K4DU_r}$ is strongly complete with respect to the class of unbounded dense irreflexive trees. We do this by building a model for every set of formulas consistent with $\mathbf{K4DU_r}$ based on a unbounded dense irreflexive tree, by building a network for a $\mathbf{K4DU_r}$-consistent set of formulas, as in Section 3.1.4.

Theorem 3.1.22. $\mathbf{K4DU_r}$ *is sound and strongly complete with respect to the class of right unbounded dense irreflexive trees.*

Logic and Axioms	**Class of Kripke frames**	**Class of Irreflexive Trees**	**Standard Frames**
$\mathbf{K4DU_r}$: **K** as defined in Definition 2.5.1 $\mathbf{4} : \mathbf{FF}p \to \mathbf{F}p$ $\mathbf{D} : \mathbf{F}p \to \mathbf{FF}p$ $\mathbf{U_r} : \mathbf{F}\top$	Right unbounded dense, transitive frames	Right unbounded dense irreflexive trees	Right unbounded dense irreflexive trees

Table 3.5: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{K4DU_r}$

Proof. Soundness again follows from the fact that the axioms are valid on the class of right unbounded dense irreflexive trees.

We have already seen that we can build a network for a **K4D**-consistent set of formulas of which the underlying model $\mathcal{M}$ is a dense tree. Using the same method, let Δ be a $\mathbf{K4DU_r}$-consistent set and let Γ_0 be a $\mathbf{K4DU_r}$-mcs containing Δ. Starting with $\mathcal{N}_0 = (\{s_0\}, \varnothing, (s_0, \Gamma_0))$ build a network $\mathcal{N} = (N, \ll, \kappa)$ as in the proof of Theorem 3.1.21. Then this network, and hence also the underlying model, are right unbounded, since $\mathbf{F}\top$ is a theorem of $\mathbf{K4DU_r}$, $\mathbf{F}\top \in \Gamma$, and hence the process of repairing defects will add a successor for s. Hence, $\mathcal{N}$ is right unbounded. Therefore, the underlying model $\mathcal{M}_{\mathcal{N}}$ is also right unbounded. Hence, $\mathcal{M}_{\mathcal{N}}$ is a right unbounded dense irreflexive tree and satisfies Γ at s_0 (by Lemma 2.8.5).

□

Theorem 3.1.23. $\mathbf{K4DU_r}$ *is sound and strongly complete with respect to the class of irreflexive trees with branches isomorphic to the non-negative rational numbers.*

Proof. Consider $\mathcal{M}_{\mathcal{N}}$ as in the proof of Theorem 3.1.22. This model is rooted by construction with root say w_0 and removing the root leaves us with a left linear, transitive, irreflexive order, with unbounded countable branches. Cantor showed that any order which is countable, order dense, and without endpoints, is isomorphic to the rational numbers Q with the usual ordering (see e.g. [18]) and hence also to the positive rational numbers. Let H be a branch of $\mathcal{M}_{\mathcal{N}}$, and let $H_0 = H \setminus \{w_0\}$, and suppose $f : H_0 \to \mathbb{Q}^+$ is an isomorphism from H_0 to the positive rational numbers. Next, define a function $g : H \to \mathbb{Q} \cup \{0\}$ as $g(x) = f(x)$ for $x \in H_0$ and $g(w_0) = 0$. Clearly this is an isomorphism, which means that the model has branches isomorphic to the non-negative rational numbers with the usual ordering. □

Also, starting with the right unbounded model $\mathcal{M}_{\mathcal{N}}$ built in the proof of Theorem 3.1.22, we can add a tail as in Definition 3.1.9 using the rational numbers for the set I and the usual strict ordering of the rational numbers for R' as defined in Definition 3.1.9. This gives a model that is an unbounded dense irreflexive tree. Furthermore, instants in $\mathcal{M}_{\mathcal{N}}$ before the tail is added are modally equivalent to the corresponding instants in the model with the tail, as the former is a generated submodel of the latter. Therefore the model with the tail still satisfies Δ at the instant that was the root before the tail was added.

This gives the following consequence.

Corollary 3.1.24. $\mathbf{K4DU_r}$ *is sound and strongly complete with respect to the class of unbounded dense irreflexive trees.*

Since the model resulting from adding a tail to the model built in the proof of Theorem 3.1.22, is countable and unbounded, the model has branches isomorphic to the rational numbers. And hence we also have the following.

Corollary 3.1.25. $\mathbf{K4DU_r}$ *is sound and strongly complete with respect to the class of irreflexive trees with branches isomorphic to* $\langle \mathbb{Q}, < \rangle$.

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
S4: **K** as defined in Definition 2.5.1 $\mathbf{T} : p \to \mathbf{F}p$ $\mathbf{4} : \mathbf{FF}p \to \mathbf{F}p$	Transitive reflexive frames	Reflexive trees	Reflexive trees

Table 3.6: Axioms, Kripke frames, Tree Frames and Standard Frames for **S4**

3.2 Reflexive Trees

This section focusses on complete axiomatisations of the Priorean logics for the classes of reflexive trees.

3.2.1 All, Discrete, Locally Finite and Rooted Reflexive Trees

The axiom $\mathbf{T} : p \to \mathbf{F}p$ is known to be canonical for reflexivity. Adding this axiom to **K4** gives the well-known logic **S4** which is sound and complete with the respect to the class of reflexive and transitive frames (see e.g. Section 4.1 in [4]). In Table 3.6 we give a summary of the classes this logic defines.

Given a **S4**-consistent set of formulas, we may produce a satisfying model based on a reflexive tree by taking the reflexive and transitive closure of the unravelling of the canonical model around a suitable point. This strategy will be used to prove the following completeness result.

Theorem 3.2.1. **S4** *is sound and strongly complete with respect to the class of reflexive trees.*

Proof. Soundness follows from the fact that the axioms are valid on reflexive trees.

For completeness, see Lemma 4.53 and Theorem 4.54 in [4] for reflexive trees where the authors start with a **S4**-consistent set of formulas Γ and then generate a submodel of the canonical model for **S4**, from an **S4**-mcs that contains Γ. Then, the submodel is unravelled (see Definition 2.6.1) to get a tree model $\mathcal{M} = (W, R, V)$. Next, they take the reflexive transitive closure of R to get R^*. Then $\mathcal{M}^* = (W, R^*, V)$ is a reflexive tree. Furthermore, there is a bounded morphism from $\mathcal{M}^*$ to submodel of the canonical model. This shows that satisfiability is preserved. Hence, since it includes the **S4**-mcs that contains Γ, Γ is also satisfied in the unravelled model. □

As seen in Section 3.1.1, the process of unravelling produces a rooted, discrete and locally finite model, and to ensure transitivity and reflexivity, we can take the reflexive, transitive closure of the relation of the unravelling. Therefore, we start with a **S4**-consistent set of formulas Γ, and then generate a submodel of the canonical model for **S4**, from an **S4**-mcs that contains Γ. Then, unravel the submodel to get a rooted, discrete, locally finite model $\mathcal{M} = (W, R, V)$. Next, take the reflexive transitive closure of R to get R^*. Then $\mathcal{M}^* = (W, R^*, V)$ is a rooted, discrete, locally finite reflexive tree, and since it includes the **S4**-mcs that contains Γ, Γ is also satisfied in the unravelled model.

Hence, we also have the following completeness results.

Theorem 3.2.2. **S4** *is sound and strongly complete with respect to the class of rooted reflexive trees.*

Proof. We need to show that $\mathcal{M}^*$ as above is rooted. But this follows from the fact that unravelling gives a rooted model. The proof is similar to that of Theorem 3.1.2. □

Theorem 3.2.3. **S4** *is sound and strongly complete with respect to the class of discrete reflexive trees.*

Proof. $\mathcal{M}^*$ as above is discrete. The proof is similar to that of Theorem 3.1.3. □

Theorem 3.2.4. **S4** *is sound and strongly complete with respect to the class of locally finite reflexive trees.*

Proof. $\mathcal{M}^*$ as above is locally finite. The proof is similar to that of Theorem 3.1.4. □

Theorem 3.2.5. **S4** *is sound and strongly complete with respect to the class of well-founded reflexive trees.*

Proof. $\mathcal{M}^*$ as above is well-founded. The proof is similar to that of Theorem 3.1.5. □

Furthermore, by adding a tail as in Definition 3.1.9 using the negative integers as the set I and letting R' be the usual non-strict ordering of the integers, the satisfying model, call it $\mathcal{M}$, produced for the proofs of Theorems 3.2.3 to 3.2.6 can be transformed into a left unbounded, reflexive tree model $\mathcal{M}'$. Moreover, since $\mathcal{M}'$ is a generate submodel of $\mathcal{M}$, the root of $\mathcal{M}$ is modally equivalent to the corresponding instant in $\mathcal{M}'$. Hence any **S4**-consistent set may be satisfied in a left-unbounded reflexive tree model. This yields the following completeness result:

Theorem 3.2.6. **S4** *is sound and strongly complete with respect to the class of left unbounded reflexive trees.*

Notice that the process of unravelling produces a right unbounded model. Hence the model $\mathcal{M}^*$ for **S4** as defined above and used in the proofs of Theorems 3.2.2 to 3.2.5, is already right unbounded. If we add the tail as described above, the model $\mathcal{M}'$ is unbounded. Hence, we have the following completeness results for unbounded trees.

Theorem 3.2.7. **S4** *is sound and strongly complete with respect to the class of right unbounded reflexive trees and the class of unbounded reflexive trees.*

Note that the unravelled model $\mathcal{M}^*$ is locally finite and right unbounded. Hence, we can use a similar isomorphism as in the proof of Theorem 3.1.8, but taking reflexivity into account, to get a completeness result for reflexive trees with branches isomorphic to the natural numbers. Furthermore, the model $\mathcal{M}'$ used to prove Theorems 3.2.6 and 3.2.7 is locally finite and unbounded. Therefore, we can define a similar isomorphism as in Theorem 3.1.12, but taking reflexivity into account, to get a completeness result for the class of reflexive trees with branches isomorphic to the integers. Hence we have the following results.

Theorem 3.2.8. **S4** *is sound and strongly complete with respect to the class of reflexive trees with branches isomorphic to* $\langle \mathbb{N}, \leq \rangle$*, and the class of reflexive trees with branches isomorphic to* $\langle \mathbb{Z}, \leq \rangle$*.*

Note that, in [60], Segerberg, and in [32], Goldblatt used the Dummett axiom $\mathbf{G}(\mathbf{G}(\varphi \to \mathbf{G}\varphi) \to \varphi) \to (\mathbf{FG}\varphi \to \varphi)$ to axiomatise the modal logic of the natural numbers. However, in the example below, we show that this axiom can be falsified in a tree and hence is not suitable as a part of an axiomatisation of the modal logic of the class of reflexive trees with branches isomorphic to the natural numbers.

Example 3.2.9. We will falsify the Dummett axiom at the instant w in the model $\mathcal{M} = (W, R, V)$ where $\mathcal{M}$ is an locally finite reflexive tree. Let $W = \{v, w, w_0, w_1, w_2, ...\}$ where all instants are distinct. Let R be a transitive, reflexive relation satisfying the following:

- Rwv
- Rvw_i or Rw_iv for no $i \in \omega$
- Rw_iw_{i+1} for all $i \in \omega$

Let $V(p) = \{v\} \cup \{w_i \mid i = 2n, n \in \omega\}$.

Hence, in this model we have the following: $\mathcal{M}, v \Vdash \mathbf{G}p$ and hence $\mathcal{M}, w \Vdash \mathbf{FG}p$. We also have $\mathcal{M}, w \Vdash \neg p$. Hence, we have $\mathcal{M}, w \nVdash \mathbf{FG}p \to p$. Furthermore, we also have $\mathcal{M}, w_i \Vdash p$ and $\mathcal{M}, w_i \nVdash \mathbf{G}p$ for all $i = 2n$, and hence $\mathcal{M}, w_i \nVdash p \to \mathbf{G}p$ for all $i = 2n$. Therefore, we have $\mathcal{M}, w_0 \nVdash \mathbf{G}(p \to \mathbf{G}p)$ and hence $\mathcal{M}, w \Vdash (\mathbf{G}(p \to \mathbf{G}p)) \to p$. Therefore, we have $\mathcal{M}, w \Vdash \mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p)$ and hence $\mathcal{M}, w \nVdash \mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to (\mathbf{FG}p \to p)$ as required.

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
K4Grz: **K** as defined in Definition 2.5.1 **4** : $\mathbf{FF}p \to \mathbf{F}p$ **Grz** : $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$	Transitive reflexive conversely well-founded frames	Conversely well-founded reflexive trees	Conversely well-founded reflexive trees

Table 3.7: Axioms,Kripke frames, Tree Frames and Standard Frames for **K4Grz**

3.2.2 Finite Reflexive Trees

The axiom **Grz** : $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$, known as the Grzegorczyk formula, defines the class of reflexive, transitive, antisymmetric frames that contain no infinite ascending chains of distinct instants (see e.g. [15]).

Consider the logic **K4Grz**. Table 3.7 summarises the axioms together with the frame classes relevant to it.

Weak completeness of **K4Grz** for the class of finite reflexive trees was given in Section 5.5 of [15]. The main results that lead to the completeness are summarised below.

Lemma 3.2.10. *(Proposition 5.48 and Corollary 5.49 in [15]) The canonical frame $\mathcal{F}^{\mathbf{KGrz}}$ for* **K4Grz** *is reflexive and transitive and hence* **K4Grz** = **S4Grz** = **KGrz**.

Recall the definition of Φ-congruent in Section 2.3 for the following lemma.

Lemma 3.2.11. *(Lemma 5.50 in [15]) Let $\mathcal{M}^{\mathbf{KGrz}} = (W, R, V)$ be the canonical model for* **KGrz** *and let Φ be a subset of formulas from $\mathcal{L}^m_{Prior}$. Then, for every formula $\mathbf{G}\varphi$ in Φ, if $\mathcal{M}^{\mathbf{KGrz}}, w \Vdash \varphi$ and $\mathcal{M}^{\mathbf{KGrz}}, w \nVdash \mathbf{G}\varphi$ for some instant $w \in W$, then there exists a $v \in W$ with Rwv such that $\mathcal{M}^{\mathbf{KGrz}}, v \nVdash \varphi$ and, for no $u \in W$ with Rvu and $v \neq u$, is $w \cong_\Phi u$.*

Using these two results [15] build a model with the required properties as follows. For weak completeness, start with a non-theorem α of **KGrz**, and let $Cl(\alpha)$ be the set of all subformulas of α. Using the lemmas above, [15] build a model in which $\neg\alpha$ is satisfied that is a finite partial order. Hence, the following result.

Theorem 3.2.12. *(Theorem 5.51 in [15])* **KGrz** *is sound and weakly complete with respect to the class of finite partial orders.*

To construct a tree from the model built in the theorem above, we need the following lemma:

Lemma 3.2.13. *(Theorem 2.19 in [15]) Every rooted non-strictly partially ordered frame is a bounded morphic image of some reflexive tree frame, which can be chosen to be finite if the rooted frame is finite.*

Putting all these results together, we get the following result:

Theorem 3.2.14. *(Corollary 5.52 in [15])* **KGrz** *is sound and weakly complete with respect to the class of finite reflexive trees.*

Since the model built in the proof of 3.2.14 is also conversely well-founded, since it is finite, we have the following consequence.

Corollary 3.2.15. **KGrz** *is sound and weakly complete with respect to the class of conversely well-founded reflexive trees.*

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
S4: **K** as defined in Definition 2.5.1 **T** : $p \to \mathbf{F}p$ **4** : $\mathbf{FF}p \to \mathbf{F}p$	Transitive, reflexive frames	Reflexive trees	Dense reflexive trees

Table 3.8: Axioms, Kripke frames, Tree Frames and Standard Frames for **S4**

3.2.3 Dense Reflexive Trees

Consider the logic **S4**. Table 3.8 summarises the axioms together with the frame classes relevant to it.

In this section we show that **S4** is strongly complete for the future fragment of the Priorean logic of the class of dense reflexive trees. We do this by building a model for every set of formulas consistent with **S4** based on a dense reflexive tree. To show this, we build a network, as in Section 3.1.4, where this was used to build irreflexive and dense tree models for **K4D**-consistent sets of formulas.

In building a reflexive dense network, we need to ensure that if we start with a reflexive network, that this is preserved when $S_{\mathbf{F}}$ and S_d defects are repaired. For this purpose, let $\mathcal{N} = (N, \underline{\ll}, \kappa)$ be a network, where N is a set of nodes, with a binary reflexive relation $\underline{\ll}$ on N and a labelling function κ that maps nodes to **S4**-mcs's. Then we say that $\mathcal{N}$ is a **coherent network** for **S4** if it has the following properties:

1. $\underline{\ll}$ is a reflexive partial ordering.
2. $\underline{\ll}$ is left linear, i.e. for all s, t and u if $s \underline{\ll} t$ and $u \underline{\ll} t$ then either $s \underline{\ll} u$ or $u \underline{\ll} s$.
3. N is connected, i.e. for all $s, t \in N$, there is a $u \in N$ such that $u \underline{\ll} s$ and $u \underline{\ll} t$.
4. $\kappa(s) R \kappa(t)$ for all $s, t \in N$ such that $s \underline{\ll} t$ where R is the canonical relation of **S4**.

A network that is coherent and saturated will be referred to as a **d-perfect network**. Defects are defined similar to how we defined in in Section 3.1.4, but taking reflexivity into account:

A network $\mathcal{N} = (N, \underline{\ll}, \kappa)$ has an S^r_d defect if there is a pair $s, t \in N$ for which the density condition not satisfied, i.e. for no u, with $u \neq s$ and $u \neq t$, do we have $s \underline{\ll} u \underline{\ll} t$. We will refer to this S^r_d defect as (s, t). The pair $(s, \mathbf{F}\psi)$ is an $S^r_{\mathbf{F}}$ defect in $\mathcal{N}$ if $\mathbf{F}\psi \in \kappa(s)$ for $s \in N$ but there is no $t \in N$ such that $s \underline{\ll} t$ and $\psi \in \kappa(t)$.

Next we show that S^r_d and $S^r_{\mathbf{F}}$ defects in coherent networks for **S4** can be repaired in such a way that the result is again a coherent network for **S4**. As will be seen in the proof, the repair process is a slight modification of that used in Section 3.1.4.

Lemma 3.2.16. *For any defect of a finite, coherent network $\mathcal{N}$ for* **S4***, there is a finite, coherent network $\mathcal{N}'$ for* **S4** *that extends $\mathcal{N}$ and lacks the defect.*

Proof. Let $\mathcal{N} = (N, \underline{\ll}, \kappa)$ be a finite, coherent network. The following describes the methods for removing S_d and $S_{\mathbf{F}}$ defects.

S^r_d defects:

Suppose there are distinct nodes $s, t \in N$ with $s \underline{\ll} t$ for which there is no distinct intermediate node. Since $\mathcal{N}$ is coherent $\kappa(s) R \kappa(t)$ and by reflexivity there exists an **S4**-mcs Γ such that $\kappa(s) R \Gamma R \kappa(t)$. Therefore, add a new node u to N with $\kappa(u) = \Gamma$ and define $\mathcal{N}' = (N, \underline{\ll}', \kappa')$ by setting

$N' = N \cup \{u\}$
$\underline{\ll}' \ = \ \underline{\ll} \cup \{(x, u) \mid x \underline{\ll} s\} \cup \{(u, x) \mid t \underline{\ll} x\} \cup \{(u, u)\}$
$\kappa' = \kappa \cup \{(u, \Gamma)\}$

Then N' does not have the defect and, as $\underline{\ll}'$ is still a reflexive partial order, N' is reflexive, left linear and connected. To complete coherency, let $x, y \in N'$ such that $x \underline{\ll}' y$. If $x = y$ then the reflexivity of the canonical relation R gives $\kappa'(x) R \kappa'(y)$. The other cases follows similarly to that in the proof of Lemma 3.1.18.

$S^r_{\mathbf{F}}$ defects:

Suppose there is a node $s \in N$ with $\mathbf{F}\psi \in \kappa(s)$ but there is no $t \in N$ such that $s \underline{\ll} t$ and $\psi \in \kappa(t)$. Let u' be a node in N with $s \underline{\ll} u'$, such that $(u', \mathbf{F}\psi)$ is a $S_{\mathbf{F}}$ defect and for all $u \in N$ with $u' \underline{\ll} u$, $(u, \mathbf{F}\psi)$ is not a $S_{\mathbf{F}}$-defect. Since N is finite, such a u' exists and is possibly s. Insert a new node s' immediately after u' on a new branch of N and let Γ' be an mcs containing ψ in $\mathcal{M}$ such that $\kappa(u') R \Gamma'$ (guaranteed by the Existence Lemma).

Define $\mathcal{N}' = (N, \underline{\ll}', \kappa')$ by
$N' = N \cup \{s'\}$
$\underline{\ll}' \ = \ \underline{\ll} \cup \{(x, s') \mid x \underline{\ll} u'\} \cup \{(s', s')\}$
$\kappa' = \kappa \cup \{(s', \Gamma')\}$

Note that $\mathcal{N}'$ does not contain the $S_{\mathbf{F}}$ defect $(s, \mathbf{F}\psi)$. It is easy to see that it is still a connected reflexive left linear partial order. To show that it is also coherent, we just need to show that the relations are preserved.

Let $x, y \in N'$ such that $x \underline{\ll}' y$. We just need to check the cases when either x or y is the new node s'. When $x = s'$ then $s' = y$ as s' was added on a new branch. Therefore, suppose $y = s'$. By construction $\kappa'(s') = \Gamma'$ with $\kappa'(u') R \Gamma'$ and by reflexive coherency $\kappa'(x) R \kappa'(u')$. Therefore, by transitivity of the canonical model $\kappa'(x) R \Gamma'$ as required, and that shows that $\mathcal{N}'$ is coherent.

□

Hence, the extension of a finite, coherent network, resulting from the repair of a $S^r_{\mathbf{F}}$ or S^r_d defect in the way described in the proof of Lemma 3.2.16, will also be a finite, coherent network. Using the same method of building a network as in Section 3.1.4 we can build a dense model for a **S4**-consistent set of formulas, as in Theorem 3.1.21, which gives us the following completeness result.

Theorem 3.2.17. **S4** *is sound and strongly complete with respect to the class of dense reflexive trees.*

Proof. The proof is exactly the same as the proof of Theorem 3.2.17, except that we create a reflexive partial ordering in the building of the network $\mathcal{N}$ and use Lemma 3.2.16 to repair the defects. The resulting model $\mathcal{M} = (W, R, V)$ will then be a dense reflexive tree. □

If we add a tail (see Definition 3.1.9) to the model built in Theorem 3.2.17 with I the rational numbers and R' the usual ordering of the rational numbers, we can build a left unbounded dense model for **S4** modally equivalent to the original model. This gives us the following completeness result.

Theorem 3.2.18. **S4** *is sound and strongly complete with respect to the class of left unbounded dense reflexive trees.*

Proof. Let $\mathcal{M}$ be the model built in Theorem 3.2.17 that satisfies an **S4**-consistent set Γ, and add a copy of the rational numbers as a tail with their usual ordering, as done in Definition 3.1.9. The resulting model will then be dense and left unbounded, and will still satisfy the **S4**-consistent set Γ. □

To construct a satisfying model for an **S4**-consistent set of formulas based on a right unbounded, dense reflexive tree, we use the model $\mathcal{M}$ built in the proof of Theorem 3.2.17, and then replace all reflexive instants at leaves of the tree with a copy of the rational numbers. This is done in the proof of the following theorem.

Theorem 3.2.19. **S4** *is sound and strongly complete with respect to the class of right unbounded dense reflexive trees, to the class of unbounded dense reflexive trees, to the class of reflexive trees with branches isomorphic to* $\langle \mathbb{Q}, \leq \rangle$*, and to* $\langle \mathbb{Q}^+ \cup \{0\}, \leq \rangle$

Proof. Soundness follows from the fact that the axioms are valid on these classes of trees.

Let Γ be an **S4**-consistent set. We build a satisfying model for Γ as in the proof of Theorem 3.2.17.. Let $\mathcal{M} = (W, R, V)$ be the model built in the proof of Theorem 3.2.17 where w_0 is the **S4**-mcs containing Γ in $\mathcal{M}$ (also, w_0 is the root of $\mathcal{M}$). Let $\mathcal{M}' = (W', R', V')$ be the model where every leaf in $\mathcal{M}$ is replaced by a copy of the rational numbers, in the following way: Suppose $\{w_i \mid i \in I\}$, where I is an index set, is the set of leaves in $\mathcal{M}$. For each i, we replace w_i with a copy of the rational numbers $\mathbb{Q}_i$. Then

- $W' = (W - \{w_i \mid i \in I\}) \cup \bigcup_i \mathbb{Q}_i$
- For all $u, v \in W'$ it is the case that $R'uv$ iff
  - $u, v \in W$ and Ruv or,
  - $u \in W$ and $v \in \mathbb{Q}_i$ for some i and Ruw_i or,
  - $u, v \in \mathbb{Q}_i$ for some i, and $u \leq v$ where $\leq$ is the natural reflexive ordering of the rational numbers.
- $V'(p) = (V(p) - \{w_i \mid w_i \in V(p)\}) \cup \{\mathbb{Q}_i \mid w_i \in V(p)\}$

Define the function $f : W' \to W$ as follows:

- $f(w) = w$ for all $w \in W - \{w_i \mid i \in I\}$
- $f(u) = w_i$ for all $u \in \mathbb{Q}_i$

We show that f is a bounded morphism:

Clearly f is surjective and local harmony is preserved. We show that the back and forth conditions hold:

Forth Condition: Suppose $R'wv$.

Case 1: $w, v \in W - \{w_i \mid i \in I\}$. Then Rwv by definition of R'. Hence $Rf(w)f(v)$ by the definition of f.

Case 2: $w \in W - \{w_i \mid i \in I\}$ and $v \in \mathbb{Q}_i$ where $f(v) = w_i$. Then Rww_i by definition of R'. Hence $Rf(w)f(v)$ by the definition of f.

Case 3: $w, v \in \mathbb{Q}_i$. Then $f(w) = f(v) = w_i$. By reflexivity, it follows that $Rf(w)f(v)$.

Back Condition: Suppose $Rf(w)v'$ then we need to show that there exists v such that $R'wv$ and $f(v) = v'$. Then, let $v = v'$ with $f(v) = v'$. From the definition of R' it follows that $R'wv$.

Therefore, f is a bounded morphism, and it follows that $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', f(w) \Vdash \varphi$. Therefore $\mathcal{M}', f(w_0) \Vdash \Gamma$. Furthermore, since $\mathcal{M}'$ is a dense, right unbounded reflexive tree, it follows that **S4** is sound and strongly complete with respect to the class of right unbounded dense reflexive trees, and hence, since the construction is countable, it is also sound and strongly complete with respect to the class of reflexive trees with branches isomorphic to $\langle \mathbb{Q}^+ \cup \{0\}, \leq \rangle$.

Now, to get the completeness result for the class of unbounded dense reflexive trees, we add a copy of the rational numbers as a tail to $\mathcal{M}'$. Let $\mathcal{M}''$ be this model. Then $\mathcal{M}'$ is a generated submodel of $\mathcal{M}''$ as seen in Definition 3.1.9. Furthermore, $\mathcal{M}''$ is dense, left and right unbounded, and the completeness result follows. Furthermore, by [18], the branches of $\mathcal{M}''$ are also isomorphic to the rational numbers. Therefore, **S4** is sound and strongly complete with respect to the class of reflexive trees with branches isomorphic to the rational numbers.

□

Chapter 4

Complete Axiomatisations of Priorian Temporal Logics of the Class of all Trees and the Class of Dense Trees

In this chapter we aim find complete finite axiomatisations for the logics of the class of all irreflexive and reflexive trees, as well as the class of dense irreflexive and reflexive trees, in the Priorian temporal language $\mathcal{L}_{Prior}$ (see Chapter 2 Section 2.3). All formulas in this chapter, and the following on axiomatisations, will be in this language.

As the frames and models for these logics are bidirectional (as opposed to uni-directional in Chapter 3), some of the methods used in Chapter 3, for example unravelling, do not work as they do not preserve satisfaction. However, by adding some dual axioms to the logics, other methods from Chapter 3 translate well to the bidirectional case.

4.1 Basic Temporal Logic for Trees

We begin the chapter with introducing the logic we will use as a basis in most of the completeness results in this and the following chapter. Because the semantics of the Priorian temporal language and the basic tense language correspond (see Chapter 2), it makes sense to look at the logic $\mathbf{K_t}$ as a possible candidate for completeness of the logics of different the classes of irreflexive trees. Other properties that all trees have in common include transitivity and left linearity. Therefore, it makes sense to add axiom **4** and **.3$_\mathbf{l}$** defined below to get $\mathbf{K_t4.3}_l$.

Definition 4.1.1. Let $\mathbf{K_t4.3}_l$ be the temporal logic containing the following axioms:

- **K** axioms for temporal logic: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ and $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$
- Dual axioms: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ and $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$
- Converse axioms of the past-future relations: $p \to \mathbf{GP}p$ and $p \to \mathbf{HF}p$
- Transitivity axiom 4: $\mathbf{FF}p \to \mathbf{F}p$
- Left linearity axiom $.3_l$: $(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$

The logic is also closed under substitution, generalisation and Modus Ponens (see Definition 2.5.1 in Section 2.5) for precise definition of these rules). Note that this logic is an extension of the basic normal logic $\mathbf{K_t}$ (defined in Section 2.5) where we just add left linearity and transitivity as these properties are needed for the trees we are interested in.

Let $\mathbf{Pr}_{basic}$ be an abbreviation of $\mathbf{K_t4.3}_l$.

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1	Transitive, left linear frames	Irreflexive trees	Irreflexive trees

Table 4.1: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{basic}$

Note that all the axioms of $\mathbf{Pr}_{basic}$ are canonical in the temporal language for the properties they define (see e.g. Chapter 4 in [4]). Here and in the rest of the chapter, unless otherwise indicated, $\vdash$ means deducible in $\mathbf{Pr}_{basic}$ (or the extension of this logic in the given section) and consistent will mean $\mathbf{Pr}_{basic}$-consistent (or the extension of this logic in the given section).

Since these axioms listed above are canonical, it follows that the canonical model $\mathcal{M} = (W, R, V)$ is transitive and left linear. Therefore, $\mathbf{Pr}_{basic}$ is strongly complete with respect to all transitive left linear frames. Furthermore, $\mathbf{Pr}_{basic}$ defines the class of all transitive, left linear frames. Table 4.1 summarises the axioms together with the frame classes relevant to it.

4.2 All Irreflexive Trees

In Section 3.1.1, completeness for the class of irreflexive trees in the future fragment of the Priorian temporal language was obtained by unravelling a generated submodel of the canonical model of $\mathbf{K_4}$. However, unravelling does not preserve the satisfaction of **P** formulas, since this process does not ensure that all **P** formulas have witnesses. To circumvent this issue, we use bulldozing instead.

In this section we show that $\mathbf{Pr}_{basic}$ is strongly sound and complete with respect to the class of irreflexive trees. Soundness is simply a question of checking the validity of the axioms on irreflexive trees – this is immediate. To establish strong completeness, we will build an irreflexive tree model for an arbitrary $\mathbf{Pr}_{basic}$-consistent set of formulas.

Let Γ be a $\mathbf{Pr}_{basic}$-consistent set of formulas and let $\mathcal{M} = (W, R, V)$ be the canonical model for $\mathbf{Pr}_{basic}$ where $\mathcal{M}, w \Vdash \Gamma$ for some instant $w \in W$. Then, by the canonicity of the axioms, we know that $\mathcal{M}$ is transitive and left linear. We need to transform $\mathcal{M}$ into an irreflexive tree.

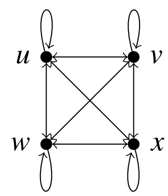
Let $\mathcal{M}' = (W', R', V')$ be the submodel of $\mathcal{M}$ bidirectionally generated by w' (cf. Definition 2.4.2). Then $\mathcal{M}'$ is transitive and connected (cf. Proposition 2.4.5), and does not branch to the left. $\mathcal{M}'$ is therefore a tree of clusters, i.e., the quotient frame (cf. Definition 2.6.2) is a tree.

We perform a version of bulldozing (originally introduced by Segerberg in [61], for other convenient presentations see also e.g. [15] or [4]) as follows: For each $v \in W'$ let $v^+ = \{(v, i) \mid i \in \mathbb{Z}\}$ if $C(v)$ is non-degenerate and let $v^+ = \{(v, 0)\}$ if $C(v)$ is degenerate. We may fix a strict well-ordering $<_C$ of each cluster C (see e.g., [39]). Let $W^b = \bigcup_{v \in W'} v^+$ and define $R^b \subseteq W^b \times W^b$ such that

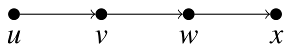
$$R^b(v,i)(u,j) \iff \begin{cases} C(v) = C(u) \quad \text{and} \quad i < j \\ \text{or} \\ C(v) = C(u) \quad \text{and} \quad i = j \quad \text{and } v <_{C(v)} u \\ \text{or} \\ C(v) \neq C(v) \quad \text{and} \quad R'vu \end{cases} \tag{4.1}$$

Lastly, define the valuation V^b such that $V^b(p) = \{(v, i) \mid v \in V'(p)\}$ and set $\mathcal{M}^b = (W^b, R^b, V^b)$. This process of bulldozing $\mathcal{M}'$ to obtain $\mathcal{M}^b$ is illustrated in Figure 4.1. $\mathcal{M}^b$ is therefor just like $\mathcal{M}'$, but with each non-degenerate cluster C "straightened out" into a strict linear ordering, unbounded both from below and from above, in which copies of each point in C repeat unboundedly both upwards and downwards.

Starting with a cluster C:



Well-order the cluster C:



Make a copy of well-ordered C for each $i \in \mathbb{Z}$ showing correspondence with dashed lines:

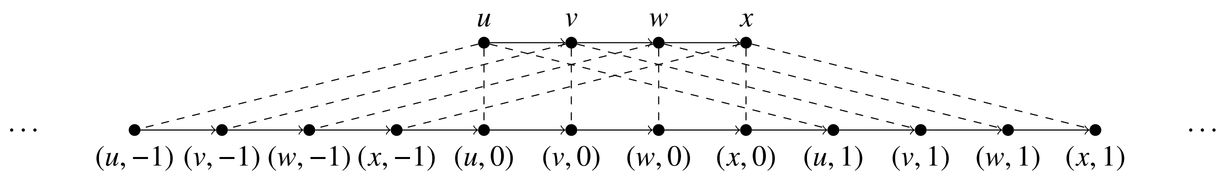


Figure 4.1: Bulldozing a non-degenerate cluster into a strict, unbounded linear order.

Lemma 4.2.1. (W^b, R^b) *is an irreflexive tree.*

Proof. We need to show that (W^b, R^b) is transitive, irreflexive, left linear and connected. Irreflexivity is immediate by the definition of R^b. For the sake of proving the other three properties, we begin by noting that for $R^b(u,i)(v,j)$ it is necessary (but not sufficient) that $R'uv$.

For the sake of transitivity, suppose that $R^b(u,i)(v,k)$ and $R^b(v,k)(x,j)$. Hence $R'uv$ and $R'vx$, and so $R'ux$, by the transitivity of R'. If $C(u) \neq C(x)$ we have $R^b(u,i)(x,j)$ by the third clause of (4.1). If, on the other hand, $C(u) = C(x)$ it follows that $C(u) = C(x) = C(v)$ and hence one can check the four possibilities corresponding to the first two clauses of (4.1) to verify that $R^b(u,i)(x,j)$.

For the sake of left-linearity, suppose that $R^b(u,i)(v,k)$ and $R^b(x,j)(v,k)$ with $(u,i) \neq (x,j)$. We may assume without loss of generality that $i \leq j$. By the remark above, $R'uv$ and $R'xv$ and so, by the left linearity of R', it must be the case that $R'ux$ or $R'xu$. In the case that $C(u) = C(x)$, if $i < j$, then $R^b(u,i)(x,j)$ or, if $i = j$, then either $u <_{C(u)} x$ or $x <_{C(u)} u$ and hence $R^b(u,i)(x,j)$ or $R^b(x,j)(u,i)$. In the case that $C(u) \neq C(x)$, the fact that $R'ux$ or $R'xu$ implies that $R^b(u,i)(x,j)$ or $R^b(x,j)(u,i)$, as desired.

For the sake of connectedness, suppose that (u,i) and (v,k) are R^b incomparable. It follows from (4.1) that u and v are R'-incomparable and hence, by the connectedness of (W', R'), that there are an instant $x \in W'$ such that $R'xu$ and $R'xv$ and, moreover, that $C(x) \neq C(u)$ and $C(x) \neq C(v)$ (or else u and v would have been comparable). Thus, by clause 3 of (4.1), $R^b(x,0)(u,i)$ and $R^b(x,0)(v,k)$. □

Next, it is easy to check that the map $f : W^b \to W'$ given by $f : (v,i) \mapsto v$ is a bounded morphism from $\mathcal{M}^b$ onto $\mathcal{M}'$. It follows that $\mathcal{M}^b, (w,0) \Vdash \Gamma$, as desired. We have therefore proved the following theorem:

Theorem 4.2.2. $\mathbf{Pr}_{basic}$ *is sound and strongly complete with respect to the class of irreflexive trees.*

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
$\mathbf{Pr}_{basic}\mathbf{T}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$	Transitive, left linear, reflexive frames	Reflexive trees	Reflexive trees

Table 4.2: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{basic}\mathbf{T}$

4.3 All Reflexive Trees

Let $\mathbf{Pr}_{basic}\mathbf{T}$ be the extension of $\mathbf{Pr}_{basic}$ with reflexivity axiom $\mathbf{T}$: $p \to \mathbf{F}p$. In this section we show that $\mathbf{Pr}_{basic}\mathbf{T}$ is strongly complete with respect to the class of reflexive trees. We do this by building a model for every set of formulas consistent with $\mathbf{Pr}_{basic}\mathbf{T}$ based on a reflexive tree, using the methods of Section 4.2. Table 4.2 summarises the axioms together with the frame classes relevant to it.

Let Γ be a $\mathbf{Pr}_{basic}\mathbf{T}$-consistent set of formulas and w an instant in the canonical model $\mathcal{M}$ for $\mathbf{Pr}_{basic}\mathbf{T}$ satisfying Γ. Let $\mathcal{M}' = (W', R', V')$ be the submodel of $\mathcal{M}$ bidirectionally generated from w. Then, since $\mathcal{M}$ is transitive, reflexive and left linear, we may appeal to Proposition 2.4.5 to conclude that $\mathcal{M}'$ is a reflexive tree of clusters. Note that this means, in particular, that $\mathcal{M}'$ has no degenerate clusters.

As in the previous subsection, let let $v^+ = \{(v, i) \mid i \in \mathbb{Z}\}$ for each $v \in W'$, noting that there are now no degenerate clusters to be treated differently. We may fix a (reflexive) well-ordering $\leq_C$ of each cluster C. Let $W^b = \bigcup_{v \in W'} v^+$ and define $R^b \subseteq W^b \times W^b$ such that

$$R^b(v,i)(u,j) \quad \Longleftrightarrow \quad \begin{cases} C(v) = C(u) \quad \text{and} \quad i < j \\ \text{or} \\ C(v) = C(u) \quad \text{and} \quad i = j \quad \text{and } v \leq_{C(v)} u \\ \text{or} \\ C(v) \neq C(v) \quad \text{and} \quad R'vu \end{cases} \tag{4.2}$$

Lastly, define the valuation V^b such that $V^b(p) = \{(v, i) \mid v \in V'(p)\}$ and set $\mathcal{M}^b = (W^b, R^b, V^b)$. Figure 4.1 may again be taken as a visual illustration of this construction, except that all instants are now reflexive. The proof of the following lemma is verbatim the same as that of Lemma 4.2.1, except for substituting references to irreflexivity with with references to reflexivity and substituting $\leq_{C(u)}$ for $<_{C(u)}$ in the proof of left-linearity.

Lemma 4.3.1. (W^b, R^b) *is a reflexive tree.*

It is again easy to check that the map $f : W^b \to W'$ given by $f : (v, i) \mapsto v$ is a bounded morphism from $\mathcal{M}^b$ onto $\mathcal{M}'$. It follows that $\mathcal{M}^b, (w, 0) \Vdash \Gamma$, as desired. We have therefore proved the following theorem:

Theorem 4.3.2. $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and strongly complete with respect to the class of all reflexive trees.*

Remark 4.3.3. Note that $\mathcal{M}^b$ has no leaves or roots, as all clusters in $\mathcal{M}'$ are bulldozed into linear orders which are unbounded both from below and above. Hence, the $\mathcal{M}^b$ is unbounded.

4.4 Unbounded Irreflexive and Reflexive Trees

In this section, we show that the addition of (the appropriate subset of) the well-known seriality axioms $\mathbf{U_l} : \mathbf{P}\top$ and $\mathbf{U_r} : \mathbf{F}\top$ to $\mathbf{Pr}_{basic}$ produces complete axiomatisations of the classes of left and/or right unbounded irreflexive

Logic and Axioms	Class of Kripke Frames	Class of (Ir)reflexive Trees	Standard Frames
$\mathbf{Pr}_{basic}\mathbf{U_l}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{U_l} : \mathbf{P}\top$	Left unbounded, transitive, left linear frames	Left unbounded irreflexive trees	Left unbounded irreflexive trees
$\mathbf{Pr}_{basic}\mathbf{U_r}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{U_r} : \mathbf{F}\top$	Right unbounded, transitive, left linear frames	Right unbounded irreflexive trees	Right unbounded irreflexive trees
$\mathbf{Pr}_{unbnd}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{U_l} : \mathbf{P}\top$ $\mathbf{U_r} : \mathbf{F}\top$	Unbounded, transitive, left linear frames	Unbounded irreflexive trees	Unbounded irreflexive trees
$\mathbf{Pr}_{basic}\mathbf{T}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$	Transitive, left linear, reflexive frames	Reflexive trees	Unbounded reflexive trees

Table 4.3: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{basic}\mathbf{U_l}$, $\mathbf{Pr}_{basic}\mathbf{U_r}$, $\mathbf{Pr}_{unbnd}$ and $\mathbf{Pr}_{basic}\mathbf{T}$

trees (See Table 4.3). In fact, we argue that, in the presence of these axioms, the construction of a satisfying model in Section 4.2 produces models based on appropriately unbounded trees. Moreover, by noting that the model constructed in the proof of Theorem 4.3.2 is in fact unbounded as we did in Remark 4.3.3, we immediately obtain the corollary that $\mathbf{Pr}_{basic}\mathbf{T}$ is sound and strongly complete with respect to the classes of (left/right) unbounded reflexive trees.

Theorem 4.4.1. *The following logics are sound and strongly complete with respect to the indicated classes of trees:*

- $\mathbf{Pr}_{basic}\mathbf{U_l}$ *with respect to the class of left unbounded irreflexive trees.*
- $\mathbf{Pr}_{basic}\mathbf{U_r}$ *with respect to the class of right unbounded irreflexive trees.*
- $\mathbf{Pr}_{basic}\mathbf{U_l}\mathbf{U_r}$*, denoted by* $\mathbf{Pr}_{unbnd}$*, with respect to the class of unbounded irreflexive trees.*

Proof. The model $\mathcal{M}^b$ constructed in Section 4.2 for the completeness proof of $\mathbf{Pr}_{basic}$ is an irreflexive tree, so it is sufficient to show that, in the presence of the additional axiom $\mathbf{P}\top$ ($\mathbf{F}\top$), it is left (right) unbounded. Indeed, in the presence of $\mathbf{P}\top$ ($\mathbf{F}\top$), the canonical model will be left (right) unbounded and hence this will also be true of the bidirectionally point generated submodel $\mathcal{M}'$. We showed that there exited a surjective bounded morphism $f : \mathcal{M}^b \to \mathcal{M}'$. Let $(v, i) \in W^b$. Then $f(v, i) = v$. If $\mathcal{M}'$ is left-unbounded, then v has a predecessor, say u. But then, by the back condition, (v, i) must have a predecessor which, by the irreflexivity of $\mathcal{M}^b$, must be different from (v, i). Hence $\mathcal{M}^b$ will be left unbounded if $\mathcal{M}'$ is. The argument for the transfer of right-unboundedness is similar. □

Now, let us turn to reflexive trees. By Remark 4.3.3, the satisfying model $\mathcal{M}^b$ constructed in the completeness for $\mathbf{Pr}_{basic}\mathbf{T}$ is an unbounded, reflexive tree. This immediately yields the following theorem:

Theorem 4.4.2. $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and strongly complete with respect to the following classes of trees:*

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{dense}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{D} : \mathbf{F}p \to \mathbf{FF}p$	Transitive, left linear, dense frames	Dense irreflexive trees	Dense irreflexive trees

Table 4.4: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{dense}$

- *the class of left unbounded reflexive trees.*
- *the class of right unbounded reflexive trees.*
- *the class of unbounded reflexive trees.*

4.5 Dense Irreflexive Trees

Let $\mathbf{Pr}_{dense}$ be the logic obtained by adding the density axiom $\mathbf{D} : \mathbf{F}p \to \mathbf{FF}p$ (which is canonical for density; see for example Chapter 4 in [4]) to $\mathbf{Pr}_{basic}$. Table 4.4 summarises the axioms together with the frame classes relevant to it.

In this section we show that $\mathbf{Pr}_{dense}$ is strongly complete with respect to the class of dense irreflexive trees. We do this by building a model for every set of formulas consistent with $\mathbf{Pr}_{dense}$ based on a dense irreflexive tree, using a step-by-step process.

Let $\mathcal{M} = (W, R, V)$ be the canonical model for this $\mathbf{Pr}_{dense}$. Borrowing concepts and terminology from [4], [49], [12] and [11], we will build the required model using modified versions of networks (see Section 2.8).

To make sure we build the correct network, we will require the network to be strict coherent and saturated, as well as dense (see 2.8). All networks in the remainder of this section will be networks for a $\mathbf{Pr}_{dense}$-consistent set of formulas. Since we are interested in building a dense model, we need to ensure that the network is dense. Therefore, as in Section 3.1.4, an extra condition is needed: Let $(N, \ll, \kappa)$ be a network. Then we add the following to the saturation conditions:

- $\ll$ is dense, i.e. for all distinct $s, t \in N$ with $s \ll t$ there exists a node $u \in N$ different from s and t such that $s \ll u \ll t$.

A saturated network satisfying the above conditions will be called a **d-saturated network**. If a network is both strict coherent and d-saturated, it is called a **strict d-perfect network**. Notice that strict d-perfect networks will give rise to frames and models that are dense trees.

Recall from Section 3.1.4 that we call the violations of the conditions of d-saturation in a network $\mathcal{N} = (N, \ll, \kappa)$ **defects**, and formalise them as follows: The pair (s, t) is an S_d-defect if the density condition is not satisfied, i.e., $s \ll t$, and for no $u \in N$ do we have $s \ll u \ll t$. The pair $(s, \mathbf{F}\psi)$ is an $S_\mathbf{F}$ defect if $\mathbf{F}\psi \in \kappa(s)$ for $s \in N$ but there is no $t \in N$ such that $s \ll t$ and $\psi \in \kappa(t)$. The pair $(s, \mathbf{P}\psi)$ is an $S_\mathbf{P}$ defect if $\mathbf{P}\psi \in \kappa(s)$ for $s \in N$ but there is no $t \in N$ such that $t \ll s$ and $\psi \in \kappa(t)$.

From Lemma 2.8.5, it follows that if we can construct a strict d-perfect network for a given consistent set of formulas, we will have the required model. For this purpose we have to repair S_d, $S_\mathbf{P}$ and $S_\mathbf{F}$ defects.

As in Section 3.1.4 Lemma 3.1.18, we need to show that if we start with a (finite) strict coherent network and repair a defect, the resulting extended network is also (finite and) strict coherent. Any of the above defects can be repaired without losing the coherency of the network. The proof is similar to the proof given in Section 4.6. in [4] for countable linear orders.

Lemma 4.5.1 (Repair Lemma)**.** *For any defect of a finite, strict coherent network $\mathcal{N}$ for a* $\mathbf{Pr}_{dense}$*-consistent set of formulas, there is a finite strict coherent network $\mathcal{N}'$ for the* $\mathbf{Pr}_{dense}$*-consistent set of formulas that extends $\mathcal{N}$ and lacks the defect.*

Proof. S_d and $S_{\mathbf{F}}$ defects have already been covered in Lemma 3.1.18, hence we only do the case of $S_{\mathbf{P}}$ defects in this proof.

$S_{\mathbf{F}}$ *defects:*

Let $\mathcal{N} = (N, \ll, \kappa)$ be a finite, strict coherent network for a $\mathbf{Pr}_{dense}$-consistent set of formulas.

Suppose there is a node $s \in N$ with $\mathbf{P}\psi \in \kappa(s)$ but there is no $t \in N$ such that $t \ll s$ and $\psi \in \kappa(t)$. Let u' be a node in N such that $u' \underline{\ll} s$ and $(u', \mathbf{P}\psi)$ is a $S_{\mathbf{P}}$ defect and for all $u \in N$ with $u \ll u'$, $(u, \mathbf{P}\psi)$ is not a $S_{\mathbf{P}}$ defect. Since $\mathcal{N}$ is finite and $\ll$ is acyclic, such a u' exists. Insert a new node s' immediately before u' and let Γ' be an mcs containing ψ in $\mathcal{M}$ such that $\Gamma' R \kappa(u')$ (guaranteed to exist by the Existence Lemma for the canonical model). More formally, let $\mathcal{N}' = (N', \ll', \kappa')$ where

$$N' = N \cup \{s'\}, \; \ll' = \ll \cup \{(x, s') \mid x \ll u'\} \cup \{(s', x) \mid u' \underline{\ll} x\}, \text{ and } \kappa' = \kappa \cup \{(s', \Gamma')\}.$$

Note that $\mathcal{N}'$ does not contain the $S_{\mathbf{P}}$ defect $(s, \mathbf{P}\psi)$. It is easy to see that $\ll'$ is a connected, left linear, strict partial order on N', given that $\ll$ had these properties. So to establish strict coherency, we just need to verify clause 4 of Definition 2.8.2. To this end, let $x, y \in N'$ such that $x \ll' y$. We just need to check the cases when either x or y is the new node s'. When $x = s'$ the situation is similar as for the S_d defect, so suppose $y = s'$. By construction $\kappa'(s') = \Gamma'$ with $\Gamma' R \kappa'(u')$ and by strict coherency $\kappa'(x) R \kappa'(u')$. Then $\kappa'(x) R \kappa'(y)$ by the transitivity of the canonical relation R.

□

Since $(s, \mathbf{P}\psi)$ is not a defect in $\mathcal{N}'$ any more, and we showed in Lemma 3.1.18 that the same holds for S_d and $S_{\mathbf{F}}$ defects, we have the following result.

Lemma 4.5.2. *Once a defect has been repaired in a strict coherent network for a* $\mathbf{Pr}_{dense}$*-consistent set of formulas, no strict coherent extension of that network will have that defect again.*

Proof. The proof is similar to that of (Lemma 3.1.19), but also taking $S_{\mathbf{P}}$-defects into account. □

Therefore, starting with a finite strict coherent network, the proof below (already done for the linear case in e.g. [4]) shows that it can be turned into a strict d-perfect network using this process of repairing defects.

Lemma 4.5.3. *Any finite, strict coherent network $\mathcal{N} = (N, \ll, \kappa)$ for a* $\mathbf{Pr}_{dense}$*-consistent set of formulas can be extended to a strict d-perfect network.*

Proof. Let $\mathcal{N} = (N, \ll, \kappa)$ be a finite strict coherent network for a $\mathbf{Pr}_{dense}$-consistent set of formulas Γ. Let $S = \{s_i \mid i \in \omega\}$ be a countable set of nodes disjoint from N, i.e., a supply of new nodes to be used in repairing, step by step, the defects of $\mathcal{N}$. Next, let

$$\mathcal{D} = [(N \cup S) \times (N \cup S)] \cup [(N \cup S) \times \{\mathbf{F}\varphi \mid \varphi \in \mathcal{F}\wr\nabla\}] \cup [(N \cup S) \times \{\mathbf{P}\varphi \mid \varphi \in \mathcal{F}\wr\nabla\}],$$

i.e., $\mathcal{D}$ is the set of all possible defects of $\mathcal{N}$ and of any network extending it with nodes from S. Note that $\mathcal{D}$ is countable and let $D_0, D_1, D_2, \ldots$ be an arbitrary but fixed enumeration of its elements.

We will construct a sequence of finite, coherent extensions of $\mathcal{N}$, namely $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots$ as follows: let $\mathcal{N}_0 = \mathcal{N}$ and suppose that $\mathcal{N}_n$ has already been defined and is finite and coherent. Since $\mathcal{N}_n$ is finite, it has at least a density defect. Let D be the defect of $\mathcal{N}_n$ that is minimal in the enumeration of the set of potential defects $\mathcal{D}$. Let

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
$\mathbf{Pr}_{basic}\mathbf{T}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$	Transitive, left linear, reflexive frames	Reflexive trees	Dense reflexive trees

Table 4.5: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{dense}\mathbf{T}$

$\mathcal{N}_{n+1} = (N_{n+1}, \ll_{n+1}, \kappa_{n+1})$ be the network obtained by repairing D — by Lemma 4.5.1 $\mathcal{N}_{n+1}$ exists, is finite and coherent. Define $\mathcal{N}^* = (N^*, \ll^*, \kappa^*)$ by setting:

$$N^* = \bigcup_{0 \le i} N_i \qquad \ll^* = \bigcup_{0 \le i} \ll_i \qquad \kappa^* = \bigcup_{0 \le i} \kappa_i.$$

We claim that $\mathcal{N}^*$ is a d-perfect network. To verify coherency, it is sufficient to note that a violation of any of the conditions in Definition 2.8.2 would imply a violation of the same condition in some $\mathcal{N}_i$, contradicting the coherency of all $\mathcal{N}_i$. Now, suppose that $\mathcal{N}^*$ is not saturated, i.e., it contains a defect, say D. Then there must by some network $\mathcal{N}_i$ with defect D. Indeed, indeed, if D is a future/past defect $(s, \mathbf{F}\psi)/(s, \mathbf{P}\psi)$, it would be present from the step in the construction in which node s is added; if D is a density defect (s, t) it would be present from the step in the construction where last of the two nodes s and t are added. But that would mean that D never got repaired in any $\mathcal{N}_j$, $i < j$, which would imply the existence of infinitely many defects prior to D in the enumeration of $\mathcal{D}$, contradicting its well-foundedness.

□

We are now ready to prove have the following completeness result.

Theorem 4.5.4. $\mathbf{Pr}_{dense}$ *is sound and strongly complete with respect to the class of dense irreflexive trees.*

Proof. Soundness follows from the fact that the axioms are valid on the class of dense trees.

Given a $\mathbf{Pr}_{dense}$-consistent set Γ, extend it to a $\mathbf{Pr}_{dense}$-mcs Γ_0. Let $\mathcal{N}_0$ be the network $(\{s_0\}, \varnothing, (s_0, \Gamma_0))$. Then, trivially, $\mathcal{N}_0$ is a finite, strict coherent network for Γ_0. By Lemma 4.5.3, this can be extended to a strict d-perfect network $\mathcal{N} = (N, \ll, \kappa)$.

Now, since $\mathcal{N}$ is a strict d-perfect network, it follows that the underlying model $\mathcal{M}_{\mathcal{N}}$ is a dense tree, and satisfies Γ at s_0 (By Lemma 2.8.5).

Hence, $\mathbf{Pr}_{dense}$ is sound and strongly complete with respect to all dense trees as required.

□

4.6 Dense Reflexive Trees

Let $\mathbf{Pr}_{basic}\mathbf{T}$ be the logic $\mathbf{Pr}_{basic}$ that contains the reflexivity axiom $\mathbf{T}$. Table 4.5 summarises the axioms together with the frame classes relevant to it.

In this section we show that $\mathbf{Pr}_{basic}\mathbf{T}$ is strongly complete with respect to the class of dense reflexive trees. We do this by building a model for every set of formulas consistent with $\mathbf{Pr}_{basic}\mathbf{T}$ based on a dense reflexive tree.

We have already seen in Section 3.2 that the step-by-step process of building a network translates well to reflexive trees for the future fragment of the Priorian temporal language, and hence can be adapted to the language that includes past operators. For this purpose we will work with $\mathcal{M} = (W, R, V)$, the canonical model for $\mathbf{Pr}_{basic}\mathbf{T}$

and the labelling function κ that relates nodes in the network to mcs's in W. The aim, as in Section 4.5, is to build a d-perfect network for a $\mathbf{Pr}_{basic}\mathbf{T}$-consistent set of formulas, with a reflexive partial ordering $\leqq$.

The repair of $S_{\mathbf{F}}$ and S_d defects in a reflexive network has already been done in Lemma 3.2.16. The repair process of $S_{\mathbf{P}}$ defects is described in the lemma below.

Lemma 4.6.1. *For any $S_{\mathbf{P}}$ defect of a finite, coherent network $\mathcal{N}$ for a $\mathbf{Pr}_{dense}\mathbf{T}$-consistent set of formulas there is a finite, coherent network $\mathcal{N}'$ for the $\mathbf{Pr}_{dense}\mathbf{T}$-consistent set of formulas that extends $\mathcal{N}$ and lacks the defect.*

Proof. Choosing a new node s' to repair an $S_{\mathbf{P}}$ defect is similar to that of Lemma 4.5.1, but to ensure that the relation is reflexive in the extended network, we define it as follows:

Define $\mathcal{N}' = (N, \leqq', \kappa')$ by

$N' = N \cup \{s'\}$

$\leqq' \ = \ \leqq \cup \{(x, s') \mid x \leqq u'\} \cup \{(s', x) \mid u' \leqq x\} \cup \{(s', s')\}$

$\kappa' = \kappa \cup \{(s', \Gamma')\}$

Again as in Lemma 4.5.1, we can show that the extended network does not have the defect any more, and is coherent.

□

We hence also have an equivalent to Lemma 4.5.3.

Lemma 4.6.2. *Any finite, coherent network $\mathcal{N} = (N, \leqq, \kappa)$ for a $\mathbf{Pr}_{dense}\mathbf{T}$-consistent set of formulas can be extended to a d-perfect network.*

With these lemmas we can now build the required network as in Section 4.5, which gives us the following result.

Theorem 4.6.3. $\mathbf{Pr}_{dense}\mathbf{T}$ *is sound and strongly complete with respect to the class of dense reflexive trees.*

Next, we look at unbounded dense trees.

4.7 Unbounded Irreflexive and Reflexive Dense Trees

Starting with $\mathbf{Pr}_{dense}$ in the irreflexive cases, we can add the left and right seriality axioms $\mathbf{U_l} = \mathbf{P}\top$ and $\mathbf{U_r} = \mathbf{F}\top$, to get $\mathbf{Pr}_{dense}\mathbf{U_l}\mathbf{U_r}$ (denoted by $\mathbf{Pr}_{\mathbb{Q}}$), $\mathbf{Pr}_{dense}\mathbf{U_l}$ and $\mathbf{Pr}_{dense}\mathbf{U_r}$. We will also show that $\mathbf{Pr}_{basic}\mathbf{T}$ in the reflexive cases, can be used to build unbounded models for completeness of the respective classes of unbounded, dense, reflexive trees. Table 4.6 summarises the axioms together with the frame classes relevant to it.

To show that these logics are strongly complete with respect to unbounded dense trees, left unbounded dense trees and right unbounded dense trees, we need to confirm that the networks built on these logics have the desired unboundedness properties.

Theorem 4.7.1. *The following logics are sound and strongly complete with respect to the indicated classes of trees:*

- $\mathbf{Pr}_{dense}\mathbf{U_l}$ *to the class of left unbounded dense irreflexive trees.*
- $\mathbf{Pr}_{dense}\mathbf{U_r}$ *to the class of right unbounded dense irreflexive trees.*
- $\mathbf{Pr}_{\mathbb{Q}}$ *to the class of unbounded dense irreflexive trees.*

Logic and Axioms	Class of Kripke Frames	Class of (Ir)reflexive Trees	Standard Frames
$\mathbf{Pr}_{dense}\mathbf{U_l}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{D} : \mathbf{F}p \to \mathbf{FF}p$ $\mathbf{U_l} : \mathbf{P}\top$	Left unbounded, transitive, left linear dense frames	Left unbounded dense irreflexive trees	Left unbounded dense irreflexive trees
$\mathbf{Pr}_{dense}\mathbf{U_r}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{D} : \mathbf{F}p \to \mathbf{FF}p$ $\mathbf{U_r} : \mathbf{F}\top$	Right unbounded, transitive, left linear dense frames	Right unbounded irreflexive dense trees	Right unbounded irreflexive dense trees
$\mathbf{Pr}_{\mathbb{Q}}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{D} : \mathbf{F}p \to \mathbf{FF}p$ $\mathbf{U_l} : \mathbf{F}\top$ $\mathbf{U_r} : \mathbf{F}\top$	Unbounded, transitive, left linear dense frames	Unbounded dense irreflexive trees	Unbounded dense irreflexive trees
$\mathbf{Pr}_{basic}\mathbf{T}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$	Transitive, left linear, reflexive frames	Reflexive trees	Dense reflexive trees

Table 4.6: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{dense}\mathbf{U_l}$, $\mathbf{Pr}_{dense}\mathbf{U_r}$, $\mathbf{Pr}_{\mathbb{Q}}$ and $\mathbf{Pr}_{dense}\mathbf{T}$

Proof. Soundness follows from the fact that the axioms are valid on the respective classes of trees.

We have already seen that we can build a network for a $\mathbf{Pr}_{dense}\mathbf{T}$-consistent set of formulas of which the underlying model $\mathcal{M}$ is a dense irreflexive tree. Using the same method, we see that we can build a left unbounded, dense irreflexive tree for $\mathbf{Pr}_{dense}\mathbf{U_l}$. Let Γ_0 be a $\mathbf{Pr}_{dense}\mathbf{U_l}$ consistent set and let w be an mcs in the canonical model for $\mathbf{Pr}_{dense}\mathbf{U_l}$ containing Γ_0. Starting with $\mathcal{N}_0 = (\{s_0\}, \emptyset, (s_0, \Gamma_0))$ build a network $\mathcal{N} = (N, \ll, \kappa)$ as in the proof of Theorem 4.5.4. To show that this network, and hence also the underlying model, are left unbounded, let $s \in N$ and $\kappa(s) = \Gamma$. Then, as Γ is an mcs in the canonical model for $\mathbf{Pr}_{dense}\mathbf{U_l}$, it has a predecessor Γ'. Now let φ be any formula in Γ'. Then, we have $\mathbf{P}\varphi \in \Gamma$. Hence, after the process of repairing defects, there must be a $t \in N$ such that $t \ll s$, $\kappa(t) = \Lambda$, where Λ is an mcs with $\varphi \in \Lambda$. Hence, s has a predecessor. Hence, $\mathcal{N}$ is left unbounded.

Similarly, starting with $\mathbf{Pr}_{dense}\mathbf{U_r}$, we can build a dense model that is right unbounded.

Lastly, we can also build a dense model for $\mathbf{Pr}_{\mathbb{Q}}$ that is unbounded. □

To show completeness in the reflexive case, we will replace the root (for left unboundedness), and any leaves (for right unboundedness), with a copy of the rational numbers similar to the proof of Theorem 3.2.19.

Theorem 4.7.2. $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and strongly complete with respected to the following classes of trees:*

- *left unbounded dense reflexive trees.*
- *right unbounded dense reflexive trees.*
- *unbounded dense reflexive trees.*

Proof. Soundness follows from the fact that the axioms are valid on the respective classes of trees.

Soundness follows from the fact that the axioms are valid on the respective classes of trees.

We start with right unbounded dense reflexive trees. Let Γ be an $\mathbf{Pr}_{dense}\mathbf{T}$-consistent set. By Theorem 4.6.3 there exists a dense, reflexive tree model $\mathcal{M} = (W, R, V)$ and $w_0 \in W$ such that $\mathcal{M}, w_0 \Vdash \Gamma$. Let $\mathcal{M}' = (W', R', V')$ be the model where every leaf in $\mathcal{M}$ is replaced by a copy of the rational numbers, in the following way: Suppose $\{w_i \mid i \in I\}$, where I is an index set, is the set of leaves in $\mathcal{M}$. For each $i \in I$, let $\mathbb{Q}_i = \{x_i \mid x \in \mathbb{Q}\}$ be an indexed copy of the rational numbers. Then let

- $W' = (W - \{w_i \mid i \in I\}) \cup \bigcup_i \mathbb{Q}_i$
- Define $R' \subseteq W' \times W'$ such that, for all $u, v \in W'$ it is the case that $R'uv$ iff
  - $u, v \in W$ and Ruv or,
  - $u \in W$ and $v \in \mathbb{Q}_i$ for some i and Ruw_i or,
  - $u, v \in \mathbb{Q}_i$ for some i, i.e. $v = x_i$ and $u = y_i$ for some $x, y \in \mathbb{Q}$, and $x \leq y$ in $\mathbb{Q}$.
- $V'(p) = (V(p) - \{w_i \mid i \in I\}) \cup \{\bigcup \mathbb{Q}_i \mid w_i \in V(p)\}$

Define the function $f : W' \to W$ as follows:

- $f(w) = w$ for all $w \in W - \{w_i \mid i \in I\}$
- $f(u) = w_i$ for all $u \in \mathbb{Q}_i$

It is straightforward to check f is a bounded morphism. Therefore $\mathcal{M}', f(w_0) \Vdash \Gamma$ and moreover, $\mathcal{M}'$ is a dense, right unbounded reflexive tree.

For the left unbounded dense reflexive trees, we can do a similar construction by adding replacing the root of $\mathcal{M}$, if it has one, with a copy of the rational numbers as a 'tail'. Therefore, $\mathbf{Pr}_{basic}\mathbf{T}$ is sound and strongly complete with respect to the class left unbounded dense trees.

By combining these two constructions, we can transform $\mathcal{M}$ into and unbounded dense reflexive tree. Hence we obtain that $\mathbf{Pr}_{basic}\mathbf{T}$ is sound and strongly complete with respect to the class unbounded dense reflexive trees.

Hence, the completeness results follow. □

4.8 Trees with Branches Isomorphic to the Rational Numbers

Note that in the repair process of building networks in Section 4.7, there are countably many steps and therefore only countably many nodes are added, hence it will create a model that is also countable. Hence, we have the following theorem.

Theorem 4.8.1. $\mathbf{Pr}_{\mathbb{Q}}$ *is sound and strongly complete with respect to the class of irreflexive trees with branches isomorphic to* $\langle\mathbb{Q}, <\rangle$ *and* $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and strongly complete with respect to the class of reflexive trees with branches isomorphic to* $\langle\mathbb{Q}, \leq\rangle$.

Proof. Soundness follows since the axioms are valid on the respective trees with branches isomorphic to the rational numbers.

To show completeness in the irreflexive case, we recall a theorem proved by Cantor (see e.g. Section 8.4. of [18]) that states that any order which is countable, order dense, and without endpoints is isomorphic to the rational numbers Q with the usual ordering. Hence, $\mathbf{Pr}_{\mathbb{Q}}$ is sound and strongly complete with respect to the class of irreflexive trees with branches isomorphic to $\langle\mathbb{Q}, <\rangle$.

For completeness of the reflexive case, we build an unbounded dense reflexive tree for $\mathbf{Pr}_{dense}\mathbf{T}$ as described in the proof of Theorem 4.7.2. Again, by Cantor (see e.g. Section 8.4. of [18]) it follows that this tree has branches isomorphic to the rational numbers. Hence, $\mathbf{Pr}_{basic}\mathbf{T}$ is sound and strongly complete with respect to the class of reflexive trees with branches isomorphic to $\langle\mathbb{Q}, \leq\rangle$. □

Chapter 5

Complete Axiomatisations of the Priorian Temporal Logics of Discrete Trees

In this chapter, we establish complete axiomatisations of the logics of the classes of discrete trees. We start with the class of discrete irreflexive trees in Section 5.1 where we use the idea of anti-axioms to establish additional rules for the logic. Then, in Section 5.2, we use filtrations, unfolding and bulldozing to build a model for the logic of discrete reflexive trees. In Section 5.3 we find completeness results for the logics of various classes of unbounded discrete trees. In the irreflexive case, we have the seriality axioms to ensure unboundedness in our models, however, these axioms do not necessarily give us an unbounded model in the reflexive case. However, the model we built in Section 5.3, is already unbounded as we bulldozed all the clusters. In Sections 5.4 to 5.9 we use the step-by-step process to build networks for the respective logics. These sections are similar with minor changes to account for the different properties of the respective classes of trees. In Sections 5.10 to 5.13, we use selective filtrations to build the models for the respective classes of trees. The most challenging proof was to show that the model we built for the logic of the class of finite reflexive trees, is indeed finite. The reason for this challenge in the reflexive case, is the axioms used, **Grz** and $\mathbf{Grz_l}$, are not as strong as the axioms used for the irreflexive trees, $\mathbf{L_l}$ and $\mathbf{L_r}$. This will become clear in the respective sections.

5.1 All Discrete Irreflexive Trees

It is well known that irreflexivity is a frame property not definable by modal or temporal axioms. In [28] Gabbay and Reynolds use an additional rule to axiomatise the temporal logic of the class of irreflexive frames. Taking inspiration form that work, we propose the following as an axiomatisation of the logic of the class of discrete irreflexive trees.

- Irreflexivity Rule: If $\vdash \neg p \wedge \mathbf{H}p \rightarrow \varphi$ then $\vdash \varphi$ where $p \notin var(\varphi)$. (IRR)
- Forwards Discrete Rule: If $\vdash ((\mathbf{F}p \wedge \neg\mathbf{FF}p) \vee \mathbf{G}\bot) \rightarrow \varphi$ then $\vdash \varphi$ where $p \notin var(\varphi)$. (FDR)
- Backwards Discrete Rule: If $\vdash ((\mathbf{P}p \wedge \neg\mathbf{PP}p) \vee \mathbf{H}\bot) \rightarrow \varphi$ then $\vdash \varphi$ where $p \notin var(\varphi)$. (BDR)

Let $\mathbf{Pr}_{discr}$ denote the logic $\mathbf{Pr}_{basic}$[1] with the above three rules added.The first rule is to ensure irreflexivity (anti-reflexivity) and the last two are to ensure discreteness (anti-density). Table 5.1 summarises the axioms together with the frame classes relevant to it.

[1]Recall 4.1.1

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{discr}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 IRR, FDR and BDR	Transitive, left linear frames	Irreflexive trees	Discrete irreflexive trees

Table 5.1: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{discr}$

In Section 3.1.1 we saw that the basic modal logic **K4** is sound and strongly complete with respect to the class of irreflexive discrete trees. This was achieved by unravelling. However, unravelling is not an option in the temporal case, as past witnesses will not be accounted for, since this process only unravels forwards. However, bulldozing is a similar process that works in the temporal case. However, when bulldozing a model it is possible to break the discreteness of the model, since the instant of any degenerate cluster that is adjacent to a non-degenerate cluster will then not have an immediate successor/predecessor. Therefore, bulldozing is not an option in the irreflexive discrete case as there might be degenerate clusters in any model. For this reason, it makes sense to build irreflexivity into the model from the start of the process.

Another option in building the right model for the temporal case is to build a network as done in Section 4.5. If we follow this process however, there is no guarantee that we will not create any infinite descending sequences between two instants when repairing S_p defects, and thereby violating forward discreteness. To prevent this from happening, we can either introduce an extra axiom as we will do in Section 5.4, or we can introduce more rules to the temporal logic $\mathbf{Pr}_{basic}$. With the rules we introduced, we will show that we can build a discrete irreflexive tree for $\mathbf{Pr}_{discr}$.

Several authors, including [68], [35], [11] and [27] have introduced rules defining the opposite of a required property as anti-axioms. Furthermore, [26] also showed that adding IRR to any extension of $\mathbf{K_t}$ does not add any extra validities and hence the canonical models will look the same.

First we show that $\mathbf{Pr}_{discr}$ is sound with respect to the class of irreflexive, discrete frames.

Lemma 5.1.1. $\mathbf{Pr}_{discr}$ *is sound with respect to the class of discrete irreflexive trees.*

Proof. We have already shown that $\mathbf{Pr}_{basic}$ is sound on the class of all irreflexive trees (See Section 4.2). We only need to show that the rules are also sound on the class of discrete irreflexive trees.

The fact that the IRR-rule is sound on the class of irreflexive trees was already shown in [26]. We only need to show that the FDR-rule and the BDR-rule are sound on the class of discrete irreflexive trees.

FDR-rule:

Let $\mathcal{F}$ be a discrete irreflexive frame and suppose $\mathcal{F} \Vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \rightarrow \varphi$ where $q \notin var(\varphi)$. Let $\mathcal{M} = (\mathcal{F}, V)$ be any model based on $\mathcal{F}$ and let $w \in W$. Then $(\mathcal{F}, V), w \Vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \rightarrow \varphi$. We consider the following two cases:

Case 1: w has no successor. Then $(\mathcal{F}, V), w \Vdash \mathbf{G}\bot$, which means $(\mathcal{F}, V), w \Vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot)$ and hence $(\mathcal{F}, V), w \Vdash \varphi$.

Case 2: w has an immediate successor, say v (since $\mathcal{F}$ is discrete). Let $V' \sim_q V$ be a valuation with $V'(q) = \{v\}$ (i.e. V' agrees with V except maybe at q). Then, $(\mathcal{F}, V'), v \Vdash q$ and for all $x \neq v$, we have $(\mathcal{F}, V'), x \nVdash q$. Hence, it follows that $(\mathcal{F}, V'), w \Vdash \mathbf{F}q \wedge \neg\mathbf{FF}q$, seeing that ($\mathcal{F}$ is discrete and irreflexive, and therefore, $(\mathcal{F}, V'), w \Vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot)$. Now since $\mathcal{F} \Vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \rightarrow \varphi$, it follows that $(\mathcal{F}, V'), w \Vdash \varphi$. Since $q \notin var(\varphi)$, it follows that $(\mathcal{F}, V), w \Vdash \varphi$. But w and V were arbitrary. Therefore, $\mathcal{F} \Vdash \varphi$.

Hence, the FRD rule preserves validity on $\mathcal{F}$.

A symmetric argument shows that $\mathcal{F} \Vdash ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \rightarrow \varphi$ implies $\mathcal{F} \Vdash \varphi$ for some $r \notin var(\varphi)$. □

We define the canonical model for $\mathbf{Pr}_{discr}$ as the canonical model for $\mathbf{Pr}_{basic}$ with the three extra rules added.

In [68] Venema shows that the canonical model with the IRR rule is not irreflexive. We will now build a submodel of the canonical model of $\mathbf{Pr}_{discr}$ that is a discrete irreflexive tree. Before we can build this submodel, we need the following background.

Firstly, in some syntactical derivations that follow in the proofs below, we will use the following derivation rules and implications. These rules can be shown to be derivable (See Definition 2.5.3) by using propositional tautologies and modus ponens.

Remark 5.1.2.

- $\vdash_{\mathbf{K_t}} (p \wedge q) \to r$ iff $\vdash_{\mathbf{K_t}} p \to (q \to r)$. (∗∗)
- If $\vdash_{\mathbf{K_t}} (p \wedge q) \to \bot$ then $\vdash_{\mathbf{K_t}} p \to \neg q$. (⋆)
- If $\vdash_{\mathbf{K_t}} p \to q$ and $\vdash_{\mathbf{K_t}} p \wedge q \to r$ then $\vdash_{\mathbf{K_t}} p \to r$. (†)
- If $\vdash_{\mathbf{K_t}} p \to q$ then $\vdash_{\mathbf{K_t}} (p \wedge r) \to (q \wedge r)$. (⊛)
- If $\vdash_{\mathbf{K_t}} p \to q$ and $\vdash_{\mathbf{K_t}} q \to r$ then $\vdash_{\mathbf{K_t}} p \to r$. (⊖)

Next, the following rules following rules can be deduced using axioms of temporal logic and deduction:

- If $\vdash_{\mathbf{K_t}} \mathbf{G}(p \to q)$ then $\vdash_{\mathbf{K_t}} \neg(\mathbf{F}p \wedge \neg q)$. (‡)
- If $\vdash_{\mathbf{K_t}} p \to q$ then $\vdash_{\mathbf{K_t}} \mathbf{F}p \to \mathbf{F}q$. (⋇)
- $\vdash_{\mathbf{K_t}} p \to \Box q$ iff $\vdash_{\mathbf{K_t}} \Diamond^{-1} p \to q$ where $\Diamond^{-1}$ is $\mathbf{P}$ if $\Box$ is $\mathbf{G}$, and $\Diamond^{-1}$ is $\mathbf{F}$ if $\Box$ is $\mathbf{H}$. (∗)

Lemma 5.1.3. *If Γ is a $\mathbf{Pr}_{discr}$-consistent set*[2] *then $\Gamma \cup \{\neg p \wedge \mathbf{H}p, (\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot, (\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot\}$ is a $\mathbf{Pr}_{discr}$-consistent set for some $p, q, r \notin var(\Gamma)$.*

Proof. Let Γ be a $\mathbf{Pr}_{discr}$-consistent set and let $p, q, r \notin var(\Gamma)$. The case when $\Gamma \bigcup \{\neg p \wedge \mathbf{H}p\}$ is $\mathbf{Pr}_{discr}$-consistent is already known (In [28] this was given as an exercise). We do the case when the additional formulas are added.

Suppose $\Gamma_0 \cup \{\neg p \wedge \mathbf{H}p, (\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot, (\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot\}$ is not $\mathbf{Pr}_{discr}$-consistent. Then there is a finite subset $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\} \subseteq \Gamma_0$ such that $\vdash \bigwedge \Gamma \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \to \bot$, and hence we have the following derivation:

1. $\vdash \bigwedge \Gamma \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \to \bot$
2. $\vdash (\neg p \wedge \mathbf{H}p) \to (((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \to (((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \to \bot \to (\bigwedge \Gamma \to \bot)))$ (∗∗) in 5.1.2
3. $\vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \to (((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot \to \bot) \to (\bigwedge \Gamma \to \bot))$ (IRR)
4. $\vdash ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot \to \bot) \to (\bigwedge \Gamma \to \bot)$ (FRD)
5. $\vdash \bigwedge \Gamma \to \bot$ (BRD)

But this contradicts the fact that Γ is $\mathbf{Pr}_{discr}$-consistent. Hence, $\Gamma \cup \{\neg p \wedge \mathbf{H}p, (\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot, (\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot\}$ is $\mathbf{Pr}_{discr}$-consistent.

□

Next we define ID-sets (where ID is an abbreviation of irreflexive and discrete) which will help us select the mcs's for the submodel of the canonical model. First, for each formula of the form $\varphi = \Diamond_1(\varphi_1 \wedge \Diamond_2(\varphi_2 \wedge ... \wedge \Diamond_n\varphi_n)\cdots)$, let $\varphi(p, q, r) = \Diamond_1(\varphi_1 \wedge \Diamond_2(\varphi_2 \wedge ... \wedge \Diamond_n(\varphi_n \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot))\cdots)$ for some propositional variables p, q, r.

Definition 5.1.4. A set Γ is called an **ID-set** if the following two conditions hold:

[2] See Definition 2.5.4

- For some distinct p, q, r we have $(\neg p \wedge \mathbf{H}p), ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot), ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \in \Gamma$
- Let $\varphi = \Diamond_1(\varphi_1 \wedge \Diamond_2(\varphi_2 \wedge ... \wedge \Diamond_n\varphi_n)\cdots)$. Then, if $\varphi \in \Gamma$, then for some p, q, r we have $\varphi(p, q, r) \in \Gamma$.

where each $\Diamond_i$ is either **P** or **F**.

Remark 5.1.5. Note that, at any instant w in a model $\mathcal{M}$ for $\mathbf{Pr}_{basic}$, we have that if $\mathcal{M}, w \Vdash \varphi(p, q, r)$ then $\mathcal{M}, w \Vdash \varphi$. Hence, by the completeness of $\mathbf{Pr}_{basic}$ it follows that $\vdash_{\mathbf{Pr}_{basic}} \varphi(p, q, r) \to \varphi$. Now, since $\mathbf{Pr}_{discr}$ is an extension of $\mathbf{Pr}_{basic}$ we also have $\vdash_{\mathbf{Pr}_{discr}} \varphi(p, q, r) \to \varphi$.

The next step is to prove Lindenbaum's Lemma for ID-sets. For this we need the following proposition.

Proposition 5.1.6. *Let $\lambda_1, \lambda_2, \gamma_1, \gamma_2, ..., \gamma_n$ be formulas in the Priorian language.*
If $\vdash_{\mathbf{K_t}} \lambda_1 \to \Box_0(\gamma_1 \to \Box_1(\gamma_2 \to ...\Box_{n-1}(\gamma_n \to \Box_n\neg\lambda_2)))$
then $\vdash_{\mathbf{K_t}} \lambda_2 \to \Box_n^{-1}(\gamma_n \to \Box_{n-1}^{-1}(\gamma_{n-1} \to ...\Box_1^{-1}(\gamma_1 \to \Box_0^{-1}\neg\lambda_1)))$
where $\Box_i^{-1}$ is **H** *if $\Box_i$ is* **G**, *and $\Box_i^{-1}$ is* **G** *if $\Box_i$ is* **H**.

Proof. See Theorem 3.2.4 in [28] for the proof. □

Next, we state Lindenbaum's Lemma for ID-sets.

Lemma 5.1.7. *For any $\mathbf{Pr}_{discr}$-consistent set Γ such that there are infinitely many propositional variables not appearing in any formula in Γ, there exists a maximal consistent ID-set Γ^* such that $\Gamma \subseteq \Gamma^*$.*

Proof. Suppose Γ is a $\mathbf{Pr}_{discr}$-consistent set such that there are infinitely many propositional variables not appearing in any formula in Γ.

Let $\psi_0, \psi_1, ...$ be a list of all formulas in the Priorian language $\mathcal{L}_{Prior}$, where the numeration is done so that all formulas of the form φ in Definition 5.1.4 appear in the odd places. We will construct a sequence of finite $\mathbf{Pr}_{discr}$-consistent sets Γ_n using a recursive definition, such that $\bigcup_n \Gamma_n = \Gamma^*$ is the required maximal consistent ID-set containing φ.

Let $\Gamma_0 = \Gamma \cup \{\neg p \wedge \mathbf{H}p, (\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot, (\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot\}$ for some $p, q, r \notin var(\psi)$ for all $\psi \in \Gamma$. Then Γ_0 is $\mathbf{Pr}_{discr}$-consistent by Lemma 5.1.3.

Next, suppose that Γ_k has already been constructed and is consistent, and consider ψ_k. Then at least one of $\Gamma_k \cup \{\psi_k\}$ or $\Gamma_k \cup \{\neg\psi_k\}$ is consistent. If $\Gamma_k \cup \{\neg\psi_k\}$ is consistent, then add $\neg\psi_k$ to Γ_k to get Γ_{k+1}. If $\Gamma_k \cup \{\neg\psi_k\}$ is not consistent and $\Gamma_k \cup \{\psi_k\}$ is consistent, and k is even, then add ψ_k to Γ_k to get Γ_{k+1}.

If $\Gamma_k \cup \{\neg\psi_k\}$ is not consistent and $\Gamma_k \cup \{\psi_k\}$ is consistent, and k is odd, then ψ_k is of the form φ as in Definition 5.1.4. Then, let $p, q, r \notin var(\Gamma_k)$ and let $\Gamma_{k+1} = \Gamma_k \cup \{\psi_k\} \cup \{\psi_k(p, q, r)\}$. We will show that in this case, Γ_{k+1} is also consistent.

Suppose not, i.e. Γ_k is consistent but $\Gamma_{k+1} = \Gamma_k \cup \{\psi_k\} \cup \{\psi_k(p, q, r)\}$ is not. Then there exists a finite subset $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma_k$ such that $\vdash ((\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m) \wedge \psi_k \wedge \psi_k(p, q, r) \to \bot$. Using the fact that $\psi_k(p, q, r) = (\Diamond_1(\psi_1 \wedge \Diamond_2(\psi_2 \wedge ... \wedge \Diamond_n(\psi_n \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot)))\cdots)$, the following derivation holds:

1. $\vdash ((\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m) \wedge \psi_k \wedge (\Diamond_1(\psi_1 \wedge \Diamond_2(\psi_2 \wedge ... \wedge \Diamond_n(\psi_n \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot)))\cdots) \to \bot$
2. $\vdash ((\gamma_1\wedge\gamma_2\wedge...\wedge\gamma_m)\wedge(\Diamond_1(\psi_1\wedge\Diamond_2(\psi_2\wedge...\wedge\Diamond_n\psi_n\wedge(\neg p\wedge\mathbf{H}p)\wedge((\mathbf{F}q\wedge\neg\mathbf{FF}q)\vee\mathbf{G}\bot)\wedge((\mathbf{P}r\wedge\neg\mathbf{PP}r)\vee\mathbf{H}\bot))\cdots))) \to \bot$ (Remark 5.1.5 and † in 5.1.2)
3. $\vdash (\Diamond_1(\psi_1\wedge\Diamond_2(\psi_2\wedge...\wedge\Diamond_n\psi_n\wedge(\neg p\wedge\mathbf{H}p)\wedge((\mathbf{F}q\wedge\neg\mathbf{FF}q)\vee\mathbf{G}\bot)\wedge((\mathbf{P}r\wedge\neg\mathbf{PP}r)\vee\mathbf{H}\bot))\cdots)) \to \neg(\gamma_1\wedge\gamma_2\wedge...\wedge\gamma_m)$ (⋆ in 5.1.2)
4. $\vdash ((\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot)) \to (\psi_n \to \Box_n^{-1}(\psi_{n-1} \to \Box_{n-1}^{-1}(\psi_{n-2} \to ...\Box_1^{-1}(\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m))\cdots)$ (Proposition 5.1.6, * and ⋇ in 5.1.2)

5. $\vdash (\neg p \wedge \mathbf{H}p) \rightarrow (((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \rightarrow (\psi_n \rightarrow \Box_n^{-1}(\psi_{n-1} \rightarrow \Box_{n-1}^{-1}(\psi_{n-2} \rightarrow ...\Box_1^{-1}(\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m))\cdots))$ (**) in 5.1.2

6. $\vdash (((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot)) \rightarrow (\psi_n \rightarrow \Box_n^{-1}(\psi_{n-1} \rightarrow \Box_{n-1}^{-1}(\psi_{n-2} \rightarrow ...\Box_1^{-1}(\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m))\cdots)$ (IRR)

7. $\vdash ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \rightarrow (((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \rightarrow (\psi_n \rightarrow \Box_n^{-1}(\psi_{n-1} \rightarrow \Box_{n-1}^{-1}(\psi_{n-2} \rightarrow ...\Box_1^{-1}(\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m))\cdots))$ (**) in 5.1.2

8. $\vdash ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \rightarrow (\psi_n \rightarrow \Box_n^{-1}(\psi_{n-1} \rightarrow \Box_{n-1}^{-1}(\psi_{n-2} \rightarrow ...\Box_1^{-1}(\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m))\cdots)$ (FDR)

9. $\vdash (\psi_n \rightarrow \Box_n^{-1}(\psi_{n-1} \rightarrow \Box_{n-1}^{-1}(\psi_{n-2} \rightarrow ...\Box_1^{-1}(\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m))\cdots)$ (BDR)

10. $\vdash (\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m) \rightarrow (\Box_1(\psi_1 \rightarrow \Box_2(\psi_2 \rightarrow ...\Box_n\neg\psi_n)\cdots)$ (Proposition 5.1.6)

11. $\vdash (\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_m) \rightarrow \neg(\Diamond_1(\psi_1 \wedge \Diamond_2(\psi_2 \wedge ...\Diamond_n\psi_n)\cdots)$ (‡ and ⊖ in 5.1.2)

But this contradicts the consistency of Γ_k. Hence, Γ_{k+1} is consistent and by construction we have that Γ^* is the required maximal consistent ID-set. □

We can now define the model we need for the completeness result. Let $\mathcal{M} = (W, R, V)$ be the canonical model for $\mathbf{Pr}_{discr}$. Let $\mathcal{M}' = (W', R', V')$ be the submodel of $\mathcal{M}$ defined as follows:

- W' is the set of all the maximal consistent ID-sets.
- R' is the restriction of R to W'.
- V' is the restriction of V to W'.

Note that, since $\mathcal{M}$ is transitive and left linear, the submodel $\mathcal{M}'$ will also be. Next, we will prove the Existence Lemma for ID sets.

Lemma 5.1.8. *For any $w \in W'$, if $\mathbf{F}\delta \in w$ (alternatively $\mathbf{P}\delta \in w$), then there exists a $v \in W'$ such that $R'wv$ (alternatively $R'vw$) such that $\delta \in v$.*

Proof. Suppose $\mathcal{M}'$ is defined as above and let $\mathbf{F}\delta \in w$ for some $w \in W'$. Then for some p, q, r we have $\mathbf{F}(\delta \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot) \in w$ by the second property of an ID-set. We will build the required maximal consistent ID-set by recursively defining a sequence of sets of which the union will be the required maximal consistent ID-set.

Let $\delta' = \delta \wedge (\neg p \wedge \mathbf{H}p) \wedge ((\mathbf{F}q \wedge \neg\mathbf{FF}q) \vee \mathbf{G}\bot) \wedge ((\mathbf{P}r \wedge \neg\mathbf{PP}r) \vee \mathbf{H}\bot)$ and let $\Gamma_0 = \{\delta'\} \cup \{\gamma \mid \mathbf{G}\gamma \in w\}$. First we show that Γ_0 is consistent. Suppose not, then there is a finite subset $\gamma_1, \gamma_2, ..., \gamma_n \in \{\gamma \mid \mathbf{G}\gamma \in w\}$ such that $\vdash (\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n) \rightarrow \neg\delta'$. Then, by generalisation and the **K** axiom, it follows that $\vdash \mathbf{G}(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n) \rightarrow \mathbf{G}\neg\delta'$. Hence, we have $\vdash (\mathbf{G}\gamma_1 \wedge \mathbf{G}\gamma_2 \wedge ... \wedge \mathbf{G}\gamma_n) \rightarrow \mathbf{G}\neg\delta'$, which means that $\mathbf{G}\neg\delta' \in w$, contradicting the fact that $\mathbf{F}\delta' \in w$. We know that Γ_0 is consistent. Furthermore, Lemma 5.1.7 cannot be used because all variables occur in Γ_0.

Now let $\psi_1, \psi_2, ...$ be a list of all formulas in $\mathcal{L}_{Prior}$, where the numeration is done such that all formulas of the form φ in Definition 5.1.4 appear in the odd places.

We will now define a sequence $\Gamma_0, \Gamma_1, ...$ such that $v = \bigcup_i \Gamma_i$ is the desired maximal consistent ID-set.

Suppose that Γ_k has already been constructed and is consistent, and consider ψ_k. Then at least one of $\Gamma_k \cup \{\psi_k\}$ or $\Gamma_k \cup \{\neg\psi_k\}$ is consistent. If $\Gamma_k \cup \{\neg\psi_k\}$ is consistent, then add $\neg\psi_k$ to Γ_k to get Γ_{k+1}. If $\Gamma_k \cup \{\neg\psi_k\}$ is not consistent and $\Gamma_k \cup \{\psi_k\}$ is consistent, and k is even, then add ψ_k to Γ_k to get Γ_{k+1}.

If $\Gamma_k \cup \{\neg\psi_k\}$ is not consistent and $\Gamma_k \cup \{\psi_k\}$ is consistent, and k is odd, then ψ_k is of the form φ as in Definition 5.1.4.

First note that $\Gamma_k - \Gamma_0$ is a finite set, since only k formulas have been added up to this step, and that $\mathbf{F}(\delta' \wedge \bigwedge(\Gamma_k - \Gamma_0) \wedge \psi_k) \in w$. For suppose not, then $\mathbf{G}(\delta' \wedge \bigwedge(\Gamma_k - \Gamma_0) \rightarrow \neg\psi_k) \in w$. Hence, $(\delta' \wedge \bigwedge(\Gamma_k - \Gamma_0) \rightarrow \neg\psi_k) \in \Gamma_0 \subseteq \Gamma_k$. But $\left\{\delta'_k\right\} \cup \{\bigwedge(\Gamma_k - \Gamma_0)\} \in \Gamma_k$ and therefore $\Gamma_k \vdash \neg\psi_k$, which contradicts the consistency of $\Gamma_k \cup \{\psi_k\}$.

Therefore, since w is an ID-set, there are p', q', r' such that

$\mathbf{F}(\varphi' \wedge \bigwedge(\Gamma_k - \Gamma_0) \wedge \psi_k(p', q', r')) \in w$ (*)

Now let $\Gamma_{k+1} = \Gamma_k \cup \{\psi_k\} \cup \{\psi k(p', q', r')\}$. Then, we show that Γ_{k+1} is consistent. Suppose not, then there is a finite subset $\{\delta', \gamma_1, \gamma_2, ..., \gamma_n\}$ with $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma_0$ and $\gamma_{m+1}, \gamma_{m+2}, ..., \gamma_n \in \Gamma_k - \Gamma_0$ such that $\vdash (\delta' \wedge \gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n \wedge \psi_k \wedge \psi_k(p', q', r')) \rightarrow \bot$.

But $\mathbf{F}\delta, \mathbf{G}\gamma_1, \mathbf{G}\gamma_2, ..., \mathbf{G}\gamma_m \in w$. (by the definition of Γ_0)

Also, note that $\mathbf{F}(\delta' \wedge \gamma_{m+1}, \gamma_{m+2}, ..., \gamma_n \wedge \psi_k \wedge \psi_k(p', q', r')) \in w$. (by * and the fact that w is an ID-set)

Therefore, $\mathbf{F}(\delta' \wedge \gamma_1, \gamma_2, ..., \gamma_n \wedge \psi_k \wedge \psi_k(p', q', r')) \in w$. (since, if $\vdash \Box\alpha \wedge \Diamond\beta$ then $\vdash \Diamond(\alpha \wedge \beta)$)

Hence, $\mathbf{F}\bot \in w$, which means $\bot \in w$, which contradicts the consistency of w.

Hence, $v = \bigcup_i \Gamma_i$ is the desired maximal consistent ID-set.

A symmetrical argument shows the case for $\mathbf{P}\psi \in w$. □

The last result needed is a Truth Lemma for the submodel.

Lemma 5.1.9. *For all φ it is the case that $\mathcal{M}', w \Vdash \varphi$ iff $\varphi \in w$.*

Proof. We prove this by induction on φ. The case when φ is p follows from the definition and the boolean cases follows immediately.

Hence, suppose the statement holds for ψ and φ is $\mathbf{F}\psi$. Assume that $\varphi \in w$. Then by Lemma 5.1.8 there exists a $v \in W'$ with Rwv such that $\psi \in v$. Hence, by the induction hypothesis $\mathcal{M}', v \Vdash \psi$ and hence $\mathcal{M}', w \Vdash \varphi$.

Next, suppose $\mathcal{M}', w \Vdash \varphi$. Then there exists a v with $R'wv$ such that $\mathcal{M}', v \Vdash \psi$. Therefore, we have $\psi \in v$ by the induction hypothesis and hence $\varphi \in w$ by the definition of R'.

A symmetric argument proves the statement for when φ is $\mathbf{P}\psi$ and hence the statement is true for all φ as required. □

Hence, all we need to ensure that the submodel is the required model is to confirm that the rules give us irreflexivity and discreteness of the submodel. [28] showed that the IRR rule ensures that $\mathcal{M}'$ is irreflexive. We show in the following lemma that FDR and BDR give discreteness.

Lemma 5.1.10. *$\mathcal{M}'$ is discrete.*

Proof. Let $\mathcal{M}'$ be as defined and suppose $w \in W'$. Then, for some q we have $\mathbf{F}q \wedge \neg\mathbf{FF}q \in w$. Hence, there is a $v \in W$ with $R'wv$ such that $q \in v$. Now suppose there is a $u \in W'$ such that $wR'uR'v$. Then $\mathbf{F}q \in u$. But then $\mathbf{FF}q \in w$, which gives a contradiction. Hence, v is an immediate successor of w.

A symmetrical argument with $\mathbf{P}r \wedge \neg\mathbf{PP}r \in w$ for some r shows that w has an immediate predecessor. Now, since w was arbitrary, it follows that $\mathcal{M}'$ is discrete, as required. □

Hence, $\mathcal{M}'$ is discrete and irreflexive. Furthermore, it is a tree since it is left linear, transitive and irreflexive as a submodel of the canonical model of $\mathbf{Pr}_{discr}$. Putting all these results together gives us the desired completeness result.

Theorem 5.1.11. $\mathbf{Pr}_{discr}$ *is sound and weakly complete with respect to the class of discrete irreflexive trees.*

Proof. Soundness follows from Lemma 5.1.1 and the canonicity of the remaining axioms.

Suppose γ is a $\mathbf{Pr}_{discr}$-consistent formula with only finitely many propositional variables, and let $\mathcal{M}'$ be as defined. Then, by Lemma 5.1.7 there exists a $w \in W'$ such that $\gamma \in w$. Then, by Lemma 5.1.9 it follows that $\mathcal{M}', w \Vdash \gamma$. Hence, $\mathbf{Pr}_{discr}$ is weakly complete with respect to the class of discrete trees. □

Recall from Definition 2.5.14 that if a logic is strongly complete with respect to a class of frames, then entailment over this class of frames is compact. Conversely, if a logic is not compact, it is not canonical and hence cannot be strongly complete with respect to any class of frames. The next proposition shows this for the class of discrete irreflexive trees.

Proposition 5.1.12. *The logic of the class of discrete irreflexive trees is not compact.*

Proof. Let Λ be the logic for the class of discrete irreflexive trees.

Consider the following set of formulas:

$\Gamma = \{\mathbf{P}p_1\} \cup \{\mathbf{H}(p_i \to \mathbf{P}p_{i+1}) \mid i \in \omega\} \cup \{\mathbf{P}q\} \cup \{\mathbf{H}(q \to \mathbf{H}\neg p_i) \mid i \in \omega\} \cup \left\{\mathbf{H}(p_i \to \bigwedge_{j=1, j\neq i}^{n} \neg p_j) \mid i \in \omega, n \in \omega\right\}$

First we show that Γ is Λ-consistent and then we show that no model based on a frame for the logic can satisfy all the formulas of Γ at a single instant. Hence, the logic is not compact.

Note that a set of formulas Γ is Λ-consistent iff any finite subset Σ is Λ consistent. Let Σ be a finite set such that $\Sigma \subset \Gamma$. Then there is a finite set Ψ such that $\Psi \subset \Gamma$ and $\Sigma \subseteq \Psi \subset \Gamma$ where

$\Psi = \{\mathbf{P}p_1\} \cup \{\mathbf{H}(p_i \to \mathbf{P}p_{i+1}) \mid 1 \leq i \leq n\} \cup \{\mathbf{P}q\} \cup \{\mathbf{H}(q \to \mathbf{H}\neg p_i) \mid 1 \leq i \leq n\} \cup \left\{\mathbf{H}(p_i \to \bigwedge_{j=1, j\neq i}^{m} \neg p_j) \mid 1 \leq i \leq n\right\}$

for some n. Let $\hat{\Psi}$ be the conjunction of all the formulas in Ψ. Next consider the frame $\mathcal{F}$ consisting of $\{-(n+1), ..., -2, -1, 0\}$ with the usual irreflexive ordering of the integers. Note that this frame is discrete. Next let $\mathcal{M}$ be any model based on $\mathcal{F}$ with its valuation defined as follows: For all $1 \leq i \leq n$, $V(p_i) = \{-i\}$, $V(q) = \{-(n+1)\}$.

Next we show that $\mathcal{M}, 0 \Vdash \hat{\Psi}$. Since $V(q) = \{-(n+1)\}$ we have that $\mathcal{M}, 0 \Vdash \mathbf{P}q$ and since $-(n+1)$ has no predecessor we have $\mathcal{M}, 0 \Vdash \mathbf{H}(q \to \mathbf{H}\neg p_i)$ for $1 \leq i \leq n$. Furthermore, since $V(p_1) = \{-1\}$ we have $\mathcal{M}, 0 \Vdash \mathbf{P}p_1$ and since $V(p_i) = \{-i\}$ we have $\mathcal{M}, 0 \Vdash \mathbf{H}(p_i \to \mathbf{P}p_{i+1})$ for $1 \leq i \leq n$. Lastly $\mathcal{M}, 0 \Vdash \mathbf{H}(p_i \to \bigwedge_{j=1, j\neq i}^{m} \neg p_j)$ for $1 \leq i \leq n$ since $V(p_i) = \{-i\}$.

Hence $\hat{\Psi}$ is Λ-consistent, and since Σ was an arbitrary finite subset of Γ, it follows that Γ is Λ-consistent.

Since Λ is not the inconsistent logic, there must be some frame in the class of discrete irreflexive trees. Thus any Λ-consistent set of formulas can be satisfied at some point in a model based on a frame in this class. In particular, there is a model $\mathcal{M}'$ based on this frame such that for some instant w, $\mathcal{M}', w \Vdash \Gamma$. There exists a w_1 with Rw_1w such that $\mathcal{M}', w_1 \Vdash p_1$. Also, for each $j = 2, 3, ...$ there exists a w_j with $Rw_{j+1}w_j$ ($w_{j+1} \neq w_j$) such that $\mathcal{M}', w_j \Vdash p_j$ and $\mathcal{M}', w_{j+1} \Vdash p_{j+1}$. Furthermore, there exists a v such that $\mathcal{M}', v \Vdash q$ and Rvw_j for all j. Hence, there is an infinitely decreasing sequence of instants between w and v. Therefore $\mathcal{M}'$ is not discrete. Therefore, Γ is not satisfiable on a discrete model. Hence, Λ is not compact. □

Note that $\mathcal{M}$ constructed to satisfy $\hat{\Psi}$ in the proof of Proposition 5.1.12 is also finite, locally finite, conversely well-founded and well-founded. Furthermore, by using the usual reflexive ordering on $\mathcal{F}$ in the proof of Proposition 5.1.12, the same consequence will follow for the classes of reflexive trees. Hence, the same argument applies to the logics of the corresponding classes of trees. This is summarised in the proposition below.

Proposition 5.1.13. *The logics of the following classes of trees are not compact.*

- *locally finite irreflexive trees*
- *well-founded irreflexive trees*
- *conversely well-founded irreflexive trees*
- *finite irreflexive trees*
- *discrete reflexive trees*
- *locally finite reflexive trees*

Logic and Axioms	**Class of Kripke Frames**	**Class of Reflexive Trees**	**Standard Frames**
$\mathbf{Pr}_{basic}\mathbf{T}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$	Transitive, left linear, reflexive frames	Reflexive trees	Discrete reflexive trees

Table 5.2: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{basic}\mathbf{T}$

- *well-founded reflexive trees*
- *conversely well-founded reflexive trees*
- *finite reflexive trees*

From these propositions, it follows that none of the logics for the classes of trees in this chapter can be strongly complete with respect to any class of trees, and hence, the best we can hope for is weak completeness of these logics with respect to the respective classes of trees.

5.2 All Discrete Reflexive Trees

In this section we will show $\mathbf{Pr}_{basic}\mathbf{T}$ is a complete axiomatisation for the class of discrete reflexive trees. Table 5.2 summarises the axioms together with the frame classes relevant to it.

In constructing a satisfying model for a $\mathbf{Pr}_{basic}\mathbf{T}$-consistent formula, we will use some well known processes, including filtrations and bulldozing, and also introduce a new process, namely, unfolding. Since these processes will work on any extension of $\mathbf{Pr}_{basic}$, we will keep the exposition as general as possible, formulating results for arbitrary extensions of $\mathbf{Pr}_{basic}$, where possible. Accordingly, let Λ be an extension of $\mathbf{Pr}_{basic}$. We will show that, for each non-theorem of Λ, we can transform the canonical model of Λ into a finite tree on which the non-theorem is refuted, by first using the minimal filtration on a connected submodel of the canonical Λ, and then we will unfold this model into a tree. Then, since $\mathbf{Pr}_{basic}\mathbf{T}$ is an extension of $\mathbf{Pr}_{basic}$ we will conclude that we can transform the canonical model of $\mathbf{Pr}_{basic}\mathbf{T}$ into a finite tree, and then bulldoze the clusters of this tree to get a discrete reflexive tree on which the non-theorem is still refuted.

Let α be any non-theorem of Λ. Let w_0 be an instant in the canonical model for Λ such that $\alpha \notin w_0$ and let $\mathcal{M} = (W, R, V)$ be the submodel of the canonical model generated by w_0. Then $\mathcal{M}$ is a tree of clusters with $\mathcal{M}, w_0 \nVdash \alpha$, since Λ is an extension of $\mathbf{Pr}_{basic}$.

We will now build a finite model for Λ in which α is refuted, using the minimal filtration defined as follows. Start with $\mathcal{M} = (W, R, V)$, the submodel of the canonical model of Λ generated by w_0, and filter W through $Cl(\alpha)$ to get $\mathcal{M}' = (W', R', V')$ where the R' is defined as follows: For all $[w], [v] \in W'$, $R'[w][v]$ iff there exists a $w' \in [w]$ and a $v' \in [v]$ such that $Rw'v'$.

Lemma 5.2.1. *$\mathcal{M}'$ is connected.*

Proof. Suppose not, then there are $[w], [v] \in W$ with $[w] \neq [v]$, and neither $R'[w][v]$ nor $R'[v][w]$, and there is no $[u] \in W$ such that $R'[u][w]$ and $R'[u][v]$. But then, by the definition of R' it follows that neither Ruw nor Ruv for any u. But this contradicts the connectedness of $\mathcal{M}$ (the submodel of the canonical model of Λ generated by w_0 defined above). Hence, $\mathcal{M}'$ is connected. □

Therefore, $\mathcal{M}'$ is connected, and finite, since $Cl(\alpha)$ is finite. And, by Proposition 2.7.3, it follows that $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', [w] \Vdash \varphi$ for $\varphi \in Cl(\alpha)$, and hence $\mathcal{M}', [w_0] \nVdash \alpha$.

However, this filtration may produce, from a transitive and left linear model, one that is neither left linear nor transitive. An example of this can be found in Example 5.2.2 and shown in Figure 5.1 .

As mentioned at the start of this section, the process described here, applies to any extension of $\mathbf{Pr}_{basic}$. And therefore, the frames for these logics need not necessarily be reflexive, as is indeed the case in the example below.

Example 5.2.2. Let $\mathcal{F} = (W, R)$ be a frame with

$W = \{s, t, u, v, w, x, y, z\}$ and

$R = \{(w, u), (w, v), (w, x), (w, y), (w, z), (w, s), (w, t), (u, v), (u, s), (v, u), (u, u), (v, s), (y, t)\}$.

Let $\Phi = \{p, \mathbf{G}p, q, \mathbf{G}q, r\}$ and consider the valuation V define as follows: $V(p) = \{s, t, u, v, x, y, z\}$, $V(q) = \{s, t, z\}$ and $V(r) = \{u, z\}$. Then, $\mathcal{F}' = (W', R')$ where

$W' = \{\{w\}, \{u\}, \{v\}, \{x, y\}, \{z\}, \{s, t\}\}$. Notice that $\{s, t\}$ has two distinct predecessors, namely $\{v\}$ and $\{x, y\}$ which breaks left linearity, and that transitivity is lost because $\neg R'\{z\}\{s, t\}$.

Figure 5.1 shows the two ways that we can lose left linearity and transitivity during the filtration process. We omit arrows that can be inferred from transitivity in the sketch of $\mathcal{F}$.

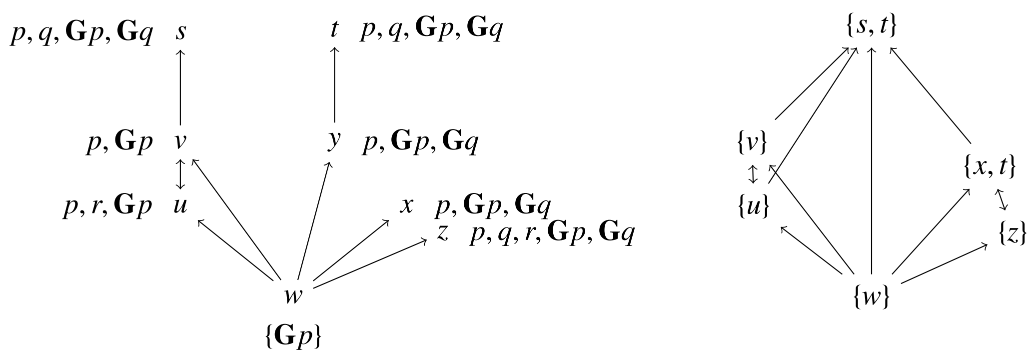


Figure 5.1: On the left is $\mathcal{F}$ and on the right is $\mathcal{F}'$ of Example 5.2.2

Transforming $\mathcal{M}'$ into a left-linear model will require a few more results. However, transitivity can be repaired by taking the transitive closure of R'. Let $\mathcal{M}^r = (W', R^r, V')$ where R^r is the transitive closure of R'. Note that $\mathcal{M}^r$ is connected, transitive and finite.

Recall Definition 2.6.2 for the quotient frame of a transitive model given in Section 2.6. For clarity, please note that the equivalence classes, which are the instants of $\mathcal{M}^r$, will be indicated with square brackets, e.g., $[w], [v]$, while clusters, which are the instants in the quotient frames of the models as defined in Definition 2.6.2, will be indicated using capital letters, e.g. C, D.

Recall the definition of a unique root cluster (see Definition 2.6.2 A quotient frame has a unique root cluster C if for all clusters $D \neq C$, $R_{\sim}CD$ and for no cluster E do we have $R_{\sim}EC$.

Remark 5.2.3. The quotient frame of any transitive ordering does not contain any non-trivial cycles. For suppose there is a cycle of clusters $C_0, C_1, ...C_n, C_0$, then, for all $[w] \in C_i$ and all $[v] \in C_j$ for $i, j = 0, ..., n$, it follows that $R^r[w][v]$ and $R^r[v][w]$ by transitivity of R^r. Hence, all instants will be in the same cluster.

The following lemma, which was stated for the linear case in [60] extends to trees as follows.

Lemma 5.2.4. *$\mathcal{M}^r$ has a single root cluster.*

Proof. Note that $\mathcal{M}^r$ is a transitive. Hence, we may consider the quotient frame $\mathcal{F}^r_{\sim} = (W'_{\sim}, R^r_{\sim})$ of $\mathcal{M}^r$ (see Definition 2.6.2).Note that, since $\mathcal{F}^r_{\sim}$ is finite has no non-trivial cycles (see Remark 5.2.3), there must be a cluster

C with no predecessors in $\mathcal{F}^r_\sim$. Otherwise, we can create an infinitely descending sequence of distinct clusters, contradicting the fact that $\mathcal{F}^r_\sim$ is finite. Hence, there is no cluster D such that $R^r_\sim DC$ and $D \neq C$.

Next, we show that for any cluster $E \neq C$, we have $R^r_\sim CE$. Firstly, we know that it is not the case that $R^r_\sim EC$. Let $[w] \in C$ and $[v] \in E$, then $[w] \neq [v]$. Since $\mathcal{M}^r$ is connected by Lemma 5.2.1, it follows that either $R^r[w][v]$ or $R^r[v][w]$, or there must be a $[u] \in W'$ with $[u] \neq [w]$ and $[u] \neq [v]$, such that $R^r[u][w]$ and $R^r[u][v]$. But, since it is not the case that $R^r_\sim EC$, it is also not the case that $R^r[v][w]$. If $R^r[w][v]$, we are done, so assume there is a $[u] \in W'$ such that $R^r[u][w]$ and $R^r[u][v]$. But $[w] \in C$, therefore $[u]$ cannot be in a cluster G with $G \neq C$ such that $R^r_\sim GC$. Therefore, $[u] \in C$ which means that $R^r[w][u]$ and since $R^r[u][v]$, it follows from transitivity that $R^r[w][v]$. Therefore, it follows that $R^r_\sim CE$. Hence, C^f is a root cluster $\mathcal{F}_\sim$. □

Hence, the filtered model $\mathcal{M}^r$ is a finite, transitive, connected frame with a single root cluster. The next step in building the required model is to remove all violations of left linearity. We will work with clusters of the quotient frame $\mathcal{F}^r_\sim = (W'_\sim, R^r_\sim)$ of the filtered model $\mathcal{M}^r$ to get a refined model that is a tree of clusters. First, we have the following results that will help in the proof of Lemma 5.2.7.

Lemma 5.2.5. *Let $\mathcal{M}'$ be as above and suppose we have a path $[w]R'[w_1]R'...R'[w_n]$ with $\mathcal{M}', [w_n] \Vdash \varphi$. Then $\mathcal{M}', [w] \Vdash \mathbf{F}\varphi$ for all $\mathbf{F}\varphi \in Cl(\alpha)$.*

Proof. Suppose we have a path $[w]R'[w_1]R'...R'[w_n]$ with $\mathcal{M}', [w_n] \Vdash \varphi$. We use induction on n (the length of the path). Suppose $n = 1$. Then $\mathcal{M}', [w_1] \Vdash \varphi$ and $[w]R'[w_1]$. Then there exist $w' \in [w]$ and $w'_1 \in [w_1]$ such that $Rw'w'_1$ and, by the filtration theorem, $\mathcal{M}, w'_1 \Vdash \varphi$. But then $\mathcal{M}, w' \Vdash \mathbf{F}\varphi$, and hence $\mathcal{M}', [w] \Vdash \mathbf{F}\varphi$.

Next, suppose the statement holds for $n = k$ and suppose there is a path of successors $[w]R'[w_1]R'...R'[w_{k+1}]$ with $\mathcal{M}', [w_{k+1}] \Vdash \varphi$. Then $[w_1]R'[w_2]R'...R'[w_{k+1}]$ is a path of length k and hence, by the induction hypothesis, it follows that $\mathcal{M}', [w_1] \Vdash \mathbf{F}\varphi$. But $[w]R'[w_1]$, therefore, there exist $w' \in [w]$ and $w'_1 \in [w_1]$ such that $Rw'w'_1$ and $\mathcal{M}, w'_1 \Vdash \mathbf{F}\varphi$. But then, since R is transitive, $\mathcal{M}, w' \Vdash \mathbf{F}\varphi$, and hence $\mathcal{M}', [w] \Vdash \mathbf{F}\varphi$. Therefore the statement holds for all n. □

The dual of Lemma 5.2.5 is given below and the proof follows symmetrically.

Lemma 5.2.6. *Let $\mathcal{M}'$ be as above and suppose we have a path of predecessors $[w_n]R'[w_{n-1}]R'...R'[w]$, and for all $\mathbf{P}\varphi \in Cl(\alpha)$, if $\mathcal{M}', [w_n] \Vdash \varphi$, then $\mathcal{M}', [w] \Vdash \mathbf{P}\varphi$.*

We are now ready to prove the following result.

Lemma 5.2.7. *For all $\varphi \in Cl(\alpha)$ and for all instants $w \in W$ we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^r, [w] \Vdash \varphi$.*

Proof. Let $\mathcal{M}$ and $\mathcal{M}^r$ be defined as above. Now, since for the minimal filtration $\mathcal{M}'$, we know that, for all $\varphi \in Cl(\alpha)$ and for all instants $w \in W$, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', [w] \Vdash \varphi$. We will use induction on φ to show that $\mathcal{M}', [w] \Vdash \varphi$ iff $\mathcal{M}^r, [w] \Vdash \varphi$.

First, we show that if $\mathcal{M}^r, [w] \Vdash \varphi$, then $\mathcal{M}', [w] \Vdash \varphi$. The base case and boolean cases follow naturally, so suppose the statement holds for ψ and φ is $\mathbf{F}\psi$ for $\mathbf{F}\psi \in Cl(\alpha)$. Now suppose that $\mathcal{M}^r, [w] \Vdash \mathbf{F}\psi$. Then there is a $[w_n]$ such that $[w]R^r[w_n]$ with $\mathcal{M}^r, [w_n] \Vdash \psi$. But, since $[w]R^r[w_n]$, there is a path of R'-successors $[w]R'[w_1]R'...R'[w_n]$. Furthermore, by the induction hypothesis, it follows that $\mathcal{M}', [w_n] \Vdash \psi$. But then, by Lemma 5.2.5, it follows that $\mathcal{M}', [w] \Vdash \mathbf{F}\psi$.

Using a symmetrical argument and Lemma 5.2.6, we can show that if $\mathcal{M}^r, [w] \nVdash \mathbf{P}\psi$, then $\mathcal{M}', [w] \nVdash \mathbf{P}\psi$. Hence, if $\mathcal{M}^r, [w] \Vdash \varphi$, then $\mathcal{M}', [w] \Vdash \varphi$ for all $\varphi \in Cl(\alpha)$.

Next, we show that if $\mathcal{M}', [w] \Vdash \varphi$, then $\mathcal{M}^r, [w] \Vdash \varphi$. The base case and disjunction follow naturally. We need to show that the transitive closure did not add connections to satisfy any formulas that was not satisfied before. Therefore, let φ be $\neg\psi$ for $\neg\psi \in Cl(\alpha)$ and suppose $\mathcal{M}', [w] \Vdash \neg\psi$.

We do induction on ψ. Now if ψ is a propositional letter or boolean case, the result follows naturally. We only need to consider the case when ψ is $\mathbf{P}\delta$ or $\mathbf{F}\delta$, and where the statement holds for δ. We do the case when ψ is $\mathbf{F}\delta$ and the other will follow symmetrically. So suppose $\mathcal{M}', [w] \nVdash \mathbf{F}\delta$, then, for all $[v] \in W'$ with $R[w][v]$, it follows that $\mathcal{M}', [v] \nVdash \delta$. By the induction hypothesis $\mathcal{M}^r, [v] \nVdash \delta$. Now, let $[u] \in W'$ such that $R^r[w][u]$. Then, since R^r is the transitive closure of R', either $R'[w][u]$ or there exists a $[s]$ such that $R'[w][s]$ and $R'[s][u]$. But then $[u]$ is an R'-successor of $[w]$, and hence $\mathcal{M}', [u] \nVdash \delta$. Therefore, $\mathcal{M}^r, [u] \nVdash \delta$. As $[u]$ was arbitrary, it follows that $\mathcal{M}^r, [w] \nVdash \mathbf{F}\delta$.

Lastly, let φ be $\mathbf{F}\psi$ for $\mathbf{F}\psi \in Cl(\alpha)$ and suppose the statement holds for ψ and that $\mathcal{M}', [w] \Vdash \mathbf{F}\psi$. Then, there exists a $[v]$ with $R'[w][v]$ such that $\mathcal{M}', [v] \Vdash \psi$, and by the induction hypothesis, it follows that $\mathcal{M}^r, [v] \Vdash \psi$. But, since $R'[w][v]$, it follows that $R^r[w][v]$, and hence $\mathcal{M}^r, [w] \Vdash \mathbf{F}\psi$. The case when φ be $\mathbf{P}\psi$ follows symmetrically. Therefore, for all $\varphi \in Cl\alpha$, we have that if $\mathcal{M}', [w] \Vdash \varphi$, then $\mathcal{M}^r, [w] \Vdash \varphi$.

Therefore, $\mathcal{M}', [w] \Vdash \varphi$ iff $\mathcal{M}^r, [w] \Vdash \varphi$, and hence, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^r, [w] \Vdash \varphi$.

□

Therefore, $\mathcal{M}^r$ still preserves satisfiability of formulas in $Cl(\alpha)$.

Before we can define the next process, namely unfolding, we need a few more concepts and results. The aim is to use these as tools to ensure that satisfiability of formulas in $Cl(\alpha)$ is preserved in the unfolded model. First, we define what is meant by a **path of immediate successors** from the root cluster C to a cluster in the quotient frame of $\mathcal{M}^r$: Let $\mathcal{F}^r_\sim = (W^r_\sim, R^r_\sim)$ be the quotient frame of $\mathcal{M}^r$ and let C be the root cluster guaranteed by Lemma 5.2.4. Then for any cluster $D \in W^r_\sim$ a path of immediate $R^r_\sim$-successors from C to D is a sequence of clusters $C_0, C_1, C_2, ..., C_n$, with $C = C_0$, such that $D = C_n$ and C_1 is an immediate $R^r_\sim$-successor of C and for each $i = 1, ..., n-1$, we have that C_{i+1} is an immediate $R^r_\sim$-successor of C_i. Let the trivial path be the path where $C_n = C$. We say that the path $\sigma = C, C_1, C_2, ..., C_n$ is of **length** n and the trivial path is of length 0.

The following lemma shows that for all $\mathbf{P}\varphi \in Cl(\alpha)$, if $\mathcal{M}^r, [w] \Vdash \mathbf{P}\varphi$, then every path of immediate successors from the root cluster to the cluster containing $[w]$ contains a witness $[v]$ in a cluster on the path such that $\mathcal{M}^r, [v] \Vdash \varphi$.

Lemma 5.2.8. *Let $\mathcal{M}^r$ be as defined above and recall that the $\mathcal{F}^r_\sim = (W^r_\sim, R^r_\sim)$ is the quotient frame of $\mathcal{M}^r$. For all clusters $C' \in W^r_\sim$ and $[w] \in C'$, if for some $\mathbf{P}\varphi \in Cl(\alpha)$ we have $\mathcal{M}^r, [w] \Vdash \mathbf{P}\varphi$, then for every path of immediate successors σ in $\mathcal{F}^r_\sim$ from C (the root cluster) to C' there is a $[v]$ in some cluster C'' on σ such that $R^r[v][w]$ and $\mathcal{M}^r, [v] \Vdash \varphi$.*

Proof. Consider the generated submodel $\mathcal{M} = (W, R, V)$ of the canonical model for Λ and $\mathcal{M}^r$ as defined above, with $\mathcal{F}^r_\sim = (W^r_\sim, R^r_\sim)$ the quotient frame of $\mathcal{M}^r$, and let $\mathbf{P}\varphi \in Cl(\alpha)$. Let C' be a cluster in $W^r_\sim$ and let σ be any path of immediate successors from the root cluster C to C'.

Suppose $\mathcal{M}^r, [w] \Vdash \mathbf{P}\varphi$ where $[w] \in C'$. Then, there exists a $[v'] \in W^r$ with $R^r[v'][w]$ such that $\mathcal{M}^r, [v'] \Vdash \varphi$. However, this does not guarantee that there will be a $[v]$ on every path from C to C' with $R^r[v][w]$ and $\mathcal{M}^r, [v] \Vdash \varphi$. However, we can use induction on the length of σ to establish this fact.

The case when σ is the trivial path follows easily, since $[v']$ must be in C. Now suppose the statement holds for any path of length $n = k$ and let σ be a path of length $k+1$ where $\mathcal{M}^r, [w] \Vdash \mathbf{P}\varphi$ and $[w] \in C'$.

First note that, for any cluster C and equivalence class $[c]$ with $[c] \in C$ we have that if $\mathcal{M}^r, [c] \Vdash \mathbf{P}\varphi$, then $\mathcal{M}^r, [c'] \Vdash \mathbf{P}\varphi$ for all $[c'] \in C$ (as $[c]$ and $[c']$ lie in the same cluster and hence $R^r[c][c']$). For the induction, we need the following claim.

Claim 1. *If $R^r_\sim C_i C'$, then there exists $[x] \in C_i$ and $[y] \in C'$ such that Rx'.*

Proof. Since $R^r_\sim C_i C'$, there must a $[u_1] \in C_i$ and a $[u_n] \in C'$ such that $R^r[u_1][u_n]$. Therefore, since R^r is the transitive closure of the relation of the minimal filtration R', there is a path of immediate successors $[u_1]R'[u_2]R'...R'[u_n]$. But C_i is an immediate predecessor cluster of C', and therefore, there must be a $[u_m]$ and $[u_{m+1}]$ such that $[u_m] \in C_i$

and $[u_{m+1}] \in C'$. But, since $R'[u_m][u_{m+1}]$, we have without loss of generality that Ru_mu_{m+1}, by the definition of R'. □

Let $[x] \in C_i$ and $[y] \in C'$ be the equivalence classes guaranteed by the claim, i.e. there is a $x' \in [x]$ and a $y' \in [y]$ such that $Rx'y'$. Now, since $\mathcal{M}^r, [w] \Vdash \mathbf{P}\varphi$ and $[y] \in C'$, it follows that $\mathcal{M}^r, [y] \Vdash \mathbf{P}\varphi$, and by definition of the filtration, $\mathcal{M}, y' \Vdash \mathbf{P}\varphi$. Hence, there is an $x'' \in W$ such that $\mathcal{M}, x'' \Vdash \varphi$ with $Rx''y'$. But $Rx'y'$, and hence, by left linearity of $\mathcal{M}$, we have that either $Rx' = x''$, $Rx'x''$ or $Rx''x'$. Now, if $Rx' = x''$ or $Rx'x''$, then $[x'']$ must be in either C_i or in C' and we are done, since $\mathcal{M}^r, [x''] \Vdash \varphi$ and $[x'']$ is on σ. Otherwise, if $Rx''x'$, then $\mathcal{M}, x' \Vdash \mathbf{P}\varphi$ and hence $\mathcal{M}^r, [x'] \Vdash \mathbf{P}\varphi$. But, by the induction hypothesis, and the fact that $\rho = C, C1, ..., C_i$ is a path of length k, it follows that there is an $[s]$ in some cluster C'' on ρ such that $R^r[s][x]$ and $\mathcal{M}^r, [x] \Vdash \varphi$.

□

Our next goal is to repair the violations of left linearity as illustrated in Example 5.2.10 and Figure 5.2. This will be done with a process called unfolding, and using the results above to show that satisfaction of formulas in $Cl(\alpha)$ is preserved. We will build a new model $\mathcal{M}^*$ informed by the structure of quotient frame $\mathcal{F}^r_\sim$ of $\mathcal{M}^r$ as follows:

- Let C be the root cluster of $\mathcal{M}^r$ guaranteed by Corollary 5.2.4 and let Σ be the set of all paths of immediate $R^r_\sim$-successors from C. Since $\mathcal{F}^r_\sim$ is finite and acyclic (see Remark 5.2.3), Σ is finite. Let $W^* = \{(\sigma, [w]) \mid \sigma \in \Sigma \text{ and } [w] \in last(\sigma)\}$.
- Define $R^* \subseteq W^* \times W^*$ such that $R^*(\sigma, [w])(\rho, [v])$ iff σ is a (not necessarily strict) initial segment of ρ and $R^r[w][v]$[3].
- Define $V^* : Prop \to \wp(W^*)$ such that $V^*(p) = \{(\sigma, [w]) \mid [w] \in V'(p)\}$.

Let $\mathcal{M}^* = (W^*, R^*, V^*)$. We call $\mathcal{M}^*$ the **unfolding** of $\mathcal{M}^r$. Intuitively, the unfolding is a forward unravelling on the cluster level, parameterized with equivalence classes in the final cluster of the paths. It is easy to check that $\mathcal{M}^*$ is left linear and transitive (by definition of R^* and the transitivity of R^r), and hence, a tree of clusters:

Lemma 5.2.9. *$\mathcal{M}^*$ is left linear and transitive.*

We illustrate unfolding in the following example.

Example 5.2.10. Consider the refined quotient frame of Example **??**. Clearly $\mathcal{F}^r_\sim$ is not left linear since C_4 has two incomparable predecessors, namely C_2 and C_3. We also see that there are two sequences, namely $\sigma = C_1, C_2, C_4$ and $\rho = C_1, C_3, C_4$, from the root cluster to C_4. Figure 5.2 illustrates the unfolding of this quotient frame resulting in a left linear quotient frame.

Next, we need to show that, satisfaction of formulas in $Cl(\alpha)$ is preserved in $\mathcal{M}^*$ at corresponding equivalence classes.

Lemma 5.2.11. *Let $\mathcal{M}^r = (W', R^r, V')$ and $\mathcal{M}^* = (W^*, R^*, V^*)$ be the models defined above and suppose $\varphi \in Cl(\alpha)$. Then $\mathcal{M}^r, [w] \Vdash \varphi$ iff $\mathcal{M}^*, (\sigma, [w]) \Vdash \varphi$, for all $[w] \in W'$ and for all $\sigma \in \Sigma$ such that $[w] \in last(\sigma)$.*

Proof. Let $\mathcal{M}^r = (W', R^r, V')$ and $\mathcal{M}^* = (W^*, R^*, V^*)$ be as above and suppose $\varphi \in Cl(\alpha)$. We proceed by induction in φ. The base case for propositional variables follow by the definition of V^* and the Boolean cases are straight forward. We consider the modal cases, which we prove contrapositively. Suppose the statement holds for ψ. Now consider the case when φ is $\mathbf{G}\psi$.

[3]The condition that $R^r[w][v]$ makes it possible to perform this process on a non-reflexive model, and for the resulting model after unfolding to preserve non-reflexivity by preserving the degenerate clusters.

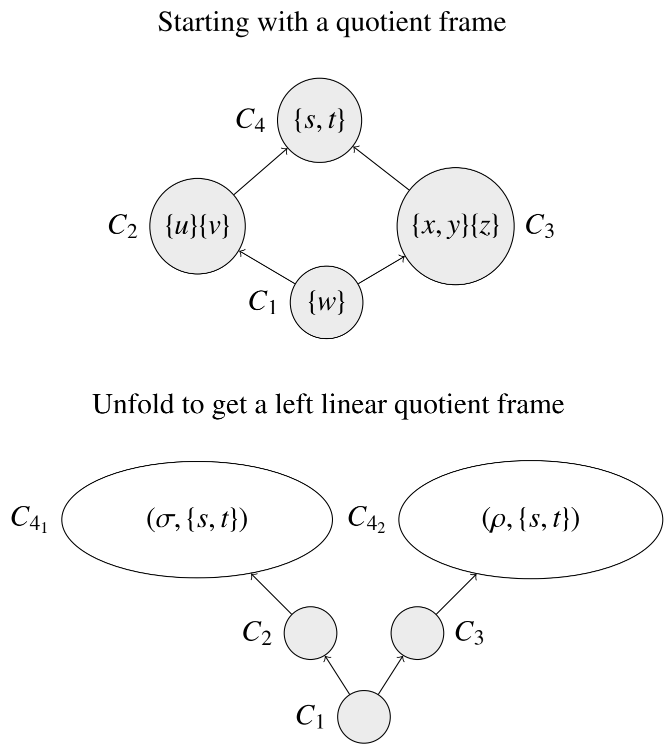


Figure 5.2: Unfolding process from Example 5.2.10

$\Rightarrow$: Suppose $\mathcal{M}^r, [w] \nVdash \mathbf{G}\psi$. Then there exists an equivalence class $[v] \in W'$ with $R^r[w][v]$ such that $\mathcal{M}^r, [v] \nVdash \psi$. Since $R^r[w][v]$, there is at least one (not necessarily proper) extension ρ of σ such that $[v] \in last(\rho)$ and $R^*(\sigma, [w])(\rho, [v])$ and, by the induction hypothesis, $\mathcal{M}^*, (\rho, [v]) \nVdash \psi$. Hence, $\mathcal{M}^*, (\sigma, [w]) \nVdash \mathbf{G}\psi$.

$\Leftarrow$: Suppose $\mathcal{M}^*, (\sigma, [w]) \nVdash \mathbf{F}\psi$. Then there exists a $(\rho, [v]) \in W^*$ such that $R^*(\sigma, [w])(\rho, [v])$ and $\mathcal{M}^*, (\rho, [v]) \nVdash \psi$. Furthermore, since $R^*(\sigma, [w])(\rho, [v])$ we have $R^r[w][v]$. By the induction hypothesis we have $\mathcal{M}^r, [v] \nVdash \psi$ and therefore $\mathcal{M}^r, [w] \Vdash \mathbf{F}\varphi$.

Next, consider the case when φ is $\mathbf{H}\psi$.

$\Rightarrow$: Suppose $\mathcal{M}^r, [w] \nVdash \mathbf{H}\psi$. Then, by Lemma 5.2.8 there exists $[v] \in W'$ such that $R^r[v][w]$ and an initial segment ρ of σ such that $[v] \in last(\rho)$ and $\mathcal{M}^r, [v] \nVdash \psi$. But then $R^*(\rho, [v])(\sigma, [w])$ and, by the induction hypothesis, $\mathcal{M}^*, (\rho, [v]) \nVdash \psi$. Therefore, $\mathcal{M}^*, (\sigma, [w]) \nVdash \mathbf{H}\psi$.

$\Leftarrow$: Suppose $\mathcal{M}^*, (\sigma, [w]) \nVdash \mathbf{H}\psi$. So there is a $(\rho, [v]) \in W^*$ such that $R^*(\rho, [v])(\sigma, [w])$ and $\mathcal{M}^*, [(\rho, [v]) \nVdash \psi$. From the definition of R^* it follows that $R^r[v][w]$ and, by the induction hypothesis, $\mathcal{M}^r, [v] \nVdash \psi$. Therefore $\mathcal{M}^r, [w] \nVdash \mathbf{P}\varphi$. □

We observe that $\mathcal{M}^*$ is finite. To see this, note that: (a) the domain of $\mathcal{M}^r$ is finite, being the same of as that of the filtration $\mathcal{M}'$; (b) the quotient frame of $\mathcal{M}^r$ contains no non-trivial cycles (see Remark 5.2.3), so there are only finitely many paths from its root, i.e. Σ is finite. This result will be used in Chapter 6.

Corollary 5.2.12. *$\mathcal{M}^*$ is finite.*

Hence, $\mathcal{M}^*$ is a finite tree of clusters and $\mathcal{M}^*, (\sigma, [w_0]) \nVdash \alpha$ for some path σ. Through this process, we have shown that we can begin with the canonical model of Λ and build a finite tree of clusters, which is a model for Λ, and on which the non-theorem α is refuted.

Now, since Λ is any extension of $\mathbf{Pr}_{basic}$, this process applies in particular to $\mathbf{Pr}_{basic}\mathbf{T}$, and the resulting model after using minimal filtration and unfolding, will be a finite reflexive (since reflexivity is preserved by filtrations) tree of non-degenerate clusters.

Let α be any non-theorem of $\mathbf{Pr}_{basic}\mathbf{T}$ and let $Cl(\alpha)$ be the set of subformulas of α closed under single negations. Let w_0 be an instant in the canonical model for $\mathbf{Pr}_{basic}\mathbf{T}$ such that $\alpha \notin w_0$ and let $\mathcal{M} = (W, R, V)$ be the submodel of the canonical model generated by w_0. Then $\mathcal{M}$ is a reflexive tree of clusters with $\mathcal{M}, w_0 \nVdash \alpha$. Let $\mathcal{M}^*$ be the resulting model after taking the transitive closure of the minimal filtration of $\mathcal{M}$ through $Cl(\alpha)$ and unfolding. Then $\mathcal{M}^*$ is a finite tree of non-degenerate clusters with $\mathcal{M}^*, (\sigma, [w_0]) \nVdash \alpha$.

Returning to our aim of building a discrete reflexive tree for $\mathbf{Pr}_{basic}\mathbf{T}$, the last step in this construction is to bulldoze all clusters. We will describe the method for reflexive bulldozing for $\mathcal{M}^p = (W^p, R^p, V^p)$, where R^p is an arbitrary reflexive left linear pre-order. Notice that, since $\mathcal{M}^p$ is reflexive, there will be no degenerate clusters. Recall that $\mathcal{F}_\sim = (W^p_\sim, R^p_\sim)$ is the quotient frame of $\mathcal{M}^p$. The process of bulldozing we will describe creates a new model $\mathcal{M}^b$ where all clusters are trivial and such that $\mathcal{M}^p$ is a bounded morphic image of $\mathcal{M}^b$.

Let $S \in W^p_\sim$ be a cluster. Since every set can be linearly ordered (See e.g., [39]), we can define a transitive and reflexive ordering $\leq_S$ on the elements of S, for each cluster $S \in W^p_\sim$. In this process of "straightening out" S, some relation links have been lost, since the linear ordering chooses only one of $v \leq_S w$ or $w \leq_S v$ for every $v, w \in S$ with $w \neq v$. To replace these lost links up to bisimulation, let $S_i = \{w_i \mid w \in S\}$ for all $i \in \mathbb{Z}$ and let $S_\mathbb{Z} = \bigcup_{i \in \mathbb{Z}} S_i$, and replace each S with $S_\mathbb{Z}$. Define a linear ordering $\leq_{S_i}$ for every copy S_i as follows: $w_i \leq_{S_i} v_i$ iff $w \leq_S v$ for all $v_i, w_i \in S_i$ and $i \in \mathbb{Z}$.

Next extend the $\leq_{S_i}$'s to $S_\mathbb{Z}$ to get $\leq_{S_\mathbb{Z}}$ as follows: $\leq_{S_\mathbb{Z}} = \bigcup_{i \in \mathbb{Z}} \leq_{S_i} \cup \{(w_i, w_j) \mid w_i \in S_i, w_j \in S_j, i \leq j\}$.

Then $\leq_{S_\mathbb{Z}}$ is a reflexive linear ordering on $S_\mathbb{Z}$. See Figure 5.3 for a visual representation of this construction using a cluster of four instants for simplicity.

We say that $w \in S$ **corresponds** to $w_i \in S_\mathbb{Z}$. (See Figure 5.3 where the dashed lines shows the correspondence). This induces correspondence classes $[[w]]$ where w is the original instant in S, i.e. $[[w]] = \{w_i \mid i \in \mathbb{Z}\}$.

Now let $\mathcal{M}^b = (W^b, R^b, V^b)$ where:

- $W^b = \bigcup_{S \in W^p_\sim} S_\mathbb{Z}$ where $S_\mathbb{Z}$ are the sets constructed from the clusters S described above.
- $R^b = (\bigcup_{S \in W_\sim} \leq_{S_\mathbb{Z}}) \cup \{(w_i, w_j) \mid w_i \in [[w]], w_j \in [[v]], R^p wv\}$
- Let $w_i \in V^b(p)$ iff $w \in V^p(p)$ where $w_i \in [[w]]$.

Note that the reflexive bulldozed model is reflexive. Also, connectedness and left linearity are preserved in the construction of R^b. Hence, $\mathcal{M}^b$ is connected, reflexive, and left linear. We only need to confirm that R^b is transitive. We can do this by splitting the possible options into the following cases:

1. Suppose $R^b v_i w_j$ and $R^b w_j s_k$ where $v_i, w_j, s_k \in S_\mathbb{Z}$ for $S \in W_\sim$. Then, $v_i \leq_{S_\mathbb{Z}} w_j$ and $w_j \leq_{S_\mathbb{Z}} s_k$, and hence, by the transitivity of $\leq_{S_\mathbb{Z}}$, it follows that $v_i \leq_{S_\mathbb{Z}} s_k$. But then $R^b v_i s_k$ by definition of R^b.
2. Suppose $R^b v_i w_j$ and $R^b w_j s_k$ where v_i, w_j and s_k do not all originate from the bulldozing of the same cluster. Assume that $v_i, w_j \in S_\mathbb{Z}$ and , $s_k \in T_\mathbb{Z}$ where $R_\sim ST$. Then, $v, w \in S$ and hence $R^p vw$ as they are in the same cluster. Also, since $R^p_\sim ST$, it follows that $R^p ws$ by definition of $R^p_\sim$. Therefore, by the transitivity of R^p it follows that $R^p vs$. Therefore, $R^b v_i s_k$ by definition of R^b. The other cases will follow similarly.

Therefore, the bulldozing $\mathcal{M}^b$ of $\mathcal{M}^p$ is a tree. Hence, we have shown that any reflexive left linear pre-order $\mathcal{M}^p$ can be bulldozed into a tree $\mathcal{M}^b$. We only have to show that corresponding instants in $\mathcal{M}^b$ and $\mathcal{M}^p$ satisfy the same formulas.

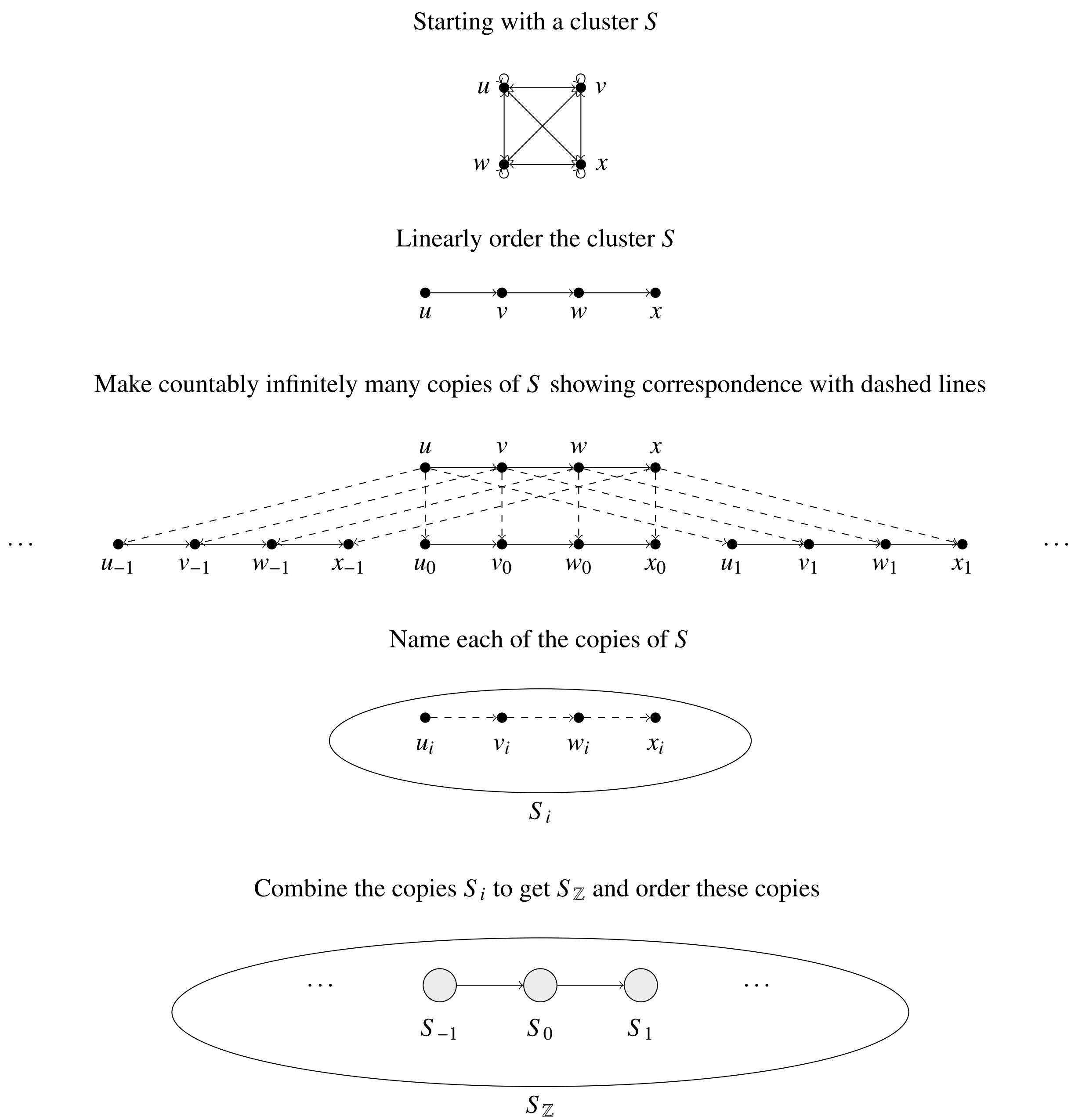


Figure 5.3: Constructing $S_{\mathbb{Z}}$

Lemma 5.2.13. *Let $\mathcal{M}^p$ and $\mathcal{M}^b$ be the models defined above and let $f : W^b \to W^p$ be the function defined by $f : w_i \mapsto w$ for all $w_i \in W^b$ with $w_i \in [[w]]$. Then f is a surjective bounded morphism.*

Proof. We will show that the conditions of a bounded morphism are met:

- Surjectivity:

  Let $w \in W^p$. Then w is in some cluster S where $S \in W^p_{\sim}$. Then, for each instant $w_i \in S_{\mathbb{Z}}$ such that $w_i \in [[w]]$ it is the case that $f(w_i) = w$.

- Local harmony:

  Let $w_i \in S_{\mathbb{Z}}$. Then $w_i \in V^b(p)$ iff $w \in V^p(p)$ where $w_i \in [[w]]$, by definition of V^b. But $w_i \in [[w]]$ implies that $f(w_i) = w$. Hence, $w_i \in V^b(p)$ iff $(w_i) \in V^p(p)$

- Back and forth properties:

  For the forth property, consider the different cases below:

  – Suppose $R^b v_i w_j$ with $v_i, w_j \in S_{\mathbb{Z}}$ and $v_i \in [[v]]$ and $w_j \in [[w]]$. But then $f(v_i) = v$ and $f(w_j) = w$, and since v and w are in the same cluster of W^p, it follows that $R^p vw$.

  – Suppose $R^b v_i w_j$ where $v_i \in S_{\mathbb{Z}}$ and $w_j \in T_{\mathbb{Z}}$ with $T \neq S$, $v_i \in [[v]]$ and $w_j \in [[w]]$. Then $f(v_i) = v$ and $f(w_j) = w$ and $R^p vw$, by definition of R^b.

  For the back property, consider the different cases below:

  – Suppose $R^p f(v_i) w$ with $v_i \in S_{\mathbb{Z}}$ and let $f(v_i) = v$. Then we need to find w_j such that $f(w_j) = w$ and $R^b v_i w_j$. Consider a w_j for some $j > i$. Then $R^b v_i w_j$ and $f(w_j) = w$.

  – Suppose $R^p v f(w_j)$ with $w_j \in S_{\mathbb{Z}}$ and let $f(w_j) = w$. Then we need to find v_i such that $f(v_i) = v$ and $R^b v_i w_j$. Consider a v_i for some $i < j$. Then $R^b v_i w_j$ and $f(v_i) = v$.

Hence, f is a bounded morphism.

□

Let $\mathcal{M}^b_*$ be the bulldozed model of $\mathcal{M}^*$. Then, since $\mathcal{M}^*$ is a left linear reflexive pre-order, the result of Lemma 5.2.13 applies and it follows that all formulas at corresponding instants in $Cl(\alpha)$ that are satisfied in $\mathcal{M}^*$, will also be satisfied in $\mathcal{M}^b_*$. Furthermore, we used the set of integers to bulldozed the clusters, and hence, $\mathcal{M}^b_*$ is a discrete reflexive tree, and in particular, $\mathcal{M}^b_*, (\sigma_i, [w_0]) \nVdash \alpha$ for some $i \in \mathbb{Z}$.

From the results above the completeness theorem follows:

Theorem 5.2.14. $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and weakly complete with respect to the class of discrete reflexive trees.*

Proof. Soundness follows from the fact that the axioms are valid on a discrete reflexive tree. Weak completeness follows from the argument above. □

Logic and Axioms	Class of Kripke Frames	Class of (Ir)reflexive Trees	Standard Frames
$\mathbf{Pr}_{discr}\mathbf{U_l}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 IRR, FDR and BDR $\mathbf{U_l} : \mathbf{P}\top$	Transitive, left linear irreflexive frames	Left unbounded discrete irreflexive trees	Left unbounded discrete irreflexive trees
$\mathbf{Pr}_{discr}\mathbf{U_r}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 IRR, FDR and BDR $\mathbf{U_r} : \mathbf{F}\top$	Transitive, left linear irreflexive frames	Right unbounded discrete irreflexive trees	Right unbounded discrete irreflexive trees
$\mathbf{Pr}_{discr}\mathbf{U_l}\mathbf{U_r}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 IRR, FDR and BDR $\mathbf{U_l} : \mathbf{P}\top$ $\mathbf{U_r} : \mathbf{F}\top$	Transitive, left linear irreflexive frames	Unbounded discrete irreflexive trees	Unbounded discrete irreflexive trees
$\mathbf{Pr}_{basic}\mathbf{T}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$	Transitive, left linear, reflexive frames	Reflexive trees	Unbounded discrete trees

Table 5.3: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{discr}\mathbf{U_l}$, $\mathbf{Pr}_{discr}\mathbf{U_r}$, $\mathbf{Pr}_{discr}\mathbf{U_l}\mathbf{U_r}$ and $\mathbf{Pr}_{basic}\mathbf{T}$

5.3 Unbounded Irreflexive and Reflexive Discrete Trees

Consider the following logics: $\mathbf{Pr}_{discr}\mathbf{U_l}\mathbf{U_r}$, $\mathbf{Pr}_{discr}\mathbf{U_l}$ and $\mathbf{Pr}_{discr}\mathbf{U_r}$, where $\mathbf{U_l}$ and $\mathbf{U_r}$ are the seriality axioms $\mathbf{P}\top$ and $\mathbf{F}\top$. Table 5.3 summarises the axioms together with the frame classes relevant to it.

In this section we obtain weak completeness of $\mathbf{Pr}_{discr}\mathbf{U_l}\mathbf{U_r}$ with respect to the class of unbounded discrete irreflexive trees, $\mathbf{Pr}_{discr}\mathbf{U_l}$ with respect to the class of left unbounded discrete irreflexive trees, and $\mathbf{Pr}_{basic}\mathbf{U_r}$ with respect to the class of right unbounded discrete irreflexive trees.

It is easy to see that we can build submodels of the canonical models for $\mathbf{Pr}_{discr}\mathbf{U_l}\mathbf{U_r}$, $\mathbf{Pr}_{discr}\mathbf{U_l}$ and $\mathbf{Pr}_{discr}\mathbf{U_r}$, similar to the process followed in Section 5.1, and that the resulting models will be an unbounded discrete irreflexive tree, a left unbounded discrete irreflexive tree and right unbounded discrete irreflexive tree, respectively. Hence, we have the following results.

Theorem 5.3.1. *The following logics are sound and strongly complete with respect to the indicated classes of trees:*

- $\mathbf{Pr}_{discr}\mathbf{U_l}$ *with respect to the class of left unbounded discrete irreflexive trees.*
- $\mathbf{Pr}_{discr}\mathbf{U_r}$ *with respect to the class of right unbounded discrete irreflexive trees.*
- $\mathbf{Pr}^{r}_{udisc}$ *with respect to the class of unbounded discrete irreflexive trees.*

Note that the bulldozed model is also unbounded as the bulldozing process turns a non-degenerate cluster into an unbounded linear ordering. Therefore, the reflexive tree built in Section 5.2 is unbounded. Hence, we have the following corollary from Theorem 5.2.14.

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{lfin}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{S}$: $\mathbf{P}q \wedge \mathbf{H}(q \rightarrow \mathbf{P}q) \rightarrow \mathbf{HP}q$	Transitive, left linear frames with no infinitely descending sequences of not necessarily distinct instants with a strict lower bound	Irreflexive trees with no infinitely descending sequences of distinct instants with a strict lower bound	Locally finite irreflexive trees

Table 5.4: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{lfin}$

Corollary 5.3.2. $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and weakly complete with respect to the class of left unbounded discrete reflexive trees, right unbounded discrete reflexive trees, and unbounded discrete reflexive trees.*

5.4 Locally Finite Irreflexive Trees

Consider the following formula that we will add as axioms to $\mathbf{Pr}_{basic}$ to get a logic that is sound and complete with respect to the class of locally finite irreflexive trees:

- $\mathbf{S} : \mathbf{P}q \wedge \mathbf{H}(q \rightarrow \mathbf{P}q) \rightarrow \mathbf{HP}q$[4]

Let $\mathbf{Pr}_{lfin}$ be the logic $\mathbf{Pr}_{basic}\mathbf{S}$. Table 5.4 summarises the axioms together with the frame classes relevant to it.

We will show that $\mathbf{Pr}_{lfin}$ is a complete axiomatisation for the class of locally finite irreflexive trees by building a locally finite model for this logic. We will do this by by building a network for a $\mathbf{Pr}_{lfin}$-consistent formula that is locally finite, i.e., a network that has no infinitely ascending or descending sequences between two nodes.

First, the following definition will be useful.

Definition 5.4.1. Given a $\mathbf{Pr}_{basic}$-frame $\mathcal{F} = (W, R)$, an instant $t \in W$ is a **strict lower bound** for a set of nodes $X \subseteq W$ if Rtx and not Rxt for every $x \in X$.

We begin by showing in Lemma 5.4.2which class of Kripke frames validates all the axioms of $\mathbf{Pr}_{lfin}$. This will also ensure soundness of the logic $\mathbf{Pr}_{lfin}$ with respect to irreflexive frames with no infinitely descending sequences between two instants.

Lemma 5.4.2. $\mathbf{S}$ *is valid in any irreflexive frame* $\mathcal{F}$ *for* $\mathbf{Pr}_{basic}$ *iff there are no infinitely descending sequences with a strict lower bound in* $\mathcal{F}$.

Proof. $\Rightarrow$: Let $\mathcal{F} = (W, R)$ be a frame for $\mathbf{Pr}_{basic}$. We will use contraposition. Suppose $\mathcal{F}$ has an infinitely descending sequence of instants $...Ru_2Ru_1Rv$ with a strict lower bound t, i.e., Rtu_i and not Ru_it, for all $i \in \omega$. Let V be a valuation with $V(p) = \{u_i \mid i \in \omega\}$. Then $(\mathcal{F}, V), v \Vdash \mathbf{P}p \wedge \mathbf{H}(p \rightarrow \mathbf{P}p)$, since for every u_i with $(\mathcal{F}, V), u_i \Vdash p$ it follows that $(\mathcal{F}, V), u_{i+1} \Vdash p$. However, $(\mathcal{F}, V), t \Vdash \neg\mathbf{P}p$ since we do not have Ru_it for any $i \in \omega$. Therefore $(\mathcal{F}, V), v \nVdash \mathbf{HP}p$, and hence, $(\mathcal{F}, V), v \nVdash \mathbf{S}$. Hence, $\mathbf{S}$ is not valid on $(\mathcal{F}, V)$.

$\Leftarrow$: We will prove the contrapositive again. Let $\mathcal{F} = (W, R)$ be a frame for $\mathbf{Pr}_{basic}$ on which $\mathbf{S}$ is not valid. Then, there is a valuation V and instant w such that $(\mathcal{F}, V), w \nVdash \mathbf{S}$. Therefore, $(\mathcal{F}, V), w \Vdash \mathbf{P}q \wedge \mathbf{H}(q \rightarrow \mathbf{P}q)$ but $(\mathcal{F}, V), w \nVdash \mathbf{HP}q$. Then, there is a v with Rvw such that $(\mathcal{F}, V), v \Vdash \mathbf{H}\neg q$. Notice that Rvu for all u with Ruw such that $(\mathcal{F}, V), u \Vdash q$. Furthermore, there is an instant u_0 with Ru_0w and Rvu_0 such that $(\mathcal{F}, V), u_0 \Vdash q$. But for all u with Ruw we have $(\mathcal{F}, V), u \Vdash q \rightarrow \mathbf{P}q$. We now use this to show that we can build an infinitely descending

[4]This axiom was used in [60], [38] and [37], and was suggested for this section by Valentin Goranko.

sequence of instants between w and v starting with u_0, and that v is a strict lower bound for this sequence. Since $(\mathcal{F}, V), u_0 \Vdash q$ and Ru_0w, it follows that $(\mathcal{F}, V), u_0 \Vdash q \to \mathbf{P}q$. Hence, there must be a u_1 with Ru_1u_0 such that $(\mathcal{F}, V), u_1 \Vdash q$, and hence $(\mathcal{F}, V), u_1 \Vdash q \to \mathbf{P}q$. Again, there must be a u_2 with Ru_2u_1 such that $(\mathcal{F}, V), u_2 \Vdash q$, and hence $(\mathcal{F}, V), u_2 \Vdash q \to \mathbf{P}q$. We can continue applying these formulas to create an infinitely descending sequence of instants $..., u_1, u_0$ with Rvu_i for all i (Note that irreflexivity gives $u_i \neq u_j$ for all i, j.). Lastly, v must be a strict lower bound for this infinitely descending sequence of predecessors u_i of w, since R is Rvu_i for each i. □

We focus the rest of the section on proving weak completeness of $\mathbf{Pr}_{lfin}$ with respect to locally finite trees. For this purpose we will build a network for a $\mathbf{Pr}_{lfin}$-consistent formula, similar to the one in Section 4.5 and make sure that, in the process of repairing defects, we do not create an infinitely ascending or descending sequence of nodes between any two nodes.

We will argue that the repair process of $S_{\mathbf{F}}$ defects (as defined in Section 4.5), cannot create any infinitely ascending or descending sequences between nodes, purely by construction. However, when repairing $S_{\mathbf{P}}$ defects (as defined in Section 4.5), all new nodes are inserted in the linear past and it is possible that we could create an infinite descending sequence between two nodes in this way. Lemmas 5.4.3 and 5.4.4 will help ensure that such an infinite sequence will not arise in the construction of the model.

We will use mcs's and the relation of the canonical model for $\mathbf{Pr}_{lfin}$ to build a network for a $\mathbf{Pr}_{lfin}$-consistent formula, such that the underlying model will be a locally finite tree. Note that all claims in the rest of this subsection refer to networks for a $\mathbf{Pr}_{lfin}$-consistent formula.

Recall the definitions of a strict coherent and saturated network, as well as a strict perfect network, from Section 2.8. We will modify the process of removing saturation defects from Section 4.5 to ensure that the process cannot create an infinitely descending sequence between two nodes. Preventing infinitely ascending sequences will be a consequence of the way we repair defects, as will be discussed later in this section. We will repair $S_{\mathbf{P}}$ defects in a particular way as stipulated below.

First we distinguish between two different types of $S_{\mathbf{P}}$ defects:

- $S_{\mathbf{P}_1}$ **defects**: The pair $(s, \mathbf{P}\psi)$ is an $S_{\mathbf{P}_1}$ defect if it is an $S_{\mathbf{P}}$ defect and there exists a t with $t \ll s$ such that $\mathbf{H}\neg\psi \in \kappa(t)$. We define such a t as a **block** for the $(s, \mathbf{P}\psi)$ defect, and note that no node can be added before t to repair the defect.

- $S_{\mathbf{P}_2}$ **defects**: The pair $(s, \mathbf{P}\psi)$ is an $S_{\mathbf{P}_2}$ defect if it is an $S_{\mathbf{P}}$ defect and there does not exists a t with $t \ll s$ such that $\mathbf{H}\neg\psi \in \kappa(t)$.

The repair process for these defects is given in the proofs of Lemmas 5.4.3 and 5.4.4.

Lemma 5.4.3. *For any $S_{\mathbf{P}_2}$ defect of a finite strict coherent network $\mathcal{N}$ for a $\mathbf{Pr}_{lfin}$-consistent formula there is a finite strict coherent network $\mathcal{N}'$ for the $\mathbf{Pr}_{lfin}$-consistent formula that extends $\mathcal{N}$ and lacks that defect.*

Proof. Let $\mathcal{N} = (N, \ll, \kappa)$ be a finite, strict coherent network for a $\mathbf{Pr}_{lfin}$-consistent formula, and let R be the canonical relation for $\mathbf{Pr}_{lfin}$. Suppose $(s, \mathbf{P}\psi)$ is a $S_{\mathbf{P}_2}$ defect, i.e., $s \in N$ with $\mathbf{P}\psi \in \kappa(s)$ but there is no $t \in N$ such that $t \ll s$ and $\psi \in \kappa(t)$, and for all t with $t \ll s$ we have that $\mathbf{H}\neg\psi \notin \kappa(t)$. Let t be such that $\mathbf{H}\neg\psi \notin \kappa(t)$. Then, since $\kappa(t)$ is an mcs, we must have $\mathbf{P}\psi \in \kappa(t)$. Hence, for all t in N such that $t \ll s$, we have that $(t, \mathbf{P}\psi)$ is an $S_{\mathbf{P}}$ defect. Also, since N is finite, irreflexive and left linear, it must have a root, say r. Now, insert a new node s' before r and let Γ' be an mcs containing ψ in $\mathcal{M}$ such that $\Gamma' R \kappa(r)$ (guaranteed by the Existence Lemma for normal logics 2.5.10, since $\mathbf{P}\psi \in \kappa(r)$).

Define $\mathcal{N}' = (N', \ll', \kappa')$ by
$N' = \{s'\} \cup N$
$\ll' = \ll \cup \{(s', x) \mid x \in N\}$

$\kappa' = \kappa \cup \{(s', \Gamma')\}$

Next we prove that that $\mathcal{N}'$ is still a strict coherent network after repairing the defect in these ways.

Note that $\mathcal{N}'$ does not contain the $S_{\mathbf{P}}$ defect $(s, \mathbf{P}\psi)$. It is still a connected strict left linear partial order. We just need to show that the fourth property of coherence holds.

Let $x, y \in N'$ such that $x \ll' y$. We just need to check the case when $x = s'$, since s' is the root of N' and we therefore cannot have $x \ll' s'$ for any $x \in N'$. Let $x = s'$. If $y = r$ we are done by construction. Otherwise, we have $\Gamma' R \kappa'(r)$ by the choice of Γ' and by the consistency of N we have $\kappa'(r) R \kappa'(y)$ (since r was the root of N). Hence, by transitivity, we have $\Gamma' R \kappa'(y)$, as required. □

Note that repairing $S_{\mathbf{P}_2}$ defects might lead to a left unbounded network, but this is not a concern for local finiteness.

Next, in Lemma 5.4.4, we show that the same $S_{\mathbf{P}_1}$ defect does not have to be repaired more than once on the same branch. However, it might happen that at each adding of a node in the repairing process, the label of the new node will create new defects involving the same formula, and so on, which could create an infinite descending sequence between two nodes. To ensure that that is not the case, we will have to work within a finite set of formulas and filter the canonical model through that set.

Let α be any non-theorem of $\mathbf{Pr}_{lfin}$. Let w_0 be an instant such that $\mathcal{M}, w_0 \nVdash \alpha$ where $\mathcal{M}$ is the canonical model for $\mathbf{Pr}_{lfin}$. Now, let $\mathcal{M}'$ be the submodel of $\mathcal{M}$ generated by w_0.

Note that the subset of formulas of the form $\mathbf{P}\psi$ in $Cl(\alpha)$ is finite. Let $\{\mathbf{P}\psi_1, \mathbf{P}\psi_2, ..., \mathbf{P}\psi_m\}$ be the set of all formulas in $Cl(\alpha)$ of the form $\mathbf{P}\psi$ and consider the set P of all $S_{\mathbf{P}_1}$ defects of the form $(u, \mathbf{P}\psi_i)$ for some $u \in N$. Define a relation on P for a network $\mathcal{N}$ as follows: $(u, \mathbf{P}\psi_i) \sim_{\mathcal{N}} (u', \mathbf{P}\psi_j)$ iff $i = j$ and one of $u \ll u'$, $u' \ll u$ or $u = u'$. Let $[(u, \mathbf{P}\psi_i)]_{\mathcal{N}}$ represent the equivalence classes for each i and each u.

We show that $\sim_{\mathcal{N}}$ is an equivalence relation:

Clearly it is reflexive and symmetric by definition. For transitivity, suppose $(u, \mathbf{P}\psi) \sim_{\mathcal{N}} (u', \mathbf{P}\psi)$ and $(u', \mathbf{P}\psi) \sim_{\mathcal{N}} (u'', \mathbf{P}\psi)$. Then one of $u \ll u'$, $u' \ll u$ or $u = u'$ and one of $u' \ll u''$, $u'' \ll u'$ or $u' = u''$ holds, and hence by the transitivity and left linearity of $\ll$ we have $(u, \mathbf{P}\psi) \sim_{\mathcal{N}} (u'', \mathbf{P}\psi)$.

Lemma 5.4.4. *For any $S_{\mathbf{P}_1}$-defect $(s, \mathbf{P}\psi)$ of a finite, strict coherent network $\mathcal{N}$ for a $\mathbf{Pr}_{lfin}$-consistent formula there is a finite strict coherent network $\mathcal{N}'$ for the $\mathbf{Pr}_{lfin}$-consistent formula that extends $\mathcal{N}$ and lacks all defects in the same equivalence class $[(s, \mathbf{P}\psi)]_{\mathcal{N}}$. Furthermore, if the defect was repaired by adding the node s', then $(s', \mathbf{P}\psi)$ is not a defect, either.*

Proof. [5] Let $\mathcal{N} = (N, \ll, \kappa)$ be a finite, strict coherent network for a $\mathbf{Pr}_{lfin}$-consistent formula and let R be the canonical relation for $\mathbf{Pr}_{lfin}$. Let $(s, \mathbf{P}\psi)$ be an $S_{\mathbf{P}_1}$ defect. We can assume that s is the $\ll$-lowest node on its branch that has that defect. Thus, there is a node t such that s is an immediate $\ll$-successor of t in $\mathcal{N}$ and t is the "blocking" state for this defect, i.e. $\neg\psi \in \kappa(t)$ and $\mathbf{H}\neg\psi \in \kappa(t)$. We are going to insert a new node u between t and s, i.e. $t \ll u \ll s$, such that $\Gamma = \kappa(u)$ satisfies the following:

1. $\Gamma R \kappa(s)$,
2. $\psi \in \Gamma$,
3. $\mathbf{H}\neg\psi \in \Gamma$.

If we can show that such Γ exists, then we are done as there will be no nodes on the same branch as s that has a defect in the same equivalence class as $(s, \mathbf{P}\psi)$. Indeed, since $\psi \in \Gamma$, it cannot be that $\Gamma R \kappa(t)$ or $\Gamma = \kappa(t)$, so, by the left linearity, it would follow that $\kappa(t) R \Gamma$, so the coherence would be preserved after adding u between t and s.

[5]The proof was provided in V Goranko, personal communication, 2023.

To prove the existence of such Γ, it suffices to show that the set $\mathbf{H}(\kappa(s)) \cup \{\psi \wedge \mathbf{H}\neg\psi\}$ is $\mathbf{Pr}_{lfin}$-consistent, where $\mathbf{H}(\kappa(s)) = \{\varphi \mid \mathbf{H}\varphi \in \kappa(s)\}$.

Suppose not. We can assume that $\mathbf{H}(\kappa(s))$ is closed under conjunctions (up to equivalence). That means that there is $\varphi_s \in \mathbf{H}(\kappa(s))$ such that $\varphi_s \wedge \psi \wedge \mathbf{H}\neg\psi$ is $\mathbf{Pr}_{lfin}$-inconsistent.

Then $\mathbf{Pr}_{lfin} \vdash \varphi_s \to \neg(\psi \wedge \mathbf{H}\neg\psi)$.

Therefore, $\mathbf{Pr}_{lfin} \vdash \mathbf{H}(\varphi_s \to \neg(\psi \wedge \mathbf{H}\neg\psi))$,

hence $\mathbf{Pr}_{lfin} \vdash \mathbf{H}\varphi_s \to \mathbf{H}\neg(\psi \wedge \mathbf{H}\neg\psi))$.

Since $\mathbf{H}\varphi_s \in \kappa(s)$, it follows that $\mathbf{H}\neg(\psi \wedge \mathbf{H}\neg\psi) \in \kappa(s)$, i.e. $\mathbf{H}(\psi \to \mathbf{P}\psi) \in \kappa(s)$.

Since also $\mathbf{P}\psi \in \kappa(s)$, by the axiom **S** we obtain that $\mathbf{HP}\psi \in \kappa(s)$. Since $t \ll s$, by coherency, it follows that $\mathbf{P}\psi \in \kappa(t)$, which contradicts the assumption that $\mathbf{H}\neg\psi \in \kappa(t)$ and the consistency of $\kappa(t)$.

□

Now that we have the repair process for defects, we show, in the next lemma that we can build a strict coherent network that has no more defects.

Lemma 5.4.5. *Let α be a non-theorem for $\mathbf{Pr}_{lfin}$. Then, there is a strict coherent network for $\neg\alpha$ that has no defects involving any formulas in $Cl(\alpha)$.*

Proof. Let α be a formula that is not a theorem of $\mathbf{Pr}_{lfin}$. We build a locally finite network for $Cl(\alpha)$, i.e. we only repair saturation defects for formulas in $Cl(\alpha)$. This is different to the networks we have build for dense trees, as we were not limited to a finite set of formulas in those cases.

Let $S = \{s_i \mid i \in \omega\}$ be a countable set of nodes disjoint from N, i.e., a supply of new nodes to be used in repairing, step by step, the defects of $\mathcal{N}$. Next, let

$$\mathcal{D} = [(N \cup S) \times \{\mathbf{F}\varphi \mid \varphi \in \mathcal{L}\}] \cup [(N \cup S) \times \{\mathbf{P}\varphi \mid \varphi \in \mathcal{L}\}],$$

i.e., $\mathcal{D}$ is the set of all possible defects of $\mathcal{N}$ and of any network extending it with nodes from S. Note that $\mathcal{D}$ is countable and let $D_0, D_1, D_2, \ldots$ be an arbitrary but fixed enumeration of its elements.

We will construct a sequence of finite, strict coherent extensions of $\mathcal{N}$, namely $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots$ as follows: Since α is not a theorem of $\mathbf{Pr}_{lfin}$, there exists an mcs w that contains $\neg\alpha$. Let $\mathcal{N}_0 = (\{s_0\}, \emptyset, (s_0, w))$.

Next, suppose $\mathcal{N}_n$ is a finite, strict coherent network for $n \geq 0$. If no more defects exist, let

$$N = \bigcup_{i=0}^{n} N_i \qquad \ll = \bigcup_{i=0}^{n} \ll_i \qquad \kappa = \bigcup_{i=0}^{n} \kappa_i \tag{5.1}$$

Otherwise, let D be the defect of $\mathcal{N}_n$ that is minimal in the enumeration $\mathcal{D}$. Next, form $\mathcal{N}_{n+1}$ by repairing the defect D. In this process, an extension of $\mathcal{N}_n$ is created without the defect, and any new extension will not have the defect D either. Define $\mathcal{N} = (N, \ll, \kappa)$ as follows:

$$N = \bigcup_{n\in\omega} N_n \qquad \ll = \bigcup_{n\in\omega} \ll_n \qquad \kappa = \bigcup_{n\in\omega} \kappa_n \tag{5.2}$$

By construction, $\mathcal{N}$ is strict coherent with no more defects.

□

In the next lemma we show that $\mathcal{N}$ is locally finite.

Lemma 5.4.6. *Let $\mathcal{N}$ be the resulting network for a $\mathbf{Pr}_{lfin}$-consistent formula after repairing all $S_{\mathbf{P}_1}$, $S_{\mathbf{P}_2}$ and $S_{\mathbf{F}}$ defects built in the proof of Lemma 5.4.5. Then $\mathcal{N}$ is locally finite, i.e. for any nodes s, t of this network such that $s \ll t$, there are at most finitely many nodes between s and t.*

Proof. We first prove that repairing each type of defect cannot create an infinitely descending sequence between two nodes.

In Lemma 5.4.3 it was shown that when repairing $S_{\mathbf{P}_2}$ defects, a new root is added to the network each time. Hence, even though the new node might still have a defect involving the same formula and will have to be repaired again, the process will not create an infinite, decreasing sequence between two nodes, as no nodes are added that have any predecessors.

The construction of an infinitely descending sequence between two distinct nodes in the repair process of $S_{\mathbf{P}_1}$ defects requires the repair of infinitely many past defects and there are only finitely many **P**-formulas to consider. Therefore, we will necessarily have a formula involved in infinitely many of these defects. This can be done in two ways. First, we have to repair infinitely many $S_{\mathbf{P}_1}$ defects $(s, \mathbf{P}\psi)$ containing the same formula ψ. Secondly, we have to repair infinitely many $S_{\mathbf{P}_1}$ defects $(s, \mathbf{P}\psi_1), (s, \mathbf{P}\psi_2), ...$ where $\psi_i \neq \psi_j$ for all $i \neq j$. Since we started with a finite set of formulas $Cl(\alpha)$, the second option cannot happen. Also, in the proof of Lemma 5.4.4 we saw that the first option cannot happen either. Therefore, such an infinitely descending sequence cannot be created in the limit of the repair process.

Suppose t was added to repair a $S_{\mathbf{F}}$ defect in a finite, strict coherent network, and suppose that the past defects $(s_i, \mathbf{P}\varphi_i)$, $i = 1, 2, ..., n$, have already been repaired in the linear past of t. Then there exist $s'_1, s'_2, ..., s'_n$ with $s'_i \ll s_i$ for $i = 1, 2, ..., n$. Furthermore, $s_i \ll t$ for $i = 1, 2, ..., n$ and hence, by transitivity $s'_i \ll t$ for $i = 1, 2, ..., n$. Hence $(t, \mathbf{P}\varphi_i)$ cannot be a defect for $i = 1, 2, ..., n$.

To ensure local finiteness, we also need to prevent infinitely ascending sequences resulting from the repair process. The only way that we can create such a sequence, is by repairing $S_{\mathbf{F}}$ defects. However, when repairing $S_{\mathbf{F}}$ defects as in Lemma 3.1.18, we add a new node on a new branch, which means that we cannot create any infinitely ascending sequences between nodes in this way. Hence, only finitely many nodes are added between any two nodes in the repair process of defects. □

Theorem 5.4.7. $\mathbf{Pr}_{lfin}$ *is sound and weakly complete with respect to the class of locally finite irreflexive trees.*

Proof. In Lemma 5.4.2 it was shown that **S** is valid on the class of frames with no infinitely descending sequences between two instants with a strict lower bound, and hence also locally finite irreflexive frames. This, and the fact that all other axioms are valid in all irreflexive trees, give soundness of the logic.

For completeness, let α be a formula that is not a theorem of $\mathbf{Pr}_{lfin}$. Let $\mathcal{N}$ be the network built in the proof of Lemma 5.4.5. We showed in Lemma 5.4.6 that $\mathcal{N}$ is locally finite. Furthermore, since $\mathcal{N}$ is a strict perfect network, it follows from Lemma 2.8.5 and the fact that $\alpha \in Cl(\alpha)$, that for the underlying model $\mathcal{M}_{\mathcal{N}}$, we have $\mathcal{M}, s_0 \not\Vdash \alpha$.

Hence, $\mathbf{Pr}_{lfin}$ is sound and weakly complete with respect to the class of locally finite trees, as required. □

By Lemma 5.4.2 we also have the following.

Theorem 5.4.8. $\mathbf{Pr}_{lfin}$ *is sound and weakly complete with respect to the class of irreflexive trees with no infinitely descending sequences between two nodes.*

Notice that the network we built above also does not have any infinitely ascending sequences between two nodes by construction and that this construction is independent of axiom **S**. Therefore, without axiom **S** we can build a network as above, that has no infinitely ascending sequences between two nodes. This leads to the following consequence.

Theorem 5.4.9. $\mathbf{Pr}_{basic}$ *is sound and weakly complete with respect to the class of irreflexive trees with no infinitely ascending sequences between two nodes.*

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
$\mathbf{Pr}^r_{lfin}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \to \mathbf{F}p$ $\mathbf{Q} : (\mathbf{F}p \wedge \mathbf{H}\neg p) \to \mathbf{F}(p \wedge \mathbf{H}(p \vee \mathbf{H}\neg p))$	Transitive, left linear, reflexive simple successor cluster frames	Reflexive trees with no infinitely descending sequences with a strict lower bound	Locally finite reflexive trees

Table 5.5: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}^r_{lfin}$

The question of whether it is possible to get strong completeness for these logics with respect to the given classes was answered in Proposition 5.1.13, where we showed that the logic for the class of locally finite irreflexive trees is not compact, and hence we have the following result.

Corollary 5.4.10. *There is no class of frames with respect to which* $\mathbf{Pr}_{lfin}$ *is sound and strongly complete.*

5.5 Locally Finite Reflexive Trees

In this section we follow the same method to build a model as we did in Section 5.4 for the locally finite irreflexive trees, but with an axiom suitable for reflexive frames.

Let $\mathbf{Q}$ be the following axiom :

- $(\mathbf{F}p \wedge \mathbf{H}\neg p) \to \mathbf{F}(p \wedge \mathbf{H}(p \vee \mathbf{H}\neg p))$

Note that the negation for this axiom is equivalent to $\mathbf{H}\neg p \wedge \mathbf{F}p \wedge \mathbf{G}(p \to \mathbf{P}(\neg p \wedge \mathbf{P}p))$. This will be useful in the proof of Lemma 5.5.1 below.

Let $\mathbf{Pr}^r_{lfin}$ denote the logic $\mathbf{Pr}_{basic}\mathbf{TQ}$. Table 5.5 summarises the axioms together with the frame classes relevant to it.

We will show that $\mathbf{Pr}^r_{lfin}$ is weakly complete with respect to the class of locally finite reflexive trees. We do this by building a model for every formulas consistent with $\mathbf{Pr}^r_{lfin}$ based on a locally finite reflexive tree, by building a locally finite reflexive network for a $\mathbf{Pr}_{lfin}$-consistent formula that is a tree. This network will have no infinitely descending or ascending sequences of distinct nodes.

We will call frames where all non-root clusters are simple (see Definition 2.6.2), and any infinitely descending sequences of distinct instants between two distinct instants are contained within the root cluster, **simple successor cluster frames**.

Lemma 5.5.1. *The formula* $\mathbf{Q}$ *is valid on a frame* $\mathcal{F}$ *for* $\mathbf{Pr}_{basic}\mathbf{T}$ *iff* $\mathcal{F}$ *is a simple successor cluster frame.*

Proof. First we show that if $\mathcal{F} = (W, R)$ is a simple successor cluster frame, then $\mathbf{Q}$ is valid on $\mathcal{F}$. Using contraposition, let $\mathcal{F} = (W, R)$ be a frame for $\mathbf{Pr}_{basic}\mathbf{T}$ on which $\mathbf{Q}$ is not valid, i.e. there is a valuation V and instant w such that $(\mathcal{F}, V), w \Vdash \mathbf{H}\neg p \wedge \mathbf{F}p \wedge \mathbf{G}(p \to \mathbf{P}(\neg p \wedge \mathbf{P}p))$. Then $(\mathcal{F}, V), w \Vdash \mathbf{H}\neg p$ and there exists a v with Rwv such that $(\mathcal{F}, V), v \Vdash p$, and $(\mathcal{F}, V), u \Vdash p \to \mathbf{P}(\neg p \wedge \mathbf{P}p)$ for all u with Rwu. But then $(\mathcal{F}, V), v \Vdash \mathbf{P}(\neg p \wedge \mathbf{P}p)$. Hence, there must be a v_1 with Rv_1v and $(\mathcal{F}, V), v_1 \Vdash \neg p \wedge \mathbf{P}p$. Notice that $v \neq v_1$ since $(\mathcal{F}, V), v \Vdash p$ and $(\mathcal{F}, V), v_1 \Vdash \neg p \wedge \mathbf{P}p$. But then there exists a v_2 with wRv_2Rv_1 with $(\mathcal{F}, V), v_2 \Vdash p$, and hence we can repeat this process to find an infinitely descending sequence ..., v_2Rv_1Rv between w and v, and not equal to w (Since $(\mathcal{F}, V), w \Vdash \mathbf{H}\neg p$). Now, if there is repetition of instants in this sequence, then those instants form a non-simple cluster which is not the root (since w comes strictly before). Otherwise, there is an infinitely descending sequence of distinct instants bounded below by w. Hence, $\mathcal{F}$ is not a simple successor cluster frame.

Next we show that if $\mathbf{Q}$ is valid on a $\mathbf{Pr}_{basic}\mathbf{T}$-frame $\mathcal{F} = (W, R)$ then $\mathcal{F}$ is a simple successor cluster frame. Again using contraposition, suppose $\mathcal{F}$ is not a simple successor cluster frame. Then, we either have a non-root cluster that is not simple, or we have an infinitely descending sequence of distinct instants between two distinct instants, that are not contained in a root cluster. We consider the two cases:

Case 1: $\mathcal{F} = (W, R)$ has an infinitely descending sequence of distinct instants between two distinct instants, that is not contained in a root cluster. Let w, v be distinct instants such that Rwv, and let $w...Rv_2Rv_1Rv$, all v_i distinct, be an infinitely descending sequence between w and v. We show that $\mathbf{Q}$ is not valid on this frame by defining a valuation V as follows: $V(p) = \{v_i \mid i \text{ is odd.}\}$. Then we have $(\mathcal{F}, V), w \Vdash \mathbf{H}\neg p$ and $(\mathcal{F}, V), w \Vdash \mathbf{F}p$, and $(\mathcal{F}, V), v \Vdash \mathbf{P}p$. Furthermore, for each even v_i we have $(\mathcal{F}, V), v_i \Vdash \neg p \wedge \mathbf{P}p$, and for each odd v_i we have $(\mathcal{F}, V), v_i \Vdash p$, i.e. the instants in the infinitely descending sequence come in pairs where $(\mathcal{F}, V), v_i \Vdash \neg p \wedge \mathbf{P}p$, and $(\mathcal{F}, V), v_{i+1} \Vdash p$. Therefore, since Rwv, we have $(\mathcal{F}, V), w \Vdash \mathbf{G}(p \to \mathbf{P}(\neg p \wedge \mathbf{P}p))$. Hence, $(\mathcal{F}, V), w \Vdash \neg\mathbf{Q}$, as required.

Case 2: $\mathcal{F} = (W, R)$ has a non-simple, non-root cluster. Let C be a non-root, non-simple cluster of $\mathcal{F}$. Then there are at least two instants w, v in C, and there is a u in a root cluster with Ruw and Ruv. Define a valuation as follows: $V(p) = \{v\}$. Then $(\mathcal{F}, V), u \Vdash \mathbf{H}\neg p$ (since it is not the case that Rvu), and $(\mathcal{F}, V), u \Vdash \mathbf{F}p$ (since Ruv and $(\mathcal{F}, V), v \Vdash p$). Also, since Rvw, it is the case that $(\mathcal{F}, V), v \Vdash p$ and $(\mathcal{F}, V), w \Vdash \neg p \wedge \mathbf{P}p$. Therefore, we have $(\mathcal{F}, V), u \Vdash \mathbf{G}(p \to \mathbf{P}(\neg p \wedge \mathbf{P}p))$. Hence, $(\mathcal{F}, V), u \Vdash \neg\mathbf{Q}$, as required.

□

Therefore, in particular, the logic $\mathbf{Pr}^r_{lfin}$ is sound on locally finite trees of clusters. We focus the rest of the section on getting weak completeness of this logic with respect to the class of locally finite reflexive trees. For this purpose we will build a network similar to the one in Section 5.4, and again make sure that in the process of repairing defects, we do not create an infinitely ascending or descending sequence of nodes between any two nodes.

The following lemma will help ensure that infinitely descending sequences will not arise in the construction of the model.

Lemma 5.5.2. *Let $\mathcal{M} = (W, R, V)$ be a model for $\mathbf{Pr}^r_{lfin}$ and suppose there are two instants $w, v \in W$ such that Rwv and $w \neq v$, with $\mathcal{M}, v \Vdash \mathbf{P}\varphi$ and $\mathcal{M}, w \Vdash \mathbf{H}\neg\varphi$. Then there is an instant x with Rwx and Rxv such that $\mathcal{M}, x \Vdash \varphi$ and for all x' with Rwx' and $Rx'x$ we have $\mathcal{M}, x' \nVdash \neg\varphi \wedge \mathbf{P}\varphi$.*

Proof. Suppose $\mathcal{M} = (W, R, V)$ is a model for $\mathbf{Pr}^r_{lfin}$ and $\mathcal{F} = (W, R)$ is the frame (i.e., $\mathcal{F}$ is the disjoint union of trees of clusters), and let $w, v \in W$ such that Rwv and $w \neq v$, with $\mathcal{M}, v \Vdash \mathbf{P}\varphi$ and $\mathcal{M}, w \Vdash \mathbf{H}\neg\varphi$. Then $\mathcal{M}, w \Vdash \mathbf{FP}\varphi$ since $\mathcal{M}, v \Vdash \mathbf{P}\varphi$. But $\mathcal{M}, w \Vdash \mathbf{H}\neg\varphi$. Therefore, there must be an instant x with Rwx and Rxv (by left linearity) and $\mathcal{M}, x \Vdash \varphi$. Hence, $\mathcal{M}, w \Vdash \mathbf{F}\varphi \wedge \mathbf{H}\neg\varphi$.

Now suppose that for all x with Rwx and Rxv such that $\mathcal{M}, x \Vdash \varphi$ there exists a x' with Rwx' and $Rx'x$, with $\mathcal{M}, x' \Vdash \neg\varphi \wedge \mathbf{P}\varphi$. But then $\mathcal{M}, w \Vdash \mathbf{G}(\varphi \to \mathbf{P}(\neg\varphi \wedge \mathbf{P}\varphi))$ and hence $\mathcal{M}, w \nVdash \mathbf{Q}$, which gives a contradiction and hence the result follows. □

Starting with $\mathbf{Pr}^r_{lfin}$ and a non-theorem of this logic, and working with mcs's and the canonical relation for this logic, we will build a network for this $\mathbf{Pr}_{lfin}$-consistent formula such that the underlying model will be a locally finite reflexive tree on which the non-theorem is refuted. To guarantee that we do not create any infinitely descending sequences of distinct instants between two instants in the repair process of building the network, we will filter the canonical model of this logic through $Cl(\alpha)$, where α is any non-theorem of the logic. Note that all claims in the rest of this subsection refer to networks for a $\mathbf{Pr}_{lfin}$-consistent formula $\neg\alpha$.

Recall the definitions of a coherent and saturated network, as well as a perfect network from Section 2.8. Notice that perfect networks will give rise to frames and models that are trees as the underlying frames and models have the same properties as the network.

We already know from Section 5.4 that there is a process of removing saturation defects. We will follow the same repair process for $S_{\mathbf{P}_2}$ and $S_{\mathbf{F}}$ defects, but we ensure that the relation is reflexive by adding the pair (u, u) to the relational pairs, for each new node u added in the repair process.

Recall the definition of $S_{\mathbf{P}_1}$ defects:

- $S_{\mathbf{P}_1}$ defects: The pair $(s, \mathbf{P}\psi)$ is an $S_{\mathbf{P}_1}$ defect if it is an $S_{\mathbf{P}}$ defect and there exists a t with $t \ll s$ such that $\mathbf{H}\neg\psi \in \kappa(t)$, i.e. t is a block for the $(s, \mathbf{P}\psi)$ defect as no node can be added before t to repair the defect.

Also, we will partition the set of defects into equivalence classes as we did in Section 5.4, using the following equivalence relation on the set P of all formulas of the form $\mathbf{P}\varphi$ in $Cl(\alpha)$: $(u, \mathbf{P}\psi_i) \sim_{\mathcal{N}} (u', \mathbf{P}\psi_j)$ iff $i = j$ and one of $u \ll u'$, $u' \ll u$ or $u = u'$. Let $[(u, \mathbf{P}\psi_i)]_{\mathcal{N}}$ represent the equivalence classes for each i and each u. We showed that $\sim_{\mathcal{N}}$ is an equivalence relation in Section 5.4.

Lemma 5.5.3. *For any $S_{\mathbf{P}_1}$ defect $(s, \mathbf{P}\psi)$ of a finite, coherent network $\mathcal{N}$ for a $\mathbf{Pr}_{lfin}$-consistent formula, there is a finite coherent network $\mathcal{N}'$ for the $\mathbf{Pr}_{lfin}$-consistent formula, that extends $\mathcal{N}$ and lacks all defects in the equivalence class $[(u, \mathbf{P}\psi)]_{\mathcal{N}}$. Furthermore, if the defect was repaired by adding the node s', then $(s', \mathbf{P}\psi)$ is not a defect, either.*

Proof. Note that this proof is similar as that of 5.4.4 , but we use Lemma 5.5.2 to select the new node.

Let $\mathcal{N} = (N, \ll, \kappa)$ be a finite, coherent network for a $\mathbf{Pr}_{lfin}$-consistent formula and let R be the canonical relation for $\mathbf{Pr}_{lfin}$. Let $(s, \mathbf{P}\psi)$ be an $S_{\mathbf{P}_1}$ defect that is minimal in N, meaning that for all $S_{\mathbf{P}_1}$ defects $(x, \mathbf{P}\psi)$ we have $s \ll x$. Then there exists a t' with $t' \underline{\ll} s$ such that $\mathbf{H}\neg\psi \in \kappa(t')$. Since N is finite and coherent, there must be a $t \ll s$ with $\mathbf{H}\neg\psi \in \kappa(t)$, such that for all $t' \ll s$ with $\mathbf{H}\neg\psi \in \kappa(t')$, we have $t' \underline{\ll} t$. To repair the defect, we add a node s' between t and s (with s' the immediate successor of t and s the immediate successor of s'). But $\mathbf{P}\psi \in \kappa(s)$ and $\mathbf{H}\neg\psi \in \kappa(t)$, and hence by Lemma 5.5.2, there is an mcs Γ' such that $\psi \in \Gamma'$ and $\kappa(t)R\Gamma' R\kappa(s)$, and for all Δ with $\kappa(t)R\Delta R\Gamma'$ we have $\neg\psi \wedge \mathbf{P}\psi \notin \Delta$ (this choice of Γ' will block the propagation of the defect - see below). Let $\kappa(s') = \Gamma'$.

Define $\mathcal{N}' = (N', \ll', \kappa')$ by

$N' = \{s'\} \cup N$

$\ll' = \ll \cup \{(s', x) \mid t \ll x\} \cup \{(x, s') \mid x \underline{\ll} t\} \cup \{(s, s)\}$

$\kappa' = \kappa \cup \{(s', \Gamma')\}$

Then, since $\psi \in \Gamma'$, we have that $(v, \mathbf{P}\psi)$ is not a defect for all v with $s' \ll' v$. Therefore, no successor of s' has a defect in the same equivalence class as $(s, \mathbf{P}\psi)$. Also, for all v' with $v' \ll' s'$ we have $\mathbf{H}\neg\psi \in \kappa'(v')$ by the maximality of t and the fact that s' is the immediate successor of t. Therefore no predecessor of s' has a defect in the same equivalence class as $(s, \mathbf{P}\psi)$.

Furthermore, by using Lemma 5.5.2 in the choice of Γ', we see that $(s', \mathbf{P}\psi)$ is also not a defect.

Therefore, no node on the same branch as s has a defect in the same equivalence class as $(s, \mathbf{P}\psi)$. Hence, once an $S_{\mathbf{P}_1}$ defect has been repaired, then all defects in the same equivalence class will also be repaired. Therefore, not only does N' not have the defect $(s, \mathbf{P}\psi)$, but it also does not have any defects in the same equivalence class as $(s, \mathbf{P}\psi)$.

For coherency, the choice of Γ' already gives us $\kappa'(t)R\Gamma' R\kappa'(s)$. The rest follows from the coherency of N and transitivity. □

Let α be any non-theorem of $\mathbf{Pr}^r_{lfin}$. In the next lemma we build a coherent network $\mathcal{N}$ for a $\mathbf{Pr}^r_{lfin}$-consistent formula where all the defects involving formulas in $Cl(\alpha)$ are repaired.

Lemma 5.5.4. *Let α be a non-theorem for $\mathbf{Pr}^r_{lfin}$. Then, there is a coherent network for $\neg\alpha$ that has no defects involving any formulas in $Cl(\alpha)$.*

Proof. The proof is the same as that of Theorem 5.4.5 just using a reflexive relation for the network. □

It is easy to see that we can argue as in the proof of Lemma 5.4.6 to get the following result.

Lemma 5.5.5. *Let $\mathcal{N}$ be the resulting network for a $\mathbf{Pr}^r_{lfin}$-consistent formula after repairing all $S_{\mathbf{P}_1}$, $S_{\mathbf{P}_2}$ and $S_{\mathbf{F}}$ defects built in the proof of Lemma 5.4.5. Then $\mathcal{N}$ is locally finite, i.e. for any nodes s, t of this network such that $s \ll t$, there are at most finitely many nodes between s and t.*

Theorem 5.5.6. $\mathbf{Pr}^r_{lfin}$ *is sound and weakly complete with respect to the class of locally finite reflexive trees.*

Proof. From Lemma 5.5.1 it follows that **Q** is valid on the class of locally finite reflexive frames. This, and the fact that all other axioms are valid in all reflexive trees, give soundness of the logic.

For completeness, let α be a formula that is not a theorem of $\mathbf{Pr}^r_{lfin}$. Let $\mathcal{N}$ be the network built in the proof of Lemma 5.5.4. We showed in Lemma 5.5.5 that $\mathcal{N}$ is locally finite. Furthermore, since $\mathcal{N}$ is a perfect network, it follows from Lemma 2.8.5 and the fact that $\alpha \in Cl(\alpha)$, that for the underlying model $\mathcal{M}_{\mathcal{N}}$, we have $\mathcal{M}, s_0 \nVdash \alpha$.

Hence, $\mathbf{Pr}^r_{lfin}$ is sound and weakly complete with respect to the class of locally finite trees, as required. □

We also have the following corollary that follows from the construction process.

Corollary 5.5.7. $\mathbf{Pr}^r_{lfin}$ *is sound and weakly complete with respect to the class of reflexive trees with no infinitely descending sequences between two nodes.*

Notice that the network we built above also does not have any infinitely ascending sequences of distinct nodes between two nodes by construction without axiom **Q**. This leads to the following consequence.

Theorem 5.5.8. $\mathbf{Pr}_{basic}\mathbf{T}$ *is sound and weakly complete with respect to the class of reflexive trees with no infinitely ascending sequences of distinct nodes between two nodes.*

Note that, by Proposition 5.1.13, the logic of the class of locally finite trees is not compact, and hence not strongly complete with respect to any class of frames.

5.6 Unbounded Irreflexive and Reflexive Locally Finite Trees

Consider the following logics: $\mathbf{Pr}_{lfin}\mathbf{U_l}\mathbf{U_r}$, $\mathbf{Pr}_{lfin}\mathbf{U_l}$ and $\mathbf{Pr}_{lfin}\mathbf{U_r}$, where $\mathbf{U_l}$ and $\mathbf{U_r}$ are the seriality axioms $\mathbf{P}\top$ and $\mathbf{F}\top$, and $\mathbf{Pr}_{lfin}$ contains **S** given by $\mathbf{P}q \wedge \mathbf{H}(q \to \mathbf{P}q) \to \mathbf{HP}q$. Table 5.6 summarises the axioms together with the frame classes relevant to it.

In this section we obtain weak completeness of $\mathbf{Pr}_{lfin}\mathbf{U_l}\mathbf{U_r}$ with respect to the class of unbounded discrete irreflexive trees, $\mathbf{Pr}_{lfin}\mathbf{U_l}$ with respect to the class of left unbounded discrete trees, and $\mathbf{Pr}_{lfin}\mathbf{U_r}$ with respect to the class of right unbounded discrete trees.

In Section 4.7 we have already seen that, in the irreflexive case, the process of building a network using mcs's that include the seriality axioms, will yield an unbounded model (defined in Section 2.2.1. However, when restricting the set of formulas to a non-theorem and its subformulas, it is possible that some branches might not be unbounded. This might happen when the seriality axioms are not the set of subformulas we repair defects for. We can rectify this problem by adding the required seriality axiom to the finite set of formulas as in Theorem 5.6.1 below.

As we will see in the next section, $\mathbf{Pr}_{basic}\mathbf{S}\mathbf{U_l}\mathbf{U_r}$ is sound and weakly complete with respect to the class of irreflexive trees with branches isomorphic to the integers. Therefore, let $\mathbf{Pr}_{\mathbb{Z}}$ denote the logic $\mathbf{Pr}_{basic}\mathbf{S}\mathbf{U_l}\mathbf{U_r}$.

Logic and Axioms	Class of Kripke Frames	Class of (Ir)reflexive Trees	Standard Frames
$\mathbf{Pr}_{lfin}\mathbf{U_l}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{S} : \mathbf{P}q \wedge \mathbf{H}(q \rightarrow \mathbf{P}q) \rightarrow \mathbf{HP}q$ $\mathbf{U_l} : \mathbf{P}\top$	Transitive, left linear, left unbounded frames	Left unbounded irreflexive trees with no infinitely descending sequences with a strict lower bound	Left unbounded locally finite irreflexive trees
$\mathbf{Pr}_{lfin}\mathbf{U_r}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{S} : \mathbf{P}q \wedge \mathbf{H}(q \rightarrow \mathbf{P}q) \rightarrow \mathbf{HP}q$ $\mathbf{U_r} : \mathbf{F}\top$	Transitive, left linear, right unbounded frames	Right unbounded irreflexive trees with no infinitely descending sequences with a strict lower bound	Right unbounded locally finite irreflexive trees
$\mathbf{Pr}_{lfin}\mathbf{U_l}\mathbf{U_r}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{S} : \mathbf{P}q \wedge \mathbf{H}(q \rightarrow \mathbf{P}q) \rightarrow \mathbf{HP}q$ $\mathbf{U_l} : \mathbf{P}\top$ $\mathbf{U_r} : \mathbf{F}\top$	Transitive, left linear unbounded frames	Unbounded irreflexive trees with no infinitely descending sequences with a strict lower bound	Unbounded locally finite irreflexive trees
$\mathbf{Pr}^r_{lfin}$: $\mathbf{Pr}_{basic}$ as defined in Definition 4.1.1 $\mathbf{T} : p \rightarrow \mathbf{F}p$ $\mathbf{Q} : (\mathbf{F}p \wedge \mathbf{H}\neg p) \rightarrow \mathbf{F}(p \wedge \mathbf{H}(p \vee \mathbf{H}\neg p))$	Transitive, left linear, reflexive simple successor cluster frames	Reflexive trees with no infinitely descending sequences with a strict lower bound	Unbounded locally finite reflexive trees

Table 5.6: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{lfin}\mathbf{U_l}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$, $\mathbf{Pr}_{lfin}\mathbf{U_l}\mathbf{U_r}$ and $\mathbf{Pr}^r_{lfin}$

Theorem 5.6.1. *The following logics are sound and strongly complete with respect to the given classes:*

- $\mathbf{Pr}_{lfin}\mathbf{U_l}$ *with respect to the class of left unbounded locally finite irreflexive trees.*
- $\mathbf{Pr}_{lfin}\mathbf{U_r}$ *with respect to the class of right unbounded locally finite irreflexive trees.*
- $\mathbf{Pr}_{\mathbb{Z}}$ *with respect to the class of unbounded locally finite irreflexive trees.*

Proof. Soundness follows from Lemmas 5.4.2, and the fact that the remaining axioms are valid on the respective classes of trees.

For all these logics, we use the same method to build a network as in Sections 5.1. However, since we are working with a finite set of subformulas of a non-theorem α closed under single negations $Cl(\alpha)$, unboundedness cannot be guaranteed for the networks built $\neg\alpha$. For this reason, we will include the appropriate seriality axioms in $Cl(\alpha)$.

We do the case for $\mathbf{Pr}_{lfin}\mathbf{U_l}$, and the others will follow similarly.

Let α be a non-theorem of $\mathbf{Pr}_{lfin}\mathbf{U_l}$ and let $Cl(\alpha)_{\mathbf{U_l}}$ be the set of all subformulas of α closed under single negations, as well as $\mathbf{U_l}$. Let w be an mcs for $\mathbf{Pr}_{lfin}\mathbf{U_l}$ where α is false. Building a network as in Theorem 5.4.5, the result will be left unbounded, since $\mathbf{U_l}$ will be in every mcs, which means the process of repairing defects will continually add on new predecessor nodes to repair any $\mathbf{P}\top$ defects. □

Theorem 5.6.2. $\mathbf{Pr}^r_{lfin}$ *is sound and weakly complete with respect to the following classes of trees:*

- *the class of locally finite, left unbounded reflexive trees*
- *the class of locally finite, right unbounded reflexive trees*
- *the class of locally finite, unbounded reflexive trees*

Proof. Soundness follows from the fact that the axioms are valid on the respective classes of trees.

Let α be a non-theorem of $\mathbf{Pr}^r_{lfin}$. Let $\mathcal{M} = (W, R, V)$ be the model built for $\mathbf{Pr}^r_{lfin}$ in Section 5.5 that refutes α at s_0. Let $\mathcal{M}' = (W', R', V')$ be the model where every leaf in $\mathcal{M}$ is replaced by a copy of the natural numbers, in the following way: Suppose $\{w_i \mid i \in I\}$, where I is an index set, is the set of leaves in $\mathcal{M}$. For each i, we replace w_i with a copy of the natural numbers $\mathbb{N}_i$. Then

- $W' = (W - \{w_i \mid i \in I\}) \cup \bigcup_i \mathbb{N}_i$
- For all $u, v \in W'$ it is the case that $R'uv$ iff
  - $u, v \in W$ and Ruv or,
  - $u \in W$ and $v \in \mathbb{N}_i$ for some i and Ruw_i or,
  - $u, v \in \mathbb{N}_i$ for some i, and $u \leq v$ where $\leq$ is the natural reflexive ordering of the rational numbers.
- $V'(p) = (V(p) - \{w_i \mid w_i \in V(p)\}) \cup \{\mathbb{N}_i \mid w_i \in V(p)\}$

Define the function $f : W' \to W$ as follows:

- $f(w) = w$ for all $w \in W - \{w_i \mid i \in I\}$
- $f(u) = w_i$ for all $u \in \mathbb{N}_i$

We show that f is a bounded morphism:

Clearly f is surjective and local harmony is preserved. We show that the back and forth conditions hold:

Forth Condition: Suppose $R'wv$.

Case 1: $w, v \in W - \{w_i \mid i \in I\}$. Then Rwv by definition of R'. Hence $Rf(w)f(v)$ by the definition of f.

Case 2: $w \in W - \{w_i \mid i \in I\}$ and $v \in \mathbb{N}_i$ where $f(v) = w_i$. Then Rww_i by definition of R'. Hence $Rf(w)f(v)$ by the definition of f.

Case 3: $w, v \in \mathbb{N}_i$. Then $f(w) = f(v) = w_i$. By reflexivity, it follows that $Rf(w)f(v)$.

Back Conditions:

Suppose $Rf(w)v'$ then we need to show that there exists v such that $R'wv$ and $f(v) = v'$. Then, let $v = v'$ with $f(v) = v'$. From the definition of R' it follows that $R'wv$. Similarly for when $Rv'f(w)$

Therefore, f is a bounded morphism, and it follows that $\mathcal{M}, f(w) \Vdash \varphi$ iff $\mathcal{M}', w \Vdash \varphi$, and in particular $\mathcal{M}', s_0 \nVdash \alpha$. Furthermore,since $\mathcal{M}'$ is a right unbounded locally finite reflexive tree, it follows that $\mathbf{Pr}^r_{lfin}$ is sound and strongly complete with respect to the class of right unbounded locally finite reflexive trees.

To show completeness for the class of left unbounded locally finite reflexive trees, we will replace the root of the model (if it exists) with a copy of the natural numbers as we did for the leaves, and define a symmetrical bounded morphism $f : W' \to W$.

Suppose w_0 is the root of $\mathcal{M}$. We replace w_0 with a copy of the negative integers $\mathbb{Z}^-_0$. Then let $\mathcal{M}' = (W', R', V')$ be defined as follows:

- $W' = (W - \{w_0\}) \cup \mathbb{Z}^-_0$
- For all $u, v \in W'$ it is the case that $R'uv$ iff
  - $u, v \in W$ and Ruv or,
  - $v \in W$ and $u \in \mathbb{N}_0$ or,
  - $u, v \in \mathbb{Z}^-_0$, and $u \leq v$ where $\leq$ is the natural reflexive ordering of the integers.
- $V'(p) = (V(p) - \{w_0\}) \cup \mathbb{Z}^-_0$ if $w_0 \in V(p)$, and $V'(p) = V(p)$ if $w_0 \in V(p)$

Then, we define $f : W' \to W$ as follows:

- $f(w) = w$ for all $w \in W - \{w_0\}$
- $f(u) = w_0$ for all $u \in \mathbb{Z}^-_0$

Using a symmetrical argument, we can show that f is a bounded morphism. Hence, $\mathbf{Pr}^r_{lfin}$ is sound and strongly complete with respect to the class of left unbounded locally finite trees reflexive trees.

By replacing both the leaves and root as above, we get an unbounded reflexive locally finite tree and hence, $\mathbf{Pr}^r_{lfin}$ is sound and strongly complete with respect to the class of unbounded reflexive trees.

□

5.7 Trees with Branches Isomorphic to the Integers

We have already seen in Theorem 5.6.1 that $\mathbf{Pr}_{\mathbb{Z}}$ is sound and weakly complete with respect to the class of unbounded locally finite irreflexive trees, and that $\mathbf{Pr}^r_{lfin}$ is sound and weakly and complete with respect to the class of unbounded locally finite reflexive trees. In this section, we will show that $\mathbf{Pr}_{\mathbb{Z}}$ is sound and weakly and complete with respect to the class of irreflexive trees with branches isomorphic to the integers, and that $\mathbf{Pr}^r_{lfin}$ is sound and weakly complete with respect to the class of reflexive trees with branches isomorphic to the integers.

We begin by proving a representation theorem for trees with branches isomorphic to the integers.

Lemma 5.7.1. *The unbounded locally finite irreflexive trees are exactly the irreflexive trees with branches isomorphic to the integers.*

Proof. Consider any branch on the model $\mathcal{M} = (W, R, V)$ built for $\mathbf{Pr}_{\mathbb{Z}}$ described in Theorem 5.6.1 and let $w \in W$ be any instant on this branch. By the construction of $\mathcal{M}$, every instant has an immediate successor and every instant has an immediate predecessor, and, as seen in Section 5.6 , it is also unbounded. Furthermore, as seen in Corollary 5.4.6, this model is a locally finite tree, and therefore any successor or predecessor of an instant can be reached through finitely many steps of immediate successors or predecessors.

Now let H be a branch in $\mathcal{M}$ with $w \in H$ and define the following isomorphism, $f : \mathbb{Z} \to H$: Let $f(0) = w$ and let $f(-n) = w'$ where w' can be reached from w by n immediate predecessor steps and let $f(n) = w''$ where w'' can be reached from w by n immediate successor steps, where $w', w'' \in H$. Clearly f preserves the order and is a bijection. Defining an isomorphism for each branch H of $\mathcal{M}$, it follows that $\mathcal{M}$ has branches isomorphic to $\langle \mathbb{Z}, < \rangle$ from which the desired completeness result follows.

Lastly, the fact that irreflexive trees with branches isomorphic to the integers are unbounded and locally finite, follows immediately. □

We also have the following result, where we use the unbounded model built in the proof of Theorem 5.6.2 and replace the strict relation with a reflexive one in the proof of Theorem 5.7.3.

Lemma 5.7.2. *The unbounded locally finite reflexive trees are exactly the reflexive trees with branches isomorphic to the integers.*

We now show that these logics are sound and weakly complete with respect to the classes of trees with branches isomorphic to $\langle \mathbb{Z}, < \rangle$ and $\langle \mathbb{Z}, \leq \rangle$ respectively, in the theorems below.

Theorem 5.7.3. $\mathbf{Pr}_{\mathbb{Z}}$ *is sound and weakly complete with respect to the class of irreflexive trees with branches isomorphic to* $\langle \mathbb{Z}, < \rangle$.

Proof. Soundness follows as before from the fact that axioms are valid on the class of irreflexive trees with branches isomorphic to the integers, and complete is a consequence of Lemma 5.7.1. □

Similarly, using Lemma 5.7.2, we have the following result.

Theorem 5.7.4. $\mathbf{Pr}^{r}_{lfin}$ *is sound and weakly complete with respect to the class of reflexive trees with branches isomorphic to* $\langle \mathbb{Z}, \leq \rangle$.

5.8 Well-Founded and Conversely Well-Founded Irreflexive Trees

The following formulas have been shown ([32]) to define well-foundedness, and conversely well-foundedness for linear orders, and for any frames in [66].

- Well-Founded $\mathbf{L_l}$: $\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$
- Conversely Well-Founded $\mathbf{L_r}$: $\mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$

We summarise this in the lemma below.

Lemma 5.8.1. $\mathbf{L_l}$ *defines the class of transitive, well-founded irreflexive frames and* $\mathbf{L_r}$ *defines the class of transitive conversely well-founded irreflexive frames. Consequently, any frame on which any of these two formulas is valid cannot contain a reflexive instant.*

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{wf}$: $\mathbf{K_G}$: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H}$: $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F}$: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P}$: $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1}$: $p \to \mathbf{GP}p$ $\mathbf{Conv_2}$: $p \to \mathbf{HF}p$ $\mathbf{.3_l}$: $(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{L_l}$: $\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$	Transitive, irreflexive, left linear well-founded frames	Well-founded irreflexive trees	Well-founded irreflexive trees

Table 5.7: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{wf}$

Proof. The proof that $\mathbf{L_l}$ defines the class of well-founded, transitive, irreflexive frames was given in [66].

Symmetrically it can be shown that $\mathbf{L_r}$ defines the class of conversely well-founded frames.

Next, observe that no frame on which $\mathbf{L_r}$ or $\mathbf{L_l}$ is valid can be reflexive, for say w is a reflexive instant in such a frame, then $...wRwRwR$ is an infinite sequence refuting well-foundedness and $wRwRwR...$ is an infinite sequence refuting conversely well-foundedness.

□

Since all frames for the basic temporal logic that contain either $\mathbf{L_l}$ or $\mathbf{L_r}$, or both, are already transitive, we do not need to add the transitivity axiom when considering logics containing these axioms.

Hence, all frames for the logic that contains the converse and dual axioms as well as $\mathbf{K_H}$, $\mathbf{K_G}$, $\mathbf{.3_l}$, and $\mathbf{L_l}$ are transitive, left linear, irreflexive, and well-founded. Likewise, all frames for the logic that contains the converse and dual axioms as well as $\mathbf{K_H}$, $\mathbf{K_G}$, $\mathbf{.3_l}$, and $\mathbf{L_r}$ are transitive, left linear, irreflexive, and conversely well-founded.

Let $\mathbf{Pr}_{wf}$ denote the logic containing the following axioms: The $\mathbf{K}$ axioms, dual and converse axioms, as well as $\mathbf{.3_l}$, and $\mathbf{L_l}$.

Let $\mathbf{Pr}_{cwf}$ be the logic containing the following axioms: The $\mathbf{K}$ axioms, dual and converse axioms, as well as $\mathbf{.3_l}$, and $\mathbf{L_r}$.

We will show that these two logics are sound and weakly complete with respect to the class of well-founded irreflexive trees and conversely well-founded irreflexive trees, respectively.

5.8.1 Well-Founded Irreflexive Trees

In this section we show that $\mathbf{Pr}_{wf}$ is weakly complete with respect to the class of well-founded irreflexive trees. We do this by building a model for every formula consistent with $\mathbf{Pr}_{wf}$ based on a well-founded irreflexive tree, using networks as in Section 4.5 but using Lemma 5.8.3 to select an mcs to repair any $S_\mathbf{P}$ defects.

Table 5.7 summarises the axioms together with the frame classes relevant to it.

Next, we give a version of Lemma 3.1.13 with the converse operators.

Lemma 5.8.2. *Let $\mathcal{M} = (W, R, V)$ be a model for Λ, where Λ is a logic containing $\mathbf{L_l}$, and suppose that for some instant $w \in W$, $\mathcal{M}, w \nVdash \mathbf{H}\varphi$. Then there exists an irreflexive instant $v \in W$ such that Rvw and, $\mathcal{M}, v \nVdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{H}\varphi$.*

Proof. Let $\mathcal{M} = (W, R, V)$ be a model for Λ, where Λ is a logic containing $\mathbf{L_l}$, and suppose that for some instant $w \in W$, $\mathcal{M}, w \nVdash \mathbf{H}\varphi$. Since all substitution instances of $\mathbf{L_l}$ are true in $\mathcal{M}$, we have $\mathcal{M}, w \Vdash \mathbf{H}(\mathbf{H}\varphi \to \varphi) \to \mathbf{H}\varphi$ and hence $\mathcal{M}, w \nVdash \mathbf{H}(\mathbf{H}\varphi \to \varphi)$. Therefore, there is an instant $v \in W$ such that Rvw and, $\mathcal{M}, v \nVdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{H}\varphi$. Furthermore, v is irreflexive, otherwise $\mathcal{M}, v \Vdash \varphi$. □

Lemma 5.8.2 implies the following:

Corollary 5.8.3. *Let $\mathcal{M} = (W, R, V)$ be a model for $\mathbf{Pr}_{wf}$ and suppose that for some instant $w \in W$, $\mathcal{M}, w \Vdash \mathbf{P}\varphi$. Then there exists an (irreflexive) instant $v \in W$ such that Rvw and, $\mathcal{M}, v \Vdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{H}\neg\varphi$.*

We will use this result to show that we will not create any infinite, descending sequences when constructing the network.

Let α be a non-theorem of $\mathbf{Pr}_{wf}$. We will build a network for $\neg\alpha$ where we only repair defects of formulas in $Cl(\alpha)$.

For the rest of this section, networks will refer to networks for $\neg\alpha$ and defects will only be those that are in $Cl(\alpha)$. Note that the proof of the following lemma is largely similar to that of Lemma 5.5.3 but we use Corollary 5.8.3 to select the new node.

Lemma 5.8.4. *In any strict coherent network $\mathcal{N}$ for $\neg\alpha$ with an $S_\mathbf{P}$ defect on a given branch, there exists an extension of this network that does not contain a defect involving the same formula on the same branch.*

Proof. Let $\mathcal{N} = (N, \ll, \kappa)$ be a strict coherent network for $\neg\alpha$ and suppose that for some $s \in N$, $(s, \mathbf{P}\psi)$ is a defect.

First we describe how we will repair this defect. Let u' be a node in N such that $u' \underline{\ll} s$ and $(u', \mathbf{P}\psi)$ is a $S_\mathbf{P}$ defect, and for all $u \in N$ with $u \ll u'$, $(u, \mathbf{P}\psi)$ is not a $S_\mathbf{P}$-defect. Since N is finite, such a u' exists and is possibly s. Then insert a new node s' immediately before u', and let Γ' be the mcs guaranteed by Corollary 5.8.3. Note that in this case we also have $\Gamma' R \kappa(u')$.

Define $\mathcal{N}' = (N, \ll', \kappa')$ by
$N' = \{s'\} \cup N$
$\ll' = \ll \cup \{(x, s') \mid x \ll u'\} \cup \{(s', x) \mid u' \underline{\ll} x\}$
$\kappa' = \kappa \cup \{(s', \Gamma')\}$

To prove coherency, we see that N' is still a finite tree and we only need to show that the relations are preserved. Since $\Gamma' R \kappa'(u')$, coherency follows from the coherency of N and transitivity for all $u' \underline{\ll}' x$. Now, consider any y such that $y \ll' s'$. Then $y \underline{\ll}' u'$. By construction $\kappa'(s') = \Gamma'$ with $\Gamma' R \kappa'(u')$ and coherency of N, it follows that $\kappa'(y) R \kappa'(u')$. Since $\mathcal{N}'$ is a strict *total* order to the left as $\mathcal{N}'$ is left linear, we must have that either $\Gamma' R \kappa'(y)$, $\Gamma' = \kappa'(y)$ or $\kappa'(y) R \Gamma'$. However, we cannot have $\Gamma' R \kappa'(y)$, as $\psi \in \Gamma'$ would imply that $\mathbf{P}\psi \in \kappa'(y)$ which would contradict the minimality of u'. Also, we cannot have $\Gamma' = \kappa'(y)$, as $\psi \in \kappa'(y)$ would mean that $(s, \mathbf{P}\psi)$ was not a defect in the first place. Hence, $\kappa'(y) R \Gamma'$ as required, and that shows that $\mathcal{N}'$ is strict coherent.

Next, we have to show that the extension $\mathcal{N}'$ does not have an $S_\mathbf{P}$ defect involving the same formula on any branch containing s. First, note that since $\mathcal{M}, u' \Vdash \mathbf{P}\psi$, it follows that for no t with $u' \ll' t$ do we have that $(t, \mathbf{P}\psi)$ is a defect. Notice that this set of successors includes s and all successors of s.

Now let $v \in N'$ such that $v \ll' u'$. Then, by Corollary 5.8.3 we have $\mathcal{M}, v \Vdash \psi$ and $\mathcal{M}, v \Vdash \mathbf{H}\neg\psi$. Hence, $(v, \mathbf{P}\psi)$ is not a defect. □

Lemma 5.8.5. *Let α be a non-theorem for $\mathbf{Pr}_{wf}$. Then, there is a strict coherent network for $\neg\alpha$ that has no defects involving any formulas in $Cl(\alpha)$.*

Proof. Let α be a formula that is not a theorem of $\mathbf{Pr}_{wf}$. We will use $\mathbf{Pr}_{wf}$-mcs's and the canonical relation for $\mathbf{Pr}_{wf}$, considering only formulas in $Cl(\alpha)$, to build a well-founded model on which α is refuted.

Let $S = \{s_i \mid i \in \omega\}$ be a countable set of nodes disjoint from N, i.e., a supply of new nodes to be used in repairing, step by step, the defects of $\mathcal{N}$. Next, let

$$\mathcal{D} = [(N \cup S) \times \{\mathbf{F}\varphi \mid \varphi \in \mathcal{L}\}] \cup [(N \cup S) \times \{\mathbf{P}\varphi \mid \varphi \in \mathcal{L}\}],$$

i.e., $\mathcal{D}$ is the set of all possible defects of $\mathcal{N}$ and of any network extending it with nodes from S. Note that $\mathcal{D}$ is countable and let $D_0, D_1, D_2, \ldots$ be an arbitrary but fixed enumeration of its elements.

We will construct a sequence of finite, strict coherent extensions of $\mathcal{N}$, namely $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots$ as follows: Since α is not a theorem of $\mathbf{Pr}_{wf}$, there exists an instant $w \in W$ such that $\mathcal{M}, w \nVdash \alpha$. Let $\mathcal{N}_0 = (\{s_0\}, \emptyset, (s_0, w))$.

Next, suppose $\mathcal{N}_n = (N_n, \ll_n, \kappa_n)$ is a finite, strict coherent network for $n \geq 0$. If no more defects exist, let

$$N_n = \bigcup_{i=0}^{n} N_i \qquad \ll_n = \bigcup_{i=0}^{n} \ll_i \qquad \kappa_n = \bigcup_{i=0}^{n} \kappa_i \tag{5.3}$$

Otherwise, let D be the defect of $\mathcal{N}_n$ that is minimal in the enumeration of potential defects. Next, form $\mathcal{N}_{n+1}$ by repairing the defect D. In this process an extension of $\mathcal{N}_n$ is created without the defect and any new extension will also not have the defect D. Define $\mathcal{N} = (N, \ll, \kappa)$ as follows:

$$N = \bigcup_{n \in \omega} N_n \qquad \ll = \bigcup_{n \in \omega} \ll_n \qquad \kappa = \bigcup_{n \in \omega} \kappa_n \tag{5.4}$$

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By construction $\mathcal{N}$ is strict coherent with no more defects.

□

We now show that $\mathcal{N}$ is well-founded.

Lemma 5.8.6. *Let $\mathcal{N}$ be the resulting network for $\neg\alpha$ after repairing all $S_{\mathbf{P}}$ and $S_{\mathbf{F}}$ defects built in the proof of Lemma 5.8.5. Then $\mathcal{N}$ is well-founded.*

Proof. From using Corollary 5.8.3 in the repair process in the proof of Lemma 5.8.5, and the fact that $Cl(\alpha)$ is finite, it follows that repairing $S_{\mathbf{P}}$ defects cannot create any infinitely descending sequences. This can also not happen when repairing $S_{\mathbf{F}}$ defects, as this process adds nodes to new branches. □

Hence, we have the following theorem.

Theorem 5.8.7. $\mathbf{Pr}_{wf}$ *is sound and weakly complete with respect to the class of well-founded irreflexive trees.*

Proof. Soundness follows from Lemma 5.8.1 and the fact that the remaining axioms are valid on frames for this logic.

Let $\mathcal{N}$ be the network built in Lemma 5.8.5. Then by Lemma 5.8.6, this network is well-founded. Now, since $\mathcal{N}$ is a strict perfect network, it follows from Lemma 2.8.5 that for the underlying model $\mathcal{M}_{\mathcal{N}}$ we have $\mathcal{M}, s_0 \nVdash \alpha$.

Hence, $\mathbf{Pr}_{wf}$ is weakly complete with respect to the class of well-founded irreflexive trees, as required. □

The following lemma shows that the network built for the completeness result above, is actually more than well-founded. This will be useful when we show completeness for the class of irreflexive trees with branches isomorphic to the natural numbers in Section 5.11.

Lemma 5.8.8. *The constructed network in the proof of Lemma 5.8.5 is locally finite.*

Proof. Let $\mathcal{N}$ be the network built in Lemma 5.8.5. Suppose there are infinitely many linearly ordered instants $S = \{u_1, u_2, ...\}$ between nodes v and w with $v \ll w$. Since $\mathcal{N}$ is well founded there is no infinitely descending sequence between v and w. But then there has to be an infinitely ascending sequence in S, since we can choose the smallest node $s_1 \in S$, and then the smallest node $s_2 \in S \setminus \{s_1\}$, etc., to build an infinitely ascending sequence

[6] Notice that it is possible that these sets are finite if we run out of defects at some point in the construction.

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{cwf}$: $\mathbf{K_G} : \mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H} : \mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F} : \mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P} : \mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1} : p \to \mathbf{GP}p$ $\mathbf{Conv_2} : p \to \mathbf{HF}p$ $\mathbf{.3_l} : (\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{L_r} : \mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$	Transitive, irreflexive, left linear conversely well-founded frames	Conversely well-founded irreflexive trees	Conversely well-founded irreflexive trees

Table 5.8: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{cwf}$

$s_1 \ll s_2 \ll ...$ with $\{s_1, s_2, ...\} \subseteq S$. Now, at the point when w was added, the network was still finite, and adding nodes to repair $S_\mathbf{F}$ defects after that, would have resulted in adding a leaf. Therefore, the infinite sequence must have been created by repairing $S_\mathbf{P}$ defects, and in particular, since $Cl(\alpha)$ is finite, there must be some formula $\mathbf{P}\varphi \in Cl(\alpha)$ that was repaired infinitely many times. Suppose s_i was added to repair a defect involving $\mathbf{P}\varphi$. Then, there are only finitely many instants s_j preceding s_i. Also, since s_i was added to repair a defect involving $\mathbf{P}\varphi$, no successors of s_i will have a defect involving $\mathbf{P}\varphi$. Hence, such an infinitely ascending sequence cannot exists. Therefore, $\mathcal{N}$ is locally finite. □

Remark 5.8.9. From Lemma 5.8.8, we can conclude that the network constructed in the proof of Lemma 5.8.5 has branches that are either finite or isomorphic to $(\mathbb{N}, <)$.

5.8.2 Conversely Well-Founded Irreflexive Trees

In this section we show that $\mathbf{Pr}_{cwf}$ is weakly complete with respect to the class of conversely well-founded irreflexive trees. We do this by building a model for every formulas consistent with $\mathbf{Pr}_{cwf}$ based on a conversely well-founded tree, by building a network for a $\mathbf{Pr}_{cwf}$-consistent formula that has no infinitely ascending sequences, as done above for $\mathbf{Pr}_{wf}$. First recall Lemma 3.1.13.

Table 5.8 summarises the axioms together with the frame classes relevant to it.

Lemma 5.8.10. *Let $\mathcal{M} = (W, R, V)$ be a model for Λ, where Λ is a logic containing $\mathbf{L_r}$, and suppose that for some instant $w \in W$, $\mathcal{M}, w \nVdash \mathbf{G}\varphi$. Then there exists an irreflexive instant $v \in W$ such that Rwv and, $\mathcal{M}, v \nVdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{G}\varphi$.*

Proof. Let $\mathcal{M} = (W, R, V)$ be a model for Λ, where Λ is a logic containing $\mathbf{L_r}$, and suppose that for some instant $w \in W$, $\mathcal{M}, w \nVdash \mathbf{G}\varphi$. Since all substitution instances of $\mathbf{L_r}$ are true in $\mathcal{M}$, we have $\mathcal{M}, w \Vdash \mathbf{G}(\mathbf{G}\varphi \to \varphi) \to \mathbf{G}\varphi$ and hence $\mathcal{M}, w \nVdash \mathbf{G}(\mathbf{G}\varphi \to \varphi)$. Therefore, there is an instant $v \in W$ such that Rwv and, $\mathcal{M}, v \nVdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{G}\varphi$. Furthermore, v is irreflexive, otherwise $\mathcal{M}, v \Vdash \varphi$. □

We will repair $S_\mathbf{F}$ defects using the following equivalent of Lemma 5.8.10.

Corollary 5.8.11. *Let $\mathcal{M} = (W, R, V)$ be a model for $\mathbf{Pr}_{cwf}$ and suppose that for some instant $w \in W$, $\mathcal{M}, w \Vdash \mathbf{F}\varphi$. Then there exists an (irreflexive) instant $v \in W$ such that Rwv and, $\mathcal{M}, v \Vdash \varphi$ and $\mathcal{M}, v \Vdash \mathbf{G}\neg\varphi$.*

Let α be any non-theorem of $\mathbf{Pr}_{cwf}$. Note that $Cl(\alpha)$ is finite.

First we show that $S_{\mathbf{F}}$ defects need only be repaired once on a branch. The following lemma is similar to Lemma 5.8.4, but since the network is not linear, we have to consider the different branches. Recall the definition of equivalence classes for defects in Section 5.4. We can define equivalence classes for $S_{\mathbf{F}}$ defects in a similar way:

Let $\{\mathbf{F}\psi_1, \mathbf{F}\psi_2, ..., \mathbf{F}\psi_m\}$ be the set of all formulas in $Cl(\alpha)$ of the form $\mathbf{F}\psi$ and consider the set F of all $S_{\mathbf{F}}$ defects of the form $(u, \mathbf{F}\psi_i)$ for some $u \in N$. Define a relation on F for a network $\mathcal{N}$ as follows: $(u, \mathbf{F}\psi_i) \sim_{\mathcal{N}} (u', \mathbf{F}\psi_j)$ iff $i = j$ and one of $u \ll u'$, $u' \ll u$ or $u = u'$. Similarly to Section 5.4, $\sim_{\mathcal{N}}$ is an equivalence relation. Let $[(u, \mathbf{F}\psi_i)]_{\mathcal{N}}$ represent the equivalence classes for each i and each u.

Note that the proof of the following lemma is largely similar to that of Lemma 5.5.3 but we use Corollary 5.8.11 to select the new node.

Lemma 5.8.12. *In a strict coherent network for a $\mathbf{Pr}_{cwf}$-consistent formula, if an $S_{\mathbf{F}}$ defect $(u, \mathbf{F}\psi_i)$ has been repaired on a branch by adding a node u' to get an extension $\mathcal{N}'$, then $\mathcal{N}'$ is strict coherent and does not contain any defects in the same equivalence class as $(u, \mathbf{F}\psi_i)$.*

Proof. Let $\mathcal{N} = (N, \ll, \kappa)$ be a strict coherent network a $\mathbf{Pr}_{cwf}$-consistent formula and let $(u, \mathbf{F}\psi_i)$ be an $S_{\mathbf{F}}$ defect that is maximal with respect to $\ll$ in the equivalence class of $(u, \mathbf{F}\psi_i)$ modulo $\sim_u$. Now add a new node u' as an immediate successor of u on a new branch, and use Corollary 5.8.11 to find an mcs Γ' such that ψ_i and $\mathbf{G}\neg\psi_i$ are in Γ'. Let $\kappa(u') = \Gamma'$. Then we know that $\kappa(u) R \Gamma'$, where R is the canonical relation.

Define $\mathcal{N}' = (N, \ll', \kappa')$ by

$N' = \{u'\} \cup N$

$\ll' = \ll \cup \{(x, u') \mid x \ll u\} \cup \{(u, u')\}$

$\kappa' = \kappa \cup \{(u', \Gamma')\}$

Let $u'' \in N$ such that $u'' \ll' u'$. Then, since $\psi_i \in \Gamma'$, $(u'', \mathbf{F}\psi_i)$ is not a defect. Furthermore, by Corollary 5.8.11 the node u' that was added in such that $\mathbf{G}\neg\psi_i \in \Gamma'$ and hence $\mathbf{F}\psi_i \notin \Gamma'$. Hence, by the consistency of the canonical relation, $(u'', \mathbf{F}\psi_i)$ is not a defect.

Hence, after adding a node u' to repair an $S_{\mathbf{F}}$ defect in this way, no extension network will have an $S_{\mathbf{F}}$ defect in the same equivalence class as $(u, \mathbf{F}\psi_i)$.

Coherency follows from construction. □

Hence, we can prove the following lemma, using the method of Lemma 5.4.5

Lemma 5.8.13. *Let α be a non-theorem for $\mathbf{Pr}_{lfin}$. Then, there is a strict coherent network for $\neg\alpha$ that has no defects involving any formulas in $Cl(\alpha)$.*

To confirm that we will not create any infinitely ascending sequences in the repair process, we need to show that we cannot add any future defects in the same equivalence class as a previously repaired defect, when repairing additional defects. We also have to show that repairing $S_{\mathbf{P}}$ defect can also not create any infinitely ascending sequences. We do this in the following Lemma.

Lemma 5.8.14. *Let $\mathcal{N}$ be the resulting network for a $\mathbf{Pr}_{cwf}$-consistent formula after repairing all $S_{\mathbf{P}}$ and $S_{\mathbf{F}}$ defects built in the proof of Lemma 5.8.13. Then $\mathcal{N}$ is conversely well-founded.*

Proof. First note that repairing $S_{\mathbf{P}}$ defects cannot lead to infinitely ascending sequences as a new node is added in the $\ll$-past in the repair process.

We have already shown in the proof of Lemma 5.8.12 that the node added in the repair process of an $S_{\mathbf{F}}$ defect u', or any of the predecessors of u', will not have an $S_{\mathbf{F}}$ defect in the equivalence class of $(u, \mathbf{F}\psi_i)$. We just have to make sure that no future added nodes will have a defect in the same equivalence class. However, by Corollary

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
$\mathbf{Pr}^r_{wf}$: $\mathbf{K_G} : \mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H} : \mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F} : \mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P} : \mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1} : p \to \mathbf{GP}p$ $\mathbf{Conv_2} : p \to \mathbf{HF}p$ $\mathbf{.3_l} : (\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{Grz_l} : \mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$	Transitive, reflexive, left linear well-founded frames	Well-founded reflexive trees	Well-founded reflexive trees

Table 5.9: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}^r_{wf}$

5.8.11 the node u' that was added is such that $\mathbf{F}\psi_i \notin \Gamma'$ and hence $\mathbf{G}\neg\psi_i \in \Gamma'$. Therefore, for u' and all successors of u', that can be added to repair future defects, we have that $(u'', \mathbf{F}\psi_i)$ is not a defect.

Lastly, as we are working with finitely many $S_{\mathbf{F}}$ defects, each branch in the network will end in a leaf. □

This shows that the underlying model of the network after repairing all defects, will be a conversely well-founded tree, which gives us the result below:

Theorem 5.8.15. $\mathbf{Pr}_{cwf}$ *is sound and weakly complete with respect to the class of conversely well-founded irreflexive trees.*

Proof. The proof follow similarly as that of 5.8.7, but using Lemmas 5.8.13 and 5.8.14. □

Recall from Proposition 5.1.13 that the logic for the class of well-founded trees is not compact. Also, Segerberg showed in [61] that the logic of the class of conversely well-founded frames is not strongly complete. Therefore, weak completeness for $\mathbf{Pr}_{cwf}$ and $\mathbf{Pr}_{wf}$ with respect to any class of trees is the best we can do. Similarly for the logics of the well-founded and conversely well-founded reflexive trees we discuss in the next two sections.

5.9 Well-Founded Reflexive Trees

We begin this section by introducing the dual of axiom **Grz**, namely $\mathbf{Grz_l} : \mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$. $\mathbf{Grz_l}$ defines the class of reflexive, transitive frames that contain no infinitely descending chains of distinct points (since $\mathbf{Grz_l}$ is the dual of **Grz**, see e.g., Proposition 3.48 in [15]).

Let $\mathbf{Pr}^r_{wf}$ be the logic $\mathbf{K_t.3_lGrz_l}$ (where $\mathbf{K_t}$ is defined in Definition 2.5.2). Note that we do not need the transitivity axiom included in $\mathbf{Pr}_{basic}$ as $\mathbf{Grz_l}$ already ensures transitivity. Table 5.9 summarises the axioms together with the frame classes relevant to it.

In this section we show that $\mathbf{Pr}^r_{wf}$ is weakly complete with respect to the class of well-founded reflexive trees. We do this by building a model for every formula consistent with $\mathbf{Pr}^r_{wf}$ based on a well-founded tree, using the method of building a network, similar to what was done in Section 5.8.1.

We could also use the method of selective filtration as in Section 5.12, which will also give us the desired model; however, the benefit of using networks is that it not only gives us a well-founded reflexive tree, but one that is also locally finite. This will come in handy when we build model for trees with branches isomorphic to the natural numbers, in the reflexive case (See Section 5.11).

Let α be any non-theorem of $\mathbf{Pr}^r_{wf}$. Let w_0 be a $\mathbf{Pr}^r_{wf}$-mcs such that $\alpha \notin w_0$.

Let $\mathfrak{P} = \{\mathbf{P}\psi_1, \mathbf{P}\psi_2, ..., \mathbf{P}\psi_m\}$ be the set of all subformulas in $Cl(\alpha)$ of the form $\mathbf{P}\psi$.

Before describing the method of repairing defects, we give the following result, which is the dual of Lemma 3.2.11. Even though we use the equivalent form given in Corollary 5.9.2 in this section, we will use the form of the result given in Lemma 5.9.1 in Section 5.13.

Lemma 5.9.1. *Let $\mathcal{M} = (W, R, V)$ be any model for $\mathbf{Pr}^r_{fin}$ and let Φ be a set of formulas that is closed under subformulas. Then, for every formula $\mathbf{H}\varphi \in \Phi$, if $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \nVdash \mathbf{H}\varphi$ for some instant $w \in W$, then there exists a $v \in W$ with Rvw such that $\mathcal{M}, v \nVdash \varphi$, and for no $u \in W$ with Ruv, do we have $w \cong_\Phi u$ (Φ-congruent).*

Proof. Let $w \in W$ such that $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \nVdash \mathbf{H}\varphi$ for some formula $\mathbf{H}\varphi \in \Phi$. Suppose that for all v with Rvw and $\mathcal{M}, v \nVdash \varphi$, there exists a u_v with Ru_vv such that $w \cong_\Phi u_v$. Then $\mathcal{M}, u_v \Vdash \varphi$ and $\mathcal{M}, u_v \nVdash \mathbf{H}\varphi$, and hence $\mathcal{M}, u_v \nVdash \varphi \to \mathbf{H}\varphi$. But then $\mathcal{M}, v \nVdash \mathbf{H}(\varphi \to \mathbf{H}\varphi)$ and hence, $\mathcal{M}, v \Vdash \mathbf{H}(\varphi \to \mathbf{H}\varphi) \to \varphi$. Therefore, since Rvw, it follows that $\mathcal{M}, w \Vdash \mathbf{H}(\mathbf{H}(\varphi \to \mathbf{H}\varphi) \to \varphi)$.

Now, since $\mathcal{M}, w \nVdash \mathbf{H}\varphi$, there exists a v' with $Rv'w$ such that $\mathcal{M}, v' \nVdash \varphi$. But by transitivity of R, we also have $\mathcal{M}, v' \Vdash \mathbf{H}(\mathbf{H}(\varphi \to \mathbf{H}\varphi) \to \varphi)$. Hence $\mathcal{M}, v' \nVdash \mathbf{H}(\mathbf{H}(\varphi \to \mathbf{H}\varphi) \to \varphi) \to \varphi$, i.e., $\mathcal{M}, v' \nVdash \mathbf{Grz_l}$, contradicting the assumption that $\mathbf{Grz_l}$ is valid in $\mathcal{M}$.

□

Corollary 5.9.2. *Let $\mathcal{M} = (W, R, V)$ be any model for $\mathbf{Pr}^r_{wf}$ and let Φ be a set of formulas that is closed under subformulas and single negations. Then, for every formula $\mathbf{P}\varphi \in \Phi$, if $\mathcal{M}, w \nVdash \varphi$ and $\mathcal{M}, w \Vdash \mathbf{P}\varphi$ for some instant $w \in W$, then there exists a $v \in W$ with Rvw such that $\mathcal{M}, v \Vdash \varphi$, and for no $u \in W$ with Ruv, do we have $w \cong_\Phi u$.*

Note that all networks in the remainder of this section will be for a $\mathbf{Pr}^r_{wf}$-consistent formula (See Definition 2.8.2.). The proof of the following lemma is largely similar to that of Lemma 5.5.3 but we use Corollary 5.9.2 to build the network.

Lemma 5.9.3. *For any $S_\mathbf{P}$ defect $(s, \mathbf{P}\psi)$ at a node s of a finite, coherent network $\mathcal{N}$ for a $\mathbf{Pr}^r_{wf}$-consistent formula, there is a finite coherent network $\mathcal{N}'$ for $\mathbf{Pr}^r_{wf}$ that extends $\mathcal{N}$ and lacks the defect $(s, \mathbf{P}\psi)$.*

Proof. Let $\mathcal{N} = (N, \underline{\ll}, \kappa)$ (where $\underline{\ll}$ is a reflexive partial ordering as defined in Section 3.2.3), be a coherent network for a $\mathbf{Pr}^r_{wf}$-consistent formula and let $(s, \mathbf{P}\psi)$ be a $S_\mathbf{P}$ defect. Let u' be a node in N such that $u' \underline{\ll} s$ and $(u', \mathbf{P}\psi)$ is an $S_\mathbf{P}$ defect, and for all $u \in N$ with $u \ll u'$, $(u, \mathbf{P}\psi)$ is not an $S_\mathbf{P}$ defect. Since N is finite, such a u' exists and is possibly s. Insert a new node t immediately before u' and let $\kappa(t) = \Gamma$, where Γ is the predecessor of $\kappa(u')$ in the canonical model guaranteed to exists by Corollary 5.9.2.

Define $\mathcal{N}' = (N, \underline{\ll}', \kappa')$ by

$N' = N \cup \{t\}$

$\underline{\ll}' = \underline{\ll} \cup \{(x, t) \mid x \ll u'\} \cup \{(t, x) \mid u' \underline{\ll} x\} \cup \{(t, t)\}$

$\kappa' = \kappa \cup \{(t, \Gamma)\}$

Note that, by the choice of Γ, $\mathcal{N}'$ does not contain the $S_\mathbf{P}$ defect $(s, \mathbf{P}\psi)$. Furthermore, no successor of t, and t itself, have an $S_\mathbf{P}$ defect involving $\mathbf{P}\psi$.

We show that $\mathcal{N}'$ is coherent. First note that $\mathcal{N}'$ transitive, reflexive and left linear by construction. To show that the relations are preserved, suppose $t \underline{\ll}' y$ and $t \neq y$. If $y = u'$ we are done. Suppose $u' \ll' y$. Then $R\kappa'(u')\kappa'(y)$ by coherency of $\mathcal{N}$, and hence by transitivity of the canonical relation, it follows that $R\Gamma\kappa'(y)$. Next, suppose $x \underline{\ll}' t$. Then, by left linearity of the canonical relation we have that either $R\kappa'(x)\Gamma$ or $R\Gamma\kappa'(x)$, since both Γ and $\kappa'(x)$ are predecessors of $\kappa(u')$. Now, since $(x, \mathbf{P}\psi)$ is not a defect (by the $\ll'$-minimality of u'), it follows that we cannot have $R\Gamma\kappa'(x)$. Therefore $R\kappa'(x)\Gamma$. Therefore, $\mathcal{N}'$ is coherent. □

Now that we have the repair process for defects, we show, in the next lemma that we can build a coherent network that has no more defects.

Lemma 5.9.4. *Let α be a non-theorem for $\mathbf{Pr}^r_{wf}$. Then, there is a coherent network for $\neg\alpha$ that has no defects involving any formulas in $Cl(\alpha)$.*

Proof. Let α be a formula that is not a theorem of $\mathbf{Pr}^r_{wf}$. We build a well-founded, locally finite network using mcs's and the canonical relation R for $\mathbf{Pr}^r_{wf}$ by restricting the defects we repair to the formulas in $Cl(\alpha)$. Let $S = \{s_i \mid i \in \omega\}$ be the supply of nodes that will be used to build the network. Next, consider the set of potential defects $\mathcal{D} = [S \times \bigcup_i \{\mathbf{F}\varphi_i\}] \cup [S \times \{\bigcup_i \mathbf{P}\varphi_i]$, where $\{\mathbf{F}\varphi_i\}$ and $\{\mathbf{P}\varphi_i\}$ contain all the $\mathbf{F}$ and $\mathbf{P}$ formulas in $Cl(\alpha)$. Note that an enumeration of $\mathcal{D}$ exists since S is countable, and $Cl(\alpha)$ is finite.

Since α is not a theorem of $\mathbf{Pr}^r_{wf}$, there exists an instant w_0 in the domain of the canonical model of $\mathbf{Pr}^r_{wf}$ such that $\alpha \notin w_0$. Let $\mathcal{N}_0 = (\{s_0\}, (s_0, s_0), (s_0, w_0))$, where $s_0 \in S$.

Next, suppose $\mathcal{N}_n$ is a finite, coherent network for $n \geq 0$. If no more defects exist, let

$$N = N_n \quad \underline{\ll} = \underline{\ll}_n \quad \kappa = \kappa_n \tag{5.5}$$

Otherwise, let D be the defect of $\mathcal{N}_n$ that is minimal in the enumeration of $\mathcal{D}$. Next, form $\mathcal{N}_{n+1}$ by repairing the defect D. In this process an extension of $\mathcal{N}_n$ is created without the defect and any new extension will also not have the defect D.

Define $\mathcal{N} = (N, \underline{\ll}, \kappa)$ as follows:

$$N = \bigcup_{n\in\omega} N_n \quad \underline{\ll} = \bigcup_{n\in\omega} \underline{\ll}_n \quad \kappa = \bigcup_{n\in\omega} \kappa_n \tag{5.6}$$

By construction, $\mathcal{N}$ is coherent with no more defects. □

In the next lemma we show that $\mathcal{N}$ is well-founded.

Lemma 5.9.5. *Let $\mathcal{N}$ be the resulting network for $\neg\alpha$ after repairing all $S_\mathbf{P}$ and $S_\mathbf{F}$ defects built in the proof of Lemma 5.9.4. Then $\mathcal{N}$ is well-founded.*

Proof. Let $\mathcal{N} = (N', \underline{\ll}', \kappa')$ be the network built in Lemma 5.9.4. Suppose that a node t was added in the repair of a defect, say $(s, \mathbf{P}\psi)$, using Corollary 5.9.2. Then $\kappa'(t)$, and all mcs's Γ' such that $\Gamma' R \kappa'(t)$ (where R is the canonical relation), are not $Cl(\alpha)$-congruent to $\kappa'(s)$. Note that, at each stage in the building of the model, there are only finitely many nodes. So when t was added, the network was finite. To create an finitely descending sequence preceding t, we would have had to add infinitely many nodes preceding t after t was added. Hence, they must have been added to repair $S_\mathbf{P}$-defects. Suppose that $...\underline{\ll}' u_2, \underline{\ll}' u_1 \underline{\ll}' t$ is such an infinitely descending sequence of nodes in $\mathcal{N}$ preceding t that was created to repair the defects $(x_1, \mathbf{P}\psi_1), (x_2, \mathbf{P}\psi_2), \ldots$ (note that not all x_i's have to be distinct).

For each node, there are only finitely many formulas of the form $\mathbf{P}\psi$ that need to be repaired. Therefore, for each distinct x_i there is a node u_{x_i} such that $u_{x_i} = u_j$ for some j and for all u_k added to repair defects at x_i we have that $u_{x_i} \underline{\ll}' u_k$. Let $...\underline{\ll}' u_{x'_2} \underline{\ll}' u_{x'_1} \underline{\ll}' t$ be this infinitely descending subsequence of $...\underline{\ll}' u_2, \underline{\ll}' u_1 \underline{\ll}' t$, with corresponding sequence of defects $(x'_1, \mathbf{P}\psi'_1), (x'_2, \mathbf{P}\psi'_2), \ldots$. Then, for each i, $u_{x'_i}$ will not be $Cl(\alpha)$-congruent to $\kappa'(s)$, and also not $Cl(\alpha)$-congruent to $\kappa'(x'_j)$, where $j \leq i$ (since we use Lemma 5.9.4 to repair defects, which give this result, regardless of the order in which the defects are repaired). But there are only finitely many different mcs's modulo $Cl(\alpha)$, and therefore we can only add finitely many nodes before t in the repair process, contradicting the existence of such an infinitely descending sequence of nodes preceding t. □

We can now state the completeness result.

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
$\mathbf{Pr}^r_{cwf}$: $\mathbf{K_G}$: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H}$: $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F}$: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P}$: $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1}$: $p \to \mathbf{GP}p$ $\mathbf{Conv_2}$: $p \to \mathbf{HF}p$ $\mathbf{.3_l}$: $(\mathbf{P}p \wedge \mathbf{P}q) \to$ $\mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{Grz}$: $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$	Transitive, reflexive, left linear conversely well-founded frames	Conversely well-founded reflexive trees	Conversely well-founded reflexive trees

Table 5.10: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}^r_{cwf}$

Theorem 5.9.6. $\mathbf{Pr}^r_{wf}$ *is sound and weakly complete with respect to the class of well-founded reflexive trees.*

Proof. $\mathbf{Grz_l}$ is valid on the class of well-founded reflexive trees.

For completeness, let α be a formula that is not a theorem of $\mathbf{Pr}^r_{wf}$. Let $\mathcal{N}$ be the network built in the proof of Lemma 5.9.4 of which the underlying model $\mathcal{M}_\mathcal{N}$ contains w_0, with $\neg\alpha \in w_0$. We showed in Lemma 5.9.5 that $\mathcal{N}$ is well-founded.

Furthermore, since $\mathcal{N}$ is a perfect network, it follows from Lemma 2.8.5 and the fact that $\alpha \in Cl(\alpha)$, that for the underlying model $\mathcal{M}_\mathcal{N}$, we have $\mathcal{M}, w_0 \nVdash \alpha$.

Hence, $\mathbf{Pr}^r_{wf}$ is sound and weakly complete with respect to the class of well-founded trees, as required. □

Lemma 5.9.7. *The constructed network in the proof of Lemma 5.9.4 is locally finite.*

Proof. Let $\mathcal{N}$ be the network built in Lemma 5.9.4. Suppose there are infinitely many linearly ordered instants $S = \{u_1, u_2, ...\}$ between nodes v and w with $v \ll w$. Since $\mathcal{N}$ is well founded there is no infinitely descending sequence between v and w. But then there has to be an infinitely ascending sequence in S, since we can choose the smallest node $s_1 \in S$, and then the smallest node $s_2 \in S \setminus \{s_1\}$, etc., to build an infinitely ascending sequence $s_1 \ll s_2 \ll ...$ with $\{s_1, s_2, ...\} \subseteq S$. Now, at the point when w was added, the network was still finite, and adding nodes to repair $S_\mathbf{F}$ defects after that, would have resulted in adding a leaf. Therefore, the infinite sequence must have been created by repairing $S_\mathbf{P}$ defects, and in particular, since $Cl(\alpha)$ is finite, there must be some formula $\mathbf{P}\varphi \in Cl(\alpha)$ that was repaired infinitely many times. Suppose s_i was added to repair a defect involving $\mathbf{P}\varphi$. Then, there are only finitely many instants s_j preceding s_i. Also, since s_i was added to repair a defect involving $\mathbf{P}\varphi$, no successors of s_i will have a defect involving $\mathbf{P}\varphi$. Hence, such an infinitely ascending sequence cannot exists. Therefore, $\mathcal{N}$ is locally finite. □

5.10 Conversely Well-Founded Reflexive Trees

Consider the logic $\mathbf{Pr}^r_{cwf}$ that extends the basic temporal logic $\mathbf{K_t}$, with the axioms $\mathbf{.3_l}$, $\mathbf{Grz}$ (As we do not need the transitivity axiom, we start with $\mathbf{K_t}$ instead of $\mathbf{Pr}_{basic}$). Table 5.10 summarises the axioms together with the frame classes relevant to it.

In this section we show that $\mathbf{Pr}^r_{cwf}$ is weakly complete with respect to the class of conversely well-founded reflexive trees as defined in Section 2.1. We do this by building a model for every formula consistent with $\mathbf{Pr}^r_{cwf}$ based on a conversely well-founded reflexive tree, using the method of selected filtrations as done in Chapter 5 in [15].

As seen for the future fragment of the Priorian language in Lemma 3.2.10 the canonical frame for a logic that includes the axiom **Grz** is transitive and reflexive. We can show in a similar way that this also the case for $\mathbf{Pr}^r_{cwf}$. We will use the method of selective filtrations to build a conversely well-founded reflexive tree, and will use Lemma 3.2.11 to guarantee that we do not create any infinitely ascending sequences of distinct instants in the selection process.

First, we state an analogue of Lemma 3.2.11 for the backwards looking modality **H**. The proof is similar to that of Lemma 3.2.11.

Lemma 5.10.1. *Let $\mathcal{M} = (W, R, V)$ be a model for $\mathbf{Pr}^r_{cwf}$ and let Φ be a subset of formulas that is closed under single negations. Then, for every formula $\mathbf{H}\varphi$ in Φ, if $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \nVdash \mathbf{H}\varphi$ for some instant $w \in W$, then there exists an instant $v \in W$ with Rvw such that $\mathcal{M}, v \nVdash \varphi$ and, for no $u \in W$ with Ruv, is $w \cong_\Phi u$.*

The process for showing weak completeness for the class of converserly well-founded frames, as given in Chapter 5 in [15], is adapted for our purposed in the proof of the theorem below.

Theorem 5.10.2. $\mathbf{Pr}^r_{cwf}$ *is sound and weakly complete with respect to the class of conversely well-founded reflexive trees.*

Proof. Soundness follows from the fact that the axioms for the logic are valid on conversely well-founded reflexive trees.

To prove completeness, let α be a formula that is not a theorem of $\mathbf{Pr}^r_{cwf}$. We will apply a selective filtration method to the canonical model $\mathcal{M} = (W, R, V)$ for $\mathbf{Pr}^r_{cwf}$ through $Cl(\alpha)$ to build a conversely well-founded reflexive tree on which α is refuted. Since α is not a theorem of $\mathbf{Pr}^r_{cwf}$, there exists an instant $w_0 \in W$ such that $\mathcal{M}, w_0 \nVdash \alpha$. We will use w_0 as the starting point for building the required model for $\mathbf{Pr}^r_{cwf}$ that will refute α by systematically adding more reflexive instants to the model we are constructing, until a truth lemma relativised to $Cl(\alpha)$ holds.

Let w and v be arbitrary elements of W and let $\Phi^{\mathbf{G}}_w = \{\mathbf{G}\psi \in Cl(\alpha) \mid \mathbf{G}\psi \notin w$ and $\psi \in w\}$ and $\Phi^{\mathbf{H}}_v = \{\mathbf{H}\psi \in Cl(\alpha) \mid \mathbf{H}\psi \notin v$ and ψ We will refer to the elements of these sets as **G**-requirements and **H**-requirements, respectively. The systematic selection process is defined inductively as follows:

Set $W_0 = \{w_0\}$. Then, $\mathcal{F}_0 = (W_0, R_0)$ (where $R_0 = \{(w_0, w_0)\}$) is a finite (conversely well-founded) reflexive tree.

Next, suppose $W_1, ..., W_n$ have already been constructed, where each W_i consists of all the new instants that was added in the i^{th} step of the construction. Then, for each $i = 1, ..., n$, let R_i be constructed as follows:

For all $x, y \in \bigcup_{s=0}^{i} W_s$, let $R_i xy$ iff:

- y was added in the i^{th} step as a witness for a **G**-requirement at x.
- x was added in the i^{th} step as a witness for a **H**-requirement at z, where $z \in \bigcup_{s=0}^{i} W_s$, and y is an instant such that $y \in \bigcup_{s=0}^{i} W_s$ and Rxy.

If all formulas in $\Phi^{\mathbf{G}}_w$ and in $\Phi^{\mathbf{H}}_v$ have the required witnessing instants, for all $w, v \in W_n$, then let $W^* = \bigcup_{s=0}^{n} W_s$. Let R^* be the reflexive and transitive closure of $\bigcup_{s=0}^{n} R_s$, and let $V^*(p) = W^* \cap V(p)$. Let $\mathcal{M}^* = (W^*, R^*, V^*)$. Then, by construction, $\mathcal{M}^*$ is a well-founded reflexive tree, and hence is the required model.

Otherwise for each $w, v \in W_n$ consider all formulas in $\Phi^{\mathbf{G}}_w$ and in $\Phi^{\mathbf{H}}_v$ that require witnessing instants.

If $\mathbf{G}\varphi \in \Phi^{\mathbf{G}}_w$, and there is no $s \in W^*$ with R^*ws such that $\neg\varphi \in s$, use Lemma 3.2.11 to select an instant w' with Rww' such that $\varphi \notin w'$, and for no $u \in W$ with $Rw'u$, do we have $w \cong_{Cl(\alpha)} u$.

If $\mathbf{H}\varphi \in \Phi^{\mathbf{H}}_v$, and there is no $t \in W^*$ with R^*tv such that $\neg\varphi \in t$, then choose a v' with $Rv'v$ such that $\varphi \notin v'$. Such a v' must exist by the Existence Lemma 2.5.10.

Then let W_{n+1} be the set of all w' and v' selected above and repeat the process until a set W_k is reached for which all formulas in $\Phi^{\mathbf{G}}_w$ and $\Phi^{\mathbf{H}}_v$ have witnessing instants, for all w, v in W_k (Note that this process might be infinite.).

Similar to the proof of Lemma 5.9.5, and using the fact that we used Lemma 5.10.1 when adding instants for **G** formulas, we can show that the resulting model is conversely well-founded.

Therefore, let $W^{**} = \bigcup_{s=0}^{\infty} W_s$ and $\mathcal{M}^{**} = (W^{**}, R^{**}, V^{**})$, where R^{**} be the reflexive and transitive closure of $\bigcup_{s=0}^{\infty} R_s$, and let $V^{**}(p) = W^{**} \cap V(p)$. At this point all modal formulas in $Cl(\alpha)$ will be satisfied in $\mathcal{M}^{**}$. Furthermore, we can prove a version of the truth lemma relativised to $Cl(\alpha)$, using induction, as follows.

Claim 2. *Let $w \in W$. Then, for all $\varphi \in Cl(\alpha)$, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^{**}, w \Vdash \varphi$.*

Proof. The proposition and Boolean cases follow easily. So, suppose $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^{**}, w \Vdash \varphi$ for all $w \in \mathcal{M}^{**}$ and for all $\varphi \in Cl(\alpha)$.

Let φ be $\mathbf{G}\psi$.

$\Rightarrow$: Suppose $\mathcal{M}, w \Vdash \mathbf{G}\psi$. Then for all $v \in W$ with Rwv it is the case that $\mathcal{M}, v \Vdash \psi$, and hence, by the induction hypothesis, $\mathcal{M}^{**}, v' \Vdash \psi$, where $v' \in W \cap W^{**}$ and Rwv'. Hence $\mathcal{M}^{**}, w \Vdash \mathbf{G}\psi$.

$\Leftarrow$: Suppose $\mathcal{M}, w \nVdash \mathbf{G}\psi$. Then there exists a v' with Rwv' and $v' \neq w$, such that $\mathcal{M}, v \nVdash \psi$. Let v be the v' selected in the construction of $\mathcal{M}^{**}$. Therefore, by the induction hypothesis $\mathcal{M}^{**}, v \nVdash \psi$, since $v \in W^{**}$. Hence, $\mathcal{M}^{**}, w \nVdash \mathbf{G}\psi$.

Let φ be $\mathbf{H}\psi$.

The case of, if $\mathcal{M}, w \Vdash \mathbf{H}\psi$, then $\mathcal{M}^{**}, w \Vdash \mathbf{H}\psi$ follows similarly to the one when φ is $\mathbf{G}\psi$.

$\Leftarrow$: Similar to the case for **G**, suppose $\mathcal{M}, w \nVdash \mathbf{H}\psi$. Then, there exists a $v' \in W$ with $Rv'w$ such that $\mathcal{M}, v' \nVdash \varphi$. Let v be the v' selected in the construction of $\mathcal{M}^{**}$. Then, $\mathcal{M}, v \nVdash \varphi$, and by the induction hypothesis it follows that $\mathcal{M}^{**}, v \nVdash \varphi$. Hence $\mathcal{M}^{**}, w \nVdash \mathbf{H}\psi$ by construction.

□

Notice that $\mathcal{M}^{**}$ is left linear by the canonicity of axiom $\mathbf{3}_l$. It is also connected, transitive, reflexive and conversely well-founded by construction. Furthermore, by construction of R^{**}, it is also a tree, and hence is the required model. Furthermore, since $w_0 \in W^{**}$, it also refutes α. Therefore, $\mathbf{Pr}^r_{fin}$ is weakly complete with respect to the class of conversely well-founded, reflexive trees. □

5.11 Trees with Branches Isomorphic to the Natural Numbers

Let $\mathbf{Pr}_{wf}\mathbf{U_r}$ be denoted by $\mathbf{Pr}_{\mathbb{N}}$ and recall the logic $\mathbf{Pr}^r_{wf}$ from Section 5.9. Table 5.11 summarises the axioms of these logics together with the frame classes relevant to it.

We begin this section by showing that $\mathbf{Pr}_{\mathbb{N}}$ is weakly complete with respect to the class of irreflexive trees with branches isomorphic to the natural numbers with the normal strict ordering.

We have already seen that $\mathbf{Pr}_{\mathbb{Z}}$ is weakly complete with respect to the class of trees with branches isomorphic to the integers in Theorem 5.7.3, and we can use a similar method to find a completeness result for trees with branches isomorphic to the natural numbers.

In [32] and [22] the Priorian logic of the natural numbers was axiomatised with the formula formula $\mathbf{G}(\mathbf{G}p \to p) \to (\mathbf{FG}p \to \mathbf{G}p)$ (see also [60]). The completeness proof uses a process of selective filtration to build satisfying model isomorphic to the natural numbers with their strict ordering for any formula consistent with this logic. Since the model being built is so specific, the process of "slotting in" witnesses needs to be very carefully managed. In the context of branching time, however, the process becomes less complex, as the repair process for $S_{\mathbf{F}}$ defects can add new nodes on new branches, which preserves the discreteness of the model (see Section 5.6.1).

In the next lemma we prove that the branches of any right unbounded locally finite well-founded model are isomorphic to the natural numbers with the usual strict ordering.

Lemma 5.11.1. *The right unbounded, locally finite, well-founded, irreflexive linear orderings are exactly the orderings isomorphic to $\langle \mathbb{N}, < \rangle$.*

Logic and Axioms	**Class of Kripke Frames**	**Class of (Ir)reflexive Trees**	**Standard Frames**
$\mathbf{Pr}_{\mathbb{N}}$: $\mathbf{K_G}$: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H}$: $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F}$: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P}$: $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1}$: $p \to \mathbf{GP}p$ $\mathbf{Conv_2}$: $p \to \mathbf{HF}p$ $\mathbf{.3_l}$: $(\mathbf{P}p \wedge \mathbf{P}q) \to$ $\mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{L_l}$: $\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$ $\mathbf{U_r}$: $\mathbf{F}\top$	Right unbounded, transitive, irreflexive, left linear well-founded frames	Right unbounded well-founded irreflexive trees	Irreflexive trees with branches isomorphic to the natural numbers
$\mathbf{Pr}^r_{wf}$: $\mathbf{K_G}$: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H}$: $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F}$: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P}$: $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1}$: $p \to \mathbf{GP}p$ $\mathbf{Conv_2}$: $p \to \mathbf{HF}p$ $\mathbf{.3_l}$: $(\mathbf{P}p \wedge \mathbf{P}q) \to$ $\mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{Grz_l}$: $\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$	Transitive, reflexive, left linear well-founded frames	Well-founded reflexive trees	Reflexive trees with branches isomorphic to the natural numbers

Table 5.11: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{\mathbb{N}}$ and $\mathbf{Pr}^r_{wf}$

Proof. Let $(W, R,)$ be a right unbounded, locally finite, well-founded irreflexive linear ordering. We define a function $f : \mathbb{N} \to W$ as follows: Since (W, R) is well-founded, it has a least element, w. Let $f(0) = w$. Suppose $f(n)$ has already been defined. Then, by well foundedness, $W - f[0, 1, \ldots, n]$ has a least element, say v, let $f(n+1) = v$. Clearly, this construction is one-to-one as it maps successors in $\mathbb{N}$ to successors in W and no instant in W can have more than one pre-image.

Next, we show that f respects the order. If $n, m \in \mathbb{N}$ such that $n < m$ and suppose $f(n) = w$ and $f(m) = v$. Then in the construction process, n would have been assigned the least element of a set containing both w and v. Since $f(n) = w$, it follows that Rwv. Hence f respects the order.

Since (W, R) is right unbounded the process of assigning natural numbers to instants in W under f will be infinite. Hence, we will find an image for each $n \in \mathbb{N}$.

Lastly, f is onto. If, to the contrary, there is a $u \in W$ such that $u \neq f(n)$ for any $n \in \mathbb{N}$, then $u \in W - f[\mathbb{N}]$, so we have that $f(n)Ru$ for all $n \in \mathbb{N}$. Therefore, there are infinitely many instants between $f(0)$ and u. This contradicts the fact that W is locally finite.

Therefore f is an order preserving isomorphism from (W, R) to $\mathbb{N}$. □

Note that the model built in Theorem 5.8.7 is already a locally finite well-founded tree (see Lemma 5.8.8). To get a model with branches isomorphic to the natural numbers, the only missing property is right unboundedness, which we do by adding $\mathbf{U_r}$ to the logic.

Theorem 5.11.2. $\mathbf{Pr}_{\mathbb{N}}$ *is sound and weakly complete with respect to the class of irreflexive trees with branches isomorphic to* $\langle \mathbb{N}, < \rangle$.

Proof. Soundness again follows from the fact that the axioms are valid on the class of trees with branches isomorphic to the natural numbers.

For completeness, let $\mathcal{M} = (W, R, V)$ be the model for $\mathbf{Pr}_{\mathbb{N}}$ obtained by the method of Theorem 5.8.7. Then $\mathcal{M} = (W, R, V)$ is a right unbounded locally (See Lemma refremark like natural numbers and Remark 5.8.9) finite well-founded tree (See the proof of Theorem 3.1.22 for right unboundedness). Hence, completeness follows from Lemma 5.11.1. □

Lemma 5.11.3. *The right unbounded, locally finite, well-founded, reflexive linear orderings are exactly the orderings isomorphic to* $\langle \mathbb{N}, \leq \rangle$.

Proof. The proof is similar to that of Lemma 5.11.1. □

Next, we show that $\mathbf{Pr}^r_{wf}$ is sound and weakly complete with respect to the class of reflexive trees with branches isomorphic to the natural numbers. In Theorem 5.9.6 and Lemma 5.9.7 we saw that $\mathbf{Pr}^r_{wf}$ is sound and weakly complete with respect to the class of well-founded, locally finite reflexive trees. By replacing each leaf in the model built in the proof of Theorem 5.9.6 with a copy of the natural numbers, similar to what we did in the proof of Theorem 5.6.2, we can build a model that is a right unbounded well-founded reflexive tree. Hence, we have the following result. (The proof is similar to that of Lemma 5.11.2.)

Theorem 5.11.4. $\mathbf{Pr}^r_{wf}$ *is sound and weakly complete with respect to the class of reflexive trees with branches isomorphic to* $\langle \mathbb{N}, \leq \rangle$.

Remark 5.11.5. We can easily find a complete axiomatisation for the classes of (ir)reflexive trees with branches isomorphic to the negative integers in the following way. Starting with the logic $\mathbf{Pr}_{cwf}$, we add the left seriality axiom $\mathbf{U_l}$ and use a similar argument to the one used in the proof of Theorem 5.11.2. For the reflexive case, starting with the model build for $\mathbf{Pr}^r_{cwf}$, and replacing the root with a copy of the negative integers as in the proof of Theorem 5.6.2 we have a left unbounded conversely-well founded tree, and as been shown in this proof, the

Logic and Axioms	Class of Kripke Frames	Class of Irreflexive Trees	Standard Frames
$\mathbf{Pr}_{fin}$: $\mathbf{K_G}$: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H}$: $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F}$: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P}$: $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1}$: $p \to \mathbf{GP}p$ $\mathbf{Conv_2}$: $p \to \mathbf{HF}p$ $\mathbf{.3_l}$: $(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{L_l}$: $\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$ $\mathbf{L_r}$: $\mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$	Transitive, irreflexive, left linear well-founded and conversely well-founded frames	Well-founded and conversely well-founded irreflexive trees	Finite irreflexive trees

Table 5.12: Axioms,Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}_{fin}$

bounded morphism guarantees that satisfaction in this model has not been affected. We can then use a similar argument to the one used in the proof of Theorem 5.11.4 to get the completeness result.

5.12 Finite Irreflexive Trees

Let $\mathbf{Pr}_{fin}$ denote the logic containing the converse and dual axioms as well as $\mathbf{K_H}$, $\mathbf{K_G}$, $\mathbf{.3_l}$, $\mathbf{L_l}$ and $\mathbf{L_r}$. Recall from Lemma 5.8.1, that all frames on which the axioms $\mathbf{L_l}$ and $\mathbf{L_r}$ are valid, are transitive, irreflexive, well-founded and conversely well-founded. Notice, that a frame for $\mathbf{Pr}_{fin}$ is not necessarily finite, as it may contain an infinite number of branches. Table 5.12 summarises the axioms together with the frame classes relevant to it.

In this section we show that $\mathbf{Pr}_{fin}$ is weakly complete for the class of finite irreflexive trees. We do this by building a model for every formula consistent with $\mathbf{Pr}_{fin}$ based on a finite irreflexive tree, using the method of selective filtrations from [15] as in Section 5.10.

Beginning with $\mathbf{Pr}_{fin}$ and a non-theorem, we will build a finite model for this logic by constructing a model from the canonical model $\mathcal{M} = (W, R, V)$ on which the non-theorem is refuted. We need Lemma 5.8.2 and 5.8.10 for the construction. These two lemmas give us the tools to construct a finite, transitive, irreflexive model that is left linear (i.e. a finite tree) by allowing us to choose irreflexive witnesses for formulas of the form $\neg\mathbf{H}\varphi$ and $\neg\mathbf{G}\varphi$. Recall that $\mathbf{L_l}$ and $\mathbf{L_r}$ are valid on well-founded and conversely well-founded frames (See Lemma 5.8.1).

Theorem 5.12.1. $\mathbf{Pr}_{fin}$ *is sound and weakly complete with respect to the class of finite irreflexive trees.*

Proof. Soundness follows, since the axioms are valid on the class of finite irreflexive trees.

Let α be a formula that is not a theorem of $\mathbf{Pr}_{fin}$. We will use a selective filtration process on the canonical model $\mathcal{M} = (W, R, V)$ for $\mathbf{Pr}_{fin}$ through $Cl(\alpha)$ to build a finite tree on which α is refuted. Since α is not a theorem of $\mathbf{Pr}_{fin}$, there exists an instant $w \in W$ such that $\mathcal{M}, w \nVdash \alpha$. If w is reflexive, then $\mathcal{M}, w \nVdash \mathbf{G}\alpha$ and $\mathcal{M}, w \nVdash \mathbf{H}\alpha$.[7] Then by Lemmas 5.8.10 and 5.8.2, there exists $u, v \in W$ with Ruw such that $\mathcal{M}, u \Vdash \mathbf{H}\alpha$ and $\mathcal{M}, u \nVdash \alpha$, and Rwv such that $\mathcal{M}, v \Vdash \mathbf{G}\alpha$ and $\mathcal{M}, v \nVdash \alpha$. Therefore, we can always find an irreflexive instant in the canonical model where α is false.

Let w_0 be any irreflexive instant such that $\mathcal{M}, w_0 \nVdash \alpha$. We will use w_0 as the starting point for building the required model for $\mathbf{Pr}_{fin}$ that will refute α by systematically adding more irreflexive instants to the model until we

[7]Note that the fact that $\mathbf{G}\alpha$ and $\mathbf{H}\alpha$ are not in the set of subformulas does not matter, as this argument just shows that we can find an irreflexive satisfying instant from which to proceed.

can prove a truth lemma relativised to formulas in $Cl(\alpha)$.

Let $\Phi_w^{\mathbf{G}} = \{\mathbf{G}\psi \in Cl(\alpha) \mid \mathbf{G}\psi \notin w \text{ and } \psi \in w\}$ and $\Phi_v^{\mathbf{H}} = \{\mathbf{H}\psi \in Cl(\alpha) \mid \mathbf{H}\psi \notin v \text{ and } \psi \in v\}$. We will refer to the elements of these sets as **G**-requirements and **H**-requirement respectively. The systematic selection process is defined inductively as follows:

Set $W_0 = \{w_0\}$. Then, $\mathcal{F}_0 = (W_0, \emptyset)$ is a finite irreflexive tree.

Next, suppose $W_1, ..., W_n$ have already been constructed, where each W_i consists of all the new instants that was added in the i^{th} step of the construction. Then, for each $i = 1, ..., n$, let R_i be constructed as follows:

For all $x, y \in \bigcup_{s=0}^{i} W_s$, let $R_i xy$ iff:

- y was added in the i^{th} step as a witness for a **G**-requirement at x.
- x was added in the i^{th} step as a witness for a **H**-requirement at z, where $z \in \bigcup_{s=0}^{i} W_s$, and y is an instant such that $y \in \bigcup_{s=0}^{i} W_s$ and Rxy.

If all formulas in $\Phi_w^{\mathbf{G}}$ and in $\Phi_v^{\mathbf{H}}$ have the required witnessing instants, for all $w, v \in W_n$, then let $W^* = \bigcup_{s=0}^{n} W_s$. Let R^* be the transitive closure of $\bigcup_{s=0}^{n} R_s$, and let $V^*(p) = W^* \cap V(p)$. Let $\mathcal{M}^* = (W^*, R^*, V^*)$. Then, by construction, $\mathcal{M}^*$ transitive, irreflexive. It is also left linear, for suppose not. Then there are distinct instants $w, v, u \in W^*$ such that R^*uw and R^*vw. Notice that this could not have happened when adding instants to fulfil **G**-requirements, as w would have had to be added as a leaf at both u and v. Therefore, this must have happened when adding instants to fulfil **H**-requirements. But, by construction of all the R_i's, instants were only added in the linear past, and hence this could also not have happened. Hence, $\mathcal{M}^*$ is a finite irreflexive tree, and hence is the required model.

Otherwise, if not all formulas in $\Phi_w^{\mathbf{G}}$ and in $\Phi_v^{\mathbf{H}}$ have the required witnessing instants, for all $w, v \in W_n$, then, for each $w, v \in W_n$ consider all formulas in $\Phi_w^{\mathbf{G}}$ and in $\Phi_v^{\mathbf{H}}$ that require witnessing instants.

If $\mathbf{G}\varphi \in \Phi_w^{\mathbf{G}}$ and there is no $x \in W_n$ with Rwx such that $\neg\varphi \in x$, use Lemma 5.8.10 to select an instant w' with Rww' such that $\varphi \notin w'$ and $\mathbf{G}\varphi \in w'$.

If $\mathbf{H}\varphi \in \Phi_v^{\mathbf{H}}$ and there is no $u \in W_n$ with Ruv such that $\neg\varphi \in u$, use Lemma 5.8.2 to select an instant v' with $Rv'v$ such that $\varphi \notin v'$ and $\mathbf{H}\varphi \in v'$.

Then let W_{n+1} be the set of all w' and v' selected above and repeat the process until a set W_k is reach for which all formulas in $\Phi_w^{\mathbf{G}}$ and $\Phi_v^{\mathbf{H}}$ have witnessing instants, for all w, v in W_k.

Reasoning by contradiction, supposing such a k does not exist and the process does not terminate and hence produces an infinite model. Then as a result of König's Lemma, [8], we either have an infinite branch or infinitely many children for some node. We derive contradictions in both cases, and conclude that the process must terminate.

First note that, if there exists a branch in $\mathcal{M}^{**}$ with infinitely many instants, the instants must be contained between two instants, since, if we want to create an infinitely descending sequence with no lower bound, we would have to fulfil infinitely many consecutive **H**-requirements. This cannot be the case when using Lemma 5.8.2 to add instants for **H**-requirements. Similar for infinitely ascending sequences without an upper bound, using Lemma 5.8.10 to add instants for **G**-requirements. Hence, this infinite set of instants must have occurred in the interaction between adding instants for **H** and **H**-requirements. However, we can use an argument similar to that of Lemma 5.8.8, to argue that this cannot happen.

Lastly, instants added for **G**-requirements are added as a leaf on a new branch, by construction of R^{**}, and since each instant only has finitely many **G**-requirements, only finitely many immediate successors were added. Hence the process will end and will therefore give a finite model, which we define next.

Let $W^{**} = \bigcup_{s=0}^{k} W_s$ and $\mathcal{M}^{**} = (W^{**}, R^{**}, V^{**})$, where R^{**} be the transitive closure of $\bigcup_{s=0}^{k} R_s$, and let $V^{**}(p) = W^{**} \cap V(p)$. We can prove a version of the truth lemma relativised to $Cl(\alpha)$, using induction, as follows:

The proposition and Boolean cases follow easily. So suppose $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^{**}, w \Vdash \varphi$ for all $w \in \mathcal{M}^{**}$ and for some $\varphi \in Cl(\alpha)$.

Logic and Axioms	Class of Kripke Frames	Class of Reflexive Trees	Standard Frames
$\mathbf{Pr}^r_{fin}$: $\mathbf{K_G}$: $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ $\mathbf{K_H}$: $\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$ $\mathbf{Dual_F}$: $\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$ $\mathbf{Dual_P}$: $\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$ $\mathbf{Conv_1}$: $p \to \mathbf{GP}p$ $\mathbf{Conv_2}$: $p \to \mathbf{HF}p$ $\mathbf{.3_l}$: $(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$ $\mathbf{Grz_l}$: $\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$ $\mathbf{Grz}$: $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$	Transitive, reflexive, left linear well-founded and conversely well-founded frames	Well-founded and conversely well-founded reflexive trees	Finite reflexive trees

Table 5.13: Axioms, Kripke frames, Tree Frames and Standard Frames for $\mathbf{Pr}^r_{fin}$

Let φ be $\mathbf{G}\psi$.

$\Rightarrow$: Suppose $\mathcal{M}, w \Vdash \mathbf{G}\psi$. Then for all $v \in W$ with Rwv it is the case that $\mathcal{M}, v \Vdash \psi$, and hence, by the induction hypothesis, $\mathcal{M}^{**}, v' \Vdash \varphi$, where $v' \in W \cap W^{**}$ and Rwv'. Hence $\mathcal{M}^{**}, w \Vdash \mathbf{G}\psi$.

$\Leftarrow$: Suppose $\mathcal{M}, w \nVdash \mathbf{G}\psi$. Then, by Lemma 5.8.10, there exists a $v' \in W$ with Rwv' such that $\mathcal{M}, v' \nVdash \psi$ and $\mathcal{M}, v' \Vdash \mathbf{G}\psi$. Let v be the v' guaranteed by Lemma 5.8.10 that was selected in the construction of $\mathcal{M}^{**}$. Then $v \in W^{**}$, and hence by the induction hypothesis, we have $\mathcal{M}^{**}, v \nVdash \psi$ and $R_k wv$ by construction. Therefore, $\mathcal{M}^{**}, w \nVdash \mathbf{G}\psi$.

Similar for when φ is $\mathbf{H}\psi$ using the fact that Lemma 5.8.2 was used in the construction process.

Notice that $\mathcal{M}^{**}$ is left linear by the canonicity of axiom $\mathbf{3}_l$. It is also connected, transitive and finite by construction. Furthermore, the submodel of the canonical model generated by the point we start with is a tree of clusters. The model we construct is a submodel of that and, since we pick only irreflexive points it has only degenerate clusters, i.e. it is an irreflexive tree. Furthermore, since $w_0 \in W^{**}$, it also refutes α. Therefore, $\mathbf{Pr}^r_{fin}$ is weakly complete with respect to the class of finite irreflexive trees.

□

Recall from Propositions 5.1.12 and 5.1.13 that the logic for finite trees is not compact and hence cannot be strongly complete. Therefore, weak completeness is the best result we can get for the class of finite irreflexive trees.

5.13 Finite Reflexive Trees

Consider the logic $\mathbf{Pr}^r_{fin}$ that extends the basic temporal logic $\mathbf{K_t}$, with the axioms $\mathbf{.3_l}$, $\mathbf{Grz}$ and $\mathbf{Grz_l}$. Table 5.13 summarises the axioms together with the frame classes relevant to it. In this section we show that $\mathbf{Pr}^r_{fin}$ is weakly complete with respect to the class of finite reflexive trees. We do this by building a model for any formula consistent with $\mathbf{Pr}^r_{fin}$ based on a finite reflexive tree, by adapting the method of [15], from Section 5.10. The required model for the completeness proof is constructed in a way that is broadly similarly to that in Theorem 5.12.1, but using Lemma 5.9.1 which cannot guarantee that witnesses can be chosen in such a way as to ensure that the same defect will not appear again on that branch. However showing that the process terminates requires a more involved argument as become clear in the proof of Theorem 5.13.1.

For weak completeness of $\mathbf{Pr}^r_{fin}$ with respect to finite reflexive trees, we start with a non-theorem α of $\mathbf{Pr}^r_{fin}$. Then, through the process of selective filtration through $Cl(\alpha)$, we will build a finite reflexive tree on which α is refuted.

In the case of conversely well-founded frames, Lemma 5.10.1 was sufficient to ensure that there were no infinitely ascending sequences of distinct instants in the model. Furthermore, in the case of well-founded frames, Lemma 5.9.1 was sufficient to ensure that there were no infinitely descending sequences of distinct instants in the model. By adding both **Grz** and $\mathbf{Grz_l}$ in the construction of a model, we would expect to rule out both ascending and descending sequences of distinct instants. However, the interacting of both future and past instant being added in the construction could possibly create interference which could lead to infinitely descending sequences of distinct instants in the constructed model. We will address this in the proof of the theorem below. Therefore, we have to be very careful in the selection process of instants to ensure that this does not happen.

Theorem 5.13.1. $\mathbf{Pr}^r_{fin}$ *is sound and weakly complete with respect to the class of finite, reflexive trees.*

Proof. Soundness follows from the fact that the axioms for the logic are valid on finite reflexive trees.

Let α be a formula that is not a theorem of $\mathbf{Pr}^r_{fin}$. We will apply a selective filtration argument to the canonical model $\mathcal{M} = (W, R, V)$ for $\mathbf{Pr}^r_{fin}$ through $Cl(\alpha)$ to build a finite reflexive tree on which α is refuted.

Since α is not a theorem of $\mathbf{Pr}^r_{fin}$, there exists an instant $w_0 \in W$ such that $\mathcal{M}, w_0 \nVdash \alpha$. We will use w_0 as the starting point for building the required model for $\mathbf{Pr}^r_{fin}$ that will refute α by systematically adding more reflexive instants to the model until a truth lemma relativised to $Cl(\alpha)$ holds.

We define a sequence of models $\mathcal{M}_n$ inductively as follows:

Set $W_0 = \{w_0\}$ and $\mathcal{M}_0 = (W_0, (w_0, w_0), V_0)$, where $V_0(p) = V(p) \cap W_0$ for all p. Note that $\mathcal{M}_0$ is a model of the right kind, i.e., a finite reflexive tree.

Next, suppose $W_n = \{w_0, w_1, ..., w_m\}$ for some m, and R_n and V_n have already been constructed.

Let $\Phi^{\mathbf{G}}_w = \{\mathbf{G}\psi \in Cl(\alpha) \mid \mathbf{G}\psi \notin w \text{ and } \psi \in w, \text{ and for no } w' \in W_n \text{ with } R_n ww' \text{ is } \neg\psi \in w'\}$ and
$\Phi^{\mathbf{H}}_v = \{\mathbf{H}\psi \in Cl(\alpha) \mid \mathbf{H}\psi \notin v \text{ and } \psi \in v \text{ and for no } v' \in W_n \text{ with } R_n v'v \text{ is } \neg\psi \in v'\}$.

Thus $\Phi^{\mathbf{G}}_w$ and $\Phi^{\mathbf{H}}_v$ are the sets of requirements for witnessing instants at the current stage in the process of selective filtration. We will refer to the elements of $\Phi^{\mathbf{G}}_w$ and $\Phi^{\mathbf{H}}_v$ as defects in general.[8] More specifically, $\mathbf{G}\psi \in \Phi^{\mathbf{G}}_w$ is called a **G** defect at w and $\mathbf{H}\psi \in \Phi^{\mathbf{H}}_v$ is called a **H** defect at v.

If $\Phi^{\mathbf{G}}_w = \emptyset$ and $\Phi^{\mathbf{H}}_v = \emptyset$ for all $w, v \in W_n$, then let $\mathcal{M}_n = (W_n, R_n, V_n)$ where $V_n(p) = V(p) \cap W_n$. Then $\mathcal{M}_n$ is the required model, since all **G** and **H** formulas that should not be satisfied at a given instant have instants witnessing this, and since $V_n(p) = V(p) \cap W_n$ it follows, by a straight-forward induction on $\varphi \in Cl(\alpha)$, that $\varphi \in w$ iff $\mathcal{M}_n, w \Vdash \varphi$.

Otherwise, we will start by adding the witnessing instants for $\Phi^{\mathbf{G}}_{w_k}$ where w_k is such that k is the lowest index of all elements in W_n for which $\Phi^{\mathbf{G}}_{w_k} \neq \emptyset$. Now suppose $\Phi^{\mathbf{G}}_{w_k} = \left\{\mathbf{G}\psi_j \mid j = 1, 2, ..., s\right\}$. Then for $\mathbf{G}\psi_1 \in \Phi^{\mathbf{G}}_{w_k}$ we use Lemma 3.2.11 to select an instant w_{m+1} with $Rw_k w_{m+1}$ such that $\varphi \notin w_{m+1}$, and for no $u \in W$ with $Rw_{m+1}u$, do we have $u \cong_{Cl(\alpha)} w_k$. Then let $\mathcal{M}_{n+1} = (W_{n+1}, R_{n+1}, V_{n+1})$ where $W_{n+1} = W_n \cup \{w_{m+1}\}$, R_{n+1} is the reflexive transitive closure of $R_n \cup \{(w_k, w_{m+1})\}$, and $V_{n+1}(p) = V(p) \cap W_{n+1}$. That is, to repair a **G** defect, we add a leaf. The new instant w_{m+1} added is only a successor of the instant w_k and not related to any previously added instants before taking the reflexive transitive closure of the relation.

If there is no such k of lowest index such that $\Phi^{\mathbf{G}}_{w_k} \neq \emptyset$, select the lowest index of all elements in W_n, say l, for which $\Phi^{\mathbf{H}}_{v_l} \neq \emptyset$ and suppose $\Phi^{\mathbf{H}}_{v_l} = \left\{\mathbf{H}\psi_j \mid j = 1, 2, ..., m\right\}$. Then for $\mathbf{H}\psi_1 \in \Phi^{\mathbf{H}}_{v_l}$, use Lemma 5.9.1 to select an instant w_{m+1} with $Rw_{m+1}v_l$ such that $\varphi \notin w_{m+1}$, and for no $u \in W$ with Ruw_{m+1}, do we have $u \cong_{Cl(\alpha)} v_l$. Then let $\mathcal{M}_{n+1} = (W_{n+1}, R_{n+1}, V_{n+1})$ where $W_{n+1} = W_n \cup \{w_{m+1}\}$, $V_{n+1}(p) = V(p) \cap W_{n+1}$ and R_{n+1} is defined as follows:

[8] Note that, in the proof of Theorem 5.12.1, we referred to the elements of similar sets as requirements. Since the process to show that we are building a finite model is simpler in Theorem 5.12.1, we could be less strict with the way we defined the sets $\Phi^{\mathbf{G}}_w$ and $\Phi^{\mathbf{H}}_v$, and these sets could contain formulas that are not defects. This is not the case in this proof and hence, we refer to the elements of these sets as defects.

Since the canonical model is left linear, w_{m+1} will be in the linear past of v_l in some cluster. We need to ensure that we slot w_{m+1} in at the right spot in the new model, using its placement in the canonical model. Let $\{x_i \mid i = 1, 2, ..., r\}$ be the set of instants in W_n with $R_n x_i v_l$ and $R w_{m+1} x_i$. Select the $x_j \in \{x_i \mid i = 1, 2, ..., r\}$ for which $R_n x_j x_i$ for all i. This instant exists since $\mathcal{M}_n$ is a tree and thus left-linear. Then let R_{n+1} be the reflexive transitive closure of $R_n \cup (w_{m+1}, x_j) \cup \{(x, w_{m+1}) \mid x \in R_n^{-1}[x_j]\}$. That is, to repair **H** defects, we slot the new instant in directly below the earliest (in the currently constructed relation) predecessor of the instant where we are repairing, which has (in the canonical model) the new instant as a predecessor. This way of constructing the relation preserves the left linearity of the model.

Repeat the process of selecting instants. Let $\mathcal{M}^* = (W^*, R^*, V^*)$ where $W^* = \bigcup_{i=0}^{\infty} W_i$, $R^* = \bigcup_{i=0}^{\infty} R_i$, and $V^*(p) = V(p) \cap W^*$.

Notice that in the selection process and construction of the relation, no proper clusters can be created. To see, this recall that all **G** defects are repaired by adding leaves, hence even if an instant w is added as a successor of an instant v and w and v are in the same cluster in the canonical model, it will be added as a "new" instant as a successor of v only. Also, in the construction of the relation when repairing **H** defects, we ensured that the past is linear and that no proper clusters are created. This means that no bulldozing or untangling (which leads to an infinite model) is necessary to ensure that the model is a tree.

Next we show that W^* is finite:

We will use a consequence of König's Lemma [8] that states that every infinite tree contains either an instant with infinitely many immediate successors, or an infinite path of distinct instants, to show that the selection process will terminate.

For any instant $v \in W^*$ define $T(v)$ as the step in the construction in which it was added, i.e., the smallest $i \in \mathbb{N}$ such that $v \in W_i$.

We will show that $\mathcal{M}^*$ contains no infinite branches. Suppose, to the contrary, that it does contain a branch on which there are infinitely many points. This means that there is an infinitely descending or ascending sequence of distinct instants in $\mathcal{M}^*$, i.e., a sequence of instants $v_1, v_2, v_3, \ldots$ such that $R^* v_i v_{i+1}$ for all $i \in \mathbb{N}$ (ascending) or $R^* v_{i+1} v_i$ for all $i \in \mathbb{N}$ (descending).

In the ascending case, we claim that there is a subsequence $u'_1, u'_2, u'_3, \ldots$ of $v_1, v_2, v_3, \ldots$ such that for all $i, j \in \mathbb{N}$, if $i < j$ then $T(u'_i) < T(u'_j)$. To construct this sequence, let $u'_1 = v_1$. Suppose that $u'_1, \ldots, u'_n$ have already been chosen. Note that, since the sets $\{v_i \mid T(v_i) \leq T(u'_n)\}$ and $((R^*)^{-1}[u'_n] \cap \{v_i \mid i \in I\})$ are finite, the set $\{v_i \mid T(v_i) > T(u'_n)\} - ((R^*)^{-1}[u'_n] \cap \{v_i \mid i \in I\})$ is non-empty (and infinite). Let u'_{n+1} be the element v_j in the latter set with least index j.

Next, we claim that only finitely many of the elements among $\{u'_1, u'_2, \ldots\}$ were added to repair **H**-defects. We use the fact that if u'_j was added to repair an **H**-defect involving $\neg\mathbf{H}\psi$, then u'_l, with $l > j$, would not have been added to repair a defect involving $\neg\mathbf{H}\psi$. Since $Cl(\alpha)$ is finite, it follows that at most finitely many of the elements of $\{u'_1, u'_2, \ldots\}$ were added to repair **H**-defects. So we have a subsequence $\{u_1, u_2, u_3, \ldots\}$ such that all were added to repair **G**-defects and $T(u_i) < T(u_j)$ for all $i < j$.

Next, we argue that every $Cl(\alpha)$-type only occurs at most finitely many times in the sequence $u_1, u_2, \ldots$, and hence, since there are finitely many $Cl(\alpha)$-types this is a contradiction with the infinitude of $u_1, u_2, \ldots$. Indeed, suppose we have an infinite subsequence $u''_1, u''_2, \ldots$ of $u_1, u_2, \ldots$ such that $Cl(\alpha)(u''_j) = Cl(\alpha)(u''_k)$ for all $j, k \in \mathbb{N}$. Then, for all u''_j, if u''_j was added to repair a defect at an instant z_j then, by the way we select, there must be an instant z_{j+1} such that $u''_j R^* z_{j+1} R^* u''_{j+1}$ and $Cl(\alpha)(z_{j+1}) \neq Cl(\alpha)(z_k)$ for all $k < j + 1$ (by the fact that we used Lemma 5.10.1 in the construction). Therefore, we can create an infinitely ascending sequence of instant that are all of different $Cl(\alpha)$-types, which is impossible.

Hence, no infinitely ascending sequences of distinct instants can be created in the construction of the model.

A symmetrical argument using Lemma 5.9.1 can be used to show that no branch containing an infinitely descending sequence of distinct instants can be created in the construction of the model.

Therefore, the number of instants added to repair **G** defects and the number of instants added to repair **H** defects is finite and hence we cannot add infinitely many instants on a branch. Therefore, every branch in $\mathcal{M}^*$ is finite.

Lastly, we show that we cannot have any instants with infinitely many immediate successors in the model. Let $w \in W_n$ for some n. The only ways that an immediate successor can be added to w is to either select an instant to repair a **G** defect at w, or at one of its predecessors in $\mathcal{M}_n$. Since the model has finite branches as shown above, w has only finitely many predecessors. Furthermore, notice that at any given $w \in W_n$ the set $\Phi_w^{\mathbf{G}}$ is finite. Hence, whether the immediate successor was added to repair a defect at w or any of its predecessors, only finitely many immediate successors of w can be added, and hence the result follows.

Therefore, each branch of $\mathcal{M}^*$ is finite, and every instant in the model has only finitely many immediate successors. Therefore, by König's Lemma, the process will terminate and hence the model will be finite.

Notice that $\mathcal{M}^*$ is left linear by the canonicity of axiom $\mathbf{3}_l$ and by the method of repairing **H** defects in the construction. It is also connected, transitive, reflexive and finite by construction, with no proper non-degenerate clusters. Hence, it is the required model. Furthermore, since $w_0 \in W_r$, it also refutes α. Therefore, $\mathbf{Pr}^r_{fin}$ is weakly complete with respect to the class of finite, reflexive trees. □

This concludes our investigation of the axiomatisations of the classes of trees in the Priorian temporal language. The next chapter will investigate decidability of the logics we used to axiomatise the classes of trees in the Priorian language.

Chapter 6

Decidability of Priorian Temporal Logics of Trees

The overarching concern of this thesis is the study of the Priorian theories of classes of trees. The foregoing chapters have focused on the finite axiomatisations of these theories. However, this does not yet give us decision procedures for these theories and hence, this question will be the focus of this chapter.

We will use three methods to show decidability of the logics of Chapters 3, 4, and 5. We start with the finite model property and Harrop's theorem in Section 6.1. Most of the logics of trees have the finite model property and it can be easily shown, albeit not with respect to the classes of trees, but with respect to the Kripke frames. However, in Proposition 6.1.9, we will show that some logics of trees do not have the finite model property. We cover some of these logics in Section 6.2. Lastly, we use conservative extensions in Section 6.3 for the logics in the future fragment of the Priorian temporal language that contain the density axiom.

6.1 Decidability via the Finite Model Property

The first method we will use to show that logics are decidable uses the finite model property (see e.g. [15]) and Harrop's Theorem.

Definition 6.1.1. A logic Λ has the **finite model property** if there is a class of finite frames $\mathcal{C}$ such that $\Lambda = \{\varphi \mid \forall \mathcal{F} \in \mathcal{C}, \mathcal{F} \Vdash \varphi\}$.

Below is the version of Harrop's Theorem that will be used in this chapter.

Theorem 6.1.2 (Harrop's Theorem)**.** *If Λ is a finitely axiomatisable normal logic with the finite model property, then Λ is decidable.*

A proof for this theorem can be found in [4] Theorem 6.15.

Therefore, if a logic Λ is sound and complete with respect to a class of finite frames, i.e., Λ has the finite model property, we can apply Harrop's theorem ([41]) to conclude decidability of Λ. Our first approach will therefore be to find a class of finite frames for which the logics of the respective classes of trees are sound and complete, then use Harrop's theorem to conclude decidability of the logic. Note that the class of finite frames we find does not have to be a class of trees. As long as our finitely axiomatisable logic is sound and complete with respect to a given class of finite frames, it will be decidable. For example, $\mathbf{Pr}_{basic}$ defines the class of transitive, left linear frames, and hence, given a $\mathbf{Pr}_{basic}$-consistent formula/set of formulas, we only need to build a finite model for this logic that is transitive and left linear, and on which the formula/set of formulas are satisfied. This will be enough to conclude decidability of $\mathbf{Pr}_{basic}$.

We start our investigation with the logics in the future fragment of the Priorian temporal language in the following section.

6.1.1 The Future Fragment of the Priorian Temporal Logics

We already have completeness results for finite frames with respect to classes of $\mathbf{KL_r}$ and **KGrz**, as they are sound and weakly complete with respect to the class of finite irreflexive trees and finite reflexive trees respectively (see Section 3.1.3 and 3.2.2). Hence, these logics have the finite model property. It was also shown in [62] that these logics are PSPACE decidable in the polymodal case. Hence, we have the following result.

Proposition 6.1.3. $\mathbf{KL_r}$ *and* **KGrz** *have the finite model property.*

It has been shown (see e.g. [4]) that if any formula φ in the future fragment of the Priorian language (defined in Chapter 2) is satisfiable in a model, then φ is satisfiable on a finite model which may be obtained through filtration. However, not all properties of models are preserved through filtrations, which means that the filtered model might not be a model in the desired class. In fact, properties preserved under surjective homomorphisms will be preserved under a filtration (see e.g [4]). Such properties include reflexivity and unboundedness, but exclude properties such as transitivity and symmetry (see e.g [4]). To preserve transitivity, we can use the transitive filtration, and end up with a finite model for the logic that is also transitive. Using these concepts, we have the following result.

Proposition 6.1.4. $\mathbf{K4U_r}$ *and* $\mathbf{S4U_r}$ *have the finite model property.*

Proof. By using the transitive filtration on the canonical model of the logics $\mathbf{K4U_r}$ and $\mathbf{S4U_r}$, we get a finite model based on a frame that validates the axioms, i.e., the properties that the logics for these classes define, and on which all formulas consistent with the logics can be satisfied. Therefore, these logics have the finite model property. □

Decidability of **K4** and **S4** has already been proved (see e.g. [45] where Ladner first established the existence of PSPACE algorithms to decide whether a formula is a theorem of these logics or not, and [40]). The method of using the transitive filtration (also know as the Lemmon filtration) can also be used to show decidability of these logics [4]. These results, as well as the results in Propositions 6.1.3 and 6.1.4, are summarised in Theorem 6.1.5 below and in Table 6.1.1 (see 8 for the list of axioms and rules).

Theorem 6.1.5. *All logics in the future fragment of the Priorian temporal language for the classes of trees listed in Table 6.1.1 are decidable.*

Proof. As shown above, these logics all have the finite model property and by Harrop's theorem are decidable. □

Decidability of the logics for the various classes of irreflexive trees that contain the density axiom will be considered in Section 6.3, since it is difficult to show that these logics have the finite model property. In the linear case, density of the filtration could be achieved by falling back on the density of the original model, which easily gives the finite model property. However, since we allow branching, this will not work and we need to use a different approach. We could have used the transitive closure of the minimal filtration to show the finite model property for these logics, and have derived decidability this way. However, decidability will follow from the fact that the logics that contain the density axiom are conservative extensions of the corresponding temporal logics, from which decidability also follows. This is done in Section 6.3. But first, we look at logics in the Priorean temporal language that have the finite model property.

Logic	Class of Kripke frames w.r.t. the logic has the FMP	Class of trees w.r.t. which logic is complete
K4	Finite, transitive frames	Irreflexive trees (3.1.1), rooted irreflexive trees (3.1.2), discrete irreflexive trees (3.1.3), locally finite irreflexive trees (3.1.4), well-founded irreflexive trees (3.1.5), left-unbounded irreflexive trees (3.1.10)
$\mathbf{K4U_r}$	Finite, right-unbounded transitive frames	Right-unbounded irreflexive trees (3.1.6), unbounded irreflexive trees (3.1.11), irreflexive trees with branches isomorphic to the natural numbers (3.1.8) and to the integers (3.1.12)
$\mathbf{KL_r}$	Finite, transitive, conversely well-founded frames (and hence irreflexive)	Finite irreflexive trees (weak completeness) (3.1.16), conversely well-founded irreflexive trees (weak completeness) (3.1.17)
S4	Finite, transitive, reflexive frames	Reflexive trees (3.2.1), discrete reflexive trees (3.2.3), locally finite reflexive trees (3.2.4), rooted reflexive trees (3.2.2), left unbounded reflexive trees (3.2.6), well-founded reflexive trees (3.2.5), right unbounded reflexive trees (3.2.7), unbounded reflexive trees (3.2.7), reflexive trees with branches isomorphic to $(\mathbb{N}, \leq)$ (3.2.8), reflexive trees with branches isomorphic to $(\mathbb{Z}, \leq)$ (3.2.8), left unbounded dense reflexive trees (3.2.18), right unbounded dense trees (3.2.19), unbounded dense trees (3.2.19), reflexive trees with branches isomorphic to $\langle\mathbb{Q}), \leq\rangle$ (3.2.19), and $\langle\mathbb{Q}^+ \cup \{0\}, \leq\rangle$ (3.2.19).
KGrz	Finite, transitive, conversely well-founded reflexive frames	Finite reflexive trees (weak completeness) (3.2.14), conversely well-founded reflexive trees (weak completeness) (3.2.15)

Table 6.1: Logics in the future fragment of the Priorian temporal language with the Finite Model Property

6.1.2 Priorian Temporal Logics

Similar to the case for the logics in the future fragment of the Priorean temporal language, $\mathbf{Pr}_{fin}$ and $\mathbf{Pr}^r_{fin}$ are sound and complete with respect to the classes of finite irreflexive and reflexive trees, respectively (see Theorems 5.12.1 and 5.13.1 and are hence have the finite model property. Hence, we have the following result as a corollary.

Corollary 6.1.6. *$\mathbf{Pr}_{fin}$ and $\mathbf{Pr}^r_{fin}$ have the finite model property.*

To use the finite model property for other logics in the Priorean temporal language, we need to identify suitable classes of finite frames which validate the axioms of the logics and on which all formulas consistent with the logics can be satisfied. For example, the class of finite frames for $\mathbf{Pr}_{unbnd}$ must be transitive, left linear and unbounded. Some of these properties are preserved by filtrations, e.g., all filtrations preserve reflexivity and unboundedness. However, left linearity is not preserved under filtrations, as seen in Example 5.2.2. Since all the logics contain axiom **.3**, we need to come up with a strategy to turn a finite model into a left linear finite model. Fortunately, we have already done so in Section 5.2. As seen in this section, the model after using the minimal filtration and then unfolding is finite, as we have seen in Corollary 5.2.12. Therefore, given any non-theorem of any extension of $\mathbf{Pr}_{basic}$, we can take the canonical model of this logic, generate a submodel from an mcs containing the non-theorem, then use the minimal filtration and taking the transitive closure of the relation of the filtration, and the resulting model will be a finite transitive model for the logic. However, this model might not be left linear, but after unfolding the model, we will end up with a finite model (see Corollary 5.2.12), transitive, left linear model, as we have done in Section 5.2. Notice that the process works for an irreflexive and reflexive relation. In the reflexive case, we just take the transitive and reflexive closure of the relation of the filtration. Therefore, we only have to ensure that the unfolded model is of the right type - depending on which axioms were in the extension of $\mathbf{Pr}_{basic}$. Also, since we are not interested in building a tree, we do not need to bulldoze the clusters which will ensure that the model stays finite to meet the requirement of the finite model property. We do this in the proof of the following proposition and summarise the results of Propositions 6.1.6 and Theorem 6.1.7 in Table 6.2 (See 8 for the list of axioms and rules).

Theorem 6.1.7. *All logics in the Priorian temporal language for the classes of trees in Table 6.2 have the finite model property.*

Proof. Let Λ be any of the logics listed in Table 6.2 and α a non-theorem of Λ. Let $\mathcal{M}$ be the submodel of the canonical model of Λ generated from any mcs containing $\neg\alpha$. Let $\mathcal{M}''$ be be the result of refining and unfolding the transitive filtration of $\mathcal{M}$, as described in Section 5.2. Then, by Lemma 5.2.12, $\mathcal{M}''$ is finite, transitive and left linear and refutes α. We need to confirm that the frame $\mathcal{F}''$ upon which $\mathcal{M}''$ is based validates the axioms of Λ.

As noted, $\mathcal{F}''$,is transitive and left linear and hence validates the axioms of $\mathbf{Pr}_{basic}$. If $\mathbf{T} \in \Lambda$ (respectively, $\mathbf{U_l} \in \Lambda$, $\mathbf{U_r} \in \Lambda$) it follows that $\mathcal{F}''$ will be reflexive (respectively, left unbounded as a Kripke frame, right unbounded as a Kripke frame) as these properties are easily seen to be preserved by filtration, refinement and unfolding.

Next, we argue that if $\mathbf{D} \in \Lambda$, then $\mathcal{F}''$ is dense. To that end, suppose $(\sigma, [w]), (\rho, [v]) \in W''$ and $R''(\sigma, [w])(\rho, [v])$. If $C([w])$ is non-degenerate, then $R''(\sigma, [w])(\sigma, [w])$ and $R''(\sigma, [w])(\rho, [v])$ and so $(\sigma, [w])$ is the required intermediate instant. Similarly if $C([v])$ is non-degenerate. So suppose that both $C([w])$ and $C([v])$ are degenerate. If there is a cluster on the path ρ strictly between $C([w]) = last(\sigma)$ and $C([v])$ we are done. So suppose that $C([v])$ is an immediate $R^r_{\sim}$-successor of $C([w])$. Since $R''(\sigma, [w])(\rho, [v])$ the definition of R'' implies that $R^r[w][v]$. Hence, since $[w]$ and $[v]$ are the only elements in their respective clusters, the definition of R^r implies that there must be $w' \in [w]$ and $v' \in [v]$ such that $Rw'v'$. By the density of R there is a $u \in W$ such that $Rw'u$ and Ruv'. Hence $R^r[w][u]$ and $R^r[u][v]$ and so $R^r_{\sim}C([w])C([u])$ and $R^r_{\sim}C([u])C([v])$. Moreover, the degeneracy of $C([w])$ and $C([v])$ implies

Logic	Class of Kripke frames w.r.t. the logic has the FMP	Class of trees w.r.t. which logic is complete
$\mathbf{Pr}_{basic}$	Finite, transitive, left linear frames	Irreflexive trees (4.2.2)
$\mathbf{Pr}_{basic}\mathbf{U_l}$	Finite, transitive, left linear, left unbounded frames	Left unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{basic}\mathbf{U_r}$	Finite, transitive, left linear, right unbounded frames	Right unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{unbnd}$	Finite, transitive, left linear, unbounded frames	Unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{dense}$	Finite, transitive, left linear, dense frames	Dense irreflexive trees (4.5.4)
$\mathbf{Pr}_{dense}\mathbf{U_l}$	Finite, transitive, left linear, left unbound, dense frames	Left unbounded dense irreflexive trees (4.7.1)
$\mathbf{Pr}_{dense}\mathbf{U_r}$	Finite, transitive, left linear, right unbound, dense frames	Right unbounded dense irreflexive trees (4.7.1)
$\mathbf{Pr}_{\mathbb{Q}}$	Finite, transitive, left linear, dense, unbounded frames	Irreflexive trees with branches isomorphic to the rational numbers (4.8.1)
$\mathbf{Pr}_{discr}$	Finite, transitive, left linear frames	Discrete irreflexive trees (5.1.11)
$\mathbf{Pr}_{discr}\mathbf{U_l}$	Finite, transitive, left linear, left unbounded frames	Left unbounded discrete irreflexive trees (weak completeness) (5.3.1)
$\mathbf{Pr}_{discr}\mathbf{U_r}$	Finite, transitive, left linear, right unbounded frames	Right unbounded discrete irreflexive trees (weak completeness) (5.3.1)
$\mathbf{Pr}^{r}_{udisc}$	Finite, transitive, left linear, unbounded frames	Unbounded, discrete irreflexive trees (5.3.1)
$\mathbf{Pr}_{basic}\mathbf{T}$	Finite, transitive, left linear, reflexive frames	Reflexive trees (4.3.2), discrete reflexive trees (weak completeness) (5.2.14), unbounded discrete reflexive trees (5.3.2), left unbounded reflexive trees (4.4.2), right unbounded reflexive trees (4.4.2), unbounded reflexive trees (4.4.2), dense reflexive trees (4.6.3), left unbounded dense irreflexive trees (4.7.1), right unbounded dense irreflexive trees (4.7.1), irreflexive trees with branches isomorphic to the rational numbers (4.8.1)

Table 6.2: Temporal Logics with the Finite Model Property

that $C([u]) \neq C([w])$ and $C([u]) \neq C([v])$. But this contradicts the assumption that that $C([v])$ is an immediate $R^{r}_{\sim}$-successor of $C([w])$. Hence $\mathcal{F}''$ is dense as a Kripke frame.

Therefore, all the extensions of $\mathbf{Pr}_{basic}$ in Table 6.2 have the finite model property. □

Hence, we have the following corollary.

Corollary 6.1.8. *The logics in the Priorian temporal language for the classes of trees in Table 6.2 are decidable.*

This leaves us with all the classes of well-founded and conversely well-founded (ir)reflexive trees, including (ir)reflexive trees with branched isomorphic to the natural numbers, as well as all the classes of locally finite (ir)reflexive trees, including (ir)reflexive trees with branched isomorphic to the integers. In the modal case decidability of the logics for these classes (**KL** and **KGrz**) is well established. However, in the temporal case, decidability is still unknown as far as the author knows. We can show that these logics do not have the finite model property, as done in the proposition below.

Proposition 6.1.9. *The following logics do not have the finite model property:* $\mathbf{Pr}_{wf}$*,* $\mathbf{Pr}^r_{wf}$*,* $\mathbf{Pr}_{cwf}$*,* $\mathbf{Pr}^r_{cwf}$*,* $\mathbf{Pr}_{lfin}$*,* $\mathbf{Pr}_{lfin}\mathbf{U_l}$*,* $\mathbf{Pr}_{lfin}\mathbf{U_r}$*,* $\mathbf{Pr}_{\mathbb{Z}}$ *and* $\mathbf{Pr}^r_{lfin}$*.*

Proof. We start with the case for $\mathbf{Pr}_{wf}$, and $\mathbf{Pr}^r_{wf}$ will follow similarly. Note that any frame for these two logics is well-founded, transitive and asymmetric.

Consider the formula $\varphi : \neg(\mathbf{F}p \wedge \mathbf{G}(p \rightarrow \mathbf{F}\neg p) \wedge \mathbf{G}(\neg p \rightarrow \mathbf{F}p))$. We first show that this formula is not a theorem for this logic by finding a model based on a frame for $\mathbf{Pr}_{wf}$ where $\neg\varphi$ is satisfied. Let $\mathcal{M} = (W, R, V)$ where $W = \{0, 1, 2, ...\}$, and let R be the strict ordering of the natural numbers. Let $V(p) = \{i \mid i \text{ is even}\}$. Then $\mathcal{M}$ is well-founded and transitive, and hence a model based on a frame for $\mathbf{Pr}_{wf}$. Also, we have that $\mathcal{M}, 0 \Vdash \neg\varphi$.

However, it is easy to see that φ is valid on any finite well-founded, transitive frame of $\mathbf{Pr}_{wf}$, i.e. any finite frame for $\mathbf{Pr}_{wf}$, since we need an infinite sequence of instants to make $\neg\varphi$ true. Hence, $\mathbf{Pr}_{wf}$ is not identical with the validities of its finite frames, i.e. it lacks the finite model property.

The same argument using the formula φ but letting R be the reflexive ordering of the natural numbers, will work to show that $\mathbf{Pr}^r_{wf}$ also lacks the finite model property.

Next we show that $\mathbf{Pr}_{lfin}$, $\mathbf{Pr}_{lfin}\mathbf{U_l}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$ and $\mathbf{Pr}_{\mathbb{Z}}$ do not have the finite model property, and the case for $\mathbf{Pr}^r_{lfin}$ will follow similarly with a reflexive relation.

Let $\mathcal{M}' = (W', R', V')$ where $W' = \{..., 0, 1, 2, ...\}$, and where R is the strict ordering of the integers. Let $V'(p) = \{i \mid i \text{ is even}\}$. Then $\mathcal{M}'$ is locally finite and transitive, and hence a model based on a frame for$\mathbf{Pr}_{lfin}$, $\mathbf{Pr}_{lfin}\mathbf{U_l}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$ and $\mathbf{Pr}_{\mathbb{Z}}$. Also, we have that $\mathcal{M}, 0 \Vdash \neg\varphi$. However, on any finite, transitive asymmetric frame for $\mathbf{Pr}_{lfin}$ $\mathbf{Pr}_{lfin}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$ and $\mathbf{Pr}_{\mathbb{Z}}$, φ is valid. Hence again, $\mathbf{Pr}_{lfin}$ $\mathbf{Pr}_{lfin}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$ and $\mathbf{Pr}_{\mathbb{Z}}$ are not identical with the validities of their finite frames, i.e. they lack the finite model property.

The same argument using the formula φ but letting R be the reflexive ordering of the natural numbers, will work to show that $\mathbf{Pr}^r_{lfin}$ also lacks the finite model property.

Using the formula $\neg(\mathbf{P}p \wedge \mathbf{H}(p \rightarrow \mathbf{P}\neg p) \wedge \mathbf{H}(\neg p \rightarrow \mathbf{P}p))$, we can use a symmetric argument to show that $\mathbf{Pr}_{cwf}$ (with R the strict linear ordering of the natural numbers) and $\mathbf{Pr}^r_{cwf}$ (with R the reflexive linear ordering of the natural numbers) also lack the finite model property. □

In the next section, we investigate the decidability of some of the logics mentioned in Proposition 6.1.9.

6.2 Decidability via Mosaics

The use of quasi-models and mosaics stem from work by the authors of the papers [72], [71], amongst others. The terminology used in here mostly follows that in Section 6.4. in [4]. The notions and constructions we introduce in this section are inspired by those described in Section 6.4. in [4], where mosaics are used to prove decidability of the logic of the natural numbers. The branching nature of our models complicates these notions and constructions significantly. We will use a similar method here but will redefine most notions and constructions to our context. We will use a similar method here, but will adjust some definitions to suit the case for trees (partial orders). We start with the logic of class of irreflexive trees with branches isomorphic to the natural numbers and then we look at well-founded trees.

The semantics of the Priorian temporal language was given in Section 2.3.1. First, we add the definition of the $\wedge$ connective to the language, which will be use in this section in place of the $\vee$ connective: The truth of an arbitrary formula $\varphi \wedge \psi$ of $\mathcal{L}_{Prior}$ at an instant t in a model $\mathcal{M}$ is defined inductively as follows where $p \in Prop$: $\mathcal{M}, t \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, t \Vdash \varphi$ and $\mathcal{M}, t \Vdash \varphi$

In this section, we will use the operators $\mathbf{F}$ and $\mathbf{P}$ as primitive operators, as well as $\neg$, $\perp$ and $\wedge$ as primitive connectives.

Definition 6.2.1. Let Γ be a subformula closed set of formulas. A **Hintikka set** H over Γ is a maximal subset of Γ that satisfies the following conditions:

1. $\bot \notin H$
2. If $\neg\varphi \in \Gamma$, then $\neg\varphi \in H$ iff $\varphi \notin H$.
3. If $\varphi \wedge \psi \in \Gamma$, then $\varphi \wedge \psi \in H$ iff $\varphi \in H$ and $\psi \in H$.

Let $Cl(\varphi)$ denote the smallest set containing φ, $\mathbf{P}\top$ and $\mathbf{F}\top$, which is closed under subformulas and single negations. Let $|\varphi|$ denote the number of subformulas in φ. In Section 6.4 of [4] the authors used the concept of a brick for the linear case. We will adapt this concept to make it suitable for partial orders.

Definition 6.2.2. Let $\mathfrak{S}$ be the set of Hintikka sets. A **fan** is a pair $(\Phi, \mathfrak{S})$ such that Φ and each $\Psi \in \mathfrak{S}$ are Hintikka sets satisfying:

(B0) If $\mathbf{F}\varphi$ or φ belongs to Ψ for some $\Psi \in \mathfrak{S}$, then $\mathbf{F}\varphi \in \Phi$.

(B1) If $\mathbf{P}\varphi$ or φ belongs to Φ, then $\mathbf{P}\varphi \in \Psi$ for each $\Psi \in \mathfrak{S}$.

(B2) $\mathbf{F}\top \in \Psi$ for all $\Psi \in \mathfrak{S}$.

A fan $(\Phi, \mathfrak{S})$ is called **small** if it satisfies, in addition:

(B3) If $\mathbf{F}\varphi \in \Phi$, then $\varphi \in \Psi$ or $\mathbf{F}\varphi \in \Psi$ for some $\Psi \in \mathfrak{S}$.

(B4) If $\mathbf{P}\varphi \in \Psi$ for some $\Psi \in \mathfrak{S}$, then either $\varphi \in \Phi$ or $\mathbf{P}\varphi \in \Phi$.

We define a two terms to aid in the narrative of fans: The set Φ in a fan $(\Phi, \mathfrak{S})$ will be referred to as the **handle** of the fan, while the sets $\Psi \in \mathfrak{S}$ will be referred to as the **feathers** of the fan. The intuition behind the fan is that the handle contains the formulas in $Cl(\varphi)$ true at a state in a model, while the feathers contain the formulas in $Cl(\varphi)$ true at some of its successors. Intuitively, in a small fan, the feathers represent immediate successors of the handle.

What we are really interested in are sets of fans satisfying certain saturation conditions (designed specifically to define a set we will use to build a model that has branches isomorphic to $(\mathbb{N}, <)$). A set of fans $\mathcal{B}$ is a **saturated set of fans for** φ (in short: a φ-SSF) if it satisfies conditions S0, S1, S2, S3 and S4, below.

(S0) Let $\mathfrak{B}$ be the set of Hintikka sets occurring in fans in $\mathcal{B}$. Then, there is a unique Hintikka set Φ_r in $\mathfrak{B}$ such that $\neg\mathbf{P}\top \in \Phi_r$ and $\neg\mathbf{P}\psi \in \Phi_r$ for all $\mathbf{P}\psi \in Cl(\varphi)$. Moreover, $\varphi \in \Phi_r \cup \bigcup \mathfrak{S}$ for some fan $(\Phi_r, \mathfrak{S}) \in \mathcal{B}$. Intuitively, Φ_r is the handle of a fan representing the root of a model.

(S1) For all $(\Phi, \mathfrak{S}) \in \mathcal{B}$, if there is a $\Psi \in \mathfrak{S}$ with $\mathbf{F}\psi \in \Psi$, then there is a fan $(\Psi, \mathfrak{S}') \in \mathcal{B}$ and a $\Psi' \in \mathfrak{S}'$ such that $\psi \in \Psi'$.

For the next property of a SSF we need the concept of a **path of small fans** from Φ to Ψ: It is a sequence $(\Phi_0, \mathfrak{S}_0), ..., (\Phi_n, \mathfrak{S}_n)$, $n \geq 0$ of small fans such that $\Phi = \Phi_0$, $\Psi \in \mathfrak{S}_n$, and for all $i < n$ there exists a $\Psi_i \in \mathfrak{S}_i$ such that $\Psi_i = \Phi_{i+1}$.

(S2) For all $(\Phi, \mathfrak{S}) \in \mathcal{B}$ and all $\Psi \in \mathfrak{S}$ there is a path of small fans in $\mathcal{B}$ from Φ to Ψ.

(S3) For any path of small fans $(\Phi_0, \mathfrak{S}_0) \dots (\Phi_n, \mathfrak{S}_n)$ in $\mathcal{B}$ such that $\Phi_0 = \Phi_r$, if $\mathbf{P}\psi \in \Phi_i \cup \bigcup \mathfrak{S}_i$ for some $0 \leq i \leq n$, then $\psi \in \Phi_j$ for some $0 \leq j \leq i$.

(S4) For every Hintikka set Δ in a fan in $\mathcal{B}$ such that $\Delta \neq \Phi_r$, there is a fan $(\Phi_r, \mathfrak{S}')$ such that $\Delta \in \mathfrak{S}'$.

The **size of an SSF** $\mathcal{B}$ is the number of fans in $\mathcal{B}$. For any path of small fans $(\Phi_0, \mathfrak{S}_0), ..., (\Phi_n, \mathfrak{S}_n)$, the sequence $\Phi_0, \Phi_1, ..., \Phi_n, \Psi$, where $\Psi \in \mathfrak{S}_n$, will be referred to as the **spine** of the path.

The concepts of bricks and paths of small bricks were used to prove decidability of the logic of the natural numbers $(\mathbb{N}, <)$ in [4]. We will use the same method, but with fans and paths of small fans, to show decidability of $\mathbf{Pr}_{\mathbb{N}}$.[1]

Lemma 6.2.3. *Let φ be a $\mathbf{Pr}_{\mathbb{N}}$-consistent formula. Then there is a φ-SSF of size at most* $2^{2(|\varphi|+4)} \times 2^{2^{2(|\varphi|+4)}}$.

Proof. Let φ be a $\mathbf{Pr}_{\mathbb{N}}$-consistent formula. So, by the completeness of $\mathbf{Pr}_{\mathbb{N}}$ (see Theorem 5.11.2), φ is satisfiable on a tree model with branches isomorphic to $(\mathbb{N}, <)$. Specifically, let $\mathcal{M} = (W, R, V)$ be the model for $\mathbf{Pr}_{\mathbb{N}}$ built to satisfy $\neg\alpha$ using the methods in Section 5.11 and letting $\alpha = \neg\varphi$. Then φ is satisfied in $\mathcal{M}$. Also, recall that $\mathcal{M}$ has branches that are isomorphic to the natural numbers and each instant has only finitely many immediate successors. Furthermore, $\mathcal{M}$ has a root, say w_0.

For all $w \in W$, let Γ_w be defined as follows: $\Gamma_w = \{\psi \in Cl(\varphi) \mid \mathcal{M}, w \Vdash \psi\}$.

Next we define the notion of an **$\mathcal{M}$-fan** (Intuitively, a fan read off from the model $\mathcal{M}$. Indeed, as we will show below, an $\mathcal{M}$-fan is a fan.): A fan $(\Phi, \mathfrak{S})$ is an $\mathcal{M}$-fan if the following conditions are met:

1. $\Phi = \Gamma_w$ for some $w \in W$.
2. If $\mathfrak{S} = \{\Psi_1, ..., \Psi_n\}$, then there exist $v_1, ..., v_n \in W$ such that Rwv_i and $\Psi_i = \Gamma_{v_i}$ for $i = 1, ..., n$.
3. For each immediate successor w' of w there is a successor v in the subtree rooted at w' (possibly w' itself) and a $\Psi \in \mathfrak{S}$ such that $\Psi = \Gamma_v$.

We will show that the set of $\mathcal{M}$-fans $\mathcal{B}$ is a saturated set of fans for φ by showing that all the conditions of Definition 6.2.2 are met.

A **sequential $\mathcal{M}$-fan** is an $\mathcal{M}$-fan $(\Gamma_w, \mathfrak{S})$ such that there exists $w \in W$ such that $\Gamma = \Gamma_w$, and for all $\Delta \in \mathfrak{S}$, there is an immediate successor v of w such that $\Delta = \Gamma_v$.

First we show that the elements of $\mathcal{B}$ are fans. Let $(\Gamma_w, \mathfrak{S}) \in \mathcal{B}$.

(B0) Suppose $\mathbf{F}\psi$ or ψ, belongs to Γ_v for some $\Gamma_v \in \mathfrak{S}$, and hence, Rwv, with $\mathbf{F}\psi, \psi \in Cl(\varphi)$. Now, since $\mathbf{F}\psi$ or ψ belongs to Γ_v, there is an instant u with Rvu or $u = v$, and $\mathcal{M}, u \Vdash \psi$. By the transitivity of $\mathcal{M}$ it follows that Rwu and hence $\mathcal{M}, w \Vdash \mathbf{F}\psi$. Hence, by definition, we have that $\mathbf{F}\psi \in \Gamma_w$.

(B1) Suppose $\mathbf{P}\psi \in \Gamma_w$ or $\psi \in \Gamma_w$, for some $\mathbf{P}\psi \in Cl(\varphi)$. Now, since $\mathbf{P}\psi$ or ψ belongs to Γ_w, there is an instant u with Ruw or $u = w$, and $\mathcal{M}, u \Vdash \psi$. By the transitivity of $\mathcal{M}$ it follows that Ruv for each v such that $\Gamma_v \in \mathfrak{S}$, and hence $\mathcal{M}, v \Vdash \mathbf{P}\psi$. Hence, by definition, we have that $\mathbf{P}\psi \in \Gamma_v$.

(B2) This follows from the fact that $\mathcal{M}$ is right unbounded.

Next, we show that sequential $\mathcal{M}$-fans are small. Consider an arbitrary sequential $\mathcal{M}$-fan $(\Gamma_w, \mathfrak{S})$.

(B3) Suppose $\mathbf{F}\psi \in \Gamma_w$. Then, by definition, we have $\mathcal{M}, w \Vdash \mathbf{F}\psi$ and hence there is a u with Rwu such that $\mathcal{M}, u \Vdash \psi$. Now, by definition, there is a v on the same branch as u, that is an immediate successor of w, and $\Gamma_v \in \mathfrak{S}$. Now, if $u = v$ it follows that $\mathcal{M}, v \Vdash \psi$. If Rvu then $\mathcal{M}, v \Vdash \mathbf{F}\psi$. Hence, either $\mathbf{F}\psi$ or ψ is in Γ_v for $\Gamma_v \in \mathfrak{S}$, as required.

[1]We note that there are substantial similarities between the methods of networks and the methods of mosaics, e.g., both procedures constructs a model for a formula in a step-by-step way. However, while mosaics use finite Hintikka sets in this construction, networks label nodes with (generally infinite) mcs's. This implies that the set of possible brick (or fans, as used in the present chapter) is finite and this is essential to decide the existence of a saturated set of bricks/fans.

(B4) Suppose $\mathbf{P}\psi \in \Gamma_v$ for some $\Gamma_v \in \mathfrak{S}$. Then, by definition we have $\mathcal{M}, v \Vdash \mathbf{P}\psi$ and hence there is a u with Ruv such that $\mathcal{M}, u \Vdash \psi$. Now, by left linearity we have that either $u = w$ or Ruw. Now, if $u = w$ it follows that $\mathcal{M}, w \Vdash \psi$. If Ruw then $\mathcal{M}, w \Vdash \mathbf{F}\psi$. Hence either $\mathbf{P}\psi$ or ψ is in Γ_w as required.

Next we show that $\mathcal{B}$ is saturated for φ.

(S0) Let w_φ be the instant in $\mathcal{M}$ where φ is true. Now, if $w_\varphi = w_0$, consider the sequential $\mathcal{M}$-fan $(\Gamma_{w_0}, \mathfrak{S})$, where $\mathfrak{S} = \{\Gamma_v : v \text{ is an immediate successor of } w_0\}$, otherwise, if Rw_0w_φ consider the $\mathcal{M}$-fan $(\Gamma_{w_0}, \mathfrak{T})$, where $\mathfrak{T} = \{\Gamma_v : v \text{ is an immediate successor of } w_0 \text{ and } \neg(Rvw_\varphi \vee v = w_\varphi\}\cup\{\Gamma_{w_\varphi}\}$. In either case, it follows that $\neg\mathbf{P}\top \in \Gamma_{w_0}$ and $\neg\mathbf{P}\psi \in \Gamma_{w_0}$ for all $\mathbf{P}\psi \in Cl(\varphi)$ (since w_0 is the root of $\mathcal{M}$), and $\varphi \in \Gamma_{w_0} \cup \bigcup \mathfrak{S}$ or $\varphi \in \Gamma_{w_0} \cup \bigcup \mathfrak{T}$. Since all other instants $w \in W$ have predecessors, it also follows that Γ_{w_0} is the only Hintikka set occurring in an $\mathcal{M}$-fan having these properties.

(S1) Let $(\Gamma_w, \mathfrak{S}) \in \mathcal{B}$ and suppose $\mathbf{F}\psi \in \Gamma_v$ for some $\Gamma_v \in \mathfrak{S}$. Then, by definition we have $\mathcal{M}, v \Vdash \mathbf{F}\psi$ and hence there is a u with Rvu such that $\mathcal{M}, u \Vdash \psi$. Now consider any $\mathcal{M}$-fan of the form $(\Gamma_v, \mathfrak{T})$ such that $\Gamma_u \in \mathfrak{T}$. Then $\psi \in \Gamma_u$, as required.

(S2) Let $(\Gamma_w, \mathfrak{S}) \in \mathcal{B}$ and let Γ_v be an arbitrary element of $\mathfrak{S}$. Then, since the branches of $\mathcal{M}$ are isomorphic to the natural numbers, it follows that there is a path of immediate successors $w, u_1, u_2, ...u_n, v$ from w to v. Then $(\Gamma_w, \mathfrak{T}_1), (\Gamma_{u_1}, \mathfrak{T}_2), ..., (\Gamma_{u_n}, \mathfrak{T}_n)$ where each $\mathcal{M}$-fan in the path is sequential, is a path of small fans from Γ_w to any set in $\mathfrak{S}$.

(S3) Let $(\Gamma_{w_0}, \mathfrak{S}_0) \dots (\Gamma_{w_n}, \mathfrak{S}_n)$ be path of small fans such that $\neg\mathbf{P}\top \in \Gamma_{w_0}$, and suppose $\mathbf{P}\psi \in \Gamma_{w_i} \cup \bigcup \mathfrak{S}_i$ for some $0 \leq i \leq n$. Now since $\neg\mathbf{P}\top \in \Gamma_{w_0}$ it follows that $\mathbf{P}\psi \notin \Gamma_{w_0}$. Hence, there must be an instant v such that w_0RvRw_i or $v = w_0$, and $\mathcal{M}, v \Vdash \psi$, or $\psi \in \Gamma_{w_i}$. If $\psi \in \Gamma_{w_i}$ then we are done, since we have that ψ is in a Hintikka set of a fan on the path. Otherwise, we need to show that $(\Gamma_v, \mathfrak{S}_k)$, for some k with $0 \leq k < i$, is a fan in the path $(\Gamma_{w_0}, \mathfrak{S}_0) \dots (\Gamma_{w_n}, \mathfrak{S}_n)$. Now, for some $\Gamma \in \mathfrak{S}_{(i-1)}$ it follows from the definition of a path of small fans that $\Gamma_{w_i} = \Gamma$, and hence $\mathbf{P}\psi \in \Gamma$. But (B4) implies that either $\mathbf{P}\psi \in \Gamma_{w_{(i-1)}}$ or $\psi \in \Gamma_{w_{(i-1)}}$. If $\psi \in \Gamma_{w_{(i-1)}}$ we can let $v = w_{(i-1)}$ and we are done. If $\mathbf{P}\psi \in \Gamma_{w_{(i-1)}}$ we can argue similarly to get that either $\mathbf{P}\psi \in \Gamma_{w_{(i-2)}}$ or $\psi \in \Gamma_{w_{(i-2)}}$, and continuing we get that for all $0 \leq j < i$ we have $\mathbf{P}\psi \in \Gamma_{w_j}$ or $\psi \in \Gamma_{w_j}$. But $\mathbf{P}\psi \notin \Gamma_{w_0}$ and hence, we cannot have $\mathbf{P}\psi \in \Gamma_{w_j}$ for all $0 \leq j < i$, since at the least we must have $\psi \in \Gamma_{w_0}$, if not sooner. Hence, for some Γ_{w_k} we have $\psi \in \Gamma_{w_k}$ and we can let $v = w_k$.

(S4) Given any $\mathcal{M}$-fan $(\Gamma_w, \mathfrak{S})$ with $w \neq w_0$. Then, since $\mathcal{M}$ is a tree and w_0 is the root of the tree, it follows that Rw_0w. We can therefore take any $\mathcal{M}$-fan $(\Gamma_{w_0}, \mathfrak{T})$, where $w \in \mathfrak{T}$.

Furthermore, since the collection of sets $\{\Gamma_w \mid w \in W\}$ is a subset of the power set of $Cl(\varphi)$, it is of size at most $2^{2(|\varphi|+4)}$, where $2(|\varphi| + 4)$ is the size of $Cl(\varphi)$ (recall that $Cl(\varphi)$ is closed under single negations, hence multiplying by 2 and we added $\mathbf{P}\top$ and $\mathbf{F}\top$, hence adding the 4). Therefore, the number of possible $\mathcal{M}$-fans $(\Gamma_w, \mathfrak{S})$ will be bounded above by the number of ways to select a Γ_w multiplied by the number of ways we can select $\mathfrak{S}$. But then $\mathcal{B}$ is of size at most $2^{2(|\varphi|+4)} \times 2^{2^{2(|\varphi|+4)}}$. □

Lemma 6.2.4. *If there is a φ-SSF, then φ is $\mathbf{Pr}_{\mathbb{N}}$-consistent.*

Proof. Suppose $\mathcal{B}$ is a φ-SSF. We will use these fans to construct the required model for φ.

Let W be the set of all Hintikka sets occurring in members of $\mathcal{B}$ and define a relation $R \subseteq W \times W$ as follows: For all $\Phi, \Psi \in W$, $R\Phi\Psi$ iff there is a path of small fans from Φ to Ψ in $\mathcal{B}$ (notice that R, so defined, is transitive). Define a valuation V as follows: For all $\Phi \in W$, $\Phi \in V(p)$ iff $p \in \Phi$. Let $\mathcal{M} = (W, R, V)$.

Note that if $(\Phi, \mathfrak{S}) \in \mathcal{B}$, then, by (S2), there exists a path of small fans from Φ to all Ψ in $\mathfrak{S}$. Therefore, $R\Phi\Psi$ for all Ψ in $\mathfrak{S}$.

Next, we prove a Truth Lemma relative to $Cl(\varphi)$:

Claim 3. *For all $\psi \in Cl(\varphi)$ and all $\Phi \in W$, $\mathcal{M}, \Phi \Vdash \psi$ iff $\psi \in \Phi$.*

Proof. The base and boolean cases follow easily.

Suppose the claim holds for γ and suppose ψ is $\mathbf{F}\gamma$.

$\Rightarrow$: Suppose $\mathcal{M}, \Phi \Vdash \mathbf{F}\gamma$. Then there exists a $\Psi \in W$ such that $R\Phi\Psi$ and $\mathcal{M}, \Psi \Vdash \gamma$. Therefore, by the induction hypothesis, $\gamma \in \Psi$. By the definition of R, there is a path of small fans from Φ to Ψ. Then, by induction on the length of this path and using (B0), we have that $\mathbf{F}\gamma \in \Phi$.

$\Leftarrow$: Suppose $\mathbf{F}\gamma \in \Phi$ and let $(\Delta, \mathfrak{S})$ be any fan in $\mathcal{B}$ such that Φ is either the handle of this fan, or in the feathers of the fan. We consider these two cases separately:

Case 1: $\Phi = \Delta$. Then, by (S2), there is a path of small fans from Φ to any Hintikka set in $\mathfrak{S}$. Let $(\Phi, \mathfrak{S}_0)$ be the first small fan on this path. Then, by (B3), either γ or $\mathbf{F}\gamma$ is in Ψ for some $\Psi \in \mathfrak{S}_0$. If $\gamma \in \Psi$, then, by the induction hypothesis, $\mathcal{M}, \Psi \Vdash \gamma$. But $R\Phi\Psi$, and hence $\mathcal{M}, \Phi \Vdash \mathbf{F}\gamma$. If $\mathbf{F}\gamma \in \Psi$, then, by (S1), there is a fan $(\Psi, \mathfrak{S}')$ and a $\Psi' \in \mathfrak{S}'$ such that $\gamma \in \Psi'$. Then, by the induction hypothesis, $\mathcal{M}, \Psi' \Vdash \gamma$. But $R\Phi\Psi'$ by (S2) and the transitivity of R. Hence, $\mathcal{M}, \Phi \Vdash \mathbf{F}\gamma$.

Case 2: $\Phi \in \mathfrak{S}$. Then, we can again use (S1) and (S2) as in the argument in Case 1 to show that $\mathcal{M}, \Phi \Vdash \mathbf{F}\gamma$.

Suppose the claim holds for γ and suppose ψ is $\mathbf{P}\gamma$.

$\Rightarrow$: Suppose $\mathcal{M}, \Phi \Vdash \mathbf{P}\gamma$. Then there exists a $\Psi \in W$ such that $R\Psi\Phi$ and $\mathcal{M}, \Psi \Vdash \gamma$. Therefore, by the induction hypothesis, $\gamma \in \Psi$. By the definition of R, there is a path of small fans from Ψ to Φ. Then, by induction on the length of this path and using (B1), we have that $\mathbf{P}\gamma \in \Phi$.

$\Leftarrow$: Suppose $\mathbf{P}\gamma \in \Psi$. Then, by (S4), there is a fan $(\Phi_r, \mathfrak{S})$ in $\mathcal{B}$ such that $\Psi \in \mathfrak{S}$ where Φ_r is the Hintikka set specified by (S0). Then, by (S2), there is a path of small fans from Φ_r to Ψ, and by (S3), there is a Φ_i on the spine of this path such that $\gamma \in \Phi_i$. Then $\mathcal{M}, \Psi_i \Vdash \gamma$, by the induction hypothesis. But $R\Phi_i\Phi$, and hence $\mathcal{M}, \Psi \Vdash \mathbf{P}\gamma$.

Therefore, for all $\psi \in Cl(\varphi)$ and all $\Phi \in W$, $\mathcal{M}, \Phi \Vdash \psi$ iff $\psi \in \Phi$.

□

Notice that $\mathcal{M}$ is transitive with a single root Φ_r. We will next transform $\mathcal{M}$ into a left linear model by doing a one-step transitive forward unravelling of $\mathcal{M}$ as follows:

- Let $W^* = \{\langle \Phi_r, \Phi_0, ..., \Phi_n \rangle \mid \text{where } \Phi_r, \Phi_0, ..., \Phi_n \text{ is the spine of a path of small fans}\}$, i.e, $\Phi_r, \Phi_0, ..., \Phi_n$ is a path of immediate R-successors.
- For all $\sigma, \rho \in W^*$, $R^*\sigma\rho$ iff σ is a proper initial segment of ρ.
- Let $V^*(p) = \{\sigma : last(\sigma) \in V(p)\}$, where $last(\sigma)$ is the last Hintikka set in the sequence σ.

Let $\mathcal{M}^* = (W^*, R^*, V^*)$. Then $\mathcal{M}^*$ is transitive, locally finite, left linear and has a single root $\sigma_r = \langle \Phi_r \rangle$, and hence a locally finite tree. Furthermore, by (B2), each branch of $\mathcal{M}^*$ is right unbounded. Hence $\mathcal{M}^*$ has branches isomorphic to $(\mathbb{N}, <)$ (by Lemma 5.11.1).

Next, we prove that satisfaction is preserved at corresponding instants:

Claim 4. *For all $\psi \in Cl(\varphi)$ and all $\sigma \in W^*$, $\mathcal{M}^*, \sigma \Vdash \psi$ iff $\mathcal{M}, last(\sigma) \Vdash \psi$.*

Proof. The base and boolean cases follow easily.

Suppose the claim holds for γ and suppose ψ is $\mathbf{F}\gamma$.

$\Rightarrow$: Suppose $\mathcal{M}^*, \sigma \Vdash \mathbf{F}\gamma$. Then there exists a $\rho \in W^*$ such that $R^*\sigma\rho$ and $\mathcal{M}^*, \rho \Vdash \gamma$. Therefore, by the induction hypothesis, $\mathcal{M}, last(\rho) \Vdash \gamma$. But, by the definition of R^*, we have that $Rlast(\sigma)last(\rho)$, and hence $\mathcal{M}, last(\sigma) \Vdash \mathbf{F}\gamma$.

$\Leftarrow$: Suppose $\mathcal{M}, last(\sigma) \Vdash \mathbf{F}\gamma$. Then, there exists a $\Psi \in W$ such that $Rlast(\sigma)\Psi$ and $\mathcal{M}, \Psi \Vdash \gamma$. Now, consider any fan $(last(\sigma), \mathfrak{S})$ with $\Psi \in \mathfrak{S}$ (such a fan exists by definition of R). By (S2) there is a path of small fans from $last(\sigma)$ to Ψ. Adding the spine of this path of small fans to σ gives a sequence ρ with $last(\rho) = \Psi$, such that σ is a proper initial segment of ρ. Hence, by construction of R^*, it follows that $R^*\sigma\rho$, and since, by the induction hypothesis, $\mathcal{M}^*, \rho \Vdash \gamma$, we have $\mathcal{M}^*, \sigma \Vdash \mathbf{F}\gamma$.

Suppose the claim holds for γ and suppose ψ is $\mathbf{P}\gamma$.

$\Rightarrow$: Suppose $\mathcal{M}^*, \sigma \Vdash \mathbf{P}\gamma$. Then there exists a $\rho \in W^*$ such that $R^*\rho\sigma$ and $\mathcal{M}^*, \rho \Vdash \gamma$. Therefore, by the induction hypothesis, $\mathcal{M}, last(\rho) \Vdash \gamma$. But, by the definition of R^*, we have that $Rlast(\rho)last(\sigma)$, and hence $\mathcal{M}, last(\sigma) \Vdash \mathbf{P}\gamma$.

$\Leftarrow$: Suppose $\mathcal{M}, last(\sigma) \Vdash \mathbf{P}\gamma$. Then, σ is the spine of a path of small fans ending in $last(\sigma)$, and by (S3) there is a Φ_i on the spine of this path such that $\gamma \in \Phi_i$. Let ρ be the initial segment of σ such that $\Phi_i = last(\rho)$. But then, by Claim 3, $\mathcal{M}, last(\rho) \Vdash \gamma$. Then, by the induction hypothesis, $\mathcal{M}^*, \rho \Vdash \gamma$. But $R^*\rho\sigma$, and hence $M^*, \sigma \Vdash \mathbf{P}\gamma$. □

Hence, since the unravelled tree $\mathcal{M}^*$ (with branches isomorphic to $(\mathbb{N}, <)$) and $\mathcal{M}$ satisfy the same formulas at corresponding instants, we have that $\mathcal{M}^*, \sigma \Vdash \varphi$, where σ is the sequence where $last(\sigma)$ is the Hintikka set containing φ (this is given by (S0)).

Hence, φ is satisfiable in a model for $\mathbf{Pr}_{\mathbb{N}}$. By soundness of $\mathbf{Pr}_{\mathbb{N}}$ with respect to the class of irreflexive trees with branches isomorphic to the natural numbers, satisfiability in a model implies consistency in the logic, and therefore φ is $\mathbf{Pr}_{\mathbb{N}}$-consistent. □

Hence, the previous two lemmas give that there is a φ-SSF iff φ is $\mathbf{Pr}_{\mathbb{N}}$-consistent, and for each $\mathbf{Pr}_{\mathbb{N}}$-consistent formula φ, there is a φ-SSF with an upperbound on its size. Putting the results of these two lemmas together, gives us the following result.

Theorem 6.2.5. $\mathbf{Pr}_{\mathbb{N}}$ *is decidable.*

Remark 6.2.6. We have shown that from any SSF's we can build a model for a satisfiable formula based on a frame for the logic. Therefore, if we could show that for any consistent formula we could find a SSF, this could turn this process into a completeness proof, thus proving both completeness and decidability at once. Our method of proving completeness in Section 5.8.1 and Theorem 5.11.2 is less complicated, and therefore we feel that it is valuable to provide the two separate arguments.

By removing condition (B2) from the definition of a fan, we can follow the same process to prove a decidability result for $\mathbf{Pr}_{wf}$. We point out where the proofs will differ in the proof of the following theorem.

Theorem 6.2.7. $\mathbf{Pr}_{wf}$ *is decidable.*

Proof. Let φ be a $\mathbf{Pr}_{wf}$-consistent formula. We start by showing that there is a φ-SSF of size at most $2^{2(|\varphi|+4)} \times 2^{2^{2(|\varphi|+4)}}$:

To prove this we start with the model built in Section 5.8.1 and continue as in the proof of Lemma 6.2.3 but leaving out (B2). We cannot use the fact that the branches are isomorphic to the natural numbers, but we can use the fact that the branches are locally finite (since we used sequential $\mathcal{M}$-fans to construct the φ-SSF), i.e. all instants with successors have immediate successors, and similarly for predecessors (see Lemma 5.8.8). Also, note that we also cannot use the fact that every instant will have a successor and hence we have to consider the case when w_φ in the proof of (S0) does not have a successor. In this case we consider the fan $(\Gamma_{w_\varphi}, \emptyset)$, which has the required properties.

Next, we need to show that if there is a φ-SSF, then φ is $\mathbf{Pr}_{wf}$-consistent:

The proof of this statement works exactly as in the proof of Lemma 6.2.4, and the construction produces a well-founded model. □

It is possible to modify the constructions and arguments above to prove the decidability of $\mathbf{Pr}_{(\mathbb{N},\leq)}$ and $\mathbf{Pr}^r_{wf}$. In particular, we need to make the following adjustments and modify the proofs in the obvious way to incorporate a reflexive relation:

In the definition of a small fan, replace conditions (B3) and (B4) with conditions (B3^r) and (B4^r) respectively:

(B3^r) If $\mathbf{F}\varphi \in \Phi$, then either $\varphi \in \Phi$, or φ or $\mathbf{F}\varphi \in \Psi$ for some $\Psi \in \mathfrak{S}$.

(B4^r) If $\mathbf{P}\varphi \in \Psi$ for some $\Psi \in \mathfrak{S}$, then either $\varphi \in \Psi$, or φ or $\mathbf{P}\varphi$ is in Φ.

In the definition of s saturated set of fans, replace condition (S1) with condition (S1^r):

(S1^r) For all $(\Phi, \mathfrak{S}) \in \mathcal{B}$, if there is a $\Psi \in \mathfrak{S}$ with $\mathbf{F}\varphi \in \Psi$, then either $\varphi \in \Psi$, or there is a fan $(\Psi, \mathfrak{S}')$ and a $\Psi' \in \mathfrak{S}'$ such that $\varphi \in \Psi'$.

Using these changes, we can prove the following results.

Theorem 6.2.8. $\mathbf{Pr}^r_{wf}$ *is decidable.*

Proof. Let φ be a $\mathbf{Pr}^r_{wf}$-consistent formula. We start by showing that there is a φ-SSF of size at most $2^{2(|\varphi|+4)} \times 2^{2^{2(|\varphi|+4)}}$:

To prove this we start with the model built in Section 5.9 and continue as in the proof of Lemma 6.2.3 but leaving out (B2). Again, we cannot use the fact that the branches are isomorphic to the natural numbers, but we can use the fact that the branches are locally finite (by the definition of an $\mathcal{M}$-fan, and how we showed that a sequential $\mathcal{M}$-fan is a φ-SSF).

We show how to prove the modified conditions:

(B3^r) Suppose $\mathbf{F}\psi \in \Gamma_w$. Then, by definition we have $\mathcal{M}, w \Vdash \mathbf{F}\psi$ and hence there is a u with Rwu such that $\mathcal{M}, u \Vdash \psi$. Now, by definition, there is a v on the same branch as u, which is an immediate successor of w, and $\Gamma_v \in \mathfrak{S}$. Now, if $u = v$ it follows that $\mathcal{M}, v \Vdash \psi$. If Rvu and $u \neq v$ then $\mathcal{M}, v \Vdash \mathbf{F}\psi$. Hence, either $\psi \in \Gamma_w$, or $\mathbf{F}\psi$ or ψ is in Γ_v for $\Gamma_v \in \mathfrak{S}$, as required.

(B4^r) Suppose $\mathbf{P}\psi \in \Gamma_v$ for some $\Gamma_v \in \mathfrak{S}$. Then, by definition we have $\mathcal{M}, v \Vdash \mathbf{P}\psi$ and hence there is a u with Ruv such that $\mathcal{M}, u \Vdash \psi$. Now, by left linearity we have that Ruw. Now, if $u = w$ it follows that $\mathcal{M}, w \Vdash \psi$. If Ruw and $u \neq w$, then $\mathcal{M}, w \Vdash \mathbf{F}\psi$. Hence, either $\psi \in \Gamma_v$, or $\mathbf{P}\psi$ or ψ is in Γ_w, as required.

(S1^r) Let $(\Gamma_w, \mathfrak{S}) \in \mathcal{B}$ and suppose $\mathbf{F}\psi \in \Gamma_v$ for some $\Gamma_v \in \mathfrak{S}$. Then, by definition we have $\mathcal{M}, v \Vdash \mathbf{F}\psi$ and hence there is a u with Rvu such that $\mathcal{M}, u \Vdash \psi$. Now, if $u = v$ then $\psi \in \Gamma_v$. Otherwise, consider any $\mathcal{M}$-fan of the form $(\Gamma_v, \mathfrak{T})$ such that $\Gamma_u \in \mathfrak{T}$. Then $\psi \in \Gamma_u$, as required.

Next, we show that if there is a φ-SSF, then φ is $\mathbf{Pr}^r_{wf}$-consistent:

The proof of this statement works exactly as in the proof of Lemma 6.2.4, but replacing (B3), (B4) and (S1) with the reflexive counterparts (B3^r) , (B4^r) and (S1^r), and we define the relations as follows:

- Let $\Phi, \Psi \in W$, then $R\Phi\Psi$ iff $\Phi = \Psi$, or there is a path of small fans from Φ to Ψ.
- Let $\sigma, \rho \in W^*$. Then $R^*\sigma\rho$ iff $\sigma = \rho$ or σ is an initial segment of ρ.

The resulting model will then be the required reflexive well-founded model, and since the logic is sound with respect to the class of reflexive well-founded frames, the result follows. □

Theorem 6.2.9. $\mathbf{Pr}_{(\mathbb{N},\leq)}$ *is decidable.*

Proof. Let φ be a $\mathbf{Pr}_{wf}$-consistent formula. First we show that there is a φ-SSF of size at most $2^{2(|\varphi|+4)} \times 2^{2^{2(|\varphi|+4)}}$:

To prove this, we start with the model built in Section 5.11 and continue as in the proof of Lemma 6.2.3 and the proof of Theorem 6.2.8 for the reflexive conditions. This will be enough to prove this statement.

Next we show that if there is a φ-SSF, then φ is $\mathbf{Pr}_{wf}$-consistent:

The proof of this statement works exactly as in the proof of Lemma 6.2.4, but using the reflexive conditions (B3^r) , (B4^r) and (S1^r), and the relations as defined in Theorem 6.2.8, and the construction will give us a reflexive model with branches isomorphic to $(\mathbb{N}, \leq)$, since the model satisfying φ will be well-founded, transitive and reflexive. □

If might be possible that we could use the same method to show that $\mathbf{Pr}_{lfin}$, $\mathbf{Pr}_{lfin}\mathbf{U_l}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$, $\mathbf{Pr}_{\mathbb{Z}}$ and $\mathbf{Pr}^r_{lfin}$ are decidable by making the following changes to the definition of a φ-SSF as given in Definition 6.2.2: Remove condition (S0) and add a dual condition of condition (S1) for a formula $\mathbf{P}\psi$. Showing that $\mathbf{Pr}_{cwf}$ and $\mathbf{Pr}^r_{cwf}$ are decidable using this method turned out to be more complicated and might need another method.

For now, we will leave the decidability of $\mathbf{Pr}_{lfin}$, $\mathbf{Pr}_{lfin}\mathbf{U_l}$, $\mathbf{Pr}_{lfin}\mathbf{U_r}$, $\mathbf{Pr}_{\mathbb{Z}}$, $\mathbf{Pr}^r_{lfin}$, $\mathbf{Pr}_{cwf}$ and $\mathbf{Pr}^r_{cwf}$ as an open question.

6.3 Decidability via Conservative Extensions

In this section we use the concept of a conservative extension to deduce decidability of the logics in the future fragment of the Priorian temporal language in Chapter 3 , containing the density axiom, from the decidability of the dense temporal logics in Chapter 4.

Definition 6.3.1. Let $\mathcal{L}$ and $\mathcal{L}^+$ be two languages with $\mathcal{L} \subseteq \mathcal{L}^+$ and let Λ and Λ^+ be logics in the respective languages. Then Λ^+ is a **conservative extension** of Λ if for all formulas φ in the language $\mathcal{L}$ we have that $\Lambda \vdash \varphi$ iff $\Lambda^+ \vdash \varphi$.

This means that all theorems of Λ are theorems of Λ^+ and Λ^+ adds no additional theorems when restricted to the language $\mathcal{L}$. We use this fact in the proof of the following Lemma.

Lemma 6.3.2. *Let Λ be a logic and let Λ^+ be a conservative extension of Λ. If Λ^+ is decidable, then so is Λ.*

Proof. Let Λ^+ be a conservative extension of Λ and suppose Λ^+ be decidable. Given a formula φ in the language of Λ^+, but restricted to the language of Λ, then there exists a decision procedure to determine whether this formula is a theorem of Λ^+ or not, since Λ^+ is decidable. Since Λ^+ is a conservative extension of Λ, a formula is a theorem of Λ iff it is a theorem of Λ^+, and hence we can use the same decision procedure to determine whether φ is a theorem of Λ, or not. Hence, $\Lambda$is also decidable. □

In the following theorem, we show that the Priorian temporal logics for the classes of dense trees considered in this thesis are conservative extensions of the future fragment of the Priorian temporal logics for the corresponding classes of dense trees.

Lemma 6.3.3. *The following logics are conservative extensions of the respective modal counterparts:*

- $\mathbf{Pr}_{dense}$ *is a conservative extension of* **K4D**
- $\mathbf{Pr}_{dense}\mathbf{U_r}$ *is a conservative extension of* $\mathbf{K4DU_r}$

Proof. Consider the logics $\mathbf{Pr}_{dense}$ and **K4D** which are the logics in the Priorian temporal language, and future fragment of the Priorian temporal language, respectively, that are sound and complete with respect to the class of dense irreflexive trees (See Theorems 3.1.21 and 4.5.4). We will prove the result for these two logics and the proof of the remaining two logics will follow similarly.

Let φ be a theorem of **K4D**. Since $\mathbf{Pr}_{basic}\mathbf{D}$ has all the axioms and rules of inference of **K4D**, we can use the same derivation to show that φ is also a theorem of $\mathbf{Pr}_{basic}\mathbf{D}$.

For the converse, we will follow a semantic argument. Suppose ψ is not a theorem of **K4D**. Then, for some model $\mathcal{M}$ based on a dense irreflexive tree, and some instant w, we have that $\mathcal{M}, w \nVdash \psi$, by the completeness shown in Theorem 3.1.21. Hence, we can falsify ψ on a dense irreflexive tree and in particular on a model for $\mathbf{Pr}_{dense}$, by the soundness shown in Theorem 4.5.4.

Hence, the result follows.

The other case is similar, using the following theorem of completeness: Theorem 4.7.1 for $\mathbf{Pr}_{dense}\mathbf{U_r}$.

□

This gives us the following decidability result.

Theorem 6.3.4. *The logics in the future fragment of the Priorian language for the following classes of trees are decidable.*

- *dense irreflexive trees*
- *left unbounded dense irreflexive trees*
- *right unbounded dense irreflexive trees*
- *unbounded dense irreflexive trees*
- *irreflexive trees with branches isomorphic to the rational numbers*

Proof. We already know that the temporal logics for these classes of trees are decidable, as seen in Theorem 6.1.7. Furthermore, from Lemma 6.3.3 we know that these logics are conservative extensions of their counterparts in the future fragment of the Priorian language. Hence, by Lemma 6.3.2, **K4D** is decidable. But **K4D** is sound and complete with respect to the class of dense irreflexive trees, and the class of left unbounded dense irreflexive trees; $\mathbf{K4DU_r}$ is sound and complete with respect to the class of right unbounded dense irreflexive trees, the class of unbounded dense irreflexive trees, and the class of irreflexive trees with branches isomorphic to the rational numbers. □

Hence, all the logics studied in Chapters 3, 4, and 5 are decidable, except those for the conversely well-founded trees and locally finite trees. We have also not shown that the logics for these trees are not decidable. This remains as an open question.

Chapter 7

Summary and Survey of Axiomatisations of Temporal Logics of Trees

In this chapter we will give a summary of the main completeness results of this thesis, as well as a survey of some similar results in the literature. In this thesis, we have only considered axiomatisations the logics of different classes of trees in the future fragment of the Priorian temporal ($\mathcal{L}^m_{Prior}$) and Priorian temporal ($\mathcal{L}_{Prior}$) languages. We will summarise these results in the first section. The second section gives a survey of the literature on logics of classes of trees in some extensions of $\mathcal{L}_{Prior}$ and $\mathcal{L}^m_{Prior}$. We will mainly look at the Peircean and Ockhamist languages that are well suited to accommodate branching time. Lastly, we will look at some open questions in this field that could be further investigated.

7.1 Axiomatisations of the Priorian Logics of Classes of Trees

7.1.1 Summary of Axiomatisations in the Literature

Although there are few results of axiomatisations in the Priorian languages for specific trees, several authors have published results on linear frames. The most comprehensive studies were done by Segerberg in [60], as well as Burgess in [10], [13] and [13]. Other studies include [51], [32], [53], and others.

Recall that **K** is the smallest normal modal logic, **4** the axiom for transitivity and $\mathbf{.3_r}$ the axiom for right linearity. We know that $\mathbf{K4.3_r}$ is sound and strongly complete with respect to the class of all linear orders. In [60] Segerberg uses this fact to build axiomatisation for some subclasses of linear orders. In Table 7.1 we summarise the axiomatisations of the logics of linear frames in the Priorian modal language. Note that we only mention the additional axioms needed in the axiomatisations.

In the Table 7.2 is a summary of axiomatisations of the logics of linear frames in the Priorian temporal language as found by [60]. All axiomatisations include those of $\mathbf{K_t}$, the smallest normal temporal logic, **4**, $\mathbf{.3_l}$ and $\mathbf{.3_r}$ the axioms for right and left linearity, as well as those mentioned in the table.

In Table 7.3, we summarise the results for axiomatisations of linear orders found by Burgess in [13]. Burgess starts with $\mathbf{K_t}$, the smallest normal temporal logic, as well as the axiom for transitivity, **4**, and the following: $\mathbf{F}p \wedge \mathbf{F}p \rightarrow \mathbf{F}(p \wedge \mathbf{F}p) \vee \mathbf{F}(p \wedge q) \vee \mathbf{F}(\mathbf{F}p \wedge q)$ and the dual $\mathbf{F}p \wedge \mathbf{P}p \rightarrow \mathbf{P}(p \wedge \mathbf{P}p) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(\mathbf{P}p \wedge q)$. He shows that this logic is sound and complete with respect to the class of all linear orders and uses this as a basis for some subclasses as listed in the Table 7.3.

Both [60] and [13] used filtrations to show completeness of the logics with respect to discrete linear orders and linear orders isomorphic to the real numbers. However, [19] used mcs's to build a model in order to show strong

Class of Trees	Axiomatisation
Irreflexive linear orders and well-founded linear orders	No additional axioms
Reflexive linear orders, well-founded reflexive linear orders, reflexive linear orders isomorphic to the rational numbers, reflexive linear orders isomorphic to the real numbers	$\mathbf{T}$
Unbounded irreflexive linear orders	$\mathbf{U_r}$
Reflexive linear orders	$\mathbf{T}$
Unbounded reflexive linear orders	$\mathbf{T}$, $\mathbf{U_r}$
Dense irreflexive linear orders	$\mathbf{D}$
Dense, unbounded irreflexive linear orders, irreflexive linear orders isomorphic to the rational numbers, irreflexive linear orders isomorphic to the real numbers	$\mathbf{D}$
Irreflexive linear orders isomorphic to the integers, Irreflexive linear orders isomorphic to the natural numbers	$\mathbf{U_r}$, $\mathbf{G}(\mathbf{G}p \to p) \to (\mathbf{FG}p \to \mathbf{G}p)$
Finite irreflexive linear orders, conversely well-founded irreflexive linear orders	$\mathbf{L_r}$
Reflexive linear orders isomorphic to the integers, reflexive linear orders isomorphic to the natural numbers	$\mathbf{U_r}$, $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to (\mathbf{FG}p \to p)$
Conversely well-founded reflexive linear orders	$\mathbf{Grz}$

Table 7.1: Axiomatisations of Modal Logics of Linear Frames by [60]

Class of Trees	Axiomatisation
Irreflexive linear orders	No additional axioms
Reflexive linear orders, reflexive linear orders isomorphic to the rational numbers	$\mathbf{T}$
Irreflexive linear orders isomorphic to the rational numbers	$\mathbf{D}$, $\mathbf{U_l}$, $\mathbf{U_r}$
Irreflexive linear orders isomorphic to the real numbers	$\mathbf{D}$, $\mathbf{U_l}$, $\mathbf{U_r}$, $\mathbf{C_l}$, $(\mathbf{F}p \leftrightarrow \mathbf{H}p) \to (\mathbf{E}p \to \mathbf{A}p)$
Reflexive linear orders isomorphic to the real numbers	$\mathbf{T}$, $\mathbf{E}p \wedge \mathbf{E}\neg p \wedge \mathbf{A}(\mathbf{H}p \vee \mathbf{G}\neg p) \to \mathbf{E}(\mathbf{H}p \wedge (q \to \mathbf{G}(p \to q)) \wedge (\neg q \to \mathbf{G}(p \to \neg q))) \vee \mathbf{E}(\mathbf{G}\neg p \wedge (r \to (\neg p \to r)) \vee (\neg r \to \mathbf{H}(\neg p \to \neg r)))$
Finite reflexive linear orders	$\mathbf{T}$, $\mathbf{Grz}$, $\mathbf{Grz_l}$
Finite irreflexive linear orders	$\mathbf{L_l}$, $\mathbf{L_r}$
Reflexive linear orders isomorphic to the integers	$\mathbf{T}$, $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to (\mathbf{FG}p \to p)$, $\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to (\mathbf{PH}p \to p)$
Irreflexive linear orders isomorphic to the integers	$\mathbf{U_r}$, $\mathbf{U_l}$, $\mathbf{G}(\mathbf{G}p \to p) \to (\mathbf{FG}p \to \mathbf{G}p)$, $\mathbf{H}(\mathbf{H}p \to p) \to (\mathbf{PH}p \to \mathbf{G}p)$
Reflexive linear orders isomorphic to the natural numbers	$\mathbf{T}$, $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to (\mathbf{FG}p \to p)$, $\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to (\mathbf{PH}p \to p)$, $\mathbf{P}(\mathbf{P}p \to \mathbf{H}p)$, $\mathbf{F}(\mathbf{F}p \to \mathbf{G}p)$
Irreflexive linear orders isomorphic to the natural numbers	$\mathbf{U_r}$, $\mathbf{EH}\bot$, $\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to (\mathbf{FG}p \to p)$, $\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to (\mathbf{PH}p \to p)$

Table 7.2: Axiomatisations of Temporal Logics of Linear Frames by [60]

Class of Trees	**Axiomatisation**
Irreflexive linear orders	No additional axioms
Unbounded linear orders	$\mathbf{G}p \to \mathbf{F}p$, $\mathbf{H}p \to \mathbf{P}p$
Bounded linear orders	$\mathbf{G}\bot \to \mathbf{FG}\bot$, $\mathbf{H}\bot \to \mathbf{PH}\bot$
Dense linear orders	$\mathbf{D}$
Linear orders isomorphic to the rational numbers	$\mathbf{D}$, $\mathbf{G}p \to \mathbf{F}p$, $\mathbf{H}p \to \mathbf{P}p$
Linear orders where each instant has an immediate predecessor and successor	$p \wedge \mathbf{H}p \to \mathbf{FH}p$, $p \wedge \mathbf{G}p \to \mathbf{PG}p$
Discrete linear orders	$p \wedge \mathbf{H}p \to \mathbf{G}\bot \vee \mathbf{FH}p$, $p \wedge \mathbf{G}p \to \mathbf{H}\bot \vee \mathbf{PG}p$
Complete (no gaps) linear orders	$\mathbf{F}p \wedge \mathbf{FG}\neg p \to \mathbf{F}(\mathbf{HF}p \wedge \mathbf{G}\neg p)$, $\mathbf{P}p \wedge \mathbf{PH}\neg p \to \mathbf{P}(\mathbf{GP}p \wedge \mathbf{H}\neg p)$
Linear orders isomorphic to the real numbers	$\mathbf{D}$, $\mathbf{G}p \to \mathbf{F}p$, $\mathbf{H}p \to \mathbf{P}p$, $\mathbf{F}p \wedge \mathbf{FG}\neg p \to \mathbf{F}(\mathbf{HF}p \wedge \mathbf{G}\neg p)$, $\mathbf{P}p \wedge \mathbf{PH}\neg p \to \mathbf{P}(\mathbf{GP}p \wedge \mathbf{H}\neg p)$
Well-founded linear orders	$\mathbf{L_l}$
Linear orders isomorphic to the natural numbers	$\mathbf{L_l}$, $p \wedge \mathbf{G}p \to \mathbf{H}\bot \vee \mathbf{PG}p$

Table 7.3: Axiomatisations of Temporal Logics of Linear Frames by [13]

completeness for these logics. We summarise these results in Table 7.4.

The results on axiomatisation of the logics of linear frames came in useful when searching for axiomatisations of the logics of the classes of trees as each branch can be seen as a linear ordering. In some cases, only minor adjustments had to be made to translate these results into axiomatisations of the logics of the classes of trees. In other cases, new methods were developed to find these axiomatisations. We summarise these results in the next section.

When looking at the literature on axiomatisations specifically for the logics of trees in the Priorian languages, there were only a few results. Some axiomatisations for the logics of different classes of trees in the Priorian temporal languages have been established in the literature, for example [26] and [61].

Class of Trees	**Axiomatisation**
Irreflexive linear orders	$\mathbf{K_t}$, $\mathbf{4}$, $\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$
Unbounded irreflexive linear orders	$\mathbf{K_t}$, $\mathbf{4}$, $\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$
Linear orders isomorphic to the rational numbers	$\mathbf{K_t}$, $\mathbf{4}$, $\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $\mathbf{GG}p \to \mathbf{G}p$
Linear orders isomorphic to the real numbers	$\mathbf{K_t}$, $\mathbf{4}$, $\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $\mathbf{GG}p \to \mathbf{G}p$, $(\mathbf{G}p \to \mathbf{PG}p) \to (\mathbf{G}p \to \mathbf{H}p)$
Unbounded discrete irreflexive linear orders	$\mathbf{K_t}$, $\mathbf{4}$, $\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $(p \wedge \mathbf{G}p) \to \mathbf{PG}p$, $(p \wedge \mathbf{H}p) \to \mathbf{FH}p$
Linear orders isomorphic to the integers	$\mathbf{K_t}$, $\mathbf{4}$, $\mathbf{F}p \to \mathbf{G}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\mathbf{P}p \to \mathbf{H}(p \vee \mathbf{P}p \vee \mathbf{F}p)$, $\neg\mathbf{G}\bot$, $\neg\mathbf{H}\bot$, $(\mathbf{G}p \to \mathbf{PG}p) \to (\mathbf{G}p \to \mathbf{H}p)$ (weak completeness)

Table 7.4: Axiomatisations of Temporal Logics of Linear Frames by [19]

Abbr	Axiom	Property of Kripke Frames defined by Axiom
$\mathbf{K_G}$	$\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$	Valid on all frames
$\mathbf{K_H}$:	$\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$	Valid on all frames
$\mathbf{Dual_F}$	$\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$	Valid on all frames
$\mathbf{Dual_P}$	$\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$	Valid on all frames
Conv	$p \to \mathbf{GP}p$ and $p \to \mathbf{HF}p$	Valid on all frames
4	$\mathbf{FF}p \to \mathbf{F}p$	$\forall xyz(Rxy \wedge Ryz \to Rxz)$ (Transitivity)
$\mathbf{.3_l}$	$(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$	$\forall xyz(Rxz \wedge Ryz \to Rxy \vee Ryx \vee x = y)$ (Left linear)
D	$\mathbf{F}p \to \mathbf{FF}p$	$\forall xy(Rxy \to \exists x(Rxz \wedge Rzy))$ (Density)
$\mathbf{U_l}$	$\mathbf{P}\top$	$\forall x \exists y Ryx$ (Left seriality)
$\mathbf{U_r}$	$\mathbf{F}\top$	$\forall x \exists y Rxy$ (Right seriality)
S	$\neg p \wedge \mathbf{H}\neg p \wedge \mathbf{F}p \to \mathbf{F}(p \wedge \mathbf{H}\neg p)$	No infinitely descending sequences between two instants (irreflexive frames)
Q	$\mathbf{P}p \vee \mathbf{GH}\neg p \vee \mathbf{F}(p \wedge \mathbf{H}(p \vee \mathbf{H}\neg p))$	No infinitely descending sequences of distinct instants between two instants (reflexive frames)
$\mathbf{L_l}$	$\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$	Well-foundedness, transitivity, irreflexivity
$\mathbf{L_r}$	$\mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$	Converse well-foundedness, transitivity, irreflexivity
T	$p \to \mathbf{F}p$	Reflexivity
Grz	$\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$	No infinitely ascending sequences of distinct instants, transitivity, reflexivity
$\mathbf{Grz_l}$	$\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$	No infinitely descending sequences of distinct instants, transitivity, reflexivity

Table 7.5: List of Axioms

7.1.2 Summary of Axiomatisations in this Thesis

Table 7.5 gives a list of axioms in the Priorian languages used in the axiomatisations for the logics of the different classes of trees in this thesis. The axioms are listed with their abbreviations and the properties they define. All the axioms are canonical with the exception of **S**, **Q**, $\mathbf{L_l}$, $\mathbf{L_r}$, **Grz** and $\mathbf{Grz_l}$.

We use $\mathbf{A}\varphi$ as an abbreviation for $\mathbf{H}\varphi \wedge \varphi \wedge \mathbf{G}\varphi$ and $\mathbf{E}\varphi$ as an abbreviation for $\mathbf{P}\varphi \vee \varphi \vee \mathbf{F}\varphi$.

Next, Table 7.6 gives a summary of the rules used in the axiomatisations. The first three rules, MP, SUB and GEN, are part of all the logics in this thesis while the last three are additional rules introduced to find a completeness result for the class of discrete trees

We are now ready to list the results. We start with the axiomatisations of classes of trees in the modal language in Table 7.7. Then, we list the results for the temporal language in Tables 7.8, 7.9 and 7.10.

Then, in Table 7.11 we list the different methods used in building a model in the process of showing that the axiomatisations are complete with respect to the respective classes of trees. In all cases, we started with the canonical model of the logic we wanted to show is complete with respect to the given class, and then modified it in different ways while preserving satisfaction of relevant formulas to construct a tree based on a frame which is a tree

Abbr	**Rule**	**Class of Frames on which Rule is Valid**
MP	If $\vdash p$ and $\vdash p \to q$ then $\vdash q$.	All frames
SUB	If $\vdash \varphi$ then $\vdash \psi$, where φ is obtained from ψ by uniformly replacing proposition letters in φ by arbitrary formulas.	All Frames
GEN	If $\vdash \varphi$ then $\vdash \mathbf{G}\varphi$ and If $\vdash \varphi$ then $\vdash \mathbf{H}\varphi$.	All frames
IRR	If $\vdash \neg p \wedge \mathbf{H}p \to \varphi$ then $\vdash \varphi$ where $p \notin var(\varphi)$.	Irreflexive frames
FDR	If $(\vdash \mathbf{F}p \wedge \neg\mathbf{FF}p \to \varphi) \vee \mathbf{G}\bot$ then $\vdash \varphi$, where $p \notin var(\varphi)$.	Forwards discrete frames
BRD	If $(\vdash \mathbf{P}p \wedge \neg\mathbf{PP}p \to \varphi) \vee \mathbf{H}\bot$ then $\vdash \varphi$, where $p \notin var(\varphi)$.	Backwards discrete frames

Table 7.6: List of Rules

from the required class. The methods we used include bulldozing, filtrations, and building a model step-by-step. We also used anti-axioms as rules as has been done before for irreflexive frames. Lastly, we introduced a new method specifically applicable to building a tree, namely unfolding.

Logic	Axioms	Class of Kripke Frames Defined by Axioms	Class of Trees for which Axiomatisation is Complete
K4	$\mathbf{K_G}$, $\mathbf{Dual_F}$, **4**	Transitive frames	Irreflexive trees (3.1.1), rooted irreflexive trees (3.1.2), discrete irreflexive trees (3.1.3), locally finite irreflexive trees (3.1.4), well-founded irreflexive trees (3.1.5), left-unbounded irreflexive trees (3.1.10)
$\mathbf{K4U_r}$	$\mathbf{K4 + U_r}$	Right-unbound transitive frames	Right-unbounded irreflexive trees (3.1.6), unbounded irreflexive trees (3.1.11), irreflexive trees with branches isomorphic to the natural numbers (3.1.8) and to the integers (3.1.12)
$\mathbf{KL_l}$	$\mathbf{K_G}$, $\mathbf{Dual_F}$, $\mathbf{L_l}$	Transitive, conversely well-founded frames (and hence irreflexive)	Finite irreflexive trees (weak completeness) (3.1.16), conversely well-founded irreflexive trees (weak completeness) (3.1.17)
K4D	**K4 + D**	Transitive, dense frames	Dense irreflexive trees (3.1.21), left unbounded dense irreflexive trees 3.1.10
$\mathbf{K4DU_r}$	**K4 + D**, $\mathbf{U_r}$	Transitive, unbounded, dense frames	Right unbounded dense irreflexive trees (3.1.22), irreflexive trees with branches isomorphic to the non-negative rational numbers (3.1.23), unbounded dense irreflexive trees (3.1.24), irreflexive trees with branches isomorphic to the rational numbers (3.1.25)
S4	**K4 + T**	Transitive, reflexive frames	Reflexive trees (3.2.1), discrete reflexive trees (3.2.3), locally finite reflexive trees (3.2.4), rooted reflexive trees (3.2.2), left unbounded reflexive trees (3.2.6), well-founded reflexive trees (3.2.5), right unbounded reflexive trees (3.2.7), unbounded reflexive trees (3.2.7), reflexive trees with branches isomorphic to $(\mathbb{N}, \leq)$ (3.2.8) and to $(\mathbb{Z}, \leq)$ (3.2.8). left unbounded dense reflexive trees (3.2.18), right unbounded dense trees (3.2.19), unbounded dense trees (3.2.19), reflexive trees with branches isomorphic to $\langle \mathbb{Q}), \leq \rangle$ (3.2.19), and $\langle \mathbb{Q}^{+} \cup \{0\}, \leq \rangle$ (3.2.19).
KGrz	$\mathbf{K_G}$, $\mathbf{Dual_F}$, **Grz**	Transitive finite reflexive frames	Finite reflexive trees (weak completeness) (3.2.14), conversely well-founded reflexive trees (weak completeness) (3.2.15)

Table 7.7: Axiomatisations in the Future Fragment of the Priorian Temporal Language of Tree Classes

Logic	Axioms/Rules	Class of Kripke Frames Defined by Axioms	Class of Trees for which Axiomatisation is Complete
$\mathbf{Pr}_{basic}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, **4**, $\mathbf{.3_l}$	Transitive, left linear frames	Irreflexive trees (4.2.2)
$\mathbf{Pr}_{basic}\mathbf{U_l}$	$\mathbf{Pr}_{basic} + \mathbf{U_l}$	Transitive, left linear, left unbounded frames	Left unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{basic}\mathbf{U_r}$	$\mathbf{Pr}_{basic} + \mathbf{U_r}$	Transitive, left linear, right unbounded frames	Right unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{unbnd}$	$\mathbf{Pr}_{basic} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, unbounded frames	Unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{dense}$	$\mathbf{Pr}_{basic} + \mathbf{D}$	Transitive, left linear, dense frames	Dense irreflexive trees (4.5.4)
$\mathbf{Pr}_{dense}\mathbf{U_l}$	$\mathbf{Pr}_{dense} + \mathbf{U_l}$	Transitive, left linear, left unbound, dense frames	Left unbounded dense irreflexive trees (4.7.1)
$\mathbf{Pr}_{dense}\mathbf{U_r}$	$\mathbf{Pr}_{dense} + \mathbf{U_r}$	Transitive, left linear, right unbound, dense frames	Right unbounded dense irreflexive trees (4.7.1)
$\mathbf{Pr}_{\mathbb{Q}}$	$\mathbf{Pr}_{dense} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, dense, unbounded frames	Irreflexive trees with branches isomorphic to the rational numbers (4.8.1)
$\mathbf{Pr}_{lfin}$	$\mathbf{Pr}_{basic} + \mathbf{S}$	Transitive, left linear, locally finite frames	Locally finite irreflexive trees (weak completeness) (5.4.7)
$\mathbf{Pr}_{lfin}\mathbf{U_l}$	$\mathbf{Pr}_{lfin} + \mathbf{U_l}$	Transitive, left linear, left unbound, locally finite frames	Left unbound locally finite irreflexive trees (weak completeness) (5.6.1)
$\mathbf{Pr}_{lfin}\mathbf{U_r}$	$\mathbf{Pr}_{lfin} + \mathbf{U_r}$	Transitive, left linear, right unbound, locally finite frames	Right unbound locally finite irreflexive trees (weak completeness) (5.6.1)
$\mathbf{Pr}_{lfin}\mathbf{U_l}\mathbf{U_r}$	$\mathbf{Pr}_{lfin} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, unbound, locally finite frames	Unbounded locally finite irreflexive trees (weak completeness) (5.6.1)
$\mathbf{Pr}_{\mathbb{Z}}$	$\mathbf{Pr}_{basic} + \mathbf{S} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, locally finite, unbounded frames	Irreflexive trees with branches isomorphic to the integers (weak completeness) (5.7.3)

Table 7.8: Axiomatisations in the Priorian Temporal Language of Tree Classes 1

Logic	Axioms/Rules	Class of Kripke Frames Defined by Axioms	Class of Trees for which Axiomatisation is Complete
$\mathbf{Pr}_{discr}$	$\mathbf{Pr}_{basic}$, IRR, FDR, BDR	Transitive, left linear	Discrete irreflexive trees (5.1.11)
$\mathbf{Pr}_{discr}\mathbf{U_l}$	$\mathbf{Pr}_{discr} + \mathbf{U_l}$	Transitive, left linear, left unbound, frames	Left unbound discrete irreflexive trees (weak completeness) (5.3.1)
$\mathbf{Pr}_{discr}\mathbf{U_r}$	$\mathbf{Pr}_{discr} + \mathbf{U_r}$	Transitive, left linear, right unbound, frames	Right unbound discrete irreflexive trees (weak completeness) (5.3.1)
$\mathbf{Pr}^{r}_{udisc}$	$\mathbf{Pr}_{discr} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, unbounded frames	Unbounded, discrete Trees (5.3.1)
$\mathbf{Pr}_{fin}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $\mathbf{.3_l}$, $\mathbf{L_l}$, $\mathbf{L_r}$	Transitive, well-founded and conversely well-founded frames	Finite irreflexive trees (weak completeness) (5.12.1)
$\mathbf{Pr}_{wf}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $\mathbf{.3_l}$, $\mathbf{L_l}$	Transitive, well-founded frames	Well-founded irreflexive trees(weak completeness) (5.8.7)
$\mathbf{Pr}_{cwf}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $\mathbf{.3_l}$, $\mathbf{L_r}$	Transitive, conversely well-founded frames	Conversely well-founded irreflexive trees(weak completeness) (5.8.15)
$\mathbf{Pr}_{\mathbb{N}}$	$\mathbf{Pr}_{wf} + \mathbf{U_r}$	Transitive, left linear, conversely well-founded, locally finite, right unbounded frames	Irreflexive trees with branches isomorphic to the natural numbers (weak completeness) (5.11.2)
$\mathbf{Pr}_{basic}\mathbf{T}$	$\mathbf{Pr}_{basic} + \mathbf{T}$	Transitive, left linear, reflexive frames	Reflexive trees (4.3.2), discrete reflexive trees (5.2.14), unbounded discrete reflexive trees (5.3.2), left unbounded reflexive trees (4.4.2), right unbounded reflexive trees (4.4.2), unbounded reflexive trees (4.4.2) dense reflexive trees (4.6.3), left unbounded dense reflexive trees (4.7.2), right unbounded dense reflexive trees (4.7.2), unbounded dense trees (4.7.2), reflexive trees with branches isomorphic to $\langle \mathbb{Q}, \leq \rangle$ (4.8.1)

Table 7.9: Axiomatisations in the Priorian Temporal Language of Tree Classes 2

Logic	Axioms/Rules	Class of Kripke Frames Defined by Axioms	Class of Trees for which Axiomatisation is Complete
$\mathbf{Pr}^r_{fin}$	$\mathbf{K_t}$, $\mathbf{.3_l}$ + **Grz** + $\mathbf{Grz_l}$	Transitive, left linear, well-founded, conversely well-founded, reflexive frames	Finite reflexive trees (weak completeness) (5.13.1)
$\mathbf{Pr}^r_{wf}$	$\mathbf{K_t}$, $\mathbf{.3_l}$ + **Grz**	Transitive, left linear, well-founded, reflexive frames	Well-founded (weak completeness) (5.9.6), reflexive trees with branches isomorphic to $(\mathbb{N}, \leq)$ (weak completeness) (5.11.4)
$\mathbf{Pr}^r_{cwf}$	$\mathbf{K_t}$, $\mathbf{.3_l}$ + $\mathbf{Grz_l}$	Transitive, left linear, conversely well-founded, reflexive frames	Conversely well-founded reflexive trees (weak completeness) (5.10.2).
$\mathbf{Pr}^r_{lfin}$	$\mathbf{Pr}_{basic}$ + **T** + **Q**	Transitive, left linear, reflexive, frames with no infinitely descending sequences between instants	Locally finite reflexive trees (weak completeness) (5.5.6), locally finite left unbounded reflexive trees (weak completeness) (5.6.2), locally finite right unbounded reflexive trees (weak completeness) (5.6.2), locally finite unbounded reflexive trees (5.6.2, reflexive trees with branches isomorphic to the integers (weak completeness) 5.7.4

Table 7.10: Axiomatisations in the Priorian Temporal Language of Tree Classes 3

7.2 Survey of Axiomatisations using Extended Languages

This section looks at current results available in the literature on axiomatisations of the logics of trees in extensions of the Priorian languages specifically suitable for describing branching time structures. The two main extensions are the Peircean and Ockhamist languages, but we will also look at variations of these that include operators like "since", "until" and "next time". We will use terminology defined in Chapter 2. Most of the other languages are, in some way, an extension of the Priorian languages defined in Chapter 2. In these following sections, when defining a new language, we assume that *Prop*is a set of propositional letters and $p \in Prop$. We also assume the semantics of the Priorian languages as defined in Section 2.3. Before going into the literature, we define the semantics of the more popular extensions of $\mathcal{L}_{Prior}$.

7.2.1 Syntax and Semantics of the Peircean and Ockhamist Temporal Languages

We start this section with the semantics of the two languages specifically used for branching time logics. The Peircean and Ockhamist languages, dating back to work done by Author Prior in 1967 [51], are more specific to branching time than the Priorian languages as they deal in more detail with a branching future. As we are working with left linear frames, the past operators **P** and **H** are still defined as in the Priorian case, but these languages define different future operators that deals with branching frames specifically.

As we have pointed out, the Peircean and Ockhamist languages are extensions of the Priorean language. While the latter can be interpreted on arbitrary Kripke frames, the former two will always be interpreted on tree models. For this reason we will work on models that are trees $\mathcal{M} = (W, \prec, V)$. Also recall that H_t is the set of all branches of the model $\mathcal{M}$ passing through the instant $t \in W$.

Class	Methods
All irreflexive/reflexive trees, left- and right unbounded irreflexive/reflexive trees, unbounded irreflexive/reflexive trees	Choose an instant w_0 in the canonical model where an arbitrary set of formulas Γ is valid, and taking the point generated submodel of the canonical model, generated by w_0, and then bulldoze the submodel.
Dense irreflexive/reflexive trees, dense left- and right unbounded irreflexive/reflexive trees, dense unbounded irreflexive/reflexive trees, irreflexive/reflexive trees with branches isomorphic to the rational numbers Locally finite irreflexive/reflexive trees, left- and right unbounded locally finite irreflexive/reflexive trees, unbounded locally finite irreflexive/reflexive trees, irreflexive/reflexive trees with branches isomorphic to the integers Well-founded / conversely well-founded irreflexive/reflexive trees, irreflexive/reflexive trees with branches isomorphic to the natural numbers	Using a labelling function to map nodes to mcs's in the canonical model for the logics, we built a network step-by-step, and the underlying model is of the right type of tree. For each of these classes, we define a coherent and perfect network to have the right properties so that the resulting network will be of the right type
Discrete irreflexive trees, discrete left- and right unbounded irreflexive trees, discrete unbounded irreflexive trees	Adding rules IRR, FDR and BDR to $\mathbf{Pr}_{basic}$ and using ID-sets to define a submodel of the canonical model with the right properties
Discrete reflexive trees, discrete unbounded reflexive trees	Choose any non-theorem α of the logic, take a submodel of the canonical model (point generated by α) and filter it through the set of subformulas of α closed under single negations, using the minimal filtration. Then unfold the filtration and bulldoze the unfolded model.
Finite irreflexive/reflexive trees	Using the canonical model as basis, choose mcs's from the canonical model to build a finite model step-by-step.

Table 7.11: Methods used to show Completeness

7.2.1.1 Syntax and Semantics of the Peircean Temporal Languages

If *Prop*is a set of propositional letters and $p \in Prop$, then the set of formulae of the Peircean language, $\mathcal{L}_{Peirce}$ can be recursively defined by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \mathbf{p}\varphi \mid \mathbf{f}\varphi \mid \mathbf{P}\varphi \mid \mathbf{F}\varphi \tag{7.1}$$

In the Peircean language, the future operator, denoted by **f** to distinguish from the Priorian future operator **F**, is intuitively interpreted as "it will necessarily be the case that sometime in the future...". While **F** only says that it might happen somewhere in the future, **f** ensures that it happens somewhere in all possible futures. [1]

The truth of a formula of $\mathcal{L}_{Peirce}$ at an instant t in a tree model $\mathcal{M}$ is defined by adding the following clause for **f**, and it's dual **g**, to the truth conditions for $\mathcal{L}_{Prior}$ as specified [51]:

- $\mathcal{M}, t \Vdash \mathbf{f}\varphi$ iff $(\forall h' \in H_t(W))(\exists t' \in h')(t \prec t'$ and $\mathcal{M}, t' \Vdash \varphi)$
- $\mathcal{M}, t \Vdash \mathbf{g}\varphi$ iff $(\exists h' \in H_t(W))(\forall t' \in h')(t \prec t' \implies \mathcal{M}, t' \Vdash \varphi)$

Note that, by left linearity, there is no branching to the left and hence, the syntax of the additional past operators of the Peircean Language coincide with that of the syntax of the past operators in the Priorian language, and will not yield a richer language. But the additional **f** operator makes it possible to talk about something that will happen somewhere in all possible futures. The Priorian language cannot express this concept. The Peircean language is therefore a richer extension of the Priorian language when considered on trees.

7.2.1.2 Syntax and Semantics of the Ockhamist Languages

If *Prop* is a set of propositional letters and $p \in Prop$, then the set of formulas of the Ockhamist language, $\mathcal{L}_{Ockham}$ can be recursively defined by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid P\varphi \mid F\varphi \mid \Diamond\varphi \tag{7.2}$$

The Ockhamist language $\mathcal{L}_{Ockham}$ introduces quantifiers $\Diamond$ and $\Box$ that quantify Priorian formulas over branches.

Given a tree model $\mathcal{M} = (W, \prec, V)$ where V is a valuation that assigns a truth value to each propositional letter at each instant, i.e. V assigns to every propositional letter, p, the set of instant-branch pairs, $V(p) = \{(t, h) \mid t \in W, h \in H_t(W), \mathcal{M}, h, t \Vdash p\}$ at which p is considered to be true.

A *branch-independent* valuation is a valuation such that $(t, h) \in V(p)$ implies $(t, h') \in V(p)$ for all $h, h' \in H_t(W)$. If a valuation is branch-independent we can think of it as a function that assigns to every propositional letter a subset of W, i.e. $V : Prop \to \mathcal{P}(T)$.

Thus, in the Ockhamist branching time semantics, a valuation can either depend on the branch or not. The truth definition for propositional letters in $\mathcal{L}_{Ockham}$ will depend on the choice of valuation.

Let $\mathfrak{M}$ be the class of triples $(\mathcal{M}, t, h)$ where $\mathcal{M}$ is a model $(W, \prec, V)$, $t \in W$ and $h \in H_t(W)$. The truth relation $\Vdash$ is a subset of the product of $\mathfrak{M}$ and the Ockhamist language, i.e.

$$\Vdash \subseteq \mathfrak{M} \times \mathcal{L}_{Ockham} \tag{7.3}$$

The truth of a Priorian formula φ with the additional operators of $\mathcal{L}_{Ockham}$ at an instant-path pair (t, h) in a model $\mathcal{M}$ is defined as follows:

[1]Note that this notation is different for the notation in the literature. We make this change to be consistent with the use of **F** and **P** in this thesis.

- $\mathcal{M}, h, t \nVdash \bot$
- $\mathcal{M}, h, t \Vdash p$ iff $(t, h) \in V(p)$
- $\mathcal{M}, h, t \Vdash \neg\varphi$ iff $\mathcal{M}, h, t \nVdash \varphi$
- $\mathcal{M}, h, t \Vdash \varphi \vee \psi$ iff $\mathcal{M}, h, t \Vdash \varphi$ or $\mathcal{M}, h, t \Vdash \psi$
- $\mathcal{M}, h, t \Vdash \mathbf{P}\varphi$ iff $(\exists t' \in h)((t' < t)$ and $(\mathcal{M}, h, t' \Vdash \varphi))$ (which is equivalent to $(\exists t' < t)(\mathcal{M}, h, t' \Vdash \varphi)$ as the past is linear)
- $\mathcal{M}, h, t \Vdash \mathbf{F}\varphi$ iff $(\exists t' \in h)((t' > t)$ and $(\mathcal{M}, h, t' \Vdash \varphi))$
- $\mathcal{M}, h, t \Vdash \Diamond\varphi$ iff $(\exists h' \in H_t(T))(\mathcal{M}, h', t \Vdash \varphi)$

If the valuation is path-independent then $\mathcal{M}, h, t \Vdash p$ iff $t \in V(p)$.

The dual operators of the modal and temporal operators are: $\mathbf{H} = \neg\mathbf{P}\neg$, $\mathbf{G} = \neg\mathbf{F}\neg$ and $\Box = \neg\Diamond\neg$. The truth conditions of these defined operators is therefore as follows:

- $\mathcal{M}, h, t \Vdash \mathbf{H}\varphi$ iff $(\forall t' \in h)((t' < t) \implies (\mathcal{M}, h, t' \Vdash \varphi))$ (which is equivalent to $(\forall t' < t)(\mathcal{M}, h, t' \Vdash \varphi)$ as the past is linear)
- $\mathcal{M}, h, t \Vdash \mathbf{G}\varphi$ iff $(\forall t' \in h)((t' > t) \implies (\mathcal{M}, h, t' \Vdash \varphi))$
- $\mathcal{M}, h, t \Vdash \Box\varphi$ iff $(\forall h' \in \mathrm{H}_t(\mathrm{T}))(\mathcal{M}, h', t \Vdash \varphi)$

The Ockhamist temporal operators introduced above can informally be interpreted as follows:

- **P** is interpreted as "sometime in the past" and **H** as "it has always been the case".
- **F** is interpreted as "sometime in the future" and **G** as "it will always be the case".
- $\Diamond$ is interpreted as "for some path passing through the current instant".
- $\Box$ is interpreted as "for all paths passing through the current instant".

Again we see that the Ockhamist language is a richer extension of the Priorian language as it has additional modalities. However, it is also a richer extension of the Piercean language, since **f** of the Peircean language is equivalent to $\Box\mathbf{F}$ in the Ockhamist language. Furthermore, in the Ockhamist language, we can talk about something always being the case on some particular future. Neither the Priorian nor the Peircean languages can express this concept.

7.2.1.3 Syntax and Semantics of the Since, Until and Next Time Operators

Before we look at the results from the literature, we introduce the semantics of three more operators that are commonly used as an extension of the three languages already defined: the next time operator, and the since and until operators, followed by the additional modal operators that extends $\mathcal{L}_{Prior}$ to the more expressive Peircean and Ockhamist languages.

For the "Next Time" operator **X** we consider the class of linear, forwards discrete, unbounded models. This operator was already considered in [51]. Let $\mathcal{L}^n_{Prior}$ be the language that expands $\mathcal{L}_{Prior}$ to include the **X** operator.

The truth of a formula with the **X** operator in $\mathcal{L}^n_{Prior}$ at an instant t with a unique immediate successor $s(t)$ in such a model $\mathcal{M}$ is defined as follows:

1.	$\mathbf{K_G}$	$\mathbf{K_H}$	$\mathbf{G}(p \to q) \to (\mathbf{f}p \to \mathbf{f}q)$
2.	$\mathbf{H}p \to \mathbf{P}p$	$\mathbf{G}p \to \mathbf{f}p$	$\mathbf{G}p \to \mathbf{g}p$
3.	$\mathbf{H}p \to \mathbf{HH}p$	$\mathbf{G}p \to \mathbf{GG}p$	$\mathbf{f}p \to \mathbf{ff}p$
4.	$p \to \mathbf{GP}p$	$p \to \mathbf{HF}p$	
5.	$\mathbf{H}p \to (p \to (\mathbf{G}p \to \mathbf{GH}p))$	$\mathbf{H}p \to (p \to (\mathbf{g}p \to \mathbf{gH}p))$	
6.	$\mathbf{fG}p \to \mathbf{Gf}p$		

Table 7.12: Burgess's Axiomatisation for Irreflexive Unbounded Trees as given in [11]

$\mathcal{M}, t \Vdash \mathbf{X}p$ iff $\mathcal{M}, s(t) \Vdash p$

A past analogue of **X**, usually denoted **Y** ("Yesterday"), can be defined likewise. Since trees are left linear, **Y** is well defined on left unbounded, backward discrete trees.

$\mathcal{L}_{Prior}$ can also be expanded to include the two binary modal operators, since and until (first introduced in [43]. Let $\mathcal{L}^{su}_{Prior}$ be the language that expands $\mathcal{L}_{Prior}$ to include the **S** and **U** operators.

The truth of an $\mathcal{L}^{su}_{Prior}$ formula at an instant t in a tree model $\mathcal{M}$ is defined by adding the following two clauses to the truth definition of L_{Prior}-formulas (cf. Section 2.3.1):

- $\mathcal{M}, t \Vdash \mathbf{U}(\varphi, \psi)$ iff $\exists t'$ such that $t < t'$ and $\mathcal{M}, t' \Vdash \varphi$, and $\forall t''$ with $t < t'' < t'$, $\mathcal{M}, t'' \Vdash \psi$
- $\mathcal{M}, t \Vdash \mathbf{S}(\varphi, \psi)$ iff $\exists t'$ such that $t' < t$ and $\mathcal{M}, t' \Vdash \varphi$, and $\forall t''$ with $t' < t'' < t$, $\mathcal{M}, t'' \Vdash \psi$

$\mathbf{U}(\varphi, \psi)$ is interpreted as "ψ will be until φ" and $\mathbf{S}(\varphi, \psi)$ is interpreted as "ψ has been since φ".

In the next section we look at some axiomatisations of the logics of some classes of trees in languages that are extensions of the Priorian languages.

7.2.2 Axiomatisation Results from the Literature

We start this section with a look are results in the standard Peircean and Ockhamist languages, and then look at some results that include additional modal operators.

7.2.2.1 Axiomatisations in the Peircean Language

Axiomatisations of the logics of classes of trees in the Peircean language are scarce in the literature. In [11], an axiomatisation for the logic of the class of irreflexive unbounded trees was given, and in [73] a few alterations were made to the original axiomatisation. Note that the logic include all propositional tautologies as well as the rules MP, GEN, SUB, as well as the following rule:

If p does not appear in φ and $\vdash \mathbf{H}\neg p \to (\neg p \to (\mathbf{G}p \to \varphi))$ then $\vdash \varphi$.

In Table 7.12 we give the axiomatisation presented in [11].

An alternative axiomatisation for the logic of irreflexive unbounded trees was presented in [73]. Here, Zanardo dispenses of the additional rule of Burgess's axiomatisation above, in favour of an infinite set of axioms. For this axiomatisation presented in [73], we need the following notation: If $A = q_1, q_2, ...q_n$ is a set of propositional letters, then we write $\mathbf{g}\neg A$ for $\bigwedge_{i,n}(\mathbf{g}\neg q_i)$. Next, we need the following definition:

Definition 7.2.1. Let Q be a non-empty set of propositions and let $S = Q_0, ..., Q_n$ be a sequence of non-empty subsets of Q such that $\bigcup_i Q_i = Q$, where $Q_i = \{q_{0,i}, q_{1,i}, ..., q_{n_i,i}\}$ for $i = 0, ..., n$. Let p, q and r be any propositional variables. Then

- $\varphi_{p,r}(S) ::= \bigwedge_{i,n}\{\bigwedge_{j,n_i} \mathbf{G}\neg(p \wedge \mathbf{F}r \wedge \mathbf{g}\neg Q_{i-1} \wedge q_{i,j})\}$

Let $S_0, ..., S_m$ be any enumeration of all sequences like S above, then

- $\psi_{p,r}(Q) ::= \bigvee_{i,m}(\varphi_{p,r}(S_i))$

Let a $(K_0, ..., K_h)$ be finite non-empty sets of propositional variables

- $\varphi_{p,r}([K_0, ..., K_h], S) ::= \bigwedge_{i,n}[\mathbf{G}\neg(p \wedge \mathbf{F}r \wedge \bigwedge_{l,h}(\neg\mathbf{g}\neg)K_l \wedge \mathbf{g}\neg Q_{i-1} \wedge q_{i,j})]$
- $\psi_{p,r}([K_0, ..., K_h], S) ::= \bigvee_{i,m}(\varphi_{p,r}([K_0, ..., K_h], S_i))$
- $\Phi_{p,r}(K_0, ..., K_h) ::= \psi_{p,r}(K_0) \wedge \psi_{p,r}([K_0], K_1) \wedge \psi_{p,r}([K_0, K_1], K_2 \wedge ... \wedge \psi_{p,r}([K_0, ..., K_{h-1}], K_h))$
- $\Xi(K_0, ..., K_h) ::= [\mathbf{g}\neg K_0 \vee (\neg\mathbf{g}K_0 \wedge \mathbf{g}\neg\neg K_1) \vee ... \vee (\bigwedge_{j,h-1} \neg\mathbf{g}\neg K_j) \wedge \mathbf{g}\neg K_h)]$

Next we give the axiomatisation of [73] by noting the differences to that in [11]. All as in Table 7.12 except

- Replace $\mathbf{f}p \to \mathbf{ff}p$ in row 3. with $\mathbf{f}(p \vee \mathbf{f}p) \to \mathbf{f}p$.
- Replace $\mathbf{H}p \to (p \to (\mathbf{g}p \to \mathbf{gH}p))$ in row 5. with $\mathbf{H}p \to (p \to (\mathbf{g}(p \wedge q) \to \mathbf{g}(\mathbf{H}p \wedge q)))$.
- Add $\mathbf{g}p \wedge \mathbf{g}q \wedge \Phi_{p,r}(K_0, ..., K_h) \to \mathbf{g}(p \wedge (\mathbf{F}r \wedge \neg q \wedge \mathbf{g}\neg q \wedge \bigwedge_{j,h-1}[\neg\mathbf{g}K_j] \wedge \mathbf{g}\neg K_h \to \mathbf{P}(q \wedge p \wedge \Xi(K_0, ..., K_h))))$.

7.2.2.2 Axiomatisations of CTL and CTL* and variations

Other axiomatisations in the Peircean and Ockhamist languages include the computational tree logics (CTL and CTL*). These logics are interpreted on the class of computation trees, where every branch has the order type of the natural numbers. These trees are naturally obtained as tree unfoldings of discrete transition systems and represent the possible infinite computations arising in such systems (see [38] and [37] for further details on these logics). In [55], an axiomatisation for the logic of trees in the Ockhamist language is proposed and a proof is outlined, but not given. As there is no published proof yet, we will focus our attention of the complete axiomatisations of the computational tree logics. These logics were developed for discrete time, and the models for these logics are trees with branches isomorphic to the natural numbers.

CTL is the temporal logic of the class of omega trees (with branches isomorphic to the natural numbers) with Peircean semantics, and it was introduced by Emerson and Clarke in [24], while CTL* is the temporal logic of the class of omega trees with Ockhamist semantics, and it was introduced by Emerson and Halpern in [25].

The language of CTL consists of the following. If *Prop* is a set of propositional letters, then the set of formulas of CTL can be recursively defined by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \to \psi \mid \Box\mathbf{X}\varphi \mid \Box\mathbf{U}(\varphi, \psi) \mid \Diamond\mathbf{U}(\varphi, \psi) \tag{7.4}$$

The truth of a formula with the new future operators of CTL at an instant t in a tree model $\mathcal{M} = (W, R, V)$ is defined as follows:

- $\mathcal{M}, t \Vdash \Box\mathbf{X}\varphi$ iff for all $s \in W$, if Rts and there is no x such that $tRxRs$, then $\mathcal{M}, s \Vdash \varphi$
- $\mathcal{M}, t \Vdash \Box\mathbf{U}(\varphi, \psi)$ iff for all $h \in H(t)$ there exists an $s \in h$ with Rts and $\mathcal{M}, s \Vdash \varphi$, and for all s' with $Rts's$ we have $\mathcal{M}, s' \Vdash \psi$
- $\mathcal{M}, t \Vdash \Diamond\mathbf{U}(\varphi, \psi$ iff there exists an $h \in H(t)$ such that there exists an $s \in h$ with Rts and $\mathcal{M}, s \Vdash \varphi$, and for all s' with $Rts's$ we have $\mathcal{M}, s' \Vdash \psi$

7.2.2.2.1 CTL

Axiomatisations for CTL can be found in [23] and [32]. We give the axiomatisation of [32] below:

Let Λ_{CTL} be the smallest logic containing the following axioms and are closed under the following rules:

Axioms:

- $\Box\mathbf{X}(p \to q) \to (\Box\mathbf{X}p \to \Box\mathbf{X}q)$
- $\Box\mathbf{X}\top$
- $\Diamond\mathbf{U}(p,q) \leftrightarrow (p \vee (q \wedge \Diamond\mathbf{X}\Diamond\mathbf{U}(p,q)))$
- $\Box\mathbf{U}(p,q) \leftrightarrow (p \vee (q \wedge \Box\mathbf{X}\Box\mathbf{U}(p,q)))$

Rules:

- If $\vdash_{\Lambda_{CTL}} p$ then $\vdash_{\Lambda_{CTL}} \Box\mathbf{X}p$
- If $\vdash_{\Lambda_{CTL}} p \vee (q \wedge \Diamond\mathbf{X}r) \to r$ then $\vdash_{\Lambda_{CTL}} \Diamond\mathbf{U}(p,q) \to r$
- If $\vdash_{\Lambda_{CTL}} p \vee (q \wedge \Box\mathbf{X}r) \to r$ then $\vdash_{\Lambda_{CTL}} \Box\mathbf{U}(p,q) \to r$

7.2.2.2.2 CTL*

The axiomatisation for CTL* we will give in this section can be found in [54], and an extension of CTL* with past operators PCTL* is established in [56]. In [54], Reynolds defines the language by starting with the propositional linear temporal logic PLTL that is axiomatised with the following set of axioms and rules:

Axioms:

- All propositional tautologies
- $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$
- $\mathbf{G}p \to (p \wedge \mathbf{X}p \wedge \mathbf{X}(\mathbf{G}p))$
- $\mathbf{X}\neg p \leftrightarrow \neg\mathbf{X}p$
- $\mathbf{X}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q)$
- $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$
- $\mathbf{U}(p,q) \leftrightarrow (p \vee \mathbf{X}\mathbf{U}(p,q))$
- $\mathbf{U}(p,q) \to \mathbf{F}p$

Rules:

- Modus ponens
- Generalisation
- Substitution

The following was then added to the above axiomatisation:

If *Prop* is a set of propositional letters, then the set of formulas of CTL* can be recursively defined by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \to \psi \mid \mathbf{X}\varphi \mid \mathbf{U}(\varphi, \psi) \mid \mathbf{E}\varphi \tag{7.5}$$

Let $\mathcal{M} = (W, R, V)$ be a model for CTL*. Before defining the truth conditions of the additional operators, we need the following definition:

Full path: A full path in $\mathcal{M}$ is an infinite sequence $s_0, s_1, ...$ of instants in W such that, for each i, we have Rs_is_{i+1}. For any full path $b = s_0, s_1, ...$, we write $b_{\geq i}$ for the full path $s_i, s_{i+1}, ...$, and b_i is just s_i.

We now define the truth of a formula in a model that includes the operators **X**, **U** and **E**, evaluated at a full path b:

- $\mathcal{M}, b \Vdash \mathbf{X}\varphi$ iff $\mathcal{M}, b_{\geq 1} \Vdash \varphi$
- $\mathcal{M}, b \Vdash \mathbf{U}(\varphi, \psi)$ iff there is an $i \geq 0$ such that $\mathcal{M}, b_{\geq i} \Vdash \varphi$ and for each j, if $0 \leq j < i$, then $\mathcal{M}, b_{\geq j} \Vdash \psi$
- $\mathcal{M}, b \Vdash \mathbf{E}\varphi$ iff there is some full path b' such that $b_0 = b'_0$ and $\mathcal{M}, b' \Vdash \varphi$

Let **A** be the dual of **E**, i.e.:

- $\mathcal{M}, b \Vdash \mathbf{A}\varphi$ iff for all full paths b' with $b_0 = b'_0$ we have $\mathcal{M}, b' \Vdash \varphi$

Next we add the following axioms and rules to that of PLTL.

Axioms:

- $\mathbf{A}(p \to q) \to (\mathbf{A}p \to \mathbf{A}q)$
- $\mathbf{A}p \to \mathbf{AA}p$
- $\mathbf{A}p \to p$
- $p \to \mathbf{AE}p$
- $\mathbf{A}\neg p \leftrightarrow \neg\mathbf{E}p$
- $p \to \mathbf{A}p$
- $\mathbf{AX}p \to \mathbf{XA}p$

Rules:

- Add generalisation of the **A** operator: If $\vdash_{CTL*} p$ then $\vdash_{CTL*} \mathbf{A}p$

Let Λ_B be the logic defined above. In [65] it was shown that this logic is sound and weakly complete with respect to the class of bundled CTL* trees (A bundle tree is defined as a triple $(T, <, B)$ where $\mathcal{T} = (T, <)$ is a tree and the bundle $B \subseteq H(T)$ is a non-empty set of branches such that every instant in T belongs to some branch in B.).

By adding the following axiom schema and rule to Λ_B, we get Λ_{CTL*}, which is the logic found to be sound and complete with respect to all CTL* trees (trees with branches isomorphic to the natural numbers) in [54]:

Axiom schema: $\mathbf{AG}(\mathbf{E}p \to \mathbf{XU}(\mathbf{E}p, \mathbf{E}q)) \to (\mathbf{E}p \to \mathbf{EGU}(\mathbf{E}p, \mathbf{E}q))$

Rule: If $\vdash_{CTL*} \varphi \to \psi$ then $\vdash_{CTL*} \psi$, provided that there are disjoint sets L and Q of atoms such that:

Language and Modal Operators	Class of Trees	Reference
CTL* (with past operators) $\Box$, **U**, **Y**	Discrete trees	[56] and [57]
BTC (Branching time with choices) **X**, **F**, **f**, $\Diamond$	Discrete trees	[17]
OSU(Ockhamist Since and Until) with the irreflexivity rule **F**, **S**, **U**	Discrete, irreflexive trees	[74]
$\mathcal{L}_{Peirce}$	Well-founded trees	[7]
CTL$_{rp}$ (CTl* with reference pointers) Although only **X**, **U** and $\Box$ is used, the reference pointers make it possible to consider past formulas as well.	CTL* trees	[36]
CTL* $\Box$, **X**, **U**	R-generable CTL* trees	[42]
CTL with path quantifiers	CTL trees	[1]
$\mathcal{L}_{Peirce}$ **G**, **H**, **f**, **X**, with the IRR rule	Irreflexive trees	[29]
X, **U**, $\Box$, first-order logic	Discrete, unbounded trees	[21]
Monadic second-order logic (MSO)	Finite trees	[30]
$\mathcal{L}^m_{Prior}$ 'mother of', 'first daughter of' and 'second daughter of'	Finite ordered binary trees	[5]

Table 7.13: Further Axiomatisations of Logics of Classes of Trees.

- ψ only used atoms from L
- φ is functionally $L + Q$- expandable (φ is $L + Q$- expandable is we can choose the valuations of the atoms in Q such that φ holds at any instant in the expanded structure).

Some other axiomatisations for trees in the languages CTL and CTL* trees, and some variations of these, were established in the literature. In Table 7.2.2.2.2 we give a brief summary of some of these.

Other interesting axiomatisation results involving bundled trees can be found in, for example, [6], [9] [14].

7.3 Open Questions for Future Study

The field of branching time logics is still mostly unexplored. Although some progress has been made in finding axiomatisations and establishing decidability of the logics of different classes of trees in the Priorian languages, there is still much scope for further research.

Some open questions in the Priorian languages, include the following:

- Is there a finite axiomatisation for the logics of the classes of trees with branches isomorphic to the real numbers in the Priorian temporal language and its future fragment, and if so can we show strong completeness or just weak completeness for these logics. We spent considerable time on trying to use Segerberg's ([60]) methods of showing that the logic of real numbers is weakly complete, but could not make it work for branching time. Based on the work done on linear frames in [19] where they showed strong completeness of the logic of the real numbers, it might be worthwhile looking into these methods as well to see if we could find an axiomatisation for the class of trees with branches isomorphic to the real numbers that is strongly complete. However, we do suspect that we might run into the same issues as before when introducing branching time. Furthermore, none of the logics for any of the real trees have been shown to be decidable or not.

- We have shown decidability of most of the logics for the different classes of trees in Chapter 6. Although we showed decidability of the logics of the classes of well-founded trees, we have not shown that the logics for the classes of conversely well-founded irreflexive/reflexive trees are decidable. It seems that the branching forward makes it difficult to make sure that we can build a model that is conversely-well founded, as we did for the well-founded trees. We also do not have any decidability results for any of the locally finite trees. To the author's knowledge, these still remain open questions.

- Although we covered most of the many interesting and applicable classes of trees in this thesis, there are yet other classes that could still be considered. Since all the classes we considered were defined in terms of properties of the individual branches, it might be interesting to consider other properties of trees, e.g. those related to branching behaviour. For example, finitely branching, or complete binary trees are important from the perspective of applications.

There are also many classes of trees that have not be axiomatised in the Peircean and Ockhamist languages and a full study of these, as well as the decidability of their logics, will also be useful.

Chapter 8

Conclusion

Though many open questions remain on axiomatisations and decidability of the logics of different classes of trees, we have made a big dent in finding axiomatisations and showing decidability for the logics of different classes of trees in the Priorian languages. We also surveyed the literature on axiomatisations of the logics of classes of trees in other languages to get a better idea of which results are still outstanding.

In Chapter 3, we looked at the logics in the future fragment of the Priorian temporal language for different classes of trees. We were able to use several well-known modal axioms to form axiomatisations of these logics. Standard techniques and constructions were then sufficient to prove that these axiomatisations were complete. For most logics we could use extension of the basic logic **K4** by adding axioms that are known to define certain properties, for example the density or seriality axioms, and simple constructions using the canonical model, for example unravelling and building networks, to build a model for a consistent set of formulas for the logics. Since the language of these trees only consider future operators, we used a construction we called "adding a tail" to show completeness with respect to the left unbounded classes of trees. In this process, we found axiomatisations of all classes of trees we listed in the preliminary chapter, in Section 2.2.1, except the classes involving trees with branches isomorphic to the real numbers. Whether the logics of these classes have finite axiomatisations in the future fragment of the Priorian temporal language remains an open question, and if so, we also do not know whether strong completeness is possible.

In Chapters 4 and 5, we made significant progress in finding axiomatisations for the logics of trees in the Priorian temporal language. Most of our logics were extensions of $\mathbf{Pr}_{basic}$, where we added a combination of know and new axioms. Although known methods like bulldozing and filtrations came in useful here, we also had to introduce a new method to build models with the properties appropriate for the class of trees we were interested in, namely unfolding. Interestingly, although unfolding is similar to forwards unravelling, which usually only works for the future fragment of the Priorian temporal language, it was structured in such a way that we preserved the necessary witnesses for past formulas. It was only for the discrete irreflexive trees where we needed "anti-axiom" style rules. The reason we opted for this approach was that we were unable to find an axiom that defined discreteness. A further complication was that it was difficult to build a model that was discrete, since bulldozing only gives a discrete model if there are no degenerate clusters. For this reason, we could use bulldozing for the discrete reflexive trees, but only after we unfolded the filtration to get a left linear model. Again, we were not yet able to find a finite axiomatisation for the logics of the classes of trees with branches isomorphic to the real numbers with their usual strict and non-strict ordering, and hence also do not know if it is possible. It thus remains an open question whether a (strongly) complete axiomatisation for this logic exists. Due the presence of branching, one of the main obstacles was finding an axiom that would ensure that each branch was a complete ordering.

The classes of trees and their axiomatisations are summarised in the Appendices in Section 8.

Looking back at Section 2.9 we see that working with trees, as opposed to linear orders, can make the process of finding axiomatisations more complex, or, in some cases, it could also make it less complex. For example, finding an axiom to capture a specific property is frequently easier in the linear case. There are even some axiomatisations in the literature to show completeness of a logic with respect to the real numbers. However, there seems to be more degrees of freedom when building trees as opposed to linear orders, for example, when adding a new node to a new branch when building a model, we build a network that has no infinitely ascending sequences with a strict upper bound. This would not have been possible in the linear case.

In Chapter 6 we showed that most of the logics for the classes of trees we are interested in were decidable. The methods we used here were mostly known and in some cases only slight adjustments had to be made to adjust for trees. We established the finite model property for several of the logics. We were able to transfer decidability to several future-fragment Priorean logics of classes of trees by showing that their full Priorean analogues were conservative extensions and decidable.The most important methodological contribution of this chapter is perhaps the development of a mosaic-like method for proving that the logics of well-founded trees and trees with with branches isomorphic to the natural numbers are decidable. Because of the way that we have defined a saturated set of fans for a formula, we could built models for the logics in Section 6.2 using forwards unravelling, ensuring that the witnesses for past formulas were preserved in the process, even thought it usually does not work in the temporal language. We have no decidability results for any class of trees with branches isomorphic to the real numbers in the Priorian temporal language and its future fragment. Other open questions here are whether the logics for the class of irreflexive and reflexive conversely well-founded trees, as well as any locally finite trees, are decidable. The method we used for the well-founded trees might still be adapted to apply to locally finite trees. However, this method failed in the case of conversely well-founded trees since branching makes it impossible to guaranty the conversely well-foundedness of the models we were building. Perhaps a different method would work here, otherwise it might be possible to show that these logics are, in fact, not decidable.

In Chapter 7, we summarised all the results of this thesis and also surveyed some related results in the literature, especially those that involved extension of the Priorean language. We specifically looked at two languages that suit trees well, namely the Peircean and Ockhamist languages. Both of these languages can be seen as extensions of the Priorean temporal languages and are more expressive than the Priorian language. Although there are not many axiomatisation results in these languages for classes of trees, there were some interesting tools used that could potentially be applied to other classes of trees to find axiomatisations and show completeness. From this chapter, however, it is clear that there is much scope for further research in the fields of axiomatisations and decidability of the logics of trees.

This leaves us with many interesting open questions to investigate. Some open questions relating to the Priorian temporal language and its future fragment, were summarised in Section 7.3.

There are also many classes of trees whose logics have not be axiomatised in the Peircean and Ockhamist languages and a full study of these, as well as the decidability of their logics, will also be useful.

Therefore, although this thesis has significantly contributed to the axiomatisations and decidability of the logics of trees in the Priorian languages, there is much scope for further investigation.

Appendices

Appendices

Although the results listed here have been noted elsewhere in the thesis, we repeat them here to have a convenient summary in one place.

List of Axioms and Rules

The axioms used in this thesis are listed in Table 1 with their abbreviations and the properties they define. All the axioms are canonical with the exception of **S**, **Q**, $\mathbf{L_l}$, $\mathbf{L_r}$, **Grz** and $\mathbf{Grz_l}$.

We use $\mathbf{A}\varphi$ as an abbreviation for $\mathbf{H}\varphi \wedge \varphi \wedge \mathbf{G}\varphi$ and $\mathbf{E}\varphi$ as an abbreviation for $\mathbf{P}\varphi \vee \varphi \vee \mathbf{F}\varphi$.

The rules that are used in the definition of the logics in this thesis are listed in Table 2. The first three rules are part of all the logics in this thesis while the last three are additional rules introduced to find a completeness result for the class of discrete trees.

Logics and Classes of Trees

Priorian Modal Language

Tables 3 and 4 list the modal logics and the classes of trees they are sound and complete with respect to. The following rules are included in all the logics in Tables 3 and 4: MP, SUB and GEN.

Abbr	Axiom	Property of Kripke Frames defined by Axiom
$\mathbf{K_G}$	$\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$	Valid on all frames
$\mathbf{K_H}$:	$\mathbf{H}(p \to q) \to (\mathbf{H}p \to \mathbf{H}q)$	Valid on all frames
$\mathbf{Dual_F}$	$\mathbf{F}p \leftrightarrow \neg\mathbf{G}\neg p$	Valid on all frames
$\mathbf{Dual_P}$	$\mathbf{P}p \leftrightarrow \neg\mathbf{H}\neg p$	Valid on all frames
Conv	$p \to \mathbf{GP}p$ and $p \to \mathbf{HF}p$	Valid on all frames
4	$\mathbf{FF}p \to \mathbf{F}p$	$\forall xyz(Rxy \wedge Ryz \to Rxz)$ (Transitivity)
$\mathbf{.3_l}$	$(\mathbf{P}p \wedge \mathbf{P}q) \to \mathbf{P}(p \wedge \mathbf{P}q) \vee \mathbf{P}(p \wedge q) \vee \mathbf{P}(q \wedge \mathbf{P}p)$	$\forall xyz(Rxz \wedge Ryz \to Rxy \vee Ryx \vee x = y)$ (Left linear)
D	$\mathbf{F}p \to \mathbf{FF}p$	$\forall xy(Rxy \to \exists x(Rxz \wedge Rzy))$ Density
$\mathbf{U_l}$	$\mathbf{P}\top$	$\forall x \exists y Ryx$ Left seriality
$\mathbf{U_r}$	$\mathbf{F}\top$	$\forall x \exists y Rxy$ Right seriality
S	$\neg p \wedge \mathbf{H}\neg p \wedge \mathbf{F}p \to \mathbf{F}(p \wedge \mathbf{H}\neg p)$	No infinitely descending sequences between two instants (irreflexive frames)
Q	$\mathbf{P}p \vee \mathbf{GH}\neg p \vee \mathbf{F}(p \wedge \mathbf{H}(p \vee \mathbf{H}\neg p))$	No infinitely descending sequences of distinct instants between two instants (reflexive frames)
$\mathbf{L_l}$	$\mathbf{H}(\mathbf{H}p \to p) \to \mathbf{H}p$	Well-foundedness, transitivity, irreflexivity
$\mathbf{L_r}$	$\mathbf{G}(\mathbf{G}p \to p) \to \mathbf{G}p$	Converse well-foundedness, transitivity, irreflexivity
T	$p \to \mathbf{F}p$	Reflexivity
Grz	$\mathbf{G}(\mathbf{G}(p \to \mathbf{G}p) \to p) \to p$	No infinitely ascending sequences of distinct instants, transitivity, reflexivity
$\mathbf{Grz_l}$	$\mathbf{H}(\mathbf{H}(p \to \mathbf{H}p) \to p) \to p$	No infinitely descending sequences of distinct instants, transitivity, reflexivity

Table 1: List of Axioms

Abbr	Rule	Class of Frames on which Rule is Valid
MP	If $\vdash p$ and $\vdash p \to q$ then $\vdash q$.	All frames
SUB	If $\vdash \varphi$ then $\vdash \psi$, where φ is obtained from ψ by uniformly replacing proposition letters in φ by arbitrary formulas.	All Frames
GEN	If $\vdash \varphi$ then $\vdash \mathbf{G}\varphi$ and If $\vdash \varphi$ then $\vdash \mathbf{H}\varphi$.	All frames
IRR	If $\vdash \neg p \wedge \mathbf{H}p \to \varphi$ then $\vdash \varphi$ where $p \notin var(\varphi)$.	Irreflexive frames
FDR	If $(\vdash \mathbf{F}p \wedge \neg\mathbf{FF}p \to \varphi) \vee \mathbf{G}\bot$ then $\vdash \varphi$ where $p \notin var(\varphi)$.	Forwards discrete frames
BRD	If $(\vdash \mathbf{P}p \wedge \neg\mathbf{PP}p \to \varphi) \vee \mathbf{H}\bot$ then $\vdash \varphi$ where $p \notin var(\varphi)$.	Backwards discrete frames

Table 2: List of Rules

Logic	Axioms	Class of Kripke frames	Class of trees
K4	**K_G**, **$Dual_F$**, **4**	Transitive frames	Irreflexive trees (3.1.1), rooted irreflexive trees (3.1.2), discrete irreflexive trees (3.1.3), locally finite irreflexive trees (3.1.4), well-founded irreflexive trees (3.1.5), left-unbounded irreflexive trees (3.1.10)
$K4U_r$	**K4 + U_r**	Right-unbound transitive frames	Right-unbounded irreflexive trees (3.1.6), unbounded irreflexive trees (3.1.11), irreflexive trees with branches isomorphic to the natural numbers (3.1.8) and to the integers (3.1.12)
KL_l	**K_G**, **$Dual_F$**, **L_l**	Transitive, conversely well-founded frames (and hence irreflexive)	Finite irreflexive trees (weak completeness) (3.1.16), conversely well-founded irreflexive trees (weak completeness) (3.1.17)
K4D	**K4 + D**	Transitive, dense frames	Dense irreflexive trees (3.1.21), left unbounded dense irreflexive trees (3.1.10
$K4DU_r$	**K4 + D**, **U_r**	Transitive, unbounded, dense frames	Right unbounded dense irreflexive trees (3.1.22), irreflexive trees with branches isomorphic to the non-negative rational numbers (3.1.23), unbounded dense irreflexive trees (3.1.24), irreflexive trees with branches isomorphic to the rational numbers (3.1.25)

Table 3: Modal Logics of Irreflexive Trees

Logic	Axioms	Class of Kripke frames	Class of trees
S4	**K4 + T**	Transitive, reflexive frames	Reflexive trees (3.2.1), discrete reflexive trees (3.2.3), locally finite reflexive trees (3.2.4), rooted reflexive trees (3.2.2), left unbounded reflexive trees (3.2.6), well-founded reflexive trees (3.2.5), right unbounded reflexive trees (3.2.7), unbounded reflexive trees (3.2.7), reflexive trees with branches isomorphic to $(\mathbb{N}, \leq)$ (3.2.8), reflexive trees with branches isomorphic to $(\mathbb{Z}, \leq)$ (3.2.8), left unbounded dense reflexive trees (3.2.18), right unbounded dense trees (3.2.19), unbounded dense trees (3.2.19), reflexive trees with branches isomorphic to $\langle\mathbb{Q}), \leq\rangle$ (3.2.19), and $\langle\mathbb{Q}^{+} \cup \{0\}, \leq\rangle$ (3.2.19).
KGrz	**K_G**, **$Dual_F$**, **Grz**	Transitive finite reflexive frames	Finite reflexive trees (weak completeness) (3.2.14), conversely well-founded reflexive trees (weak completeness) (3.2.15)

Table 4: Modal Logics of Reflexive Trees

Logic	Axioms/Rules	Class of bi-directional Kripke frames	Class of trees
$\mathbf{Pr}_{basic}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, **4**, $\mathbf{.3_l}$	Transitive, left linear frames	Irreflexive trees (4.2.2)
$\mathbf{Pr}_{basic}\mathbf{U_l}$	$\mathbf{Pr}_{basic} + \mathbf{U_l}$	Transitive, left linear, left unbounded frames	Left unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{basic}\mathbf{U_r}$	$\mathbf{Pr}_{basic} + \mathbf{U_r}$	Transitive, left linear, right unbounded frames	Right unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{unbnd}$	$\mathbf{Pr}_{basic} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, unbounded frames	Unbounded irreflexive trees (4.4.1)
$\mathbf{Pr}_{dense}$	$\mathbf{Pr}_{basic} + \mathbf{D}$	Transitive, left linear, dense frames	Dense irreflexive trees (4.5.4)
$\mathbf{Pr}_{dense}\mathbf{U_l}$	$\mathbf{Pr}_{dense} + \mathbf{U_l}$	Transitive, left linear, left unbound, dense frames	Left unbounded dense irreflexive trees (4.7.1)
$\mathbf{Pr}_{dense}\mathbf{U_r}$	$\mathbf{Pr}_{dense} + \mathbf{U_r}$	Transitive, left linear, right unbound, dense frames	Right unbounded dense irreflexive trees (4.7.1)
$\mathbf{Pr}_{\mathbb{Q}}$	$\mathbf{Pr}_{dense} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, dense, unbounded frames	Irreflexive trees with branches isomorphic to the rational numbers (4.8.1)
$\mathbf{Pr}_{lfin}$	$\mathbf{Pr}_{basic} + \mathbf{S}$	Transitive, left linear, locally finite frames	Locally finite irreflexive trees (weak completeness) (5.4.7)
$\mathbf{Pr}_{lfin}\mathbf{U_l}$	$\mathbf{Pr}_{lfin} + \mathbf{U_l}$	Transitive, left linear, left unbound, locally finite frames	Left unbound locally finite irreflexive trees (weak completeness) (5.6.1)
$\mathbf{Pr}_{lfin}\mathbf{U_r}$	$\mathbf{Pr}_{lfin} + \mathbf{U_r}$	Transitive, left linear, right unbound, locally finite frames	Right unbound locally finite irreflexive trees (weak completeness) (5.6.1)
$\mathbf{Pr}_{\mathbb{Z}}$	$\mathbf{Pr}_{lfin} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, unbound, locally finite frames	Unbounded locally finite irreflexive trees (weak completeness) (5.6.1), irreflexive trees with branches isomorphic to the integers (weak completeness) (5.7.3)

Table 5: Temporal Logics of Trees 1

Priorian Temporal Language

Tables 5, 6 and 7 list the temporal logics and the classes of trees they are sound and complete with respect to. The following rules are included in all the logics in these tables: MP, SUB and GEN.

Logic	**Axioms/Rules**	**Class of bi-directional Kripke frames**	**Class of trees**
$\mathbf{Pr}_{discr}$	$\mathbf{Pr}_{basic}$, IRR, FDR, BDR	Transitive, left linear	Discrete irreflexive trees (5.1.11)
$\mathbf{Pr}_{discr}\mathbf{U_l}$	$\mathbf{Pr}_{discr} + \mathbf{U_l}$	Transitive, left linear, left unbound, frames	Left unbound discrete irreflexive trees (weak completeness) (5.3.1)
$\mathbf{Pr}_{discr}\mathbf{U_r}$	$\mathbf{Pr}_{discr} + \mathbf{U_r}$	Transitive, left linear, right unbound, frames	Right unbound discrete irreflexive trees (weak completeness) (5.3.1)
$\mathbf{Pr}^{r}_{udisc}$	$\mathbf{Pr}_{discr} + \mathbf{U_l} + \mathbf{U_r}$	Transitive, left linear, unbounded frames	Unbounded, discrete Trees (5.3.1)
$\mathbf{Pr}_{fin}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $\mathbf{.3_l}$, $\mathbf{L_l}$, $\mathbf{L_r}$	Transitive, well-founded and conversely well-founded frames	Finite irreflexive trees (weak completeness) (5.12.1)
$\mathbf{Pr}_{wf}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $\mathbf{.3_l}$, $\mathbf{L_l}$	Transitive, well-founded frames	Well-founded irreflexive trees(weak completeness) (5.8.7)
$\mathbf{Pr}_{cwf}$	$\mathbf{K_H}$, $\mathbf{K_G}$, Dual, Conv, $\mathbf{.3_l}$, $\mathbf{L_r}$	Transitive, conversely well-founded frames	Conversely well-founded irreflexive trees(weak completeness) (5.8.15)
$\mathbf{Pr}_{\mathbb{N}}$	$\mathbf{Pr}_{wf} + \mathbf{U_r}$	Transitive, left linear, conversely well-founded, locally finite, right unbounded frames	Irreflexive trees with branches isomorphic to the natural numbers (weak completeness) (5.11.2)
$\mathbf{Pr}_{basic}\mathbf{T}$	$\mathbf{Pr}_{basic} + \mathbf{T}$	Transitive, left linear, reflexive frames	Reflexive trees (4.3.2), discrete reflexive trees (5.2.14), unbounded discrete reflexive trees (5.3.2), left unbounded reflexive trees (4.4.2), right unbounded reflexive trees (4.4.2), unbounded reflexive trees (4.4.2), dense reflexive trees (4.6.3), left unbounded dense reflexive trees (4.7.2), right unbounded dense reflexive trees (4.7.2), unbounded dense trees (4.7.2), reflexive trees with branches isomorphic to $\langle \mathbb{Q}, \leq \rangle$ (4.8.1)

Table 6: Temporal Logics of Trees 2

Logic	Axioms/Rules	Class of bi-directional Kripke frames	Class of trees
$\mathbf{Pr}^{r}_{fin}$	$\mathbf{K_t}$, $\mathbf{.3_l}$ + **Grz** + $\mathbf{Grz_l}$	Transitive, left linear, well-founded, conversely well-founded, reflexive frames	Finite reflexive trees (weak completeness) (5.13.1)
$\mathbf{Pr}_{basic}\mathbf{TGrz_l}$	$\mathbf{Pr}_{basic}$ + **T** + $\mathbf{Grz_l}$	Transitive, left linear, well-founded, reflexive frames	Well-founded (weak completeness) (5.9.6), reflexive trees with branches isomorphic to $(\mathbb{N}, \leq)$ (weak completeness) (5.11.4).
$\mathbf{Pr}_{basic}\mathbf{TGrz}$	$\mathbf{Pr}_{basic}$+**T**+**Grz**	Transitive, left linear, conversely well-founded, reflexive frames	Conversely well-founded reflexive trees (weak completeness) (5.10.2).
$\mathbf{Pr}^{r}_{lfin}$	$\mathbf{Pr}_{basic}$ + **T** + **Q**	Transitive, left linear, reflexive, frames with no infinitely descending sequences of distinct instants between instants	Locally finite reflexive trees (weak completeness) (5.5.6), locally finite left unbounded reflexive trees (weak completeness) (5.6.2), locally finite right unbounded reflexive trees (weak completeness) (5.6.2), locally finite unbounded reflexive trees, reflexive trees with branches isomorphic to the integers (weak completeness) (5.6.2, 5.7.4

Table 7: Temporal Logics of Trees 3

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