X and Y-coordinates of Pell Equation of Some Special forms

By

Faith Shadow Zottor
Supervisor: Prof Florian Luca

This thesis is submitted to the University of the Witwatersrand
in fulfilment of the requirements
for the award of PhD Mathematics.

DECLARATION

This thesis was written in the School of Mathematics, University of the Witwatersrand in fulfilment of the requirements for the award of PhD in Mathematics under the supervision Prof. Florian Luca

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of the Witwatersrand or any other University.

Signature: 
Student: Faith Shadow Zottor
fzottor@gmail.com
January 16th 2023

Abstract

This work primarily characterises the values of d in Pell equation $X^2 - dY^2 = \pm 1$ that have at least some specified number of the sequence of X or Y solutions belonging to some interesting sequence of positive integers.

Preface

The main chapters of this thesis have either been published or submitted for publication in accredited journals. The content of Chapter 2 is published in [19]. The Content of Chapter 3 has been submitted to an accredited journal for publication.

All computations were done using Mathematica.

Dedication

My sincere gratitude goes to God for taking me through this. I am heavily indebted to Prof Florian Luca, my supervisor, for giving me this opportunity. Thanks for your inspiration, guidance and patience. I am grateful to Mr Michael Cudjoe, Ms Irene Obeng, Mr Shashilan Singh,Dr Danny Mukonda, Mr Victor Zankoni, my dad Mr Christopher Zottor, my mum Madam Victoria Avornyo, Madam Witty Ahliyah,Ms Thandeka Mvelase, Mr Louis Zottor, Mr Shelter Zottor, and to the entire Zottor and Ahliyah family for their great support and love.

This is dedicated to you all.

ACKNOWLEDGEMENTS

The PhD was funded by the National Research Foundation of South Africa and The World Academy of Sciences under the NRF-TWAS African Renaissance Doctoral Scholarship. I also acknowledge receipt of the PhD Merit Award from the University of the Witwatersrand.

Contents

Organization of the Work

- Chapter 1 serves an introductory purpose, introducing some concepts that the body of the thesis uses.
- Chapter 2 discusses the problem of finding the values of d in the Pell equation that have at least two X coordinates being base-b repdigits for some integer base $b \geq 2$.
- Chapter 3 discusses the problem of finding the values of d in the Pell equation that have at least three Y coordinates being Fibonacci numbers.
- In Chapter 4, a summary of the main results is provided.

Chapter 1

Introduction and Motivation

The material in this section appears widely in the literature. Example, see [25].

Subject of Investigation

For S a special set of numbers such as; squares, Fibonacci numbers, Tribonacci numbers, rep-digits etc, we consider, for non-square $d > 1$, the X and Y-coordinates of the Pell equations $X^2 - dY^2 = \pm 1$ and impose the condition that $Y_k \in S$ or $X_k \in S$, where (X_k, Y_k) is the kth solution of the Pell equation. We then enquire about how many values of k the membership $Y_k \in S$ or $X_k \in S$ holds is satisfied by only finitely many d? As an example, if we take the set S as the Tribonacci numbers, for how many values of k do we require the membership $Y_k \in S$ to hold so that there are only finitely many d? The following discussions show that the case of requiring the membership for only two values of k is not interesting as there are infinitely many d for which this happens in the case of the $Y-$coordinates. So do we need to impose 3, 4 or 100 such k to get finitely many d?

1.1 Methodology

We applied results from the theory of Diophantine Approximations specifically due to Matveev and Baker to create absolute bounds on variables. The bounds were then reduced by some reduction algorithms like the Baker-Davenport reduction method. All computations were carried out with Mathematica.

1.2 Introduction

The Pell equation is an equation of the form $X^2 - dY^2 = 1$; to which we desire positive integer solutions (X, Y). Any pair (X, Y) solving the Pell equation for which $XY = 0$ is called a trivial solution. For any integer d, $(\pm 1, 0)$ is easily seen as a trivial solution to the Pell equation. The restriction of the pair (X, Y) to only positive integers leads us to consider only the case where d is a non-square positive integer as affirmed by the following;

Proposition 1.2.1 *The condition that d is a positive integer which is not a square is necessary for the Pell equation to have a nontrivial solution.*

It is worth noting that if (X, Y) is a solution for any positive integer d, then $(-X, -Y)$, $(-X, Y)$, and $(X, -Y)$ are also solutions.

Example 1.2.2 (The case $d = 2$) *When $d = 2$, we consider $X^2 - 2Y^2 = 1$ which we rewrite as;*

$$Y^2 = \frac{X^2 - 1}{2}.$$

We attempt to find positive integer solutions (X, Y). This suggests $X^2 - 1$ is even, and so X must be necessarily odd. We discard the choice $X = 1$ since it leads to the trivial solution $(1, 0)$.
With the choice $X = 3$, we obtain;

$$Y^2 = \frac{8}{2} = 4 = 2^2.$$

The pair $(3, 2)$ is thus seen to be the smallest positive integer solution for the case $d = 2$. Continuing in this manner, the next smallest value of X to produce a square is $X = 17$ with the corresponding $Y^2 = 144 = 12^2$. But will a further search for results be fruitful? Yes! and as will be shown later, we shall encounter solutions infinitely many times in a further search. We will look at a closed form formula for all the solutions.

Lagrange answered the question of the existence of nontrivial solutions in;

Theorem 1.2.3 (Lagrange) *For every positive integer d that is not a perfect square, $X^2 - dY^2 = 1$ has a non-trivial (integral) solution.*

1.2.1 Generating more Solutions from one Solution

Now that the question of the existence of a non-trivial solution is out of the way, we ask how many non-trivial solutions there are for a particular non-square positive d. Let's first get back to some basics in our quest to answer this question. As will be shown later, we can conveniently choose d in the Pell equations as a positive square free integer greater than 1.

Definition 1.2.4 (Quadratic Number Field) *This is a degree 2 field extension of the field K of rational numbers. K thus has dimension 2 as a vector space over $\mathbb{Q}$.*

The quadratic number fields are formed by adjoining $\sqrt{d}$ to $\mathbb{Q}$ for a squarefree $d \neq 0, 1$. We denote this by $\mathbb{Q}\left(\sqrt{d}\right) := \left\{a + b\sqrt{d} : a, b \in \mathbb{Q}\right\}$. For any $\alpha = a + b\sqrt{d} \in \mathbb{Q}\left(\sqrt{d}\right)$, $\alpha' := a - b\sqrt{d} \in \mathbb{Q}[\sqrt{d}]$ since

$\mathbb{Q}\left(\sqrt{d}\right)$ is a field. We term α', the conjugate of α. We equip $\mathbb{Q}\left(\sqrt{d}\right)$ with a norm map $N : \mathbb{Q}\left(\sqrt{d}\right) \to \mathbb{Q}$ which associates $\alpha\alpha'$ to every $\alpha \in \mathbb{Q}\left(\sqrt{d}\right)$. Consider the ring;

$$\mathbb{Z}[\sqrt{d}] := \left\{a + b\sqrt{d} : a, b \in \mathbb{Z}\right\} \subset \mathbb{Q}(\sqrt{d}).$$

It is easily seen that the norm map applied to elements of $\mathbb{Z}[\sqrt{d}]$ outputs integers. For any $\alpha = a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$,

$$N(\alpha) = \left(a + b\sqrt{d}\right)\left(a - b\sqrt{d}\right) = a^2 - db^2.$$

We then see very interestingly that every positive integer solution (a, b) to the Pell equation corresponds to an element $a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ of norm 1.

Whenever $\alpha \in \mathbb{Z}[\sqrt{d}]$ has norm 1, we have $\alpha\alpha' = 1$. We view α' as the multiplicative inverse of α. Every norm 1 element of $\mathbb{Z}[\sqrt{d}]$ is thus a unit. We can form a multiplicative group of units of the ring $\mathbb{Z}[\sqrt{d}]$. We can therefore construct a norm 1 unit from any two norm 1 units of the group of units of $\mathbb{Z}[\sqrt{d}]$ since it obeys closure under multiplication and the norm map is multiplicative. For consider (X_1, Y_1), (X_2, Y_2) solutions to $X^2 - dY^2 = 1$ corresponding to the units $X_1 + Y_1\sqrt{d}$ and $X_2 + Y_2\sqrt{d}$ of $\mathbb{Z}[\sqrt{d}]$, we apply the norm map to their product;

$$\begin{aligned} N\left(X_1 + Y_1\sqrt{d}\right)\left(X_2 + Y_2\sqrt{d}\right) &= N\left(X_1 + Y_1\sqrt{d}\right)N\left(X_2 + Y_2\sqrt{d}\right) \\ &= \left(X_1^2 - dY_1^2\right)\left(X_2^2 - dY_2^2\right) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

The product is thus a norm 1 unit and hence a solution. The generation of more norm 1 units means ultimately more integer solutions to the the Pell equation.

1.2.2 The case of $d = 2$ revisited.

The solution $(X_1, Y_1) = (3, 2)$ to $X^2 - 2Y^2 = 1$ corresponds to the unit $3 + 2\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. The computation;

$$\left(3 + 2\sqrt{2}\right)^2 = \left(3 + 2\sqrt{2}\right)\left(3 + 2\sqrt{2}\right) = 17 + 12\sqrt{2}$$

leads us to identify $17 + 12\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ as a unit and the corresponding $(17, 12)$ generated as a solution to $X^2 - 2Y^2 = 1$. We readily agree that we can produce more solutions beyond these two by performing the computation $\left(3 + 2\sqrt{2}\right)^n$ for integer values of $n \geq 3$. We thus see that the identification of a single nontrivial norm 1 element $3 + 2\sqrt{2}$ of $\mathbb{Z}[\sqrt{2}]$ leads us to produce infinitely many more norm 1 elements of $\mathbb{Z}[\sqrt{2}]$ with the corresponding infinitely many more integer solutions to $X^2 - 2Y^2 = 1$. It is easy to see that this is the case in general for any positive squarefree integer $d > 1$.

Theorem 1.2.5 *If (a, b) and (s, t) satisfy the Pell equation, then (u, v) which satisfy*

$u+v\sqrt{d}=(a+b\sqrt{d})(s+t\sqrt{d})$ *also satisfies the Pell equation.*

Proof. We have $u=as+btd$ and $v=at+sb$ from the equation $u+v\sqrt{d}=(a+b\sqrt{d})(s+t\sqrt{d})$. This gives;

$$\begin{aligned} u^2-dv^2 &= (as+btd)^2-d(at+sb)^2 \\ &= (as)^2-d(at)^2+(btd)^2-d(sb)^2 \\ &= a^2(s^2-dt^2)-db^2(s^2-dt^2) \\ &= a^2-db^2 \\ &= 1 \end{aligned}$$

This means, all (a,b) that satisfy $a+b\sqrt{d}=(x+y\sqrt{d})^n$ for positive integer values of n for any non-trivial (x,y) that satisfies the Pell equation also satisfies Pell equation. So the Pell equation has infinitely many solutions if it has a non-trivial solution.
Granted that the computations $\left(3+2\sqrt{2}\right)^n$ for integer values of $n\geq 1$ yield infinitely many positive integer solutions (a,b) to $X^2-2Y^2=1$, a legitimate question is of all the positive integer solutions are of the form $\left(3+2\sqrt{2}\right)^n$. This is answered relievingly affirmatively as the following argument shows;
Suppose (a,b) is a positive integer solution to $X^2-2Y^2=1$ but not of the form $\left(3+2\sqrt{2}\right)^n$. Consider the corresponding norm 1 unit $a+b\sqrt{2}$. We have necessarily that $a+b\sqrt{2}>3+2\sqrt{2}$. There exists a largest n such that $a+b\sqrt{2}>\left(3+2\sqrt{2}\right)^n$. For this n, the inequality $a+b\sqrt{2}>\left(3+2\sqrt{2}\right)^{n+1}$ does not hold. Now consider the following product of norm 1 units $\alpha=\left(a+b\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{-n}=\left(a+b\sqrt{2}\right)\left(3-2\sqrt{2}\right)^n$. The product is necessarily of the form $\alpha=X'+Y'\sqrt{d}$ and (X',Y') is necessarily a positive integer solution to $X^2-2Y^2=1$. We must have $\alpha<3+2\sqrt{2}$ since otherwise, we would have $\alpha=(a+b\sqrt{2})\left(3-2\sqrt{2}\right)^n\geq 3+2\sqrt{2}$. And so $a+b\sqrt{2}\geq\left(3+2\sqrt{2}\right)^{n+1}$. Since $a+b\sqrt{2}$ cannot be a power of $3+2\sqrt{2}$, we have that $a+b\sqrt{2}>\left(3+2\sqrt{2}\right)^{n+1}$; a contradiction. But $\alpha<3+2\sqrt{2}$ is a contradiction since $(3,2)$ is the smallest positive integer solution and thus any other positive integer solution(a,b) must satisfy $a+b\sqrt{2}>3+2\sqrt{2}$. So all the positive integer solutions are indeed of the given form.

Explicit formulas for the solutions to $X^2-2Y^2=1$

Let (X_n,Y_n) be a positive integer solution to $X^2-2Y^2=1$. Then $X_n+Y_n\sqrt{2}=(3+2\sqrt{2})^n$. Since conjugation in $\mathbb{Q}\left(\sqrt{2}\right)$ is a homomorphism, we have that $X_n+Y_n\sqrt{2}$ and $(3+2\sqrt{2})^n$ have the same image under conjugation so $X_n-Y_n\sqrt{2}=(3-2\sqrt{2})^n$. Writing $V=3+2\sqrt{2}$ and $V'=3-2\sqrt{2}$, we have that $X_n=\frac{1}{2}\left(V^n+(V')^n\right)$ and $Y_n=\frac{1}{2\sqrt{2}}\left(V^n-(V')^n\right)$. Note that $\frac{1}{2}(V')^n$, $\frac{1}{2\sqrt{2}}(V')^n<\frac{1}{2}$. So we could write $X_n=\left\lceil\frac{V^n}{2}\right\rceil$ and $Y_n=\left\lfloor\frac{V^n}{2\sqrt{2}}\right\rfloor$. As a reward for this, we see that $\frac{X_n}{Y_n}$ is a very close rational approximation to $\sqrt{2}$.
We have so far taken for granted the existence if a nontrivial solution to the Pell equation. We now take steps to establish this. Our starting step is the following result due to Dirichlet.

Theorem 1.2.6 (Box Principle) *If there are n objects to be placed in $m < n$ boxes, then at least $\lceil \frac{n}{m} \rceil$ objects must be in the same box.*

We denote by $[\alpha]$ and $\{\alpha\}$ the integer and fractional parts respectively, of a real number α.

Lemma 1.2.7 *For α an irrational number and any positive integer s, the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{sb} \tag{1.1}$$

has an integer solution (a, b) with $1 \leq b \leq s$.

Proof. Consider the following partition of the unit interval $[0, 1)$ of the real number line into s intervals of equal length $\frac{1}{s}$;

$$\left[0, \frac{1}{s}\right), \left[\frac{1}{s}, \frac{2}{s}\right), \ldots, \left[\frac{s-1}{s}, 1\right). \tag{1.2}$$

We list the numbers $\{\alpha t\}$ for $t = 1, 2, \ldots, s+1$; a list of $s+1$ numbers. The box principle suggests that any fitting of the $s+1$ numbers on the list into the intervals of (1.2) will have at least 2 of them in one interval. Remember that the fractional part of any real number α lies in the interval $[0, 1)$. Suppose that $\{\alpha t_1\}$ and $\{\alpha t_2\}$ lie in the same interval, then it must be true since the intervals are open at one end that;

$$|\{\alpha t_1\} - \{\alpha t_2\}| < \frac{1}{s}. \tag{1.3}$$

Since any real sum is the sum of the fractional and integer parts, we have that;

$$|(\alpha t_1 - [\alpha t_1]) - (\alpha t_2 - [\alpha t_2])| < \frac{1}{s}. \tag{1.4}$$

We choose $b = t_1 - t_2$ and $a = [\alpha t_1] - [\alpha t_2]$ and write $|b\alpha - a| < \frac{1}{s}$ and so $\left|\alpha - \frac{a}{b}\right| < \frac{1}{sb}$ and $b \leq s + 1 - 1 = s$.
Additionally, we see that by increasing the values of s, we produce newer solutions (a, b).
As a bonus, realising that $1 \leq b \leq s$, we see that $\frac{1}{sb} \leq \frac{1}{b^2}$ and so any solution of (1.4) is also a solution to;

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2}. \tag{1.5}$$

So we have that

Corollary 1.2.8 *For any irrational number α, the inequality;*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2} \tag{1.6}$$

has infinitely many integer solutions (a, b).

We also have the following result;

Lemma 1.2.9 *Let d be a positive nonsquare integer. There exists some integer K satisfying $|K| < 2\sqrt{d} + 1$ such that the equation $X^2 - dY^2 = K$ has infinitely many integer solutions.*

Note that $K \neq 0$ since $\sqrt{d}$ is irrational.

Proof. Choose α in (1.6) as $\sqrt{d}$ and let (a, b) be a solution to (1.6.) Then, $\frac{a}{b} < \sqrt{d} + 1$ and

$$\left|a^2 - db^2\right| = b^2 \left|\sqrt{d} - \frac{a}{b}\right| \left|\sqrt{d} + \frac{a}{b}\right| < b^2 . \frac{1}{b^2} \left|\sqrt{d} + \frac{a}{b}\right| < 2\sqrt{d} + 1.$$

This means all the infinitely many solutions of (a, b) of (1.6) are such that $a^2 - db^2$ takes on one of the finitely many integer values in the interval $\left(-\left(2\sqrt{d}+1\right), 2\sqrt{d}+1\right)$. Then we see very easily by the box principle that for one value K such that $|K| < 2\sqrt{d} + 1$, there must be infinitely many (X, Y) such that $X^2 - dY^2 = K$.

There is just one more result before we arrive at our target.

Lemma 1.2.10 *There exists a nonzero integer K such that $X^2 - dY^2 = K$ has at least two distinct positive integer solutions (X_1, Y_1), (X_2, Y_2) satisfying $X_1 \equiv X_2 \pmod{|K|}$ and $Y_1 \equiv Y_2 \pmod{|K|}$.*

Proof. Choose K such that $X^2 - dY^2 = K$ has infinitely many solutions. The existence of such a K is guaranteed by Lemma 1.2.9. For any 2 positive integers X and Y satisfying $X^2 - dY^2 = K$, we reduce X and Y modulo $|K|$. Note that $X \pmod{|K|}$ and $Y \pmod{|K|}$ has a finite number of possibilities for the infinitely many (X, Y) that satisfy $X^2 - dY^2 = K$. So it must be the case by the box principle that there are at least 2 distinct positive integer solutions (X_1, Y_1) and (X_2, Y_2) such that $X_1 \equiv X_2 \pmod{|K|}$ and $Y_1 \equiv Y_2 \pmod{|K|}$.

After the hardwork, we are finally there!

Proof of Theorem 1.2.3 (Lagrange's theorem). Let $(X_1, Y_1$ and (X_2, Y_2) be two distinct solutions to $X^2 - dY^2 = K$ that the previous lemma guaranteed. Then we must have that for $U_1 = X_1 + Y_1\sqrt{d}$ and $U_2 = X_2 + Y_2\sqrt{d}$, $N(U_1) = N(U_2) = K$. The product $U_1 U_2^{-1}$ is a norm 1 unit in $\mathbb{Z}\left(\sqrt{d}\right)$ and thus a solution to $X^2 - dY^2 = 1$. Let's write $M = U_1 U_2^{-1} = a + b\sqrt{d}$. But hey! are a and b indeed integers? If they are, is the solution nontrivial? We now check that;

$$M = U_1 U_2^{-1} = \left(X_1 + Y_1\sqrt{d}\right) \frac{\left(X_2 - Y_2\sqrt{d}\right)}{K}.$$

So,

$$a = \frac{X_1 X_2 - dY_1 Y_2}{K}, \quad b = \frac{X_1 Y_2 - X_2 Y_1}{K}.$$

Recall by the the previous Lemma, we have $X_1 \equiv X_2 \pmod{|K|}$ and $Y_1 \equiv Y_2 \pmod{|K|}$. So we have that

$$aK = X_1 X_2 - dY_1 Y_2 \equiv X_1^2 - dY_1^2 \equiv 0 \pmod{|K|}$$

giving us that a is an integer. Similarly,

$$bK = X_1 Y_2 - X_2 Y_1 \equiv X_1 Y_1 - X_1 Y_1 \equiv 0 \pmod{|K|}.$$

So we have that b is also an integer. Assume for a contradiction that $b = 0$, then $M = U_1 U_2^{-1} = 1 + 0\sqrt{d} = 1$. This gives $U_1 = U_2$. So we have that a nontrivial solution to $X^2 - dY^2 = 1$ always exists for any positive non-square integer d.

Since $x^2 - (x^2 - 1) = 1$, we have by setting $d = x^2 - 1$ that $(x, 1)$ is a solution to Pell equation. Note that this choice of d requires the least value of x to be 2. That d is positive is then obvious. That d is non-square is also seen by observing that $(x-1)^2$, and x^2, are consecutive squares and $(x-1)^2 < x^2 - 1 < x^2$. So, $d = x^2 - 1$ cannot be square, since there is no square between two consecutive squares. This means for every $x \geq 2$, we can find a positive non-square d such that $(x, 1)$ is a solution to Pell equation; and so for any sequence $\{U_n\}_{n\geq 1} := S$ chosen from the set of positive integers greater than 1, there is always a d for every member x of S such that $(x, 1)$ satisfies the Pell equation. The solution $(x, 1)$ is very easily seen as the smallest positive solution for the chosen d for x from the given set. But once the d is chosen, how many more of the infinitely-many x-coordinates that satisfy $X^2 - dY^2 = 1$ are also from the set S?
Consider the case where the sequence S is the sequence of rep-digits in base b for $b \geq 2$ and we ask; what is the cardinality of the set W containing values of d which are such that, for each d, there is more than one base b rep-digit x-coordinate satisfying the Pell equation?
Remember that every non-square positive integer is either square-free or non-square-free. We see by identifying the biggest m such that $m^2 \mid d$ for any non-square-free non-square positive integer d that, $d = am^2$ for some square-free integer $a > 1$. We can then transform the Pell equation into $X^2 - aK^2 = 1$, where $K = mY$. We can then now just require d in the Pell equation to be a positive square-free integer.

1.3 Literature Review

In their paper, "On X-coordinates of Pell equations which are rep-digits"[16], B. Faye and F. Luca considered this question and proved that there can be only finitely many d such that the Pell equation has 2 solutions with x-coordinates being base-b rep-digits. This means for infinitely many d, there is at most one base-b rep-digit x-coordinate of the solutions (x, y).

Their result is based on the following theorem;

Theorem 1.3.1 *Let $b \geq 2$ be fixed. Let $d \geq 2$ be square-free and let $(X_n, Y_n) := (X_n(d), Y_n(d))$ be the nth positive integer solution of the Pell equation $X^2 - dY^2 = 1$. If the Diophantine equation;*

$$X_n = a\left(\frac{b^m - 1}{b - 1}\right)$$

with $a \in \{1, 2, \ldots, b-1\}$ has two positive integer solutions (n, a, m), then

$$d \leq exp((10b)^{10^5})$$

This result actually served as a generalization of a similar result by Dossavi-Yovo, Luca and Togbe in the paper "On the X-coordinates of Pell equations which are rep-digits"[13] in which they considered the specific case of base-10 rep-digits and demonstrated that except the cases $d = 2$ and $d = 3$, in which there are two x-coordinates of the solutions of the Pell equations being

rep-digits base-10, solutions that correspond to all other values of d have at most one x-coordinate being a rep-digit base-10. They in fact actually identified the pair of pairs $((X_1 = 3, X_3 = 99), (X_1 = 2, X_2 = 7))$ as the rep-digit x-coordinates corresponding to the cases $d = 2$ and $d = 3$ respectively.
F. Luca again teamed up with Togbé to study the case where the x-coordinates of the solutions to the equation $X^2 - dY^2 = \pm 1$ is a Fibonacci number for a square-free d in the paper "On the x-coordinates of Pell equations which are Fibonacci numbers"[23]. Here again, the question of interest is how many square-free $d's$ there are such that two of their infinitely many solutions have Fibonacci number x-coordinates.

They showed that when we require values of d for which there are two Fibonacci number values of $x \geq 1$ satisfying $X^2 - dY^2 = \pm 1$, only $d = 2$ stands; producing the x-coordinates, $X = 1 = F_1 = F_4$ and $X = 3 = F_4$ where $\{F_m\}_{m \geq 1}$ is the sequence of Fibonacci numbers.
This result was based on the following theorem;

Theorem 1.3.2 *Let $d \geq 2$ be square-free. The Diophantine equation*

$$x_n \in \{F_m\}_{m \geq 1}$$

has at most one solution (n, m) in positive integers except for $d = 2$. In this case,we have;

$$(n, m) \in \{(1,1), (1,2), (2,4)\}$$

In another paper "On the x-coordinates of Pell equations which are Tribonacci numbers"[22], F. Luca et al. considered the sequence $U := T$ of tribonacci numbers given by $T_0 = T_1 = 0$, $T_1 = 1$ and $T_{m+3} = T_{m+2} + T_{m+1} + T_m$, for all $m \geq 0$ and proved the following;

Theorem 1.3.3 *Let $d \geq 2$ a square-free integer. The Diophantine equation*

$$X_n = T_m$$

has at most one solution (n, m) in positive integers with the following exceptions:

- $(n_1, m_1) = (1,3)$ *and* $(n_2, m_2) = (2,5)$ *in the case of* $X^2 - dY^2 = 1$.
- $(n_1, m_1) = (1,1)$, $(n_2, m_2) = (1,2)$ *and* $(n_3, m_3) = (3,5)$ *in the case of* $X^2 - dY^2 = -1$.

1.3.1 The Case of the Y-coordinate

How do the above arguments play out in the case where we shift interest to the Y-coordinate of the Pell equation?

Here, if we suppose our sequence $\{U_n\}_{n \geq 1} := S$ is the set of base-2 rep-digits, that is $S = \left\{2^n - 1\right\} : n \geq 1$, and choose $d = 2^{2l} - 1$ for some positive integer l, it is readily verified that

$(X_1, Y_1) = (2^l, 2^1 - 1 = 1)$ and $(X_3, Y_3) = (2^{3l+2} - 3.2^l, 2^{2l+2} - 1)$ satisfy the Pell equation. The cardinality of the set of values of d for which there are two base-2 rep-digit Y-coordinates of the Pell equation is thus the same as that of the set of positive integers. We can thus construct infinitely many d such that there are two Y-coordinates of the Pell equation that are base-2 rep-digits.
But are there any values of d for which there are 3 Y-coordinates of the Pell equation that are base-2 rep-digits? B. Faye and F. Luca in the paper "On Y-coordinates of Pell equations which are base-2 rep-digits"[**?**] answered in the negative. This was based on the following result;

Theorem 1.3.4 *Let $d > 1$ be an integer which is not a square and let (X_k, Y_k) be the sequence of positive integer solutions to $X^2 - dY^2 = 1$. Then the equation $Y_k = 2^n - 1$ has at most two positive integer solutions (k, n).*

The property of having two Y-coordinates, satisfying the Pell equation for each of infinitely many d , from some special set of interest is not exclusive to values of d constructed in the above manner only. For, suppose our sequence $\{U_n\}_{n\geq 1} := S$ contains 1 and infinitely many even numbers, then by choosing $d = u^2 - 1$ where u is chosen such that $2u \in S$, we see that $(X_1, Y_1) = (u, 1)$ and $(X_2, Y_2) = (2u^2 - 1, 2u)$ both satisfy $X^2 - dY^2 = 1$. Therefore there are infinitely many values of d for which the Pell equation is satisfied by two Y-coordinates from the set S constructed in this manner.

1.4 Some Preliminaries

1.4.1 Binary Recurrent sequences

A sequence $(U_n)_{n\geq 0}$ is called a binary recurrent sequence if it satisfies the recurrence

$$u_{n+2} = a_1 u_{n+1} + a_2 u_n$$

for some fixed constants $a_1, a_2 \in \mathbb{C}$. We also fix u_0, u_1 as initial values of the sequence. For the particular case where $a_1, a_2, u_0, u_1 \in \mathbb{Z}$, we see that $(U_n)_{n\geq 0}$ is a sequence of integers. Associated with the binary recurrent sequence $(U_n)_{n\geq 0}$ is the polynomial $f(x) = x^2 - a_1 x - a_2$ called the characteristic polynomial.
Suppose $f(x) = x^2 - a_1 x - a_2 = (x - \alpha_1)(x - \alpha_2)$ where α_1 and α_2 are distinct roots of $f(x)$, then we have that

Proposition 1.4.1 *(see [28]) There exists some constants $c_1, c_2 \in \mathbb{Q}(\alpha_1, \alpha_2)$ such that the formula*

$$u_n = c_1 \alpha_1^n + c_2 \alpha_2^n$$

holds for all $n \geq 0$.

We can set $\alpha_1 = \frac{1}{2}\left(a_1 + \sqrt{a_1^2 + 4a_2}\right)$ and $\alpha_2 = \frac{1}{2}\left(a_1 - \sqrt{a_1^2 + 4a_2}\right)$. Then by fixing u_0 and u_1 as initial values of the sequence, we can calculate c_1 and c_2 from the equations $u_0 = c_1 + c_2$ and $u_1 = c_1\alpha_1 + c_2\alpha_2$ to obtain

$$c_2 = \frac{u_1 - \alpha_1 u_0}{\alpha_2 - \alpha_1}$$

and

$$c_1 = \frac{u_1 - \alpha_2 u_0}{\alpha_1 - \alpha_2}.$$

In the case where $a_1 = a_2 = 1$, we have $\alpha_1 = \frac{1}{2}\left(1 + \sqrt{5}\right)$ and $\alpha_2 = \frac{1}{2}\left(1 - \sqrt{5}\right)$. If in addition, we set $u_0 = 0$ and $u_1 = 1$, then we obtain $c_2 = -\frac{1}{\sqrt{5}}$ and $c_1 = \frac{1}{\sqrt{5}}$. This yields $u_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}$. The sequence $u_n := F_n$ so obtained is called the Fibonacci sequence. The Fibonacci sequence is thus a binary recurrence sequence.
If we now take $u_0 = 2$ and $u_1 = 1$, we obtain $u_n := L_n = \alpha_1^n + \alpha_2^n$. The sequence L_n is called the Lucas sequence.

X and Y coordinates of Pell Equations as binary recurrences.

Theorem 1.2.5 and the discussion following it shows that if (x_1, y_1) is the minimal solution of $X^2 - dY^2 = 1$, then all positive integer solutions (x, y) are of the form $(x, y) = (x_n, y_n)$ for some positive integer n where $x_n + \sqrt{d}y_n = \left(x_1 + \sqrt{d}y_1\right)^n$. It is widely known in the theory of Pell equations that the result is also true in the case of the more general Pell equation $X^2 - dY^2 = \pm 1$. Consider the equation $X^2 - dY^2 = \pm 1$ with minimal solution (x_1, y_1) and set

$$\alpha = x_1 + \sqrt{d}y_1 \text{ and } \beta = x_1 - \sqrt{d}y_1.$$

All the solutions (x_n, y_n) satisfy

$$x_n + \sqrt{d}y_n = \left(x_1 + \sqrt{d}y_1\right)^n = \alpha^n \tag{1.7}$$

for some positive integer n. From equation (1.7), we also obtain

$$x_n - \sqrt{d}y_n = \left(x_1 - \sqrt{d}y_1\right)^n = \beta^n. \tag{1.8}$$

We solve equations (1.7) and (1.8) to obtain

$$x_n = \frac{\alpha^n + \beta^n}{2} \text{ and } y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}} \quad \text{for all} \quad n \geq 1. \tag{1.9}$$

We define $(x_0, y_0) = (1, 0)$ so that formula (1.9) holds with $n = 0$. We then see that the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfy the binary recurrences with characteristic polynomial whose roots are α and β. The characteristic polynomial is thus

$$f(x) = (x - \alpha)\left(x - \beta\right) = x^2 - \left(\alpha + \beta\right)x + \alpha\beta = x^2 - (2x_1)x \pm 1.$$

This yields the binary recurrences

$$x_{n+2} = 2x_1x_{n+1} \pm x_n, \quad y_{n+2} = 2x_1y_{n+1} \pm y_n \quad \text{with} \quad (x_0, y_0) = (1,0).$$

1.4.2 Results from transcendental number theory and diophantine approximations

We list a couple of results which we use to obtain lower bounds for linear forms in logarithms.

Height of an algebraic number

For an algebraic number λ of minimal polynomial over $\mathbb{Z}$ with positive leading coefficient

$$f(X) := a_0X^d + a_1X^{d-1} + \cdots + a_d = a_0(X - \lambda^{(1)}) \cdots (X - \lambda^{(d)}) \in \mathbb{Z}[X]$$

(here, $\lambda^{(1)} = \lambda$), we put

$$h(\lambda) := \frac{1}{d}\left(\log a_0 + \sum_{\substack{1\le i\le d\\ |\lambda^{(i)}|>1}} \log|\lambda^{(i)}|\right)$$

for the logarithmic height of λ.

The LLL-algorithm.

At one occasion, we need to find a lower bound for linear forms with bounded integer coefficients in three variables for which methods based on continued fractions are not applicable. Instead, we will use the LLL–algorithm which we now briefly describe.

Let $\tau_1, \ldots, \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \le X_i. \tag{1.10}$$

We set $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \mathbf{e}_t \qquad \text{for} \qquad 1 \le j \le t-1 \qquad \text{and} \qquad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 1.4.2 *Let $X_1, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed constant. With the above notation on Ω, we consider a reduced basis $\{\mathbf{b}_i\}$ to Ω and its associated Gram–Schmidt $\{\mathbf{b}_i^*\}$ basis. We set*

$$c_1 := \max_{1\le i\le t} \frac{||\mathbf{b}_1||}{||\mathbf{b}_i^*||}, \quad \mathbf{m}_\Omega := \frac{||\mathbf{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2 \qquad \textit{and} \qquad T := \left(1 + \sum_{i=1}^{t} X_i\right)/2.$$

If the integers x_i satisfy that $|x_i| \le X_i$, for $i = 1, \dots, t$ and $\mathbf{m}_\Omega{}^2 \ge T^2 + Q$, then we have

$$\left|\sum_{i=1}^{t} x_i \tau_i\right| \ge \frac{\sqrt{\mathbf{m}_\Omega^2 - Q} - T}{C}.$$

For more details, see Proposition 2.3.20 in [11, Section 2.3.5].

The following result is referred to in the literature as Baker's lower bound for a non-zero linear form in logarithms (see [5], [24]).

Theorem 1.4.3 *(Matveev's theorem). Assume that $\lambda_1, \dots, \lambda_t$ are positive real algebraic numbers in a number field $\mathbb{F}$ of degree D, $b_1, \dots, b_t$ are rational integers, and*

$$\Lambda := \lambda_1^{b_1} \cdots \lambda_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right)$$

where

$$B \geqslant \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geqslant \max\{Dh(\lambda_i), |\log \lambda_i|, 0.16\}, \textit{ for all } i = 1, \dots, t.$$

When $t = 2$ and λ_1, λ_2 are positive and multiplicatively independent, we can do better. Namely, let in this case let B_1, B_2 be real numbers larger than 1 such that

$$\log B_i \ge \max\left\{h(\lambda_i), \frac{|\log \lambda_i|}{D}, \frac{1}{D}\right\} \quad \text{for} \quad i = 1, 2,$$

and

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Lambda = b_1 \log \lambda_1 + b_2 \log \lambda_2.$$

Note that $\Lambda \neq 0$ since λ_1 and λ_2 are multiplicatively independent. The following inequality is Corollary 2 in [21].

Theorem 1.4.4 *With the above notations and conventions, assuming that λ_1, λ_2 are positive algebraic numbers which are multiplicatively independent, then*

$$\log |\Lambda| > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2.$$

When $t = 3$ a better bound than the general one given by Theorem 1.4.3 in some special case is due to Mignotte [26, Proposition 5.2]; see also [27].

Theorem 1.4.5 *Let* $\Lambda := b_2 \log\gamma_2 - b_1 \log\gamma_1 - b_3 \log\gamma_3 \neq 0$ *with* b_1, b_2, b_3 *positive integers with* $\gcd(b_1, b_2, b_3) = 1$ *and* $\gamma_1, \gamma_2, \gamma_3$ *positive real algebraic numbers* > 1 *which are multiplicatively independent in a field* $\mathbb{K}$ *of degree* D. *Let*

$$d_1 := \gcd(b_1, b_2) = b_1/b_1' = b_2/b_2', \quad d_3 := \gcd(b_2, b_3) = b_2/b_2'' = b_3/b_3''.$$

Let a_1, a_2, a_3 *be real numbers such that*

$$a_i \geq \max\{4, 4.296\log\gamma_i + 2Dh(\gamma_i)\}, \quad i = 1,2,3, \quad \Omega := a_1 a_2 a_3 \geq 100.$$

Put

$$b' := \left(\frac{b_1'}{a_2} + \frac{b_2'}{a_1}\right)\left(\frac{b_3''}{a_2} + \frac{b_2''}{a_3}\right), \quad \log\mathscr{B} \geq \max\{0.882 + \log b', 10/D\}.$$

Then one of the following holds:

(i)

$$\log|\Lambda| > -790.95\Omega D^2(\log\mathscr{B})^2;$$

(ii) *there exist nonzero integers* r_0, s_0 *with* $r_0 b_2 = s_0 b_1$ *satisfying the inequalities*

$$|r_0| < 5.61(D\log\mathscr{B})^{1/3} a_2 \quad and \quad |s_0| < 5.61(D\log\mathscr{B})^{1/3} a_1;$$

(iii) *there exist integers* $r_1 \neq 0, s_1 \neq 0, t_1, t_2$ *satisfying*

$$\gcd(r_1, t_1) = \gcd(s_1, t_2) = 1, \qquad (t_1 b_1 + r_1 b_3)s_1 = r_1 b_2 t_2,$$

and also

$$\begin{aligned}
|r_1 s_1| &< 5.61\delta(D\log\mathscr{B})^{1/3} a_3,\\
|s_1 t_1| &< 5.61\delta(D\log\mathscr{B})^{1/3} a_1,\\
|r_1 t_2| &< 5.61\delta(D\log\mathscr{B})^{1/3} a_2,
\end{aligned}$$

where $\delta := \gcd(r_1, s_1)$. *If* $t_1 = 0$, *we can take* $r_1 = 1$ *and if* $t_2 = 0$ *we can take* $s_1 = 1$.

1.4.3 Reduction method

During the course of our calculations, we get some upper bounds on our variables which are too large, thus we need to reduce them. We carry out the reductions by means of the following reduction algorithms.

The following result is Lemma 5 (a) in [14], which is a slight variation of a result of Baker and Davenport (see [3]).

Lemma 1.4.6 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon := ||\mu q|| - M||\tau q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$|m\tau - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$m \le M \quad and \quad k \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following result is criterion of Legendre (see Theorem 8.2.4 in [29]).

Lemma 1.4.7 (Legendre) *Let τ be an irrational real number and x, y be integers.*

(i) *If*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2},$$

then $x/y = p_k/q_k$ is a convergent of τ. Furthermore,

$$\left|\tau - \frac{x}{y}\right| \ge \frac{1}{(a_{k+1}+2)y^2},$$

where $[a_0, \ldots, a_k, \ldots]$ is the continued fraction expansion of τ.

(ii) *If $y < q_{k+1}$, then*

$$\frac{1}{(A+2)y^2} \le \left|\tau - \frac{x}{y}\right|,$$

where $A := \max\{a_j : j \le k+1\}$.

1.4.4 An elementary inequality

The following appears as Lemma 7 in [20].

Lemma 1.4.8 *If $s \ge 1$, $T > (4s^2)^s$ and $T > x/(\log x)^s$, then*

$$x < 2^s T(\log T)^s.$$

The statement of the above lemma is formulated in [20] only for an integer parameter s but a close look at its proof there shows that it works for any real parameter $s \ge 1$.

Chapter 2

X-coordinates of Pell Equation which are repdigits

Chapter Note: The result of this chapter is published in [19].

In this chapter, an algorithm is given which finds, for an integer base $b \geq 2$, all squarefree integers $d \geq 2$ such that sequence of X-components $\{X_n\}_{n\geq 1}$ of the Pell equation $X^2 - dY^2 = \pm 1$ has two members which are base b–repdigits. We implement this algorithm and find all the solutions to this problem for all bases $b \in [2, 100]$.

2.1 Introduction

For a positive integer base $b \geq 2$ a *repdigit* is a positive integer N whose base b-representation has a unique repeating digit. Letting $a \in \{1, \ldots, b-1\}$ be the value of the repeating digit and m be the number of digits of N we have

$$N = a\left(\frac{b^m - 1}{b-1}\right).$$

When $b = 10$, we omit the base and simply call N a repdigit. Recently there has been a flurry of activity regarding finding all members of some classical sequence (Fibonacci numbers [23] and its generalisations [8], perfect powers [9], etc.) which are repdigits for some particular base b. Two classical sequences are the sequences of X- or Y-coordinates of the Pell equation $X^2 - dY^2 = \pm 1$ associated to a squarefree integer $d \geq 2$. The first paper where the presence of X–coordinates of Pell equations was studied that we are aware of is [13]. There it is shown that if $\{X_n\}_{n\geq 1}$ is the sequence of X–coordinates of the Pell equation $X^2 - dY^2 = 1$, then X_n is a repdigit for at most one value of n except for $d = 2, 3$, each of which has two solutions n for which X_n is a repdigit. In his Math Sci Net review [30] of [13], the reviewer writes: "*The techniques* (used in [13]) *appear to be very specific to the base*-10 *case so that generalising to other bases would require a separate work of equal magnitude.*" The problem was revisited in [16] where the general base b was treated. There it was shown that if X_n is a base b–repdigit for two values of n then

$$d \le \exp((10b)^{10^5}). \tag{2.1}$$

While the result has the theoretical merit of showing that the problem has a finite answer (even in effective form), the bound (2.1) is useless for practical computations. A close look at the proof from [16] shows that the bound (2.1) appeared when studying the equation

$$X_n = a\left(\frac{b^m - 1}{b-1}\right) \quad \text{for} \quad a \in \{1,\ldots,b-1\}$$

with n even. Since $X_n = 2X_{n/2}^2 - 1$, the above equation with n even implies

$$2X_{n/2}^2 - 1 = \frac{a(b^m - 1)}{b-1}.$$

Writing $m = 3m_0 + r$ for some $r \in \{0,1,2\}$, and putting $(x, y) := (X_{n/2}, b^{m_0})$, the above equation becomes

$$2x^2 - 1 = \frac{a(b^r y^3 - 1)}{b-1}. \tag{2.2}$$

When $a \neq b-1$, the above equation represents an elliptic curve and the integer points (x, y) on it can be effectively bounded using results of Baker [2]. These bounds lead to the inequality (2.1). In practice, for a fixed a and b one can use Magma to find all the particular integer solutions (x, y) of (2.2). However, if one were to loop over all $b \in [2,100]$ and all $a \in \{1,\ldots,b-1\}$, then this would become unfeasible as there are too many curves and some have rather large coefficients.

In this note, we revisit this problem. While all previous papers addressed only the positive solutions (X, Y) to $X^2 - dY^2 = 1$, we allow also for the right–hand side to be -1. That is, we let $\{X_n\}_{n\ge1}$ be the sequence of X–coordinates of the Pell equation $X^2 - dY^2 = \pm1$. Our results are the following.

Theorem 2.1.1 *Let $b \ge 2$ be an integer. Let $d \ge 2$ be an integer which is not a square and (X_n, Y_n) be the nth positive integer solution to the Pell equation $X^2 - dY^2 = \pm1$. Assume that $1 \le n_1 < n_2$ are such that*

$$X_{n_1} = a_1\left(\frac{b^{m_1} - 1}{b-1}\right) \quad \textit{and} \quad X_{n_2} = a_2\left(\frac{b^{m_2} - 1}{b-1}\right) \quad \textit{with} \quad a_1, a_2 \in \{1,\ldots,b-1\}. \tag{2.3}$$

Then $B := \max\{n_1, m_1, n_2, m_2\}$ satisfies

$$B \le 6 \times 10^{27} (\log(2b))^6.$$

We present the following numerical corollary of Theorem 2.1.1

Corollary 2.1.2 *Let $b \in [2,100]$. All integer positive solutions of equation (2.3) have $1 \le n_1 < n_2 \le 5$*

and d in the set:

$$\{2,3,5,8,10,15,17,24,26,35,37,48,50,63,65,80,101,120,122,143,170,195,226,255,257,325,399,\\ 401,485,528,677,728,842,1023,1224,1226,1370,1601,1682,1935,2117,3248,3250,3968,4095\}.$$

We do not list the actual solutions to (2.3) for $b \in [2,100]$. They can be reconstructed from the table at the end. Or, one can just take any d appearing in the above list and find the minimal positive integer solution $(X_1(d), Y_1(d))$ to the equation $X^2 - dY^2 = \pm 1$. Then one can generate $(X_i(d), Y_i(d))$ for $i = 1, \ldots, 5$, write these five numbers in base b for all $b \in [2,100]$ and select those instances for which 2 of the above X–coordinates are base b-repdigits.

Before proceeding to the proofs we mention a related result of Bennett and Pintér [6]. Their result is more general but for our situation it implies the following result.

Theorem 2.1.3 *Let $\mathbf{U} := \{U_n\}_{n\geq 0}$ be a sequence such that $U_m = cb^m + d$ holds for all $m \geq 1$, where $b \geq 2$ is a fixed integer and $c \in \mathbb{Q}^*, d \in \mathbb{Q}$ are fixed rational numbers. Then there exist only finitely many positive integers d which are not squares such that if (X_n, Y_n) is the nth solution of the Pell equation $X^2 - dY^2 = \pm 1$, then $X_n = U_m$ has two solutions (n, m). Furthermore, all such d, as well as the solutions (n, m) are effectively computable.*

The above Theorem 2.1.3 is related to the problem but it has the defect that in Theorem 2.1.3, the coefficient c of b^m is fixed, whereas for our problem a can be any of member of the set of nonzero digits $\{1, \ldots, b-1\}$, and then the repdigit is indeed of the form $cb^m + d$ with $(c, d) := (a/(b-1), -a/(b-1))$.

2.2 Bounding the Variables

2.2.1 Bounding n, m in the equation $X_n = a(b^m - 1)/(b-1)$

We let (X_1, Y_1) be the minimal solution of the equation $X^2 - dY^2 = \pm 1$. We define:

$$\delta := X_1 + \sqrt{d}Y_1 \quad \text{and} \quad \eta := X_1 - \sqrt{d}Y_1 = \varepsilon\delta^{-1}, \quad \text{where} \quad \varepsilon := X_1^2 - dY_1^2 \in \{\pm 1\}.$$

Then

$$X_n = \frac{\delta^n + \eta^n}{2} \geq \frac{\delta^n - \delta^{-n}}{2} > \frac{\delta^n}{3},$$

where the last inequality holds since $\delta \geq 1 + \sqrt{2} > \sqrt{3}$. We thus have

$$\frac{\delta^n}{b+1} < X_n < \delta^n \quad \text{for all} \quad n \geq 1. \tag{2.4}$$

We also have that

$$b^{m-1} \leq a\left(\frac{b^m - 1}{b-1}\right) < b^m. \tag{2.5}$$

We now assume as in the statement of Theorem 2.1.1 that (n_1, a_1, m_1) and (n_2, a_2, m_2) are such that

$$X_{n_1} = a_1\left(\frac{b^{m_1}-1}{b-1}\right) \quad \text{and} \quad X_{n_2} = a_2\left(\frac{b^{m_2}-1}{b-1}\right). \tag{2.6}$$

We also assume that $n_1 < n_2$ and $a_1, a_2 \in \{1, \ldots, b-1\}$. Setting $(n, a, m) := (n_i, a_i, m_i)$ for $i \in \{1,2\}$ and using inequalities (2.4) and (2.5), we get

$$\frac{\delta^n}{b+1} < X_n = a\left(\frac{b^m-1}{b-1}\right) < b^m \quad \text{and} \quad b^{m-1} < a\left(\frac{b^m-1}{b-1}\right) = X_n < \delta^n. \tag{2.7}$$

Next, we set

$$X_n = \frac{1}{2}\left(\delta^n + \eta^n\right) = a\left(\frac{b^m-1}{b-1}\right). \tag{2.8}$$

So, we get

$$\delta^n - \left(\frac{2a}{b-1}\right)b^m = -\eta^n - \frac{2a}{b-1} \quad \text{for} \quad (n,a,m) = (n_i, a_i, m_i) \quad \text{and} \quad i = 1,2.$$

Hence,

$$\begin{aligned}\left|\delta^n\left(\frac{2a}{b-1}\right)^{-1} b^{-m} - 1\right| &\leq \frac{1+2a/(b-1)}{(2a/(b-1))b^m}\\ &= \frac{1+(b-1)/(2a)}{b^m}\\ &< \frac{b+1}{2b^m} \quad \text{for} \quad i = 1,2.\end{aligned}$$

Thus,

$$\left|\delta^n\left(\frac{2a}{b-1}\right)^{-1} b^{-m} - 1\right| < \frac{b+1}{2b^m} \leq \frac{3}{4}. \tag{2.9}$$

Set

$$\Lambda := n\log\delta - \log\left(\frac{2a}{b-1}\right) - m\log b.$$

Since $\left|e^\Lambda - 1\right| < \frac{3}{4}$, it follows that

$$|\Lambda| < e^{|\Lambda|}\left|e^\Lambda - 1\right| < 4|e^\Lambda - 1| < \frac{2(b+1)}{b^m} < \frac{4}{b^{m-1}}.$$

Recalling that $(n, a, m) := (n_i, a_i, m_i)$, we get

$$\left|n_i \log\delta - m_i \log b - \log(2a_i/(b-1))\right| < \frac{4}{b^{m_i-1}} \quad \text{for} \quad i = 1,2. \tag{2.10}$$

We apply Matveev's Theorem 1.4.3 on the left–hand side of (2.9). We need to justify that $\Gamma := e^\Lambda - 1 \neq 0$. Well, if $\Gamma = 0$, we then get that $\delta^n \in \mathbb{Q}^*$, which is impossible for any positive integer n. We take

$$t := 3, \quad \lambda_1 := \frac{2a}{b-1}, \quad \lambda_2 := b, \quad \lambda_3 := \delta, \quad b_1 := -1, \quad b_2 := -m, \quad b_3 := n.$$

We put $B := \max\{m, n\}$. We have

$$h(\lambda_1) \le \log(2b), \quad h(\lambda_2) = \log b, \quad \text{and} \quad h(\lambda_3) = \frac{1}{2}\log\delta.$$

Further, the field containing $\lambda_1, \lambda_2, \lambda_3$ is $\mathbb{K} := \mathbb{Q}(\sqrt{d})$ has degree $D := 2$. Hence, we can take

$$A_1 := 2\log(2b), \qquad A_2 := 2\log b \qquad \text{and} \qquad A_3 := \log\delta.$$

Now Matveev's theorem tells us that

$$\begin{aligned}\log|\Gamma| &> -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log B)(2\log(2b))(2\log b)(\log\delta)\\ &> -3.9\times 10^{12}(\log\delta)(\log b)(\log(2b))(1+\log B).\end{aligned}$$

Comparing the above inequality with (2.9, we get

$$(m-1)\log b - \log 4 < 3.9\times 10^{12}(\log\delta)(\log b)(\log(2b))(1+\log B).$$

Thus,

$$m+2 < 3+\frac{\log 4}{\log b}+3.9\times 10^{12}(\log\delta)(\log(2b))(1+\log B) < 4\times 10^{12}(\log\delta)(\log(2b))(1+\log B).$$

Also, since $b^{m+2} > b^m(b+1) > \delta^n$ (see (1.3)), we have

$$n\log\delta < (m+2)\log b < 4\times 10^{12}(\log\delta)(\log b)(\log(2b))(1+\log B),$$

so

$$n < 4\cdot 10^{12}(\log b)(\log(2b))(1+\log B).$$

Let us record what we have proved so far.

Lemma 2.2.1 *If the equation*

$$X_n = a\left(\frac{b^m-1}{b-1}\right) \quad \textit{holds with} \quad a\in\{1,2,\ldots,b-1\},$$

then putting $\delta := X_1+\sqrt{d}Y_1$ *and* $B := \max\{m,n\}$, *we have*

$$n < 4\times 10^{12}(\log b)(\log(2b))(1+\log B), \quad \textit{and} \quad m < 4\times 10^{12}(\log\delta)(\log(2b))(1+\log B). \tag{2.11}$$

2.2.2 Bounding all variables n_1, m_1, n_2, m_2 in terms of b

Next, we return to the two inequalities given by (2.10). Multiply the one for $i = 1$ by n_2 and the one for $i = 2$ by n_1, subtract the results and apply the triangle inequality to get

$$\begin{aligned}
\left|(n_2 m_1 - n_1 m_2)\log b + n_2 \log\left(\frac{2a_1}{b-1}\right) - n_1 \log\left(\frac{2a_2}{b-1}\right)\right| &= \\
\left|n_1\left(n_2\log\delta - m_2\log b - \log\left(\frac{2a_2}{b-1}\right)\right) - n_2\left(n_1\log\delta - m_1\log b - \log\left(\frac{2a_1}{b-1}\right)\right)\right| &\le \\
n_2\left|n_1\log\delta - m_1\log b - \log\left(\frac{2a_1}{b-1}\right)\right| + n_1\left|n_2\log\delta - m_2\log b - \log\left(\frac{2a_2}{b-1}\right)\right| &< \\
\frac{4n_2}{b^{m_1-1}} + \frac{4n_1}{b^{m_2-1}} &< \\
\frac{8n_2}{b^{m_1-1}}.&
\end{aligned}$$

That is

$$\left|(n_2 m_1 - n_1 m_2)\log b + n_2 \log\left(\frac{2a_1}{b-1}\right) - n_1 \log\left(\frac{2a_2}{b-1}\right)\right| < \frac{8n_2}{b^{m_1-1}}. \tag{2.12}$$

We shall consider separate arguments for the case when the left–hand side of (2.12) is non–zero and the case when it is zero.

The case when the left–hand side of (2.12) is non–zero

We use again Matveev's Theorem 1.4.3 in order to get a lower bound on the left–hand side of (2.12). To this end, let us denote the left–hand side of (2.12) by

$$\Gamma_1 := (n_1 m_2 - n_2 m_1)\log b + n_1 \log\left(\frac{2a_2}{b-1}\right) - n_2 \log\left(\frac{2a_1}{b-1}\right) \neq 0.$$

By the absolute value inequality, we have

$$\begin{aligned}
|n_1 m_2 - n_2 m_1| &< \frac{8n_2}{b^{m_1-1}\log b} + \frac{n_1\left|\log((2a_2)/(b-1))\right|}{\log b} + \frac{n_2\left|\log((2a_1)/(b-1))\right|}{\log b} \\
&\le \frac{8n_2}{\log 2} + n_1 + n_2 < 14n_2. \qquad (2.13)
\end{aligned}$$

So we can take $B := 14n_2$ and then this choice of B satisfies $B \geq \max\{|n_1 m_2 - n_2 m_1|, n_1, n_2\}$. We also take

$$t := 3, \quad \lambda_1 := b, \quad \lambda_2 := \frac{2a_2}{b-1}, \quad \lambda_3 := \frac{2a_1}{b-1}, \quad b_1 := n_2 m_1 - n_1 m_2, \quad b_2 := n_2, \quad b_3 := -n_1.$$

Further, $\lambda_1, \lambda_2, \lambda_3$ are all rational, so $D := 1$. Thus, we can take

$$A_1 := \log b, \quad A_2 := \log(2b), \quad A_3 := \log(2b).$$

We assume that

$$b^{m_1-1} > 16n_2. \tag{2.14}$$

In this case, the right–hand side of estimate (2.12) is smaller than 1/2. Thus, $|\Gamma_1| < 1/2$, and so, by (2.12), we get

$$\left|\lambda_1^{b_1}\lambda_2^{b_2}\lambda_3^{b_3} - 1\right| = |e^{\Gamma_1} - 1| < 2|\Gamma_1| < \frac{16n_2}{b^{m_1-1}}. \tag{2.15}$$

Since $\Gamma_1 \neq 0$, the left–hand side above is nonzero and by Theorem 1.4.3, its logarithm is at least as large as

$$\begin{aligned}\log\left|e^{\Gamma_1} - 1\right| &> -1.4 \times 30^6 \times 3^{4.5}(1+\log(14n_2))(\log b)(\log(2b))(\log(2b)) \\ &> -1.44 \times 10^{11}(1+\log(14n_2))(\log b)(\log(2b))^2.\end{aligned}$$

Comparing this with (2.15), we get;

$$(m_1 - 1)\log b - \log(16n_2) < 1.44 \times 10^{11}(1+\log(14n_2))(\log b)(\log(2b))^2.$$

Thus,

$$m_1 + 2 < 3 + \frac{\log(16n_2)}{\log b} + 1.44 \times 10^{11}(1+\log(14n_2))(\log(2b))^2 < 1.5 \times 10^{11}(1+\log(14n_2))(\log(2b))^2. \tag{2.16}$$

We thus have,

$$m_1 + 2 < 1.5 \times 10^{11}(1+\log(14n_2))(\log(2b))^2. \tag{2.17}$$

This was under the assumption that (2.14) holds. But in case (2.14) fails, then

$$m_1 + 2 = 3 + (m_1 - 1) \leq 3 + \frac{\log(16n_2)}{\log b} < 3(1+\log(14n_2)),$$

so estimate (2.17) holds as well. Hence, estimate (2.17) holds regardless of whether estimate (2.14) holds or not. Also, since $b^{m_1+2} > \delta^{n_1}$ (see (1.3)), we have;

$$n_1 \log\delta < 1.5 \times 10^{11}(1+\log(14n_2))(\log(2b)))^3. \tag{2.18}$$

Since $n_1 \geq 1$ and $\delta \geq 1+\sqrt{2}$, this yields

$$n_1 \leq 2 \cdot 10^{11}(1+\log(14n_2))(\log(2b))^3 \quad \text{and} \quad \log\delta < 1.5 \times 10^{11}(1+\log(14n_2))(\log(2b))^3. \tag{2.19}$$

We need to extract absolute bounds for n_i, m_i for $i = 1, 2$. For this we distinguish two cases according to $B_2 := \max\{n_2, m_2\}$.

Case 1. $B_2 = n_2$.

In this case, the first equation (2.11) tells us that

$$14n_2 < 14 \times 4 \times 10^{12}(\log(2b))^2(1+\log n_2) < 5.6 \times 10^{13}(\log(2b))^2 \log(14n_2).$$

Taking $x := 14n_2$ and applying Lemma 1.4.8 with $s := 1$ and $T := 5.6\times 10^{13}(\log(2b))^3 > 4$, we get

$$\begin{aligned}14n_2 &< 2T\log T = 11.2\times 10^{13}(\log(2b))^3(\log(5.6\times 10^{13})+2\log(\log(2b)))\\ &< 11.2\times 10^{13}(\log(2b))^2(2\log(2b))\left(1+\frac{\log(5.6\times 10^{13})}{2\log(2b)}\right)\\ &< 3\times 10^{15}(\log(2b))^3.\end{aligned}$$

In the above, we used that $\log(2b) < 2b$ and the right–most factor $1+\log(5\times 10^{13})/(2\log(2b))$ is smaller than 12.42 (and maximal at $b := 2$).

Case 2. $B_2 = m_2$.

In this case, the second equation (2.11) together with the second equation (2.19) tells us that

$$\begin{aligned}14m_2 &< 14\times 4\times 10^{12}(\log\delta)(\log(2b))(1+\log m_2)\\ &< (14\times 4\times 10^{12})\times(1.5\times 10^{11})(\log(2b)))^4(1+\log(14m_2))(1+\log m_2)\\ &< 9\times 10^{24}(\log(2b))^4(\log(14m_2))^2.\end{aligned}$$

We take $x := 14m_2$ and apply again Lemma 1.4.8 with $s := 2$ and $T := 9\times 10^{24}(\log(2b))^4 > (4\times 2)^2$, to get that

$$\begin{aligned}14m_2 &< 4T(\log T)^2 = 4\times 9\times 10^{24}(\log(2b))^4(\log(9\times 10^{24})+4\log(\log(2b)))^2\\ &= 4\times 9\times 10^{24}(\log(2b))^4(4\log(2b))^2\left(1+\frac{\log(9\times 10^{24})}{4\log(2b)}\right)^2\\ &< 8\times 10^{28}(\log(2b))^6.\end{aligned}$$

Let us record what we have proved.

Lemma 2.2.2 *Let $b \geq 2$. Assume that the equations*

$$X_{n_i} = a_i\left(\frac{b^{m_i}-1}{b-1}\right)\quad \textit{hold for}\quad i = 1,2$$

with $1 \leq n_1 < n_2$, $a_1, a_2 \in \{1,\dots,b-1\}$ and the left-hand side of (2.12) is non-zero. Then putting $B := \max\{m_2, n_2\}$, we have

$$14B \leq 8\times 10^{28}(\log(2b))^6.$$

In addition,

$$\begin{aligned}m_1 &< 1.5\times 10^{11}(1+\log(14B))(\log(2b))^2;\\ n_1 &< 2\times 10^{11}(1+\log(14B))(\log(2b))^3;\\ n_2 &< 4\times 10^{12}(1+\log B)(\log b)(\log(2b)).\end{aligned}$$

Note that Lemma 2.2.2 already implies Theorem 2.1.1. As for the numerical Corollary 2.1.2, we have

Lemma 2.2.3 *Keep the notations and assumptions from Lemma 2.2.2. If in addition* $b \le 100$*, then* $B \le 2\times 10^{32}$ *and furthermore*

$$\begin{array}{rcl} m_1 & < & 4\times 10^{14};\\ n_1 & < & 3\times 10^{15};\\ n_2 & < & 8\times 10^{15}. \end{array}$$

The case when the left–hand side of (2.12) is zero

Consider the $\mathbb{Q}$–span of the numbers

$$\zeta_1 := \log\lambda_1 = \log b,\quad \zeta_2 := \log\lambda_2 = \log(2a_1/(b-1)),\quad \zeta_3 := \log\lambda_3 = \log(2a_2/(b-1)).$$

Since $n_1 n_2 \ne 0$, but $\Gamma_1 = 0$, it follows that the dimension r of the $\mathbb{Q}$–vector space $\mathbb{Q}\zeta_1 + \mathbb{Q}\zeta_2 + \mathbb{Q}\zeta_3$ cannot be 3. So, it can only be 1 or 2. In what follows, we study these possibilities.

Case 1. $r = 1$.

Write $b = b_1^x$ with the maximal integer exponent x and a positive integer b_1, which is no longer a perfect power. Since $r = 1$, it follows that $2a_1/(b-1) = b_1^{y_1}$. Since $b-1$ is coprime to b, it follows that $y_1 \ge 0$. Thus, $b-1 \mid 2a_1$. This only gives the possibilities $a_1 = (b-1)/2$ (and b is odd) or $a_1 = b-1$ and in this case $b_1 = 2$ (so, b is a power of 2). The same holds for $2a_2/(b-1)$. Thus, we must have $a_1 = a_2 = (b-1)/2$ or $a_1 = a_2 = b-1$ and $b_1 = 2$. In case $a_1 = a_2 = (b-1)/2$, we get $m_1 n_2 = n_1 m_2$. We may assume that n_1 and n_2 are coprime, for if not, by denoting $n_0 := \gcd(n_1, n_2)$, we can write $n_1 = n_1' n_0,\ n_2 = n_2' n_0$ and replace d by $dY_{n_0}^2$ and the pair (n_1, n_2) by the pair (n_1', n_2'). Thus, we may assume that $n_0 = 1$, with which we have $m_1 = n_1 t$ and $m_2 = n_2 t$ for some positive integer t. Hence, we get

$$X_{n_1} = \frac{a_1(b^{m_1}-1)}{b-1} = \frac{b^{n_1 t}-1}{2},\quad \text{and}\quad X_{n_2} = \frac{a_2(b^{m_2}-1)}{b-1} = \frac{b^{n_2 t}-1}{2}.$$

Thus,

$$\delta^{n_i} + \eta^{n_i} = 2X_{n_i} = b^{n_i t} - 1, \qquad \text{holds for} \qquad i = 1, 2.$$

Thus,

$$|\delta^{n_i} - b^{n_i t}| = |1 + \eta^{n_i}|, \quad \text{which gives} \quad |\delta - b^t| = \frac{|1+\eta^{n_i}|}{\delta^{n_i-1} + \cdots + b^{t(n_i-1)}}. \tag{2.20}$$

On the other hand, $|\delta - b^t||\eta - b^t|$ is a nonzero integer, so it is at least 1. Thus, $|\delta - b^t| > 1/(1+b^t)$. We evaluate (2.20) in $i = 2$ and get

$$\frac{1}{1+b^t} < |\delta - b^t| = \frac{|1+\eta^{n_2}|}{b^{t(n_2-1)} + \cdots + \delta^{n_2-1}}.$$

Assuming $n_2 \geq 3$, since $|\eta| = \delta^{-1} < 1/2$, we get that

$$\frac{1}{1+b^t} < \frac{1+1/8}{b^{2t}} = \frac{9}{8b^{2t}},$$

so $b^{2t} < (9/8)(b^t+1)$, which does not have an integer solution $b^t \geq 2$. So, we must have $n_1 = 1, n_2 = 2$. We then get

$$\delta+\eta = b^t - 1, \quad \delta^2+\eta^2 = b^{2t}-1.$$

Since

$$b^{2t}-1 = \delta^2+\eta^2 = (\delta+\eta)^2 - 2\delta\eta = (b^t-1)^2 - 2\varepsilon = b^{2t} - 2b^t + 1 - 2\varepsilon,$$

we get $b^t = 1-\varepsilon$. Hence, $\varepsilon = -1$, $b^t = 2$, so $b = 2$, $t = 1$ and this is not the case since b must be odd. Thus, there are no solutions in case $a_1 = a_2 = (b-1)/2$ and b is odd.

Assume next that $a_1 = a_2 = b-1$ and $b = 2^x$. In this case,

$$\Gamma_1 = (n_2m_1 - n_1m_2)\log 2^x + (n_2-n_1)\log 2 = ((n_2m_1-n_1m_2)x + (n_2-n_1))\log 2,$$

so since $\Gamma_1 = 0$, we have $(n_2m_1-n_1m_2)x + n_2 - n_1 = 0$. This can be rewritten as

$$n_2(m_1x+1) = n_1(m_2x+1).$$

Again, we assume that n_1 and n_2 are coprime. Thus, $m_1x+1 = n_1t$, $m_2x+1 = n_2t$ for some integer t. Thus, $m_i = (n_it-1)/x$ for $i = 1,2$. We have

$$X_{n_i} = b^{m_i} - 1 = (2^x)^{(n_it-1)/x} - 1 = 2^{n_it-1} - 1 \quad \text{for} \qquad i = 1,2.$$

Thus,

$$\delta^{n_i} + \eta^{n_i} = 2X_{n_i} = 2^{n_it} - 2 \quad \text{for} \quad i = 1,2.$$

As in the previous case, this gives

$$|\delta^{n_i} - 2^{n_it}| = |2+\eta^{n_i}|, \quad \text{therefore} \quad |\delta - 2^t| = \frac{|2+\eta^{n_i}|}{2^{t(n_i-1)} + \cdots + \delta^{n_i-1}} \quad \text{for} \quad i = 1,2.$$

Again $|\delta-2^t||\eta-2^t| \geq 1$ since it is a non–zero integer. Thus, $|\delta-2^t| > 1/(2^t+1)$. Assuming $n_2 \geq 3$, we get

$$\frac{1}{2^t+1} < |\delta-2^t| = \frac{|2+\eta^{n_2}|}{2^{t(n_2-1)} + \cdots + \delta^{n_2-1}} < \frac{2+1/8}{2^{(n_2-1)t}},$$

so $2^{(n_2-1)t} < (17/8)(2^t+1)$. This implies $t = 1$ and $n_2 = 3$. Further,

$$|\delta-2| = \frac{|2+\eta^3|}{2^2+2\delta+\delta^2} < \frac{2+1/8}{2^2} < 0.6,$$

so $\delta < 2.6$. The only such value is $\delta = 1+\sqrt{2}$, for which we have $X_{n_2} = X_3 = 7 = 2^3 - 1 \neq 2^{n_2t-1} - 1$. Thus, there is no solution in the case $n_2 \geq 3$. Hence, we must have $n_1 = 1$, $n_2 = 2$, and therefore

$X_{n_1} = X_1 = 2^{t-1} - 1$ and $X_{n_2} = X_2 = 2^{2t-1} - 1$. Since $X_2 = 2X_1^2 - \varepsilon$, we get

$$2(2^{t-1} - 1)^2 - \varepsilon = 2^{2t-1} - 1, \quad \text{so} \quad 2^{2t-1} - 1 = 2^{2t-1} - 2^{t+1} + 2 - \varepsilon,$$

which gives $2^{t+1} = 3 - \varepsilon$. Hence, $\varepsilon = -1$, $t = 1$, so $X_1 = 2^{t-1} - 1 = 0$, a contradiction.

Let us record what we have proved.

Lemma 2.2.4 *There is no solution to the system of equations*

$$X_{n_1} = a_1 \left(\frac{b^{m_1} - 1}{b - 1} \right) \qquad \text{and} \qquad X_{n_2} = a_2 \left(\frac{b^{m_2} - 1}{b - 1} \right)$$

with $1 \le n_1 < n_2$ *and* $a_1, a_2 \in \{1, \ldots, b-1\}$ *for which* $\Gamma_1 = 0$ *and* $r = 1$.

Case 2. $r = 2$.

The treatment here is inspired of the treatment for $r = 1$, but first we need to eliminate some easy cases. If $a_1 = a_2$, then

$$\Gamma_1 = (n_2 m_1 - n_1 m_2) \log b + (n_2 - n_1) \log \left(\frac{2a}{b-1} \right),$$

where $a = a_1$ and since $r = 2$, it follows that $\log b$ and $\log(2a/(b-1))$ are linearly independent over $\mathbb{Q}$. Since $\Gamma_1 = 0$, we get $n_1 = n_2$, a contradiction. Similarly, if $2a_1 = b - 1$, then

$$\Gamma_1 = (n_2 m_1 - n_1 m_2) \log b - n_1 \log \left(\frac{2a_2}{b-1} \right),$$

and since $r = 2$, we get that $\log b$ and $\log(2a_2/(b-1))$ are linearly independent over $\mathbb{Q}$. Since $\Gamma_1 = 0$, we get $n_1 = 0$, a contradiction. A similar contradiction is obtained if $2a_2 = b - 1$. So, we assume that $a_1 \neq (b-1)/2$, $a_2 \neq (b-1)/2$ and $a_1 \neq a_2$. Write $b := b_1^x$, where b_1 is not a perfect power of some other integer. Next we write

$$\frac{2a_i}{b-1} = b_1^{y_i} c_i \qquad \text{for} \qquad i = 1, 2. \tag{2.21}$$

It remains to explain how to find y_i for $i = 1, 2$. Let $p \mid b_1$ and put $\alpha_p := \nu_p(b_1)$, where the notation $\nu_p(k)$ is the exponent of the prime p in the factorisation of k. We then put $y_i := \nu_p((2a_i)/(b-1))/\alpha_p$ for $i = 1, 2$. Since p does not divide $b - 1$, it follows that $y_i \ge 0$ for $i = 1, 2$. With this definition, we see that the exponent of p in $2a_i/(b-1)$ and in $b_1^{y_i}$ is the same. Hence, c_i does not contain p in its factorisation. Let us see that $\log c_1$ and $\log c_2$ are linearly dependent over $\mathbb{Q}$. Indeed, since $\log c_1 = \log(2a_1/(b-1)) - (y_1/x) \log b$ and $\log c_2 = \log(2a_2/(b-1)) - (y_2/x) \log b$, and $r = 2$, it follows that $\log c_1, \log c_2$ and $\log b$ are linearly dependent. Write

$$A \log c_1 + B \log c_2 + C \log b = 0$$

for some coefficients A, B, C rational and not all zero. Since p does not appear in the factorisations of c_1 and c_2 but it appears in the factorisation of b, it follows that $C = 0$. Thus, $A \log c_1 = -B \log c_2$. Thus, $\log c_1$ and $\log c_2$ are linearly dependent. Further, none of c_1 or c_2 is 1. Indeed, say if $c_1 = 1$, then $c_2 \neq 1$ (since $r = 2$), and then

$$\Gamma_1 = ((n_2 m_1 - n_1 m_2) + (n_2 y_1 - n_1 y_2)/x) \log b - n_1 \log c_2,$$

and since $r = 2$, we get that $\log b$ and $\log c_2$ are linearly independent over $\mathbb{Q}$. Since $\Gamma_1 = 0$, we get $n_1 = 0$, a contradiction. Thus, none of c_1, c_2 is 1. Let us show that there is a prime q dividing $b - 1$ which appears in the factorisation of c_1. Assume this is not so. Since c_1 and c_2 are multiplicatively dependent, it follows that no prime q dividing $b - 1$ appears in the factorisation of c_2 either. It then follows that $b - 1 \mid 2a_1$ and $b - 1 \mid 2a_2$. Since $2a_1 \leq 2(b - 1)$, it follows that either $2a_i = b - 1$ for some $i = 1, 2$, a situation which we saw is not allowed, or $a_i = b - 1$ for both $i = 1, 2$, so $a_1 = a_2$, which is again not allowed. Hence, indeed there is some prime q dividing $b - 1$ which divides c_1. It then follows that $\log c_1 / \log c_2 = \nu_q(2a_1/(b-1))/\nu_q(2a_2/(b-1)) := u/v$, where u/v is a reduced fraction. Thus, $c_1 = c^u$, $c_2 = c^v$ for some number c. It then follows that

$$\Gamma_1 = ((n_2 m_1 - n_1 m_2) + (n_2 y_1 - n_1 y_2)/x) \log b + (n_2 u - n_1 v) \log c.$$

Since $r = 2$, it follows that $\log b$ and $\log c$ are linearly independent over $\mathbb{Q}$, so $n_1/n_2 = u/v$. Thus, since $\Gamma_1 = 0$ and $\gcd(n_1, n_2) = 1$, we get that $n_1 = |u|$, $n_2 = |v|$. In particular, u and v must have the same sign. Thus, (n_1, n_2) is determined uniquely in terms of (b, a_1, a_2). Next, we look at the coefficient of $\log b = x \log b_1$ in Γ_1 and impose that it is zero to get

$$(n_2 m_1 - n_1 m_2)x + n_2 y_1 - n_1 y_2 = 0. \tag{2.22}$$

Thus,

$$\frac{m_1 x + y_1}{n_1} = \frac{m_2 x + y_2}{n_2}.$$

Recall that y_1, y_2 are not necessarily integers, but αy_1 and αy_2 are integers where again $\alpha = \alpha_p$. Multiplying both sides above by α, we get

$$\frac{m_1 x \alpha + (y_1 \alpha)}{n_1} = \frac{m_2 x \alpha + (y_2 \alpha)}{n_2}.$$

The above equality represents an equality between a fraction with denominator dividing n_1 and a fraction with denominator dividing n_2, and since n_1 and n_2 are coprime, we get that the common value of the above fraction is an integer let's call it t. Since $y_i \alpha \in \mathbb{Z}$, we get $c_i^{\alpha} = (2a_i/(b-1))^{\alpha} b_1^{-y_i \alpha} \in \mathbb{Q}$. Since $c_i = c^{n_i}$ for $i = 1, 2$, it follows that $(c^{\alpha})^{n_1}$ and $(c^{\alpha})^{n_2}$ are both rational numbers and since n_1 and n_2 are coprime, we get that $c^{\alpha} \in \mathbb{Q}$.

Now let us go back to our equations and write

$$\delta^{n_i} + \gamma^{n_i} = 2X_{n_i} = \left(\frac{2a_i}{b-1}\right) b^{m_i} - \left(\frac{2a_i}{b-1}\right) = b_1^{m_i x + y_i} c_i - \left(\frac{2a_i}{b-1}\right) = (b_1^{t/\alpha} c)^{n_i} - \left(\frac{2a_i}{b-1}\right)$$

for $i = 1,2$. Thus,

$$\delta^{n_i} - \left(b_1^{t/\alpha} c\right)^{n_i} = -\eta^{n_i} - 2a_i/(b-1). \tag{2.23}$$

Note that

$$1 \le n_1 < n_2 = \left| v_q \left(\frac{2a_2}{b-1} \right) \right| \le \frac{\log(b-1)}{\log 2} < 2\log b.$$

It remains to bound m_1, m_2. If $\delta < b^3$, then

$$b^{m_i - 1} < a_i \left(\frac{b^{m_i} - 1}{b-1} \right) = X_{n_i} < \delta^{n_i} < b^{3n_i},$$

so $m_i < 3n_i$ for $i = 1,2$. Thus, we assume that $\delta > b^3$. Let $M := \max\{\delta, b_1^{t/\alpha} c\}$. Then

$$\left|\delta - b_1^{t/\alpha} c\right| = \frac{|2a_1/(b-1) + \eta^{n_1}|}{\delta^{n_1 - 1} + \cdots + (b_1^{t/\alpha} c)^{n_1 - 1}} > \frac{2a_1/(b-1) - 1/\delta}{n_1 M^{n_1 - 1}} > \frac{2/b - 1/\delta}{n_1 M^{n_1 - 1}} > \frac{1}{b n_1 M^{n_1 - 1}}, \tag{2.24}$$

since $\delta > b^3 > b$. On the other hand, since $n_2 \ge 2$ and $|\eta| = 1/\delta < 1/2$,

$$\left|\delta - b_1^{t/\alpha} c\right| = \frac{|2a_2/(b-1) + \eta^{n_2}|}{\delta^{n_2 - 1} + \cdots + (b_1^{t/\alpha} c)^{n_2 - 1}} < \frac{2.5}{M^{n_2 - 1}}. \tag{2.25}$$

Comparing (2.24) and (2.25), we get

$$\frac{1}{b n_1 M^{n_1 - 1}} < \frac{2.5}{M^{n_2 - 1}},$$

which gives $M^{n_2 - n_1} < 2.5 b n_1$. In particular,

$$\delta \le M \le M^{n_2 - n_1} < 2.5 b n_1 < 5 b \log b < b^3,$$

a contradiction. Let us summarise what we proved.

Lemma 2.2.5 *Assume that*

$$X_{n_1} = a \left(\frac{b^{m_1} - 1}{b-1} \right) \qquad \text{and} \qquad X_{n_2} = a_2 \left(\frac{b^{m_2} - 1}{b-1} \right)$$

hold for $1 \le n_1 < n_2$, *that* $\Gamma_1 = 0$ *and that* $r = 2$. *Then*

$$1 \le n_1 < n_2 < 2\log b \qquad \text{and} \qquad m_i \le 3n_i \quad \text{for} \quad i = 1,2.$$

2.3 Reducing the bounds

We need to reduce the bounds. For this, we will need to reduce the value of m_1 from inequality (2.12), which for us it is

$$|\Gamma_1| < \frac{8n_2}{b^{m_1 - 1}}.$$

We start with the case when Γ_1 is a linear form in logarithms, where the $\mathbb{Q}$–span of the numbers $\zeta_1 = \log\lambda_1$, $\zeta_2 = \log\lambda_2$ and $\zeta_3 = \log\lambda_3$ has rank $r < 3$.

Case 1. $r = 1$.

We know that in this case $\Gamma_1 \neq 0$. Furthermore, the only possibilities are $a_1 = a_2 = (b-1)/2$ when b is odd, and $a_1 = a_2 = b-1$ when b is a power of 2. In particular, $|\Gamma_1| \geq \log 2$. We thus get that

$$b^{m_1-1} \leq \frac{8n_2}{\log 2} < 16n_2 < 16 \times 8 \times 10^{15} < 2 \times 10^{17}. \tag{2.26}$$

Case 2. $r = 2$ *and* $\Gamma_1 = 0$.

We know that in this case we cannot have $a_1 = a_2$, or $2a_1 = b-1$ or $2a_2 = b-1$. Further, we also cannot have that $a_1 = b-1$ and b is a power of 2. Indeed, for in this case, $b_1 = 2$, $b = 2^x$, $2a_1/(b-1) = 2 = b_1$ so $y_1 = 1$ and $c_1 = 1$. Writing then $2a_2/(b-1) = 2^{y_2}c_2$, we have

$$\Gamma_1 = ((n_1m_2 - n_2m_1)x + n_2 - n_1y_2)\log 2 - n_1\log c_2$$

and $\log 2$ and $\log c_2$ are linearly independent because $r = 2$. Since $\Gamma_1 = 0$, we get that $n_1 = 0$, which is not allowed. Similarly, we cannot have that b is a power of 2 and $a_2 = b-1$. We now wrote a computer program in Mathematica which went though all triples (b, a_1, a_2) with $b \in [2, 100]$, $a_1, a_2 \in \{1, \ldots, b-1\}$ and picked the ones such that the vector space $\mathbb{Q}\zeta_1 + \mathbb{Q}\zeta_2 + \mathbb{Q}\zeta_3$ has $\mathbb{Q}$–dimension $r = 2$. Of those triples, it eliminated the following ones:

(i) $a_1 = a_2$;

(ii) $2a_1 = b-1$ or $2a_2 = b-1$;

(iii) b is a power of 2 and one of a_1 or a_2 is $b-1$.

There were 788 triples left. Let us denote by S the set of these left–over triples. For the triples in S, we asked Mathematica to generate $x, b_1, y_1, y_2, c_1, c_2$. Some interesting examples of quintuples (b, a_1, a_2, c_1, c_2) produced are

$$\left(46, 30, 40, \frac{1}{1587}, \frac{1}{2518569}\right), \left(50, 28, 32, \frac{1}{109375}, \frac{1}{11962890625}\right), \left(82, 32, 36, \frac{1}{384758443521}, \frac{1}{620289}\right),$$

with c_1, c_2 rational numbers of rather large heights, as well as

$$\left(28, 4, 9\frac{1}{189\sqrt{7}}, \frac{1}{3\sqrt{7}}\right), \quad \left(76, 45, 54, \frac{3}{5\sqrt{19}}, \frac{9}{475}\right), \quad \left(99, 42, 36, \frac{2}{7\sqrt{11}}, \frac{4}{539}\right),$$

where not both c_1, c_2 were rational. Of course not all of them gave viable examples because for some $c_1 = c_2$, or $\log c_1$ and $\log c_2$ have opposite signs, etc. It turns out that the valid ones, namely the ones for which $\log c_1/\log c_2$ is a positive rational number in $(0,1)$ of the form n_1/n_2 with coprime n_1, n_2 all have $1 \leq n_1 < n_2 \leq 5$. So, by the results from Lemma 2.2.5, we must have

$m_1 \leq 12$ and $m_1 \leq m_2 \leq 15$. Thus, we computed, for all the 788 triples (b, a_1, a_2), and for all $1 \leq n_1 < n_2 \leq 5$ and $m_1 \leq 12,\ m_1 \leq m_2 \leq 15$, the resultant the polynomials

$$R := \operatorname{Res}(P_{n_1}^{\varepsilon}(X_1) - a_1(b^{m_1}-1)/(b-1), P_{n_2}^{\varepsilon}(X_1) - a_2(b^{m_2}-1)/(b-1), X_1), \quad \varepsilon \in \{\pm 1\},$$

where

$$P_n^{\varepsilon}(X) = \frac{1}{2}\left((X+\sqrt{X^2-\varepsilon})^n + (X-\sqrt{X^2-\varepsilon})^n\right) \quad \text{for} \quad \varepsilon \in \{\pm 1\},$$

which is the Chebyshev polynomial that gives the nth coordinate X_n of the Pell equation $X^2 - dY^2 = \pm 1$ provided that $X_1^2 - dY_1^2 = \varepsilon$ and selected those situations for which $R = 0$ (so the above two polynomials have a root X_1 in common, which might be a positive integer or not). This computation lasted a few minutes and gave the following 16 examples $(b, a_1, a_2, n_1, n_2, m_1, m_2, \varepsilon)$:

$$\begin{aligned}
&(5,4,1,1,2,1,3,1), \quad (5,4,1,2,4,1,3,1), \quad (5,4,1,2,4,1,3,-1), \quad (8,2,1,1,2,1,2,-1),\\
&(9,1,3,1,2,1,1,-1), \quad (19,1,3,1,2,1,1,-1), \quad (27,1,3,1,2,1,1,-1), \quad (32,4,1,1,2,1,2,-1),\\
&(49,1,7,1,3,1,1,-1), \quad (50,2,7,1,2,1,1,1), \quad (50,2,7,2,4,1,1,1), \quad (50,2,7,2,4,1,1,-1),\\
&(55,1,3,1,2,1,1,-1), \quad (81,1,3,1,2,1,1,-1), \quad (82,2,9,1,2,1,1,-1), \quad (99,1,7,1,3,1,1,-1).
\end{aligned}$$

For example, for the first one

$$X_1 = \frac{4(5^1-1)}{5-1} = 4, \quad X_2 = \frac{5^3-1}{5-1} = 31, \quad X_2 = 2X_1^2 - 1.$$

This corresponds to $d = 15$. In fact, here is the full list of (d, n_1, n_2, b) for $n_1 < n_2$ which are coprime. We have $(n_1, n_2) = (1, 2)$ for

$$(d, b) \in \{(2,19),\ (2,27),\ (2,55),\ (2,81),\ (3,50),\ (5,8),\ (5,32),\ (5,82),\ (15,5)\},$$

and $(n_1, n_2) = (1, 3)$ for $(d, b) \in \{(2,49),\ (2,99)\}$. However, we calculated Γ_1 for all the above choices and we did not get any example with $\Gamma_1 = 0$. We conclude that there is no instance with $\Gamma_1 = 0$.

Case 3. $r = 2$ and $\Gamma_1 \neq 0$.

We distinguish the following possibilities:

(i) $2a_1 = b - 1$. This only appears for b odd. In this case,

$$|\Gamma_1| = \left|(n_2 m_1 - m_2 n_1)\log b - n_1 \log\left(\frac{2a_2}{b-1}\right)\right| < \frac{8n_2}{b^{m_1-1}}.$$

Here, $\log b$ and $\log(2a_2/(b-1))$ are linearly independent. The above can be written as

$$\left|\tau - \frac{(n_2 m_1 - n_1 m_2)}{n_1}\right| < \frac{8n_2}{n_1(\log b) b^{m_1-1}}, \qquad \text{with} \qquad \tau := \frac{\log(2a_2/(b-1))}{\log b}. \tag{2.27}$$

Similar considerations apply when $2a_2 = b - 1$, and $\log b$ and $\log(2a_1/(b-1))$ are linearly independent.

(ii) $a_1 = a_2$. In this case,

$$|\Gamma_1| = \left|(n_2 m_1 - n_1 m_2)\log b + (n_2 - n_1)\log\left(\frac{2a}{b-1}\right)\right| < \frac{8n_2}{b^{m_1-1}},$$

where $a = a_1$. Here, $\log b$ and $\log(2a/(b-1))$ are linearly independent. As in case (i) above, this implies

$$\left|\tau - \frac{n_1 m_2 - n_2 m_1}{n_2 - n_1}\right| < \frac{8n_2}{(n_2 - n_1)(\log b) b^{m_1-1}}, \qquad \text{with} \qquad \tau := \frac{\log(2a_2/(b-1))}{\log b}. \tag{2.28}$$

(iii) $b = 2^x$ is a power of 2 and $a_1 = b - 1$. In this case,

$$|\Gamma_1| = \left|((n_2 m_1 - n_1 m_2)x + n_2)\log 2 - n_1 \log\left(\frac{2a_2}{b-1}\right)\right| < \frac{8n_2}{b^{m_1-1}},$$

and $2a_2/(b-1)$ is not a power of 2. This can be written as

$$\left|\tau - \frac{(n_2 m_1 - n_1 m_2)x + n_2}{n_1}\right| < \frac{8n_2}{n_1(\log 2) b^{m_1-1}} \qquad \text{with} \qquad \tau := \frac{\log(2a_2/(b-1))}{\log 2}. \tag{2.29}$$

The same considerations apply to the symmetric situation when $a_2 = b - 1$ and $2a_1/(b-1)$ is not a power of 2.

(iv) The set S of 788 triples (b, a_1, a_2) for which $r = 2$ and which are not in any of the cases (i), (ii), (iii) above. For each of these triples, we already generated the corresponding c_1, c_2. Sometimes they are not rational but their squares are always rational (this was confirmed computationally). Thus, $c_1^2 = (c^2)^u$ and $c_2^2 = (c^2)^v$, where u, v are integers and c^2 is rational. Since $2a_i/(b-1) = b_1^{y_i} c_i$, we get that $b_1^{2y_i} = (2a_i/(b-1))^2 c_i^{-2} \in \mathbb{Q}$, and since b_1 is not a power of a rational number, it follows that $2y_i \in \mathbb{Z}$ for $i = 1, 2$. Thus,

$$|2\Gamma_1| = |(2(n_2 m_1 - n_1 m_2)x + n_2(2y_1) - n_1(2y_2))\log b_1 - (n_2 u - n_1 v)\log(c^2)| < \frac{16n_2}{b^{m_1-1}},$$

and b_1 and c^2 are rational numbers which are multiplicatively independent and the coefficients

$$(N, M) := \big(2(n_2 m_1 - n_1 m_2)x + n_2(2y_1) - n_2(2y_2), n_2 u - n_1 v\big)$$

of $\log b_1$ and $\log c^2$, respectively, are integers. By simultaneously changing the signs of (u, v) (so, replacing (u, v) by $(-u, -v)$), we may assume that $c^2 > 1$. The above inequality can be written as

$$\left|\tau - \frac{M}{N}\right| < \frac{16n_2}{N(\log c^2) b^{m_1-1}}, \qquad \text{with} \qquad \tau := \frac{\log b_1}{\log(c^2)}. \tag{2.30}$$

Now each of the estimates (2.27), (2.28), (2.29), (2.30) is of the form

$$\left|\tau - \frac{P}{Q}\right| < \frac{16n_2}{Q(\log \kappa) b^{m_1-1}}, \tag{2.31}$$

for a certain irrational τ which is a ratio of two logarithms of rational numbers, and where $P, Q > 0$

are integers with $Q \in \{n_1, n_2, n_2 - n_1, |2(n_2 m_1 - n_1 m_2)x + n_2(2y_1) - n_2(2y_2)|\}$ and $\kappa \in \{2, b_1, b, c^2\}$. The minimal value of c^2 computed is 64/49 (and the maximal is 384758443521). A uniform upper bound on Q is

$$Q \le 2 \times 6 \times 14 n_2 + (2 \times 7) n_2 + (2 \times 7) n_2 < 200 n_2,$$

where we used the fact that $x \le 6$, $y_i \le 7$ and $|n_2 m_1 - n_2 m_2| \le 14 n_2$ (see (2.13)). We would like that in all instances, the right–hand side of (2.31) is smaller than $1/(2Q^2)$. This would be so if

$$b^{m_1 - 1} > \frac{32 n_2 Q}{\log \kappa},$$

which from the above remarks holds if

$$b^{m_1 - 1} > 2 \times 10^{36} > \frac{32 \times 200 \times (8 \times 10^{15})^2}{\log(64/49)} > \frac{32 n_2 \times (200 n_2)}{\log \kappa} > \frac{32 n_2 Q}{\log \kappa}.$$

Thus, assuming $b^{m_1 - 1} > 2 \times 10^{36}$, it follows, by Lemma 1.4.7, that $P/Q = p_k/q_k$ is a convergent of τ. Now $Q < 200 n_2 < 1.6 \times 10^{18} < F_{90}$, where F_{90} is the 90th Fibonacci number, so we deduce that $k \le 89$. We computed, for all potential τ, the first 100 terms of its continued fraction expansion $[a_0, a_1, \ldots, a_{99}, \ldots]$. In the first three cases (i)–(iii), we got that $a_i \le 682970$ for all $i = 0, \ldots, 99$. For the last case, we got that $a_i \le 13107$ for $i = 0, \ldots, 99$. By Lemma 1.4.7, it follows that

$$\left| \tau - \frac{P}{Q} \right| > \frac{1}{7 \times 10^5 Q^2}.$$

Comparing the above inequality with (2.31), we get

$$b^{m_1 - 1} \le \frac{16 \times 7 \times 10^5 n_2 Q}{\log \kappa} < \frac{16 \times 7 \times 10^5 \times 200 \times n_2^2}{\log \kappa} < 6 \times 10^{41}.$$

Thus, we conclude that having assumed that the inequality $b^{m_1 - 1} > 2 \times 10^{36}$ holds, we concluded that $b^{m_1 - 1} < 6 \times 10^{41}$ must hold. We record what we just proved.

Lemma 2.3.1 *If $r = 1$, then $\Gamma_1 \neq 0$ and*

$$b^{m_1 - 1} < 2 \times 10^{17}.$$

If $r = 2$, then $\Gamma_1 \neq 0$ and

$$b^{m_1 - 1} < 6 \times 10^{41}.$$

Note that the above lemma shows that if $r = 1, 2$, then $m_1 < 90$ for all $b \ge 3$. If $b = 2$, then $a_1 = a_2 = 1$ so we are in the case $r = 1$, for which we get $m_1 \le 58$, which is even better.

Case 4. $r = 3$.

As in the Case 2, we found all triples (b, a_1, a_2) with $b \in [2, 100]$, $a_1, a_2 \in \{1, \ldots, b - 1\}$ and $a_1 < a_2$, such that the vector space $\mathbb{Q}\zeta_1 + \mathbb{Q}\zeta_2 + \mathbb{Q}\zeta_3$ has $\mathbb{Q}$–dimension $r = 3$, so $\Gamma_1 \neq 0$. We keep these triples in a set denoted by S. A similar calculation in Mathematica as in Case 2, reveals that $|S| = 158791$.

For each triple (b, a_1, a_2) in S, we calculate a lower bound for $|\Gamma_1|$ through Lemma 1.4.2. By inequality (2.13) and Lemma 2.2.3, we set $X_1 := 1.2\times 10^{17}$ and $X_2 := 8\times 10^{15}$, as upper bounds for $|n_2 m_1 - n_1 m_2|$ and $n_1 < n_2$, respectively.

We take $C := (3X_1)^3$ and consider the lattice Ω spanned by

$$v_1 := \left(1, 0, \lfloor C\log b\rfloor\right), \quad v_2 := \left(0, 1, \lfloor C\log(2a_1/(b-1))\rfloor\right), \quad v_3 := \left(0, 0, \lfloor C\log 2a_2/(b-1)\rfloor\right).$$

Using Mathematica, we estimate a reduced basis $\{\mathbf{b}_i\}$ (LLL–algorithm) for Ω and its associated Gram–Schmidt $\{\mathbf{b}_i^*\}$ basis. So, we calculated the parameters

$$Q = X_1^2 + X_2^2, \qquad T = (1 + X_1 + 2X_2)/2, \qquad c_1 = \max_{1\le i\le 3} \frac{||\mathbf{b}_1||}{||\mathbf{b}_i^*||} \qquad \text{and} \qquad \mathbf{m}_\Omega = \frac{||\mathbf{b}_1||}{c_1}.$$

By the conclusion of Lemma 2.2.3 together with inequality (2.12), we obtain that

$$3.1\times 10^{-42} < |\Gamma_1| < 8n_2 \times b^{-(m_1-1)},$$

which leads to

$$b^{m_1-1} < 2.1\times 10^{58}.$$

Note that the above inequality includes the inequalities of Lemma 2.3.1.

So, by inequality (1.3) with $i = 1$, we have a bound for n_1:

$$n_1 < (m_1\log b + \log(b+1))/\log\delta =: l_b.$$

At this point, we consider the polynomial equations:

$$P_{n_1}^{\varepsilon}(X_1) = \frac{1}{2}\left(\left(X_1 + \sqrt{X_1^2 - \varepsilon}\right)^{n_1} + \left(X_1 - \sqrt{X_1^2-\varepsilon}\right)^{n_1}\right) = a_1\frac{b^{m_1}-1}{b-1} \quad \text{for} \quad \varepsilon\in\{\pm 1\}, \tag{2.32}$$

with $b\in[2,100]$, $a_1\in[1,b-1]$, $m_1\in[1, 1+\log(2.1\times 10^{58})/\log b]$. In order to have some control over n_1 and X_1, we assume that $\delta > 10^3$, so $n_1 < l_b < 20$, reducing in this way both the complexity and the running time of the calculations. Furthermore, given that $\delta = X_1 + \sqrt{X_1^2-\varepsilon} < 2X_1 + 1$ also we have that $X_1 \ge 500$.

A quick computer search on the above equations (2.32) showed that all solutions in this range have $n_1\in\{1,2\}$. Note that for $n_1 = 1$, we obtain the polynomials identity $P_{n_1}^{\pm}(X_1) = X_1$ and therefore there is no equation to solve. In this case, we keep the set $A_b^{(\varepsilon)}$ as the set of all the X_1's taken as b–repdigits on the right–hand side of (2.32) with $b\in[2,100]$, $a_1\in[1,b-1]$, $m_1\in[1, 1+\log(2.1\times 10^{58})/\log b]$. For $n_1 = 2$, we obtain

	b's	$\lvert B_b^{(\varepsilon)}\rvert$
$\varepsilon = 1$	2, 8, 18, 32, 50, 72, 98	88, 30, 22, 18, 16, 14, 14
$\varepsilon = -1$	55	1

where here we put $B_b^{(\varepsilon)}$ for the set of solutions X_1 of (2.32), for each b and ε fixed. On the other hand, if $\delta \le 10^3$, then

$$1.86 \times X_1 < X_1(1+\sqrt{1-\varepsilon X_1^{-2}}) = \delta. \tag{2.33}$$

So, $X_1 \le 540$. We keep the set $C^{(\varepsilon)}$ to be the set of values of X_1, where $X_1 \in [x_\varepsilon, 540]$ with $x_\varepsilon = 1$ if $\varepsilon = -1$ or $x_\varepsilon = 2$ if $\varepsilon = 1$.

We now, consider the set of pairs related of integers to $A_b^{(\varepsilon)}, B_b^{(\varepsilon)}$ and $C^{(\varepsilon)}$ as follows:

$$D = \left(\bigcup_{\substack{b=2\\ \varepsilon=\pm1}}^{100} \{b\} \times A_b^{(\varepsilon)} \right) \cup \left(\bigcup_{\substack{b=2,8,18,32,50,55,72,98\\ \varepsilon=\pm1}} \{b\} \times B_b^{(\varepsilon)} \right) \cup \left(\bigcup_{\substack{b=2\\ \varepsilon=\pm1}}^{100} \{b\} \times C^{(\varepsilon)} \right).$$

For each pair (b, X_1) in D, we have a pair (b,δ), where $\delta := \delta_{X_1,\varepsilon} = X_1 + \sqrt{X_1^2 - \varepsilon}$; we denote by D^* the set of all such pairs (b,δ).

We return to inequality (2.10), which we rewrite for $i = 2$ as

$$\left| n_2 \frac{\log \delta}{\log b} - m_2 - \frac{\log(2a_2/(b-1))}{\log b} \right| < \frac{4}{(\log b) b^{m_2-1}}. \tag{2.34}$$

We need to distinguish two cases according to the parameter $\mu_b := \log(2a_2/(b-1))/\log b$:

(i) $b = 2$ (so $a_2 = 1$), or b is odd and $a_2 = (b-1)/2$.

In this case, $\mu_b \in \{0,1\}$ and

$$\left| \tau_{b,\delta} - \frac{m_2^*}{n_2} \right| < \frac{4}{n_2(\log b) b^{m_2-1}}, \qquad \text{with} \qquad m_2^* \in \{m_2, m_2+1\} \quad \text{and} \quad \tau_{b,\delta} := \frac{\log \delta}{\log b}. \tag{2.35}$$

We assume $b^{m_2-1} > 10^{17}$, so $4 \times (n_2(\log b) b^{m_2-1})^{-1} < (2n_2^2)^{-1}$. By Lemma 1.4.7, m_2^*/n_2 is a convergent of $\tau_{b,\delta}$. For each pair $(b,\delta) \in D^*$, where b satisfies (i), we calculated the continued fraction expansion $[a_0^{(b,\delta)}, a_1^{(b,\delta)}, \ldots]$ and the convergents $p_k^{(b,\delta)}/q_k^{(b,\delta)}$ of the corresponding $\tau_{b,\delta}$, then found the first $k_{b,\delta} \in \mathbb{Z}^+$ such that $q_{k_{b,\delta}}^{(b,\delta)} > 8 \times 10^{15} \ge n_2$ and we calculated, according inequality (2.35) and to the conclusion of Lemma 1.4.7,

$$m_2^{(b,\delta)} := \frac{\log\left(4\left(a_{k_{b,\delta}}^{(b,\delta)} + 2\right)(8 \times 10^{15})/\log b\right)}{\log b} + 1.$$

This allowed us to dramatically reduce the bound on m_2. A few hours of computational work revealed that $\max\{m_2^{(b,\delta)}\} \le 72$. The maximum was obtained for $(b, \varepsilon, n_1) = (73, 1, 1)$ and $\delta > 10^3$.

(ii) $b \ge 4$ is even, or b is odd and $a_2 \ne (b-1)/2$.

In this case, $\mu_b \neq 0$. We apply Lemma 1.4.6 with $(m,n,k) := (n_2, m_2, m_2 - 1)$,

$$(\tau_{b,\delta}, \mu_b) := \left(\frac{\log\delta}{\log b}, \frac{\log(2a_2/(b-1))}{\log b}\right), \qquad (A_b, B_b) := \left(\frac{4b}{\log b}, b\right).$$

For $p_k^{(b,\delta)}/q_k^{(b,\delta)}$, the kth convergent of $\tau_{b,\delta}$, we take $\varepsilon_{b,\delta} := ||\mu_b q_k|| - M||\tau_{b,\delta} q_k||$, $M := 8 \times 10^{15}$ being an upper bound to n_2 according to Lemma 2.2.3. After several days of computational work, we obtained that for all $(b,\delta) \in D^*$, where b satisfies (ii), $q_k^{(b,\delta)} > 6M$, it holds that

$$q_k^{(b,\delta)}/\varepsilon_{b,\delta} < 10^{70}, \qquad \text{for both } \varepsilon \in \{\pm 1\}.$$

Thus,

$$m_2 < \frac{\log\left(A q_k^{(b,\delta)}/\varepsilon_{b,\delta}\right)}{\log B} < \frac{\log\left(4b q_k^{(b,\delta)}/(\varepsilon_{b,\delta}\log b)\right)}{\log b} < \frac{\log(8\times 10^{70}/\log 2)}{\log 2} \le 240.$$

Hence, in all cases we have that $m_1 < m_2 \le 240$.

Now that our upper bound on m_2 (and so also on n_2) has been substantially reduced, we can do a new cycle of reduction. Performing two more reduction cycles, we obtain that $m_1 \le m_2 \le 85$ and using the left–hand side of inequality (1.3), we also get that $n_1 < n_2 < (m_2 \log b + \log(b+1))/\log\delta$.

Gathering all the information obtained, our problem is reduced to searching for solutions to (2.3) in the range: $b \in [2,100]$, $a_1, a_2 \in [1, b-1]$ and

$$1 \le n_1 < n_2 \le (m_2 \log b + \log(b+1))/\log\delta, \qquad 1 \le m_1 \le m_2 \le 85.$$

This is equivalent to determining for which b and d fixed, the equation

$$X_n = a\left(\frac{b^m - 1}{b-1}\right) \quad \text{with} \quad a \in \{1, \ldots, b-1\}. \tag{2.36}$$

has at least two solutions (n, m, a).

An extensive computational search allows us to verify that for $\delta > 10^4$ (here $n \le 5$ and $X_1 > 5000$), there is no pair of polynomial equations as (2.32) with a common solution X_1, for all $b \in [2,100]$. Hence, we conclude that (2.36) has at most one solution (n, m, a), for each $b \in [2,100]$ and $\delta > 10^4$.

It remains to deal with the existence of multiple solutions to (2.36) when $\delta \le 10^4$, $X_1 \in [x_\varepsilon, 5380]$ according to (2.33). Since we know the values of $\varepsilon \in \{\pm 1\}$, we can set $X_0 := 1$, and generate the first few terms of the sequence $\{X_n\}_{n\ge 1}$ via the recurrence

$$X_n = 2X_1 X_{n-1} - \varepsilon X_{n-2} \quad \text{for all} \quad n \ge 2. \tag{2.37}$$

Therefore our last step is as following:

(i) We take R_b to be the set of all base b–repdigits with at most 85 digits (given that $m_2 \le 85$), where $b \in [2,100]$.

(ii) For each $X_1 \in [x_\varepsilon, 5380]$ and $\varepsilon = \pm 1$, we use (2.37) to generate the set

$$X^{(\varepsilon, X_1)} := \{X_n : 1 \le n \le (m_2 \log b + \log(b+1)) / \log \delta\}.$$

(iii) We search computationally for solutions to $|X^{(\varepsilon,X_1)} \cap R_b| \ge 2$, $b \in [2,100]$, $X_1 \in [x_\varepsilon, 5380]$ and $\varepsilon = \pm 1$.

Before listing the actual solutions, we make some comments. We denote $a(b^m-1)/(b-1)$ by $R(b,a,m)$. The solutions of (2.3) such that $\max\{X^{(\varepsilon,X_1)} \cap R_b\} \ge b$, for some b, will be called *nontrivial solutions* for the base b.

The same solutions found nontrivial for b might appear again for a base $b' > b$ if the elements of $X^{(\varepsilon,X_1)} \cap R_b$ remain b'–repdigits. If $b' > \max\{X^{(\varepsilon,X_1)} \cap R_b\}$ such solutions become trivial. If $b' \le \max\{X^{(\varepsilon,X_1)} \cap R_b\}$ such solutions are still non-trivial but we no longer list them since they yield neither a new value for d, nor a larger value for n_2.

An exhaustive search in Mathematica reveals the following lists.

ε	b	X_1	d	δ	$X_n = R(b,a,m)$
1	3	2	3	$2+\sqrt{3}$	$X_1 = 2 = R(3,2,1)$ $X_3 = 26 = R(3,2,3)$
	5	4	15	$4+\sqrt{15}$	$X_1 = 4 = R(5,4,1)$ $X_2 = 31 = R(5,1,3)$
	6	2	3	$2+\sqrt{3}$	$X_2 = 7 = R(6,1,2)$
	10	3	8	$3+2\sqrt{2}$	$X_1 = 3 = R(10,3,1)$ $X_3 = 99 = R(10,9,2)$
	15	11	120	$11+\sqrt{120}$	$X_1 = 11 = R(15,11,1)$ $X_2 = 241 = R(15,1,3)$
	16	3	8	$3+2\sqrt{2}$	$X_2 = 17 = R(16,1,2)$
	22	9	80	$9+4\sqrt{5}$	$X_1 = 9 = R(22,9,1)$ $X_2 = 161 = R(22,7,2)$
		14	195	$14+\sqrt{195}$	$X_1 = 14 = R(22,14,1)$ $X_2 = 391 = R(22,17,2)$
	32	23	528	$23+\sqrt{528}$	$X_1 = 23 = R(32,23,1)$ $X_2 = 1057 = R(32,1,3)$
	40	12	143	$12+\sqrt{143}$	$X_1 = 12 = R(40,12,1)$ $X_2 = 287 = R(40,7,2)$
	45	8	63	$8+3\sqrt{7}$	$X_1 = 8 = R(45,8,1)$ $X_3 = 2024 = R(45,44,2)$
	46	6	35	$6+\sqrt{35}$	$X_1 = 6 = R(46,6,1)$ $X_3 = 846 = R(46,18,2)$
		20	399	$20+\sqrt{399}$	$X_1 = 20 = R(46,20,1)$ $X_2 = 799 = R(46,17,2)$
		27	728	$27+2\sqrt{182}$	$X_1 = 27 = R(46,27,1)$ $X_2 = 1457 = R(46,31,2)$
	48	5	24	$5+2\sqrt{6}$	$X_1 = 5 = R(56,5,1)$ $X_2 = 49 = R(48,1,2)$
	58	3	8	$8+2\sqrt{2}$	$X_5 = 3363 = R(58,57,2)$
	60	4	15	$4+\sqrt{15}$	$X_3 = 244 = R(60,4,2)$
	70	6	35	$6+\sqrt{35}$	$X_2 = 71 = R(70,1,2)$
	72	16	255	$16+\sqrt{255}$	$X_1 = 16 = R(72,16,1)$ $X_2 = 511 = R(72,7,2)$
	78	35	1224	$35+6\sqrt{34}$	$X_1 = 35 = R(78,35,1)$ $X_2 = 2449 = R(78,31,2)$
		44	1935	$44+3\sqrt{215}$	$X_1 = 44 = R(78,44,1)$ $X_2 = 3871 = R(78,49,2)$
	88	32	1023	$32+\sqrt{1023}$	$X_1 = 32 = R(88,32,1)$ $X_2 = 2047 = R(88,23,2)$
		57	3248	$57+4\sqrt{203}$	$X_1 = 57 = R(88,57,1)$ $X_2 = 6497 = R(88,73,2)$
	90	64	4095	$64+3\sqrt{455}$	$X_1 = 64 = R(90,64,1)$ $X_2 = 8191 = R(90,1,3)$
	96	5	24	$5+2\sqrt{6}$	$X_3 = 485 = R(96,5,2)$
		7	48	$7+4\sqrt{3}$	$X_2 = 97 = R(96,1,2)$
	100	63	3968	$63+8\sqrt{62}$	$X_1 = 63 = R(100,63,1)$ $X_3 = 999999 = R(100,99,3)$

ε	b	X_1	d	δ	$X_n = R(b,a,m)$
-1	2	1	2	$1+\sqrt{2}$	$X_1 = 1 = R(2,1,1)$ $X_2 = 3 = R(2,1,2)$ $X_3 = 7 = R(2,1,3)$
	4	2	5	$2+\sqrt{5}$	$X_1 = 2 = R(4,2,1)$ $X_5 = 682 = R(4,2,5)$
	6	1	2	$1+\sqrt{2}$	$X_3 = 7 = R(6,1,2)$
	8	2	5	$2+\sqrt{5}$	$X_2 = 9 = R(8,1,2)$
		6	37	$6+\sqrt{37}$	$X_1 = 6 = R(8,6,1)$ $X_2 = 73 = R(8,1,3)$
	10	1	2	$1+\sqrt{2}$	$X_3 = 99 = R(10,9,2)$
		4	17	$4+\sqrt{17}$	$X_1 = 4 = R(10,4,1)$ $X_2 = 33 = R(10,2,2)$
	12	3	10	$3+\sqrt{10}$	$X_3 = 117 = R(12,9,2)$
	16	1	2	$1+\sqrt{2}$	$X_4 = 17 = R(16,1,2)$
		5	26	$5+\sqrt{26}$	$X_1 = 5 = R(16,5,1)$ $X_2 = 51 = R(16,3,2)$
	18	2	5	$2+\sqrt{5}$	$X_3 = 38 = R(18,2,2)$
		3	10	$3+\sqrt{10}$	$X_2 = 19 = R(18,1,2)$
	22	2	5	$2+\sqrt{5}$	$X_4 = 161 = R(22,7,2)$
	26	11	122	$11+\sqrt{122}$	$X_1 = 11 = R(26,11,1)$ $X_2 = 243 = R(26,9,2)$
		16	257	$16+\sqrt{257}$	$X_1 = 16 = R(26,16,1)$ $X_2 = 513 = R(26,19,2)$
	35	13	170	$13+\sqrt{170}$	$X_1 = 13 = R(35,13,1)$ $X_3 = 8827 = R(35,7,3)$
	40	1	2	$1+\sqrt{2}$	$X_5 = 41 = R(40,1,2)$
		15	226	$15+\sqrt{226}$	$X_1 = 15 = R(40,15,1)$ $X_2 = 451 = R(40,11,2)$
		26	677	$26+\sqrt{677}$	$X_2 = 1353 = R(40,33,2)$
	41	6	37	$6+\sqrt{37}$	$X_3 = 882 = R(41,21,2)$
	42	8	65	$8+\sqrt{65}$	$X_1 = 8 = R(42,8,1)$ $X_2 = 129 = R(42,3,2)$
	49	35	1226	$35+\sqrt{1226}$	$X_1 = 35 = R(49,35,1)$ $X_2 = 2451 = R(49,1,3)$
	50	22	485	$22+\sqrt{485}$	$X_1 = 22 = R(50,22,1)$ $X_2 = 969 = R(50,19,2)$
		29	842	$29+\sqrt{842}$	$X_1 = 29 = R(50,29,1)$ $X_2 = 1683 = R(50,33,2)$
	55	8	65	$8+\sqrt{65}$	$X_3 = 2072 = R(55,37,2)$
	58	18	325	$18+\sqrt{325}$	$X_1 = 18 = R(58,18,1)$ $X_2 = 649 = R(58,11,2)$
		41	1682	$41+\sqrt{1682}$	$X_2 = 3363 = R(58,57,2)$
	64	10	101	$10+\sqrt{101}$	$X_1 = 10 = R(64,10,1)$ $X_3 = 4030 = R(64,62,1)$
	66	4	17	$4+\sqrt{17}$	$X_2 = 268 = R(66,4,2)$
	76	6	37	$6+\sqrt{37}$	$X_5 = 128766 = R(76,22,3)$
	82	37	1370	$37+\sqrt{1370}$	$X_1 = 37 = R(82,37,1)$ $X_2 = 2739 = R(82,33,2)$
		46	2117	$46+\sqrt{2117}$	$X_1 = 46 = R(82,46,1)$ $X_2 = 4233 = R(82,51,2)$
	88	20	401	$20+\sqrt{401}$	$X_1 = 20 = R(88,20,1)$ $X_2 = 801 = R(88,9,2)$
	96	40	1601	$40+\sqrt{1601}$	$X_1 = 40 = R(96,40,1)$ $X_2 = 3201 = R(96,33,2)$
		57	3250	$57+\sqrt{3250}$	$X_1 = 57 = R(96,57,1)$ $X_1 = 6499 = R(96,67,2)$

For example, for the triple $(\varepsilon, b, X_1) = (-1, 38, 3)$, we obtained $X^{(-1,3)} \cap R_{38} = \{3, 19, 117\}$. However, note that $b = 38$ isn't included in the above list. The reason is that this solution was already found nontrivial for $b = 2,\ 18$. However, $117 = R(38,3,2)$. The values of d appearing in the above table appear in the statement of Corollary 2.1.2. The largest n_2 appearing in the above table is $n_2 = 5$.

This completes the proof of our computational result.

Chapter 3

Y-Coordinates of Pell Equation which are Fibonacci Numbers

Chapter Note: The results of this chapter have been submitted to an accredited journal for publication.

Let $d \geq 2$ be an integer which is not a square. In this chapter, the sequence $(F_n)_{n\geq 0}$ of Fibonacci numbers is considered with the pair $(X_m, Y_m)_{m\geq 1}$ set to be the mth solution of the Pell equation $X^2 - dY^2 = \pm 1$. It is then demonstrated that except for $d = 2$ when the equation $Y_m = F_n$ has the three solutions $(m, n) = (1,2), (2,3), (3,5)$, there are at most two positive integer solutions (m, n) for all d.

3.1 Introduction

For a non-square positive integer $d \geq 2$, we consider the Pell equation

$$X^2 - dY^2 = \pm 1. \tag{3.1}$$

If m is some positive integer, then all the positive integer solutions (X, Y) to 3.1 are of the form $(X, Y) = (X_m, Y_m)$ with the pair (X_m, Y_m) satisfying

$$X_m + Y_m\sqrt{d} = (X_1 + Y_1\sqrt{d})^m$$

where (X_1, Y_1) is designated as the minimal solution. We recall that the sequences $(X_m)_{m\geq 1}$ and $(Y_m)_{m\geq 1}$ are binary recurrent with characteristic polynomial $x^2 - (2X_1)x + \lambda$, where $\lambda = X_1^2 - dY_1^2 \in \{\pm 1\}$.

Several recent papers have investigated the following problem. Assume that $\mathbf{U} := (U_n)_{n\geq 0}$ is some interesting sequence of positive integers. What can one say about the number of solutions of the containment $X_m \in \mathbf{U}$ for a generic d? What about the number of solutions of the containment $Y_m \in \mathbf{U}$? For the sequence of X-coordinates, the answer is that for most binary recurrent sequences (Fibonacci numbers [23], repdigits [19, 16] in some given integer base $b \geq 2$, etc.), the equation $X_m \in \mathbf{U}$ has at most one positive integer solution m for any given d except for a few (finitely many) values of d, which have been explicitly calculated for the above examples $\mathbf{U}$.

For the sequence of Y-coordinates of the Pell equation corresponding to some d, it was shown in [18] that if $\mathbf{U} := (U_n)_{n\geq 0}$ is a fixed binary recurrent sequence of integers whose characteristic equation has real roots, then provided d exceeds some effectively computable bound depending on $\mathbf{U}$, we have that the containment $Y_m \in \mathbf{U}$ has at most two solutions m. It has exactly two solutions for infinitely many d's in the case where the fixed binary recurrent sequence $\mathbf{U}$ has 1 together with infinitely many even integers. The cases of the specific binary recurrent sequences with general terms $U_n = 2^n - 1$ and $U_n = L_n$, the nth Lucas number, respectively, were treated in an elementary way in [17], and following the non-elementary procedure described in [18] in [15], respectively. We set $\mathbf{U} := (F_n)_{n\geq 0}$ with $F_0 = 0,\ F_1 = 1$. with recurrence $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$ and study the equation $Y_m = F_n$ in integers (m, n) with $m \geq 1,\ n \geq 1$. Our result is the following.

Theorem 3.1.1 *Let $d \geq 2$ be an integer which is not a square, $(X_m, Y_m)_{m\geq 1}$ be the mth positive integer solution to the Pell equation (3.1). The Diophantine equation*

$$Y_m = F_n$$

has at most two positive solutions (m, n) with $m \geq 1,\ n \geq 1$ except for $d = 2$ when it has three solutions $(m, n) \in \{(1, 2), (2, 3), (3, 5)\}$.

We follow the genera approach from [18] with some details borrowed from [15] in proving the result.

3.2 Pell equation and Fibonacci sequence

We set (X_1, Y_1) to be the minimal solution of (3.1). We first note the following facts about Fibonacci numbers and Pell equations.

$$\gamma := X_1 + \sqrt{d}Y_1 \qquad \text{and} \qquad \delta := X_1 - \sqrt{d}Y_1.$$

Then

$$X_m = \frac{\gamma^m + \delta^m}{2} \qquad \text{and} \qquad Y_m = \frac{\gamma^m - \delta^m}{2\sqrt{d}} \qquad \text{hold for all} \quad m \geq 1.$$

Note that $\delta = \lambda\gamma^{-1}$, where $\lambda \in \{\pm 1\}$. We put $\mathbb{L} := \mathbb{Q}(\gamma)$.

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given by $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$ with the initial values $F_0 = 0$ and $F_1 = 1$. By setting $(\alpha, \beta) := ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$, the Binet formula for the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

We note that $\beta = -\alpha^{-1}$.

Lemma 3.2.1 *The inequalities*

$$\gamma^{m-1} Y_1 / \sqrt{2} < Y_m < \sqrt{2}\gamma^{m-1} Y_1$$

hold for all $m \geq 1$. *Further, the inequalities*

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{3.2}$$

hold for all $n \geq 1$.

In the above, the inequalities involving Y_m are from Section 2.1 in [15] and the inequalities involving F_n are well–known.

3.3 The proof of Theorem 3.1.1

As in [17] and [15], we put $\mathbb{M} := \mathbb{K}\mathbb{L}$, where $\mathbb{K}$ is the number field $\mathbb{Q}(\alpha)$ and $\mathbb{L}$ is $\mathbb{Q}(\gamma)$. There are infinitely many d such that $Y_m = F_n$ has two positive integer solutions (m, n) since $F_1 = F_2 = 1$ and F_n is even whenever $3 \mid n$ (thus there are infinitely many even Fibonacci numbers). Assume $Y_m = F_n$ has 3 positive integer solutions (m_i, n_i) for $i = 1, 2, 3$. Suppose further that $m_1 < m_2 < m_3$. Then we have that $n_1 < n_2 < n_3$. We also assume $n \geq 2$ since the instance $n = 1$ produces the same Fibonacci number as the instance $n = 2$.

3.3.1 α and γ are multiplicatively dependent

In this case, $\mathbb{K} = \mathbb{L}$ and the fundamental unit in the ring of algebraic integers,$\mathcal{O}_{\mathbb{L}}$, of $\mathbb{L}$ is α. Thus, $\gamma = \left(\left(1 + \sqrt{5}\right)/2\right)^m$ for some positive integer m. We thus have that $d = 5 \cdot d_1^2$ for some positive integer $d_1 \geq 1$. We rewrite equation (3.1) as

$$(2X_{m_i})^2 - 5(2d_1 Y_{m_i})^2 = \pm 4.$$

As is widely known, this equation is solved in positive integers only by the pair (X, Y) of the form $(X, Y) = (L_t, F_t)$ where $t \geq 1$ is some integer. The sign in the right-hand side is determined as the sign of $(-1)^t$. Thus, there are positive integers t_i for $i = 1, 2, 3$ such that the relations

$$(2X_{m_i}, 2d_1 Y_{m_i}) = (L_{t_i}, F_{t_i}),$$

hold.

It is therefore the case that $2d_1F_{n_i} = F_{t_i}$ for $i = 1,2,3$ and $t_1 < t_2 < t_3$. The above equation gives in particular that $n_3 \mid t_3$ with the instance $n_3 = t_3$ being clearly impossible. So, only the instance $n_3 < t_3$ is possible. If we assume that $d_1 \ge 2$, then the index of appearance, $z(2d_1)$, of $2d_1$ in the Fibonacci sequence (that is, the smallest positive integer k such that $2d_1 \mid F_k$) is at least 5 and it is a multiple of 3, so it is at least 6. Since $2d_1 \mid F_{t_i}$, it is also the case that $z(2d_1) \mid t_i$. In particular, $t_3 = k_3 z(2d_1)$ for some positive integer $k_3 \ge 3$. We thus see that $t_3 \ge 18$. From $2d_1F_{n_i} = F_{t_i}$, we have that $F_{t_3}F_{n_2} = F_{t_2}F_{n_3}$. The factor F_{t_3} appearing in the left hand side of the last equation has, by Carmichael's Primitive Divisor Theorem (see [7] and [10]), a primitive divisor, that is a prime factor $p \mid F_{t_3}$ such that $p \nmid F_k$ for any $k < t_3$. In particular, the factor p does not appear in the right–hand side of the above equation since both inequalities $t_2 < t_3$ and $n_3 < t_3$ hold. The case $d_1 = 1$ yields the equation $2F_{n_3} = F_{t_3}$, so again F_{n_3} does not have a primitive prime factor. Since $t_3 \ge 4$ (because $t_3 > n_3$), the only possibilities are $t_3 = 6, 12$, which do not lead to solutions since none of $F_6/2 = 4$ or $F_{12}/2 = 72$ is a Fibonacci number. Thus, no solution exists in this case.

3.3.2 A useful inequality

We proceed with the proof of Theorem 3.1.1 by supposing now that γ and α are multiplicatively independent. From the equation $Y_m = F_n$, we have that

$$\frac{\gamma^m - \delta^m}{2\sqrt{d}} = \frac{\alpha^n - \beta^n}{\sqrt{5}}. \tag{3.3}$$

This yields

$$\gamma^m (2\sqrt{d})^{-1}\sqrt{5}\alpha^{-n} - 1 = \frac{\delta^m\sqrt{5}}{2\sqrt{d}\alpha^n} - \frac{\beta^n}{\alpha^n} = \frac{\lambda^m\sqrt{5}}{2\sqrt{d}\alpha^n\gamma^m} - \frac{(-1)^n}{\alpha^{2n}}. \tag{3.4}$$

Equation (3.3) gives that

$$\frac{\gamma^m}{\sqrt{d}} \ge \frac{\gamma^m + 1}{2\sqrt{d}} \ge F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \ge \alpha^{n-2}.$$

So,

$$\sqrt{d}\gamma^m \ge d\alpha^{n-2}. \tag{3.5}$$

Taking the absolute value in (3.4), we have that

$$\left|(2\sqrt{d})^{-1}\sqrt{5}\gamma^m\alpha^{-n} - 1\right| < \left(1 + \frac{\sqrt{5}\alpha^2}{2d}\right)\frac{1}{\alpha^{2n}} < \frac{2.5}{\alpha^{2n}},$$

We thus record

$$\left|(2\sqrt{d})^{-1}\sqrt{5}\gamma^m\alpha^{-n} - 1\right| \le \frac{2.5}{\alpha^{2n}}. \tag{3.6}$$

We note that since $4d/5$ is not a unit but $\gamma^{2m}\alpha^{-2n}$ is a unit, the equality $4d/5 = \gamma^{2m}\alpha^{-2n}$ does not hold. This ensures that the left–hand side of (3.6) is not zero. We next suppose that $\gamma > 10^6$ and conclude that $d \ge 94$ (see Section 3.2 in [15]). From the first inequality of (3.5), we have that

$$\sqrt{d}\gamma^m \ge \left(\frac{d}{\alpha^2}\right)\alpha^n > 35.9\alpha^n.$$

We apply absolute values in (3.4), to obtain

$$\left|(2\sqrt{d})^{-1}\sqrt{5}\gamma^m\alpha^{-n}-1\right| < \left(1+\frac{\sqrt{5}}{71.8}\right)\frac{1}{\alpha^{2n}} < \frac{1.04}{\alpha^{2n}}.$$

The rightmost quantity in the above inequality is less than $1/2$. We use the fact that if $|e^x-1|<y<1/2$ for real x, then $|x|<2y$ to obtain

$$|m\log\gamma - n\log\alpha - \log(2\sqrt{d}) + \log\sqrt{5}| \le \frac{2.08}{\alpha^{2n}} \quad \text{for} \quad n\ge 2. \tag{3.7}$$

Next, we rewrite (3.3) as

$$\gamma^m - 2\sqrt{d}(\sqrt{5})^{-1}(\alpha^n-\beta^n) - (\lambda\gamma^{-1})^m = 0.$$

We multiply through by γ^m to obtain

$$\gamma^{2m} - 2\sqrt{d}(\sqrt{5})^{-1}(\alpha^n-\beta^n)\gamma^m - \lambda^m = 0.$$

This yields

$$\gamma^m = \sqrt{d}(\sqrt{5})^{-1}(\alpha^n-\beta^n) \pm \sqrt{d(5)^{-1}(\alpha^n-\beta^n)^2+\lambda^m}.$$

We rule out

$$\gamma^m = \sqrt{d}(\sqrt{5})^{-1}(\alpha^n-\beta^n) - \sqrt{d(5)^{-1}(\alpha^n-\beta^n)^2+\lambda^m},$$

since it is less than 1. We thus have

$$\begin{aligned}\gamma^m &= \sqrt{d}(\sqrt{5})^{-1}(\alpha^n-\beta^n) + \sqrt{d(5)^{-1}(\alpha^n-\beta^n)^2+\lambda^m}\\ &= \sqrt{d}(\sqrt{5})^{-1}\alpha^n - \sqrt{d}(\sqrt{5})^{-1}\beta^n + \sqrt{d}(\sqrt{5})^{-1}\alpha^n(1+x)^{1/2},\end{aligned}$$

$$x := -2\left(\frac{\beta}{\alpha}\right)^n + \left(\frac{\beta}{\alpha}\right)^{2n} + \frac{\lambda^m}{5^{-1}d\alpha^{2n}}.$$

Taking into account the fact that $d\ge 94$, we can bound x as

$$|x| \le \frac{1}{\alpha^{2n}}\left(2+\frac{1}{\alpha^{2n}}+\frac{5}{d}\right) \le \frac{1}{\alpha^{2n}}\left(2+\frac{1}{\alpha^4}+\frac{5}{94}\right) < \frac{2.08}{\alpha^{2n}} < 0.31.$$

In the case $x<0$, we denote $f(x)=\sqrt{1+x}$ and note by the Mean Value Theorem for derivatives that

$$\sqrt{1+x} = 1+f'(\zeta)x, \quad f'(\zeta)=\frac{1}{2\sqrt{1+\zeta}}, \quad \zeta\in(x,0).$$

Similarly, the case of $x>0$ yields

$$\sqrt{1+x} = 1+f'(\zeta)x, \quad f'(\zeta)=\frac{1}{2\sqrt{1+\zeta}}, \quad \zeta\in(0,x).$$

So, in either case, we record

$$\sqrt{1+x} = 1 + f'(\zeta)x, \quad f'(\zeta) = \frac{1}{2\sqrt{1+\zeta}}, \quad \zeta \in (-|x|, |x|).$$

The fact that $|x| < 0.31$ yields $|1+\zeta| > 0.69$, so $f'(\zeta) < 0.61$. Thus,

$$\sqrt{1+x} = 1 + O_{0.61}(|x|) = 1 + O_{1.4}(\alpha^{-2n}),$$

where the notation $O_C(|x|)$ is the Landau symbol and the constant implied by it can be taken to be C. The last equality follows from the fact that

$$|x| < \frac{2.08}{\alpha^{2n}}.$$

Thus,

$$\begin{aligned}
\gamma^m &= \sqrt{d}(\sqrt{5})^{-1}\alpha^n(1-(\beta/\alpha)^n + 1 + O_{1.4}(\alpha^{-2n})) \\
&= 2\sqrt{d}(\sqrt{5})^{-1}\alpha^n(1-(1/2)(\beta/\alpha)^n + O_{0.7}(\alpha^{-2n}) \\
&= 2\sqrt{d}(\sqrt{5})^{-1}\alpha^n(1+O_{1.2}(\alpha^{-2n})).
\end{aligned}$$

Hence,

$$\gamma^{-m} = (2\sqrt{d})^{-1}\sqrt{5}\alpha^{-n}\left(1+O_{1.2}\left(\alpha^{-2n}\right)\right)^{-1} = (2\sqrt{d})^{-1}\sqrt{5}\alpha^{-n}(1+O_{1.5}(\alpha^{-2n})).$$

The last estimate is justified by putting y such that $|y| \le 1.2/\alpha^{2n}$ and note that

$$(1+y)^{-1} = 1 - y + y^2 - \cdots = 1 - \frac{y}{1+y}, \tag{3.8}$$

and

$$\left|\frac{1}{1+y}\right| \le \frac{1}{1-|y|} < \frac{1}{1-1.2/\alpha^{2n}} < \frac{1}{1-1.2/\alpha^4}.$$

Since $|y| < 1.2\alpha^{-2n}$ and

$$\left|\frac{y}{1+y}\right| = |y|\left|\frac{1}{1+y}\right| < \frac{1.2\alpha^{-2n}}{1-1.2/\alpha^4} < 1.5\cdot\alpha^{-2n},$$

the desired estimate is obtained. We now use this in equation (3.4) to get

$$\begin{aligned}
(2\sqrt{d})^{-1}\sqrt{5}\gamma^m\alpha^{-n} - 1 &= \frac{(-1)^{n+1}}{\alpha^{2n}} + \left(\frac{\lambda^m\sqrt{5}}{2\sqrt{d}\alpha^n}\right)\gamma^{-m} \\
&= \frac{(-1)^{n+1}}{\alpha^{2n}} + \left(\frac{\lambda^m\sqrt{5}}{2\sqrt{d}\alpha^n}\right)(2\sqrt{d})^{-1}\sqrt{5}\alpha^{-n}\times\left(1+O_{1.5}\left(\alpha^{-2n}\right)\right) \\
&= \frac{(-1)^{n+1}}{\alpha^{2n}} + \frac{5\lambda^m}{4d\alpha^{2n}}\left(1+O_{1.5}\left(\alpha^{-2n}\right)\right) \\
&= \left((-1)^{n+1} + \frac{5\lambda^m}{4d}\right)\frac{1}{\alpha^{2n}} + O_{7.5/4d}(\alpha^{-4n}).
\end{aligned}$$

We put

$$z := m\log\gamma - n\log\alpha - \log(2\sqrt{d} + \log(\sqrt{5})).$$

By (3.7), we have that $|z| < 2.08/\alpha^{2n} < 1$. We see that

$$|e^z - 1 - z| < (e-2)z^2 < 0.719z^2.$$

This shows that

$$e^z - 1 = z + O_{0.719}(z^2) = z + O_{0.719\cdot 2.2^2}(\alpha^{-4n}) = z + O_{3.5}(\alpha^{-4n}).$$

Hence, we get

$$z + O_{3.5}(\alpha^{-4n}) = \left((-1)^{n+1} + \frac{5\lambda^m}{4d}\right)\alpha^{-2n} + O_{7.5/4d}(\alpha^{-4n}),$$

which implies that

$$\left| m\log\gamma - n\log\alpha - \log(2\sqrt{d}) + \log(\sqrt{5}) - \left((-1)^{n+1} + \frac{5\lambda^m}{4d}\right)\alpha^{-2n} \right| < \frac{3.6}{\alpha^{4n}}. \tag{3.9}$$

The left–hand side of the above inequality is non-zero since if it were zero, then the exponential of the non-zero algebraic number $\left((-1)^{n+1} + 5\lambda^m/4d\right)\alpha^{-2n}$ is an algebraic number. This contradicts Baker's reformulation of the Lindermann-Weierstrass theorem. We summarise the main results of this section in the following result which is the analog of Lemma 5 in [15]. Recall that $\lambda = X_1^2 - dY_1^2 \in \{\pm 1\}$.

Lemma 3.3.1 *Assume that $Y_m = F_n$ for $m \geq 1$ and $n \geq 2$. The following holds.*

(i)

$$0 < \left|(2\sqrt{d})^{-1}\sqrt{5}\gamma^m\alpha^{-n} - 1\right| \leq \frac{2.5}{\alpha^{2n}}. \tag{3.10}$$

(ii) *If $\gamma > 10^6$, then*

$$0 < |m\log\gamma - n\log\alpha - \log(2\sqrt{d}) + \log\sqrt{5}| \leq \frac{2.08}{\alpha^{2n}} \quad \text{for} \quad n \geq 2. \tag{3.11}$$

(iii) *If $\gamma > 10^6$, then*

$$0 < \left| m\log\gamma - n\log\alpha - \log(2\sqrt{d}) + \log(\sqrt{5}) - \left((-1)^{n+1} + \frac{5\lambda^m}{4d}\right)\alpha^{-2n} \right| < \frac{3.6}{\alpha^{4n}}. \tag{3.12}$$

3.3.3 Bounds on n and m in terms of γ

From Lemma 3.2.1, we have

$$(m-1)\log\gamma - \log\sqrt{2} \leq \log F_n \leq (n-1)\log\alpha. \tag{3.13}$$

We apply Theorem 1.4.3 on the left-hand side of (3.10) with the following choice of parameters; $t := 3$ and

$$\gamma_1 := 2\sqrt{d/5}, \quad \gamma_2 := \alpha, \quad \gamma_3 := \gamma.$$

We take $b_1 := -1$, $b_2 := -n$ and $b_3 := m$. We assume $n \geq 3$. Here, $\mathbb{M} := \mathbb{Q}(\sqrt{d}, \alpha)$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := [\mathbb{M} : \mathbb{Q}] = 4$. We have

$$h(\gamma_1) = h(4d/5)/2 \leq \log(4d)/2 < 1.5 \log \gamma.$$

The last inequality holds as $\gamma = X_1 + \sqrt{d}Y_1 \geq \sqrt{d-1} + \sqrt{d}$ and $(\sqrt{d-1} + \sqrt{d})^3 > 4d$ holds for all $d \geq 2$. Thus, $h(\gamma_1) < 1.5 \log \gamma$. Next, we have $h(\gamma_2) = (\log \alpha)/2$ and $h(\gamma_3) = (\log \gamma)/2$. Therefore, we can take

$$A_1 := 6 \log \gamma, \ A_2 := 2 \log \alpha, \ A_3 := 2 \log \gamma.$$

We note that

$$Y_m \geq \frac{\gamma^m - \delta^m}{2\sqrt{d}Y_1} = \frac{\gamma^m - \delta^m}{\gamma - \delta} > \gamma^m - 1 \geq \gamma^{m-2}.$$

So, we have that $\gamma^{m-2} < Y_m < \alpha^{n-1}$. This implies $m \leq n$ since $\gamma > \alpha$. We thus choose $B := n$. We set

$$\Lambda := (2\sqrt{d})^{-1}\sqrt{5}\gamma^m \alpha^{-n} - 1.$$

With the above observations, the calculations preceding Lemma 6 in [15] apply to our situation and yield the following bounds.

Lemma 3.3.2 *If the pair (m, n) with $m \geq 1$ and $n \geq 2$ is an integer solution to $Y_m = F_n$ then*

- (i) $(m-1)\log \gamma < (n+1)\log \alpha + \log \sqrt{2}$;
- (ii) $n < 8 \cdot 10^{15} \max\{1, (\log \gamma)^3\}$;
- (iii) $m < 8 \cdot 10^{15} \max\{1, (\log \gamma)^2\}$.

3.3.4 The case $\gamma < 10^2$

In this case, the relation $\gamma = X_1 + \sqrt{X_1^2 - \lambda}$ gives that $X_1 \leq 50$. We also use Lemma 3.3.2 to obtain the bounds $m < 3.5 \cdot 10^{19}$ and $n < 1.61 \cdot 10^{20}$. We refer to the right-hand side of (i) in Lemma 3.3.1 and note that $4.7/\alpha^{2n} < 1/2$ if $n \geq 3$. Thus, we apply (i) in Lemma 3.3.1 in the case of $n = n_3$ and $m = m_3$ and pass to logarithmic form to obtain the inequality

$$\left| m_3 \log \gamma - n_3 \log \alpha - \log \frac{2\sqrt{d}}{\sqrt{5}} \right| < \frac{5}{\alpha^{2n_3}}.$$

We apply Lemma 1.4.6 to sharpen the bound on n_3. To that end, we rewrite the above inequality as

$$\left| m_3 \tau - n_3 + \mu \right| < \frac{A}{B^{n_3}}, \tag{3.14}$$

with the following data:

$$\tau := \frac{\log\gamma}{\log\alpha}, \quad \mu := -\frac{\log(2\sqrt{d/5})}{\log\alpha}, \quad A := 10.4 > \frac{5}{\log\alpha}, \quad B := \alpha^2. \tag{3.15}$$

We also set $M := 10^{20} > m_3$. For each value of τ resulting from the pair (X_1, λ) (discarding the values for which τ is rational), we see that the denominator of the 149th convergent of τ, q_{149}, in all cases satisfy $q_{149} \geq F_{149} > 6 \cdot 10^{30}$. We look for a uniform lower bound on $\varepsilon := ||\mu q|| - M||\tau q||$ for all pairs (X_1, λ). The inequality

$$\left|\tau q_{149} - p_{149}\right| < \frac{1}{q_{149}}$$

holds viewing p_{149}/q_{149} as a convergent of τ. Therefore,

$$M\|\tau q\| \leq \frac{M}{q} \leq \frac{10^{20}}{6\times 10^{30}} < 0.5 \cdot 10^{-10}.$$

Across all pairs (X_1, λ), we found that $\|q\mu\| > 1.66 \cdot 10^{-9}$. We thus used $\varepsilon > 1.61 \cdot 10^{-9}$ and obtained the bound $n_3 < 115$.

By means of the relation $\gamma^{m-1}/\sqrt{2} < F_n$, which follows from Lemma 3.2.1, we obtained a good bound on m. To search for solutions to $Y_m = F_n$, we went back for all pairs (X_1, λ) and generated all values of the candidates d such that $X_1^2 - \lambda = dY_1^2$ for some integer Y_1. For each such (d, Y_1), we set $Y_2 = 2X_1Y_1$ and generated the sequence $\{Y_m\}_{m\geq 1}$ where $Y_{m+2} = 2X_1Y_{m+1} - \lambda Y_m$ with m satisfying $\gamma^{m-1}/\sqrt{2} < F_{115}$. We sought to find those values of d for which the set

$$\{F_n : 1 \leq n \leq 115\} \cap \left\{ Y_m : 1 \leq m \leq \frac{\log(\sqrt{2}\gamma F_{115})}{\log\gamma} \right\}$$

has cardinality 3. We found only the value $d = 2$ for which $Y_1 = F_2, Y_2 = F_3, Y_3 = F_5$.

3.3.5 The case $10^2 \leq \gamma < 10^6$

The bound on γ yields $X_1 \leq 5 \cdot 10^5$. We also apply Lemma 3.3.2 to obtain $m < 3.2 \cdot 10^{20}$. We set M of Lemma 1.4.6 to be $3.3 \cdot 10^{20}$. Since $q_{149} \geq F_{149} > 6 \cdot 10^{30} > 6M$, we again work with the 149th convergent of τ for each choice of the pair (X_1, λ). For all choices, we found again that $\|q\mu\| > 1.66 \cdot 10^{-9}$. The inequality

$$M\|\tau q\| \leq \frac{M}{q} \leq \frac{3.3 \cdot 10^{20}}{6\times 10^{30}} < 1.7 \cdot 10^{-10}.$$

also holds. The value of ε was thus uniformly chosen as $\varepsilon > 1.61 \cdot 10^{-9}$ which yields $n_3 < 115$.
A search was conducted for solutions in this range over all pairs (X_1, λ) to no fruition. From now on, we assume $\gamma > 10^6$.

3.3.6 Inequalities among solutions

Suppose (m,n) and m',n' are solutions to $Y_m = F_n$. We can thus write

$$\frac{Y_m}{Y_{m'}} = \frac{F_n}{F_{n'}}.$$

This gives us

$$\gamma^{m-m'}\left(1-\frac{\lambda^m}{\gamma^{2m}}\right)\left(1-\frac{\lambda^{m'}}{\gamma^{2m'}}\right)^{-1} = \alpha^{n-n'}\zeta \quad \text{where} \quad \zeta \in [\alpha^{-1},\alpha].$$

The right-hand side follows from the fact that $\alpha^{n-2} \le F_n \le \alpha^{n-1}$. We note that

$$\left|\log\left(1-\frac{\lambda^{m_i}}{\gamma^{2m_i}}\right)\right| \le \frac{1}{\gamma^2} \le \frac{1}{10^{12}}$$

and the inequality

$$\left|(m-m')\log\gamma-(n-n')\log\alpha\right| < \log\alpha+\frac{1}{\gamma^2}+\frac{1}{\gamma^2} < \log\alpha+\frac{2}{10^{12}} < 1$$

therefore holds. We record this as follows. This is an analogue of Lemma 7 in [15].

Lemma 3.3.3 *If* $10^6 < \gamma$ *and* $(m,n),(m',n')$ *satisfy the equations* $Y_m = F_n$ *and* $Y_{m'} = F_{n'}$*, then*

$$|(m-m')\log\gamma-(n-n')\log\alpha| < 1.$$

3.3.7 Bounding n_1

This section is concerned with the proof of the following lemma.

Lemma 3.3.4 *Assume* $\gamma > 10^6$ *and* $(n,m) \in \{(n_1,m_1),(n_2,m_2),(n_3,m_3)\}$ *are all solutions of the equation* $Y_m = L_n$ *with* $1 \le m_1 < m_2 < m_3$*. Then the inequality*

$$n_1 < 41.7+2.08\log\log\gamma$$

holds.

proof We follow the proof of Lemma 8 in [15]. Consider the matrix

$$A := \begin{pmatrix} n_1 & m_1 & 1 \\ n_2 & m_2 & 1 \\ n_3 & m_3 & 1 \end{pmatrix}.$$

Assume first that $\det A \neq 0$. Writing Lemma 3.3.1(ii) for $(m,n) := (m_\ell, n_\ell)$, for $\ell = 1,2,3$, subtracting the one for $\ell = 1$ from the ones for $\ell \in \{2,3\}$ and using the absolute value inequality, we get

$$|(m_\ell - m_1)\log\gamma - (n_\ell - n_1)\log\alpha| < \frac{4.16}{\alpha^{2n_1}} \qquad \text{for} \quad \ell \in \{2,3\}. \tag{3.16}$$

Next, we consider an application of the absolute value inequality to a suitable combination of the above inequality to obtain;

$$|(m_3-m_1)I_2-(m_2-m_1)I_3| < \frac{8.32(m_3-m_1)}{\alpha^{2n_1}}.$$

In the above inequality, I_ℓ for $\ell = 2,3$ represents the expression in the absolute value sign of (3.16) for $\ell = 2$ and 3 respectively. This yields

$$\Delta \log\alpha \leq \frac{8.32(m_3-m_1)}{\alpha^{2n_1}},$$

where

$$\Delta := |(m_3-m_1)(n_2-n_1)-(m_2-m_1)(n_3-n_1)| = |\det A| \geq 1.$$

So, by Lemma 3.3.2(iii), we get

$$\alpha^{2n_1} < \left(\frac{8.32}{\log\alpha}\right)(m_3-m_1) < 1.39\cdot 10^{17}(\log\gamma)^2.$$

We thus get

$$n_1 < \left(\frac{\log(1.39\cdot 10^{17})}{2\log\alpha}\right) + \left(\frac{1}{\log\alpha}\right)\log\log\gamma < 41.02 + 2.08\log\log\gamma,$$

which proves the inequality in the case det $A \neq 0$. Now suppose that det $A = 0$. Then A has rank less than 3. Let l_1, l_2, l_3 be the rows of the above matrix. Note that A has rank 2 since otherwise l_i and l_j should be multiples of each other, which is not the case since their third component is equal to 1 but their first components are different. Let u, v be rational numbers such that $l_1 = ul_2 + vl_3$. The numbers u, v solve the system

$$\begin{cases} u + v = 1, \\ un_2 + vn_3 = n_1, \end{cases}$$

whose solution is $(u, v) = ((n_3-n_1)/(n_3-n_2), (n_1-n_2)/(n_3-n_2))$. Since u, v also solve the system

$$\begin{cases} u + v = 1; \\ um_2 + vm_3 = m_1, \end{cases}$$

we also have $(u, v) = ((m_3-m_1)/(m_3-m_2), (m_1-m_2)/(m_3-m_2))$. We further have that $uv \neq 0$. We next go to Lemma 3.3.1(iii) which is

$$0 < \left| m\log\gamma - n\log\alpha - \log(2\sqrt{d}) + \log(\sqrt{5}) - \left((-1)^{n+1} + \frac{5\lambda^m}{4d}\right)\alpha^{-2n} \right| < \frac{3.6}{\alpha^{4n}}. \qquad (3.17)$$

We multiply the above estimate for $i = 2$ with u, for $i = 3$ with v, and subtract the one for $i = 1$, we get

$$\left| u\left((-1)^{n_2+1} - \frac{5\lambda^{m_2}}{4d}\right)\frac{1}{\alpha^{2n_2}} + v\left((-1)^{n_3+1} - \frac{5\lambda^{m_3}}{4d}\right)\frac{1}{\alpha^{2n_3}} - \left((-1)^{n_1+1} + \frac{5\lambda^{m_1}}{4d}\right)\frac{1}{\alpha^{2n_1}} \right|$$

$$< \frac{3.6(1+|u|+|v|)}{\alpha^{4n_1}} \le \frac{3.6(1+m_3+m_2)}{\alpha^{4n_1}} < \frac{8m_3}{\alpha^{4n_1}}.$$

Multiplying across by α^{2n_1}, we get

$$\left| u\left((-1)^{n_2+1} - \frac{5\lambda^{m_2}}{4d}\right)\frac{1}{\alpha^{2(n_2-n_1)}} + v\left((-1)^{n_3+1} - \frac{5\lambda^{m_3}}{4d}\right)\frac{1}{\alpha^{2(n_3-n_1)}} - \left((-1)^{n_1+1} + \frac{5\lambda^{m_1}}{4d}\right)\right|$$

$$< \frac{8n_3}{\alpha^{2n_1}}. \tag{3.18}$$

In the left-hand side of (3.18) we have 3 terms. Assume first that the first is $\ge 1/6$ in absolute value. We then have that

$$\alpha^{2(n_2-n_1)} < 6u\left(1+\frac{5}{4d}\right) < 7u.$$

Hence, by Lemma 3.3.3, we get

$$\gamma^{2(m_2-m_1)} < 7e^2 u.$$

If $u \le 2$, then the right-hand side is less than $104 < \gamma$, a contradiction. Hence, $u \ge 2$. Thus, $m_3 - m_1 \ge 2(m_3 - m_2)$, so $m_2 - m_1 \ge m_3 - m_2$. In particular, $m_2 - m_1 \ge (m_3 - m_1)/2$. Then the above inequality becomes

$$\gamma^{m_3-m_1} \le \gamma^{2(m_2-m_1)} < 7e^2 u < 52(m_3 - m_1) < \gamma(m_3 - m_1),$$

so $\gamma^{k-1} < k$ with $k := m_3 - m_1 \ge 2$, a contradiction. If the second term in the left-hand side of (3.18) is $\ge 1/6$ in absolute value, we then get by a similar argument that

$$\alpha^{2(n_3-n_1)} \le 6|v|\left(1+\frac{5}{4d}\right) < 7(m_3-m_1) \quad \text{since} \quad |v| < m_3 - m_1,$$

so

$$\gamma^{m_3-m_1} < \gamma^{2(m_3-m_1)} < e^2\alpha^{2(n_3-n_1)} < 7e^2(m_3-m_1) < \gamma(m_3-m_1),$$

and we get again the same contradiction as in the previous case. If both the first and second are less than $1/6$ in absolute value, then the left-hand side of (3.18) exceeds

$$1 - \frac{5}{4d} - \frac{1}{6} - \frac{1}{6} > \frac{1}{4},$$

so we get that

$$\alpha^{2n_1} < 32n_3. \tag{3.19}$$

Instance (3.19) together with Lemma 3.3.2(ii) gives

$$n_1 < \left(\frac{\log(32\cdot 8\cdot 10^{15})}{2\log\alpha}\right) + \left(\frac{1}{\log\alpha}\right)\log\log\gamma < 41.7 + 2.08\log\log\gamma,$$

which is what we wanted.

3.3.8 The case $m_1 > 1$

In this case, by Lemma 3.3.2(i), we have

$$\begin{aligned} \log\gamma - \log\sqrt{2} &\leq (m_1-1)\log\gamma - \log\sqrt{2} \leq (n_1+1)\log\alpha \\ &\leq (42.7 + 2.08\log\log\gamma)\log\alpha, \end{aligned}$$

which gives $\log\gamma < 31$, so $\gamma < 2.91 \cdot 10^{13}$. This gives $n_1 < 62$. Hence,

$$10^{6(m_1-1)} < \gamma^{m_1-1} \leq \sqrt{2}F_{n_1} \leq \sqrt{2}F_{61} < 3.6 \times 10^{12},$$

so $m_1 = 2$ or 3. In the case $m_1 = 2$, we have $Y_2 = 2X_1Y_1 = F_{n_1}$, which shows that F_{n_1} is even so $3 \mid n_1$. Now $Y_2 = 2X_1Y_1 = F_{n_1}$. Hence, $X_1 \mid F_{n_1}$. For each $n_1 \leq 61$ which is a multiple of 3 we took X_1 to be a divisor of $F_{n_1}/2$ with $5 \cdot 10^5 < X_1 < 1.5 \cdot 10^{13}$. For each one of these choices we generated $\gamma := X_1 + \sqrt{X_1^2 - \lambda}$ for $\lambda \in \{\pm 1\}$ and applied the method from Section 3.3.4. We obtained $\varepsilon > 1.10 \cdot 10^{-6}$ and $n < 252$. To search for solutions to $Y_m = F_n$, we again went back for all pairs (X_1, λ) satisfying the above condition and generated all values of candidate d such that $X_1^2 - \lambda = dY_1^2$ for some integer Y_1. For each such (d, Y_1), we set $Y_2 = 2X_1Y_1$ and generated the sequence $\{Y_m\}_{m\geq 1}$ where $Y_{m+2} = 2X_1Y_{m+1} - \lambda Y_m$ with m satisfying $\gamma^{m-1}/\sqrt{2} < F_{252}$. We sought to find those values of d for which the set

$$\{F_n : 1 \leq n \leq 252\} \cap \left\{ Y_m : 1 \leq m \leq \frac{\log(\sqrt{2}\gamma F_{252})}{\log\gamma} \right\}$$

has cardinality 3. This took a few minutes and produced no solution. There is therefore no solution in this case. In the case $m_1 = 3$, we write $Y_3 = (4X_1^2 - \lambda)Y_1 = F_{n_1}$. That is, $(2X_1)^2 = F_{n_1}/Y_1 + \lambda$. For each $n_1 \in [1, 61]$ and choice of $\lambda \in \{\pm 1\}$, we asked Mathematica to list all divisors Y_1 of F_{n_1} for which $F_{n_1}/Y_1 + \lambda$ is a square greater than 10^{12}. No such value of Y_1 found. So again, there is no solution in this case. From now on, we assume that $m_1 = 1$.

3.3.9 The case $\gamma \leq 10^{10}$

Then $X_1 \leq 5 \times 10^9$. We write inequalities (3.11) for $(m, n) = (m_i, n_i)$ and $i = 2, 3$, and combine them to get

$$\left|(m_3 - m_2)\log\gamma - (n_3 - n_2)\log\alpha\right| < \frac{4.16}{\alpha^{2n_2}}.$$

We divide across by $(m_3 - m_2)\log\alpha$ to get

$$\left|\frac{\log\gamma}{\log\alpha} - \frac{n_3 - n_2}{m_3 - m_2}\right| < \frac{4.16}{(m_3 - m_2)(\log\alpha)\alpha^{2n_2}} < \frac{8.7}{(m_3 - m_2)\alpha^{2n_2}}.$$

There are 6 values of γ in the range $\gamma < 10^{10}$ for which $\log\gamma/\log\alpha$ is rational (in fact, an integer) and they appear in Section 3.8 in [15]. We ignore such values since for us we know that γ and α

are multiplicatively independent. Since $\gamma < 10^{10}$, Lemma 3.3.2 shows that

$$m_3 - m_2 < m_3 < 8.7 \cdot 8 \cdot 10^{15} (\log \gamma)^2 < 3.7 \cdot 10^{19} < F_{99}.$$

For all $X_1 \in [5 \cdot 10^5, 5 \cdot 10^9]$, we generated the first 100 partial quotients $[a_0, a_1, \ldots, a_{99}]$ of each $(\log \gamma)/\log \alpha$. The maximum A of the a_k all $k \in [0, 99]$ satisfies $A < 10^{20}$. This follows from a calculation explained in Section 3.8 in [15].

Hence, by Lemma 1.4.7(ii), the left-hand side above exceeds

$$\frac{1}{(A+2)Q^2},$$

where $Q = q_k$ is the maximum denominator of some convergent to $\log \gamma / \log \alpha$ which is smaller than $m_3 - m_2 < 8 \cdot 10^{15} (\log \gamma)^2 < 4.3 \cdot 10^{18}$ (see Lemma 3.3.2 (iii)) since $\gamma \le 10^{10}$. Thus, we get that

$$\alpha^{2n_2} \le \frac{8.7(A+2)Q^2}{m_3 - m_2} \le 8.7(10^{20} + 2)(4.3 \cdot 10^{18}) < 4 \times 10^{39}. \tag{3.20}$$

This gives, $n_2 \le 50$. Further, from Lemma 3.3.2(i), we see that

$$m_2 - 1 \le \frac{51 \log \alpha + \log \sqrt{2}}{\log 10^6} < 1.9.$$

Thus, $m_2 = 2$. In the case $m_2 = 2$, we have $Y_2 = 2X_1 Y_1 = F_{n_2}$, which shows that F_{n_2} is even so $3 \mid n_2$. Now $Y_2 = 2X_1 Y_1 = F_{n_2}$. Hence, $X_1 \mid F_{n_2}$. For each $n_2 \le 50$ which is a multiple of 3 we took X_1 to be a divisor of $F_{n_2}/2$ with $5 \cdot 10^5 < X_1$. For each one of these choices we generated $\gamma := X_1 + \sqrt{X_1^2 - \lambda}$ for $\lambda \in \{\pm 1\}$ and applied the method from Section 3.3.4. We obtained $n < 230$. To search for solutions to $Y_m = F_n$, we again went back for all pairs (X_1, λ) satisfying the above condition and generated all values of candidate d such that $X_1^2 - \lambda = dY_1^2$ for some integer Y_1. For each such (d, Y_1), we set $Y_2 = 2X_1 Y_1$ and generated the sequence $\{Y_m\}_{m \ge 1}$ where $Y_{m+2} = 2X_1 Y_{m+1} - \lambda Y_m$ with m satisfying $\gamma^{m-1}/\sqrt{2} < F_{230}$. We sought to find those values of d for which the set

$$\{F_n : 1 \le n \le 230\} \cap \left\{ Y_m : 1 \le m \le \frac{\log \sqrt{2} \gamma F_{230}}{\log \gamma} \right\}$$

has cardinality 3. This took a few minutes and produced no solution. There is therefore no solution in this case.

3.3.10 Better bounds on m_3

We refer to the left-hand side of (3.11) which we rewrite as

$$0 < \left| m \log \gamma - n \log \alpha - \log \frac{2\sqrt{d}}{\sqrt{5}} \right| \le \frac{2.08}{\alpha^{2n}} \quad \text{for} \quad n \ge 2$$

for the purpose of seeking a better bound on m_3. We note that $4d > 5$. We apply Mignotte's Theorem 1.4.5 to bound the absolute value above. We have

$$(\gamma_2, \gamma_1, \gamma_3, b_2, b_1, b_3) := \left(\gamma, \frac{2\sqrt{d}}{\sqrt{5}}, \alpha, m, 1, n\right).$$

Here, we consider the equation $Y_m = F_n$ with $m > 1$. Since γ is large, we have $m < n$. We have $D = [\mathbb{Q}(\sqrt{5}, \sqrt{d}) : \mathbb{Q}] = 4$.

We can take

$$a_2 := 8.296 \log\gamma, \quad a_1 := 17.297 \log\gamma \ \ (\geq 17.297 \log\sqrt{d}), \quad a_3 := 4.$$

Further,

$$b' := \left(\frac{1}{8.296\log\gamma} + \frac{m}{17.297\log\gamma}\right) \cdot \left(\frac{n}{8.926\log\gamma} + \frac{m}{4}\right).$$

Since

$$\begin{aligned}(n-2)\log\alpha < \log F_n &= \log Y_m < \log((2\gamma^m)/(2\sqrt{d})) \\ &= m\log\gamma - \log\sqrt{d},\end{aligned}$$

and $\sqrt{d} > \alpha^2$ (in fact $d \geq 94$), we get that $n/\log\gamma < m/\log\alpha$. Thus,

$$\frac{n}{8.296\log\gamma} + \frac{m}{4} < m\left(\frac{1}{4} + \frac{1}{8.296\log\alpha}\right) < 0.51m.$$

Next, assuming $m > 1000$, we have

$$\frac{1}{8.296\log\gamma} + \frac{m}{17.297\log\gamma} < \frac{0.06m}{\log\gamma}.$$

Thus,

$$b' \leq (0.51m)\left(\frac{0.06m}{\log\gamma}\right) < \frac{0.031m^2}{\log\gamma}.$$

Since $e^{0.882} \cdot 0.031 < 0.075$, we can take

$$\log\mathcal{B} \geq \max\{2.5, \log(0.075m^2/\log\gamma)\}.$$

In case (i) of Theorem 1.4.5, we get

$$\log|\Lambda| > -790.95 \cdot 4^2 \cdot (a_1 a_2 a_3)(\log\mathcal{B})^2.$$

Thus,

$$2n\log\alpha - \log(2.08) < 7.27 \cdot 10^6 (\log\gamma)^2 (\log\mathcal{B})^2,$$

so

$$(m-1)\log\gamma < (n+1)\log\alpha + \log\sqrt{2} < 3.64 \cdot 10^6 (\log\gamma)^2 (\log\mathcal{B})^2,$$

giving

$$m < 3.7 \cdot 10^6 (\log\gamma)(\log\mathscr{B})^2. \tag{3.21}$$

In case the maximum involved in $\log\mathscr{B}$ is 2.5, we get $0.075m^2/\log\gamma \le e^{2.5}$, so

$$m < \left(\frac{e^{2.5}}{0.075}\right)^{1/2} \sqrt{\log\gamma} < 13\sqrt{\log\gamma}. \tag{3.22}$$

In the other case $\log(0.075m^2/\log\gamma) < 2\log(0.28m/\sqrt{\log\gamma}) := \log\mathscr{B}$, we get that

$$\begin{aligned} m &< 2.7\cdot 10^6 (\log\gamma)\left(2\log\left(0.28m/\sqrt{\log\gamma}\right)\right)^2 \\ &< 1.1\cdot 10^7 (\log\gamma)\left(\log(0.28m/\sqrt{\log\gamma})\right)^2. \end{aligned}$$

We get with $y := 0.28m/\sqrt{\log\gamma}$ that

$$y < (0.28\cdot 1.1)\cdot 10^7 \sqrt{\log\gamma}\left(\log y\right)^2 < 4\cdot 10^6 \sqrt{\log\gamma}\left(\log y\right)^2.$$

Hence,

$$\frac{y}{(\log y)^2} < 4\cdot 10^6 \sqrt{\log\gamma}.$$

Letting T be the right-hand side above, we have that $T > (4\cdot 2^2)^2 = 256$, so we can apply Lemma 1.4.8 with $s = 2$ to get

$$\begin{aligned} y &< 4\cdot 4\cdot 10^6 \sqrt{\log\gamma}\left(\log(4\cdot 10^6) + 0.5\log\log\gamma\right) \\ &< 1.6\cdot 10^7 \sqrt{\log\gamma}(0.5\log\log\gamma)^2 \left(\frac{\log(4\cdot 10^6)}{0.5\log\log\gamma} + 1\right)^2 \\ &< 1.6\cdot 10^7\cdot 0.25\cdot 122\sqrt{\log\gamma}(\log\log\gamma)^2 \\ &< 4.9\cdot 10^8 \sqrt{\log\gamma}(\log\log\gamma)^2. \end{aligned}$$

Hence,

$$m < \left(\frac{4.9}{0.28}\right)\cdot 10^8 \log\gamma(\log\log\gamma)^2 < 1.8\cdot 10^9 \log\gamma(\log\log\gamma)^2. \tag{3.23}$$

Between (3.21), (3.22) and (3.23), the inequality (3.23) always holds.

We now study Case (ii) of Theorem 1.4.5. We may assume that r_0 and s_0 are coprime if not we simply cancel their greatest common divisor. Since $b_1 = 1$, we get that $|r_0| = 1$, $|s_0| = b_2 = m$. Hence,

$$m = b_2 < 5.61\cdot 17.297\log\gamma(4\log\mathscr{B})^{1/3} < 110(\log\gamma)(\log\mathscr{B})^{1/3}.$$

If $\mathscr{B} \le e$, then $m < 110\log\gamma$, so inequality (3.23) holds, and if $\mathscr{B} > e$, then

$$m < 110\log\gamma(\log\mathscr{B})^{1/3} < 110(\log\gamma)(\log\mathscr{B})^2,$$

so estimate (3.21) holds and in particular estimate (3.23) holds.

We now study Case (iii) of Theorem 1.4.5. In this case, we have a relation

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2.$$

In particular, r_1 divides $s_1 t_1 b_1$. Since $b_1 = 1$ and $\gcd(r_1, t_1) = 1$, we get $r_1 \mid s_1$. Thus, $\delta = |r_1|$. Putting $s_1 =: r_1 s_1'$, we get

$$(t_1 s_1') + (r_1 s_1') n = t_2 m, \tag{3.24}$$

where

$$\begin{aligned} |r_1 s_1'| &< 5.61 \cdot 4 \cdot 4^{1/3} (\log \mathscr{B})^{1/3} < 36 (\log \mathscr{B})^{1/3}; \\ |t_1 s_1'| &< 5.61 \cdot 17.297 (\log \gamma) \cdot 4^{1/3} (\log \mathscr{B})^{1/3} < 155 (\log \gamma)(\log \mathscr{B})^{1/3}; \\ |t_2| &< 5.61 \cdot 8.296 (\log \gamma) \cdot 4^{1/3} (\log \mathscr{B})^{1/3} < 74 (\log \gamma)(\log \mathscr{B})^{1/3}. \end{aligned}$$

If $t_2 = 0$, we take $s_1 = 1$. In particular, $m < n \le |t_1| < 155(\log\gamma)(\log\mathscr{B})^{1/3}$. Again estimate (3.21) holds and in particular estimate (3.23) holds. Assume next that we are in the case $t_2 \ne 0$. We multiply both sides of inequality (3.11) by $|r_1 s_1'|$ and get

$$\left| (r_1 s_1') m \log \gamma - (r_1 s_1') \log\left(\frac{2\sqrt{d}}{\sqrt{5}}\right) - (r_1 s_1') n \log \alpha \right| < \frac{2.08 \times 36 (\log \mathscr{B})^{1/3}}{\alpha^{2n}}. \tag{3.25}$$

Replacing $(r_1 s_1') n$ by $t_2 m - (t_1 s_1')$, the left-hand side becomes

$$\begin{aligned} & \left| (r_1 s_1') m \log \gamma - (r_1 s_1') \log\left(\frac{2\sqrt{d}}{\sqrt{5}}\right) - (t_2 m - t_1 s_1') \log \alpha \right| \\ = & \left| m \log(\gamma^{r_1 s_1'} / \alpha^{t_2}) - \log\left(\left(\frac{2\sqrt{d}}{\sqrt{5}}\right)^{r_1 s_1'} / \alpha^{t_1 s_1'} \right) \right|. \end{aligned}$$

This has become a linear form in two logarithms to which we can apply Theorem 1.4.4. Here,

$$\lambda_1 := \gamma^{r_1 s_1'} / \alpha^{t_2}, \quad \lambda_2 := \left(\frac{2\sqrt{d}}{\sqrt{5}}\right)^{r_1 s_1'} / \alpha^{t_1 s_1'}.$$

They are multiplicatively independent (since γ and α are so), and they satisfy

$$\begin{aligned} h(\lambda_1) &\le h(\gamma^{r_1 s_1'}) + h(\alpha^{t_2}) \\ &\le \frac{1}{2} \left(|r_1 s_1'| \log \gamma + |t_2| \log \alpha \right) \\ &< \frac{1}{2} \left(36 + 74 \log \alpha \right) (\log \gamma)(\log \mathscr{B})^{1/3} \\ &< 36 (\log \gamma)(\log \mathscr{B})^{1/3}, \end{aligned}$$

and

$$
\begin{aligned}
h(\lambda_2) &\le h((2\sqrt{d})^{r_1 s_1'}) + h(\alpha^{t_1 s_1'}) + h((\sqrt{5})^{r_1 s_1'}) \\
&\le |r_1 s_1'|\log(2\sqrt{d}) + \frac{1}{2}|t_1 s_1'|\log\alpha + |r_1 s_1'|\log(\sqrt{5}) \\
&< 38(\log\gamma)(\log\mathscr{B})^{1/3} + \frac{1}{2}(\log\alpha)\cdot 155(\log\gamma)(\log\mathscr{B})^{1/3} \\
&< 76(\log\gamma)(\log\mathscr{B})^{1/3}.
\end{aligned}
$$

Thus, we can take

$$
\log B_1 := 36(\log\gamma)(\log\mathscr{B})^{1/3}, \quad \log B_2 := 76(\log\gamma)(\log\mathscr{B})^{1/3}
$$

and

$$
b' = \frac{m}{4\cdot 76(\log\gamma)(\log\mathscr{B})^{1/3}} + \frac{1}{4\cdot 36(\log\gamma)(\log\mathscr{B})^{1/3}}.
$$

Since $m > 1000$, the above is bounded by

$$
b' < \frac{0.004m}{(\log\gamma)(\log\mathscr{B})^{1/3}}.
$$

Thus,

$$
\log b' + 0.14 < \log\left(\frac{e^{0.14}\cdot 0.004m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right) < \log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right). \tag{3.26}
$$

Assume first that the right-hand side above is smaller than $21/D = 21/4$. Then

$$
m < \left(\frac{e^{21/4}}{0.005}\right)(\log\gamma)(\log\mathscr{B})^{1/3} < 40000(\log\gamma)(\log\mathscr{B})^{1/3}.
$$

In particular, estimate (3.21) holds and in particular estimate (3.23) holds. So, we assume that the right-hand side of (3.26) exceeds $21/D$. We then get by setting

$$
\Lambda := m\log(\gamma^{r_1 s_1'}/\alpha^{t_2}) - \log\left(\left(\frac{2\sqrt{d}}{\sqrt{5}}\right)^{r_1 s_1'}/\alpha^{t_1 s_1'}\right)
$$

that

$$
\begin{aligned}
\log|\Lambda| &> -24.34\cdot 4^4\cdot 36\cdot 76(\log\gamma)^2(\log\mathscr{B})^{2/3} \\
&\times \left(\log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right)\right)^2 \\
&> -1.71\cdot 10^7(\log\gamma)^2(\log\mathscr{B})^{2/3}\left(\log\left(\frac{0.005m}{(\log\gamma))(\log\mathscr{B})^{1/3}}\right)\right)^2.
\end{aligned}
$$

Comparing this with the upper bound in (3.25), we get

$$\begin{aligned} 2n\log\alpha &- \log(2.08\cdot 36) - (1/3)\log\log\mathscr{B} \\ &< 1.71\cdot 10^7(\log\gamma)^2(\log\mathscr{B})^{2/3}\left(\log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right)\right)^2, \end{aligned}$$

We recall that

$$\log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right) > \frac{21}{4}$$

and so

$$(n+1)\log\alpha < 8.7\cdot 10^6(\log\gamma)^2(\log\mathscr{B})^{2/3}\left(\log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right)\right)^2.$$

Since the left-hand side exceeds $(m-1)\log\gamma - \log\sqrt{2}$, we get

$$\begin{aligned} (m-1)\log\gamma - \log\sqrt{2} &< 8.7\cdot 10^6(\log\gamma)^2(\log\mathscr{B})^{2/3} \\ &\times \left(\log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right)\right)^2, \end{aligned}$$

so

$$m < 8.71\cdot 10^6(\log\gamma)(\log\mathscr{B})^{2/3}\left(\log\left(\frac{0.005m}{(\log\gamma)(\log\mathscr{B})^{1/3}}\right)\right)^2.$$

We can assume $2\log(0.28m/\sqrt{\log\gamma}) := \log\mathscr{B}$; otherwise (3.23) holds.

Further,

$$\frac{0.28m}{\sqrt{\log\gamma}} > \frac{0.005m}{\log\gamma(\log\mathscr{B})^{1/3}}.$$

Hence, we get

$$m < 8.71\cdot 10^6\cdot 4(\log\gamma)\left(\log(0.28m/\sqrt{\log\gamma})\right)^{2+2/3},$$

which gives again with $y = 0.28m/\sqrt{\log\gamma}$, that

$$y < (8.71\cdot 4\cdot 0.28)\cdot 10^6\sqrt{\log\gamma}(\log y)^{8/3} < 10^7\sqrt{\log\gamma}(\log y)^{8/3}.$$

Hence,

$$\frac{y}{(\log y)^{8/3}} < 10^7\sqrt{\log\gamma}.$$

Denoting by T right-hand side above, we have $T > (4\cdot(8/3)^2)^{8/3}$, so we can apply Lemma 1.4.8 with $s = 8/3$ getting

$$\begin{aligned} y &< 2^{8/3}\cdot 10^7\sqrt{\log\gamma}\left(\log(10^7) + (1/2)\log\log\gamma\right)^{8/3} \\ &< 2^{8/3}\cdot 10^7\sqrt{\log\gamma}(1/2\log\log\gamma)^{8/3}\left(\frac{\log(10^7)}{1/2\log\log(10^9)}+1\right)^{8/3} \\ &< 7\cdot 10^9\sqrt{\log\gamma}(\log\log\gamma)^{8/3}. \end{aligned}$$

Hence,

$$m < \left(\frac{7}{0.28}\right)\cdot 10^9(\log\gamma)(\log\log\gamma)^{8/3} = 2.5\cdot 10^{10}(\log\gamma)(\log\log\gamma)^{8/3}. \qquad (3.27)$$

This is a bit worse than (3.23) (by a factor of $25/1.8 \approx 14$), but we have the additional linear equation (3.24). Recording that

$$\begin{aligned}\log\mathscr{B} &= \max\left\{2.5, 2\log\left(0.28m/\sqrt{\log\gamma}\right)\right\}\\ &\le 2\log\left(7\cdot 10^9\sqrt{\log\gamma}(\log\log\gamma)^{8/3}\right),\end{aligned} \qquad (3.28)$$

we can record the following conclusion.

Lemma 3.3.5 *Let $\gamma > 10^{10}$ and assume that (m,n) is a positive integer solution to $Y_m = F_n$. Then one of the following holds:*

(i)

$$m < 1.8\cdot 10^9(\log\gamma)(\log\log\gamma)^2;$$

or

(ii)

$$m < 2.5\cdot 10^{10}(\log\gamma)(\log\log\gamma)^{8/3},$$

but additionally

$$a + bn + cm = 0,$$

for some integers a, b, c all nonzero (except in the case $t_1 = 0$ which has $a = 0$) with

$$|a| \le 110\log\gamma(\log\mathscr{B})^{1/3}, \quad |b| \le 36(\log\mathscr{B})^{1/3}, \quad |c| < 74(\log\gamma)(\log\mathscr{B})^{1/3},$$

where $\log\mathscr{B}$ *is bounded above as shown in (3.28).*

In the above, we used $(a,b,c) := (t_1 s_1', r_1 s_1', -t_2)$.

3.3.11 A bound on m_2

We write inequalities (3.11) for $(m,n) = (m_i, n_i)$ and $i = 2,3$, and combine them to get

$$|(m_3 - m_2)\log\gamma - (n_3 - n_2)\log\alpha| < \frac{4.16}{\alpha^{2n_2}}. \qquad (3.29)$$

We apply Theorem 1.4.4 with

$$(\lambda_1, \lambda_2, b_1, b_2) = (\gamma, \alpha, m_3 - m_2, n_3 - n_2).$$

We have $D = 4$, $\log B_1 = 0.5\log\gamma$, $\log B_2 = 0.5\log\alpha$ and

$$b' = \frac{m_3 - m_2}{2\log\alpha} + \frac{n_3 - n_2}{2\log\gamma}.$$

Now

$$\begin{array}{rcl} (m_3 - m_2)\log\gamma & \leq & (m_3 - 1)\log\gamma - \log\gamma \\ & \leq & ((n_3 + 1)\log\alpha + \log\sqrt{2}) - \log\gamma \\ & = & n_3\log\alpha + \log(\sqrt{2}\alpha/\gamma) \\ & < & n_3\log\alpha. \end{array}$$

Hence,

$$\begin{array}{rcl} b' & < & \frac{n_3}{2\log\gamma} + \frac{n_3 - n_2}{2\log\gamma} \leq \frac{2n_3 - 2}{2\log\gamma} = \frac{n_3 - 1}{\log\gamma} \\ & \leq & \frac{m_3\log\gamma + \log\alpha + \log\sqrt{2}}{(\log\alpha)\log\gamma} < 2.2m_3. \end{array}$$

where the third inequality above was obtained from the second inequality of Lemma (3.2.1). Putting Λ_1 for the left-hand side of (3.29) we get

$$\begin{array}{rcl} \log|\Lambda_1| & > & -24.34 \cdot 4^4 \cdot (0.5)^2 \log\gamma\log\alpha\max\{21/4, 0.14 + \log(2.2m_3)\}^2 \\ & > & -750\log\gamma\max\{21/4, 0.14 + \log(2.2m_3)\}^2. \end{array}$$

Comparing with (3.29), we get

$$(2n_2)\log\alpha - \log(4.16) < 750\log\gamma\max\{21/4, 0.14 + \log b'\}^2.$$

Hence,

$$(m_2 - 1)\log\gamma - \log\sqrt{2} \quad < \quad 375.1\log\gamma\max\{21/4, \log(2.2m_3)\}^2.$$

Thus,

$$m_2 < 376\max\{21/4, \log(2.42m_3)\}^2.$$

If the maximum is $21/4$, we then get $\log(2.42m_3) < 21/4 = 5.25$, so $m_2 < m_3 < e^{5.25}/(2.42) < 80$. If the maximum is $\log(2.42m_3)$, we then get

$$m_2 < 376(\log(2.42m_3))^2. \tag{3.30}$$

Since $m_3 \geq 3$ and $376(\log(2.42 \cdot 3))^2 > 80$, inequality (3.30) always holds. Let us record it.

Lemma 3.3.6 *Let $\gamma > 10^{10}$ and assume that (m, n) is a positive integer solution to $Y_m = F_n$. Then*

$$m_2 < 376(\log(2.42 m_3))^2.$$

3.3.12 Bounding γ

We return to (3.11) which we rewrite as

$$\left| m \log\gamma - n \log\alpha - \log\left(\frac{2\sqrt{d}}{\sqrt{5}} F_{n_1}\right) + \log F_{n_1} \right| \le \frac{2.08}{\alpha^{2n_2}}.$$

Now

$$\begin{aligned}\log\left(\frac{2\sqrt{d}}{\sqrt{5}} F_{n_1}\right) &= \log\left(\frac{2\sqrt{d}}{\sqrt{5}} Y_1\right) = \log\left(\frac{2\gamma\sqrt{X_1^2-\lambda}}{\gamma\sqrt{5}}\right) \\ &= \log\gamma + \log\left(\frac{2\sqrt{X_1^2-\lambda}}{\sqrt{5}\left(X_1+\sqrt{X_1^2-\lambda}\right)}\right) \\ &= \log\gamma - \log\sqrt{5} + \log(1+\zeta),\end{aligned}$$

where

$$\zeta := \frac{\sqrt{X_1^2-\lambda} - X_1}{X_1 + \sqrt{X_1^2-\lambda}} = -\frac{\lambda}{\gamma^2}.$$

Note that $|\log(1+\zeta)| < 1.01|\zeta|$ for $|\zeta| < 10^{-6}$.

We also have that

$$\frac{\alpha^{n_2}+1}{\sqrt{5}} > F_{n_2} = Y_{m_2} \ge Y_2 = 2X_1Y_1 \ge 2X_1 > 0.99\gamma$$

and so $\alpha^{n_2} > 2.2\gamma$, which gives $2.08/\alpha^{2n_2} < 0.42/\gamma^2$. Thus,

$$|(m-1)\log\gamma - n\log\alpha + \log\sqrt{5} + \log F_{n_1}| < \frac{1.5}{\gamma^2}.$$

We write the above inequality for $(m, n) = (m_i, n_i)$ with $i = 2, 3$, multiply the one for $i = 2$ with $m_3 - 1$ and the one for $i = 3$ with $m_2 - 1$ and apply the absolute value inequality to get

$$\begin{aligned}&\left|((m_3-1)n_2 - (m_2-1)n_3)\log\alpha + (m_2-m_3)\log F_{n_1} + (m_2-m_3)\log\sqrt{5}\right| \\ < \quad & \frac{3(m_3+m_2)}{\gamma^2}.\end{aligned}$$

We rewrite the above as

$$\left|((m_3-1)n_2 - (m_2-1)n_3)\log\alpha + (m_2-m_3)\log\sqrt{5}F_{n_1}\right| < \frac{3(m_3+m_2)}{\gamma^2}, \tag{3.31}$$

and treat it as a linear form in two logarithms.

The right-hand side in (3.31) is $< 6m_3/\gamma^2$. Assume that

$$\gamma^{2-1/10} \le 6m_3.$$

By Lemma 3.3.5, we get

$$\gamma^{2-1/10} < 6\cdot 2.5\cdot 10^{10}(\log\gamma)(\log\log\gamma)^{8/3},$$

which implies $\gamma < 1.5\cdot 10^7$, a case already treated (see Section (3.3.9)). Thus, we may assume that $\gamma^{2-1/10} > 6m_3$, so the right-hand side of (3.31) is smaller that $1/\gamma^{1/10}$. Since $\sqrt{5}F_{n_1}$ and α are multiplicatively independent (this is true since no rational power of α is a positive integer larger than 1), we can apply Theorem 1.4.4 to bound the left-hand side of (3.31). Here, we have $\lambda_1 := \alpha,\ \lambda_2 := \sqrt{5}F_{n_1}$. Thus, $D = 2$, $\log B_1 = 1/2$. Since

$$h(\lambda_2) = h(\sqrt{5}F_{n_1}) = h(\alpha^{n_1} - \beta^{n_1}) \le n_1 h(\alpha) + n_1 h(\beta) + \log 2 < (n_1+2)\log\alpha.$$

We choose $\log B_2 = (n_1+2)\log\alpha > \log\sqrt{5}F_{n_1}$. Further,

$$b' = (m_3 - m_2) + \frac{|(m_3-1)n_2 - (m_2-1)n_3|}{2(n_1+2)\log\alpha}.$$

Note that

$$\begin{aligned}|((m_3-1)n_2 - (m_2-1)n_3)\log\alpha| &< (m_3-m_2)\log\sqrt{5}F_{n_1} + \frac{1}{\gamma^{1/10}}\\ &< (m_3-m_2+1)\log\sqrt{5}F_{n_1}\\ &< m_3\log\sqrt{5}F_{n_1},\end{aligned}$$

where we used the fact that $1/\gamma^{1/10} < 1/10 < \log 2 \le \log F_{n_1}$ together with the fact that $m_2 \ge 2$. We recall $(n_1+2)\log\alpha > \log\sqrt{5}F_{n_1}$ to obtain

$$b' < (m_3-m_2) + \frac{m_3\log\sqrt{5}F_{n_1}}{2(n_1+2)(\log\alpha)^2} < \left(1 + \frac{1}{2\log\alpha}\right)m_3 < 2.04m_3.$$

Thus, Theorem 1.4.4 together with (3.31) give that

$$0.1\log\gamma < 24.34\cdot 2^4\cdot(1/2)\cdot(n_1+2)(\log\alpha)\max\{0.14+\log(2.04m_3), 10.5\}^2.$$

Hence,

$$\log\gamma < 1000(n_1+2)\left(\max\{0.14+\log(2.04m_3), 10.5\}\right)^2.$$

If the maximum above is at 10.5, we then get

$$m_3 < e^{10.36}/2.04 < 16000. \tag{3.32}$$

If the maximum above is at $0.14+\log b'$, then

$$\log\gamma < 1000(n_1+2)(0.14+\log 2.04 m_3)^2 < 1000(n_1+2)(\log(2.4 m_3))^2.$$

Using Lemmas 3.3.4 and 3.3.5, we get

$$\log\gamma < 1000(43.7+2.08\log\log\gamma)(\log(2.4\cdot 2.5\cdot 10^{10}\left(\log\gamma)(\log\log\gamma)^{8/3}\right))^2,$$

which gives $\log\gamma < 2.5\cdot 10^8$. Hence, by Lemma 3.3.5 again we get

$$m_3 < 2.5\cdot 10^{10}\cdot(2.5\cdot 10^8)\cdot(\log(2.5\cdot 10^8))^{8/3} < 2\cdot 10^{22}. \tag{3.33}$$

This bound is about 10^{10} times sharper than what Lemma (3.3.2) gives on m_3. Between estimates (3.32) and (3.33), we conclude that estimate (3.33) always holds. Thus, by Lemma 3.3.6, we have

$$m_2 < 376\left(\log(2.42\cdot 2\cdot 10^{22})\right)^2 < 1.1\cdot 10^6.$$

Further, by Lemma 3.3.4, we get

$$n_1 < 41.7+2.08\log\log\gamma < 41.7+2.08\log(2.5\cdot 10^8) < 82,$$

so $n_1 \le 81$. Let us record what we have proved so far.

Lemma 3.3.7 *Assuming* $\gamma > 10^{10}$ *and* $m_1 = 1$, *we have*

$$\log\gamma < 2.5\cdot 10^8, \quad m_3 < 2\cdot 10^{22}, \quad m_2 < 1.1\cdot 10^6, \quad n_1 \le 81.$$

We need to reduce the above bounds. Returning to (3.31), we get

$$\left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{|(m_2-1)n_3-(m_3-1)n_2|}{m_3-m_2}\right| < \frac{6(m_3+m_2)}{(m_3-m_2)\gamma^2}. \tag{3.34}$$

Assume first that $m_3 \le 2m_2$. Then $m_3 \le 2.2\cdot 10^6$ and the right-hand side above is at most $6\cdot(3m_2)/\gamma^2 < 2\cdot 10^7/\gamma^2$. The quantity on the left-hand side in (3.34) exceeds 10^{-68} in this range. To compute it, we used the fact that if τ is irrational and $m < x$, then

$$\left|\tau - \frac{n}{m}\right| > \frac{1}{(a_{k+1}+2)q_k^2}$$

according to Lemma 1.4.7(i), where $[a_0,\ldots,a_k,\ldots]$ is the continued fraction expansion of τ, $p_k/q_k = [a_0,\ldots,a_k]$ is the kth convergent to τ and k is maximal such that $q_k < x$. In a matter of seconds, *Mathematica* generated for each $n_1 \in [2,100]$ (the bound on n_1 is 81 in fact) the largest k such that $q_k < 2.2\cdot 10^6$ for $\tau = \log\sqrt{5}F_{n_1}/\log\alpha$. The smallest value of the left-hand side in (3.34) is larger than 10^{-68}. We thus get

$$\frac{1}{10^{68}} < \frac{2 \cdot 10^7}{\gamma^2},$$

which gives $\gamma < 6 \cdot 10^{37}$. Assume next that $m_3 \geq 2m_2$. In this case, we have that $(m_3 + m_2)/(m_3 - m_2) \leq 3$, and the right-hand side in (3.34) is at most $18/\gamma^2$. A similar calculation as before (with x replaced by $2 \cdot 10^{22}$) gives that the left-hand side of (3.34) exceeds 10^{-100}. Hence, we get

$$\frac{1}{10^{100}} < \frac{18}{\gamma^2},$$

so $\gamma < 4.3 \cdot 10^{50}$. Lemma 3.3.5 shows that

$$m_3 < 2.5 \cdot 10^{10} \log(4.3 \cdot 10^{50})(\log(\log 4.3 \cdot 10^{50}))^{8/3} < 1.9 \cdot 10^{14}.$$

Thus,

$$m_2 < 376(\log(2.42 \cdot 1.9 \cdot 10^{14}))^2 < 428590.$$

Further,

$$n_1 < 41.7 + 2.08 \log\log \gamma < 41.7 + 2.08 \log\log(4.3 \cdot 10^{50}) < 52.$$

Hence, $n_1 \leq 51$. We perform another reduction cycle with the already improved bounds. If $m_3 \leq 2m_2$, then $m_3 < 857180$, the right-hand side in (3.34) is at most $1.1 \cdot 10^7/\gamma^2$ and the left-hand side exceeds 10^{-41}. Thus, we get

$$\frac{1}{10^{41}} < \frac{1.1 \cdot 10^7}{\gamma^2},$$

so $\gamma < 1.1 \cdot 10^{25}$. Next, assuming $m_3 > 2m_2$, the left-hand side in (3.34) is at most $18/\gamma^2$. We calculate again a lower bound on the left-hand sides of (3.34) (now with $x = 1.9 \cdot 10^{14}$), getting that it exceeds $3 \cdot 10^{-57}$. We thus get

$$\frac{3}{10^{57}} < \frac{18}{\gamma^2},$$

so $\gamma < 5.5 \cdot 10^{28}$. Thus,

$$m_3 < 2.5 \cdot 10^{10} \log(9.4 \cdot 10^{28})(\log\log(9.4 \cdot 10^{28}))^{8/3} < 7.6 \cdot 10^{13},$$

and

$$m_2 < 376(\log(2.42 \cdot 7.6 \cdot 10^{13}))^2 < 405640.$$

Further, by Lemma 3.3.4, we have

$$n_1 < 41.7 + 2.08 \log\log \gamma < 41.7 + 2.08 \log\log(5.5 \cdot 10^{28}) < 51,$$

so $n_1 \leq 50$. After another cycle of reduction, we only obtained the following bounds

$$\gamma < 6.05 \cdot 10^{27}, \quad m_3 < 7.16 \cdot 10^{13}, \quad m_2 < 404168 \quad \text{and} \quad n_1 \leq 50.$$

Let us summarise what we have proved.

Lemma 3.3.8 *Assuming* $\gamma > 10^{10}$, $m_1 = 1$, *we have*

$$\gamma < 6.05 \cdot 10^{27}, \quad m_3 < 7.16 \cdot 10^{13}, \quad m_2 < 404168, \quad n_1 \le 50, \quad \textit{and } n_2 < 5.4 \cdot 10^7.$$

The only inequality to still justify is the one about n_2 but it follows from

$$(n_2 - 2)\log\alpha < (m_2 - 1)\log\gamma + \log\sqrt{2} + \log F_{n_1} < (404168 - 1)\log(6.05 \cdot 10^{27}) + R,$$

where $R = \log\sqrt{2} + \log 1.1 \cdot 10^{13}$. This implies that $n_2 < 5.4 \cdot 10^7$.

3.3.13 The case $m_2 = 2$

Here, $2X_1 = Y_2/Y_1 = F_{n_2}/F_{n_1} > \alpha^{n_2-n_1-1}$(see Lemma (3.2.1)). Since $2X_1 < \gamma + 1 \le 6.06 \cdot 10^{27}$, we get that $n_2 - n_1 \le 133$. We get $n_2 \le 195$. Since F_{n_2} is even, we have that n_2 is a multiple of 3.

For each pair of integers $n_1 \le 50, n_2 \le 195$ such that $X_1 = F_{n_2}/(2F_{n_1})$ is an integer satisfying $5 \cdot 10^8 < X_1 < 3.1 \cdot 10^{27}$, we generated $\gamma := X_1 + \sqrt{X_1^2 - \lambda}$ for $\lambda \in \{\pm 1\}$ and applied the method from Section 3.3.4. We obtained and $n_3 < 240$. To search for solutions to $Y_m = F_n$, we again went back for all pairs (X_1, λ) satisfying the above condition and generated all values of possible candidate d such that $X_1^2 - \lambda = dY_1^2$ for some integer Y_1. For each such (d, Y_1), we set $Y_2 = 2X_1Y_1$ and generated the sequence $\{Y_m\}_{m\ge 1}$ where $Y_{m+2} = 2X_1Y_{m+1} - \lambda Y_m$ with m satisfying $\gamma^{m-1}/\sqrt{2} < F_{240}$. We sought to find those values of d for which the set

$$\{F_n : 1 \le n \le 240\} \cap \left\{ Y_m : 1 \le m \le \frac{\log\sqrt{2}\gamma F_{240}}{\log\gamma} \right\}$$

has cardinality 3. This took a few minutes and produced no solution. There is therefore no solution in this case. From now on, $m_2 > 2$.

3.3.14 The final calculations

We return to (3.11) which we rewrite as

$$\left| m\log\gamma - n\log\alpha - \log\left(\frac{2\sqrt{d}}{\sqrt{5}}F_{n_1}\right) + \log F_{n_1} \right| \le \frac{2.08}{\alpha^{2n_2}}.$$

We rework $\log(2\sqrt{d}F_{n_1})$ in order to get additional terms.

$$\begin{aligned}
\log\left(\frac{2\sqrt{d}}{\sqrt{5}}F_{n_1}\right) &= \log\left(\frac{2\sqrt{d}}{\sqrt{5}}Y_1\right) = \log\left(\frac{2\gamma\sqrt{X_1^2-\lambda}}{\gamma\sqrt{5}}\right)\\
&= \log\gamma + \log\left(\frac{2\sqrt{X_1^2-\lambda}}{\sqrt{5}\left(X_1+\sqrt{X_1^2-\lambda}\right)}\right)\\
&= \log\gamma - \log\sqrt{5} + \log\left(1-\frac{\lambda}{\gamma^2}\right)\\
&= \log\gamma - \log\sqrt{5} - \frac{\lambda}{\gamma^2} + O_{0.51}\left(\frac{1}{\gamma^4}\right),
\end{aligned}$$

Since $m_2 \geq 3$, we have

$$\frac{\alpha^{n_2}+1}{\sqrt{5}} \geq F_{n_2} = Y_{m_2} \geq \gamma^{m_2-1}/\sqrt{2} \geq \gamma^2/\sqrt{2},$$

so

$$\alpha^{n_2} \geq 1.57\gamma^2.$$

Hence, (3.11) yields

$$\begin{aligned}
&\left|m\log\gamma - n\log\alpha + \log F_{n_1} - \log\gamma + \log\sqrt{5} + \frac{\lambda}{\gamma^2} + O_{0.51}\left(\frac{1}{\gamma^4}\right)\right|\\
< \quad &\frac{2.08}{1.57^2\gamma^4} < \frac{1}{\gamma^4},
\end{aligned}$$

which gives

$$\left|(m-1)\log\gamma - n\log\alpha + \log\sqrt{5}F_{n_1} + \frac{\lambda}{\gamma^2}\right| < \frac{1.51}{\gamma^4}. \tag{3.35}$$

We write the above estimates for $(m,n) = (m_i, n_i)$ for $i = 2,3$ and take a linear combination of them to get, via the absolute value inequality

$$\left|((m_2-1)n_3 - (m_2-1)n_2)\log\alpha + (m_3-m_2)\log\sqrt{5}F_{n_1} + \frac{\lambda(m_3-m_2)}{\gamma^2}\right|$$

$$< \frac{1.51(m_3+m_2-2)}{\gamma^4}.$$

This yields,

$$\begin{aligned}
((m_3-1)n_2 - (n_2-1)m_3)\log\alpha + (m_3-m_2)\log\sqrt{5}F_{n_1} &= -\frac{\lambda(m_3-m_2)}{\gamma^2}\\
&+ O_{1.51}\left(\frac{m_3+m_2-2}{\gamma^4}\right),
\end{aligned}$$

so the expression in the left–hand side above has sign $-\lambda$. Furthermore,

$$\begin{aligned} - & \lambda\left(\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}\right) \\ = & \frac{1}{\gamma^2\log\alpha}\left(1+O_{1.51}\left(\frac{m_3+m_2-2}{(m_3-m_2)\gamma^2}\right)\right). \end{aligned}$$

We study the ratio $(m_3+m_2-2)/(m_3-m_2)$.

(i) If $m_3 \le 2m_2$, by Lemma 3.3.8, we get $m_3 \le 2\cdot 404168 < 808336$. Thus,

$$\frac{m_3+m_2-2}{m_3-m_2} < 2m_3 < 1.7\cdot 10^6$$

in this case. Hence,

$$O_{1.51}\left(\frac{m_3+m_2-2}{(m_3-m_2)\gamma^2}\right) = O_{1.51}\left(\frac{1.7\cdot 10^6}{\gamma^2}\right) = O_{0.001}\left(\frac{1}{\gamma}\right)$$

in this case since $1.51\cdot 1.7\cdot 10^6 < 10^7 < \gamma/1000$.

(ii) If $m_3 > 2m_2$, then

$$\frac{m_3+m_2-2}{m_3-m_2} < \frac{1.5m_3}{m_3/2} = 3,$$

so

$$O_{1.51}\left(\frac{m_3+m_2-2}{(m_3-m_2)\gamma^2}\right) = O_{4.53}\left(\frac{1}{\gamma^2}\right) = O_{1.5\cdot 10^{-9}}\left(\frac{1}{\gamma}\right).$$

Hence, in the first case, we get

$$\left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}\right| = \frac{1}{\gamma^2\log\alpha}\left(1+O_{0.001}\left(\frac{1}{\gamma}\right)\right). \tag{3.36}$$

We use the fact that $|(1+x)^{1/L}-1| < 1.001|x|/L$ valid for all $x\in(-1/10^8, 1/10^8)$ and extract square roots getting

$$\begin{aligned} \left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}\right|^{1/2} & = \frac{1}{\gamma\sqrt{\log\alpha}}\left(1+O_{0.001}\left(\frac{1}{\gamma}\right)\right)^{1/2} \\ & = \frac{1}{\gamma\sqrt{\log\alpha}}\left(1+O_{0.001}\left(\frac{1}{\gamma}\right)\right). \end{aligned}$$

Finally, we can take reciprocals getting

$$\begin{aligned} \frac{1}{\sqrt{\log\alpha}}\left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{((m_2-1)n_3-(m_3-1)n_2)}{m_3-m_2}\right|^{-1/2} & = \gamma\left(1+O_{0.001}\left(\frac{1}{\gamma}\right)\right)^{-1} \\ & = \gamma+O_{0.1}(1). \end{aligned}$$

By the same argument, in the second case we get that the above estimate holds with $O_{0.1}(1)$ replaced by $O_{1.5\cdot 10^{-8}}$. Further, since $\gamma = 2X_1 + O_1(1/\gamma)$, we get that

$$\frac{1}{\sqrt{\log\alpha}}\left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}\right|^{-1/2} = 2X_1+\zeta, \tag{3.37}$$

where $\zeta = O_{0.2}(1)$ in the first case and $\zeta = O_{1.6\cdot 10^{-8}}(1)$ in the second case. In both cases,

$$X_1 = \left\lfloor \frac{1}{2\sqrt{\log\alpha}}\left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}\right|^{-1/2}\right\rceil \tag{3.38}$$

and the expression under the absolute value has the sign $-\lambda$. It remains to find an efficient process to detect all possible fractions of the form

$$\frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}.$$

Well, we return to estimate (3.36) and note that its right-hand side is

$$< \frac{2.1}{\gamma^2}.$$

We distinguish several cases.

Case 1. *The case $m_3 \le 2m_2$.*

In this case, we have $m_3 < 808336, \gamma > 10^{10}$, so $\gamma > 3(m_3-m_2)$. Thus, the right-hand side in (3.36) is smaller than

$$\frac{2.1}{(3(m_3-m_2))^2} < \frac{1}{2(m_3-m_2)^2}.$$

By a well-known criterion of Lagrange (see Lemma 1.4.7(i)), the fraction

$$\frac{(m_2-1)n_3-(m_3-1)n_2}{m_3-m_2}$$

is a convergent p_k/q_k of $\log\sqrt{5}F_{n_1}/\log\alpha$. Here, k is a nonnegative integer such that $q_k < m_3 - m_2 < m_3 < 808336$. Thus, we generate all such possibilities, then we calculate X_1 using formula (3.38). Having obtained X_1, we use the method from Section 3.3.9 to find a small bound on n_2. The maximum value of the number A there was at most $A < 5.18\cdot 10^{25}$. Thus, inequality (3.20) gives

$$\begin{aligned}\alpha^{2n_2} &\le \frac{8.4(A+2)Q^2}{m_3-m_2} \le 9.2(A+2)Q \\ &< 9.2\cdot(5.18\cdot 10^{25}+2)(404168) < 1.93\cdot 10^{32},\end{aligned}$$

so $n_2 < 78$. This leads to $m_2 = 2$ because $\gamma > 10^{10}$, but this case was already covered with no solution found.

From now on, we may assume that $m_3 > 2m_2$.

Case 2. *Considerations about γ and m_3.*

If $\gamma > 2.1m_3$, then again the right-hand side of (3.36) is smaller than

$$\frac{1}{2(m_3 - m_2)^2},$$

so Legendre's result applies leading again to $n_2 \leq 78$ and $m_2 = 2$.

It remains to deal with the case when $\gamma < 2.1m_3$. Note that in this case both $m_3 > 10^{10}/2.1 > 4.7 \cdot 10^9$ and $m_3 - m_2 > m_3/2 > 2.3 \cdot 10^9$ are large making direct computations with them ineffective. We write

$$\frac{(m_2 - 1)n_3 - (m_3 - 1)n_2}{m_3 - m_2} = \frac{a}{b},$$

where a and b are coprime. Thus,

$$(m_2 - 1)n_3 - (m_3 - 1)n_2 = Da \quad \text{and} \quad m_3 - m_2 = Db,$$

for some positive integer D. Write

$$\frac{2.1}{\gamma^2} = \frac{1}{b^2}\left(\frac{2.1b^2}{\gamma^2}\right). \tag{3.39}$$

Case 3. *The case when $m_3 \leq 73\gamma$ and $m_3 > 2m_2$.*

Then the right-hand side in (3.39) above is at most

$$\frac{2.1(m_3 - m_2)^2}{D^2b^2\gamma^2} < \frac{2.1 \cdot 72^2}{D^2b^2} < \frac{10900}{D^2b^2}.$$

Hence, returning to estimate (3.36), we have

$$\left|\frac{\log\sqrt{5}F_{n_1}}{\log\alpha} - \frac{a}{b}\right| < \frac{10900}{D^2b^2}.$$

If $D \geq 150$, then the right-hand side above is smaller than $1/(2b^2)$, so Legendre's criterion applies, which was already covered in Case 1. Thus, we may assume that $D < 150$. In this case

$$b = \frac{m_3 - m_2}{D} > \frac{2.3 \cdot 10^9}{150} > 1.5 \cdot 10^7. \tag{3.40}$$

The above is a particular instance of an inequality of the form

$$\left|\tau - \frac{a}{b}\right| < \frac{K}{b^2}, \tag{3.41}$$

where $\tau := \log\sqrt{5}F_{n_1}/\log\alpha$, $a/b = ((m_2 - 1)n_3 - (m_3 - 1)n_2)/(m_3 - m_2)$ and $K = 10900$. At this stage we use the following theorem of Worley [31] for the irrational τ, which generalises Legendre's result.

Theorem 3.3.9 *Assume (3.41) holds. There exist* r, s *with* $r > 0$, $s \geq 0$, $rs < 2K$ *and* $k \geq 1$ *such that*

$$a = rp_k + \eta sp_{k-1} \qquad \text{and} \qquad b = rq_k + \eta sq_{k-1}, \quad \eta \in \{\pm 1\},$$

or $1 \leq rs < K$, k *is such that* $a_{k+1} = 1$ *and*

$$a = rp_{k+1} + sp_{k-1}, \qquad \text{and} \qquad b = rq_{k+1} + sq_{k-1}.$$

In case $s = 0$, we can take $r = 1$ (since a and b are coprime) and then we have $a/b = p_k/q_k$. This is the only case possible when $K \leq 1/2$ since then $2K \leq 1$ so $rs < 1$ giving $s = 0$. This is Legendre's result. Since we already did the calculation corresponding to this case, we assume that $rs \geq 1$. Worley describes the number k. In case $rs \neq 0$, then let $\ell \geq -1$ be the unique positive integer such that if we write $\tau = [a_0, a_1, \ldots, a_\ell, a_{\ell+1}, \ldots]$ and $a/b = [a_0, a_1, \ldots, a_\ell, b_1, \ldots, b_u]$ then $b_1 \neq a_{\ell+1}$. In Worley's notation, let $\beta = [b_1, \ldots, b_u]$ and $\gamma = [a_{\ell+1}, \ldots]$. Then $\beta = b_1 + \rho/d$, where $0 \leq \rho < d$ and $b_1 \neq a_{\ell+1}$. Worley distinguishes between the cases $\beta > \gamma$ and $\beta < \gamma$. In case $\beta > \gamma$, he shows that $k = \ell + 1$, and $(r, s, \eta) = (d, d(b_1 - a_{\ell+1}) + \rho, +1)$ satisfies the required inequality $rs < 2K$ except when $b_1 = a_{\ell+1} + 1$ and $a_{\ell+2} = 1$, in which case $k = \ell + 1$ and $(r, s, \eta) = (d, \rho, +1)$ and $b = rq_{\ell+2} + sq_\ell$. Thus, in this case k is an index such that $p_k < b < 2Kp_{k+1}$ except when $a_{\ell+2} = 1$, in which case the upper bound is slightly worse namely $b < 2Kp_{k+2} < 4Kp_{k+1}$. In the case $\beta \in (0, \gamma)$, Worley distinguishes cases according to the value of $\min\{\beta, a_{\ell+1}/2\}$. If $\beta \leq a_{\ell+1}/2$, then he writes $\beta = m + \rho/d$ and shows that $k = \ell$ and $(r, s) = (dm + \rho, d)$ have the required property. This also works when $m = 0$. If $\beta > a_{\ell+1}/2$, then he writes $\beta = a_{\ell+1} - m - \rho/d$, and shows that $k = \ell + 1$ and $(r, s, \eta) = (dm + \rho, d, -1)$ have the required properties. This is the only case of the negative sign. In particular, in this case $k = \ell$ and $k = \ell + 1$ according to whether the sign is $+1$ or -1 and at any rate $q_{\ell-1} + q_\ell/(2K) < b < q_{\ell+1}$. In practice this helps to corner k when doing the calculations. Thus, for $n_1 \in [2, 50]$, we apply Worley's Theorem 3.3.9 with $\tau := \log \sqrt{5} F_{n_1} / \log \alpha$ and $K := 10900$. We calculate the continued fraction and the convergents p_k/q_k, p_{k-1}/q_{k-1} of τ such that $k \leq k_\tau$, where k_τ is maximal such that $q_{k_\tau} \leq m_3 \leq 7.16 \cdot 10^{13}$. A simple calculation reveals that $k_\tau \leq 32$, which is reached at $n_1 = 4$. Then we generated the fractions

$$\frac{a}{b} = \frac{rp_k + s\eta p_{k-1}}{rq_k + s\eta q_{k-1}}, \quad \text{with} \quad 1 \leq rs < 2K, \; k \leq k_\tau \text{ and } \quad \eta \in \{\pm 1\}.$$

If $a_{k+1} = 1$ we must also consider the fractions of the form

$$\frac{a}{b} = \frac{rp_{k+1} + sp_{k-1}}{rq_{k+1} + sq_{k-1}}, \quad \text{with} \quad 1 \leq rs < K \quad \text{and} \quad k \leq k_\tau.$$

In order to reduce the amount of fractions a/b we consider also the condition (3.40). Then we used equation (3.38) to generate all possible candidates X_1 obtainable in this way. As we obtain them we check for the extra condition namely that in estimate (3.37) we have that the distance for the formula we use to round up to X_1 (distance to the nearest integer X_1) is smaller than $(1.6/2) \cdot 10^{-8} < 10^{-8}$. Furthermore, we are left only with those X_1's that make

$\gamma = X_1 + \sqrt{X_1^2 - \lambda} > 10^{10}$, for $\lambda \in \{\pm 1\}$. After a few hours of computation we obtained the following list of X_1's:

$$15048438972, 34810465774, 58298763342, 25042219341229, 727111625435,$$
$$3375923511388, 8581755725011, 7538130851888704$$

Only to these candidates we apply the method from Section 3.3.9. The exact quadruple (η, n_1, k, X_1) resulting from the code is captured in the table below.

η	n_1	k	X_1	λ	$\max\{a_i : i \leq 68\}$
-1	5	24	15048438972	-1	116
-1	10	13	58298763342	-1	91
-1	14	8	727111625435	-1	55
-1	29	4	8581755725011	-1	294
$+1$	3	23	86632618310552	-1	323
$+1$	10	16	25042219341229	-1	69
$+1$	16	15	3375923511388	-1	279
$a_{k+1} = 1$	30	9	7538130851888704	-1	82

With the values of X_1 so obtained, we used the method from Section 3.3.9 to obtain a small bound on n_2. From the first 68 partial quotients $[a_0, a_1, \ldots, a_{67}]$ of each $(\log\gamma)/\log\alpha$, we obtained the maximum A of the a_k for all $k \in [0, 67]$ satisfies $A < 323$. So

$$\alpha^{2n_2} \leq \frac{9.2(A+2)Q^2}{m_3 - m_2} \leq 9.2(A+2)Q < 9.2 \cdot (323+2)(7.16 \cdot 10^{13}) < 2.141 \times 10^{17}. \tag{3.42}$$

By Lemma 3.2.1, the above implies

$$\gamma^{2(m_2-1)} < 2Y_{m_2}^2 = 2F_{n_2}^2 < 2\alpha^{-2}\alpha^{2n_2} < 2 \cdot \alpha^{-2} \cdot 2.141 \cdot 10^{17} < 1.64 \cdot 10^{17},$$

which is impossible for $m_2 > 2$ and $\gamma > 10^{10}$.

From now on, $m_3 > 73\gamma$. If $m_3 < 1.8 \cdot 10^9 (\log\gamma)(\log(\log\gamma))^2$, then we get

$$73\gamma < 1.8 \cdot 10^9 (\log\gamma)(\log\log\gamma)^2,$$

which gives $\gamma < 5.4 \cdot 10^9$, a contradiction.

Hence, we only have to treat:

Case 4. *The case when $m_3 > 73\gamma$ and $m_3 > 1.8 \cdot 10^9 (\log\gamma)(\log(\log\gamma))^2$.*

Then Lemma 3.3.5 shows that,

$$73\gamma < m_3 < 2.5 \cdot 10^{10} (\log\gamma)(\log\log\gamma)^{8/3},$$

so $\gamma < 2.1\cdot 10^{11}$. We then get

$$\log\mathscr{B} \le 2\log(7\cdot 10^9\sqrt{\log(2.1\cdot 10^{11})}(\log\log(2.1\cdot 10^{11}))^{8/3}) < 55.$$

Thus, we get that

$$a + bn_3 + cm_3 = 0,$$

for some nonzero integers a, b, c with

$$|a| \le 110(\log\gamma)(\log\mathscr{B})^{1/3}, \quad |b| \le 36(\log\mathscr{B})^{1/3}, \quad |c| \le 74(\log\gamma)(\log\mathscr{B})^{1/3}.$$

In particular, we have the system

$$\begin{array}{rcl} c(m_3-1)+bn_3 & = & -a-c; \\ (m_3-1)\log\gamma - n_3\log\alpha & = & -\log\sqrt{5}F_{n_1} + O_1\left(\dfrac{1.01}{\gamma^2}\right). \end{array}$$

The second equation follows from (3.35). Note that since $\gamma > 10^{10}$, it follows that the error term in the second equation above is $< 10^{-17}$. We solve it with Cramer's rule for $m_3 - 1$ getting

$$m_3 - 1 = \frac{\begin{vmatrix} -a-c & b \\ -\log\sqrt{5}F_{n_1} + O_{0.01}(1) & -\log\alpha \end{vmatrix}}{\begin{vmatrix} c & b \\ \log\gamma & -\log\alpha \end{vmatrix}}.$$

The numerator is in absolute value at most

$$\begin{array}{rcl} (|a|+|c|)\log\alpha & + & |b|(\log\sqrt{5}F_{n_1} + 0.01) \\ & < & ((110+74)(\log\alpha)\log\gamma + (36+0.36/\log\sqrt{5}F_{n_1})\log\sqrt{5}F_{n_1})(\log\mathscr{B})^{1/3} \\ & < & (88.6 + (36.52)(\log\gamma))(\log\mathscr{B})^{1/3} \\ & < & 126(\log\gamma)(\log\mathscr{B})^{1/3}. \end{array}$$

In the above, we used the fact that $2 < \sqrt{5}F_{n_1} = \sqrt{5}Y_1 < \gamma$ which is true since $d \ge 94$ for $\gamma \ge 10^{10}$. Hence,

$$\begin{vmatrix} c & b \\ \log\gamma & -\log\alpha \end{vmatrix} = O_{127}\left(\frac{(\log\gamma)(\log\mathscr{B})^{1/3}}{m_3}\right).$$

We thus get

$$-\frac{c}{b}\log\alpha = (\log\gamma)\left(1 + O_{127}\left(\frac{(\log\mathscr{B})^{1/3}}{bm_3}\right)\right).$$

The amount above that involves O_{127} is at most $127\cdot 55^{1/3}\cdot m_3^{-1} < 10^{-9}$ since $m_3 > 73\gamma > 7.3\cdot 10^{11}$.

Exponentiating we get

$$\begin{aligned}\alpha^{-c/b} &= \gamma\exp\left(O_{127}\left(\frac{(\log\gamma)(\log\mathscr{B})^{1/3}}{|b|m_3}\right)\right)\\ &= \gamma\left(1+O_{128}\left(\frac{(\log\gamma)(\log\mathscr{B})^{1/3}}{|b|m_3}\right)\right)\\ &= \gamma+O_{128}\left(\frac{\gamma(\log\gamma)(\log\mathscr{B})^{1/3}}{|b|m_3}\right).\end{aligned}$$

The second equality holds true because

$$e^x = 1+x+O_1(x^2) = 1+O_{1.001}(x) \quad \text{for} \quad 0<x<10^{-9},$$

and $1.001 < 128/127$. Since $m_3 > 73\gamma$, we have

$$O_{128}\left(\frac{\gamma(\log\gamma)(\log\mathscr{B})^{1/3}}{|b|m_3}\right) = O_{128}\left(\frac{(\log\gamma)(\log\mathscr{B})^{1/3}}{73|b|}\right) < \frac{173.85}{|b|}.$$

In the above, we used that $\gamma < 2.1\cdot 10^{11}$ and $\log\mathscr{B} < 55$. Since

$$\gamma = 2X_1 + O_1(1/10^{10})$$

and $|b| \le 36(\log\mathscr{B})^{1/3} < 140$, we get that

$$\alpha^{-c/b} = 2X_1 + O_{175}(1/|b|).$$

Thus,

$$|X_1 - 0.5\alpha^{-c/b}| < 88/|b|.$$

This shows that X_1 is determined by pairs of integers $|b|, |c|$ with

$$\begin{aligned}1 &\le |b| \le 36(\log\mathscr{B})^{1/3} < 140,\\ 1 &\le |c| \le 74(\log\gamma)(\log\mathscr{B})^{1/3} < 7337,\end{aligned}$$

and then X_1 is one of the integers of the form $\lfloor 0.5\alpha^{|c|/|b|}\rceil + \ell$, where

$$|\ell| \in [-88/|b|, 88/|b|] \cap \mathbb{Z}.$$

With these values of X_1, after checking that $X_1 \in [5\cdot 10^9, 1.1\cdot 10^{11}]$ (the upper bound follows since $X_1 < (\gamma+1)/2$ and $\gamma < 2.1\cdot 10^{11}$), we used the method of Section 3.3.9. We got $A < 9\cdot 10^{21}$. This calculation took a few minutes. So,

$$\alpha^{2n_2} \le \frac{9.2(A+2)Q^2}{m_3-m_2} \le 9.2(A+2)Q < 9.2\cdot(9\cdot 10^{21}+2)(7.16\cdot 10^{13}) < 5.93\times 10^{36}. \tag{3.43}$$

Thus, $n_2 < 100$. However, by Lemma 3.2.1

$$\gamma^{2(m_2-1)} < 2 \cdot \alpha^{-2} \cdot 5.93 \cdot 10^{36} < 4.54 \cdot 10^{36}.$$

Since $\gamma > 10^{10}$, this gives $m_2 = 2$, which is a case already treated.

This finishes the proof of the theorem.

Chapter 4

Conclusion

It was demonstrated through the work in the Chapter 2 that there are finitely many values of d of the Pell equation $X^2 - dY^2 = \pm 1$ that have at least two members of their $X-$sequence of solutions being base-b rep-digits for any fixed base $b \geq 2$.
We see for instance that the main result of Chapter 2 shows that $d = 2$ has two of its $X-$coordinates being base-b repdigits for some $b \in [2, 100]$. The way to find the actual solution is first to find the minimal solution corresponding to $d = 2$. The minimal solution is $(1, 1)$. We then generate the rest of the $X-$solutions using the binary recurrence $X_n = 2X_1X_{n-1} + X_{n-2}$ until we get to $n = 5$. We then obtain the base-b representation of each of the $X-$coordinates taking turns to do it for each $b \in [2, 100]$. We obtain $X_1 = 1, X_2 = 3, X_3 = 7, X_5 = 41$. One then confirms that $1 = (1)_2, 3 = (11)_2, 7 = (111)_2$. This means that the value $d = 2$, has at least two(exactly three) of its X sequence of solutions being base-2 repdigits.

In Chapter 3, it was demonstrated that, only the value $d = 2$ has at least three of its Y sequence of solutions being Fibonacci numbers. The solutions to $Y_m = F_n$ is this case are

$$(m, n) = (1, 2), (2, 3), (3, 5),$$

4.0.1 Possible Next Step

One direction that could be explored is the problem of determining the values of d that have at least three of their Y sequence of solutions being base-b repdigits for any fixed integer base $b \geq 2$.

Chapter 5

Bibliography

[1] M. A. Alekseyev, *On the intersections of Fibonacci, Pell, and Lucas numbers*, Integers 11A (2011), 239–259.

[2] A. Baker, On the representation of integers by binary forms, *Philos. Trans. A* **263** (1968), 173–208.

[3] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, The Quarterly Journal of Mathematics, **20** (1969), 129–137.

[4] A. Baker and G. Wüstholz, *Logarithmic forms and Diophantine Geometry*, vol. 9, Cambridge University Press, 2008.

[5] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math. **442** (1993), 19–62.

[6] M. A. Bennett and A. Pintér, Intersections of recurrence sequences, *Proc. Amer. Math. Soc.* **143** (2015), 2347–2353.

[7] Yu. F. Bilu, G. Hanrot and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte*, J. Reine Angew. Math. **539** (2001), 75–122.

[8] J. J. Bravo and F. Luca, On a conjecture about repdigits in k-generalized Fibonacci sequences, *Publ. Math. Debrecen* **82** (2013), 623–639.

[9] Y. Bugeaud and M. Mignotte, On integers with identical digits, *Mathematika* **46** (1999), 411–417.

[10] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. Math. (2) **15** (1913), 30–70.

[11] H. Cohen, *Number Theory. Volume I: Tools and Diophantine Equations*, New York: Springer, 2007.

[12] J. H. E. Cohn, *The Diophantine equation* $x^4 - Dy^2 = 1$*, II*, Acta Arith. **78** (1997), 401–403.

[13] A. Dossavi-Yovo, F. Luca and A. Togbé, On the x-coordinates of Pell equations which are rep-digits, *Publ. Math. Debrecen* **88** (2016), 381–399.

[14] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 291–306.

[15] C. A. Gómez, B. Edjeou, B. Faye and F. Luca, *On Y-coordinates of Pell equations which are Lucas numbers*, The Ramanujan J., to appear.

[16] B. Faye and F. Luca, On X-coordinates of Pell equations that are repdigits, *Fibonacci Quart.* **56** (2018), 52–62.

[17] B. Faye and F. Luca, *On the Y-coordinates of Pell equations which are base 2 rep-digits*. Glasnik Matematicki. vol. 25**75** (2020), 1–12.

[18] B. Faye and F. Luca, *Y-coordinates of Pell equations in a fixed binary reccurence*, New York J. Math. **26** (2020), 184–206.

[19] C.A Gómez, F. Luca and F.S Zottor, On the X-coordinates of Pell equations which are repdigits, Res. Number Theor. **6:41** (2020).

[20] S. Guzmán Sanchez and F. Luca, *Linear combinations of factorials and S-units in a binary recurrence sequence*, Ann. Math. Qué. **38** (2014), 169–188

[21] M. Laurent, M. Mignotte and Yu. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory **55** (1995), 285–321.

[22] F. Luca, A. Montejano, L. Szalay and A. Togbé, *On the X-coordinates of Pell equations which are Tribonacci numbers*, Acta Arith. **179** (2017), 25–35.

[23] F. Luca and A. Togbé, On the x-coordinates of Pell equations which are Fibonacci numbers, *Math. Scand.* **122** (2018), 18–30.

[24] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (6) (2000), 125–180, in Russian; English translation in *Izv. Math.* **64** (2000), 1217–1269.

[25] M. Jacobson and H. Williams. *Solving the Pell Equation.* CMS Books in Mathematics, Springer-Verlag New York 2009.

[26] M. Mignotte and P. Voutier, *A kit for linear forms in three logarithms*, *Preprint* 2008, available at http://irma.math.unistra.fr/ bugeaud/travaux/kit.pdf.

[27] M. Mignotte, *Linear forms in two and three logarithms and interpolation determinants*, in “Diophantine equations" (edited by N. Saradha), Tata Inst. Fund. Res. Stud. Math. **20**, Mumbai, 2008, pp. 151–166.

[28] M. Mignotte and N. Tzanakis, "Arithmetical study of recurrence sequences", Acta Arith., LVII(1991,)357 – 364.

[29] M. R. Murty and J. Esmonde. *Problems in algebraic number theory,* Second edition. Graduate Texts in Mathematics, **190** Springer–Verlag, New York, 2005.

[30] J. A. Vandehey, MR3491748, Math Sci Net.

[31] R. T. Worley, *Estimating* $|\alpha - p/q|$, J. Austral. Math. Soc. (Series A) **31** (1981), 202–206.