THE APPROPRIATION OF MATHEMATICAL OBJECTS BY UNDERGRADUATE MATHEMATICS STUDENTS: A STUDY

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DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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(Signature of candidate)

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ABSTRACT

In this thesis I consider how mathematics students in a traditional first–year Calculus course at a South African university appropriate mathematical objects which are new to them but which are already part of the official mathematics discourse. Although several researchers have explained mathematical object appropriation in process-object terms (for example, Sfard, 1994; Dubinsky, 1991, 1997; Tall, 1991, 1995, 1999), my focus is largely on what happens prior to the object-process stage. In line with Vygotsky (1986), I posit that the appropriation of a new mathematical object by a student takes place in phases and that an examination of these phases gives a language of description for understanding this process. This theory, which I call "appropriation theory", is an elaboration and application of Vygotsky's (1986) theory of concept formation to the mathematical domain.

I also use Vygotsky's (1986) notion of the functional use of a word to postulate that the mechanism for moving through these phases, that is, for appropriating the mathematical object, is a functional use of the mathematical sign. Specifically, I argue that the student uses new mathematical signs both as objects with which to communicate (like words are used) and as objects on which to focus and to organise his mathematical ideas (again as words are used) even before he fully comprehends the meaning of these signs. Through this sign usage the mathematical concept evolves for that student so that it eventually has personal meaning (like the meaning of a new word does for a child); furthermore, because the usage is socially regulated, the concept evolves so that its usage is concomitant with its usage in the mathematical community.

I further explicate appropriation theory by elaborating a link between the theoretical concept variables and their empirical indicators, illustrating these links with data obtained from seven clinical interviews. In these interviews, seven purposefully– chosen students engage in a set of specially–designed tasks around the definition of an improper integral. I utilise the empirical indicators to analyse two of these interviews in great detail. These analyses further inform the development of appropriation theory and also demonstrate how the theory illuminates the process of mathematical object appropriation by a particular student.

In memory of my parents, Boetie and Lily Berger, who taught me to value and cherish learning.

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CONTENTS

CHAPTER 1: INTRODUCTION

§1.1 RESEARCH FOCUS

In this thesis I am primarily concerned with examining the ways in which mathematics students construct personal meanings and develop usages of mathematical signs which are compatible with the culturally established meanings and usages of those signs.

In particular, I am concerned with the ways in which tertiary level students studying a traditional first–year[1] mathematics course at the University of Witwatersrand in South Africa appropriate mathematical objects which are presented to them in the form of a written definition. These mathematical objects are new to the students but are already part of the official mathematical discourse.

The problem is this: How does an individual learner come to know or understand a mathematical object[2] to which the initial access is the various signs (such as words and symbols) of the definition? This question is particularly problematic at a tertiary level in that the undergraduate learner, presented with the definition of a mathematical object (as is common practice in the Mathematics I Major course at the University of the Witwatersrand) has no perceptual access to the mathematical object. All the learner has are the symbols and words which are intended to somehow signify that object. In many instances, diagrams and other graphical representations of the mathematical object are not presented alongside the definition. Indeed the student at undergraduate level is often expected to construct all the properties of the mathematical object from the definition (Tall, 1995).

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[1] These students are freshmen, in American terms.

[2] Like Sfard (2000), I use the word ‘object’ in a metaphorical rather than an ontological sense.

On a philosophical note, an added complication to a consideration of how a student appropriates a mathematical object is the circularity implied by my (still-to be-presented) argument that the learner gains epistemological access to a mathematical object through his[3] use of that object. The 'meaning-through-use' paradox is as follows: if the way to understand a new object is through its use, then one has to use the object to understand it. But how can one use an object if one does not yet know what it is? In a similar vein, Sfard (1998: 10) argues that all discourses, and mathematical discourse in particular, suffer from an inherent circularity[4]:

> Signifieds can only be built through discursive activities with signifiers, whereas the existence of the signifieds is a prerequisite for the successful use of the signifiers.

Broadly speaking, my intention in this thesis is to elaborate and explore the applicability of a specific framework, which is based on Vygotsky's (1986) thesis about the functional use of a concept and his related theory of concept formation, to the question of how an undergraduate mathematics student appropriates a new mathematical object. In the course of my elaboration, I will show how the Vygotskian framework deals with the circular problem implied by the 'meaning through use' paradox. In Chapter 4, after I have defined all the necessary terms and described the requisite theoretical constructs with which I will be dealing, I explicate and articulate my research questions more precisely.

Before immersing myself in these tasks, I need to explain why I consider the issue of mathematical object appropriation as fundamental to

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[3] Throughout this thesis I use masculine pronouns (he, his or him) to refer to both male and female learners. (This usage is based on the fact that most of the students whom I interview are male.)

[4] Indeed this very discussion suffers from its own circularity: I am and will be using terms, such as sign, signifier, signified, mathematical object before I have clarified what I mean by these terms. Hopefully the reader will begin to understand what I mean by these terms through her or his use of these terms (in the activity of reading and reflecting on what I have said). Notwithstanding this expectation, and since this is an academic thesis rather than a teaching and learning experiment, I will elaborate on my use of these and other terms in Chapter 2.

mathematical education both in South Africa and internationally and to indicate the broad context of my study.

§1.2 RATIONALE

The case of a school-level competent mathematics student entering a first-year mathematics course with enthusiasm and confidence but departing a year later with low marks and a sense of mathematical confusion and disillusion, is a fairly common situation at many South African and overseas universities. At the University of the Witwatersrand, for example, the pass rate for the Mathematics I Major course in 2000 (the year and course in which this study is set) was 65%. That statistic is particularly shocking since it excludes those students who dropped out of the course during the year (often as a result of an inability to cope with the course); furthermore, it is alarming in that only students who performed relatively well in the Matriculation Higher Grade Mathematics examination[5] were accepted into the course at the beginning of the year. Such high drop–out and failure rates in university mathematics courses are not peculiar to South Africa. Confrey and Costa (1996: 139) state that even at major American research universities, the failure rate in introductory calculus courses may range from 17% to 22%.

One fairly widespread institutional response to this scenario, particularly in the United States of America (USA), has been the Calculus Reform movement. This movement has directed itself to a revision of the calculus curriculum with a focus on raising student’s conceptual understandings, problem solving skills, analytic and transference skills and reducing ‘tedious’ calculations (NSF, cited in Ganter, 2001: 4). Certainly these aims

[5] The matriculation mathematics examination is an external national mathematics examination written by all South African pupils in Grade 12 (about 17 years old) who choose to do mathematics in their final year of school. Matriculation level mathematics is offered at two levels: Higher Grade and Standard Grade. A student has to receive a minimum of 60% for the matriculation Higher Grade mathematics examination to be accepted into the Mathematics I Major course at the University of the Witwatersrand.

are commendable and worthy, but to what extent and how they can be achieved is still hotly debated in the international mathematics educational community[6].

> Discussions related to what a student should know and understand after completing a calculus course… are heated and have yet to be resolved. Also being debated are which basic mathematical computations and skills should *go along* with conceptual understanding. These are issues that never will be resolved completely…. Nonetheless, it is important to investigate student learning in calculus and collect data that will contribute to the knowledge base on how students learn calculus. (Ganter, 2001: 23; my italics)

I intend that my research will contribute significantly to these debates, specifically those which focus on student learning.

In particular, many of the proponents of the Calculus Reform Movement believe that skills are somehow separate from understandings; this is certainly implied in Ganter's statement above that "mathematical computations and skills should *go along* with conceptual understanding". In this thesis I wish to consider the contrary view: whether computational and manipulative skills are useful and productive of mathematical meaning–making for undergraduate mathematics students. Indeed, I intend to explore the possibility that activities such as imitation, association, template-matching and manipulation, which activities are often described pejoratively in the Calculus Reform milieu, are necessary activities for the student's construction of a personal and socially-acceptable meaning of a specific mathematical object.

At the same time, but in opposition to the Back–to–the–Basics Movement, I will also argue that these activities are not sufficient for the development of understanding and meaningful, flexible mathematical thinking in a particular learner.

[6] Other institutional responses include the introduction of remediation classes, bridging courses (for example, at the University of the Witwatersrand and the University of Cape Town in South Africa), pre–calculus courses, and so on.

Indeed my primary argument in this thesis will be that the use of a mathematical object by a learner within a socially regulated milieu, even before mature and 'proper' understanding, ultimately enables the development of a personally meaningful concept for that learner and a usage of that concept which is ultimately meaningful to the wider mathematical community.

On a more general pedagogical level, so–called theoretical debates about the construction of mathematical meaning and what constitutes understanding (debates which are fundamental to this thesis and within which I take a position in Chapter 2) are fundamental to the practical activities of teaching and learning mathematics.

> The basis for judging students' understanding depends heavily on the teacher's preconceptions of understanding and of the meaning of the subject matter in question. In addition, the teaching itself is strongly influenced, if not regulated, by those views held by the teachers. This clarifies the didactic impact of theories of meaning and understanding on the theory and practice of teaching and learning. (Dörfler, 2000: 100)

Certainly in the traditional environment within which I operate, much debate and discussion around what constitutes meaningful mathematical learning and teaching is urgently required. To illustrate this context it is perhaps sufficient to quote an influential member of the University of the Witwatersrand Mathematics School executive committee[7] who, while exhorting the staff to try and improve pass rates (a laudable sentiment with which I empathise) advised staff to "just drum it into them[8]".

Given the highly contested nature of what constitutes effective mathematical learning at undergraduate level (and the pedagogical strategies these

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[7] The executive committee of the School of Mathematics consists of the Head of Department, six senior mathematics academic staff appointed by her, and one mathematics academic staff member elected by the rest of the academic staff of the School of Mathematics.

[8] "It" presumably refers to various mathematical ideas, algorithms, theorems and definitions; "them" are the students.

different theories imply) it is unsurprising that several researchers have attempted to address these issues (for example, Tall, 1991, 1995, 1999; Tall et al, 2000a, 2000b; Dubinsky, 1991, 1997; Czarnocha et al, 1999).

Interestingly the focus of most of these theories of mathematical learning is the relationship between “process” and “object” (Confrey and Costa, 1996). In brief, these theories posit that in order for a mathematical concept to be known by the student, a mathematical process needs to get transformed into a mathematical object[9].

However I regard a student who is able to use a particular mathematical process as already quite far along the path of mathematical concept construction. To this extent I regard what happens **prior** to the process–object stage as crucial to a theory of mathematical learning. And this stage has largely been ignored in theories dealing with mathematical concept construction (as I will argue in my Literature Review in Chapter 3).

Indeed, over my years of teaching calculus to first–year students, I have frequently had the sense that much of what I was observing was not easily or usefully explained in process–object terms.

For example, how did I explain or even describe what was happening when a student was introduced to a new mathematical object via a definition? In that situation there are no physical or graphical objects for the student to work with, and often there are no explicit processes. There is only the signifier, referring to an abstract object, which at that stage is unknown to the learner.

Furthermore, the process–object theories do not illuminate what is happening when a student seems to muddle his way through a new definition and/or a theorem, apparently making arbitrary connections between different signifiers and between signifiers and hazy signifieds. What is determining the student’s apparent associations and how do these associations enable or constrain the student? How do some of these

[9] This transformation is called reification (Sfard, 1994) or encapsulation (Dubinsky, 1991).

‘initially–muddled’ students get to use these new mathematical objects or processes meaningfully (which it certainly appears that some of them do)?

In short, is there something else to mathematics learning not addressed by the process–object theories?

Secondly these researchers (eg Tall, 1991, 1995, 1999; Dubinsky, 1991, 1997; and their many followers) use elaborations of Piaget’s framework to focus primarily on the conceptual side of cognition. In contrast my focus is on the relationship between the student and the sign, and on symbolising activities. This focus is consequent upon my assumption that academic mathematics, like any scientific knowledge, is semiotically mediated (Vygotsky, 1978; Sierpinska et al, 1999a) and that a focus on symbolising activities is thus bound to be illuminating and revelatory.

Indeed Becker and Varelas (1993) argue that a lack of focus on semiotic phenomena in mathematical activity may result in the researcher missing significant aspects of cognition. They quote experiments by Becker (1989) and Sophian (1988) in which young children were able to reason about one-to-one correspondences (between dolls and cups) when the researcher and children used number phrases (like five cups, six dolls and so on). In contrast the same-age children regularly failed Piaget’s number conservation task (a task involving reasoning about one-to-one correspondences that avoids the use of number words or perceptual cues). Consequently they argue that “if we do not attend to children’s sign use, we may miss significant aspects of their cognition, including structural-conceptual aspects of their cognition” (p. 421).

Certainly there are several researchers in mathematics education who do address the relationship between mathematical signs and their meanings within a socio-cultural context (for example, Walkerdine, 1988; Vile and Lerman, 1996; Cobb et al, 1997; Presmeg, 1997; Sierpinska et al, 1999a, Sierpinska et al, 1999b, Sierpinska, 2000; Gravemeijer et al, 2000; Radford, 2000, 2001; Dörfler, 2000; Sfard, 2000; van Oers, 1996, 2000). But these semiotically–orientated studies are not in the main directly applicable to the individual student trying to come to grips with mathematical ideas at the post-school level. Indeed much of this latter body of literature derives from studies concerning school or pre–school children negotiating mathematical meanings in classroom or group situations (for example, Walkerdine, 1988; Jones and Thornton, 1993; Maksimov, 1993; van Oers, 2000; Cobb et al, 1997; Gravemeijer et al, 2000; Radford, 2000, 2001)[10].

I interrogate the mathematics education literature, referred to above, in Chapter 3. This literature both informs and enriches the delineation of both my empirical field (undergraduate mathematical learners) and my theoretical framework (semiotic mediation in the mathematical domain).

A further but significant motivation for my research focus is a desire to contribute directly to Vygotskian theory. As alluded to above, and as I discuss in much detail in Chapter 2, I use Vygotsky's (1986, 1994) notion of the functional use of a concept and his allied theory of concept formation to explain how the individual learns mathematics. Given that there is not much literature in the Vygotskian mould which focuses on the construction of a mathematical concept *by an individual* rather than a group of learners or a dyad (Van der Veer and Valsiner, 1994), I hope to contribute directly to Vygotskian scholarship. Furthermore, given that so much theory in the mathematics education world has been generated and elaborated on using various Vygotskian notions, the development of a framework compatible with these mathematics educational theories, but whose focus is the (socially-constituted) individual rather than a group of learners, seems apposite and worthwhile.

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[10] Important exceptions to this are Sierpinska (1993, 1994) who explicates aspects of Vygotsky's theory of concept formation with illustrations from the mathematical domain; Sfard (2000) and Dörfler (2000) who both focus on the individual learner, and Sierpinska (2000) who focuses on undergraduate mathematical learning in a technological environment.

§1.3 THE CONTEXT OF MY STUDY

In this thesis I will be focusing on the mathematical thinking of specific Mathematics I Major students at the University of the Witwatersrand in South Africa. I will be analysing this thinking (or, as I shall argue in Chapter 2, the students' usage of mathematical signs) as it manifests while these students perform particular mathematical tasks in a clinical-interview type situation.

Since the contexts within which the students operate and learn mathematics (for example, their home and school backgrounds, the pedagogic style of the course and the lecture room) shape and profoundly affect the mathematical thinking of the individual students, I will briefly describe certain aspects of that context here. I will then return to relevant aspects and details throughout the thesis, as required. In particular, I will describe my method of selection of the individual students whom I have interviewed, plus pertinent details about these students, in Chapters 5 and 6.

As indicated above, my study is set within the milieu of a first-year (freshman) mathematics course (Mathematics I Major) at the University of the Witwatersrand. The University of the Witwatersrand is a major research-orientated South African institution which draws its students from diverse socio-economic backgrounds and a wide range of high schools[11]. For example, some students come from schools which for the last several years have had close to 100% matriculation pass rate; others come from schools where the overall pass rate at matriculation level over the last few

[11] For an excellent description of schools and schooling in post–apartheid South Africa see Chapter 2 in Adler (2001).

years[12] is less than 60%.

In order to enrol in the Mathematics I Major course, students have to obtain at least a C symbol (ie at least 60%) for the matriculation Higher Grade mathematics examination (or equivalent). Students who do not obtain such a minimum mark can do other first-year mathematics courses in the Science Faculty although Mathematics I Major is the only course (excluding mathematics courses for engineering students) which allows students to proceed to a second year of mathematics. The Mathematics I Major course is intended both for students who wish to become professional mathematicians or high school mathematics teachers and for students who need to complete the course as a co-requisite to other courses in the Science Faculty such as Physics or Computer Science[13]. Students who are studying the biological sciences do not generally do the Mathematics I

[12] As a quantitative illustration of the extreme differences between schools (largely as a result of the legacy of apartheid) the reader has only to consider the following shocking statistic: in 2000, of the 19300 students who *passed* the matriculation Higher Grade mathematics examination in South Africa only 3128 (16%) were African. Of the 38500 students who *wrote* the matriculation Higher Grade mathematics examination, 20243 were African. Hence 15% of African students who wrote the matriculation Higher Grade mathematics examination passed, whereas for Whites, Coloureds and Indians, this percentage was 89%. (Department of Education, 2001). Please note: the terms ‘African’. ‘Coloured’, ‘Indian’ and ‘White’ are apartheid–constructed terms used to name apartheid–defined groups. Although the usage of these terms is generally undesirable, I use these terms when they contribute significantly to the clarification of the context of my study, which is, of course, largely informed by the after–effects of apartheid. The Department of Education uses these terms in a similar way.

[13] As an indication of this range of students in the Mathematics I Major course, consider the following statistics based on responses to two surveys I conducted in 2000. At the beginning of the year, 24.4% of Mathematics I Major students said that the major reason that they were studying Mathematics I Major was because it was a co–requisite for other subjects (such as Computer Science, Physics, and so). By the end of the year, this percentage had increased to 35.2% (an 11% increase). Thus most students (75.6%) initially chose to do mathematics because they liked it, regarded it as a challenge or thought it was a useful subject for career purposes; by the end of the year, this percentage had dropped to 64.8%, in itself an interesting statistic warranting further research.

Major course; they do a less theoretical, more skill-orientated first-year course and they cannot proceed to a second year of mathematics.

Over the last few years, and particularly with regard to the first and second-year level mathematics major courses, the School of Mathematics at the University of the Witwatersrand has veered towards a traditional and conservative approach to the learning of mathematics. For example, most pedagogic practices introduced by the Calculus Reform movement, such as the use of technology, group-work, exploratory approaches to content and long-term projects are heavily discouraged and attempts to introduce such innovations have, in recent years, been met with much resistance by the executive committee of the School. (For example, in the high-status mathematics examinations for engineering students, the use of scientific calculators is banned.) Given this environment, the textbook (Larson, Hostetler and Edwards; 1998) which we use in the Mathematics I Major course has surprisingly many features advocated by the Calculus Reform Movement: for example, multiple representations of mathematical objects are presented in the textbook as are real-life applications of many mathematical concepts. But we use the textbook in a traditional way: inter alia, students are not allowed to use technology such as graphic calculators or computers in problem–solving or in examinations, and long-term group projects are not considered acceptable components of the course.

As with many first year courses in the developed world, the curriculum is comprised of two components, Calculus and Linear Algebra (Harel and Trgalová, 1996). Notwithstanding this composition, in this thesis I will only be looking at the learning of concepts from the Calculus curriculum.

In 2000 (the year of my study) there were two Mathematics I Major classes, each on a different slot on the time-table, but both of which covered the same curriculum and wrote the same Calculus and Algebra tests and examinations. The reason for two slots was that students who could not attend the set of lectures in one slot (because of timetable clashes or because they did not like the lecturer) were able to attend the lectures on the other slot. In 2000 I lectured Calculus to one slot. The students whom I chose for my research study were all from the other slot although they were all in my tutorial group.

Hopefully this broad outline of aspects of the Mathematics I Major course at the University of the Witwatersrand, and my role in teaching the course, is sufficient to provide the reader with some meaningful background to my study. As mentioned previously I will describe other aspects of the context as required.

§1.4 OUTLINE OF REPORT

In the current chapter, I indicated that my primary research focus is on how mathematical students at an undergraduate level appropriate mathematical objects which are new to them but which are already part of the official academic discourse. In particular, I specified that my concern is with how first–year undergraduate students majoring in mathematics at the University of the Witwatersrand in South Africa learn to use mathematical signs so that this use is both personally meaningful and also meaningful to the broader mathematical community. Furthermore, I indicated that my interest is with the early stages of object appropriation, prior to the student's reification of the mathematical process into an object (Sfard, 1994).

In order to develop this research focus, I will need to clarify my theoretical assumptions about the nature of mathematics and my theoretical assumptions about how we learn. This I will do in Chapter 2. In that chapter I will also explain why I regard an elaboration of Vygotsky's theory of concept formation as desirable and appropriate to an examination of the appropriation of mathematical objects at tertiary level.

Since my research will draw on other mathematics education research concerning the relationship between mathematical signs and meanings as well as research around mathematical concept construction at tertiary level, I will critically review this literature in Chapter 3. Furthermore I will indicate which themes I will use in my account and I will draw attention to significant gaps in the literature concerning mathematical object appropriation at the tertiary level.

Having made the necessary theoretical clarifications, I will restate my research question in Chapter 4 in more focussed way than I have done in this introductory chapter. In particular, I will signal that I intend to develop an elaborated language of description (Brown and Dowling: 1998) with which to talk about the appropriation of a mathematical object by a learner and that I will name my elaborations of Vygotsky's theory to the mathematical domain, 'appropriation theory'.

I will then look at how I can use my theoretical expositions (articulated in Chapters 2, 3 and 4) to interrogate the empirical field (Brown and Dowling; 1998) in Chapter 5. That is, I will indicate how I plan to develop appropriation theory. This discussion constitutes the methodology of my account.

In accordance with the methodology I will set about developing appropriation theory in Chapter 6. To do this I will first forge a link between the theoretical and empirical fields of my thesis by developing both my theoretical concept variables and their empirical indicators.

I will use these indicator variables to analyse (in much detail) the protocols of two very different students in Chapter 7. These protocols will derive from the mathematical activities of each student as he goes about appropriating a mathematical object (the improper integral) while engaging on a mathematical task in an interview setting. In Chapter 8 I will discuss these analyses, paying particular attention to how they inform appropriation theory.

In Chapter 9 I will discuss issues surrounding the descriptive, interpretative and theoretical validity (Maxwell; 1992) and applicability of my account as well as its reliability.

Finally, in Chapter 10 I will draw conclusions from my research about the usefulness and illuminating power of appropriation theory to an understanding of the nature of mathematical learning and I will signal some pedagogical implications of my research and the need for further investigations.

CHAPTER 2: THEORETICAL BACKGROUND

In this chapter, I wish to focus on some of the underlying assumptions, terms and themes which inform or underpin my research into the problem of mathematical object appropriation at tertiary level.

- First, I wish to explicate my position with regard to various basic issues concerning the epistemology and ontology of that body of knowledge we call mathematics.
- Secondly, I wish to discuss why I regard an elaboration of a Vygotskian theory of concept formation as desirable and apposite to an examination of the appropriation of mathematical objects at tertiary level.
- Thirdly I wish to discuss my conception of the zone of proximal development (ZPD) and how this relates to the appropriation of knowledge. In this regard, I argue that the appropriation of a mathematical object necessarily involves the personal *construction* of a mathematical concept. Furthermore, through this construction the social mathematical idea is *internalised* for the individual learner.
- Fourthly I wish to undertake some lexical explications. In particular I want to elucidate my use of potentially ambiguous terms such as sign, symbol, signifier, mathematical object, mathematical concept, and so on, which appear throughout this thesis. The meaning of some of these terms is problematic in that they have been used in different ways over the centuries and their usage has often depended on the ontological and ideological assumptions of their users. Consistent with Vygotsky's maxim that the meaning of a word derives from its functional use in communication (elaborated on in §2.2.3.1), I will clarify the meaning of these terms as I use them in this chapter.

§2.1 THE NATURE OF MATHEMATICS

Questions about the nature of mathematics have challenged philosophers and mathematicians over the centuries (for example, Plato, 5 BC; Kant, 1724–1804; Frege, 1848–1925; Russell, 1872–1970). Some of the questions that have baffled and intrigued these philosophers and mathematicians are: Do mathematical objects[1], such as numbers, functions, derivatives, limits, etc., exist independently of the human mind? If so, are they just waiting to be discovered by human beings? Alternatively are mathematical objects pure creations of the human mind? If so, how are they constructed?

Given that these questions have been the source of so much debate and controversy over the centuries and given that an in–depth consideration of such philosophical issues is way beyond the remit of this report, I merely want to outline the major responses to these questions, and to indicate which of these positions I assume in my research. My outline derives largely from Sfard (2000) and Rotman (1993, 2000).

In summary, there are three major schools of thought each of which articulates a different position concerning the nature of mathematical objects (their ontology) and how we know them (their epistemology). Following on from this, and of particular interest to me, each school implies or explicates a different relationship between symbols, words and other inscriptions (the signifier) and their meaning (the signified).

The traditional view concerning the nature of mathematics, which is called Platonism or metaphysical realism, subscribes to the position that mathematical objects exist independently of the human mind and are just waiting to be ‘discovered’ by human beings. According to this view, which was originally propagated by Plato (fifth century BC), but given its modern formulation by Frege (mid–nineteenth century), mathematical objects and their properties exist independently of language and conversations about the objects. Although language may be used to communicate about the objects and to make them personally meaningful, the actual nature and

[1] As previously stated, I use the term ‘object’ in its metaphorical sense (Sfard, 2000). This use is no way reflects any particular commitment to the ontological status of a mathematical object.

properties of the mathematical objects and their relationship to other mathematical objects are independent of any talk about them. Rotman (1993: 19) summarises this position thus:

> Mathematical languages refer to and make sense of these objects and generate assertions about their properties and interconnections, but in no sense can any aspect of mathematical discourse impinge on their constitution, origin, nature, properties, or existential status.

According to the Platonic view, once mathematical objects are discovered, they are named and words and symbols are created to point (or refer) to these mathematical objects. Thus meaning derives from the mathematical objects themselves and is independent of the usage of the related symbols and language. Significantly this means that meaning (the signified) exists prior to the symbol or word (the signifier).

An alternate view holds that mathematical objects are creations of the human mind. According to this view which was developed by the intuitionists led by Jan Brouwer in the early twentieth century, mathematical objects only come into being after they are constructed in the human mind. Most significantly this construction is deemed to be unconstrained by any particular use of language or symbols.

This school of thought (and other forms of intuitionism which later developed from it) rejects the a priori existence of mathematical objects which Platonism propagates, but, like Platonism, it does not admit a linguistic dimension to the construction of mathematics. Rotman (1993: 23) explains that Brouwer’s argument was that the meaning of a mathematical concept was shared by all individuals not because of language or communication but because all people operate with the same Kantian categories. Thus “what I intuit so must you, regardless of how you or I might express, describe, symbolise or articulate such intuitions” (Rotman, ibid.). Accordingly, the position of the intuitionist school is that meaning is independent of symbols and language and that meanings may even develop and evolve prior to the introduction of any kind of symbol or word.

A third view of the nature of mathematics has evolved during the twentieth century and is based on the writings of semioticians such as Peirce (1839–1914) and Saussure (1857–1913). Although these semiotic accounts and their derivative versions are very different in their formulations and details, the essential basis of all these explications is the claim that the sign and its meaning form a unity.

Before proceeding with an account of the semiotic position, I need to clarify how I intend to use the terms sign, signifier and signified throughout this thesis. These terms have all been used in different ways by various authors and thus the potential for confusion is great.

Like Sfard (2000), I will use these words in the sense similar to that of Saussure. According to Deely (1990), a sign in the Saussurean sense is anything that tells us about something other than itself. Furthermore, an essential aspect of a sign is that it is experienced meaningfully. For example, a green traffic light is a sign that tells one to go; it is not there to make one think of greenness. In the phrase '*a*=*b*', '=' is a sign which tells us that *a* and *b* are equal; it is not there to make us think of the shape '–' or the combination of shapes '=' . The word, cat, is a sign which stands for the concept of catness. It is not there to make us think of a written inscription with shapes 'c', 'a' and 't'.

All signs have two parts: a signifier which is a perceptually accessible image (an inscription, a flag, a written word, and so on) and a signified which is an idea or meaning or thought coupled to that signifier. Examples of mathematical signifiers are symbols, words, graphs, notations, diagrams and so on. Examples of mathematical signifieds are the idea of a derivative, the idea of a limit, the idea of a function, the idea of a rectangle and so on.

In this thesis I will use the term 'concept' to refer to the mental idea of a (possibly metaphorical) object.

Continuing with my account of the semiotic position: Sfard (2000: 43–47) neatly summarises the change in the conception of the relationship between the sign (the symbol or word) and its meaning (the signified) that occurs from the Platonic and constructivist theories to the semiotic accounts: ”The conception of a sign and its meaning as independent entities was replaced with the claim of an indissoluble unity of the two” (op.cit.: 44).

Semiotic theories of meaning have underpinned several recent accounts concerning the learning or development of mathematical thinking. For example, Walkerdine (1988), Vile and Lerman (1996), Cobb et al (1997), Presmeg (1997, 1998), Gravemeijer et al (2000), Radford (2000, 2001), Sfard (2000), van Oers (1996, 2000). In Chapter 3 I examine this literature.

A particularly useful formulation of mathematics as a semiotic activity is given by Rotman (1993). He claims that mathematics is a language[2], that is, a particular mode of discourse. He argues most poetically (p. 25) that mathematics is a written discourse in which symbols and other signifiers are manipulated in various ways according to a large and complex set of rules:

> (The discourse of mathematics is) a business of making and remaking permanent inscriptions – symbols, figures, notations, graphs, marks, diagrams, equations – written down on paper, blackboards, and screens and manipulated – that is, operated upon, transformed, indexed, amalgamated, arrayed, rearranged, juxtaposed, sequenced, and ordered – according to a vast, highly developed and complex body of rules and procedures.

Essentially Rotman’s (1993) claim is that mathematical thinking and mathematical inscriptions are “co–creative and mutually originative” (p. 33). In semiotic terms, this means that the signified (the thought or meaning) and the signifier (the written inscription) are mutually constitutive. Neither can exist without the other and both evolve with each other.

> No account of mathematical practice that ignores the signifier-driven aspects of that activity can be acceptable. It is simply not plausible – either historically or conceptually – to ignore the way notational systems, structures and assignments of

[2] Rotman (1993) argues that mathematics is not a natural language. This is because it is self–consciously produced both as an instrument of technology and as an intellectual exercise in abstract reasoning and truth–production.

> names, syntactical rules, diagrams and modes of representation are constitutive of the very ‘prior’ signifieds they are supposedly describing. The upshot [is] that symbol and idea, writing and thinking, signifier and signified have to be understood as co-creative and mutually originative. (Rotman: 33)

Rotman (1993) eloquently disputes the notion that mathematicians construct mathematical thoughts and objects in their heads independently of the symbols and words that already exist in the social world, a notion which is prevalent amongst many mathematicians today. He argues that these mathematicians fail to realise that their mental constructions are only possible because of the existence of signifiers which are coupled with mathematical objects whose use is already sanctioned by the mathematical community.

> Certainly, mathematicians write down what they take to be prior meanings, but they can only realize and construct such intersubjective signifieds in relation to written signifiers already sanctioned by mathematical discourse, which in turn can only be thought in relation to prior signifieds, . . . (Rotman, ibid).

I claim that Rotman’s (ibid) framework is compatible with a position whereby the rules, procedures, signifiers and objects of official (rather than still–to–be–created) mathematics are socially and historically legitimated and that the mathematical signs derive their meaning from the particular meaning already given to them by the community of mathematicians in mathematical discourse. I justify my assertion about these compatibilities with two related arguments: Rotman’s view is that mathematics is a language whose signifiers are manipulated in ways which are “sanctioned by mathematical discourse” (see above quotation). But discourse is a social phenomenon and hence its sanction is necessarily social and/or historical. Furthermore Rotman talks of “intersubjective signifieds” (see above quotation again) implicitly acknowledging that the meaning of various mathematical objects is a social meaning.

Finally the reader should note that my usage of the term ‘sign’ is compatible with Vygotsky’s usage of this term. I will be discussing Vygotsky’s notion of a sign and how it mediates in the appropriation of mathematical objects in much more detail in §2.2.2. Furthermore, although Vygotsky was not directing his attention to mathematics specifically, Vygotsky considered language and symbols to be constitutive of meaning, rather than just representative of it: “Thought is not merely translated in words; it comes into existence through them” (cited in Sfard, 2000: 45).

§2.2 VYGOTSKIAN THEMES

In order to discuss why I regard an elaboration of a Vygotskian theory of concept formation as suited to an explication of mathematical object appropriation[3] at tertiary level, I need to refer to several key Vygotskian themes. These are the themes which I use as theoretical tools in my examination of the development of specific mathematical concepts by various students, all of whom have different social biographies and different prior learning experiences (see Chapter 6 and 7).

My goals in this section are fourfold:

- to provide a critical overview of the key Vygotskian themes which I will embrace in my elaborated theory of how a student appropriates a mathematical object.
- to explain why these themes resonate with my observations and experiences of undergraduate mathematics students as they go about using various mathematical signs.
- to indicate how I plan to apply these themes to the learning of mathematics at a tertiary level.
- to show how the different Vygotskian themes interrelate to give a unified and coherent theory of concept formation.

Specifically the themes that I wish to highlight and explicate are: (1) the relationship between the individual and the social; (2) the central role that semiotic mediation plays in the communication and construction of mathematical ideas; (3) the different phases that a learner goes through as

[3] The appropriation of a mathematical object by an individual necessarily involves the formation (or construction) of a concept. See §2.3 for further discussion on this point.

he constructs a concept which is both personally meaningful and whose use is congruent with that of the relevant social community (for example, the community of mathematicians). Also, and as an extension to (3), I wish to elaborate on the use of the pseudoconcept as a connecting link between a personally meaningful concept and the socially codified meaning of a concept (for example, in mathematics).

Whereas the use of both (1) and (2) as theoretical tools for explicating the learning of mathematics have been recognised, elaborated on and applied to greater or lesser extents (Radford, 2000, 2001; Vile and Lerman, 1996; Bartolini Bussi, 1998; Bartolini Bussi and Mariotti, 1998), Vygotsky's theory of concept formation has, to my knowledge, never been applied to the learning of mathematics at an advanced level. Thus I shall spend rather more time and space on (3).

Furthermore, Vygotsky's account of the formation of a concept (as explicated in Chapter 5 of Thought and Language) is controversial in the Vygotsky–orientated community. It has been criticised by post–Vygotskians such as Wertsch (1991) and Wertsch and Tulviste (1996) for its focus on intramental processes and its lack of regard for the intermental plane or the institutional setting. Within the domain of mathematics education, Confrey (1995: 40) claims that Vygotsky's theory of concept formation "is probably the arena of his research program most in need of modification". In contrast, my position, which I argue in this chapter is that the theory, by virtue of its basic assumptions, lends itself to an interpretation which does address the intermental aspects of knowledge construction (§2.2.3.4) and that it can be extended to deal with the construction of one concept via the mediation of another as occurs in the institutionalised academic setting (see §2.2.3.3). Indeed my interpretation and elaboration of this theory and its application to the construction of a mathematical concept by a learner is the intended contribution of this thesis to the mathematics education literature.

Before proceeding I wish to remind the reader that, besides my claim (which I argue below) of the appositeness and relevance of Vygotsky's account to an elaboration of the processes whereby an individual appropriates a mathematical object, an important motivation for my use of a Vygotskian framework is a distinct absence of similar studies in the mathematics education literature. To this extent, Van der Veer and Valsiner (1994), the well–respected interpreters and translators of Vygotsky, claim that the use of Vygotsky in the West has been highly selective. In particular they note that "*the focus on the individual developing person* which Vygotsky clearly had … has been persistently overlooked" (p. 6; italics in original). Hopefully my thesis will contribute to rectifying this situation.

Throughout this section, I have assumed that the reader is knowledgeable about the basic tenets of Vygotsky's theory. Accordingly I have only highlighted those themes which are particularly relevant to my purposes, and I have sometimes skipped over details, and indeed large sections of Vygotsky's theory. An exploration of the major themes in Vygotsky, interesting as that may be, is beyond the scope of this thesis.

§2.2.1 THE RELATIONSHIP BETWEEN THE INDIVIDUAL AND THE SOCIAL

In the twentieth century, there have been many attempts to find a way of linking social analyses with psychological analyses (Daniels, 1996: 4). The Vygotskian framework has allowed for such a link in that it places the relationship between the individual and the social at its core.

Vygotsky regarded the individual as being both biologically and socially constituted and he stated this quite explicitly:

> The idea (is) that human behaviour in its present form is not only the product of biological evolution, which has resulted in the creation of a human type with all its existing psycho–physiological functions, but is equally a product of a historical development of behaviour or cultural development. (1994 :193)

Vygotsky further expanded on the mechanism whereby the development of an individual is shaped by the socio–cultural conditions in his "general genetic law of cultural development":

> Any function in the child's cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears

> between people as an interpsychological category, and then within the child as an intrapsychological category….. Social relations or relations among people genetically underlie all higher functions and their relationships. (1981: 163)

In other words, all higher–order mental functions of the individual such as goal–directed behaviour, logical memory, voluntary attention and the formation of concepts are acquired through the mastery and internalisation of social processes.

Significantly this internalisation is not a direct mapping of the social to the individual: the social processes evolve as they become internalised. As Vygotsky (1981: 163) himself stated: “It goes without saying that internalisation transforms the process itself and changes its structure and functions”.

At this juncture it is necessary to spend a moment considering the meaning of internalisation since my study is concerned with how different students appropriate to a greater or lesser extent mathematical ideas which exist in the social world (on the chalkboard, in textbooks, in the minds and activities of their lecturers and fellow students) and internalise them (that is, make them their own). To this extent, Leont’ev (1981: 55), a student and collaborator of Vygotsky, offers a useful description of internalisation. He stresses that the social processes are mutated and developed by the individual, not just absorbed in their original form:

> Internalization is the term applied to the transition that results in the conversion of external processes with external material objects into processes carried out on the mental plane, on the plane of consciousness. In the transition these processes often undergo specific transformations – they are generalized, verbalised, abbreviated; most importantly, they can be developed further.

Before proceeding further with my discussion of key Vygotskian themes I wish to state why I find a theory which explicitly links the external (ie the social) with the internal (ie the individual) intellectually appealing and relevant to a study of undergraduate students at a tertiary institution (University of Witwatersrand) in South Africa.

In South Africa, many of apartheid's effects have been institutionalised and, despite some private and government sector attempts to redress imbalances and to promote equity, there are still very large differences in the cultural and social experiences and educational opportunities of different groups of people (these groups are largely distinguished by race). For example, many African[4] students come from under–resourced schools where there are few books and few adequately qualified mathematics teachers. As a result the knowledge base and strategies for learning of these students is very different from those of students who come from well–resourced schools with libraries of books and well–qualified teachers[5]. Furthermore, English is not the main language of many of the African students and this contributes enormously to difficulties these students experience as they study mathematics. (For an interesting discussion of the sorts of linguistic difficulties such students experience as they study mathematics at tertiary level, see Bohlmann, 2001). In these circumstances it is very difficult to try and apply a theory which ignores or at most regards the social dimension as an *influence* on the development of that individual. What is required[6] is a theory which explicitly acknowledges the dual constitution of the individual, the biological and the social. Or to put in Leont'ev's terms, a theory which directly addresses the notion that "if we removed human activity from the system of social relationships and social life, it would not exist." (1981: 47). As discussed above, Vygotsky's theory is such a theory.

At this point, it is also pertinent to note that for analytic purposes I have foregrounded the individual rather than the social in my research study.

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[4] See Chapter 1, Footnote 12, for an explanation of how I use the term 'African'.

[5] In the well–resourced schools (primarily urban or suburban schools) most teachers have a degree in mathematics and a professional teaching qualification. In the under–resourced schools (primarily non–urban or township schools) most teachers have a diploma, with limited tertiary exposure to mathematics.

[6] My argument is that such a theory is required in any circumstance but that its value is particularly obvious where there are marked social differences between groups of people.

This foregrounding of the individual is an analytic strategy that I employ, not a theoretical stance. To understand my motivation for such a move, I need to employ an analogy.

Consider the task of a specialist medical doctor such as a cardiologist. The focus of such a doctor is the heart. At medical school, the cardiologist–in–training will have dissected disembodied hearts, compared disembodied hearts and so on. In her practice, the doctor will again foreground the heart in any clinical examination, listening to it, giving electrocardiograph tests and so on. This does not mean that the cardiologist regards the other organs in the body and indeed their relationship with the heart as irrelevant or merely an influence on the heart of the patient. Indeed the cardiologist will be very aware that the heart is part of the whole and that what happens elsewhere in the body (for example, if the kidneys or lungs fail) is crucial to the health of the heart. But the focus of this cardiologist is the heart and a focus on the whole body or even on the relationship of the heart to all the other organs, would inevitably result in a less detailed and less in depth account of what was happening to the heart itself.

Analogous to this and desirous of an in–depth look at what happens as particular individuals appropriate a specific mathematical object, I will focus on the individual functioning in a particular mathematical environment (while cognisant of the social circumstances surrounding both the individual and the context).

§2.2.2 THE MECHANISM OF SEMIOTIC MEDIATION

Vygotsky (1978) considered all higher human mental functions as products of mediated activity: the role of the mediator is played by a psychological tool or sign, such as words, graphs, algebra, or a physical tool, such as a hammer. These forms of mediation which are themselves products of the socio-historical context, do not just facilitate activity; they define and shape inner processes. Thus Vygotsky saw action mediated by signs as the fundamental mechanism which links the external social world to the internal human mental processes and he argued that it is “by mastering semiotically mediated processes and categories in social interaction that human consciousness is formed in the individual" (Wertsch and Stone, 1985: 166).

Considering Ernest’s (1997) description of mathematics as the “quintessential study of abstract sign systems” and mathematics education as “the study of how persons come to master and use these systems” (ibid.) it seems to me that a framework which postulates semiotic mediation as the mechanism whereby learning takes place is well–suited for explicating the learning of a mathematical concept.

Within mathematics and within the Vygotskian framework, Radford (2001: 241) describes the dual nature of signs as follows :

> On the one hand, they function as tools allowing the individual to engage in cognitive praxis. On the other hand, they are part of those systems transcending the individual and through which a social reality is objectified.

In advanced mathematics learning, where there are no Cuisenaire rods or Dienes blocks or other such concrete objects to work with, it is easy to see how it is all “written in the signs” [7]. (I am also assuming a context in which there is an absence of technological resources such as occurs in the University of the Witwatersrand Mathematics I Major course.) Here a new mathematical object is frequently introduced via definition and there are no physical or even graphical or procedural objects with which to work. Thus the student’s only access to a new mathematical object (the signified) is through its signifier (the symbols and words written on the page), signifiers which usually bear no perceptual relationship to the mathematical object they signify (ie the symbols do not usually look like, sound like or in any way perceptually resemble the mathematical objects they signify).

For example, in a fairly standard undergraduate Algebra textbook, the minor and cofactor of a square matrix are defined, prior to the student

[7] Of course this is also true in elementary and high school mathematics when the mathematics is not mediated by concrete objects.

knowing how to compute a 3 x 3 or larger determinant of a matrix, as follows:

> Suppose A is an n x n matrix and it has been specified how to compute determinants of (n –1) x (n –1) matrices. Then the **(i,j) – minor** of A, denoted $M_{ij}(A)$, is defined to be the determinant of the (n –1) x (n –1) matrix formed from A by deleting row i and column j. Next the **(i,j) – cofactor** of A, denoted $C_{ij}(A)$, is defined by
>
> $$C_{ij}(A) = (-1)^{i+j} M_{ij}(A).$$
>
> Clearly $C_{ij}(A)$ equals either $M_{ij}(A)$ or $-M_{ij}(A)$, depending on the choice of i and j. The number $(-1)^{i+j}$ is called the **sign** of the (i,j) position. (Nicholson, 1993: 115, 116)

This example illustrates what the undergraduate student frequently has to cope with: a plethora of signifiers with their interrelated meanings, referring to mathematical objects some of which may not yet have any meaning for the particular student.

Thus the ways in which a student relates to and gives meaning to the signifiers are of primary significance in understanding how a learner appropriates mathematical objects for himself and this issue is a central focus of my study.

More generally, when a student is working with a mathematical object, the student's activities largely revolve around the deliberate manipulations and considerations of different signifiers. The importance of such motivated symbolic manipulations for meaning–making in mathematics is argued by several mathematics education researchers (for example, Pimm, 1995; Dörfler, 2000; Sfard, 2000). In Chapter 3 I discuss some of this literature further.

Wertsch (1998: 28, 29) gives a illuminating example of how semiotic mediation is central to the doing of even elementary mathematics. He asks the reader to solve the following multiplication problem:

$$\begin{array}{r} 343 \\ \times\ \underline{822} \end{array}$$

He then asks the reader to explain how he or she got an answer. Wertsch suggests that the reader would say “I just multiplied 343 by 822!” and that such a reader may then demonstrate the calculations as shown below:

```
   343
 x 822
   686
  686
2744
281946
```

Now Wertsch asks if the reader would be able to solve the problem if he or she were asked to multiply 343 by 822 without using the form of the vertical array given above. Wertsch suggests that such a reader would be unable to do so (particularly if the numbers were even larger).

So Wertsch’s argument is that the semiotic mediation afforded by the syntax of the vertical array spatial organisation shown above is crucial to the doing of the large–number multiplication.

Wertsch (p.33) also points out that the use of a calculator (a tool for mediation), or the use of Roman rather than Arabic numerals (signifiers as mediators) would also substantially change the way a reader approached the “same” problem.

The above few examples are intended to illustrate that the way in which the signifiers (ie the symbols and words) of mathematics mediate between the mathematical knowledge as it exists in the social world and the individual’s construction of a personally meaningful signified (ie the concept) is a central problem in mathematics education.

My argument about the central role that signs play in the learning and doing of mathematics is supported by the fact that other researchers in advanced mathematics education are currently problematising the same issue, albeit using different psychological paradigms. For example, researchers such as Hegedus, Tall, and Eisenberg set up a discussion group on Symbolic Cognition in Advanced Mathematics at the Psychology of Mathematics Education (PME) Conference in Utrecht in 2001. One of the aims of this group was an exploration of “the role of symbol in mathematical thought and meaning making" (Hegedus, Tall, and Eisenberg, 2001: 268) Another aim was to discuss "symbolic processing with reference to specific mathematical topics" (ibid.). My claim is that Vygotsky's theory is well poised to address the central role of signs in the individual's construction of a mathematical concept.

In the next section (§2.2.3), I begin explicitly to develop my theory of how a student makes personal meaning of a mathematical object which is new to him but which is already part of the official mathematics discourse; in doing so, I elaborate on my use of Vygotsky's notion of semiotic mediation.

§2.2.3 THE CONSTRUCTION OF A CONCEPT

§2.2.3.1 OVERARCHING NOTIONS

There are several linked and overarching notions in Vygotsky's explication of a child's concept formation (as described in Chapter 5 of Thought and Language) which together and singly resonate with my sense of how a student goes about appropriating a new mathematical object[8].

Before elaborating on these themes, it is important to note how Vygotsky used the term 'word'. He regarded a word as embodying a generalisation and hence a concept and used it as such.

Vygotsky postulated that the child uses a word (in the sense given above) for communication purposes before that child has a fully developed understanding of that word. As a result of this use in communication, the meaning of that word (ie the concept) evolves for the child

> At any age, a concept embodied in a word represents an act of generalization. But word meanings evolve. When a new word has been learned by the child, its development is barely starting; the word at first is a generalization of the most primitive type; as the child's intellect develops it is replaced by generalizations of a higher and higher type – a process that leads in the end to the formation of true

[8] In this thesis my focus is on how a learner appropriates a 'new' mathematical object. It is not on how already personally and socially meaningful usages of mathematical signs are extended.

concepts. (Vygotsky, 1986 : 149)

According to Wertsch and Stone (1985), the basis of Vygotsky's argument was that agreement on reference, as opposed to word meaning, provides an entry point for children to participate in social interaction with older members of the culture (p. 169). To understand what is meant by this, the reader needs to know how Vygotsky distinguished between reference and meaning. This is best illustrated by Vygotsky's example in which he examined the two phrases “the victor at Jena” and “the loser at Waterloo”. He argued that these phrases (the sign) had same person as referent, but that the meaning of the two phrases differed. In a similar way he claimed that children's and adults' words (the signs) coincided in their referents but differed in their meanings.

An example from mathematics are the two different signs: $y = 2x + 1$ and the graph in Figure 1.

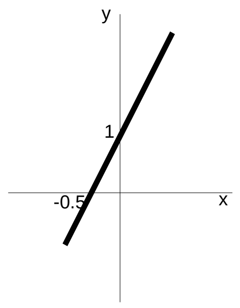

Figure 1

Here the object referred to (ie the referent) is the same in both signs (ie a straight line with gradient 2 and a y–intercept of 1) but the meaning of each sign may be very different for many students. Another mathematical example where we see how meaning may be distinguished from referent is given by the expression: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ Such an expression may have different meanings for, say, an expert and a student although the expression refers to one referent or object (the derivative of f(x)).

The use of a word or sign to refer to an object (real or virtual) prior to ‘full’ understanding resonates with my sense of how undergraduate students make new mathematical objects meaningful to themselves. In practice, the student starts communicating about this new mathematical object with peers, with lecturers or the potential other (when writing), via signifiers before he has full comprehension of the mathematical object[9]. It is this communication about the object that gives an initial access point to the new object. And this communication is only possible through the use of various signs such as symbols and words.

Vygotsky put this most succinctly (1986: 106) :

> It is a functional use of the word, or any other sign, as a means of focusing one's attention, selecting distinctive features and analysing and synthesizing them, that plays a central role in concept formation.

Secondly and closely linked to the above notion is Vygotsky's argument that the child does not spontaneously develop concepts independent of their meaning in the social world. "He does not choose the meaning of his words… The meaning of the words is given to him in his conversations with adults " (Vygotsky, 1986: 122). Vygotsky argued that the social world, with its already established meanings[10] of different words determines the way in which the child's generalisations need to develop.

> Adults, through their verbal communication with the child, are able to predetermine the path of the development of generalizations and its final point – a fully formed concept. (Vygotsky, 1986 : 120)

So the meaning of a concept (as expressed by words) is 'imposed' upon the child and this meaning is not assimilated in a ready–made form. Rather it undergoes substantial development for the child as he uses the word in his communication with more socialised others.

The notion that a word does not have meaning beyond that which has been adopted by the relevant community is certainly applicable to the mathematical domain. As I discussed in §2.1, meaning and communication

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[9] Indeed my argument is that these usages of the mathematical sign necessarily precede the development of its meaning for that student.

[10] For example, as defined in a dictionary.

about mathematical objects is only possible because the use of symbols and signs in mathematical discourse is socially sanctioned. Thus for a student to construct a mathematical concept whose meaning and use is compatible with its meaning and use in the mathematics community, that student needs to make sense for himself of a socially legitimated use[11]. How he does this is the subject of this study.

The two aspects of concept formation discussed above are epistemologically possible because the word or mathematical object can be expressed and communicated via a word or sign whose meaning is already established in the social world. According to Vygotsky, it is the learner's focused use of these signs within the community (for example the mathematics community) even prior to 'full' understanding, that enables the meaning of the word or concept to develop for that learner in a way that is compatible with that of the community.

In summary, the *same* mathematical signs mediate two processes: the development of a mathematical concept in the individual and that individual's interaction with the already codified and socially sanctioned mathematical world. In this way, the individual's mathematical knowledge is both cognitively and socially constituted.

This dual use of a mathematical sign by a learner before 'full' understanding is not well understood by the mathematics education community; indeed, its frequent manifestation in the form of activities such as manipulations, imitations and associations is often regarded disparagingly by many mathematics educators. That is, they regard such activities as 'meaningless' and without worth. Conversely, other mathematics educators may regard adequate use of a mathematical sign as sufficient evidence of a student's understanding of the relevant mathematical concept. Hopefully the theory I develop in this thesis will go some way to illuminating the dual role of the mathematical sign.

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[11] I am not denying that new meanings or creative uses of already established mathematical signs can emerge; however my focus in this thesis is on the enculturation of

§2.2.3.2 THE FORMATION OF CONCEPTS

> The central moment in concept formation, and its generative cause, is a specific use of words[12] as functional tools (Vygotsky, 1986: 107)

At the heart of Vygotsky's theory of concept formation is his view that "language is the most decisive element in systematizing perception" (Luria, 1976: 49). Language gives the tools for formulating abstractions and generalisations and so enables the transition from unmediated sensory reflection to mediated, rational thinking.

Abstraction and generalisation

Since abstraction and generalisation are crucial processes in both advanced mathematical thinking (Dreyfus, 1991) and in Vygotsky's theory of concept formation, I would like to elaborate on the meaning of these terms, with my examples drawn from mathematics.

Abstraction involves the isolating of particular attributes or relationships while ignoring other attributes. For example, the property of continuity of a function can be abstracted from a host of functions such as polynomials, certain transcendental functions and some piecewise functions.

> Achieving this capability to abstract may well be the single most important goal of advanced mathematical education. (Dreyfus, 1991: 34).

These abstracted properties contribute to the construction by the learner of an abstract concept: continuity.

Generalisation refers to the ability to move from a few cases to a large class of cases. Generalisation is the mental activity of forming a mental construct which subsumes a set of objects or concepts all of which have a common set of properties. For example, suppose a student knows about the conditions under which a two–by–two or three–by–three matrix is invertible. If he uses this knowledge to conjecture about the conditions

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the learner into the dominant mathematical culture.

[12] For mathematical concept formation one should replace the word 'words' with 'words or mathematical signs'

under which an n–by–n matrix is invertible, he is generalising. (Of course, many generalisations may be mathematically incorrect which is why proof and verification are such important aspects of mathematical activity.)

The mental construct which results from the process of generalisation is called (by Vygotsky) a generalised category. For example, the generalised category of integers includes numbers such as 3, –4, 8 since they all have certain properties in common (eg addition of these numbers is commutative, addition of any two of these numbers results in another number which has many of the same properties as the original numbers; division of one number by another may yield a number with different properties to the original numbers, and so on); the generalised category of rational numbers includes integers plus other numbers such as 2/7, –3/8, 5½ since they all have certain properties in common (for example: they can all be written in the form p/q where p and q are integers and $q \neq 0$). The generalised category of real numbers includes the rational numbers plus numbers such as $\sqrt{2}, \pi$ since these real numbers all have certain properties in common (they consist of numbers with the properties of rational numbers plus numbers which cannot be written in the form p/q where p and q are integers and $q \neq 0$). So integers, rational numbers, real numbers and so on, are all generalised categories. Clearly there may be several different ways to generalise a particular set of mathematical objects; which way a mathematician chooses to do this, depends on the task and the mathematical context.

In the same way that a word is a generalised category for Vygotsky, so a mathematical sign, for example $\int f(x)dx$ or $\lim_{x \to a} f(x) = f(a)$, may represent a generalised category[13] in mathematics.

Having discussed these terms, I can proceed with my task of explaining how Vygotsky elaborated his theory of concept formation.

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[13] These mathematical signs represent the set of all antiderivatives of f(x), and the set of all the limits of all continuous functions, respectively.

Artificial concepts

Vygotsky observed more than three hundred people (of all ages) doing a particular task in an experimental context. The task was to sort a set of twenty–two blocks of various shapes, heights, colours and sizes into distinct and exclusive categories (the subjects had to uncover for themselves what these categories were). The method of the experiment was known as the method of double stimulation in that each block was connected with two signs, a non–verbal and a verbal sign. The non–verbal signs were the physical properties of the block (eg large, small, blue, red, etc.) and the verbal sign was one of four nonsense words written on the underside of the block. These nonsense words[14] were in fact the categories into which the blocks needed to be sorted. Thus the task required the individual to abstract certain properties of the blocks (eg tall, short, red, green, etc.) and to form generalised categories from these abstracted traits. For example, all objects which were tall and large were supposed to be placed in the generalised category described by the word *lag*; all objects which were flat and large were to be placed in the generalised category described by the word *bik* and so on.

Vygotsky used his observations of how the different individuals seemed to develop meanings for the nonsense words to assert that the principles and operations which the subjects used in generating categories differed from one age group to another.

The different phases in concept formation

Vygotsky claimed that there are several major types of preconceptual thinking (heaps, complexes, pseudoconcepts and potential concepts) each of which roughly corresponds to a different stage of maturation and development of the individual (ontogenesis). According to Kozulin (1990: 159), the different representations (heap, complex, pseudoconcept, potential concept) “should not be mistaken for natural ‘stages’ in a child’s

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[14] The nonsense words were *lag* which meant tall and large, *bik* which meant flat and large, *mur* which meant tall and small and *sev* which meant flat and small.

cognitive development; rather they are methodological devices for distinguishing what seems to be the most pronounced form of concept formation at any given age".

Significantly for my purposes, Kozulin (ibid.) goes on to argue correctly that Vygotsky's position was that these "preconceptual types of representation are retained by older children and adults, who quite often revert to these more 'primitive' forms depending on their interpretation of a given task and on their chosen strategy for solution".

My claim, based on my observations over the years of undergraduate mathematics students encountering a new mathematical object in the form of a set of signifiers, is that an undergraduate mathematics student zigzags through these various preconceptual phases as he constructs a meaningful mathematical concept. The student's path through these phases is not linear; he may go back and forth between the different phases. And each phase may occur for that student in attenuated or extended form, depending on the student, the particular mathematical object, the task and the context.

I will now briefly explain the meaning of each category (heap, complex, pseudoconcept, potential concept) and where relevant, give an example of this form of thinking in the mathematics domain. Although I present some mathematically relevant examples of the different phases here, I will problematise some of the issues raised by the application of Vygotsky's theory of the formation of artificial concepts to the formation of mathematical concepts in §2.2.3.3 and §2.2.3.4 below.

In Chapter 6, I will further develop, illustrate and exemplify these different phases within the mathematical domain. My aim now is limited to an indication of the appositeness and usefulness of Vygotsky's phases as a means with which to interpret how a student goes about appropriating a mathematical object.

During the **syncretic heap** phase, the child groups together objects or ideas which are objectively unrelated. This grouping takes place according to chance, circumstance or subjective impressions in the child's mind.

In the mathematical domain, a student is using heap thinking if he uses the layout of some mathematical text on a page to justify some mathematical property, or if he associates one mathematical sign with another because of their proximity on the page.

The syncretic heap phase gives way to the **complex** phase. In this phase individual objects or ideas (ie elements) are linked in the child's mind by associations or common attributes which exist objectively between the objects or ideas. Complexes are formed according to rules that are different from those of concepts. Most importantly, in a complex the bonds between elements are *associative and factual* whereas in a concept the bonds between the elements are *abstract and logical*. Furthermore in a complex, no attribute is privileged over another whereas in a concept a particular set of attributes is privileged over others.

Complex thinking is crucial to the formation of concepts in that it allows the learner to think in coherent and objective terms and to communicate via words and symbols about a mental entity. Furthermore, in complex thinking, the learner abstracts or isolates different attributes of the objects or ideas, and the learner organises objects or ideas with particular properties into groups thus creating the basis for later generalisations.

Vygotsky distinguished between five different types of complexes.

- In the **associative type of complex**, one object or sign forms the nucleus. When the child notices a similarity between an attribute of the nucleus and an attribute of another object or sign, this new object or sign gets included in the group. Vygotsky, referring to the block–grouping task (discussed above) exemplifies the associative complex as one in which

  > the child may add one block to the nuclear object because it is of the same color, another because it is similar to the nucleus in shape or size, or in any other attribute

that happens to strike him. (Vygotsky, 1986: 114).

In the mathematical arena, the learner chooses one mathematical sign as the nucleus (usually because it is familiar or epistemologically accessible) and associates a less familiar sign with the more familiar sign due to some type of similarity between the signs. In terms of categorisations of ways of thinking, the most important feature of this association is that it is based on concrete or non–logical connections.

An example of complex thinking of the associative type occurs when the student uses a specific instance of a mathematics object (ie an exemplar) as a nucleus or core around which to build a new concept. The learner adjoins other mathematical objects, signs or examples to the nucleus because they have some attribute in common with the nucleus, for example a similar signifier.

This use of an exemplar as the nucleus around which the student tries to build the concept may be enabling or disabling of concept construction.

For example, suppose the student uses the example $\int x^2 dx = {x^3}/{3} + c$ as a nucleus around which to build the concept of the indefinite integral. If the student's focus is on the rule (add 1 to the power of x and divide by this same number when integrating), his concept development would be seriously impeded and he would, for example, state that $\int x^{-1} dx =$ $x^{-1+1}/(-1+1) + c = 1/0 + c$ which is, of course, incorrect.

This type of complex association occurs frequently in mathematics because the mathematical object is not perceptually accessible and the initial access to the object is through its signifier[15]. Accordingly some

[15] This is similar to Sfard's (2000) argument that the student's initial use of a new sign is template–driven, ie the form of the signifier reminds the student of some other similar–looking signifier with which he is already familiar. Accordingly he starts using the new signifier in a way which is similar to his use of the familiar signifier.

learners conflate the signifier with the object and end up isolating perceptual attributes of the signifier rather than attributes of the object.

- In the second type of complex thinking, the **collection complex,** objects are placed in the same group because they complement or contrast with each other in a functional way. For example, a child may group together a knife, a fork and a spoon because they complement each other in terms of their functions.

  In mathematics learning, the collection complex is often manifested by the association of mathematical objects which have opposing rather than complementary functions. For example, first–year students often differentiate rather than integrate, because they associate the differentiation process with its inverse process, integration. Children may multiply rather than divide in a problem such as:

  An insect travels 200 miles in 10 hours. What is its speed?

- In the **chain complex**, an object is included in a group because it shares an attribute with another object in the group. The new object then enters the group with all its attributes and the learner may use any of these attributes to include yet another object with a shared attribute in the group.

  Vygotsky gave the example of a child who, while trying to pick out objects which are in the same category as a yellow triangle, picks out triangular blocks of various colours. At some point this child focuses on the colour of the blue triangle (rather than the shape) and he starts picking out blue shapes which are not necessarily triangles. Vygotsky argued (1986: 116) that the original block has no special significance for the child. "Each link, once included in a chain complex, is as important as the first and may become the magnet for a series of other objects". In this way, the criteria for the selection of new objects are continually in flux.

  In the mathematical domain, the student associates one mathematical sign with another because of some similarity (for example, a shared word or a shared signifier). This new sign (or an aspect or attribute of it) is then linked to yet another sign possibly by virtue of another attribute, thus forming a chain of signs. Furthermore, the chaining may be within the mathematical discourse, or the chaining may be to a sign outside the mathematical discourse. In Chapter 6 I give an example of the former. Here I give an example of the latter, taken from Cornu (1991).

Although operating in a different theoretical framework, Cornu (1991) describes how many students carry on associating certain everyday meanings of the words ‘tends to’ and ‘limit’ with these same terms in the limit concept even after this mathematical notion has been formally defined. He lists some of the different meanings that students latch onto when they see the expression ‘tends towards’ in the mathematical context: to approach, eventually staying away from; to approach, without reaching; to approach, just reaching; to resemble (such as “this blue tends towards violet”) (Cornu, 1991:154).

In my framework, these sorts of activities may be regarded as manifestations of chain thinking. The student decontextualises the mathematical phrase ‘tends to c’ in the mathematical expression and links it to a phrase from everyday language such as ‘tends towards but never actually reaches’.

A chain complex in the Vygotskian sense is not the same as the notion of “chains of signification” (Walkerdine, 1988) although some features are similar. In the “chains of signification” approach (eg Presmeg, 1997, 1998; Cobb et al, 1997; Whitson, 1997) the primary focus is on how a signified in one cultural domain, say home life, gets transformed into a signifier in another cultural domain such as the mathematics classroom so carrying over some of the meanings and affects from one context into another.

In the Vygotskian sense, the signified in one cultural domain is not necessarily transformed into a signifier in the mathematical domain. Indeed the signifier may merely be shifted from one mathematical context to another mathematical context, signifying different signifieds in these different contexts. That is, a sign is chained to another sign *inside* or *outside* the mathematical domain.

Furthermore, a chain complex is just one category of complex thinking in Vygotsky's theory. In the "chains of signification" approach, the chaining of one signified from one context to a new signifier in a different context is a central tool for explaining how the learner makes meaning of particular signs.

- In a **diffuse complex**, the child uses vague or remote similarities (for example a similarity or sameness of signifiers) to make connections between different objects or ideas. For example, in the block–sorting experiment some children placed trapezoids and triangles in the same category, presumably latching on to a vague similarity (a trapezoid is like a triangle with the top cut off). Vygotsky classified this as a manifestation of the diffuse complex.

  In the mathematical domain, the learner uses vague similarities between signs to connect one sign with another. For example, many first–year mathematics students assume that if f(x) is differentiable, then so is $f'(x)$ (my argument is that this connection is based on a similarity of the signs f(x) and $f'(x)$). But this is not a logical or correct connection (for example f(x) = x|x| is differentiable for all $x \in \nabla$, but $f'(x)$ is not differentiable at x = 0).

  The fifth type of complex, the **pseudoconcept,** is a very special type of complex. Vygotsky argued that it is the very existence and use of pseudoconcepts that makes a learner's transition from complexes to concepts possible. Pseudoconcepts resemble true concepts in their use, but the thinking behind these pseudoconcepts is still complex in character. This is because the bonds between the different elements in a pseudoconcept are associative and factual rather than logical and abstract. But the learner is able to use the pseudoconcept in communication and activities *as if* it were a true concept. Vygotsky argued that it is through this functional use of the complex (its use as if it were a concept) that the learner transforms the complex into a concept and the bonds between the different elements of the concept become abstract and logical. I will later argue (see § 2.2.3.4) that the pseudoconcept also acts as a bridge between the intramental and the intermental. The use of pseudoconcepts is ubiquitous in mathematics and is analogous to the way in which a child uses a word in conversation with adults before he fully understands the meaning of that word. Pseudoconcepts occur whenever a student uses a particular mathematical object in a way that coincides with the use of a genuine concept, even though the student has not fully constructed that concept for himself. For example, a student may use the definition of the derivative of a function f(x), ie $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ if it exists, to compute the derivative of the function $f(x) = x^2 + 1$ before he understands the nature of the derivative or its properties.

Finally, I need to mention **potential concepts**. Vygotsky (1986: 135) argued that complex thinking creates the basis for later generalisations in that the learner classifies different objects into groups (or complexes) on the basis of particular characteristics. However classification would not be possible without abstraction of particular characteristics of these objects. Thus the learner engages in abstractions concurrently with complex thinking. While complex thinking predominates, the abstracted property is unstable and transient and has no privileged position over other properties of the object. But as the learner begins to engage in more advanced complex thinking, the abstracted traits become more stable. Vygotsky called the mental formations that result from these abstractions, **potential concepts.**

In Chapter 6, I will argue that in the mathematical domain the abstraction of attributes is so intertwined with the formation of complexes that it is impossible to distinguish potential concepts from most mathematical complex thinking. As such I suggest that potential concepts are not useful analytic constructs in an elaborated theory of object appropriation for the mathematical domain.

For ease of reference I present a summary of Vygotsky's stages of concept formation in diagrammatic form (Figure 2).

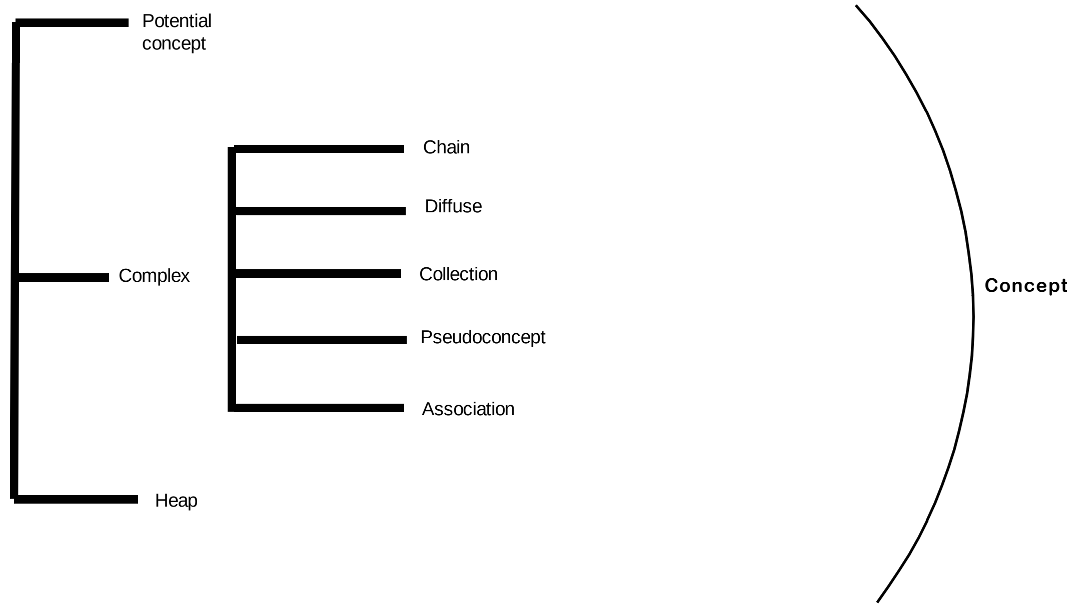

Figure 2: Stages in Vygotsky's theory of concept formation

§2.2.3.3 CONCEPTS FROM CONCEPTS

Vygotsky's experiment related to the way in which individuals develop concepts by abstracting properties of *concrete objects* (for example, one particular block is tall, another particular block is large) and using these abstract properties to generalise to a group of objects (the set of all objects which are large and tall, and so on). But my task is to investigate the learning of advanced mathematics, an activity in which there are no concrete objects, only virtual objects represented by written (or possibly spoken) signifiers. Hence a central theoretical challenge is to apply Vygotsky's theory to the abstraction and generalisation of properties of virtual objects.

A related theoretical challenge is to situate Vygotsky's theory of the formation of artificial concepts in the academic context in which mathematical concepts are formed. Although Vygotsky certainly implied that the phases of concept formation which derived from his experiment with artificial concepts were relevant to the formation of concepts in an academic environment[16], he argued (1986:161) that the latter concepts required special treatment. He called the concepts formed in an academic environment, scientific concepts, and claimed that the construction of these concepts is deliberate, conscious and systematic. Furthermore he argued that scientific concepts are part of a system with its own internal hierarchical system of interrelationships between different concepts. Thus in order to construct a new concept a learner needs to use already internalised concepts (not just concrete objects) as mediating tools (ibid.).

Certainly the use of one or more concepts to mediate in the construction of further concepts is inherent in the learning of mathematics. As Skemp (1986: 24, 25) argued, higher–order concepts are always abstracted from lower–order concepts[17] in mathematics. For example, the numbers 3, –14, 20 are lower–order concepts from which the higher–order concept of integer can be abstracted.

Kozulin (1996: 167) asserts that Vygotsky ”was fully aware that for all its importance the study of artificial concepts had some built–in limitations”. He explains that Vygotsky acknowledged that in his experiment leading to the formation of artificial concepts, the very nature of the sorting test did not allow for the formation of a hierarchical conceptual system. Also since the learner in the task had to discard his generalisations after each incorrect solution, Vygotsky recognised that the experiment was unable to show the building of one concept from another.

In summary, my problem is how to extend Vygotsky’s theory of concept formation (derived from his observation of people working with concrete objects and artificial or contrived words) to a theory which can be applied to

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[16] For example, Vygotsky (1986: 198) referred to the structures of generalisation (syncretism, complex, preconcept and concept proper) as he talked about scientific concepts.

[17] Lower–order concepts are examples of higher–order concepts. For example, red, blue, green are lower–order concepts from which the higher–order concept, ‘colour’ can be abstracted.

people learning about virtual objects and 'real'[18] concepts, where the real concepts are part of an interrelated and hierarchically organised system of concepts.

Wertsch (1985: 103) argues that Vygotsky's claim about the systematic nature of learning in the school environment was "taking an important step in his argument about generalisation, semiotic mechanisms, and word meaning". Wertsch explains that Vygotsky's original account of concept formation (based on his experiment with the formation of artificial concepts) involved a relationship between signs and objects, whereas Vygotsky's later discussion about scientific concepts involved a relationship between different signs.

> Vygotsky's claim about systems of concepts goes into the issue of how signs relate to other signs[19]. Vygotsky believed that at higher levels of development, *both* relationships must be considered in order to provide an adequate account of word meaning[20]." (Wertsch, 1985: 103)

Thus my task is to incorporate the use of 'old' signs as mediators of 'new' signs into an account of how students construct a new mathematical concept. To do this, I appeal to Minick's notion of the word (ie the sign) as the "object of communication activity" (1996: 41) in an academic environment and his account of Vygotsky's conceptualisation of the formation of scientific concepts.

To elaborate: Minick (1996: 41) asserts that Vygotsky's argument re scientific concepts was that the instruction of the child in school involves a unique form of communication in which the word assumes a function which is quite different from that characteristic of other forms of communication. The new function is that the word itself (ie the sign) and its relation to other words (ie signs) become the focus of attention. The reasons for this new

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[18] I mean 'real' (as opposed to artificial) in the sense that these words (ie concepts) are part of the mathematical discourse of academic environments.

[19] I interpret this as the issue of how concepts relate to other concepts.

[20] Word meaning may be interpreted as concept formation.

function of the word are twofold. First the child at school learns word meanings (ie concepts) not only as a means of communication but also as part of a system of knowledge. And this learning occurs not through direct experience with things or phenomena but *through other words* (ie signs or concepts).

> As it is used in these communicative contexts then, the word begins to function not only as a means of communication but as the *object of communication activity*, with the child's attention being directed explicitly toward word meanings and their interrelationships. (ibid; my italics.)

Likewise Wertsch (1985) acknowledges that Vygotsky's situating of the process of concept formation in the school environment was "an important force in the emergence of scientific concepts" (p.103). Wertsch argues that Vygotsky's point was that in the school environment, language is used to talk about language[21]. Thus Wertsch, like Minick, is saying that in the school environment signs are used to talk about other signs. So signs rather than concrete objects, become the focus of the learner's attention. Words (ie signs) are not only used for communication but they themselves and their interrelationships, become the objects of attention.

In this thesis I aim to theorise the appropriation of a mathematical object using analogous notions.

Specifically I claim that the students use mathematics signs and signifiers prior to 'full' understanding not only as a way of communicating their thoughts to their peers or the teacher but also as objects to manipulate, act with and combine in various ways (this is analogous to the use of word as the object of communication activity). Furthermore, this use of the signs is only made possible by the learner's use of related signs or concepts some of which may be called forth by particular attributes of the new sign (this is analogous to the appropriation of a new word or concept via the mediation of an 'old' word or concept). Most significantly, given that in advanced mathematics there are few material or representative objects which the

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[21] Wertsch calls this use of language to talk about language, "decontextualized metalinguistic reflection" (1985: 103)

student can directly perceive, I employ Vygotsky's argument that "the decisive role is played by the word, deliberately used to direct all the subprocesses of advanced concept formation" (Vygotsky, 1986: 139), to argue that mathematical signs (initially in the form of signifiers) serve as foci for the student's intellectual activities, enabling them to abstract and generalise.

§2.2.3.4 INTERMENTAL AND INTRAMENTAL PROCESSES

Vygotsky's theory of concept formation (heaps, complexes, concepts) has been criticised for its de–emphasis of social processes[22].

For example Wertsch, who is one of Vygotsky's major interpreters and proponents in the West, explicitly states that in Chapter 5 of Thought and Language, "concept development is treated primarily in terms of individual psychology" (1991 : 47). Wertsch goes on to argue that it is only later in Vygotsky's life, when Vygotsky specifically situated concept formation in school (scientific concepts) or in the everyday (spontaneous concepts) that Vygotsky's theory started to take on board the intermental aspects.

> "Because he was concerned with how the forms of speaking encountered in the social institution of social schooling provide the framework with which concept development occurs, he focused on the forms of teacher–child intermental functioning in this setting *rather than on children's intramental functioning alone* " (ibid.; my italics).

Wertsch goes on to say that Vygotsky's writing at this stage (Chapter 6 of Thought and Language) was "only a beginning, his writing lacks any detailed explication" (1991: 48).

I agree with Wertsch that Vygotsky's account of social processes in concept formation lacks detail; however I take issue with Wertsch's assertion that Vygotsky's account of concept formation ignores intermental processes. My disagreement with Wertsch is based on two arguments: (1) Vygotsky's overarching notions of concept development (see §2.2.3.1.) make it abundantly clear that Vygotsky regarded social processes as a fundamental aspect of the individual's concept formation (without which such formations

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[22] This is probably a major reason for why it has been overlooked as a useful theory.

could not exist). (2) Vygotsky developed a mechanism (the pseudoconcept) to function as a bridge between intramental and intermental processes.

As evidence for statement (1), Vygotsky argued that the meaning of a word (ie a concept) evolves through its use in communication with adults (or by extension, with a more knowledgeable teacher or peer). Furthermore, he argued that the meaning of a word is not chosen by the child; rather it is a socially codified meaning which the learner needs to appropriate for himself. Thus the intramental processes of concept construction are inextricably linked with the intermental processes.

To elaborate on (2): According to Vygotsky, the transition from complexes to concepts is made possible by the existence and hence use of pseudoconcepts (see §2.2.3.2). As I discussed in §2.2.3.1 (although I did not refer to pseudoconcepts specifically there) Vygotsky (1986) argued that the use of pseudoconcepts enables adults to communicate with children *as if* they were referring to the same concept: “the child’s and the adult’s words coincide in their referents but not in their meanings” (p. 131). This communication (the intermental aspect) is necessary for the transformation of the complex into a genuine concept (the intramental aspect) for the learner.

Vygotsky (1986) argued further that pseudoconcepts are a necessary phase in the child’s or student’s development of true concepts in that “complexes corresponding to word meanings are not spontaneously developed by the child: The lines along which a complex develops are predetermined by the meaning a given word already has in the language of adults” (p. 120). (Again, I discussed this in §2.2.3.1 but without specifically mentioning pseudoconcepts.) Thus the pseudoconcept has a double nature: on the one hand there is the stability and “permanence” of word meanings that are already in the social milieu and that direct the path of the child’s generalisations; on the other hand, the child’s thinking has to proceed along this preordained path in a manner appropriate to the child’s own stage of intellectual development (p. 120).

Thus the pseudoconcept functions as the bridge between concepts whose meaning is more or less fixed and constant in the social world (such as that body of knowledge we call mathematics) and the learner's need to make and shape these concepts so that they become personally meaningful. This bridging function of the pseudoconcept is the basis for my contention that the pseudoconcept can be regarded as the link between intermental functioning (ie that which occurs on the social plane) and intramental functioning (ie that which occurs on the psychological plane).

This notion of the pseudoconcept ties in extremely well with what I have observed over the years in my mathematics classes. Here students grapple with new mathematical objects via their signifiers, using these signifiers to worse or better effect, even before they have much sense of what the object is. At the pseudoconceptual phase (which may be of very short or very long duration) the student is able to use the signifiers (in algorithms, definitions, theorems, problem–solving, and so on) in ways that are commensurate with that of the mathematical community even though the student may not fully 'understand' the signified. Eventually (after a shorter or greater time depending on the student, the concept, the activities and the context) the pseudoconcept gets transformed into a concept.

At this juncture I need to confront the problem of how to elaborate on the nature and detail of the social processes which are essential to the very existence of concept formations. Even if Vygotsky had elaborated on these social processes, I would still need to situate my study in its own social arena (which obviously differs from that of Russia in the 1920's and 1930's).

Minick's (1996) contention that Vygotsky's conceptualisation of the learner's formation of scientific concepts means that "the analysis of the development of word meaning must be carried out in connection with the analysis of the development of the *function of the word*[23] *in communication*"

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[23] As previously stated, in the mathematical context the 'word' may be a symbol or a word, ie a mathematical sign.

(p. 40; my italics) and his related claim that Vygotsky now linked “the emergence of a new type of word meaning to the child’s participation in a new form of social practice” (p. 41) suggests the way forward. That is, in my analyses of students attempting to appropriate a new mathematical object in a clinical interview situation, I will need to pay attention to the function of the mathematical sign for the individual in the broader context of their mathematical studies.

For example, an aspect of the broader context is the mathematical biography of the particular student. To illustrate this, consider a first–year mathematics student at the University of the Witwatersrand who comes from a school where success is measured by performance in tests and examinations. Assume further that this student was moderately successful at school (obtaining a B–symbol for the matriculation Higher Grade mathematics examination) and that he wishes to perform well in tests and examinations at university. Such a student may well be quite satisfied with a pseudoconceptual approach to concept construction. His own measure of success (possibly internalised as a result of his results–oriented schooling) is getting the right answers to problems. Accordingly, his deliberate and mindful efforts at coming to know the new mathematical object (in which he uses memory and attention while engaging in various activities with the new mathematical signs) may well stop when he is able to use the signs correctly (ie in procedurally–oriented exercises, applications and so on). To elaborate further: suppose that the newly introduced mathematical object is the derivative of a function. After a time of deliberate and conscious activities with that object, the student may well be able to compute the derivative (using the definition) of various functions and he may be able to use the derivative to find, say, the minimum dimensions of a cylinder which can hold a fixed volume of cold drink, and so on. But this student may not have a conceptual understanding of the derivative (for example, he may not understand why the derivative of a function measures the rate of change of one variable with respect to another, or why the derivative of this function at a particular point gives the slope of the curve at that point). Indeed such a scenario would indicate that the student has a pseudoconceptual approach. Furthermore, the lecturer (for her own reasons) may well collude with that student by setting exercises and examinations which ask questions which do not test conceptual knowledge but rather test pseudoconceptual knowledge (for example standard questions testing applications and use of algorithms) so prolonging this pseudoconceptual phase (often indefinitely).

Other aspects of the wider context may include the primary language of the student, the social relations between the lecturer and student in the teaching–learning environment, the goals and motives of the student and affective dimensions of that student.

§2.3 KNOWLEDGE APPROPRIATION

In this section I would like to outline my position briefly with regard to what it means for an individual to appropriate scientific knowledge (for example of a mathematical object) in the particular context which is relevant to my study.

As indicated in §2.2.3.4, this context[24] is comprised of the individual student (with his own particular social history, language skills, prior knowledge and motivations) working on a set of specially designed mathematical tasks[25] in a clinical interview setting. The student has access to a textbook and there is a teacher/ interviewer (myself) who interacts with the students, sometimes probing the student's thinking (when assuming the role of the researcher) and sometimes guiding the student's activities (when assuming

[24] The true context is comprised of a multitude of social, economic and historical factors. Since it is not possible to take all these aspects into account in my empirical analyses and theoretical elaborations I am defining context in a narrower sense. This does not mean that I do not think that other factors are not relevant to the context, just that I will not be examining them.

[25] Specific pedagogical and research intentions are embedded in these tasks. See Chapter 5 for a discussion of the tasks.

the role of the teacher)[26]. The learner is actively involved in his own activities in the interview situation, largely choosing what to look up in the textbook, which definitions to read or re–read, what questions to ask of the textbook and what tasks to re–do or re–examine.

My purpose here is to clarify how a zone of proximal development (ZPD) is created in this context. By ZPD I am referring to Vygotsky's term for the difference between the actual development level of the learner (what the learner can do alone) and the level of potential development (what the learner can do when guided by a teacher or more capable other).

In order to understand how a ZPD is constituted I remind the reader that a basic Vygotskian idea (see §2.2.1 above) is that

> the individual, through participation in interpersonal interactions in which cultural ways of thinking are demonstrated in action, is able to appropriate them so that they become transformed from being social phenomena to being part of his or her own intrapersonal mental function (Chang and Wells, 1993: 63.).

This move from the interpersonal to intrapersonal activities takes place in the ZPD.

In this regard Cole (1985: 155) argues that Vygotsky used the ZPD to describe the shifting control or responsibilities within activities as the learner comes to adopt the role of adults in culturally organised activities. Accordingly Cole (ibid.) suggests that a useful treatment of the ZPD is

> in terms of its general conception as the structure of joint activity in any context where there are participants who exercise differential responsibility by virtue of differential expertise.

I would like to extend this conception of the ZPD so that the participants include not only human beings but also those artefacts (such as texts and software) which have been written or created by people *with particular pedagogic intentions*. After all, such artefacts are social in their origins (they

[26] Of course the effects of the teacher/interviewer's two roles are heavily intertwined: probing how a student is thinking may lead the student to reflect on his activities which in turn affects the direction of his future activities and thinking.

comprise of ideas which exist in the social arena) and *they are designed to guide the student* in his appropriation of new knowledge[27]. Indeed they may be regarded as a reification of a more knowledgeable other.

Furthermore, some people **do** learn new concepts in local contexts which consist of that learner alone with a textbook. This view of a socially constituted interaction does not contradict Vygotsky's notion of social interaction. A situation consisting of a student and a prescribed textbook is necessarily social: the textbook has been written by an expert and prescribed by the lecturer, both with pedagogic intent.

This broadened view of the ZPD is compatible with Moll and Whitmore's (1993) conception of the zone as a characteristic not of the learner nor of the teacher but of the child involved in activity in a sociocultural system where this system is "*mutually and actively* created by teachers and students" (p. 20; my italics). Moll and Whitmore further argue that semiotic mediations together with social interactions are key components of Vygotsky's theory.

> This semiotic emphasis brings with it a focus on meaning as central to human activity… that is often ignored during discussions of the zone. The emphasis is usually on the transmissions of skills from adult to child, as is the case with typical classrooms." (1993: 39)

In this regard, a textbook is a tool of semiotic mediation and so plays a central role in knowledge appropriation (see §2.2.2 for a discussion on semiotic mediation).

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[27] In particular the textbook we use in our first–year Mathematics course and which was used in the interview setting (Calculus of a Single Variable by Larson, Hostetler and Edwards, 6th Edition, 1998) is designed with pedagogic intentions. For each new mathematical topic, the textbook presents information in a particular sequence which is expected to aid and guide the learner's comprehension of the topic. This sequence is: informal description of the mathematical object, graphical representation of the object where possible, definition, theorem(s) and illustrative examples with further graphical representations.

With reference to the active nature of the learner, Moll and Whitmore argue that "the emphasis on the active child developing cultural means to assist her or her own development" is central to this formulation. As discussed above, the student in the interview context is largely responsible for decisions about which activities (such as looking up topics in the textbook, re–reading definitions and so on) to do or re–do, within the parameters of the set tasks.

Thus I am arguing for a conception of ZPD which recognises the social role of a pedagogically designed textbook in the ZPD, stresses the importance of both semiotic and social mediations in the ZPD and assumes a learner playing an active role in decisions leading to his appropriation of mathematical objects.

As previously stated, I assume that the appropriation of a mathematical object involves the personal *construction* of a mathematical concept. It is through this construction that an internalisation of a social mathematical idea is effected for the individual learner.

My assumption rests on the epistemological belief that knowledge cannot be transmitted directly from one person (or textbook) to another person. In order for a cultural transmission of ideas, the learner has to re–construct that idea for himself. Adopting a converse position (that the individual does not necessarily construct concepts for himself) leads to an untenable epistemological position whereby the passive internalisation of mathematical ideas is deemed possible.

The recognition that a learner needs to re–construct "old" knowledge for himself is entirely compatible with a socio–cultural position. Van Oers (2000), a rigorous Vygotskian, talks of the importance of "pupils collaboratively *reconstructing* 'old' mathematical means and meanings" (p. 137; my italics). Sierpinska (1998: 49) argues that Vygotsky believed no more than Piaget in the verbal transmission of knowledge.

In contrast to Hatano (1993), who asserts that the assumption that knowledge is *constructed* by the individual is a radical extension of

Vygotsky's ideas, I suggest that the reason that many writers in the socio-cultural field do not specifically refer to the construction of a concept, is twofold. First, in common translations of Vygotsky's writings such as Thought and Language (1986), Vygotsky speaks of the *formation* of a concept (rather than the *construction* of a concept). Secondly, in mathematical education literature the notion of the construction of a concept is closely tied to a constructivist position (eg Piaget, van Glasersfeld). And a constructivist position has traditionally been put in opposition to a Vygotskian position (Sierpinska, 1998, Sfard, 1998).

§2.4 SUMMARY

Throughout this thesis, I take the position that mathematical objects do not exist prior to language; rather they are co-created and mutually constituted by the human mind in mathematical discourse. Furthermore I contend that meaning and communication about mathematical objects is only possible because a particular use of symbols and signs in mathematical discourse is socially sanctioned.

As previously indicated, I will not attempt to argue the 'truthfulness' of my position. Such an argument would constitute a thesis on its own. All I wish to do is to make explicit (which I have) my position regarding the nature of mathematics since this position underpins this entire thesis.

With regard to my use of a Vygotskian framework, Vygotsky's notion of how the meaning of a word (ie a concept) evolves for a child resonates strongly with my postulate about how the meaning of a mathematical concept evolves for an undergraduate mathematics student .

Vygotsky's position was that the meaning of a word or sign is not chosen by the learner; its meaning is predetermined by its meaning in the social arena. Given this assumption, he argued that the functional use of the word or sign (for communication, for solving problems, etc) prior to 'full' understanding of what the word or sign means, enables the individual to transform the word or sign into a concept whose meaning makes personal sense and whose use is commensurate with that required by the relevant social community (for example the community of mathematicians).

Vygotsky's detailed account of the actual evolution of a concept derived from an experiment with the formation of artificial concepts. He postulated that a learner goes through different phases (heaps, complex thinking, pseudoconcepts, potential concepts) as he constructs a concept and that these phases correspond to the different ontogenetic phases of the learner. He further argued that these phases co–exist in the adult and adults may use all these phases in their construction of a concept. It is in the latter sense that I find a resonance with my experience of undergraduate mathematics learners.

Vygotsky's theory of concept formation has been criticised for two main reasons: First some post– Vygotskians argue that the theory focuses on intramental processes without due regard for intermental processes. Secondly the theory has been criticised for its focus on the formation of artificial concepts which are not part of a system of hierarchical concepts (such as exist in the academic environment).

With regard to the first point, I have argued that the social constitution of the individual is implicit in Vygotsky's overarching framework of concept formation and that his notion of the pseudoconcept is a mechanism which bridges the intermental and the intramental.

With regard to the second point, I have argued that the theory can be extended to deal with the construction of one concept via the mediation of another as occurs in the academic setting. In such a setting the function of a sign or concept is no longer just that of communication. Rather it itself becomes the "object of communication activity" (Minick, 1996). Hence the function of the sign or concept in the academic environment is to focus attention on itself and on its relation to other signs or concepts. In this way, it functions as part of a hierarchical system and one concept or sign mediates in the appropriation of another concept or sign.

My application of the above notions to an explanation of the ways in which mathematics students construct personal meanings and develop usages of mathematical signs which are compatible with the culturally established meanings and usages of those signs, is largely informed by my observations of mathematics students over the years. Basically I maintain that students use mathematics signs (symbols, words, graphs, diagrams, etc.) prior to ‘full’ understanding not only as a way of communicating their thoughts to their peers or the teacher but also as objects to focus on, manipulate and act with (this is analogous to the use of the word as the object of activity). This use of the signs is mediated by the learner’s knowledge of related signs or concepts which are called forth for that learner by particular attributes of the new object or its symbols (this is analogous to the appropriation of a new word or concept via the mediation of an ‘old’ concept or word). Sometimes direct social intervention in the ZPD is required to effect this linking of old signs with new signs. Furthermore for socially acceptable mathematical learning, the meaning of the signs with which the student engages cannot be idiosyncratically determined; they need to evolve in a way that is compatible with the use of those same signs in the mathematical community.

I also argue for a conception of ZPD which recognises the active role of a learner and the centrality of pedagogically designed resources (such as textbooks) as tools of both semiotic and social mediations for knowledge appropriation.

It also must be said that Vygotsky’s notion of a single sign system which mediates both the construction of a concept in the individual and that individual’s social interaction with the world, is a brilliant theoretical and explanatory device. It fundamentally links intramental and intermental processes and its central role in Vygotsky’s theory enables coherence and harmony between the different themes.

In the above chapter, I have set out the theoretical background for my intended explanations of how a first–year mathematics student appropriates a new mathematical object from a definition. As indicated this theoretical view has been informed primarily by my knowledge of Vygotskian theory and by my own practice as a mathematics lecturer at the University of the Witwatersrand.

Having acquainted the reader with the theoretical principles and terms of this thesis, my task now is to apply and elaborate these principles to the mathematical domain. But before embarking on this task, I need to make further preparations. Specifically I will first interrogate the literature surrounding the appropriation of a new mathematical object at tertiary level (in Chapter 3). And then in Chapter 4, I will refine my research question.

CHAPTER 3: REVIEW OF LITERATURE

§3.1 INTRODUCTION

In order to give a broader context to my concern with how students at the tertiary level appropriate mathematical objects[1], and to contextualise my research within the socio–cultural milieu within which it has developed[2], I wish to discuss 'similar' or relevant published research which somehow relates to my project. To be sure many erudite thinkers and mathematics education researchers have pondered and investigated the problem of how students appropriate new mathematical objects and I have drawn many of my ideas from these thinkers, consciously or unconsciously.

In the mathematics education literature specifically, there are three almost distinct sets of scholarship which focus on, to a smaller or greater extent, the ways in which learners appropriate mathematical objects. As alluded to in Chapter 1, there is a body of research which attends to mathematical concept construction in the tertiary setting (for example Tall, 1991, 1995, 1999; Tall et al, 2000a, 2000b; Dubinsky, 1991, 1997; Czarnocha et al, 1999; Sierpinska, Dreyfus and Hillel, 1999a; Sierpinska, 2000), so sharing an empirical field with mine. There is also that body of research which shares some of my theoretical assumptions concerning semiotics (for example Vile and Lerman, 1996; Cobb et al, 1997; Presmeg, 1997; Gravemeijer et al, 2000; Radford, 2000, 2001; Sfard, 2000; van Oers, 2000) but which is not necessarily applicable to a consideration of how the individual appropriates mathematics objects at the tertiary level. And there is also a very small body of research which uses Vygotsky's theories to

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[1] Throughout this thesis, I assume that the appropriation of mathematical objects necessarily involves the student's construction of personal meanings for these objects and a usage of the relevant mathematical signs which is compatible with that of the mathematical community.

[2] The mathematics education community and the Mathematics I Major course at the University of the Witwatersrand are important components of this socio–cultural context.

examine learning in the mathematical domain. This body of research includes writings by van Oers (ibid) and Radford (ibid), both already mentioned above. Less relevant to my research in terms of its focus on elementary mathematics, but also within the Vygotskian paradigm, is the research by Jones and Thornton, 1993 (which focuses on first–grade children in a classroom situation); the research of Maksimov, 1993 (also focusing on the learning of elementary mathematics by young children) and the writings of Lambdin, 1993 (whose work focuses on co–operative learning). Within this latter body of Vygotskian research, there are also the writings of Sierpinska (1993, 1994), who focuses on the formation of mathematical concepts by learners and the writings of Schmittau (1993), who looks at the implications of Vygotsky's notion of scientific concepts for mathematical learning. Both these latter two sets of literature are relevant to my research. But their presentation is broad and unelaborated (albeit extremely interesting), so requiring further development and further application to the mathematical domain.

Notwithstanding the usefulness of much of this literature as a basis on which to develop and extend my own ideas, many questions remain unanswered and unasked in the relevant literature. Indeed much of I what I observe amongst the Mathematics I Major students whom I teach, as they try to come to terms with new mathematical notions, cannot be satisfactorily explained (or described) by the literature which currently exists. In this regard I also intend to highlight certain gaps in the literature which are particularly relevant to an understanding of how mathematical objects are appropriated at the tertiary level.

The reader should note that my literature review is highly selective: I only examine research which contributes directly to my understanding of mathematical object appropriation in terms of its theoretical approach or empirical setting. Certainly there are many other worthy studies of mathematical learning at tertiary level (for example, Nardi, 2000; Alcock and Simpson, 1999) which I do not look at in this review since they deal neither with the mechanics of concept construction nor with semiotics. Also certain notions, for example, a fore–conception (Sierpinska, 1992), which directly inform my theoretical elaborations are more usefully integrated into those chapters where they fit naturally. Their absence in this chapter is thus a consequence of their presence elsewhere in this thesis.

§3.2 THEORETICAL THEMES

In my review I will consider certain themes that cut across the bodies of literature referred to above and which inform the theoretical elaborations or empirical investigations in my thesis. Many of these cross–cutting themes derive from the theories of Piaget or Vygotsky (for example the role of semiotic mediation in learning and the role of socio–cultural factors in learning derive from Vygotsky). Since my purpose here is not to discuss general theories of learning but rather to show how certain principles from these overarching theories have been used in mathematics education by other researchers, I do not focus on these general theories here. In Chapter 2, in which I elaborated on the overarching theoretical framework of my research, I discussed certain principles of Vygotskian theory in much detail and I explained why their application to the learning of mathematics at tertiary level is desirable and appropriate. I also explained how I planned to apply those principles to my theoretical elaborations. Although many of those principles overlap with themes in this review, my focus here is on the themes as they occur in the mathematics education literature specifically.

§3.2.1. MEANING AS USE

The idea that the only way to grasp the meaning of a mathematical sign is through the use of this sign (Sfard, 2000; van Oers, 1996, 2000; Dörfler, 2000) and that the use of this sign is informed by what it means (on whatever level) to the user, is gaining ground at the turn of the twenty–first century in mathematics education. This reflexive relationship between meaning and use is explained succinctly by Cobb (2000: 19):

> This idea implies that student's use of symbols involves some type of meaning, and that the development of meaning involves modifications in ways of symbolizing.

This relationship between meaning and use of symbols is the basis of much research which admits a semiotic dimension (for example, Nemirovsky and Monk, 2000; Sfard, 2000; van Oers, 1996, 2000; Dörfler, 2000). These authors all “reject the view that the process of constructing meaning for symbols involves associating them with separate, self–contained referents.” (Cobb, 2000: 18).

The idea of a reflexive relationship is in contrast to the Platonic or intuitionist position that holds that meaning is independent of the usage of the related symbols and language and may even develop independently of mathematical language or notation. For example, Sierpinska, Dreyfus and Hillel (1999a), drawing on Duval who in turn bases his views on Frege’s formulation of Platonism, view the objects of scientific knowledge neither as mind independent entities nor as the content of mental representations in the psychological subject. Rather they regard mathematical objects as “the invariants in the reference of several semiotic representations” (p. 15). This means that different representations of a mathematical object (eg a graphical, numeric, algebraic, or verbal representation) are taken to represent the same mathematical object. This contrasts with the reflexive view, in which the representation of an object (the signifier) and the object (the signified) are regarded as mutually constitutive (see also Rotman, 1993; Sfard, 2000).

Dörfler (2000) uses the reflexive relationship rigorously to focus on the ways in which use and meaning are related. To this extent, he declares that it is senseless to ask a question such as “what is meaning?” Rather Dörfler takes a view, reminiscent of Wittgenstein’s famous maxim that the meaning of a word is equivalent to its use in language, that

> the only available and observable indicator that a subject has grasped the meaning – whatever this is – of a linguistic or symbolic entity is that the subject has a thorough command of its social use. (p. 101)

This is the view that I take throughout this thesis. One cannot look inside a learner’s head to see whether a particular mathematical idea is meaningful or not, and if it is meaningful, what it means. The most one can do as a researcher is to look at the student's *use* of the idea as expressed through symbols, words or actions. As will be seen in my analyses of protocols (see Chapter 7) taken from clinical interviews in which the student had to perform various mathematical tasks, I look at the student's use of symbols and words when interpreting what the symbols or words may mean for that student.

The claim that meaning and use are reflexively related also resonates strongly with Vygotsky's (1986) notion of the "functional use" of a word or sign, a notion which is fundamental to my account of how a student appropriates a new mathematical object. By functional use I am referring to Vygotsky's thesis that

> words take over the function of concepts and may serve as means of communication long before they reach the level of concepts characteristic of fully developed thought. (Uznadze, cited in Vygotsky, 1986: 101)

which I discussed in Chapter 2.

In my formulation of the research question (see Chapter 4) I postulate that the functional use of the mathematical sign in communication and activity even before the sign has reached the status of a mature or fully–fledged concept, enables the production of meaning for the user. Furthermore the meaning given to the sign by the learner determines the use of that sign by the learner.

§3.2.2 THE ROLE OF SYMBOLS IN THE LEARNING OF MATHEMATICS

> I dare say that this is the last effort of the human mind, and, when this project[3] shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect not less than the telescope serves to perfect their vision. (Leibniz, cited in Cajori, 1993 : 185)

In Chapter 2, I argued that mathematical activity is mediated activity and that the role of the mediator is played by a symbol, a graph or a word (see

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[3] Leibniz (1646–1716) was one of the greatest developers of many of the mathematical notations that we still use today. The 'project' refers to his work in the development of suitable mathematical notations.

§2.2.2 for an extended discussion). Furthermore I argued that mathematical objects do not exist independently of signs and symbols; rather they are co-created and mutually constituted through the use of symbols. Indeed meaning and communication about mathematical objects is only possible because a particular use of symbols and signs in mathematical discourse is socially sanctioned (see §2.1 for further elaboration).

Although certain mathematics educators have embraced a similar position in recent years (Vile and Lerman, 1996; van Oers, 1996, 2000: Dörfler, 2000; Radford, 2000, Sfard, 2000), there has been remarkably little emphasis on semiotic mediation in the mathematics education literature in general (Cobb et al, 1997, Yackel, 2000).

Amongst mathematics educators who do acknowledge the central role of symbolising in mathematical activity, there are three not necessarily exclusive approaches: the Vygotskian approach (van Oers, 1996, 2000: Dörfler, 2000; Radford, 2000; Sfard, 2000), the emergent approach (Cobb et al, 1997, Gravemeijer et al, 2000, Lehrer et al, 2000; Lesh et al, 2000; Nemirovsky and Monk, 2000), and the "chains of signification" approach (Walkerdine, 1988, Cobb et al, 1997; Presmeg, 1997, 1998; Whitson, 1997). I will elaborate briefly on each of these approaches as used in the mathematics education literature and then explain briefly why the Vygotskian approach is most suited to my needs.

In the emergent approach, symbolising activity is seen as "supporting the emergence of mathematical meaning *in the classroom*" (Cobb et al, 1997: 163; my italics). Cultural practices and conventional uses of symbols provide direction to this emergent process although the emphasis is on using meaningful and authentic experiences as a starting point for the mathematical learning process (Gravemeijer et al, 2000).

My concern is the learning of mathematics at the tertiary level *as it happens in the here and now* at institutions like the University of the Witwatersrand. In such environments, teaching takes place in large classrooms. The lecturer presents the concepts as finished products on the chalkboard and the students do not actively participate in mathematising in the classroom.

Indeed most of the students' mathematical activity takes place outside the classroom as the students work (on their own or with friends) on the prescribed mathematical 'homework' problems using their lecture notes and textbook for reference. The emergent approach, with its emphasis on the negotiation of shared meanings in the classroom, is not apposite to such a context. Furthermore, in the emergent approach the intended starting point is problem situations that are experientially real to students, such as problems involving everyday or fictitious life settings (Gravemeijer et al, 2000: 237); in the tertiary context of this study, the intended starting point is the conventional symbolisations as they occur on the blackboard, lecture notes or text book. Although Gravemeijer et al (2000) note that mathematics itself may become experientially real for the student, in which case activities with the conventional symbolisations may constitute authentic experiences, the negotiation of mathematical meaning which derives from within mathematical discourse is not attended to specifically in the emergent approach.

In the "chains of signification" approach, semiotic analysis is used to explore how, for a particular learner, the meaning of a sign in one cultural domain, say the home background, carries through to its meaning in a new domain, such as the mathematics classroom. The primary focus is on the process whereby a signified which exists in one cultural practice with its own concerns and affects becomes a signifier in a different cultural domain (such as a mathematics classroom). This signifier in turn gives rise to a new signified which in turn may become the signifier for yet another signified and so on, constituting a chain of signification (Walkerdine, 1988). As Whitson (1997) explains, this form of analysis is based on an elaboration of Lacan's inversion of Saussure's diadic model in the context of mathematics education. (That is, in Saussure's model, the signified implicitly takes some form of priority over the signifier; in Lacan's model, this is inverted so that the signifier takes priority over the signified.) In line with Walkerdine (1988), Presmeg (1998: 146) argues that the significance of this semiotic chaining is that any individual is positioned very differently in different discursive environments and thus will carry over certain subjective meanings and feelings associated with the one environment to the other.

Although such a form of analysis is very interesting and theoretically applicable to any level of mathematical learning, I suggest that most of the chains of signification from the discourses of non–mathematical cultural domains to the discourse of tertiary–level mathematics are so long, that even if we are able to reconstruct them accurately, they may not provide useful and illuminating insights into the student's production of meaning. In a similar vein (although for a different purpose), Sfard, (2000: 92) argues that given the mathematical content of high schools as it is today,

> sequences of template transplants[4] and chains of signification forged on the way from AR discourse to VR discourse[5] of mathematics may be so long that their AR sources are no longer visible from the distant VR end.

As I argued in Chapter 2, the Vygotskian approach, in which mathematical activity is conceived of as the appropriation of conventional mathematical signs and their conventional uses, is apposite to an understanding of the appropriation of mathematical objects at tertiary–level. Within this approach the signs (symbols, words, graphs and so on) have a dual role: they are used by the individual to organise his own mathematical thinking, and they give access to the official mathematical discourse which transcends the particular individual. (For a more extended discussion of this approach, see §2.1 and §2.2.)

A discussion of the literature around the role of symbols in the learning of mathematics would be incomplete without a consideration of the procept.

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[4] Template transplants refers to the mechanism whereby the learner uses a new word or sign in the linguistic template normally used for the old sign (Sfard, 2000). I discuss this template–driven use of mathematical signs in § 3.2.3 and § 6.2.2.6

[5] Sfard (2000) distinguishes between actual reality (AR) discourse and virtual reality (VR) discourse. She explains that in actual reality discourse, "communication may be perceptually mediated by the objects that are being discussed, whereas in the virtual reality discourse perceptual mediation is scarce and is only possible with the help of what is understood as symbolic substitutes of objects under consideration". (p. 39)

The procept is a theoretical construct (rather than an approach) which was developed by Gray and Tall (1994) to describe an amalgam of process, concept and symbol. Unlike most of the other approaches and theoretical constructs around symbols and meaning (such as those discussed above), this notion has been used extensively in much neo–Piagetian literature concerned with an examination of mathematical learning at the undergraduate level (for example, Tall, 1991, 1995, 1999; Tall et al, 2000a, 2000b; Crowley and Tall, 1999, Ali and Tall, 1996).

What is a procept? According to Gray and Tall (2001), cognitive growth occurs through recurring cycles of activity with symbolic, numeric and visual representations of the mathematical objects. These cycles comprise of three different stages: procedures, processes and procepts. A procedure refers to a specific sequence of steps (like an algorithm) whereas a process is used more generally to include any number of procedures which have the same outcome. For example, $\int \frac{x^3}{\sqrt{4+x^2}}dx$ can be solved using any of the following procedures: trigonometric substitution (let x = 2 tan θ), u–substitution (let $u^2 = 4 + x^2$) or integration by parts (integrate the term $x/\sqrt{4+x^2}$). But a single process, integration, is involved. The most sophisticated use of the symbol occurs when the learner is able to use the symbol as a process or as a concept. At this point, the concept has become compressed into a manipulable entity (Tall, 1995); Gray and Tall (1994) call this entity a 'procept'. They emphasise that a procept has a dual role: it is a symbol which can be used to refer to a concept (for example, antiderivative) or to a process (for example, integration).

Further reflection on the procepts and the objects constructed from them (in arithmetic, algebra or calculus) leads the learner to the formation of further procepts. This derivation of new mental procepts by reflection on old mental procepts corresponds to Piaget's notion of *reflective abstraction* (Tall et al, 2000a). In the Vygotskian framework it resonates with the notion of sign–sign (Wertsch, 1985; Minick, 1996) mediation whereby one concept or sign mediates in the construction of another concept or use of sign.

Sign–sign mediation is a notion which I use, albeit implicitly, in my theoretical elaborations.

The problem with the notion of the procept and indeed with much of the mathematics education literature surrounding mathematical object appropriation at the tertiary level[6] is that it is rooted in a framework in which conceptual understanding is regarded as deriving solely from interiorised actions; the crucial role of language in this object appropriation is not acknowledged. As previously argued (see Chapter 1), this undue focus on conceptual rather than semiotic aspects, may lead the researcher to ignore important aspects of cognition (Becker and Varelas, 1993).

Furthermore, this non–attendance to language use leads to an insufficient acknowledgement of the social–regulation or social–constitution of mathematical activity. Consequently there is no discussion of the negotiation of meaning in this body of work (Confrey and Costa, 1996: 145). The idea, central to my framework, that one symbol may mean different things to many people or different things to the same person, depending on the context and its usage, is absent from the notion of a procept.

Notwithstanding the above remarks, Gray et al (1999), introduce some interesting elaborations of the procept. For example, they describe the procept as having a “distilled essence that can be held in the mind as a single entity, it can act as a *link* to *internal* action schemas to carry out computations, and it can be *communicated* to others” (p.114; my italics). This description has a similar ring to Vygotsky’s description of a sign as the mediator of internal and social processes. But Gray et al do not elaborate further on this internal–external duality nor root it in their framework.

[6] See, for example, Tall (1991,1995, 1999), Crowley and Tall (1999), Pinto and Tall (2001), Ali and Tall (1996), Dubinsky (1991), Czarnocha et al (1999).

§3.2.3 'AS–IF' ATTITUDE

An interesting theme which appears in the mathematics education literature surrounding semiotics, is the 'as–if' tenet (in particular, see Dörfler, 2000; Sfard, 2000).

Dörfler's (2000) thesis is that in order to appropriate a new mathematical object, the mathematics student has to be willing to adopt an attitude whereby he participates in the discourse of mathematics **as–if** the discourse is meaningful and coherent, even if he does not experience it as such. In other words, in order to learn the student needs be willing to enter into a state of suspended belief as he goes about doing mathematics and communicating mathematically.

Dörfler argues that this as–if attitude is necessary if the student is to ultimately understand advanced mathematical discourse and to experience it as meaningful. He explains this clearly:

> The conventions of mathematical discourse, beyond a cognitive understanding, need to be accepted and agreed on. Learners must indulge in the discourse, and this participation cannot be forced on them by cogent arguments. Indulgence in mathematical conventions and ways of speaking is partly an emotional willingness. And it proves sensible and justifiable only after hard, demanding work within the discourse and only after obeying its (often only) implicit rules….. I assert that a specific view, called an *as–if* attitude …can be of much support for accepting mathematical discourse. (Dörfler, 2000: 122, italics in original).

Dörfler explains that his 'as–if' tenet reflects his epistemological stance on the nature of mathematical objects. According to this stance, mathematical objects come into being through and within mathematical discourse. As such mathematical meaning resides in the discursive use of mathematical objects (or signs). Hence the way for the student to 'discover' the socially–accepted meaning of the object is through socially–regulated discursive use of the object even if meaning is initially suspended. This is a position I embrace in this thesis.

Sfard, too, employs a notion of 'as–ifness' in her theory about how students construct meanings for new mathematical signs (1997, 2000), although she does not label it as such. According to her, when a new symbol is introduced, the learner may not have any idea of what the symbol represents. Nevertheless such a learner does not become inactive. Rather he starts using the new symbol in a linguistic template into which it somehow seems to fit, **as if** the rules normally associated with that template apply to the new symbols. Sfard (1997, 2000) refers to this usage of the symbol as template–driven use.

Using the example of a fraction such as 2/5, which is introduced to a child for the first time, Sfard (1997: 350; my italics) explains:

> one can hardly expect the young students to be able to think about it[7] as a self–sustained disembodied object for which the symbol itself is but a 'representation'… In spite of this, the learners do not remain idle until the more abstract meaning of the symbol is constructed. They do use the new symbol *as if* it signified a number, even though the meaning–rendering links in the perceptual interpretations are still vague or practically non–existent.

A new stage in which new discursive forms and hence new meanings are generated begins when the new symbol starts slipping out of its slot in the initial template due to some basic misfit. In this new stage, students develop meaning by referring to mathematical objects **as if** they signified independently existing mathematical entities. Sfard (2000) calls this object–mediated discourse.

I use Sfard's notion of template–driven use directly in my theoretical elaborations (see Chapter 6). However, unlike Sfard (2000), I do not claim that template–driven use is the only opening move in meaning construction. Indeed I suggest other ways in which students use new symbols (for example the surface association phase in which students focus unduly on part of the signifier and associate this one part with other more familiar signifiers).

In Chapter 6, I also argue that since I am using the term mathematical concept to refer to the mental idea of a mathematical object, mathematical

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[7] 'It' refers to a fraction such as 2/5.

conceptual thinking has many of the same features as Sfard's (2000) object–mediated phase. I elaborate extensively on these common features in Chapter 6.

Furthermore, I too use a notion of 'as–if–ness' in my theoretical elaborations of the pseudoconceptual stage. As I discussed in Chapter 2, Vygotsky's notion of a pseudoconcept is central to my elaborated theory of mathematical object appropriation. According to this notion, the learner initially uses an immature form of a concept (called a complex) in communication and activities **as if** it were a true concept. Through this functional and socially–regulated use of the complex, the complex gets transformed into a concept in which the bonds between the different elements of the concept and the bonds between different mathematical notions (be they complexes or concepts) become logical and abstract.

As a final thought, I suggest that these notions of 'as–ifness' are consequent upon the peculiar nature of mathematics, whose subject matter are objects which we cannot ever really know. We can only act **as if** we know. Bertrand Russell (1904, cited in Sfard, 2000: 37) expressed this idea thus: "Mathematics may be defined as a subject in which we never know what we are talking about, nor whether what we are saying is true" .

§3.2.4 THE RELATIONSHIP BETWEEN THE INDIVIDUAL AND THE SOCIAL

In this section I wish to briefly discuss the way in which the individual is dealt with in the relevant bodies of literature: that body of research relating to my empirical field and that body of research relating to my theoretical field.

Broadly speaking, it seems that most of the former body of research (Tall, 1991, 1995; Tall et al, 2000a; Dubinsky, 1991, 1997; Czarnocha et al[8], 1999 and the various followers of these schools) is Piagetian or

[8] Czarnocha has worked extensively with Dubinsky. Her research is based on the APOS framework articulated by Dubinsky (1991).

neo–Piagetian in perspective, whereas most of the latter body of research is Vygotskian (Sierpinska, 1993, 1994; Schmittau, 1993; Vile and Lerman, 1996; Radford, 2000, 2001; van Oers, 2000) or admits to a strong social dimension (Cobb et al, 1997; Presmeg, 1997, 1998; Gravemeijer et al, 2000; Dörfler, 2000; Sfard, 2000).

Sfard's (1998) summary of the different foci of Piaget and Vygotsky is apposite to a consideration of how these different theoretical perspectives inform the mathematics education literature in terms of the relationship between the individual and the social.

> While Piaget regarded human intellectual development as a biologically determined phenomenon which can be influenced by culture only marginally, Vygotsky gave primacy to socio–cultural factors. Moreover, while Piaget implied that the knowledge we build is mainly a function of the world around us, and is quite literally created anew by every learner, Vygotsky saw both knowledge and meaning as collective creations which are preserved within culture and are appropriated time and again by individual children in the process of learning….(According to Vygotsky) the concepts we learn do not come directly from nature and are not uniquely determined by nature. Conceptual thinking is a byproduct of human communication and is only possible within language. The language, in turn, is a social creation, and the concepts themselves are therefore essentially social. (1998: 26)

In §3.2.2, I referred to the Piagetian underpinnings of the Tall and Dubinsky schools. I also mentioned that, in line with their respective frameworks, understanding is regarded as a consequence of the individual's interiorised actions rather than as a result of any particular use of language or mathematical discourse (which position underpins my theoretical elaborations).

A further problem for me in both the Tall and Dubinsky positions is that, not only is the individual constructed as if he were independent of the social world (in terms of both the social regulation of individual learning and the social origins and development of concepts which need to be appropriated) but also the thinking process is presented as if it were independent of the person and his prior learning, attitudes and particular circumstances.

In this regard, Confrey and Costa (1996: 143) claim:

> The central approach of these theorists ….is a focus on a transformation of thought processes into a cognitive structure which is then referred to as an object. However, this transformation is cast as requiring only ‘action’ and ‘operation’, and seldom do we see any discussion of the purposes of the constructor, the determination of satisfaction of a goal, or the use of context of the construction beyond reference to other mathematical objects.

In contrast to the Piagetians, the emergent theorists are primarily concerned with the “development of both collective meanings and the understandings of individual students who contribute to their emergence” (Gravemeijer et al, 2000: 226). According to Bowers (2000) researchers such as Bransford et al (2000), Gravemeijer et al (2000), Lehrer et al (2000) and Lesh and Doerr (2000) suggest that learning can be viewed as a process whereby individuals negotiate meanings as they participate and contribute to communal practices. Cobb et al’s (1997) study on the emergence of chains of signification in a first–grade classroom also falls into this emergent category. In that study the concern was

> to describe the evolution of the mathematical practices established by the classroom community and (to) consider the mathematical learning of individual children as they participated in and contributed to the development of these practices. (ibid.: 153)

Similarly, but within the context of a Vygotskian approach[9], Radford (2000, 2001) and van Oers (2000), both examine how high–school and pre–school children respectively, working in groups and within a classroom situation, appropriate mathematical symbols whose meanings are framed by culturally and socially constituted rules of use.

Dörfler (2000) and Sfard (2000) both examine how the individual constructs meanings by participation in mathematical discourse, which discourse has been historically and socially constituted. Sierpinska (1992, 1993) explicitly looks at the genesis of concepts by the socially–constituted individual with

[9] The emergent theorists focus on both the enculturation of the student and their contributions to the emergent classroom culture; the Vygotskians focus on how the individual appropriates mathematical objects which have been institutionalised by the wider society. (Cobb et al, 1997)

illustrations from the mathematical domain. The positions of these latter theorists in many ways mirror my perspective.

Since I have already carefully explained (in Chapter 2) why I regard the Vygotskian approach to the relationship between the individual and the social as suitable to my purposes, I will not repeat that discussion here. However I will remind the reader that, as I noted in Chapter 2 with support from Van der Veer and Valsiner (1994), the development of the individual has been continually backgrounded in Vygotskian studies looking at the learning of mathematics, especially at an advanced level[10]. I thus contend that an explication of how the (socially–constituted) individual constructs advanced mathematical concepts in the (necessarily socially–constructed and socially–regulated) world, would fill of a major gap in the mathematics education literature.

Furthermore I suggest that the premise (which I accept) that both the individual and knowledge are socially constituted, has been interpreted by some in the mathematics education world to elevate the value of research which looks at mathematical learning in groups above the value of research which looks at mathematical learning by the individual armed with a textbook and other such resources[11]. Indeed as Sfard (1998: 27, 28) so cogently puts it:

> Learning is social regardless of the way it occurs. Indeed, learning does not have to be interactive to be social. Interaction, in its turn, does not have to be between peers. The teacher's intervention is also a form of interaction[12].

She also argues, on a pedagogical level, that solitary work in mathematics and a teacher's substantial interventions may be as vital as collaborative groupwork for effective learning (1998: 27, 28). Indeed, a survey of the

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[10] A notable exception to this is Sierpinska (1993, 1994, 2000).

[11] This may also be due to an assumption that effective learning only takes place through direct social interaction. In contrast, I foreground the mediating role of semiotic tools such as pedagogically designed tasks and textbooks.

[12] Interaction with a textbook is also social.

Mathematics I Major class in which this study is rooted, showed that, at the beginning of the academic year, 48% of the class agreed or strongly agreed that it was helpful to study mathematics with a friend; 18% were not certain and 34% disagreed or strongly disagreed that it was helpful to study mathematics with a friend. With regard to the same question with the same class at the end of year, the figures were respectively 44%, 25% and 31% (that is, 7% more students were uncertain about the helpfulness of studying mathematics with a friend).

In this regard, my study is directed particularly to an examination of mathematical knowledge appropriation by an individual learner studying and working on his own with some mathematical text, with or without a teacher's interventions (with the implicit understanding that such an activity is necessarily social).

3.2.5 THE GENESIS OF CONCEPTS

As previously mentioned, Sierpinska (1993, 1994) has given an account of Vygotsky's theory of concept formation with examples taken from children's behaviour when confronted with mathematical tasks. In some sense, my thesis is an extension of Sierpinska's unelaborated account and indeed, many of Sierpinska's reflections resonate with and support my observations and theoretical elaborations.

For example, Sierpinska (1993) asserts (as I do throughout this thesis) that Vygotsky's theory of the genesis of concepts

> could be used to explain some of the curious ways in which students understand mathematical notions, and why, at certain stages of their construction of these notions, they simply cannot understand in a different or more elaborated or more abstract way (p. 87).

She also eloquently makes the point (which I argued in Chapter 2) that

> the general pattern of development of conceptual thinking from early childhood to adolescence seems to be recapitulated each time a student embarks on the understanding of something new or constructing a new concept. (p. 88)

Interestingly, but not surprisingly, her research reveals that pedagogical mediation is frequently necessary for the adequate appropriation of mathematical objects.

With respect to this latter point, Schmittau (1993) argues that the unification of scientific concepts (such as mathematical concepts) with everyday concepts requires carefully structured pedagogical mediation, such as that developed by Davydov. In Davydov's curricula, the student move from the abstract to the concrete rather than vice versa. Certainly the relationship between everyday concepts and scientific concepts (in the mathematical arena) is worthy of much research. It is, however beyond the scope of this thesis.

§3.3 RESEARCH IN THE EMPIRICAL FIELD

Literature relevant to the empirical field of this thesis comprises studies which focus on the appropriation of mathematical objects at undergraduate level.

Other than the work by Sierpinska and her colleagues (Sierpinska et al, 1999a; 1999b; Sierpinska, 2000) and Dörfler (2000), most of the literature surrounding the appropriation of mathematical objects at tertiary level (for example, Tall, 1991, 1995; Tall et al, 2000a; Dubinsky, 1991, 1997; Czarnocha et al, 1999) fits into a Piagetian or neo–Piagetian framework, as previously discussed. Before examining this latter body of work I want to focus on that work with which I share several theoretical premises.

Dörfler (2000) maintains that a theory devoted to an explication of students' development of mathematical meaning should be based on an account of how these students were inducted into mathematical discourse. He supports his theoretical elaborations with empirical evidence derived from introspection on his own experience as a practicing mathematician. (For this reason I regard this work as being applicable to the tertiary level.)

As previously noted, Sierpinska's work (Sierpinska et al, 1999a, Sierpinska, 2000) is more directly relevant to my research. This research is based on observations of groups of university students involved in a linear algebra course at a university in Canada (Sierpinska et al, 1999a). The students worked with linear algebra notions in a particular learning and teaching program within a multi-representational environment afforded by CABRI–geometry II. The primary aim of the CABRI–geometry II environment was to assist students in avoiding the ‘obstacle of formalism’ (for which goal Sierpinska et al (1999a) claimed a measure of success). This obstacle manifests in students using expressions that make no sense to them but superficially resemble the discourse of ‘official’ mathematics and in students confusing categories of objects (eg treating co–ordinates of vectors as vectors, functions as equations, and so on).

Sierpinska’s (1999a, 2000) study has certain features in common with mine: she assumes that all mathematical knowledge is semiotically mediated and that connections between mathematical concepts need to be made in terms of their relations to more general concepts rather than on empirical grounds. She also looks at mathematical learning at the tertiary level.

Furthermore Sierpinska (2000) proposes an interesting distinction between two different types of mathematical thinkers: theoretical and practical thinkers. She relates this distinction to Vygotsky’s distinction between everyday and scientific concepts .

Sierpinska characterises theoretical thinking as:

> conscious reflection on the semiotic means of representation of knowledge, systems of concepts rather than aggregates of ideas, and reasoning based on logical and semantic connections between concepts within a system. (2000: 4)

In line with Vygotsky, she argues that in theoretical thinking

> connections between concepts are made on the basis of their relations to more general concepts of which they are special cases rather than on empirical associations. (ibid.)

This interrelationship between different concepts is consistent with the way in which I describe conceptual thinking (see §6.2.4).

In contrast, practical thinkers use whatever technique or method seems easier at a given time and situation: “there is no concern for the concepts and techniques to belong to one consistent system” (op cit. : 37). I suggest that this type of thinking, with its lack of concern for a system of concepts linked together logically, is reminiscent of complex thinking, which I described in Chapter 2.

Since much of the other literature concerning the appropriation of mathematical objects at tertiary level excludes semiotic mediation and socio–cultural factors in its elaborations, I have not used many of its constructs or elaborations directly. The exception to this is Tall’s (1995) and Tall et al’s (2000b) insightful distinction between elementary and advanced mathematical thinking. I find this distinction pertinent to my understanding of mathematical object appropriation at tertiary level and have used it in my design of the mathematical task for the clinical interview (I discuss this further below).

According to Tall (1995) elementary mathematical thinking is distinguished from advanced mathematical thinking in terms of the changed status of both the mathematical object and its properties in the school curriculum and the university curriculum respectively. Essentially, in elementary mathematics, the object is *described* but in advanced mathematics the object is *defined.*

> In elementary mathematics the description is constructed from experience of the object, in advanced mathematics, the properties of the object are constructed from the definition – a reversal which causes great difficulties of accommodation for novices in advanced mathematical thinking. (Tall, 1995: 67).

Tall et al (2000b) claim that this didactic reversal requires a fundamental shift by the student. In elementary mathematical thinking, the student encounters “objects that have properties and symbols that can be manipulated “ (2000b: 19) and the student gets meaning from manipulating and playing with symbols as concepts or processes. But in advanced mathematical thinking this process is reversed. “The introduction of definitions and axioms leads to a new kind of cognitive concept – one which is *defined* by a concept definition and (has) its properties *deduced* from the definition " (Gray et al, 1999: 117; italics in original). Students now have to *deduce* properties of mathematical objects formally from mathematical definitions and construct mathematical objects using reflective abstraction on these properties (Tall et al, 2000a).

The first–year level mathematics course at the University of the Witwatersrand requires features of both elementary and advanced mathematical thinking. For example, we deal with the limit concept and the concepts of continuity and integration largely in terms of algebraic manipulations and procedures (elementary mathematical thinking). However, other mathematical objects are introduced via their definition from which the student is expected to construct the concept (advanced mathematical thinking). For example the natural logarithm is introduced via definition, ie $\ln x = \int_1^x \frac{1}{t}dt, \quad x > 0$, from which the student needs to construct the concept, the natural logarithm, for himself. In 2000 (when my study took place) complex numbers were also introduced via a particular set of properties (equality, addition and multiplication of ordered pairs of numbers). From these properties the student was expected to construct the concept of a complex number (advanced mathematical thinking). In addition, several existence theorems (advanced mathematical thinking) such as the Mean Value Theorem and Rolle's Theorem are part of the curriculum.

The mathematical task in the interviews of my research were developed specifically to include aspects of both elementary and advanced types of mathematical thinking. Thus in certain tasks, the manipulation of symbols and properties is intended to enable the construction of meaning; in other tasks the student has to use deduction in order to progress. See Chapter 5 for a full discussion of the tasks in my interview and how features relevant to both elementary and advanced mathematical thinking are included.

This distinction between elementary and advanced mathematical thinking (so useful for understanding the nature of mathematical activity at tertiary level) has also been theorised and used by both Sfard (1994) and Sierpinska (2000), but in different ways.

Sfard (1994) explains this distinction using Lakoff and Johnson's theory of metaphorical projection. Specifically she suggests that metaphorical projection from actual reality to the universe of abstract mathematical objects is the source of our understanding, imagination and reasoning. However, she argues that in advanced mathematics "it may well be that the immediate source of the basic metaphor is another, lower–level mathematical structure" (1994: 47). This lower–level mathematical structure may in turn derive from a metaphor of yet another lower–level mathematical structure and so on. Ultimately the chain of metaphors originates from our physical and bodily experiences in actual reality.

I suggest that this idea of long chains of metaphors in advanced mathematics is what makes the understanding of mathematics so difficult at a tertiary level (even for students who were 'good' at mathematics at school). For example, in primary school the child is usually able to use his bodily experience with the manipulation of concrete objects, such as marbles, to enable an understanding of the concept of number or sums or multiplication, and so on. In high school, the immediate source of metaphor may also be directly rooted in the learner's physical experience of the material world. For example, the notion of gradient of a straight line can be experienced directly when walking up a steep hill; the notion of a maximum value of a function may be experienced bodily when drawing a function which has a maximum value. But at a more advanced level (be it in high school or university), the bodily source of the metaphor may be very indirect, it may be many links away[13]. For example consider the chains and combinations of chains of metaphors (concerning limits, partitions, functions, signed areas under curves, areas of rectangles, sums and so on) involved in understanding the notion of the limit of a Riemann sum

[13] Of course, I am not denying that certain concepts introduced at higher levels, such as the limit of a sequence or the derivative of a function may have relatively short chains of metaphors linking them to a bodily experience.

($\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i$ where f is defined on [a,b] and Δ is a partition of [a,b] given by $a=x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Δx_i is the length of the ith subinterval. $x_{i-1} \leq c_i \leq x_i$).

Sfard's implicit criterion (1994) for distinguishing elementary mathematical notions from advanced mathematical notions contrasts in an interesting way with Tall (1995) and Tall et al's (2000b) distinction. It seems to me that Sfard's distinction implies a *quantitative* change from elementary mathematical thinking to advanced mathematical thinking (in elementary mathematics we construct concepts using metaphors based *immediately* on an embodied experience; in advanced mathematics we construct concepts using *chains of metaphors* which ultimately are based on an embodied experience). In contrast Tall's (1995) and Tall et al's (2000b) distinction between elementary and advanced mathematical notions is *qualitative*. Their argument is that there is a didactic reversal from elementary to advanced level: at elementary level, the definition is constructed from the object; at advanced level, the object is constructed from the definition.

Although Sierpinska (2000) does not relate her distinction between practical thinking and theoretical thinking to the distinction between elementary and advanced mathematical thinking, I suggest that these distinctions relate to each other. Sierpinska claims that for the practical thinker, "mathematical objects are 'natural objects' not 'discursive objects': definitions and theories can only describe them, not create or construct them" (op. cit: 24). That is, definitions and theories create new mathematical objects for the theoretical thinker whereas definitions and theories describe the mathematical objects for the practical thinker. Comparing this to Tall's split between the epistemological status of mathematical objects at an advanced level compared to their status at the primary level, I suggest that Sierpinska's distinction may be used to imply that only theoretical thinkers will succeed at advanced mathematical thinking.[14]

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[14] A study with such an hypothesis would make for interesting research.

Another very important theme in the literature surrounding the construction of mathematical concepts at tertiary level is that in order to do and understand mathematics, the learner needs to be able to transform a *process* into a mental *object* and that such a transformation is fundamental to the construction of a mathematical concept[15] (Tall, 1991, 1995; Tall et al, 2000a; Dubinsky, 1991, 1997; Czarnocha et al, 1999). Sfard (1991, 1994) although not specifically focussing on undergraduate learning, also has this duality at the heart of her early theories of concept construction. In her distinction between operational and structural conceptions, she asserts that the same mathematical object may be conceived structurally or operationally. For example, a rational number may be conceived as a pair of integers (structural conception) or as the result of division of integers (operational conception); a function may be conceived as a set of ordered pairs (structural) or a computational process (operational), and so on.

According to Tall et al (2000a), this idea of a process–object duality originated in the 1950's in the work of Piaget who spoke of how "actions and operations become thematized objects of thought or assimilation" (cited in Tall et al, 2000a: 1). Tall et al further recount that in the 1960's, this duality of mathematical concepts was expanded on by Dienes and in the 1980's it was further elaborated on by Davis and Greeno.

Currently and with reference to undergraduate learning, the process–object duality of mathematics is captured by Gray's and Tall's (1994) notion of a procept which I discussed earlier.

In adopting a neo–Piagetian perspective, these researchers (Tall, 1991, 1995; Tall et al, 2000a; Sfard, 1991, 1994; Dubinsky, 1991, 1997; Czarnocha et al, 1999 and the various followers of these schools) successfully extend Piaget's work regarding elementary mathematics to

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[15] Although the object–process duality is a central theme in the literature surrounding the construction of mathematical objects at tertiary level, it also applies to the construction of mathematical objects at school level.

advanced mathematical thinking. For example, Czarnocha et al (1999) theorise that in order to understand a mathematical concept, the learner needs to move between different stages. He has to manipulate previously constructed objects to form actions. "*Actions* are then interiorised to form *processes* which are then encapsulated to form *objects*" (1999: 98). Processes and objects are then organised in *schemas*. Czarnocha et al (1999) stress that this process is not linear, rather the learner moves back and forth between the different levels. To give practical expression to this theory of learning, which Dubinsky has named APOS[16] theory, he and his colleagues have designed mathematics courses in which students have to write (the action) computer programs (process becoming object) to represent various calculus or algebraic concepts.

Importantly, much of this process–object research does not resonate with a great deal of what I see in my mathematics classroom. For example it does not help me explain or describe what is happening when a learner fumbles around with 'new' mathematical signs making what appear to be arbitrary connections (but which I will argue in Chapter 6 are not arbitrary) between these new signs and other apparently unrelated signs.

Indeed, as I argued in Chapter 1, the process–object theories focus on the conceptions of learners who are already quite far along the path of mathematical object appropriation. In contract, my focus is largely on the learner who is not yet able to use an appropriate process or object, that is, the preconceptual learner whose usage of signs is characterised by various non–logical associative links.

Interestingly, Bowie's (2000) study resonates with my experience. In her study, also set in South Africa, she looks at the ways in which students who do not 'have' meaningful mathematical concepts on which to build new concepts, cope mathematically. In particular she examines and classifies those strategies which students adopt as they try to build conceptual

[16] APOS is an acronym for the different stages of the learning process: **a**ction, **p**rocess, **o**bject, **s**cheme.

schemas (as described by Dubinsky) using mathematical objects which lack meaning or have a distorted meaning for that student. I use some examples from Bowie's work in my theoretical elaborations (Chapter 6).

A further point regarding much Piagetian and neo–Piagetian research surrounding mathematical object appropriation at a tertiary level is that much of this research treats the individual as a purely cognitive machine without aims, motives or goals. In this way, the theory fails to take into account the affective dimension or the social and institutional world which the student inhabits. In this sense its explanations are necessarily limited, which limitation is exaggerated in post-apartheid South Africa in which the social, cultural and schooling contexts of groups of students are so varied.

§3.4 SUMMARY

In this review I have looked at how other researchers in mathematical education have examined my central concern: that is, how a learner appropriates a new mathematical object whose meaning has already been socially codified. In this regard I have identified three specific sets of literature. One is rooted in the same empirical field as my research (undergraduate mathematical learning) and has a strongly Piagetian bias (for example, the work of Tall and students and colleagues, the work of Dubinsky and students and colleagues). Another body of research focuses on the role of symbols in the learning of mathematics (for example Vile and Lerman, 1996; Cobb et al, 1997; Presmeg, 1997; Sierpinska et al, 1999, Sierpinska, 2000; Gravemeijer et al, 2000; Radford, 2000, 2001; Sfard, 2000; van Oers, 2000) but derives largely from studies concerning school or pre–school children negotiating mathematical meanings in classroom or group situations (for example van Oers, 1996, 2000; Cobb et al, 1997; Gravemeijer et al, 2000; Radford, 2000, 2001). Indeed, other than the work of Sfard (2000), Sierpinska et al (1999a), and Dörfler (2000), much of this latter body of research is not directly applicable to the individual student trying to come to grips with mathematical ideas at the tertiary level.

Finally there is also a very small body of work which relates directly to Vygotsky's theory of concept formation as it applies in the mathematical domain (Sierpinska, 1993, 1994, 2000; Schmittau, 1993).

Notwithstanding the fact that much of the above research does not strongly resonate with what seems to be happening as Mathematics I Major students grapple with new mathematical signs, my theory of how a student constructs a mathematical concept is paradoxically built upon and informed by many aspects of the existing bodies of literature.

To this extent, I have examined debates and tensions around issues such as the relationship between meaning and use, the role of symbols in mathematical learning, the notion of doing mathematics 'as–if' and the tension between research which focuses on the individual and research which focuses on the social. I have also briefly examined and critiqued the approach of the main body of research which is devoted to explaining the appropriation of mathematical objects at tertiary level (the writings of Tall and Dubinsky and their respective followers).

Overall I have concluded that there is a distinct gap in the literature: what is required is an elaborated theory which explains how the individual, who I assume to be socially constituted, constructs personal meanings and develops usages of mathematical signs which are compatible with the culturally established meanings and usages of those signs. I call this "appropriation theory".

CHAPTER 4: RESEARCH QUESTION

For ease of reference and focus, I re-state my basic intentions for this thesis in a summarised and succinct form. These are:

- to explore the descriptive power of Vygotsky's stages of concept formation in relation to the appropriation of a new mathematical object
- to elaborate on the different stages (or phases) in the appropriation of a mathematical object
- to demonstrate that the mechanism for moving through the different phases of mathematical object appropriation is the functional use of new words (or mathematical signs in the mathematical context)
- to develop an "elaborated language of description" with which to talk about the appropriation of a mathematical object by a learner.

These explorations and elaborations constitute my "problem" (Brown and Dowling, 1998); that is, they comprise my research question.

At this juncture it also apposite to remind the reader of Vygotsky's (1986, 1994) postulate about the functional use of a word and my initial application of this notion to the mathematical domain. In the remaining chapters of this thesis, I will elaborate on Vygotsky's postulate, and its realisation through concept formation in the mathematical domain.

§4.1 VYGOTSKY'S FUNCTIONAL USE OF A WORD AND FORMATION OF A CONCEPT

"Functional use" refers to Vygotsky's (1986, 1994) tenet that children use words[1] for communication purposes and for organising their own activities before they have a full understanding of what these words mean. The ability to use these words prior to full understanding gives rise to the impression that the child absorbs ready–made meanings of words in one go. But

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[1] Vygotsky regarded a word as embodying a generalization and hence a concept. For example, words such as 'bird', 'colour', etc. embody the concepts of "birdness", "colour", etc. Thus the meaning of a word is a concept.

Vygotsky argues that this impression is an illusion: the meaning of a word evolves for a child through that child's use of the word.

Furthermore the child does not develop meaning for words independently of their meaning in the social world. Indeed the meaning of the word is given to the child by the use of that word in socially and historically sanctioned discourse. In this way the social world determines the way in which the child's use of that word needs to develop.

Thus the meaning of a word evolves for the child through the child's use of that word with more socialised others.

On an epistemological level, this communication is only possible because the child and adult (or more socialised other) are able to write or speak various signs such as words. (We cannot read each other's minds or thought; we can only hear or read the words the child or adult uses.)

In summary, it is the child's use of words within a social community even prior to 'full' understanding, that enables the meaning of that word or concept to develop for that child in a way that is compatible with that of the community.

Related to this functional use of the word, Vygotsky argued that the evolution of the word or concept takes place through a variety of phases (ie heaps, complexes, potential concepts and concepts). This is his theory of concept formation which I outlined in Chapter 2.

§4.2 FUNCTIONAL USE AND THEORY OF CONCEPT FORMATION IN THE MATHEMATICAL DOMAIN

I apply Vygotsky's postulate about the "functional use" of a word to the learner's appropriation of a new (that is, new to the learner) mathematical object in the tertiary mathematical domain, as follows:

When a student encounters a new mathematical sign (for example, in a mathematical definition) he does not yet know to what that sign refers. But, possibly because of social and pedagogical interventions, the student does not remain inactive; rather he begins to use this mathematical sign (which may be in the form of symbols, graphs, words or diagrams) in communication with others and in mathematical activities even while his understanding of this sign is unformed and immature. Indeed it is this communication and these activities with the mathematical signs (activities comprising of manipulations, imitation, reflection on, associations with, template–matching) that give an initial access point to the new object (vague and remote as the concept of that object may be).

Furthermore and as with a word, the mathematical sign has a meaning which derives from its already established usage in mathematical discourse. Thus the student needs to acquire a mathematical concept whose meaning and use is compatible with its socially–sanctioned meaning and use in the mathematics community.

Another fundamental aspect of functional use relates to the context in which it takes place. That is, in the formal academic context the communicative function of a word (or mathematical sign) is different from its communicative function in the everyday context (Minick, 1996); in the academic context the sign itself becomes the “object of communication activity” (Minick, 1996: 41). Hence the function of the mathematical sign or concept in the academic environment is to focus attention on itself and on its relation to other signs or concepts as well as to communicate. In this regard, the mathematical concept functions as part of a hierarchical system and one concept or sign mediates in the acquisition of another concept or sign.

Related to this functional use of a mathematical sign, I suggest that Vygotsky’s theory of concept formation (1986) can be elaborated to describe the stages or phases through which a student moves as he appropriates a new mathematical object (I do this in Chapter 6). For now I need to highlight certain aspects pertinent to this elaboration.

Vygotsky built his theory of concept formation on his observations of subjects abstracting attributes of concrete blocks and organising and generalising these through the use of words (signs). But the formation of a mathematical concept is based on the abstraction of attributes of the sign (signifier or signified depending on the stage of learning) and their organisation and systematisation via **those same** signs. Thus in order to describe how the meaning of a mathematical sign evolves for a student, I need to elaborate on Vygotsky's theory of concept formation as it applies to the construction of a concept which is organised via those same signs which constitute the concept.

I call this elaborated theory of mathematical concept formation, appropriation theory.

Thus, in terms of appropriation theory, the learner zigzags his way through many phases as he goes about appropriating a new mathematical object. He does this by immersing himself in different mathematical activities with mathematical signs even before he has a mature understanding of these signs. He uses these mathematical signs as objects of communication (like he uses words in the everyday domain) and he uses these signs as objects to manipulate and imitate and as tools with which to systematise his mathematical thinking. This functional use of the mathematical sign is mediated by the learner's knowledge of related signs or concepts (these related signs may be called forth for that learner by particular attributes of the new sign) or by social interventions of a teacher, peer or text.

§4.3 SUMMARY

Appropriation theory is my elaboration of Vygotsky's theory of concept formation to the mathematical domain. It postulates that the appropriation of a new mathematical object by a learner takes place in phases (categorised as heaps, complexes and concepts) and that these phases give a language of description for understanding this process.

As an extension to Vygotsky's theory of the functional use of a word, I postulate that the mechanism for moving through these phases, ie for appropriating the mathematical object, is a functional use of the mathematical sign. In the mathematical domain this functional use manifests in the form of manipulations, reflections on, template–matchings, associations and imitations of the mathematical sign.

CHAPTER 5: METHODOLOGY AND METHODS

§5.1 INTRODUCTION

In this chapter I describe how I went about investigating my research question (posed in Chapter 4). Moreover, I explain how I moved from an informal position, based on my unrecorded observations and interpretations over many years as a mathematical lecturer of undergraduate students to a formal research-oriented position. By speaking of 'how' I moved, I am referring to my methods of doing formal research and collecting 'relevant' data, and to my justification for the appositeness of these methods. These methods, together with their motivations and characterisations, constitute the methodology of my research.

However, before discussing this methodology, I need to clarify which ontological and epistemological positions underpin my mode of research.

In Chapter 2, I dealt extensively with my ontological and epistemological assumptions regarding the nature of mathematics and what it means for a learner to construct a new mathematical concept. In the current chapter my ontological and epistemological concerns are much broader: my attention here is on the nature of the world out there (the ontological question) and how we can get to know it (the epistemological question). In other words my concern in this chapter is with how I, as researcher, am positioned in relation to the researched; on how I am able or not to gain access to the different learners' modes of thinking or usage of signs.

Paradoxically, because epistemological and ontological questions are so complex and profound, and since there is no way I can do justice to these questions in at most a few pages (such questions have plagued and fascinated philosophers for centuries), I will be brief in my clarification of my position regarding these questions. I do this in §5.2.

Having prepared the ontological and epistemological ground, I then proceed, in §5.3, to a discussion of how I went about addressing the problem of my research, ie how and why I went about investigating my research question as I did. This consideration constitutes my methodology. As the reader will see, I base my methodology on Brown and Dowling's (1998) view of research as "the continuous application of a particular coherent and systematic and reflexive way of questioning, a mode of interrogation" (p.1).

In accordance with this methodology, I need to interrogate the empirical domain using constructs from the theoretical field. But in order to do this, I need to develop and delineate both my theoretical and empirical fields. To this extent, I have already focused extensively on my theoretical field in Chapters 2 and 3. But I still need to direct my attention to defining and delineating the empirical domain. This I do in §5.4.

In §5.5, I further refine the empirical setting by describing and justifying the methods I used for my data collection. In §5.5.1, I describe aspects of a survey which I conducted so as to obtain background information about all the students in the Mathematics I Major course and also to select key students from whom ultimately I could select my critical cases for interviewing. In §5.5.2, I describe how I selected these critical cases from these key students. In §5.5.3, I discuss the clinical interview as a site for data collection and I describe and motivate the format of my interviews. These clinical interviews with individual students constituted the main site for data–gathering in my research project. In §5.5.4, I explain why I designed the mathematical task for the clinical interview as I did; how I expected that the different activities in the task would contribute to answering my research problem.

Finally in §5.6 I summarise the entire chapter.

§5.2 ONTOLOGICAL AND EPISTEMOLOGICAL ASSUMPTIONS

As a researcher, I take a constructivist position as my ontological starting point. This means that I assume that order in the world does not exist independently of the human mind; rather I assume that we (and other educational researchers) impose order on the world through our own theoretical constructions.

In line with this ontological stance, I assume that observation is always theory-laden. Due to the work of Hanson, Popper and Wittgenstein,

> researchers are aware that when they make observations they cannot argue that these are objective in the sense of being 'pure', free from the influence of background theories or hypotheses or personal hopes and desires. (Phillips, 1990: 25).

This means that I (or any researcher) can not just look at any situation ‘as it is’. The researcher’s view of the world, what he sees and what he recognises, is necessarily mediated by his particular perspective. In the context of educational research, this perspective largely takes the form of a theoretical framework or paradigm (in my case, a Vygotskian framework).

This problem of theoretical mediation is further compounded by the following consideration: even if I accept that the researcher (in this case myself) is working within a specific paradigm (in a Kuhnian sense), can that researcher achieve objectivity within this paradigm? My answer must be no. Unlike in the natural sciences where the researcher stands in a subject-object relation to that which is being observed, in the educational arena the researcher stands in a subject-subject relation. This is a crucial consideration since the researcher’s insider status in the educational process is a given, unalterable fact. For example, I have my own beliefs, prejudices and no doubt idiosyncratic ways of approaching mathematical learning and any new mathematical definition. This must influence not only *how* I measure, but also *what* I measure. It must affect what I see and what I recognise.

Furthermore, I have my own needs and hopes which necessarily must contribute to how and what I see in the world.

Related to these issues, we also have the epistemological paradox (Brown and Dowling: 47). This paradox refers to the argument that the very act of taking up the position of observer necessarily transforms the practice which is being observed. For example, the researcher’s act of watching a student perform a mathematical activity (even if he does not interact overtly with the student) must transform the student's practice. To wit, there is a very high probability that such a student would behave and indeed think differently if he were doing the mathematical task alone at home or in a library with a textbook compared to what he does when a teacher or researcher is watching. Thus, in my observations and inferences, I need to acknowledge and include the affect of the researcher on the researched[1].

Given all these epistemological arguments, I cannot hope to attain objective truth. The most I can hope for is a convincing, trustworthy and coherent account of what I have observed, regulated by a striving for objectivity and rigour. And in the end, any account that I give must be necessarily an interpretation of the world, an attempt to impose order and structure on the world. As such my interpretation (the theory I develop to describe and explain what I see) is potentially refutable and open to future modifications and elaborations.

In Chapter 9 I address issue of credibility and integrity more directly when I discuss what sort of validity and reliability claims I can make about my account.

§5.3 METHODOLOGY

Brown and Dowling (1998: 137) claim that the fundamental criterion according to which research should be judged is coherence[2]. In order to achieve this coherence they have developed a systematic procedure, a mode of interrogation particular to the needs of an educational researcher.

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[1] Interestingly the epistemological paradox is not only relevant to the social sciences; it also occurs in the physical sciences. For example Heisenberg's Uncertainty Principle asserts that the physicist's measurement of the position of a particle affects the value of the momentum of that particle, and vice versa. Thus those who decry the value of research in which a researcher's effect is unavoidable, need also to critically consider their attitude to research in particle physics.

[2] According to the Shorter Oxford English Dictionary (1964) coherence refers to a thought or speech, "of which all the parts are consistent and hang together".

Furthermore they have developed a "language of description" (p. 2) to describe this procedure.

As with much educational research, Brown and Dowling's concern is with research that takes places in an empirical context (for example, an undergraduate mathematics course). However, since the researcher cannot look at this empirical context without preconceptions (see §5.2), they justifiably argue that the researcher needs to explicate these preconceptions. This explication constitutes the theoretical field of the research (for example, socio-cultural learning theory).

Furthermore, to enable the production of coherent findings, the researcher needs to interrogate the empirical context in terms which derive from her theoretical context. But to do this effectively, she first needs to progressively and systematically refine both the empirical and theoretical contexts so that they can articulate with each other.

I will illustrate and elaborate the nature of these refinements below, rather than continue this discussion in these general terms.

In Chapter 6, I will further show how these refinements enable me to use concepts from my theoretical domain to interrogate data from my empirical domain in a rigorous fashion.

§ 5.3.1 REFINING THE THEORETICAL AND THE EMPIRICAL CONTEXTS

Brown and Dowling (1998) assert that the researcher needs to specialise the theoretical context by moving her attention from the broad theoretical field within which the research is located (for example, Vygotsky's theory of functional use of a word and his theory of concept formation[3]) to a narrower context consisting of key antecedent work. They call this narrower context, the problematic. In my research, the problematic consists of those studies which focus on the role of symbols in the appropriation of mathematical

[3] In Chapter 2, the Theoretical Framework, I examined aspects of the broad theoretical field of my research.

objects and those studies which look at the learning of mathematics concepts at the undergraduate level[4].

Crucially the theory then needs to be specialised so that it can be expressed in the form of a research question or a set of propositions which will ultimately be used to interrogate the empirical domain. Brown and Dowling refer to the set of research questions or propositions as the problem. In Chapter 4, I stated that my problem concerned the applicability of Vygotsky's theory regarding the functional use of a word, to the appropriation of a mathematical object.

In Chapter 6, I continue with this specialisation of the theoretical context by developing my theoretical concept variables (these are the different phases of concept formation such as pseudoconceptual thinking, diffuse complex thinking, heap thinking and so on). I want to ultimately see whether these theoretical concept variables are a useful and possible tool with which to interpret or interrogate the data.

Simultaneous to this theoretical specialisation, the empirical field needs to be localised. Broadly speaking, the general empirical field for my research is the appropriation of mathematical objects by undergraduate students enrolled in the Mathematics I Major course at the University of the Witwatersrand. More specifically, since my formal observations of this appropriation take place as these students perform specially designed mathematical activities in a clinical interview setting, my empirical field can be localised to the clinical interview setting. (In Brown and Dowling's terms, the clinical interview context is the local empirical setting of this research.) In §5.4, I broadly describe aspects of both the Mathematics I Major course and the clinical interview setting.

In §5.5, I further localise the empirical setting by explicating and justifying the particular methods which I use to access data.

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[4] In chapter 3, the Review of Literature, I examined aspects of the problematic of my research.

In Chapter 6, I refine my empirical setting yet again by developing the empirical indicator variables. These indicator variables are elaborated descriptions of how the theoretical concept variables (ie the phases of concept formation) are made manifest in the empirical setting, the clinical interview.

It is this systematic and dialectically constructed link between the theoretical concept variables (from the theoretical domain) and the indicator variables (from the empirical domain) that ultimately allows me to interrogate the empirical setting in terms of my theoretical constructs in a rigorous and coherent fashion.

§5.4 EMPIRICAL DOMAIN

§5.4.1 EMPIRICAL FIELD

As I discussed in Chapter 1, this study is set within the context of the Mathematics I Major course at the University of the Witwatersrand. In that chapter I indicated that the course has a mixed and heterogeneous student population: students come from both the economically and culturally advantaged sector of the population (for example, both parents may be university graduates) as well as from the economically and culturally disadvantaged sector (for example, one or more parents may be illiterate or innumerate) [5]. Although all students in the course have studied matriculation Higher Grade mathematics, the students emanate from a range of schools and thus have a range of mathematical backgrounds. For example, some students have studied Additional Mathematics at school (and obtained A symbols for this subject) [6] whereas other students have obtained an E symbol (40% - 49%) in Higher Grade Mathematics[7].

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[5] I am using the terms ‘culturally advantaged’ or ‘culturally disadvantaged’ vis–a–vis the dominant academic culture at the University of the Witwatersrand.

[6] Additional Mathematics is a Grade 12 subject offered only by certain schools who have the necessary staff and resources to teach the course. It is a course of advanced mathematics taken by the better mathematical students in addition to the normal

In 2000 there were 310 Mathematics I Major students. These students were allocated, subject to timetable constraints, to one of two parallel courses given at different times by different lecturers. The lectures took place six times a week (45 minutes per lecture) in a large lecture hall[8]. In addition each student was assigned to one tutorial class. These tutorial classes are weekly periods during which about 25 students come together in a class with a lecturer and student assistant. The tutorial classes are primarily periods in which the student can consult the lecturer or student assistant on particular tutorial problems or mathematical concepts. The tutorial problems are mathematical exercises which have been set, prior to the tutorial period, by the course co-ordinator (myself, in this instance), and are usually from the prescribed textbook. Students are expected to have engaged with the tutorial problems on their own or with friends before the tutorial. A tutorial class is about 45 minutes long.

I lectured one set of Calculus classes, a colleague lectured the other parallel course. At the beginning of the year, I selected 27 students from my colleague’s class and assigned these students to my once-a-week tutorial class. (Details of the selection of this group are given below in §5.5.2.1). These students constituted the group (which I refer to as the critical group) from whom I chose my critical cases (again details of this selection are given below in §5.5.2.2).

I constituted the critical group at the beginning of the year with myself as tutor for several reasons. First I wanted those students, some of whom I knew I was going to be interviewing later in the year, to get used to me. Secondly, I wished to meet the critical group students so that I could screen out those students whom I thought would not be suitable for interviewing

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Mathematics course.

[7] These students have been allowed to enter the Mathematics I Major course after having passed a different but easier first-year mathematics course (frequently the Mathematics I Auxiliary course at the University of the Witwatersrand. This is a course which is orientated towards students studying the biological sciences).

[8] Slightly more than half of the lectures were devoted to Calculus; the rest to Algebra.

(for example, because they spoke too softly or because they were intimidated by me). Additionally, I wanted to habituate myself to these students.

Another important aspect of the Mathematics I Major course (and indeed of the clinical interview setting) was the prescribed calculus textbook (Larson et al, 1998)[9]. The textbook is geared towards the multiple representation approach, and as such has many examples of graphs throughout its prose and the exercises. The textbook was used as the basis for the lectures as well as providing a wealth of exercises from which the tutorial exercises were selected by the course co-ordinator.

In Chapter 1, I discussed other crucial aspects of the Mathematics I Major course at the University of the Witwatersrand in 2000. For example, I discussed the focus on traditional teaching methods and assessments in the course and I also noted that lecturers in both classes covered the same curriculum and that students in both classes wrote the same tests and examinations. Since I do not wish to replicate what I have already written, I will not repeat details here; however the reader may wish to refresh her or his memory with that discussion.

§5.4.2 EMPIRICAL SETTING

I interviewed nine[10] different students, all chosen from my critical group, twice during the year 2000 (in April and October). Each interview was video-taped for later transcription. I was the interviewer in each instance.

I will not go into details about the interviews here (for example, how students were selected and the nature of the mathematical task); that I will do in the Methods section, §5.5. For now, I just wish to give a broad outline of the purpose of the interviews, and how I eventually used the interview experience and the data.

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[9] Although the sixth edition (1998 edition) of this textbook was prescribed, students could also use the fifth edition (1994 edition) of this same textbook.

[10] I actually began with ten students but one dropped out of the course during the year.

Both interviews were mathematically-oriented interviews. This meant that in each interview the student was given a particular set of specially designed mathematical activities which he was asked to do, talking out loud as he did. The student had some access to the normal course textbook during the interview although I regulated his use of the textbook (see the section on the interview guidelines below). Throughout the interview the student was encouraged to write and to speak out loud.

The first interview was primarily a practice run for the second interview. It gave me an opportunity to practice and refine my interviewing techniques and to learn from this experience. Also, it gave students the opportunity to practise their talking-aloud skills. Furthermore, my initial analyses of the first interviews helped me develop my categories of theoretical concept variables (ie the phases of thinking in mathematical object appropriation) and their indicator variables. The format and style of both interviews were the same but the mathematical task was necessarily different.

In this thesis, I have only used data from the second interview directly. Hence I call this second interview, the Mathematics Interview.

§5.5 METHODS

§5.5.1 THE SURVEY

In the first week of the academic year 2000, I conducted a survey amongst Mathematics I Major students. The primary purpose of this survey was to collect data about the students which I could then use to purposefully choose students for the critical group. I give details of the selection criteria in §5.5.2.1 where I discuss my choice of students.

In addition to the survey's purpose as a selection tool for choosing students for the critical group, I wanted to obtain background information about the students in the Mathematics I Major class (for example, whether students liked to study on their own or with other students; whether students were primarily doing the Mathematics I Major course for its own sake or as a co-requisite or pre-requisite to other courses). I have used some of this information to inform various discussions throughout this thesis (for example, in Chapter 1 and Chapter 3).

At the end of the academic year I did another survey where my questions were either the same as in the first survey or were modified so that they were appropriate to the end of the year. The primary purpose of this second survey was to see whether attitudes had changed over the year. Again I have used some of this information to support certain of my arguments throughout this thesis.

I will now explain how I used data from the first survey to select 27 students for the critical group.

§5.5.2 SELECTION OF STUDENTS

§5.5.2.1 CRITICAL GROUP

As previously stated, the critical group was constituted at the beginning of the year so that it could serve as a small pool (twenty–seven students) from which I could eventually select ten students for interviewing.

My selection of the twenty-seven students was based on the data obtained in the first survey, which was given to all students who attended the first mathematical tutorial of the year.

The selection was purposeful in that I wanted to obtain a spread of students in terms of academic standing and in terms of the type of school the student had attended (well-resourced or poorly resourced).

My methods for selecting the twenty-seven students were as follows:

First I split all the survey forms into three groups: students who chose to remain anonymous when filling in the survey questionnaire, students who attended my calculus lectures (slot E) and students who attended my colleague’s calculus lectures (slot D). I only selected (non–anonymous) students from Slot D. There were 159 students in this category.

Within the Slot D class, I sorted the students by two categories: academic status and type of school last attended.

Academic status referred to the student's matriculation Higher Grade mathematics examination symbol or the nature of the last mathematics course they had completed (if it was not the same standard as the matriculation Higher Grade examination). Students who had obtained an A, B or C for the matriculation Higher Grade mathematics examination were further divided into two sub-categories: those who had done Additional Mathematics at matriculation level and those who had not. In the end, there were nine sub-categories within the academic status category: Students who obtained an A, a B or a C for the matriculation Higher Grade mathematics examination and had not done Additional Mathematics at school; students who obtained an A, a B or a C for the matriculation Higher Grade mathematics examination and had done Additional Mathematics at school; students who were repeating Mathematics I Major; students who had obtained less than a C for the matriculation Higher Grade mathematics examination but had subsequently passed the easier Mathematics I Auxiliary course at the University of the Witwatersrand; other students. All of this information is summarised in Table 1.

Within each of the academic status sub-categories, I further distinguished between students who had gone to township or non–urban schools (coded as T/N students) and those who had gone to historically white, progressive or private schools (coded as H/P students)[11]. In South Africa the legacy of apartheid still lives on in that the historically white, progressive or private schools are generally much better resourced than the township and non–urban schools. Indeed, these latter schools are frequently under-resourced in terms of teachers, books, general infrastructure such as desks and even electricity supplies.

In my selection of the critical group students I wanted to make certain that some of the students from township and non–urban schools were represented.

Having split the students by academic status and type of school, I then decided how many students to select in each of these categories. As can be seen, I picked disproportionately more students from each B or C symbol category compared to each A symbol category. This was consequent upon my belief that the use of mathematical signs by weaker or slower students would reveal more about their thinking than this usage by quicker and more able students (who would abbreviate more processes).

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[11] Adler (2001: 162) distinguishes between these sorts of schools as follows:

- Non–urban schools refers to schools outside the urban areas of South Africa.
- Township schools are schools in those areas designated for separate development during the apartheid era. Townships were generally situated just outside the major cities. For example, in the apartheid era Soweto was a township of Johannesburg. Today Soweto is part of Greater Johannesburg.
- Historically white schools are those schools designated 'white' during the apartheid era. These schools are typically in the suburbs of major cities, and now have multiracial student bodies.
- Progressive schools refer to those few schools that were explicitly non–racial, and offered alternate approaches to the prescribed state curriculum. They were not state–run schools.
- Private schools (a category not used by Adler but relevant for my study) refers to schools which are not part of the state–run school system but which are not necessarily progressive in the sense used above. For example, religious Jewish schools, religious Muslim schools.

Also I picked proportionately fewer students from the Repeat category (since I was interested in observing how students dealt with mathematical objects which were new to them, and Repeat students would have been exposed to mathematical objects like the improper integral previously). Where possible, I also selected proportionally more students from township and non–urban schools than from historically white, progressive or private schools since these represent an important category of students at the University of the Witwatersrand, disadvantaged as they have been by South Africa's history. But within each of the academic and school subcategories, my selection of the actual students was random.

In the first tutorial, I explained to the critical group students that I was doing a research project looking at how students thought about certain mathematical concepts. I further explained that I would want to interview a number of students from that tutorial group over the course of the year. I stressed that being part of this group was voluntary. If a student did not wish to be interviewed, he or she could change the tutorial group (an administrative formality); the student would be allocated to a different tutorial group without further discussion. I emphasised that the decision to remain in the critical group was entirely voluntary.

In the event, none of the students (save one who never arrived at the first tutorial and who soon after deregistered from the course) moved from the critical group to other tutorial groups. But later in the year, one student left the university and two students left the course.

Table 1: Numbers of students doing Mathematics I Major (Slot D) selected for critical group by academic status and type of high school attended

Academic status	**Total number of students**	**Students by type of school**	**Number of students selected for critical group**	**Total number of students selected for critical group**
Matriculation A grade plus Add. Math.	24	0 T/N	0 T/N	4
		24 H/P	4 H/P	
Matriculation A grade, no Add. Math.	29	3 T/N	1 T/N	5
		26 H/P	4 H/P	
Matriculation B grade plus Add. Math.	9	0 T/N	0 T/N	2
		9 H/P	2 H/P	
Matriculation B grade, no Add. Math.	19	4 T/N	1 T/N	5
		15 H/P	4 H/P	
Matriculation C grade plus Add. Math.	0			0
Matriculation C grade, no Add. Math.	21	5 T/N	2 T/N	5
		16 H/P	3 H/P	
Repeats	27	8 T/N	1 T/N	2
		19 H/P	1 H/P	
Other first-year maths courses	18	3 T/N	2 T/N	4
		15 H/P	2 H/P	
Others, eg post-matriculation level, A–level math.	12			0

§5.5.2.2 CRITICAL CASES

After about five weeks, I selected ten students from the critical group as my critical cases. This selection was purposeful in two ways: first I wanted to get a cross-section of students in terms of academic levels and in terms of types of school background. Secondly there were practical considerations.

With regard to the former point, I wanted to test and elaborate my proposition (see Chapter 4) in diverse cases, diverse in terms of both school and academic background and achievement level. For this purpose I decided on how many students to select in each of the different sub-categories (see Table 2). As can be seen I chose proportionally more students from historically disadvantaged backgrounds.

With regard to the practical issues, I only wanted to select students who seemed to be willing to discuss their mathematics with me and students who did not seem unduly intimidated by me. After all, it seemed pointless to try and extract information from students who seemed unwilling or afraid to talk to me. Accordingly I first removed the 'incommunicative' students[12] from the pool before randomly choosing which students to interview by each sub-category. (I have indicated the number of 'incommunicative' students in each sub–category by the label: Incom)

In Table 2 I present a summary of the number of students selected by sub-category. As can be seen there were ultimately five students from T/N schools and five students from H/P schools.

[12] I categorised two H/P students and one T/N student as incommunicative.

Table 2: Numbers of students selected from critical group as critical cases (for interviewing) by academic status and type of high school attended

Academic status	Students by type of school	Number of students selected for critical group	Number of students selected as critical cases
Matriculation A grade plus Add. Mathematical	0 T/N	0 T/N	0 T/N
	24 H/P	4 H/P	1 H/P
Matriculation A grade, no Add. Math.	3 T/N	1 T/N	1 T/N
	26 H/P	4 H/P (1 Incom)	1 H/P[13]
Matriculation B grade plus Add. Math.	0 T/N	0 T/N	0 T/N
	9 H/P	2 H/P	0 H/P
Matriculation B grade, no Add. Math.	4 T/N	1 T/N	1 T/N
	15 H/P	4 H/P (1 Incom)	1 H/P
Matriculation C grade plus Add. Math.	0		0
Matriculation C grade, no Add. Math.	5 T/N	2 T/N	1 T/N
	16 H/P	3 H/P	2 H/P
Repeats	8 T/N	1 T/N	1 T/N
	19 H/P	1 H/P	0 H/P
Other first-year maths courses	3 T/N	2 T/N (1 Incom)	1 T/N
	15 H/P	2 H/P	0 H/P
Others, eg post-matriculation level, A–level math.	12	0	0

[13] This student left the course in mid-year.

§5.5.3 THE MATHEMATICAL INTERVIEW

§5.5.3.1 OVERVIEW

The mathematical interview was structured along certain dimensions, and semi-structured along others. It was structured in that all students were given exactly the same set of predetermined mathematical activities to perform (see §5.5.4 for a discussion of these activities); it was semi-structured in that my responses and prompts, as interviewer, depended to a large extent on the actions of the interviewee and on my relationship with that particular student.

In a Vygotskian sense, the zone of proximal development (ZPD) differed from student to student in the interview situation. Hence my role in that zone, how I interacted with the student, what I did and what I said (ie my prompts and probes), differed from student to student. The reader is also reminded that in terms of my conception of the ZPD (see §2.3), the actual written task (which in this case was designed with both pedagogic and research intentions) and the textbook (which was designed with pedagogic intention) are important elements of the ZPD.

Although the social interactions (including interactions between student and interviewer, and between student and pedagogically-designed text) were necessarily different, I strove for consistency on certain dimensions in all interviews. In particular, since I was trying to observe how students appropriated a new mathematical object which was presented to them in the form of a written definition, each interview was framed by the same set of guidelines. These guidelines (see §5.5.3.2) also provided a loose type of structure.

Despite these commitments to a measure of consistency, the clinical interviews in this study (and in any other educational research type study) are necessarily not neutral. This is because clinical interviews, just like any other learner-teacher engagement, are social productions and they cannot be anything else. As alluded to above, I was able to engage far more effectively with some students rather than others in the interview situation (in the sense of being able to generate more penetrative probes into those students' usage of signs or knowing when there was little point in letting the student struggle on without assistance, and so on[14]). Indeed this difference in mode of relationship between a teacher or interviewer and one or other student, is not unique to the clinical interview setting; it exists in all learning-teaching situations and, in terms of a Vygotskian framework, directly affects how and what a student learns. In this regard, Minick, Stone and Forman (1993: 6) assert:

> Educationally significant human interactions do not involve abstract bearers of cognitive structures but real people who develop a variety of interpersonal relationships with one another in the course of their shared activity in a given institutional context. … For example, appropriating the speech or actions of another person requires a degree of identification with that person and cultural community he or she represents.

To compound the argument that all learning-teaching situations are unique and cannot ever be the same for all students, there are other major differences between the interview setting of my study and other sites where a student may engage with a 'new' written mathematical definition: the task in the interview was designed with both research and pedagogical purpose (rather than just a pedagogical bias); my responses (as interviewer) were also more geared to finding out what the student was thinking (the research role) rather than assisting (the teaching role); the very fact that I was just present (rather than the student sitting on her or his own) must also have profoundly affected the thinking of that student.

Overall, I suggest that the interview setting of my project can be viewed as an extreme version of a typical learning-teaching situation. It does not mirror what happens in a typical learning-teaching situation, but it has several features in common. It is like a caricature of such a situation: presumably it exposes certain aspects of the learner's thinking (or usage of signs) and obfuscates others. With this in mind, I now proceed to a discussion of the actual format of the interview.

[14] For example, with certain students whose primary language was not English, much of my energy was spent in interpreting what they said.

§5.5.3.2 FORMAT OF THE INTERVIEW

As previously stated, nine students were interviewed, one at a time, at a mutually convenient time, over a period of about two weeks in October 2000[15]. This was the second mathematics interview for each student, and, as previously stated, my analyses in this thesis are based on the data from these interviews. Each interview took place in my office and was video-taped and later transcribed. The maximum duration of the interview was 1.5 hours (consequently some students did not complete all the activities which I had designed for the interview).

At the commencement of the interview, I reminded the student that I was doing research into how Mathematics I Major students did mathematics and how they thought about mathematics. I explained that I needed the student to talk out loud as he went about doing a set of mathematical activities, to explain what he was doing and why. I also told the student to write as much as he wanted[16].

The student controlled the pace of the interview: each mathematical activity was on a new page and the student was told to tell me when he was ready for the next page. As the task progressed, the student was allowed to refer to previous activities or definitions whenever he wanted. The student was also permitted limited access to the textbook (Larson et al, 1998 or Larson et al, 1994) provided that I agreed, and he could also ask me for assistance (although I did not always oblige). My regulations of the student's access to these resources were in accordance with my research questions and related guidelines.

The guidelines (which I describe below) were designed to support my research needs. Basically I wanted to see how the student went about

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[15] This is close to the end of the academic year in South Africa.

[16] This was the same procedure as I adopted in the first interview (with the same students).

appropriating a ‘new’ mathematics object, the improper integral[17], which had been given to him in the form of a definition[18]. The basic questions were: Was the student able to use the new mathematical object immediately after reading the definition? Was he able to use the new object effectively even when he did not know what this object was? If not, what did he do? If yes, how did he use the object? In theoretical terms, was Vygotsky’s notion of the functional use of a word a useful and illuminating way of understanding how a student used a new mathematical object? What were the stages, if any, in concept formation (as made manifest through the student’s usage of signs)? Specifically, could and should Vygotsky’s stages be elaborated and adapted to the mathematical domain? If so, how?

On a finer level, I wanted to see how the student used the object and its signifiers before he had worked with examples of the object and before he had seen graphical representations of that object. Was he able to generate exemplars of the object, to draw it, to manipulate it? If so, how did he do this? How did he use objects with which he was still coming to terms, in non-procedural problems? How did he use these in procedural problems?

I will look at how the activities in my task were designed to address these questions in §5.5.4 below. Here I look broadly at how the guidelines were intended to support this endeavour.

- In the first instance, the student was given a mathematical activity which he was expected to attempt with no assistance from the textbook or myself. For example, in the first activity he was given a definition of an improper integral and asked to generate exemplars of such an object.

[17] When considering the improper integral as an object, the characteristics of that object (such as its convergence or divergence) obviously need to be considered as fundamental aspects of that object.

[18] See Appendix A for a copy of the task given to the student in the interview.

- If he was unable to move forward at all, he could proceed to the next activity and return to the problematic activity later (for example, relatively later activities involved exposure to exemplars of the object).
- If he was still unable to move forward, despite having attempted later activities, the student was allowed to access the textbook (Larson et al, 1998 or Larson et al, 1994). This textbook contains many resources relevant to the mathematical activities. For example, it has definitions of the two types of improper integrals which include different cases (in the definitions given in the task only one case of each type of improper integral is presented); it has exemplars of different types of improper integrals and it has graphical representations of these different exemplars.
- If the student was still unable to move forward, I gave the student direct guidance on how to do the problem.
- At any stage, the student was allowed to modify or refer to a previous answer or a previous definition or question (these returns to previous answers or definitions or questions may have indicated a perturbation, or they may have indicated how a certain usage of signs led to a changed understanding of the object, and so on).

Within the framework of these guidelines, I also pre-prepared a set of standard responses specific to each question in the task.

Finally, I note that although it may sometimes appear that my interventions with each student were idiosyncratic and determined only by the immediate situation, the above guidelines gave an underlying structure and motivation for these interventions.

§5.5.4 THE TASK

Before examining the research and pedagogic intentions of each of the activities in the task, I would like to make two introductory comments.

- The mathematical task which I designed for the clinical interview has characteristics in common with tasks in both advanced mathematics and elementary mathematics. (This mathematical task, as it was presented to the students, is reproduced in Appendix A.)

  As I discussed in Chapter 3, Tall's (1995) distinction between elementary and academic mathematics hinges on the status of a mathematical definition. Specifically he argues that in elementary mathematics, the student is expected to be able to describe a new mathematical object after having experienced it; in advanced mathematics, the student is expected to deduce properties of the object from the definition. With reference to the task in this study, Questions 5 and 7 (procedural questions) primarily require skills in elementary mathematical thinking[19]; Questions 1, 2, 3, 4 and 6 primarily require skills in advanced mathematical thinking (questions in which the student has to generate exemplars or definitions or graphical representations of particular objects).

- Although I designed the task with both research and pedagogical foci, the structure of the task is pedagogically consistent with the method of theoretical generalisation (Kozulin, 1990). According to this method, which is espoused by various post-Vygotskians such as Engeström (1996) and Davydov (cited in Kozulin, 1990: 258)

  > the task of educators and psychologists is to design special scenarios of learning activity that can lead to theoretical reasoning. Such an activity should guide students from the abstract to the concrete, that is from the most general relationship characteristic of the given educational subject to its concrete empirical manifestations. (Kozulin, 1990: 258)

  This movement from the abstract to the concrete hinges on Vygotsky's distinction between scientific knowledge (ie abstract knowledge) and everyday knowledge (ie concrete knowledge).

  Sierpinska (1998: 48) further clarifies this pedagogical perspective:

[19] I discuss each question below.

> In the Vygotskian vision of the mathematics classroom, it is normal to have the students study a concept's definition as a starting point for the acquisition of this concept: they would be expected to analyse its logical structure; to find examples and nonexamples of the concept; and to figure out its place in the structure of a theory – its possible applications and so on.

§5.5.4.1 STRUCTURE AND ACTIVITIES OF THE TASK

In order to acquaint the reader with the different activities of the task and their purpose, I will first discuss the overall structure, rationale and content of the task.

The task concerned the appropriation of the improper integral, with its various properties. Although the students had never studied improper integrals before, the students had previously studied objects 'similar' to the improper integral (such as the indefinite integral and the definite integral) in lectures. They had also studied limits and they were currently studying the convergence and divergence of sequences.

The different activities in the task were designed and ordered so as to give the students opportunities to use the mathematical signs in a functional way. That is, although I obviously did not know how each student would approach the different activities in the task[20], I wanted a set of activities which allowed for the possibility of reflections, manipulations, associations, template–matchings, imitations, perturbations and so on.

To this extent, the first two activities were both definitions from which the student was expected to generate exemplars (in Question 1 an improper integral was defined; in Question 2 an improper integral with an upper infinite limit was defined). In Question 3 the student was asked to represent the mathematical object defined in Question 2 graphically. In Question 4 the student was asked to generate a definition of an improper integral with a lower infinite limit. In Question 5 the student was asked to determine whether certain improper integrals with infinite limits converged or not (these were procedural questions). The questions which followed

[20] If I had known this, the entire research project would have been redundant.

(Questions 9, 10 and 11)[21], were all non–procedural questions concerning the improper integral with an infinite limit. Questions 6, 7 and 8 were similar to Questions 5, 2 and 3 respectively except that they dealt with an improper integral with an infinite discontinuity (improper integral Type II) and the order of questions was altered. That is, in question 6 the definition of an improper integral with an infinite discontinuity was presented and the student was immediately shown two examples of this object and asked to determine whether they converged or not; in Question 7, the student was asked to generate an example of an improper integral with an infinite discontinuity and in Question 8 the student was asked to represent the improper integral with an infinite discontinuity graphically. I wanted to see how the student's exposure to examples of the mathematical object, prior to his having to generate an exemplar of the object, affected his usage of signs.

§5.5.4.2 QUESTION BY QUESTION

Question 1

Improper Integrals

The definition of a definite integral $\int_a^b f(x)dx$ requires that the interval [a, b] is finite and that f is continuous on [a, b].
If one of the limits of integration is infinite, or the function f has an infinite discontinuity on [a, b], we call the integral an improper integral.

1. Can you make up an example of an **improper integral**?

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[21] Unfortunately, due to an administration error, the questions were numbered incorrectly. Question 9, 10 and 11, in fact, came directly after Question 5. Question 6, 7, 8 came directly after Questions 9, 10, 11. Thus, the order of questions was: Q1, Q2, Q3, Q4, Q5, Q9, Q10, Q11, Q6, Q7, Q8. Naturally I handed the questions out in the corresponding order.

This was the first time most of the critical case students[22] had ever seen a reference to an improper integral.

I wanted to see what sort of object the student generated from this definition given that the student was given no perceptual or experiential access to the object. Indeed I expected that the student would gain access to this object via prior knowledge of a definite integral, a limit of integration and an infinite discontinuity.

In the end, however, it transpired that the students had not previously studied the notion of an infinite discontinuity in class (unlike in previous years); accordingly I could not and did not analyse any of the students' responses to Question 1[23].

Improper integral Type I

If f is continuous on the interval [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$.

If $\lim_{b\to\infty}\int_a^b f(x)dx$ exists, we say that the improper integral converges. Otherwise the improper integral diverges.

The above definition is of an **improper integral with an infinite integration limit.**

Here the student was presented with the definition of an improper integral Type I. This definition was given in terms of an equivalence of signifiers, ie "$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$". (This definition was mathematically necessary

[22] Exceptions were Mary, who was a repeat student, and Fred who had studied Maths I Auxiliary the previous year.

[23] Fortunately this unanticipated hitch did not in any way impede the students' activities in the remaining questions of the task.

because $\int_a^b f(x)dx$ had until that stage only been defined where a and b are both finite.)

As can be seen, only the case of the improper integral with an upper infinite integration limit was given; later (question 4) I asked the students to generate a definition of an improper integral with a lower infinite limit.

Question 2

(a) Can you make up an example of an improper integral with an infinite integration limit?

(b) Can you make up an example of a convergent improper integral with an infinite integration limit?

I wanted to see what sort of object the student constructed from the definition of an improper integral with an infinite limit; whether he attended to the whole definition, or just part of it or whether he distorted aspects of the definition into a form with which he was somehow comfortable[24].

Question 3

Can you explain what an improper integral with an infinite integration limit represents graphically.

Since both Questions 1 and 2 could be done purely through template–matching, without any real 'understanding' of what the improper integral is, Question 3 was designed to see how the student graphically represented the mathematical object defined in Question 2 (an adequate representation would require some sort of 'understanding' of what an improper integral was). Thus Question 3 was a window into how the student

[24] Kozulin (1990: 250) noted how some American students (in the context of a study geared towards finding out whether students were able to spontaneously recognise contradictory information in texts) distorted certain information that they were reading, when it contradicted other information with which they were already comfortable. In other words, contradictory information did not cause a cognitive perturbation; rather it resulted in the student's distortion of what he was reading.

conceived of an improper integral (when directly asked to do so). I was particularly curious to see if some students were able to deal with Questions 1 and 2 adequately, despite having a distorted sense of the object (the improper integral) as revealed by their answer to Question 3.

Question 4

4. How would you define $\int_{-\infty}^{b} f(x)dx$?

In Question 4 the student was asked to generate a definition of an improper integral with a lower infinite limit, so giving the student further opportunity to work with and reflect on the definition of Question 2. That is, the student could use the definition of $\int_{a}^{\infty} f(x)dx$ to generate $\int_{-\infty}^{b} f(x)dx$ by analogy.

If the student got stuck, he could return to Question 4 after Question 5(b) (in Question 5(b) he was asked to calculate an improper integral with a lower infinite limit). In that case, I was interested to see how manipulating an improper integral with a lower infinite limit (in Question 5(b)) affected the student's generation of an appropriate definition of an improper integral with a lower infinite limit..

Question 5

(a) Determine whether $\int_{1}^{\infty} \frac{dx}{x^3}$ converges or diverges.
(b) Determine whether $\int_{-\infty}^{1} xdx$ converges or diverges.
(c) Determine whether $\int_{1}^{\infty} \frac{dx}{x}$ converges or diverges.

I expected that certain students would struggle to generate exemplars of an improper integral (Questions 1 and 2) or to represent an improper integral graphically (Question 3) or to define a similar object (Question 4) without having seen an exemplar of the object. Thus in Question 5, I presented several exemplars of improper integrals with infinite limits.

The student was asked to decide whether each of these examples was convergent or divergent, thus giving him the opportunity both to see examples of an improper integral with an infinite limit, and to manipulate these examples.

Because I did not want the student to be diverted by the need to use complicated integration techniques, I specifically chose functions which were very easy to integrate.

Questions 9, 10, 11[25]

Determine whether the following statements are true or false. Justify your answer in each case.

9. If f ′ is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f'(x)dx = -f(0)$.

10. If the graph of f is symmetric with respect to the y-axis, then $\int_0^{\infty} f(x)dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)dx$ converges.

NOTE: If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$, where c is any real number.

The improper integral on the left diverges if either of the improper integrals on the right diverges; otherwise it converges.

11. If f is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)dx$ converges.

Questions 9, 10 and 11 were all taken from Larson et al (1998) and typify the sort of non-procedural questions which Mathematics I Major students occasionally have to deal with. Since I expected that several students

[25] I remind the reader that Questions 9, 10 and 11 preceded Questions 6, 7 and 8. See Footnote 21 in this chapter.

would be able to do Question 5 adequately (a procedural question) despite not having conceptual knowledge of the improper integral, I inserted Questions 9, 10 and 11. These latter questions require conceptual knowledge of an improper integral to be done successfully and so allowed me to distinguish between pseudoconceptual and conceptual thinking about an improper integral. They also provided the student with further opportunity to use (manipulate, reflect, imitate and so on) the improper integral.

In addition to knowledge of the improper integral, Questions 9, 10 and 11 all require conceptual knowledge of diverse mathematical concepts. For example, in Question 9, the student had to apply various limit laws. In Question 10 the student had to deal with the notion of symmetry and also the logic of the 'if and only if' statement. In Question 11, the student had to generate a counterexample; he had to use his answer to Question 5(c) as a basis from which to generate a function f(x) for which f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$.

Although Question 9 was non-standard, it could be solved deductively. To wit (and without explicitly justifying any of my steps): $\int_0^\infty f'(x)dx =$

$$\lim_{b\to\infty}\int_0^b f'(x)dx = \lim_{b\to\infty}[f(x)]_0^b = \lim_{b\to\infty}[f(b)-f(0)] = \lim_{b\to\infty}f(b) - \lim_{b\to\infty}f(0) = 0 - \lim_{b\to\infty}f(0) =$$

$-f(0)$.

As can be seen from this solution to the problem, Question 9 can be solved by manipulating each mathematical expression according to the relevant rules (for example, the rule that the sum of two limits equals the limit of two sums where these are defined).

I was interested to see how a student, who did not seem to have a good sense of the improper integral (as revealed by previous usages in the task) approached the problem. For example, could the student deal with the deductive reasoning required in Question 9 even if he was unable to generate exemplars of improper integrals and vice versa? If yes, what did this mean about his mathematical knowledge of improper integrals?

Question 10 was rather more complex than Question 9. Although deduction and manipulation were required, certain inferences also needed to be made. For example, the student needed to infer that, since f was symmetric with respect to the y-axis, $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{\infty} f(x)dx$ were both convergent or both divergent.

Furthermore, answering the question successfully required an understanding of the correct usage of the 'if and only if' statement (which we assume Mathematics I Major students have).

Question 11 was probably the most challenging question in the task. I expected that some students, not knowing where to go with this problem, would try and apply properties of f(x) to $\int_{0}^{\infty} f(x)dx$ (such an approach would get the students nowhere but, from prior experience, I was aware that students sometimes use such associations).

In fact, to solve this problem successfully, the student had to use his answer (if it was correct) to Question 5(c) to generate a suitable counter-example. That is, $\int_{1}^{\infty} \frac{dx}{x}$ is divergent (see Question 5(c)). But the integration interval in Question 5(c) is [1, ∞) rather than [0, ∞) as required in Question 11. Furthermore the function in Question 5(c) is $\frac{1}{x}$ and $\frac{1}{x}$ is not continuous on [0, ∞). However by shifting $\frac{1}{x}$ one unit to the left, the function becomes $\frac{1}{x+1}$. Now $\frac{1}{x+1}$ is continuous on [0, ∞) and $\lim_{x\to\infty} \frac{1}{x+1} = 0$. Furthermore $\int_{0}^{\infty} \frac{1}{x+1}dx$ is divergent and hence provides a counter-example to the given statement.

QUESTIONS 6, 7, 8

Question 6, 7 and 8 were very similar to questions 2, 3 and 4, except that the order of questions here was changed and the object was an improper integral Type II. With questions 6, 7 and 8, exemplars of an improper integral Type II were given immediately after the definition (Question 6) but *before* the student was asked to generate exemplars of the object (Question 7) or to represent it graphically (Question 8).

By altering the order of questions (as compared to Questions 2, 3, 5), I hoped to see whether seeing an exemplar of the object affected the way a student appropriated a new object[26].

First the student was given a definition of an improper integral Type II (see below). Also, a definition of an infinite discontinuity (already referred to in the definition of an improper integral and which students were supposed to have studied in class) was presented. As with the definition of an improper integral Type I, only one case (where the discontinuity is at the left limit) was given. Also, as with the definition of the improper integral Type I, the improper integral was defined in terms of a change in notation rather than in terms of verbally-named concepts.

Improper integral Type II

If f is continuous on the interval (a,b], and has an infinite discontinuity at a, then

$$\int_a^b f(x)dx = \lim_{c \to a+} \int_c^b f(x)dx \cdot$$

If $\lim_{c \to a^+} \int_c^b f(x)dx$ exists, we say the improper integral converges.

Otherwise the improper integral diverges.

The above definition is of an improper integral with an infinite discontinuity at one of its limits.

Reminder: A function f is said to have an infinite discontinuity at c, if, from the right or left, $\lim_{x \to c} f(x) = \infty$, or $\lim_{x \to c} f(x) = -\infty$.

[26] In the event though, only three out of the seven students whose work I analysed reached Questions 6, 7, 8; two students reached Question 6 only and two students got no further than Question 11.

Question 6

6. (a) Determine whether $\int_0^3 \frac{dx}{x}$ converges or diverges.
(b) Determine whether $\int_{-8}^0 \frac{dx}{x^{2/3}}$ converges or diverges.

As in Question 5, the student needed to compute the integral and then to determine whether it converged or not. But, unlike Question 5, Question 6 (a procedural question) preceded questions surrounding the generation of exemplars of the object (Question 7) or its graphical representation (Question 8).

Question 7

7. Can you make up an example of an improper integral with an infinite discontinuity?

Question 7 was analogous to Question 2(a), except that, in this case, the student would have worked with exemplars of the object (in Question 6).

Question 8

8. Can you explain what an improper integral with an infinite discontinuity represents graphically.

Question 8 was analogous to Question 3, except that, as above, the student would have worked with exemplars of the object (in Question 6).

§5.6 SUMMARY

In this chapter I described how I went about investigating my proposition concerning the applicability of Vygotsky's thesis about the functional usage of a new sign and the different stages in concept formation, to the appropriation of a new mathematical object by a university student.

First I explained which epistemological and ontological positions underpinned my methodology. Specifically I explained that my assumption was that we impose order on the world through our theoretical constructions. In line with this assumption I argued that all observations are necessarily mediated by theoretical perspectives and by the researcher's own needs and personal history. Furthermore, the very presence of an observer affects what is observed. Despite this inability to see the world without filter, I suggested that a striving for objectivity and coherence needs to regulate any educational research. Also, and consistent with this world-view, I acknowledged that my research is at most an attempt to impose order on the world; accordingly my account is an interpretation of the world and necessarily open to modifications and elaborations.

I explained how my methodology is based on Brown and Dowling's (1998) view of research as a particular mode of interrogation. According to this mode the theoretical and empirical domains need to be progressively refined so that they can articulate coherently with each other. In terms of my thesis, I explained how the theoretical field (in my research this is Vygotsky's theory of functional use and his theory of concept formation) needs to be developed into its theoretical concept variables. The theoretical concept variables are the theoretical structures which I hope to measure or observe in the empirical setting (in my research these are the phases of object appropriation, for example heap thinking, pseudoconceptual thinking and so on). I pointed out that many of my theoretical refinements had already occurred in previous chapters, such as Chapters 2 and 3.

In order to measure or observe the theoretical structures in the world-out-there, the researcher needs empirical data. In order to get the data, the empirical field needs to be delineated. In my research the broad empirical field is the appropriation of new mathematical objects by students in the Mathematics I Major class at the University of the Witwatersrand in 2000.

I reminded the reader briefly about aspects of my empirical field and then moved on to localising this field. In my research, this localised field is the clinical interview setting in which students are a given a specially designed (in terms of its pedagogical and research intentions) mathematical task to perform, talking-out loud and writing as they do.

In the rest of the chapter I discussed aspects of the clinical interview setting and how I went about collecting my data (my methods). In terms of the clinical interview setting, I argued that an interview (like any learner-teacher setup) is a social arena and so cannot be neutral. The interviewer necessarily interacts differently with each student and this interaction necessarily affects the appropriation of mathematical objects by a student. Accordingly these interactions (implicit or explicit) need to be built into any analysis of the clinical interview setting. Furthermore I reminded the reader that in my conception of the ZPD, socially-designed materials (such as the task and the textbook) are important components of the interview situation.

Finally, I discussed my methods. Specifically I described how I went about selecting the ten students for the clinical interviews, ie my critical cases. I then explained my design of and motivations for the mathematical task and the principles which guided my interview format.

In the following chapter (Chapter 6), I will further refine my empirical data by developing the (empirical) indicator variables. These indicator variables are descriptions of how the theoretical concept variables are made manifest in the clinical interview setting. The elaborated description of the indicator variables allows me to interrogate the data with my theoretical concept variables (which is what I do in both Chapters 7 and 8). Indeed it is this linking of the theoretical domain with the empirical domain which constitutes the heart of my research.

I extend my methodological focus in Chapter 9. There I use Maxwell's (1992) distinction between the different types of validity (descriptive, interpretative and theoretical validity) to examine the extent to which my account of how a student appropriates a new mathematical object, can be considered valid. In particular I characterise theoretical validity as the relationship between the empirical indicators and the theoretical concept variables (Brown and Dowling; 1998). In that chapter I also address the issue of reliability, where reliability is characterised as the extent to which the linkages between the empirical indicators and the theoretical concept variables can be replicated at other times and by other people.

CHAPTER 6: DEVELOPING APPROPRIATION THEORY

§6.1 INTRODUCTION

§6.1.1 PLAN OF ACTION

My primary aim in this chapter is to develop and illustrate the different phases that a student zigzags through as he goes about appropriating a new mathematical object presented to him in the form of a written definition.

I argued in Chapter 2 that with certain modifications and adaptations, Vygotsky's stages of concept formation (ie heaps, complexes and concepts) could be used to describe how a learner goes about constructing a mathematical concept. In this chapter I modify and adapt these stages so that they are applicable to the construction of advanced mathematical concepts[1] and I illustrate these refinements with exemplars from the mathematics domain.

My plan of action in this chapter is:

- to particularise the theoretical characteristics and nature of each phase[2] or sub–phase to the mathematical domain;
- to indicate how each phase or sub–phase is made manifest in a setting in which mathematical activity is taking place, ie to describe the empirical indicators of each phase;

[1] Given that the appropriation of a mathematical object necessarily involves the construction of a mathematical concept and vice versa, I have used the terms 'appropriate a mathematical object' and 'construct a mathematical concept' interchangeably in this chapter (and the rest of the thesis).

[2] Although Vygotsky refers to the stages of concept formation, I prefer to refer to the phases in the construction of a concept. This is because 'stages' seems to denote a linear movement (from stage 1 to stage 2, etc); phases seems to have less of a linear connotation.

- to illustrate and elaborate the relationship between the theoretical phase or sub–phase and its empirical indicators via examples taken from seven of the mathematics interviews[3].

This plan is in line with Brown and Dowling's (1998) distinction between theoretical concept variables (ie the phases or sub–phases) and their empirical indicator variables, and the use of elaborated description to articulate a relationship between these variables. See Chapter 5 for more details about this methodological approach.

§6.1.2 UNDERLYING THEORY (IN BRIEF)

My purpose in this chapter is neither to demonstrate the evolution of a particular mathematical concept for a specific student nor to show how an individual student may move back and forth between different phases[4] as he constructs a concept. However, since my theory[5] of how a particular mathematical concept evolves for a particular student underlies my classification of the different phases in mathematical object appropriation, I need to remind the reader briefly of this theory (that is, appropriation theory).

Essentially I contend that the learner zigzags his way through many different phases as he goes about appropriating a new mathematical object. Some of these phases are extremely short and hence difficult or impossible to observe. Nonetheless, in line with Vygotskian theory I maintain that the student engages in different mathematical activities with the different mathematical signs (symbols, words, graphs, diagrams, etc.) before he has 'full' understanding of these signs. That is, he uses the mathematical signs both as objects with which to communicate (like words

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[3] I will explain how and why I chose these seven interviews from the nine interviews I originally conducted in §6.1.4.

[4] I do that in Chapter 7 where I follow the path of two particular students (John and David) as they go about appropriating a particular mathematical object for themselves.

[5] This theory is primarily informed by Vygotsky's theory and my own experience as a mathematical teacher and learner.

are used by a person) and as objects on which to focus, to manipulate or to organise his mathematical world (again as words are used in language) before he fully comprehends the meaning of these signs. In the mathematical domain this use of signs is mediated by the learner's knowledge of related signs or concepts which are called forth for that learner by particular attributes of the new sign, or this use is mediated by the intervention of a teacher, peer or text.

The ability to use these signs prior to full understanding gives rise to the impression that

> *the end point in the development of the meaning of a word*[6] *coincides with the starting point*, that the ready–made meaning is given right at the beginning and that, consequently, there is no room for development. (Vygotsky, 1994: 229; italics in original)

Furthermore, when a learner encounters a new mathematical sign in the form of a definition, I contend that the learner initially focuses on the signifiers, since these are the aspects of the sign which are immediately accessible to the learner. Accordingly I have distinguished between signifier and signified–orientated usages of signs in my categorisations of the phases. For example, a first –year student familiar with an antiderivative but not with the definite or improper integral may focus on the integration signifier or the f(x) signifier or the limits of integration when he first sees the sign $\int_a^\infty f(x)dx$. After all he does not yet know the mathematical object referred to by $\int_a^\infty f(x)dx$.

Since Vygotsky's classification of the different phases in thinking derives from an experiment in which people of all ages had to group together concrete objects for which two sets of stimuli were given (the block itself and the signs used to organise the activity) it does not address certain

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[6] As previously indicated, a mathematical sign serves the same purpose in the mathematical domain as a word in language.

activities which are prevalent when abstract objects (mathematical objects) with concrete representations (the signifiers) are presented to learners. Furthermore, unlike the construction of concepts involving the abstraction of attributes of the concrete blocks and their organisation via signs, the construction of a mathematical concept is based on the abstraction of attributes of the sign (signifier or signified) and its organisation via those same signs. Thus I have had to particularise each of Vygotsky's stages in terms relevant to the construction of a mathematical concept.

The reader also needs to bear in mind that the pathway through the phases is not linear. So a learner may move from a usage of signs indicative of chain complex thinking to a usage of signs indicative of associative complex thinking back to chain complex thinking and so on.

Whilst reading this chapter, the reader may wish to consult Figure 4 at the end of this chapter. This figure provides a useful map of the different phases. Also the reader may wish to refer to §2.2.3.2 where I previously discussed Vygotsky's stages of concept formation (the purpose of that discussion was both to introduce the reader to these stages and to show how these stages could be applied to the activities of a learner appropriating a mathematical object). Indeed, for the purposes of completeness and continuity, I may need to repeat aspects of that discussion in this chapter.

§6.1.3 METHODOLOGICAL NOTE

An important methodological consideration is that one cannot see how a person is thinking: it is epistemologically impossible to gain direct access to someone else's thoughts. Indeed the only access to someone else's thinking processes is through that person's usage of signs (including body language, gestures, words, symbols, etc).

In the mathematical domain in particular, mathematical thinking is made manifest by the person's usage of signs (such as symbols and words) and it is these usages that constitute the empirical indicators of the theoretical phases.

Before commencing with my elaborated and illustrated account of the different phases, I wish to note that it is generally easier to observe mathematically incorrect usage rather than mathematically correct usage. This is for two reasons: first, mathematical usage which is correct appears normal and unremarkable whereas incorrect usage is often idiosyncratic and thus more noticeable and interesting to the reader. Secondly mathematical usage which is correct is often indistinguishable from conceptual thinking (even if it does not truly reflect conceptual thinking) and this does not make for illuminating exemplars of heap or complex thinking.

Since my main purpose in this chapter is to illustrate and exemplify the different phases that a learner may meander through as he goes about constructing a personally meaningful mathematical concept, many of the exemplars I have chosen involve inappropriate reasoning or obviously poor usage of mathematical signs.

This does not mean that less idiosyncratic (but possibly less illuminating) exemplars of many of the phases cannot be found in the mathematical interview data. Indeed if the reader wishes to witness these less extreme exemplars of the phases, she or he is advised to consult Chapter 7 where I present analyses of all David's and John's mathematical activities as they go about constructing particular mathematical concepts.

§6.1.4 THE STUDENTS

Although I interviewed nine[7] different students, I analysed only seven of these protocols. It is from these seven protocols that I take my illustrative examples of the different theoretical concept variables in this chapter.

I decided not to analyse two of the nine protocols because the students involved in these interviews (William and Nora), seemed to be unengaged in both the Mathematics I Major course and the interview task. Furthermore they both seemed very confused during the interview and were unable to

[7] I originally planned to interview ten students but one left the course mid–year. I was thus unable to interview her.

proceed in any intelligible way with the mathematical activities of the task. Indeed I felt that an analysis of their protocols would contribute very little to the development of my theory of mathematical object appropriation. At the end of the year these two students received grades of 22% and 26% thus supporting my decision not to analyse their protocols[8].

Of the remaining seven students, I analysed the protocols of David and John comprehensively, with much justification and discussion of my interpretations of these students' sign usages (see Chapter 7 for these analyses and for an explanation of why I selected these two students for the in–depth analyses). I analysed the protocols of the other five students more informally, without explicitly justifying each of my interpretations of their sign usage.

For this chapter, I have selected episodes from the protocols of each of these seven students for the purpose of illustrating the different theoretical concept variables and their empirical indicators.

Space does not permit me to reproduce all the transcripts in this thesis; but because the transcripts of David and John are largely reproduced in Chapter 7, I have tried to use illustrative episodes from these protocols where possible. Notwithstanding this bias towards using episodes from David and John's protocols, if a particular episode from one of the other five students seemed particularly illuminating of a theoretical concept variable, I selected this episode as my illustrative example. (Certainly the reader may request a copy from me of any of these five protocols if he or she so desires. For this exigency I have included the relevant line numbers when quoting from these protocols.)

In order to contextualise (to some extent) the nine students whom I interviewed, I present Table 3. Table 3 indicates the type of school, the academic grade for the matriculation Higher Grade mathematics

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[8] A research study devoted to an investigation of the causes and nature of breakdowns in the mathematical object appropriation process (as evidently has occurred with William and Nora) is an interesting future project.

examination and the academic grade for the Mathematics I Major course at the end of 2000 for each of these students.

Table 3: Grade for matriculation mathematics examination, final grade for Mathematics 1 Major and type of school attended for each interviewee

Student	Type of School[9]	Matriculation Higher Grade mathematics examination or equivalent examination grade	Final grade for Mathematics I Major
John	T/N	A	41%
David	H/P	A	81%
Mary	T/N	Repeat	71%
Fred	T/N	Other first year Wits Maths course	64%
Brian	H/P	B	52%
Tom	T/N	B	47%
Peter	T/N	C	68%
William	H/P	C	22%
Nora	H/P	C	26%

§6.2 THE PHASES

§6.2.1 THINKING IN HEAPS

The heap phase is characterised by the grouping together of disparate, inherently unrelated objects which are linked by chance in the child's perception (Vygotsky, 1986: 110). For example, in a classification task requiring a child to group together all 'similar' objects, a heap–thinking child may group together all objects which are physically close to each other. The learner does not isolate particular attributes of the objects; rather he links objects according to circumstantial or chance criteria which may have no relevance to the task at hand.

In the context of mathematics I interpret the heap phase as that phase in which the learner associates one sign with another through context or circumstance rather than through any inherent or mathematical property of the signs. Connections or associations appear as arbitrary and

[9] T/N refers to township or non–urban schools; H/P refers to historically white, progressive or private schools. See 5.5.2 for a description of these different sorts of schools.

idiosyncratic; they depend on circumstantial criteria (such as the layout of signs on the page) rather than any objective or actual links.

Bowie (2000), although working in a very different theoretical framework from mine, gives an example of sign usage which she characterises as “drawing on contextual information” (p. 12) but which in my framework illustrates heap thinking. She describes how certain first–year mathematics student in a bridging course at a South African university used the date given on the examination paper as a specific value for P_0 in the equation $P = P_0(½)^{t/29}$. This replacement of P_0 by a readily–available but irrelevant piece of numerical information enabled the students to find numerical solutions to the given problem involving $P_0(½)^{t/29}$ even though the students were never supposed to deal with P_0 as a specific value.

I suggest that heap thinking has certain characteristics in common with inner speech. As with inner speech, sense[10] predominates over meaning in heap thinking. Similarly to the word in inner speech which “acquires its sense from the context in which it appears” (Vygotsky, 1986: 245), so the sign acquires its sense from the context in which it appears during the heap thinking phase. I illustrate this phenomenon in Examples 1 and 2 below.

In summary, the crucial feature of heap thinking is that the learner’s use of signs is determined by circumstantial or context–bound criteria. In terms of my analysis of data from the mathematical interviews, I regard those instances in which a student seems to use signs in ways suggested by their physical context, as cases of heap thinking.

In terms of the interviews only two of the seven students, John and Tom, engage in obvious heap–type thinking. I will present an episode from each of their interview protocols as illustrative exemplars of thinking in heaps.

Example 1: Contrasting signifiers have contrasting properties

[10] According to Vygotsky (1986: 245) the sense of a word is “the sum of the psychological events aroused in our consciousness by the word. It is a dynamic, fluid, complex whole” in which objective meaning is only one aspect (and in heap thinking, often absent).

When discussing the notions of convergence and divergence as they apply to an improper integral with a lower infinite limit (Question 4)[11], John[12] seems to draw an analogy between the difference in the appearance of the two signifiers $\int_{-\infty}^{b}$ and $\int_{a}^{\infty}$, to a difference in the conditions for convergence or divergence.

Although he correctly recognises that the improper integral $\int_{-\infty}^{b} f(x)dx$ represents the area from negative infinity to b [line 60] and that, by analogy [line 62] with the definition of $\int_{a}^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$ is equivalent to $\lim_{a\to-\infty}\int_{a}^{b} f(x)dx$ [line 62, 65], his reasoning becomes illogical when he starts discussing conditions for convergence or divergence of these integrals.

Specifically John argues that since $\int_{-\infty}^{b} f(x)dx$ is the "opposite" of $\int_{a}^{\infty} f(x)dx$ [line 63], $\int_{-\infty}^{b} f(x)dx$ diverges if the limit exists, otherwise it converges [lines 63, 66]. Of course, this is incorrect: the concept of convergence and divergence necessitates that $\int_{-\infty}^{b} f(x)dx$ diverges if the limit *does not* exist, otherwise it converges.

John's argument that $\int_{-\infty}^{b} f(x)dx$ is the "opposite" of $\int_{a}^{\infty} f(x)dx$ seems to derive from the contrasting position (upper or lower) and sign (positive or negative) of the infinite limit on the integral signifier. Furthermore he seems to use this contrast in appearance of the signifiers to explain a contrast in conditions for convergence of the improper integral.

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[11] In Question 4, the student is asked to define $\int_{-\infty}^{b} f(x)dx$.

[12] The reader is reminded that much of the transcript from each of John's and David's interviews are reproduced in Chapter 7.

Since this analogy has no basis in logic or fact and depends entirely on the layout of the signifiers, I categorise it as indicative of heap thinking.

Example 2: The layout of some text determines its meaning

Tom is unable to proceed successfully with Questions 1, 2, 3, 4 or 5 in the interview task. Consequently I tell him that he may consult the textbook.

His response to the definition of an improper integral with infinite integration limits in the textbook (Larson, 1994: 536) is a classic example of thinking in heaps. Since Tom's response is dependent on the form and layout of the definition, I need to describe how the definition is presented in the textbook. To do this, it is probably simplest to reproduce the textbook definition.

Definition of improper integrals with Infinite Integration Limits

1. If f is continuous on the interval [a, ∞) then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$.

2. If f is continuous on the interval [–∞, b) then $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$.

3. If f is continuous on the interval (–∞, ∞) then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ where c is any real number.

In the first two cases, the improper integral **converges** if the limit exists– otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Tom studies this definition and, at my prompting, explains what he has gathered from the definition [line 101].

Pointing to Case 1 and 2 in the definition, he tells me that if one of the limit values is infinite then the integral converges. Pointing to Case 3, he explains that if both limit values are infinite then the improper integral diverges [line 101].

Tom's explanation is objectively meaningless. He seems to have created an idiosyncratic link between a single infinite limit and convergence and two infinite limits and divergence. I suggest that this link corresponds to the **order** of presentation of information about convergence and divergence in the last paragraph and the order in which the three cases (one with upper infinite, one with lower infinite limit and one with two infinite limits) are presented.

It is as if Tom read the definition as: "In the first two cases, the improper integral converges …– otherwise the improper integral diverges", conveniently omitting the phrase "if the limit exists" after the word converges, and entirely omitting the last sentence which focuses on the third case.

This demonstrates thinking in heaps since there is no actual (concrete or abstract) association between the order of information given in the last paragraph and the convergence or not of the three cases of improper integrals, Type 1. In fact the link is entirely dependent on the layout of the information.

(In Piagetian terms, Tom distorts the world out there to fit his current schema, rather than adapting his schema to fit the world out there.)

§ 6.2.2 COMPLEX THINKING

With complex thinking, "individual objects are united in the child's mind not only by his subjective impressions but also by *bonds actually existing between these objects*" (Vygotsky, 1986: 112; italics in original). According to Vygotsky (1986, 1994) when a person thinks in complexes, his thinking is coherent and objective, unlike in heap thinking where the connection between different elements of the group are subjective and incoherent. Also the bonds between objects in the groups formed as a result of complex thinking are factual and derive from experience; but they are not logical or abstract (as in conceptual thinking).

The sort of non–logical reasoning which is endemic to complex thinking is common in everyday life, particularly when people misinterpret or misuse cause–and–effect relationships. For example, suppose you are told that a particular person frequently gets colds and never exercises. How would you, the reader, explain that person's propensity for colds?

I suggest that there would be a strong tendency to argue that this person gets colds all the time because she does not exercise, whereas the reality may be that she does not exercise because she always has a cold. Or the two may be completely unrelated for this person.

In the mathematical arena, this type of concrete but illogical thinking occurs when, for example, specific instances of mathematical objects (ie exemplars) are used to argue or explain the general phenomenon; logical operators such as if–then conditions are used incorrectly; irrelevant mathematical facts are brought to bear on the situation and artificial connections (which have no basis in logic) between different signifiers or signifieds are made.

A common example of complex thinking in the context of the Mathematics I Major course is as follows: Near the beginning of the year the argument (theorem) is presented that if a function is differentiable at a certain point it is necessarily continuous at that point. After encountering this theorem, I have noticed how certain first–year students start associating differentiability with continuity, even arguing that if a function is continuous at a point it must be differentiable at that point (which is, of course, incorrect and not logically deducible from the theorem).

There are many different types of complex thinking in mathematics. As I discussed in Chapter 2, Vygotsky characterised complex thinking in five different ways: the chain complex, the associative complex, the collection complex, the diffuse complex and the pseudoconcept. But these types of complexes are not sufficient to characterise the type of sign usage that occurs in the mathematical domain: there is a need to distinguish signifier–orientated from signified–orientated usage and there is also a need to distinguish between different types of associative complexes in the mathematical domain.

To the latter extent I subdivide the associative complex into three categories: surface, artificial and example–centred. I will discuss each of these below. I have also developed two new phases of complex thinking:

the graphical complex and the template–orientated complex specifically for the mathematical arena. Again I illustrate and elaborate on these complexes below.

For ease of reference I now present a map of the different complexes through which I postulate that a student moves as he appropriates a new mathematical object. (The pink lines indicate complexes which I derive primarily from my own experiences and readings; the black lines indicate complexes which derive from Vygotsky.)

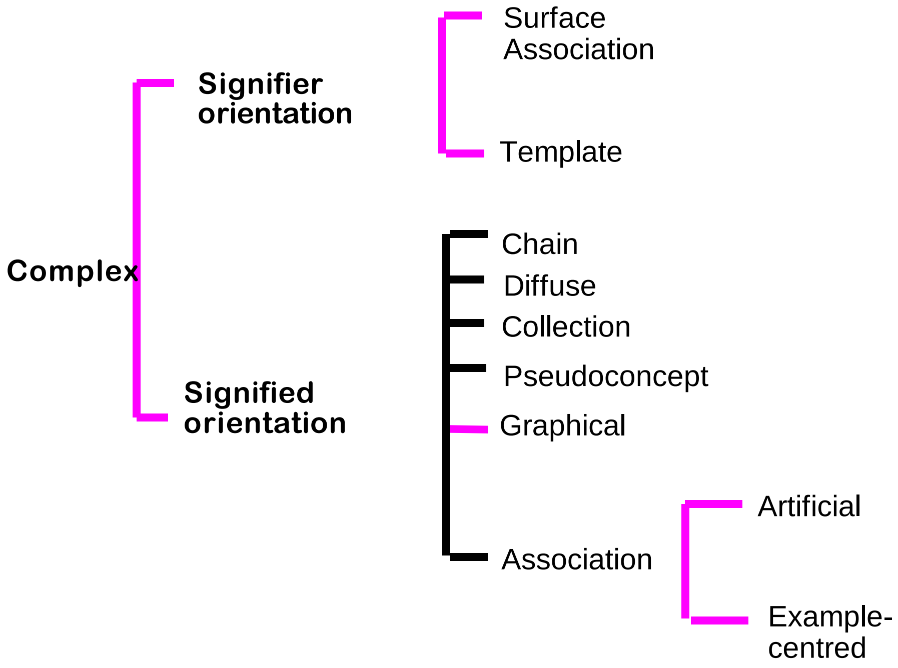

Figure 3: The different types of complex thinking in the mathematical domain

The crucial thing in all these types of complex thinking is that the learner links one mathematical sign with another because of an objective or factual connection. Such a connection is not necessarily logical as in conceptual thinking, although it may coincidentally be so. As expected, complex thinking was ubiquitous across all the interviews (as I show below).

Since complex thinking takes several forms each with its own characteristics and indicators, little purpose is served in extending this general discussion. Rather I shall proceed to describe the different forms of complex thinking as they may occur when a learner appropriates a new mathematical object.

§ 6.2.2.1: Chain complex

In the mathematical arena, chain complexes are an elaboration of Vygotsky's same–named notion.

A chain complex in the context of the appropriation of a mathematical object is characterised by the student isolating an attribute or aspect of a mathematical statement (such as a particular sign) and associating this decontextualised sign with a new sign. This new sign (or an aspect or attribute of it) is linked to yet another sign thus forming a chain of signs.

The crucial thing about a chain complex in the Vygotskian sense is that no attribute or aspect of the new sign is privileged over another. This means that when a link is formed to a new sign, the new sign enters the chain with all its attributes. And any of these attributes may then serve as a link to yet another sign. Vygotsky (1986: 116), talking about the chain complex in the context of a task in which children had to group objects into classes, put it thus:

> An object included because of one of its attributes enters the complex not just as the carrier of that one trait but as an individual, with *all* its attributes. The single trait is not abstracted by the child from the rest and is not given a special role as in a concept. (italics in original)

In both §2.2.3.2 and §3.2.2, I discuss how a chain complex in the Vygotskian sense differs from the notion of a chain of signification as used by Walkerdine (1988), Presmeg (1997, 1998), Cobb et al (1997) and Whitson (1997).

Since a chain complex involves the decontextualisation of a sign or of one of its attributes, it often manifests in the form of obtuse or idiosyncratic reasoning in which links appear as arbitrary or random although they may not be so.

Indeed, in the example I give below, I show that although the chaining of signs is not bound by the intralinguistic context, it may be rooted in the wider context of the interview task, the extralinguistic context. Thus although the links appear as arbitrary, they are not. Specifically I argue that David links the notion of convergence or divergence of an improper integral to the activities with which he was involved in lectures at the time of the interview.

(In §2.2.3.2 I gave an example of chaining within the mathematical discourse – the intralingual context.)

Example 3: Chaining of signifieds according to extralinguistic context

David reads Question 3 quietly to himself [line 41] [13]. He then proceeds to explain how he sees an improper integral as a "graph with a repeating pattern" [line 43] which "repeats to infinity" [ibid]. He further explains that the graph oscillates around the x–axis with the gaps (between graph and x–axis) getting "smaller and smaller" [ibid] when the improper integral is convergent or "bigger and bigger and bigger" [ibid] when it is divergent. David then draws the two graphs, Graphs B and C, to illustrate these cases respectively. See Appendix C, page C3, for these two graphs.

Although David at first does not talk about the area between the integrand and the x–axis, he is implicitly referring to this area. This is evidenced by the way in which he speaks about the size of the "gaps" between the graph and the x–axis [ibid]. Moreover at the end of his explanation he specifically mentions area: "the area between the graph and the axis" [ibid]. And a little later, when justifying his explanation, he explicitly refers to the area between the function and the x–axis [line 50].

[13] Question 3 requires the student to explain what an improper integral with an infinite integration limit represents graphically.

David's notion that the integrand is "a graph with a repeating pattern" or an oscillating graph is inappropriate; the function in an improper integral does not need to be oscillating, nor repeating.

David's discussion about the "gap" between the x–axis and each repetition of the graph getting smaller and smaller or bigger and bigger as x tends to infinity and its relationship to the convergence or divergence of the improper integral [line 43], is incorrect. In particular I suggest that David is linking properties of oscillating convergent or divergent sequences with general properties of convergent or divergent improper integrals.

Indeed I put forward that David's interpretation is triggered by the words convergent and divergent in the definition of an improper integral, Type I. In particular, at the time of the interview the Mathematics I Major class had just studied convergent and divergent sequences. The oscillating graph may have been brought to mind by examples (recently done in lectures) of alternating sequences which converged or diverged depending on whether the *n*th term tended to a limit or not.

In summary, I maintain that David's answer to Question 3 is contingent on the following chain: **convergent or divergent** in definition of improper integral $\subset$[14] **convergence or divergence** of sequences $\subset$ graphical representation of **converging or diverging sequences** $\subset$ graphical representation of **alternating sequences** $\subset$ **oscillating graph** $\subset$ graphical representation of improper integral as **the area between an oscillating graph and the x–axis.**

In conclusion, although David correctly recognises that an improper integral represents the area between a curve and the x–axis on an interval with an infinite limit, his focus on the oscillating nature of the curve and a repeating pattern is indicative of a concrete, non–logical link. Specifically it is indicative of complex thinking using chain complexes in which the associations are rooted in the extralinguistic context.

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[14] $\subset$ stands for : "is linked to".

In Chapter 7 we see how David, after engaging with further activities in the interview task, is able to move beyond this use of chain complexes to the use of a graphical complex in his explanation of the convergence or divergence of an improper integral.

§6.2.2.2: Complex thinking: association

Vygotsky explained that with the associative complex, one object forms the nucleus. When the child notices a similarity between an attribute of the nucleus and an attribute of another object, this new object gets included in the group.

In the mathematical arena, the learner uses one mathematical sign as a nucleus and then associates other mathematical signs which have attributes in common with the nucleus, to the nucleus. In this way, a complex is constructed. The connections and associations have some objective and factual justification (as opposed to heap thinking) although they are not based on logical thinking (as in conceptual thinking).

Indeed, as I discussed earlier, the overarching characteristic of complex thinking of any kind is the use of arguments and connections which are not logical. To quote Vygotsky (1986: 113; italics in original): "In a complex, the bonds between its components are *concrete and factual* rather than abstract and logical."

With reference to the associative complex in mathematical thinking specifically, the learner may latch onto one or more familiar signs in a mathematical statement and use these as a nucleus around which to build a new complex (that is, he uses these isolated signs to make sense of the entire statement) or he may use one example of a mathematical object as a nucleus around which to build a new complex.

But in each of these cases of complex thinking, there a nucleus to which all other mathematical signs are associated. Below I illuminate these statements via examples[15].

[15] These examples may thus serve as the nuclei around which the reader can construct

Since complex thinking of the associative type is ubiquitous in the early phases of concept construction, I have divided it into three subcategories each of which has a different character with a corresponding different indicator.

(i) Surface association

With a surface association, the student isolates a particular aspect or part of the mathematical expression (a set of signifiers or words) and associates these signifiers or words (the nucleus) with a new sign. I call this “surface” because it appears to be the result of a superficial reading of the original set of signifiers (the statement).

In the context of mathematics learning, the student seems to grasp onto particular words or signifiers without regard for their importance or their intralinguistic context. Often significant parts of the mathematical expression are ignored. Because the learner’s focus is on a signifier rather than on its meaning, this is a signifier–orientated form of complex thinking.

For example, when dealing with the greatest integer function: $\lceil\!\lceil x \rceil\!\rceil$= greatest integer $\leq x$, many students latch onto the word ‘greatest’ ignoring the condition $\leq x$. They then link the word ‘greatest’ (the nucleus) to the idea of ‘greater than’ and accordingly state that, say, $\lceil\!\lceil 4.3 \rceil\!\rceil$= 5 (whereas of course, the answer should be 4).

A surface association is empirically indicated by the student’s undue focus on a set of symbols or words and their consequent non–attendance to other objectively significant signifiers in the mathematical expression.

In mathematics a common form of surface association is indicated when the student executes a procedure as soon as he sees a particular signifier, regardless of whether such activity is required or not. For example many students write 2x (ie the student computes the derivative of x^2) immediately

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her or his understanding of what I am talking about.

on seeing the sign $\frac{d\left(x^2\right)}{dx}$. This form of complex thinking is identified most easily in those cases where the student applies a procedure despite it not being required. Generally, if it is apposite to apply a procedure, the sign usage is classified as indicative of pseudoconceptual thinking.

I suggest that surface associations are likely to flourish in an environment in which the student is unable to make sense of a given mathematical statement; in such a context the student may grasp onto a familiar signifier and try to generate meaning by linking that signifier to another sign. Indeed the student's focus on the familiar signifier may be the only strategy available (even if not feasible) to that student.

Example 4: Undue focus on part of the signifier

When reading the definition of an improper integral Type 1, David underlines the words 'converge' and 'exist' [line 17]. Soon after, in response to Question 2(a), in which he is asked to generate an example of an improper integral Type I, he presents the example: $\lim_{b \to \infty} \int_a^b \frac{1}{x^4} dx$ [16] [line 24] and immediately evaluates this integral [line 25].

I ask David to explain what he is doing [line 26]. From his explanation it becomes clear that he thinks that an improper integral with an infinite integration limit must converge, it must equal a "number that exists" [line 27]. Further evidence of David's conviction that an improper integral must converge is given by his justification of the improper integral he generated: "By the definition **that integral exists. Therefore** that is the improper integral with an infinite integration limit" [line 28].

Even more evidence of this undue focus is David's statement after reading Question 2(b), in which he is specifically asked to generate a **convergent** improper integral. At that point he states that he has "just realised

[16] Actually this example is doubly improper if $a \leq 0$.

something" [line 31]. In line 36, he explains that in Question 2(a) he had made up a convergent improper integral whereas this was not required[17].

I suggest that David's undue focus on the words 'exist' and 'converge' (evidenced by his underlining these words) when reading the definition of an improper integral Type I, leads him to associate an improper integral with a **convergent** improper integral (for which the limit necessarily **exists**). So the nucleus for this particular association are the words 'converge' and 'exist'.

This episode is an example of complex thinking using a surface association. It begins with an undue focus on certain words or signs; these signs then become the nucleus around which the student builds a concept of a particular mathematical object.

Example 5: Ignoring part of signifier

Mary's implicit belief in the first half of her mathematics interview that an integral is only improper if it has a discontinuity on the interval of integration shows how she has ignored the sufficient condition (given in the definition of an improper integral) that an integral is improper if one of its limits is infinite.

Clear evidence for this belief may be found in Mary's explicit statement that the integral given in Question 5(a), ie $\int_1^\infty \frac{dx}{x^3}$, is not improper because "looking at the range from 1 to ∞ there is nowhere where that function does not exist" [line 129], followed by a similar argument about Question 5(b), ie $\int_{-\infty}^{1} x\,dx$ [line 130].

Additional evidence of this undue focus on the discontinuity of the function on the interval of integration may be found in lines 43, 80, 83, 89, 90, 107,

[17] Of course, if a student does generate a convergent improper integral in response to Question 2(a), that is fine. What is not okay is if that student believes (as David evidently did until his perturbation when reading Question 2(b)) that an improper integral **has to be** convergent.

109, 113. For example, when generating an exemplar of an improper integral Type I in response to Question 2(a), Mary writes $\lim_{b \to -\infty} \int_b^1 \frac{1}{x} dx$. She then justifies her example by stating "This is improper because there is zero involved inside. Ya. And it is improper **and**[18] it has an infinite integration limit." [line 43].

Furthermore when, as a result of several perturbations Mary realises her error, she explicitly acknowledges that she has been ignoring parts of the given statement (the set of signifiers). "Oh…**I've being missing something** very important…**I only looked at one part of the question, not the whole thing.** This is very important. That if one of the limits of integration is infinite, that is, it tends to infinity…. oh." [lines 168, 170]

In this example the notion that the function in an improper integral must have a discontinuity on the interval of integration is the nucleus around with which the concept of an improper integral is constructed.

In trying to understand this surface reading, it is instructive to note the wider context which frames Mary's reading of the definition in the mathematics interview. Mary is a repeat student and has thus encountered improper integrals before. It is this prior experience with these integrals that presumably frames her reading.

Example 6: A sign triggers the execution of a procedure

An example of complex thinking involving the uncalled for execution of a procedure is given by John's activities when doing Question 2(a).

Question 2(a) requires the student to generate an example of an improper integral with an infinite integration limit. John does this, offering the example of $\int_0^\infty f(x)dx$ where $f(x) = \sqrt{x}$. Immediately on writing this integral, John tries to evaluate the integral even though this is not required [lines 27, 28, 32].

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[18] Throughout this chapter, I use bold–face to highlight particularly significant or revealing utterances when quoting from the transcript.

In fact, after completing the calculation and stating that he can see that the limit of the improper integral exists [line 34], John realises that he did not need to calculate the integral: “ Did I have to calculate or just give an example?” [line 34].

(ii) Using an example as a nucleus round which to construct a concept

A very common strategy used by learners when appropriating a new mathematical object is to use an example of the mathematical object to construct a new concept. In terms of Vygotsky’s theory, the example is the nucleus or core around which the new concept is built.

What happens is this: one particular example (perhaps the only available example) constitutes the core of the complex; attributes of other mathematical examples or mathematical objects are then compared to certain attributes[19] of the core; if they correspond on one or more aspects, the learner adjoins specific attributes of the new example or mathematical object to the core. In this way the learner constructs a new concept.

However since the example is an instantiation of the object (rather than the object itself) irrelevant or extraneous attributes may be included in the core as if they were intrinsic to that concept. If that happens, this use of an example may be disabling of concept construction.

For example, in our Mathematics I Major course we often introduce the notion of a function which is not differentiable at a point through the example $f(x) = |x|$, $x \in \nabla$. Students often use this example to assume that all absolute value functions are not differentiable at a particular point. (This is not correct. For example $f(x) = |x^2|$ is, of course, differentiable for all $x \in \nabla$.)

Sierpinska (2000: 14), although working in a different theoretical framework, gives convincing examples of how some students doing a linear algebra

[19] These attributes are those which are epistemologically available to the learner or which the learner regards as important for one of many reasons such as familiarity, explanatory power and so on.

course within the CABRI–geometry II technological environment were impeded by “thinking in terms of prototypical examples, rather than definitions”.

From my set of interviews, I do not have such direct and unambiguous evidence of a student being disabled by the use of an example (although this does not imply in any way that such situations do not occur). On the contrary I have evidence of students being enabled by the use of an example as a core around which to construct a particular concept.

For example, neither Tom nor Brian are able to construct suitable examples of improper integrals from the definition of an improper integral (as required by Questions 1 and 2) until they look at examples of other improper integrals in the textbook. In fact, after looking at these textbook examples, Brian exclaims: ”The examples show what the theory means” [line 73]. Tom indicates that the examples have helped him understand what is happening: “**the examples helped me very much rather than the definitions**.” [line 237].

On a more extreme level, John seems to be unable to engage in concept construction without the use of an example as a core around which to build. To wit, in Example 7, we see how John is unable to work with a mathematical statement for which he has no exemplar. In Example 8, John is only able to explain what he understands by the graphical representation of an improper integral through the use of a specific example.

Example 7: The need for an example disables a student

In this example we see how John tries to generate an example of a particular mathematical statement before attempting to work with that statement.

What happens is this: John’s first and only response to Question 9 [20] is to try and generate a function f for which f′ is continuous on [0, ∞), $\lim_{x\to\infty} f(x) = 0$,

[20] In Question 9 the student is asked to state, with justification, whether the following statement is true or false:

and $\int_0^\infty f'(x)dx = -f(0)$ [lines 283, 294, 285, 287]. However he is unable to do so [line 289] and so abandons any attempt to interpret, understand or even use deduction on the given statement.

John [lines 289–291] sums up the situation himself, indicating that he cannot proceed with the task if he does not have an example around which to base his argument (ie to construct his concept):

289. John: I cannot find an example that I can use.

290. MB: Do they want you to use an example?

291. John: No, but actually I don't understand this. **I can't explain, because I don't even understand. I just wanted to get an example.** So that I can see if it's right, what I'm going to explain.

Example 8: The use of an example enables the construction of a concept

In answering Question 3, John uses the example he generated in Question 2(b), viz. $\int_0^\infty \sqrt{x}dx$, to explain what an improper integral with an infinite integration limit represents graphically.

He sketches the function $y = \sqrt{x}$ and explains that the improper integral $\int_0^\infty \sqrt{x}dx$ represents the area between the graph and the x–axis from zero to infinity [line 58]. Using this particular example, he then argues that the area is infinite and does not have a limit [line 58].

What is particularly interesting about John's explanation is that it shows how aspects of his concept of convergence are evolving through his use of

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If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)dx = -f(0)$.

an example. Whereas in Question 2(b) John had incorrectly stated that this same improper integral was convergent because the limit equalled infinity [line 53], he now refers to the fact that the area (between the function $y = \sqrt{x}$ and the x–axis from zero to infinity) tends to infinity and so does not have a limit. In other words he is now using the mathematically correct notion that an infinite limit does *not* exist (in the sense of existence as a real number).

(iii) Unfamiliar objects are artificially associated with a familiar object

When confronting a new mathematical object or expression, part of the object or expression (perhaps a sign or symbol) may remind the learner of another mathematical sign with which he is more familiar and which is epistemologically more accessible. This more familiar sign may then become the nucleus of the new concept.

The crucial thing is that there needs to be a factual connection between the new object and the nucleus (for example the signifiers may be similar in appearance) although this connection may be artificial, irrelevant or illogical.

For example, several students at various points in the interview associate the properties of $\int_a^\infty f(x)dx$ with the properties of f(x). In that instance the properties of f(x) are far more familiar than properties of $\int_a^\infty f(x)dx$ which the student is trying to uncover. So f(x) becomes the nucleus and its properties are immediately and artificially connected to the properties of $\int_a^\infty f(x)dx$.

Clearly this is not logical; indeed it is mathematically incorrect. See Example 9 below.

Sometimes the associations, although starting off as artificial, turn out to be logical and mathematically correct. However this is more difficult to categorise as indicative of complex thinking since it looks to the observer like conceptual thinking.

Vygotsky, although referring to the block–sorting task (see §2.2.3.2) rather than to the construction of a mathematical concept, described this association thus:

> Any actual relationship which the child discovers, any associative connection between the nucleus and the element of the complex, is enough reason for the child to include this object in a group selected by him. (Vygotsky, 1994: 220, 221)

I present two examples below; in each of these we see how a learner uses one familiar mathematical sign as a nucleus to which other less familiar signs are associated.

Example 9: An artificial connection which is false

David's first response to Question 11 [21] is to search for a counter–example [line 116]. This approach is fine and David appropriately tries to generate a function f(x) for which $\lim_{x\to\infty} f(x) = 0$ holds [line 116], f is continuous on [0, ∞) [line 118] and $\int_0^\infty f(x)dx$ is ∞ [line 122]. But he is unable to generate a function which simultaneously satisfies all three conditions [line 124].

A little later he returns to Question 11. He now decides that the statement is true [line 192] and sets out to justify this assertion.

His reasoning, although not always explicit, seems to be as follows: The statement is true [line 192] because $\lim_{x\to\infty} f(x)$ and $\lim_{x\to 0} f(x)$ exist [line 192]. The reason why $\lim_{x\to 0} f(x)$ exists is because 0 is included in the domain [lines 195, 197]. Therefore "it is possible to work out the area" [line 197], ie the given integral converges.

[21] In Question 11 the student is asked to state, with justification, whether the following statement is true or false: If f is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)dx$ converges.

Although David's initial statements are correct (ie $\lim_{x\to\infty} f(x)$ and $\lim_{x\to 0} f(x)$ both exist), his conclusion is false: the existence of these two limits does *not* imply that one can work out the area.

An interpretation of this episode in line with my theoretical framework is: David is unable to find a counter–example to the given statement. He thus latches onto certain signs with which he is familiar, namely the existence of the limits of the function at both 0 and ∞ . The **existence of these limits** become the nucleus of his complex to which he associates the **existence of the limit of the area** [line 191, 197]. This is clearly not logical, and is in fact quite wrong[22].

This episode is interesting in that it shows how, under certain circumstances, a student such as David (who is usually able to justify and motivate what he does in terms of logical, procedural, graphical or historical/ prior knowledge associations) starts making associations which have neither a logical, procedural, graphical nor historical basis. I conjecture (which conjecture certainly requires further research) that the cognitively stressful circumstances in which David finds himself, provoke a response analogous to a drowning man grasping for a straw. In David's case, for example, logic, due procedure and graphical associations are replaced by spurious associations.

Example 10: An irrelevant connection

Brian argues that $\lim_{b\to\infty}\int_0^b \frac{1}{x}dx$ is not a good counter–example to the statement in Question 11 because the antiderivative of 1/x, ln x, is undefined at x=0 [line 212 – 215].

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[22] Note: A good counter–example to David 's argument is f(x) = $\frac{1}{x+1}$. Here $\lim_{x\to\infty} f(x)$ = 0 and $\lim_{x\to 0} f(x)$ = 1, but $\int_0^\infty \frac{1}{x+1}$ is divergent.

Although it is true that ln x is undefined at x= 0, this is not the reason why $\lim_{b\to\infty}\int_0^b \frac{1}{x}dx$ is not a good counter–example. It is not a good counter–example because f(x)=1/x is not defined at x = 0; thus f is not continuous on [0, ∞) (a required condition for the function f).

I suggest that Brian is associating desirable conditions for 1/x with desirable conditions for ln x where ln x (ie $\int \frac{1}{x}dx$) is the nucleus. As such Brian's argument is illustrative of complex thinking with an artificial association.

§6.2.2.3 Complex thinking: collection complex

In this form of thinking the child groups together objects on the basis of one trait in which they differ but in relation to which they are functionally complementary. (Vygotsky, 1986: 115)

> Objects become unified on the basis of a mutual complementing of one another according to some feature, and they form a single whole which consists of different, mutually complementary parts. (Vygotsky, 1994: 221)

Vygotsky (1986: 115) argued that this form of thinking is rooted in practical experience. For example, objects like a knife, fork, spoon and plate are grouped together in everyday life because they have a functional complementarity.

In mathematical thinking I suggest that this type of complex is indicated when one sign is associated with another because the two signs performs inverse (ie a contrasting) functions to each other.

For example, the student may differentiate a function rather than integrate it. See Example 11 below.

Example 11: Using an inverse operation

When testing whether the integral which he generated in response to Question 2(b) is convergent, Tom differentiates rather than integrates [line

125]. Specifically he writes that $\int_1^b \frac{1}{x^2+1}dx$ equals $\left.\frac{-2x}{\left(x^2+1\right)^2}\right|_1^b$. This is incorrect: $\frac{-2x}{\left(x^2+1\right)^2}$ is the derivative, not the antiderivative of $\frac{1}{x^2+1}$.

In terms of the collection complex, Tom associates the antiderivative with the derivative by virtue of their contrasting functions or, mathematically speaking, by virtue of the fact that they perform inverse operations.

§6.2.2.4 Complex thinking: diffuse complex

Sometimes a learner creates a bond or connection between two different objects on the basis of a remote or vague similarity between these objects or a property of these objects. Vygotsky called this form of thinking, ‘the diffuse complex’. For example, in the block–sorting experiment Vygotsky noted how some children selected trapezoids to go with triangles because of a remote similarity between the trapezoid and the triangle (a trapezoid is like a triangle with the top cut off).

Vygotsky [1994: 225] argued further that the diffuse complex manifests in “the generalizations which the child creates precisely in those realms of his thinking which do not easily submit to practical scrutiny, or in other words, in the non–visual and non–practical realms of thinking”.

In the mathematical domain, the indicator for this sort of thinking is the linking of two mathematical signs because of a vague similarity between the signs. For example, the learner may associate the properties of $f'(x)$ with the properties of f(x).

Notwithstanding this example, I will argue in Chapter 8 that I prefer to categorise this sort of thinking as indicative of complex thinking with an artificial association (in which an unfamiliar sign is associated with a familiar sign even though the connection may not be logical or relevant). For instance, Example 10 above could arguably be classified as an instance of diffuse thinking but I prefer to classify it as an instance of complex thinking with an artificial association.

§6.2.2.5 Complex thinking: graphical association

Complex thinking involving a graphical association is a type of thinking which occurs specifically in mathematics.

With this form of complex, the graphical or numerical representation of the mathematical object becomes identified by the learner as the mathematical object itself. Thus any properties which can be abstracted from the graphical or numerical representation of the mathematical object are deemed by the student to be properties of the mathematical object itself.

This may well enable and facilitate the evolution of the mathematical concept for the learner. Indeed, most of the proponents of Calculus Reform strongly advocate the use of multiple representations (ie symbolic, graphical and numerical representations) in the learning of new mathematical concepts.

However because a graphical or numeric representation is a concrete representation of an abstract object rather than the object itself, the use of graphical or numeric representations may lead to false and mathematically incorrect conclusions. For example, the graphical representation of a particular function on a graphic calculator or computer may obscure the existence of (removable) discontinuities if the resolution of the screen is too low or the number of pixels too small. Likewise it is impossible to represent graphically or numerically certain “monster” functions such as

$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ due to the oscillating nature of the sine function near

x = 0. (Lakoff and Nunez, 1997)

Moreover a graphical representation is but one instance or one manifestation of an abstract object and it is ontologically impossible to represent graphically all instances of an abstract object. Similarly a numeric representation of an abstract object involving infinitely many (countable or uncountable) numbers, necessarily requires the selection of a subset of these numbers. Again it is ontologically impossible to represent numerically all instances of the object.

In summary, the use of graphical or numeric representations may be very helpful although in certain instances it may obfuscate the nature of the mathematical object or the representation may be an ontologically impossible task. But even if the representation is helpful or enabling it is still but a concrete representation of an abstract object, not the abstract object itself. Thus the type of reasoning a student can employ when using a graphical or numerical representation of an abstract object is necessarily factual or visual rather than logical or abstract.

Accordingly, I have classified thinking or reasoning which is based on a graphical or numerical representation as a form of complex thinking. Even though the use of a graphical association is a form of complex thinking, I suggest that such usage, free as it is from obvious contradictions and inconsistencies, stands at the cusp of conceptual thinking.

The indicator for complex thinking with a graphical association is the student's use of a graphical or numerical representation of a mathematical object as a basis for an argument or investigation.

I present two examples of complex thinking using graphical association here. In Example 12 we see Brian using the graph of ln x to investigate $\lim_{x\to\infty} \ln x$. In Example 13 we see David using a graphical representation of an improper integral to discuss the nature of convergence or divergence as they relate to an improper integral.

Example 12: Using a graph for investigation

In order to determine whether $\int_1^\infty \frac{1}{x}dx$ converges or not (Question 5(c)), Brian needs to evaluate the integral. He performs the computations correctly but stops after writing that the integral equals $\lim_{b\to\infty} \ln|b| - \ln|1|$ [line 140]. He is clearly not sure about how to deal with $\lim_{b\to\infty} \ln|b|$ and he asks if he can use his calculator “to see what the limit of ln infinity would be” [line 145].

In spite of his own request he does not use the calculator; rather he draws the graph of y = ln x as if by rote [line 147]. Using this graph he is able to confidently tell me that $\lim_{b\to\infty} \ln|b|$ is infinity (and so his integral diverges).

This is an example of a learner using complex thinking: Brian has extrapolated particular properties of the ln function from its graph.

The reader may want to argue that without a concept of ln x one cannot sketch its graph. But I suggest that some people (possibly Brian) have better visual memories than memories of abstract properties and so are able to memorise the graph of ln x even if they are unable to memorise or understand the properties of ln x.

Example 13: Using a graphical representation as the basis for an argument

Question 8 asks for an explanation of what an improper integral with an infinite discontinuity (say at x=c), represents graphically. David’s explanation reveals that he thinks about convergence and divergence in terms of the addition of increasing or decreasing amounts of area between the function of the improper integral and the x–axis, for intervals of constant width, as x tends to c. This is an example of complex thinking using graphical associations.

To elaborate: Unlike his response to Question 3 (see Example 3 above), David’s explanation of convergence and divergence for an improper integral with an infinite discontinuity no longer revolves around the graphical notion of an oscillating curve. To this extent, his notion of convergence or divergence has evolved (see Chapter 7 for a full discussion of David’s evolving concept of convergence and divergence).

However his notions are still dominated by graphical associations rather than abstract or logical links. To wit, David explains convergence and divergence in terms of increasing or decreasing amounts of added area:

“As you increase ever so slightly the increments are getting smaller, the added area is getting smaller and smaller. And if it diverges the area is getting bigger and bigger and bigger. So therefore you can’t work out…. If it converges you can work out the area because its basically a sum to infinity of the small areas. And if it diverges you can’t because those areas will eventually add up to infinity” [line 179].

The problem with David’s notion of convergence is that the areas may be getting larger and larger for successive intervals as x tends to the discontinuity but the cumulative area may be getting smaller and smaller.

For example consider f(x) = $x^{-\frac{1}{3}}$. Here the areas for successive intervals between f(x) and the x–axis are getting larger and larger as x tends to 0 but the improper integral $\int_0^1 x^{-\frac{1}{3}}dx$ converges.

In this sense, David’s explanation is not logical or abstract; rather it hinges on a visual argument in which increasing or decreasing sizes of added areas imply divergence or convergence of the relevant improper integrals respectively.

I thus classify David’s thinking re convergence and divergence of an improper integral Type II as complex thinking using a graphical association.

§6.2.2.6 Template–orientation

Like a usage of signs which is based on a surface association (see §6.2.2.2) a template–orientated usage of signs is signifier–orientated.

What is a template–orientation? When a student is presented with new mathematical sign, the student may associate the new sign with another more familiar sign due to a similarity of the templates of the respective signifiers. As a result of this association, properties of the more familiar sign are applied to the new sign. In this way, the student is able to use the new signifier even before he has constructed the concept referred to by the new signifier.

This characterisation of template–orientation derives from Sfard (2000) who argues that the first use of a new mathematical symbol is necessarily template–driven, even if it is of very short duration. She argues that the learner, when confronted with a new mathematical signifier, slots this signifier into an “old discursive slot” and uses it in a way suggested by the familiar template. Gradually, “due to some basic misfit” (ibid) and discursive interactions, the new signifier slips out of its initial slot and starts to acquire a meaning and life of its own.

In my theoretical framework, template–orientated use of a mathematical sign is not necessarily the first step in mathematical object appropriation. Indeed, as I illustrate in Chapters 7 and 8, students may also use surface associations as their first step in mathematical object appropriation. Furthermore they may revert to a template–orientated use of the mathematical sign even when quite far down the path of appropriating the relevant mathematical object.

A template–orientated usage of signs may be enabling or disabling of mathematical object appropriation. For example, many university students link the matrix multiplication statement ‘AB = C’ (where A, B, C are matrices) to the statement ‘ab = c’ (where a, b, c $\in \nabla$) because of a similarity of template. Using the familiar fact that the multiplication of the reals is commutative (ie that ab = ba = c) and that ‘AB = C’ has the same template as ‘ab = c’, they may write: ‘AB = BA = C’. This is, in fact, incorrect: multiplication of matrices is not commutative.

On the other hand, the fact that the statement ‘A + B = C’ has the same template as the familiar statement ‘a + b = c’, may enable the student to write and use ‘A + B = B + A = C’, ie to extend the commutative property of the addition of reals to the addition of matrices (which is correct).

In the empirical setting, template–association is recognised fairly easily. It occurs when the student transfers properties of a sign with which they are familiar to a new or unfamiliar sign which has the same template as the original sign.

Each and every student in the mathematics interviews uses template–orientation at some point as they perform the mathematical activities in the task.

I will illustrate this form of thinking by two examples, chosen specifically because they illustrate how a template–orientation may be initially enabling or disabling of object appropriation. Examples 14 and 15 both involve the use of ∞ as if it was a real number. In the first case (Example 14) such use is enabling; in the second case (Example 15) such use is initially disabling.

Example 14: Template of a definite integral enables use of improper integral

John's initial determination of whether the improper integral, $\int_1^\infty x^{-3}dx$, in Question 5(a) converges or diverges shows how John associates the template of an improper integral with an infinite limit with the familiar template of a definite integral. With this template–orientation, he is able to use the improper integrals in mathematical activities (even though he is initially unsure about many of its properties such as its convergence and divergence).

(Brian uses a very similar template–orientation for Questions 5(a), 5(b) and 5(c) as does Peter in Question 2(b)).

In Question 5(a) John performs the following computations: He writes $\int_1^\infty x^{-3}dx = -2x^{-2}\Big|_1^\infty = \frac{-2}{x^2}\Big|_1^\infty$ = 0–2 = –2 [line 68][23]. By treating ∞ as if it is a real number, he is using the template of a definite integral to compute the improper integral.

Although this treatment is not problematic in this instance, this sort of template–orientation in which ∞ fits into a 'slot' normally occupied by a real number, can lead to grave errors as we see in Example 15.

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[23] This computation is inaccurate. $\int_1^\infty x^{-3}dx = \frac{-1}{2}x^{-2}\Big|_1^\infty$

Notwithstanding these dangers, this use of the definite integral template when working with an improper integral allows John, Brian and Peter to develop aspects of their concept of an improper integral and its convergence or divergence. Later they are able to correct their notation so that their use of ∞ becomes commensurate with its conventional usage.

Example 15: Template '$\lim_{x\to c} f(x)=a$' disables use of statement $\lim_{x\to c} f(x) = \infty$

In Example 14 we saw how a template–orientation enables the functional use of the improper integral template. But in Example 15 we see how a template–orientation distorts the meaning of the statement $\lim_{x\to c} f(x) = \infty$ and leads to serious errors involving the concepts of divergence and convergence.

I will use Fred's response to Question 5(b)[24] to illustrate. In order to determine whether $\int_{-\infty}^{1} x dx$ converges or diverges, Fred computes the integral. He does this using the correct notation (unlike John in Example 14) and correctly finds that the integral equals $\lim_{b\to -\infty} \frac{1}{2} - \frac{b^2}{2}$ = $-\infty$ [line 67]. He then states: "Therefore this one converges because the limit exists" and he writes "converges because limit exists" [line 68].

I suggest that this is an example of a template–orientation to a mathematical sign. Fred is using the template $\lim_{x\to c} f(x) = -\infty$ as he would use the template $\lim_{x\to c} f(x) = a$. Since the latter limit exists (if a is a real number), Fred assumes that the former limit must exist. In other words he has placed ∞ into a slot or template normally reserved for a real number.

[24] In Question 5(b), the student has to determine whether $\int_{-\infty}^{1} x dx$ converges or diverges.

Fred uses $\lim_{x \to c} f(x) = \infty$ consistently in this way. Later in the interview when I explain that if the limit is infinite, it does not exist as a real number [line 142], Fred accepts my explanation [line 143]. Consequently he goes back and corrects each answer in which he argued that a particular improper integral was convergent because the infinite limit existed.

§6.2.2.7 Pseudoconcept

As previously suggested, a pseudoconcept is a form of complex thinking which is particularly prevalent in mathematical thinking. Its distinctive feature is its duality: from the outside (ie to the observer) it has the appearance of a concept, but on the inside (in terms of its genesis, the conditions under which it develops and the causal associations of these conditions) it is actually a complex (Vygotsky, 1994: 226).

In mathematical activity, a learner is using a pseudoconcept when he is able to use a mathematical notion *as if* he fully understands that notion, even though his knowledge of that notion may be riddled with contradictions and illogical connections. For instance, consider the student who correctly argues that the function $f(x)=x^3$ is increasing for $x \in \nabla$, because $f'(x) = 3x^2 \geq 0$ for all $x \in \nabla$. The teacher may justifiably believe that this student has a true conceptual understanding of the concepts of the derivative and of increasing functions and of the relationship between them (viz. if $f'(x) \geq 0$, $x \in \nabla$, then f is increasing for $x \in \nabla$). But the student may have little understanding of *why* a non–negative derivative guarantees an increasing function; he may not even know what a derivative is or what an increasing function is. In this case I categorise the student's knowledge as pseudoconceptual.

Before continuing with this discussion of the pseudoconcept as it manifests in the context of a learner constructing a mathematical concept, I would like to discuss Vygotsky's argument about the role of the pseudoconcept in the development of the meaning of a word (or concept) for a child.

Basically, Vygotsky (1994: 227, 228; my italics) argued that:

> Childhood complexes which correspond to the meaning of words, do not develop freely and spontaneously along the lines marked out by the child himself, but in *certain definite directions which have been predetermined* for the developmental process of the complex by previously established meanings which have been assigned to the words in adult speech … By engaging the child in verbal communication, an adult can influence the further progress of this generalization process, as well as the end outcome of that journey which will be the result of the child's generalizations. But adults cannot pass on their method of thinking to children.

As with a word, a mathematical object is a discursively constituted and socially sanctioned object (see §2.1). Thus, paraphrasing and adapting Vygotsky's argument to the usage of mathematical signs rather than the meaning of words, I claim that:

> Learner's complexes whose usage corresponds to the usage of specific mathematical signs in the mathematical community, do not develop freely and spontaneously along the lines marked out by the learner herself or himself, but in *certain definite directions which have been predetermined* for the developmental process of the complex by previously established usages which have been assigned to the signs in the mathematical discourse. By engaging the learner in verbal or written communication, a teacher or text can influence the further progress of this generalization process, as well as the end outcome of that journey which will be the result of the learner's generalizations. But the teacher or text cannot pass on their method of thinking to children.

In other words, I am arguing that the pseudoconcept enables the learner to communicate about and engage in activities with the mathematical sign even before he knows what the relevant mathematical object is. This communication and these activities, which take place with people and texts whose usage of the relevant sign is concurrent with that of the mathematical community, guide and influence the development of the learner's generalisation processes.

In this way the pseudoconcept acts as a bridge between the mathematical complex (whose use allows for communication and activities about and with the mathematical object) and the mathematical concept (which has internal consistency and logical connections within itself and to other mathematical concepts).

In summary, the use of a pseudoconcept gives the learner a point of entry into the community of mathematicians, it allows him to engage with mathematical ideas and to communicate mathematically with teachers, texts and peers. It is this use that enables and guides the development for that learner of a personally meaningful concept whose use is commensurate with that of the wider community.

Pseudoconcepts are plentiful in mathematical thinking. An example of a pseudoconcept which abounds in mathematical activity occurs when a student uses a procedure or algorithm *as if* he has full conceptual understanding of the object and notions referred to by that procedure.

Pseudoconcepts are often difficult to detect. This is because they appear as if they are concepts[25]. In my analysis I have distinguished between a concept and a pseudoconcept by examining the integrity of the student's prior or post activity. For example, if a student is able to detect whether an improper integral is convergent or divergent but soon after, or immediately preceding the task, reveals an incoherent or inconsistent idea of what convergence or divergence is, I have classified that student's usage of signs as pseudoconceptual.

In the mathematical interviews, there are many instances of the use of pseudoconcepts. All of the students (Tom, John, Brian, Peter, Mary, Fred, David) are, at some stage of their interviews, able to do Question 5 in a reasonably satisfactory way[26]. But in each of these cases, later activities show that the student's notions of convergence and divergence are muddled or incorrect.

I will present two examples of the use of a pseudoconcept here. In the first example David does Question 5 successfully even though later activities show that his conception of the convergence or divergence of an improper integral is inadequate. In the second example Mary explains why the

[25] I describe the characteristics and indicators of a concept in §6.2.4..

[26] In Question 5, the student has to decide whether each of three given improper integrals are convergent or divergent.

statement in Question 9 is true. Although her utterances, her deductive reasoning and manipulation of the signifiers make it appear that she has a well–developed concept of an improper integral, this is not the case (as I will show).

Both of these examples illustrate how the use of a pseudoconcept gives the student access to mathematical discourse and activities, allowing the learner to engage with mathematical ideas and to communicate mathematically with teachers, texts and peers even before he or she has a fully–developed conceptual understanding of the mathematical object.

Example 16: Use of pseudoconcept in a procedure

In this example I will look at David's response to Question 5(b). I will argue that, although his activities and utterances when answering the question appear to be based on conceptual thinking, his later utterances belie this. Indeed I contend that David's notion of convergence and divergence, as used in Question 5(b), is pseudoconceptual.

Question 5(b) requires the student to determine whether the integral $\int_{-\infty}^{1} x dx$ is convergent or divergent. To do this, the student needs to compute the integral; if the result is ∞, the integral diverges, otherwise it converges.

I will now describe David's activities in answering Question 5(b).

David computes $\int_{-\infty}^{1} x dx$. He writes $\lim_{x \to -\infty} \int_{x}^{1} x dx = \lim_{x \to -\infty} \left[\frac{x^2}{2} \right]_x^1 = \lim_{x \to -\infty} \left[\frac{x^2}{2} - \frac{1}{2} \right]$ [line 73]. This use of x is ambiguous (but not wrong) in that x is used both as a function variable and as a variable limit of integration[27]. Although such a dual usage of x is potentially confusing, David manages to perform his

[27] Such ambiguity can easily be avoided by determining the integral on a finite interval, say [a, b], and then letting the relevant limit, ie a or b tend to infinity. Indeed this avoidance of ambiguity is implicit in the statement $\int_{a}^{\infty} f(x)dx = \lim_{b \to \infty} \int_{a}^{b} f(x)dx$ given in the definition of an improper integral with an infinite limit.

computations reasonably efficiently and without confusion notwithstanding that he makes a mistake in the order of terms, writing $\lim_{x\to-\infty}\left[\frac{x^2}{2}\right]_x^1 =$ $\lim_{x\to-\infty}\frac{x^2}{2}-\frac{1}{2}$, rather than $\lim_{x\to-\infty}\frac{1}{2}-\frac{x^2}{2}$.

The mistaken order of terms, however, makes no difference to the validity of David's argument that the limit does not exist and so the improper integral diverges [line 74].

David also explains that L'Hopital's rule does not apply to the determination of $\lim_{x\to-\infty}\frac{x^2}{2}-\frac{1}{2}$. That he even considers using L'Hopital's rule suggests that he is using the signifiers mindfully.

Although David is able to do Question 5(b) *as if* he has a conceptual understanding of the convergent and divergent improper integral, later activities with the convergence or divergence of an improper integral reveal otherwise (see the in–length study of David in Chapter 7, particularly Episode 5 for justification of this assertion).

Accordingly I retrospectively categorise David's thinking with regard to convergence and divergence as pseudoconceptual.

Example 17: Use of pseudoconcept in a problem

Unlike Question 5, where a particular procedure is invoked, Question 9 needs to be solved using deduction. Although Question 9 seems like a very simple problem in that the deductive steps are straightforward, not all students are able to solve it successfully.

In fact, only Mary, Brian and David solve the problem without any obvious struggle. Fred struggles a little but is finally able to solve the problem. Tom gets stuck partly because he is using inappropriate notation (he talks of $f(\infty)$) and John is unable to begin the problem without a particular example around which to base his argument (see Example 7 above). Peter does not even look at the problem because of time constraints.

In this example I will look at Mary's activities around Question 9.

Mary appears to solve the problem easily. Her deductive steps and her notation are fine [lines 157 –159]. To this extent we would be forgiven for believing that Mary is using a well–developed concept of an improper integral.

But in fact, the opposite is true. Mary believes that the integral $\int_0^\infty f'(x)dx$ is not improper because the function f' is continuous[28]. As she states: "f prime is continuous. That means not improper" [line 156].

Mary articulates this incorrect idea (that an integral is improper only if the relevant function which is being integrated has a discontinuity on the interval of integration) several times in the interview (see lines 43, 80, 83, 89, 90, 107, 109). Also see Example 5 above for discussion of this incorrect conception. Indeed it is only as a result of a later perturbation (when doing Question 10), that Mary is forced to re–look at the definition of an improper integral and to modify her notion of an improper integral [lines 168, 170].

Thus, although Mary correctly and confidently uses the improper integral in Question 9 *as if* she has a conceptual understanding of this object, her conception of an improper integral when doing Question 9 is incorrect. For this reason I categorise her use of the improper integral in Question 9 as pseudoconceptual.

§6.2.3 POTENTIAL CONCEPTS

This account of the learner's appropriation of a mathematical object would not be complete if mention were not made of Vygotsky's third preconceptual phase, the potential concept.

According to Vygotsky (1986, 1994), concept formation requires both generalisation and abstraction. The principal function of the complex is "to organise discrete elements of experience into groups" which can create a

[28] The fact that f' is continuous is irrelevant to whether the integral is improper or not. The given integral is improper because its upper limit of integration is infinite.

“basis for later generalizations” (Vygotsky, 1986: 135). But in order to do this, the learner also has to abstract out certain elements or attributes from the experience. Vygotsky called the formation that results from the grouping together of objects on the basis of a single attribute or a set of attributes, a ‘potential concept’.

Certainly, the processes of generalisations and abstractions are inherent in the construction of any mathematical concept (see §2.2.3.2 for a discussion of these processes) and so potential concepts in the Vygotskian sense abound in mathematical thinking.

An example of a mathematical potential concept is as follows:

In elementary school, the notion of multiplication hinges on the idea of repeated addition. That is, repeated addition is the abstracted out attribute of multiplication. This idea of multiplication is a potential concept. It is not an actual or fully–developed concept since it cannot be applied to the operations of multiplication by a fraction. Furthermore the notion of multiplication as repeated addition does not fit into the hierarchical system of interrelated mathematical concepts.

Potential concepts play a fundamental role in advanced mathematical complex thinking in that the formation of all mathematical complexes presupposes the abstraction of particular traits. But the abstraction of attributes is so profoundly intertwined with the formation of complexes in mathematical thinking that it is impossible to distinguish potential concepts from most mathematical complex thinking. For this reason I suggest that the potential concept is not a particularly useful or appropriate category of analysis, in the advanced mathematical domain.

In summary, abstraction and generalisation both play fundamental roles in the construction of a mathematical concept. However their use is so intertwined that the formation of complexes would be both ontologically and epistemologically impossible without both these processes. Accordingly I contend that an empirical examination of the formation of a potential concept as distinct from a complex (if that were possible) is neither illuminating nor desirable in the context of advanced mathematical activity.

§6.2.4. CONCEPTS

What is a mathematical concept?

As I discussed in §2.1, I use the term mathematical concept to refer to the mental idea of the mathematical object.

In terms of my usage, and consistent with Vygotsky's theory of concept construction (1986, 1994), an idea of a mathematical object becomes a concept (as opposed to a complex or potential concept) when its internal links, ie the links between the different properties and attributes of the concept are consistent and logical, and the external links, ie the links of that concept to other concepts, are consistent and logical.

Vygotsky's distinction between a well–developed concept and a developing concept (ie a complex or a potential concept) further illuminates the nature of a concept. After all, concepts "just start their development, rather than finish it, at a moment when the child learns the term or word meaning denoting the new concept" (Vygotsky, 1986: 159). According to Vygotsky the bonds between the elements of a concept are abstract and logical whereas the bonds between a complex are associative and factual. Furthermore in a concept a particular set of attributes is privileged over others whereas in a complex no attribute is privileged over another.

Within the mathematical (or any other academic) domain, the evolution of concepts is systematic and deliberate (Vygotsky, 1986: 148, 149). Moreover scientific[29] concepts are interrelated within a system:

> Concepts do not lie in the child's mind like peas in bag, without any bonds between them. If that were the case, no intellectual operation requiring coordination of thoughts would be possible, nor would any general conception of the world. Not even separate

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[29] Vygotsky distinguished between scientific concepts and everyday concepts. Unlike scientific concepts, everyday concepts are saturated with experience and constructed unintentionally.

> concepts as such could exist: their very nature presupposes a system. (Vygotsky: 1986: 197)

With regard to scientific concepts, “the relation to an object is mediated from the start by some other concept” (Vygotsky, 1986: 172).

Since I am using the term ‘mathematical concept’ to refer to the mental idea of a mathematical object, mathematical conceptual thinking has many of the same features as Sfard’s (2000) object–mediated phase. In particular, since my ‘concept’ refers to the mental idea of a mathematical object whose existence is indicated by a specific use of signs and Sfard’s object–mediation refers to the discursive use by the learner of an abstract object as if it was a concrete and material object, many of her notions about object–mediation can be applied to my conceptual thinking phase.

Sfard characterises object–mediation as the transformation of the “signifier–as–an–object–in–itself” to the “signifier–as–a representation–of–another–object” (Sfard, 2000: 79). I suggest that in conceptual thinking the signifier is similarly transformed. That is, for the preconceptual learner, the signifier rather than the signified is often the focus of attention (see examples 1, 2, 4, 5, 6, 14 and 15). In contrast, for the conceptual learner the signifier of the mathematical object needs to be invisible so that the signified (ie the idea of the mathematical object) becomes manifest. Sfard (2000: 79) argues as follows:

> the transition from signifier–as–an–object–in–itself to the signifier–as–a–representation–of–another–object is a quantum leap in a subject’s consciousness….. Once the symbol takes the role of representation, the whole discourse undergoes a modification. The old ways of expressing mathematical truths give way to new formulations… The ongoing mathematical conversation, perhaps more than any other type of discourse, is similar to a living organism that incessantly grows and mutates without losing its identity.

Finally and again consistent with Sfard’s idea of object–mediated sign usage, the learner in the conceptual phase is able to attend to the mathematical object in its entirety, not just to isolated or fragmented aspects of the object (although this may sometimes also be required) or just to its signifier. Indeed, “the new sign gets life of its own and becomes an integrated signifier–signified unit “ (Sfard, 2000; 59).

For example, consider the mathematical object, the natural logarithm function. In the Mathematics I Major course this object is introduced via its definition, viz. $\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0, x \in \nabla$. From this definition, the student is expected to deduce various properties of the natural logarithm. For example,

- the domain is (0, ∞) and the range is (–∞,∞):
- the function is continuous, increasing and one–to–one;
- the graph is concave down.

All these properties can be logically deduced from the definition of the ln function and their deduction is, of necessity, mediated by prior knowledge of various other mathematical concepts, such as domain, range, continuity, increasing functions, one–to-one, concavity and so on.

Similarly other properties of the natural log function (eg ln ab = ln a + ln b, $\ln a^b = b \ln a$) can be proved using properties of differentiable functions, the Second Fundamental Theorem of Calculus, and so on. These properties are also consistent with the properties established at school level (where the log function is treated as the inverse of the exponential function which is defined only for integer powers).

So a fully–fledged concept of the natural logarithm function requires logical links between the different elements of the concept (as made manifest in deductive reasoning), logical links to other concepts (eg to the differentiability of a function, the Second Fundamental Theorem of Calculus and so on) and consistency between concepts (ie the properties of one concept should not lead to contradictions between the properties of another concept).

Furthermore in order to develop the properties of the natural logarithm function from its definition and to link these properties into one coherent and consistent whole which itself relates in a logical way to other mathematical concepts, the student needs to be able to attend to all aspects of this object (many of which are mediated by yet other concepts). He cannot just focus on isolated properties or features of the natural logarithm function without regard for other properties. Such a fractured focus would inevitably lead to contradictions and gaps in the concept.

Finally, and most importantly, in conceptual thinking the signifier 'ln x' acts as a sort of trigger so that the learner becomes aware of the specific mathematical object, the 'ln function' (and is able to access its properties and links to other mathematical concepts explicitly if required) when he sees those symbols: 'l', 'n', 'x' side–by–side. It is as if conceptual thinking allows the learner to see through the signifier, 'ln x', to the signified, the 'natural ln function'.

Empirical indicators of a mathematical concept

What then are the empirical indicators of conceptual mathematical thinking, ie how are the mathematical signs used when the learner is using conceptual thinking?

- As with all the other phases, the only way to observe a student using a mathematical concept is through his usage of signs. In trying to observe the use of conceptual thinking by a student, one must bear in mind that logical linking between different elements of the concept and between different concepts may ultimately (and desirably) be invisible and attenuated.

  Since the links within and between mathematical concepts are logical links, a necessary (but not sufficient) condition for conceptual thinking is the *use of logical reasoning*. This use of logic may be explicit or implicit. Since implicit logical reasoning cannot be seen or heard, a necessary indicator of conceptual thinking is the *absence of non–logical reasoning.* In other words, if a learner reasons via association or similar non–logical forms of reasoning, his thinking cannot be classified as conceptual.

- The student needs to be flexible in his use of the signifier. Vygotsky (1986: 201), while talking about how the structure of generalisation changes as the child matures, discussed how

> a young child must reproduce the exact words in which a meaning was conveyed to him. A schoolchild can already render a relatively complex meaning in his own words.

Sierpinska (2000) refers to the same phenomenon when she describes how one particular student who is trying to "survive" the Cabri–Geometry course, attempts to use those terms which the teacher stresses as important in his solutions even when, spontaneously, he would use different words to express similar ideas. Sierpinska's (2000: 9) point is that

> by using borrowed language, he was not able to express his own ideas, which, not mediated by verbally elaborate and explicit language, lacked the necessary support to develop into fully fledged concepts.

The purpose of my point is similar: I am trying to argue that the use of borrowed language and a rigid use of signifiers indicates the non–use of conceptual thinking. In a similar vein, Sfard (2000), while describing the characteristics of object mediation (which I take as an aspect of conceptual thinking) states: "The symbol is often inadvertently exchanged with other signifiers" (p. 80).

For example, a student who is able to define the derivative of a function f(x) at x = c as either $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ or as $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, is flexible in his use of signifiers. Thirdly, an indication of conceptual thinking occurs when the learner is able to use various different signs (signifier plus signified) to talk of one mathematical object. For example, the student who talks interchangeably about the slope of a curve f(x) at a point x = a, the rate of change of f(x) with respect to x at x=a, and the derivative of f(x) at x=a is able to use all these different mental ideas (the signifieds) with their different signifiers (the words) to look at the mathematical object of the derivative.

Vygotsky referred to this phenomenon as the equivalence of concepts:

> The higher levels in the development of word meaning[30] are governed by the law of equivalence of concepts according to which any concept can be formulated in terms of other concepts in a countless number of ways (Vygotsky, 1986: 199) .

Methodological problems

Despite this list of empirical indicators of conceptual thinking in the mathematical domain, it is often very difficult to decide whether the student is using pseudoconceptual or conceptual thinking. There are two principal reasons for this:

- First, by its very nature, the pseudoconcept mirrors the true concept in its outer manifestations. Thus the only way to distinguish one from the other is to examine activities and utterances spoken just before or just after the mathematical usages in question. If the prior or post usages reflect non–conceptual or non–logical reasoning, the mathematical usages in question can be retrospectively classified as pseudoconceptual[31]. Otherwise the usage may be classified as conceptual.
- Secondly conceptual thinking is often expressed (and I suggest experienced) as an abbreviated usage of signs, which makes it very difficult to detect. Even if its existence is suspected, the observer cannot always justify her classification; there is much scope for ambiguity of intent when the learner uses signs (words or symbols) in a truncated fashion.

Indeed in my analysis of data from the mathematical interviews, I found very little explicit and non–ambiguous evidence of conceptual thinking[32].

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[30] Vygotsky's use of the term 'word meaning' is equivalent to my use of the term 'concept'.

[31] Such a strategy is based on the assumption that a concept is more stable than a complex; this stability, in turn, is a function of my conviction that a concept which is part of a logically connected set of ideas is somehow more 'fixed' than a complex with its often vague and transient links to other ideas.

[32] This does not mean that there was little conceptual thinking; rather that it is difficult to detect.

Nevertheless, I interpret David's use of the improper integral sign, and his use of infinity (in Question 10), as indicative of conceptual thinking.

Example 18: Two concepts

David reads the statement in Question 10 out loud. Without reading the attached note, he starts explaining why the statement is true. He argues that "the **area** for the integral from infinity to infinity of f(x) equals 2 times **integral** from zero to infinity of f(x)" [line 97], and he writes

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx.$$

Given that David speaks about area interchangeably with an integral and reveals no sign of non–logical reasoning in this episode, I suggest that David is using an equivalence of concepts (the improper integral is an area) in his argument.

With regard to the notion of infinity, David argues that $\int_{-\infty}^{\infty} f(x)$ "must be a number … it must not be an infinite number so that you can divide by two to get to this number over here" (referring to $\int_{0}^{\infty} f(x)dx$) [line 98].

David's argument is interesting in that it rests on two ideas, both of which are mathematically correct: First, if an improper integral is convergent, its numeric value is a real number; if it is divergent, its numeric value is infinite. Secondly, it is logically impossible to manipulate an infinite number to get a finite number.

I suggest that the latter idea requires a flexible notion of infinity which is logically linked to other mathematical concepts (such as operations on real numbers). I thus interpret David's usage of infinity as indicative of conceptual thinking about infinity.

§6.3 SUMMARY

In this chapter I have described and illustrated the different phases through which a learner moves as he goes about appropriating a mathematical object from a written definition.

In accordance with the research mode of interrogation (Brown and Dowling, 1998), I have explicated the theoretical characteristics of each phase and the indicator variables of each phase in the context of a student appropriating a new mathematical object. Furthermore I have used data from my mathematical interviews to illustrate and elaborate each phase.

The theoretical concept variables (ie the phases) originate in two sources: I have elaborated and particularised Vygotsky's stages in his theory of concept formation to the mathematical domain and I have developed other phases specific to the appropriation of a mathematical object. These latter phases derive from my experience as a lecturer and tutor of undergraduate mathematics students and from my readings in mathematical education research.

Below I summarise how I have elaborated and applied each of the broad categories (heaps, complexes, potential concept and concepts) in Vygotsky's theory of concept formation to the mathematical domain.

I have taken a concept to mean the mental idea of a mathematical object. Using Vygotsky, I have elaborated on this notion stressing that the internal links of a concept, ie the links between the different properties and attributes of a concept are consistent and logical, and that the external links, ie the links of that concept to other concepts are also consistent and logical. Furthermore a scientific concept (such as a mathematical concept) is always a part of a system of interrelated concepts so that the usage of one concept is always mediated by other concepts in the system.

According to Vygotsky (1986, 1994), the preconceptual phase consists of three components: the heap thinking phase in which chance and incoherent connections abound; the complex phase in which the learner gathers discrete elements into groups so creating a basis for later generalisations and the potential concept phase in which the primary activity of the mathematical learner is the abstraction of particular attributes of the relevant signs.

In this chapter I have demonstrated (through argument and examples) that the heap thinking phase and most of the complex phases are applicable and illuminating of various types of mathematical thinking.

In contrast, the formation of potential concepts (that is, the abstraction of attributes and properties of mathematical objects) are inextricably interwoven with the formation of complexes in advanced mathematical concept construction. Accordingly, I have argued that potential concepts are not a useful stand–alone analytic category in the advanced mathematical domain.

Furthermore, given my theoretical assumption that all mathematical knowledge is semiotically mediated, I have subdivided the complex phases into two principal categories: complexes which are dominated by the learner's engagement with signifiers and complexes which are dominated by the learner's engagement with signifieds.

The signified–orientated complexes are further split into six types of complexes: chain, association, collection, diffuse, pseudoconcept and graphical. The first five of these are elaborations and adaptations of Vygotsky's similarly–named complexes; graphical complexes are peculiar to the mathematical domain. Similarly I found it useful and illuminating to further distinguish three sub–types of the association complex in the mathematical domain, ie artificial, example–centred and surface association complexes (the latter is a signifier–orientated complex).

Within the signifier–orientated complexes, I found it useful to further distinguish two types of mathematical complexes: surface–association complexes (mentioned just above) and template–orientated complexes.

All this information is summarised in Figure 4 below.

In Chapter 7 I use the empirical indicators that I have developed in this chapter to analyse the protocols of two individual students in great detail. Hopefully this will further demonstrate the illuminating power and applicability of appropriation theory to an understanding of the learner's appropriation of a mathematical object presented via a written definition.

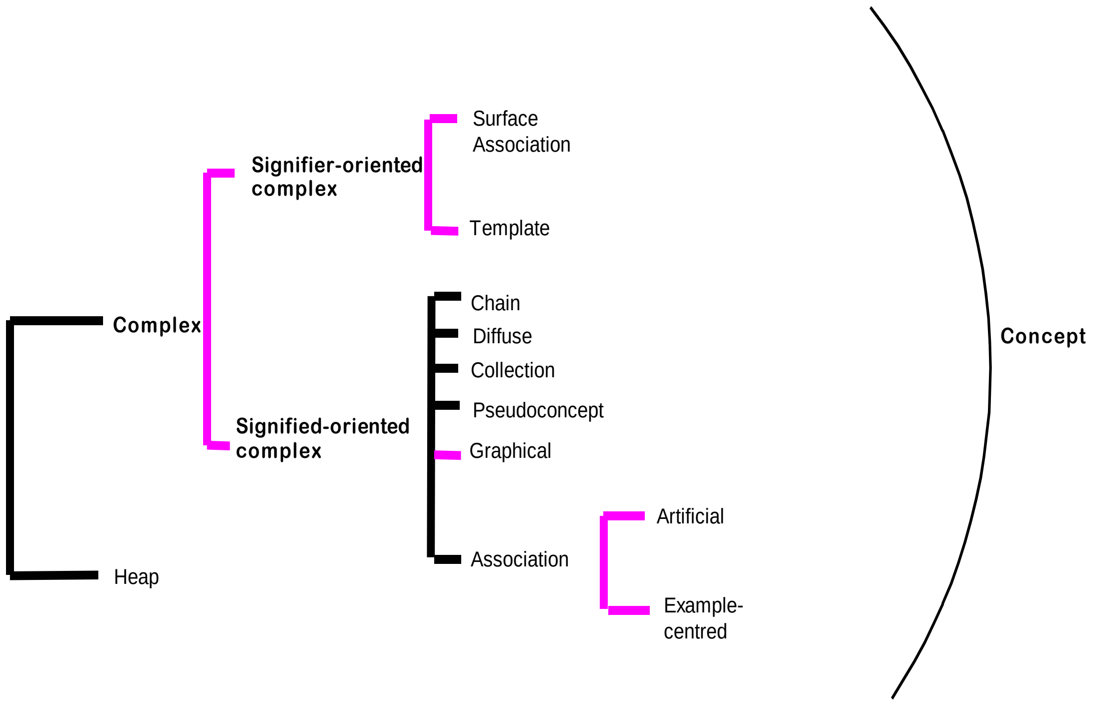

Figure 4: Phases in the appropriation of a mathematical object

Note: The phases shown with black lines derive from Vygotsky's theory of concept formation; the phases shown with pink lines derive from my own mathematical experiences or readings.

CHAPTER 7: ANALYSES OF TWO CLINICAL INTERVIEW PROTOCOLS

§7.1 INTRODUCTION

In this chapter I present the analyses of two students' protocols. Each analysis comprises a description and an interpretation of the mathematical activities of two students in a mathematical interview according to a specific format.

In the following chapter (Chapter 8) I discuss these analyses and use them as a basis for further illustrations and elaborations of various aspects of appropriation theory.

Accordingly Chapter 7 and Chapter 8 constitute two related aspects of the empirical component of this thesis and need to be considered together. Indeed the separation of Chapters 7 and 8 into two chapters is an artificial device intended to facilitate the reading of this thesis. That is, the separation is based on my belief that the reader needs to bring different analytical and critical skills to bear in perusing each component of the empirical effort and that this is promoted by separating these two chapters.

§7.1.1 PURPOSE OF ANALYSES

In the current chapter I will use appropriation theory to analyse how a student goes about appropriating a specific mathematical object whose meaning has already been sanctioned by the mathematical community[1]. My purpose in doing this is fourfold:

- First, I want to illustrate, via example, how different students move back and forth between the various phases posited by appropriation theory as

[1] Modifications to this theory are required when a person or group of people develops a new symbol or mathematical object.

they go about appropriating a new mathematical object (episodic analysis).

- Secondly I wish to show (in both Chapters 7 and 8) how appropriation theory illuminates the processes whereby a student comes to appropriate (or partially appropriate) a new mathematical object.
- Thirdly, I intend to use these episodic analyses to demonstrate (in Chapter 8) that the mechanism for moving through the different phases is the functional use of the mathematical sign.
- Finally I wish to use these episodic analyses to support certain of my elaborations of appropriation theory (Chapter 8).

§7.1.2 TWO CRITICAL CASES

For the purpose of both illustrating and elaborating appropriation theory, I have chosen to focus on the mathematical activities of two students, John and David, as each of them goes about appropriating a mathematical object, the improper integral, in a clinical interview.

In Brown and Dowling's terms (1998: 31) the activities in the interview setting of these two students constitute my critical cases.

I have chosen these two students as critical cases for several reasons:

- First, if I wish to claim that appropriation theory is an illuminating tool for the analysis of mathematical concept construction and usage, I need to show, at the least, that I can usefully apply this theory to the analysis of the mathematical activities of two students with contrasting profiles.

  Certainly John and David represent two very different sorts of students:

  David comes from a relatively wealthy urban background. English is his home language and he went to a very good private school[2]. In contrast John comes from a poor rural background and attended a poorly

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[2] David's school had a pass rate of 100% for its Matriculation level Higher Grade Mathematics examination at the end of 1999.

resourced non–urban school[3]. Furthermore John's home language is Tsonga and he is not fluent in his use of English. In fact, during a conversation with John in June, 2000, (during the mathematical tutorial time) John told me how confused he had been in his first few weeks at University by inter alia, the language of instruction, English.

David studied Additional Mathematics at school. He achieved an A symbol for the subject. Additional Mathematics was not even offered as a possible matriculation subject at John's school.

John is struggling with the first–year mathematics course and, despite much dedication, ends up failing the course with an aggregate of 41%. (He repeats the course in 2001 and passes with the excellent mark of 95%[4]). David, on the other hand, seems to find the course relatively accessible and achieves an 81% mark for the Mathematics I Major course at the end of the academic year. In the context of the University of the Witwatersrand and South Africa this is a very good mark.

- Secondly, and in line with my argument in Chapter 5 that a clinical interview is a social event and so differs on certain dimensions for different students, the clinical interviews in this study are necessarily different for each of David and John[5]. Given that I am arguing that appropriation theory is applicable to students in various learning contexts, I plan to use the fact eventually of these contrasts to strengthen my claims of the usefulness of appropriation theory as a tool for understanding mathematical object appropriation in various contexts.

  In particular the quality of my interactions with each of John and David differs. I will highlight and explicate some of these differences in the discussion (Chapter 8).

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[3] John's school had a pass rate of 60% – 79.9% for its Matriculation level Higher Grade Mathematics examination at the end of 1999.

[4] See §8.4 where I discuss this radical improvement in John's grades further.

[5] This is despite the fact that the mathematical task is the same for both students and that I am guided by the same set of guidelines for each.

In the larger question of how different students learn in the university environment, I suggest that these differences in interactions between students and lecturer are particularly significant and impact directly on the zone of proximal development (ZPD) of each particular student. But despite these differences, and as mentioned, I plan to demonstrate that the mathematical activities of both students can be usefully analysed in terms of appropriation theory.

- Thirdly and in contrast to some of the other students, it seems that both John and David are sincere and trying to perform well in the interview despite there being no reward or 'external' incentive for committed performance in the interview[6]. Furthermore John seems to be a particularly honest student in the intellectual arena; during all my encounters with him (in tutorials and consultations), he tells me what he does not understand and what he does understand[7].
- Finally, the two critical cases have certain features in common which implicitly contribute to the interview and my analysis of it.

  In particular, both students are first–time first–year students; both students received an A symbol for their Matriculation level Higher Grade Mathematics examinations and both students are studying Actuarial Science at University. Furthermore both students seem to be highly motivated, diligent and eager to learn. They both come to all their tutorials, ask questions and go to most lectures.

§7.1.3 THE MATHEMATICAL OBJECT

In the analyses that I present here, I examine John's and David's appropriation of the object: an improper integral. I do this by analysing their respective usages of mathematical signs relating to the improper integral. In

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[6] This is in contrast to some other students whom I interviewed. Brian, for example, tells me several times during his interview that his head is fuzzy and that he can't think properly (see for example, line 13, line 293 in Brian's interview).

[7] See for example lines 15, 34, 70, 116, 148, 166, 174, 186 in John's interview.

particular I focus on how they deal with a particularly important property of the improper integral, its convergence or divergence. I have chosen to focus on this property because convergence /divergence is a fundamental aspect of the improper integral and because it is a notion with which students have to grapple at various points in the interview.

§7.1.4 ANALYTICAL FRAMEWORK

In order to analyse the interview protocols, I used an analytic framework which I have developed over the course of this research. The purpose of this framework is to provide a structure within which I can distinguish between, and illuminate, the different phases that a student moves through as he goes about appropriating a mathematical object in a clinical interview situation.

Similarly to Schoenfeld (1985) I parse interview protocols into large chunks called episodes. Each episode represents a period of time during which the interviewee is focussed on or engaged in one particular aspect of a mathematical task (for example, answering a particular question).

I then further divide these episodes into sub–episodes so that in each sub–episode the learner is primarily engaged in a single set of activities seemingly directed to one particular end. In this way I am able to delineate and distinguish one set of homogeneous actions (and inevitably one phase) from another in the interview analysis.

For each episode I first present a description[8] of the activity in that episode (the *Summarising overview*). From the summarising overview I am generally able to isolate an empirical indicator variable for each sub-episode (the *Indicator* section). In terms of my methodology (see Chapter 5) I use this indicator variable to point to the theoretical concept variable; that is, the student's phase of mathematical thinking at that stage in time. I discuss these theoretical concept variables in the *Formation* section.

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[8] Since any description is never free from an implicit theoretical perspective, these descriptions are actually interpretations.

In summary, for each episode I use the following framework[9]:

EPISODE NUMBER AND LINE NUMBERS IN THE INTERVIEW TRANSCRIPT (eg **EPISODE 1: LINES 23-35**)

Question Number eg Question 9

Summarising overview: (Here I describe what appears to be happening.)

Transcript

Sub-episode number and line numbers (*eg* ***1a. lines 23-27)***

Indicator: (Here I highlight the empirical indicator variable.)

Formation: (Here I use the empirical indicator variable to classify the theoretical concept variable, ie the phase of thinking.)

Sub-episode number and line numbers (*eg* ***1b. line 28*)**

Indicator:

Formation:

etc.

In practice it frequently becomes appropriate to collapse the last two sections (ie indicator and formation) into one section.

In this chapter I occasionally need to describe, without analysis, a set of activities. Such a non–analytic description may be required for the purpose of giving continuity to my chronicling of events or as a backdrop to some other set of activities which I have analysed. In those cases I do not use the above analytic framework. Rather I indicate the focus of the set of activities and then give a broad description/ interpretation of the activities (see for example Section 6 in the analysis of John's protocol). I call such episodes: 'connecting episodes'.

[9] For ease of reference I am using the same fonts here as I do in the actual analysis.

§7.1.5 ORDER OF EPISODES

Since the purpose of these analyses is to show the development over time of a concept or concepts, I present the episodes in chronological order as they occurred in the interview. Since both students frequently returned to previous questions in the course of doing another question, the question numbers appear to jump around. This can be confusing if the reader is not aware that the order of the episodes is determined by the actual order during the interview.

Throughout the analyses I have called the student's first attempt at a question, the 'first pass'; the second attempt, the 'second pass', and so on. So, for example, 'Question 2(a): first pass' represents the student's first attempt at Question 2(a); 'Question 2(a): second pass' represents the student's second attempt at Question 2(a) (perhaps after he has attempted other activities such as Question 5).

I also remind the reader that the chronological order in which the questions were presented to the students for the first pass was Questions 1, 2, 3, 4, 5, 9, 10, 11, 6, 7, 8. For subsequent passes the student chose which question to look at again. (See §5.5.3 in the Methodology chapter for more details about the format of the interview.)

§7.1.6 EPISTEMOLOGICAL NOTE

I am very aware that, no matter the persuasive value of the empirical indicator variables, the researcher can never know for sure how someone else is thinking. Thus, although in the analyses I frequently refer to John's or David's "way of thinking", or to John's or David's "usage of signs being indicative of a way of thinking", and so on, these are phrases of convenience rather than of conviction. That is, these phrases signify my interpretation of what is happening, rather than assert an incontrovertible claim about the nature of John's or David's thinking at that time.

Indeed whenever I refer to John's or David's way of thinking or usage of signs being indicative of a way of thinking, I should really say: "my interpretation of John's (or David's) usage of the signifiers leads me to interpret his way of thinking about the signified as being pseudoconceptual/ associative, etc." Clearly, this is very clumsy; hence I have used more colloquial and accessible (although admittedly less rigorous) forms of expression.

Indeed, throughout this chapter I caution the reader that all my statements which appear to be claims about what John or David is thinking, are, in fact, my interpretations of what is happening. For example, statements like "John's usage of signs is indicative of pseudoconceptual thinking", "David's concern is with the template of the improper integral" , "John's statement that the limit exists indicates that he is using complex thinking with a template–orientation" and so on, as well as my various classifications of sign usage or ways of thinking are my interpretations of what is happening, not dogmatic statements of fact.

§7.1.7 LENGTH OF ANALYSES

Before beginning the in–depth analyses of John's and David's activities relating to the appropriation of the improper integral, I would like to note that these analyses are necessarily long and dense. This is largely because the appropriation of a mathematical object is a lengthy and complex process and an in–depth analysis of this process (with justifications for each classification of sign usage) reflects this length and complexity. Certainly I could have shortened these analyses by omitting the detailed justifications for my interpretations of the data. But such a move would have greatly weakened my claims concerning the validity and reliability of my analyses. (See §9.2 and §9.3 for further discussion around the relationship between the validity and reliability of my account and the level of detail in the analyses.)

Furthermore given a concern with the development of the appropriation process over time (for a particular student and a particular mathematical object), it is frequently necessary to analyse large sequential chunks of activities from the interviews (rather than a few interesting or typical moments in the student's activities). This need for an analysis of lengthy sequences of activities contributes largely to the resulting length of the analyses.

§7.2 ANALYSIS: JOHN'S APPROPRIATION OF AN IMPROPER INTEGRAL

I now proceed to analyse John's usages of signs relating to his appropriation of an improper integral. In doing this, I will, as formerly stated, pay especial attention to how John uses signs relating to the divergence or convergence of an improper integral.

A copy of John's written responses to the task is given in Appendix B.

EPISODE 1: LINES 23 – 35

Question 2(a): First pass

***Summarising overview*:**

In Question 2(a), the student is presented with the written definition of an improper integral Type I and asked to generate an example of such an integral.

John's response is to generate a sequence of expressions whose template corresponds in various ways to the template of an improper integral without apparent regard for their objective meaning. For example, he writes (or implies) objectively meaningless statements such as $\int_0^\infty f(x)dx =$

$\lim_{2\to\infty}\int_0^2 f(x)dx = \lim_{2\to\infty}\int_0^2 \sqrt{x}dx = \int_0^2 \sqrt{x}dx.$ [lines 27, 28].

Clearly the equality between integrals in this statement is invalid, and the second and third integrals have no objective meaning (although for John they presumably have the template of an improper integral, ie $\lim_{b\to\infty}\int_0^b \sqrt{x}dx$).

In contrast, the final integral $\int_0^2 \sqrt{x}dx$ has the familiar template of a definite integral ($\int_a^b \sqrt{x}dx$), a template with which John is familiar and which has objective meaning. It is this integral which John then evaluates.

As I discuss below, I categorise John's usage of signs in this episode as complex thinking with predominantly template associations. We also witness complex thinking with surface associations as John evaluates the integral he has generated.

Transcript

23. J: (*John reads Question 2(a). He then goes back to Page 2 and starts reading this silently to himself*). MB: What are you doing ?
24. J: I'm trying to re-read this thing of improper integral type I. (*John reads the definition on page 2 a few times silently*). So here I'm on page 3.
25. MB: Right.
26. J: So, here I'm asked.. . in question 2 (a). (*reading*): "Can you make up an example of an improper integral with an infinite integration limit?"
27. J: Okay. According to what I've, what I was reading here… an improper integral, Type I. Now I can make an example of f(x) equals to square root of x. (*John writes f(x) =* $\sqrt{x}$). So it will be the integral from zero to infinity of f (x) dx which is the same as the limit of …. can I choose any number, maybe 2? From 2 to infinity of the integral from zero to 2 of f(x)dx (*John writes* $\int_0^\infty f(x)dx = \lim_{2\to\infty}\int_0^2 f(x)dx$). Which is the same as the limit from 2 to infinity of zero to 2 of square root of x dx. (*John writes* $= \lim_{2\to\infty}\int_0^2 \sqrt{x}dx$).
28. J: So I think that I can just calculate the integral from zero to 2 of square root of xdx, so that I can continue with my calculation there. So it will be integral from zero to 2 of x to the power ½ … dx (*John starts writing on left hand side of page:* $\int_0^2 x^{\frac{1}{2}}dx$).
29. J: Which is the same as 2x. No, it is the same as 2/3 x to the power of 3 over 2…integral from 0 to 2 (*John writes* $\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2$). So it should be 2, 2 to the power of 3 over 2, over 3. Minus zero (*John writes* $= \frac{2(2)^{3/2}}{3} - 0$). Oh, do I have to substitute this 2 exactly…?
30. MB: Can't ask me. You do what you think is right.
31. J: Okay.
32. MB: I will explain it all afterwards, if you want.
33. J: Okay. (*John mumbles as he writes*). Two, eight, half, three. The same as 4 …square root of 2 over 3. (*John has written on left hand side of page =* $\frac{2\times 8^{1/2}}{3} = \frac{4\sqrt{2}}{3}$). So this will be the limit from 2 to infinity. Which is the same as …4 square root of 2 over 3 (*John now writes on right side of page: =* $= \lim_{2\to\infty}\frac{4\sqrt{2}}{3} = \frac{4\sqrt{2}}{3}$).
34. J: From which I can see that the limit exists. Ya, or maybe I did not understand… Did I have to calculate or just give an example?
35. MB: You could have just given an example. You did not have to calculate, it doesn't matter that you did though.

1a. Lines 23-27

Indicator: Neither $\lim_{2\to\infty}\int_0^2 f(x)dx$ nor $\lim_{2\to\infty}\int_0^2 \sqrt{x}dx$ are objectively meaningful expressions. Besides any other consideration, the number 2 is a constant and cannot tend to infinity. Furthermore the choice of 2 as a lower limit seems fairly arbitrary; as John himself says: “can I choose any number, maybe 2?”

However the template of all these expressions is *similar* to that of an improper integral Type I, ie $\int_a^\infty f(x)dx$ or $\lim_{b\to\infty}\int_a^b f(x)dx$ (except that in the definition of an improper integral Type I, b is a variable which takes on all real number values up to infinity.).

Formation: John’s usage of signs is indicative of complex thinking with a template–orientation. His focus seems to be on the template of an improper integral (ie $\int_a^\infty f(x)dx$ and $\lim_{b\to\infty}\int_a^b f(x)dx$) without a regard for the objective meaning of what he is writing (in particular the signifier $2 \to \infty$ in the expression $\lim_{2\to\infty}\int_0^2 f(x)dx$ [line 27] does not have a mathematical meaning).

1b. Line 28

Indicator: Again, John’s concern is with the template of his expression. Through a sequence of notational modifications, he has adapted the meaningless expression $\lim_{2\to\infty}\int_0^2 \sqrt{x}dx$ to a template with which he is familiar, ie $\int_0^2 \sqrt{x}dx$.

Formation: John’s primary focus is the template of the expression. He transforms the template from a new and unfamiliar form (the improper integral) to a familiar form (the definite integral) with which he is comfortable. It seems not to matter that there is no objective mathematical meaning to what he is doing.

1c. Line 29 - 35

Indicator and formation: John successfully computes the definite integral even though no computation was required. It is as if the integral signifier triggered his need to integrate the expression. I thus classify his sign usage in this sub–episode as indicative of complex thinking with a surface association.

EPISODE 2: LINES 38 - 53

Question 2(b): First pass

***Summarising overview*:**

In Question 2(b), the student is asked to generate an example of a convergent improper integral Type I.

Here John's activities are still template–orientated. As in episode 1, John still generates objectively meaningless statements, such as

$\lim\limits_{2\to\infty}\int_0^2 f(x)dx = \lim\limits_{2\to\infty}\int_0^\infty \sqrt{x}dx = \int_0^\infty \sqrt{x}dx$. However, unlike in episode 1, where his final term in the sequence of expressions has the template of a definite integral (ie $\int_0^2 \sqrt{x}dx$), his final expression in the above sequence (ie $\int_0^\infty \sqrt{x}dx$) has the template of an improper integral.

What is also noteworthy is John's apparent lack of concern for the fact that he has calculated (the objectively meaningless expression) $\lim\limits_{2\to\infty}\int_0^\infty \sqrt{x}dx$ in two different ways. In episode 1, he implied that this expression was equal to $\int_0^2 \sqrt{x}dx$ and calculated that expression; in episode 2, he implies that $\lim\limits_{2\to\infty}\int_0^\infty \sqrt{x}dx$ equals $\int_0^\infty \sqrt{x}dx$ and calculates this expression.

John also uses complex thinking with a template–orientation when dealing with the expression "= ∞"; he treats this expression as he would treat an expression of form "= a" where $a \in$ □ , and so incorrectly concludes that the improper integral is convergent.

Transcript

38. J: Yes 2(b). (*reading*): "Can you make up an example of a convergent improper integral with an infinite integration limit?" Okay. Can I refer back to my notes there? (*John looks at Definition on page 1 again*). They say that (*reading*): "an the improper integral… if one of the limits of integration is infinite or the function f has an infinite discontinuity on a, b, we call the integral an improper integral". Can I make up convergent … I was looking at activity 1. I thought maybe I would find that in Activity 1. But it was here (*referring to Definition on page 2*).
39. MB: Oh, okay.
40. J: (*John reads definition on page 2*): "If f is continuous on the interval a to infinity, then…. a to infinity of f(x) is the same as limit … f(x) dx. If limit from b to infinity of the integral from a to b of f(x) … we say that the improper integral converges. Otherwise the improper integral diverges". Okay. "The above definition is of an improper integral with an infinite integration limit". So I can go back to b. (*Reading Q2(b) again*): "Can you make up an example of …
41. MB: Can you talk a little louder especially when they make a noise outside.
42. J: Okay. Thanks. Can you make up an example of convergent improper integral with an infinite integration limit?
43. J: (*John spends some time looking at Question 2(a) and answer and Q2(b)*). Oh, can I use the first example that I give here in 2(a).
44. MB: Okay.
45. J: Okay, but I'll just first…. I'll just try to calculate. Here, so, I think it …
46. MB: If you want more paper you can write in that book (*referring to pad of paper*).
47. J: Oh, thanks.
48. MB: Okay.
49. J: So, 2(b). The integral …the limit as 2 tends to infinity of the integral from zero to 2 of the square root of x at. ... : (*John writes on new page:* $\lim_{2\to\infty}\int_0^2 \sqrt{x}dx$ *)*. Which is the same as limit as 2 tends to infinity of the integral from zero to infinity of xdx. (*John writes:* $\lim_{2\to\infty}\int_0^\infty \sqrt{x}dx$).
50. J: So which will be the same as …. the integral of…. I just break up the limit here. Which is the same as the integral from infinity, from zero to infinity of x to the half dx. …(*John writes:* $\int_0^\infty x^{\frac{1}{2}}dx$ *ie he drops the limit term*). Which is the same as ….
51. MB: A bit louder
52. J: Ja. Which is the same as 2 over 3 x to the power of 3 over 2, from zero to infinity. …(*John writes*: $\frac{2}{3}x^{\frac{3}{2}}\Big|_0^\infty$). So I can say …if 2 over 3… if x tend to infinity in this … in this equation… and say the whole equation came to infinity…If x tend to infinity, it can also tend to infinity
53. J: (*John writes: = ∞ ∴ the limit exists*). Ja, so the limit exists … I can say that this is a convergent improper integral.

2a. Lines 38–49

Indicator: As in episode 1, neither $\lim_{2\to\infty}\int_0^2 \sqrt{x}dx$ nor $\lim_{2\to\infty}\int_0^\infty \sqrt{x}dx$ nor their equality [line 49], are objectively meaningful expressions although the template of both expressions is *similar* to that of an improper integral Type I, ie $\lim_{b\to\infty}\int_0^b \sqrt{x}dx$.

Formation: As before, John's usage of signs is indicative of template–orientated complex thinking in that an improper integral is associated with the template $\lim_{b\to\infty}\int_0^b f(x)dx$ (except that according to John's template, b may be a constant).

2b. Line 50

Indicator & Formation: At this stage, John's activities are primarily directed to generating a template with which he is able to work. In the last case (ie $\int_0^\infty \sqrt{x}dx$) the template is that of the definite integral, ie $\int_a^b f(x)dx$.

These manipulations are on the signifier level in that John does not seem to be concerned with the meaning of what is being signified (evidenced by his equating certain meaningless expressions). Accordingly I classify his usage of signs in this sub–episode as indicative of complex thinking with a template–orientation.

Note: John's rewriting of $\lim_{2\to\infty}\int_0^\infty \sqrt{x}dx$ as $\int_0^\infty \sqrt{x}dx$ will allow him to evaluate his 'improper integral' as if it were a definite integral. This is an instance of Sfard's (2000: 60) template–driven use whereby "new uses originat(e) in old uses".

2c. Line 51–52

Indicator & Formation: John's treatment of ∞ as a real number [line 52] allows him to work with the improper integral according to the rules for a definite integral. Thus his manipulations are an instance of template–orientated complex thinking.

2d. Line 53

Indicator: John is using the template '=∞' as he would use the template '=a', where a is a real number. As a result, he incorrectly concludes that the limit exists.

Formation: John's statement that the limit exists indicates that he is using complex thinking with a template–orientation; he associates the template of the signifier '=∞' with the template '=a', where a is a real number.

EPISODE 3: LINES 56 - 58

Question 3: First pass

***Summarising overview*:**

In Question 3, the student is asked to explain what an improper integral with an infinite integration limit represents graphically.

John answers Question 3 by using the improper integral (ie $\int_0^\infty \sqrt{x}dx$) that he generated in Question 2(b). He sketches the function $y = \sqrt{x}$ and explains that the improper integral $\int_0^\infty \sqrt{x}dx$ represents the area between the graph $y = \sqrt{x}$ and the x–axis from zero to infinity. This is correct, but John has presented an explanation centred on a particular example of an improper integral, rather than a general explanation applicable to any improper integral Type 1.

John's use of a specific example to explain a general phenomenon illustrates complex thinking using complex thinking with the particular example as the nucleus.

In terms of John's evolving concept of a convergent or divergent improper integral, John's comments regarding the relationship of an infinite quantity and a limit are interesting in that they contradict his former comments regarding this relationship [line 53]. To elaborate: in the current episode John explains that his answer in Question 2(b) concurs with his sketch because the aforementioned area is infinite (this is correct). However he then states that the area is infinite and so does not have a limit [line 58].

Although this is correct, it directly contradicts his (incorrect) conclusion in Question 2(b) that the integral is infinite and so the limit exists (see episode 2d). Interestingly, John does not seem at all perturbed by this almost–simultaneous articulation of two logically contradictory views.

Transcript

56. J: Okay. (*reading*) 'Can you explain what an improper integral with an infinite integration limit represents graphically?' … Oh, I have to explain it (*mumbling*) and graph?
57. MB: Well you could explain it or draw it. You can use a graph.
58. J: Okay. I can use my example of $f(x) = \sqrt{x}$. So the graph of this …. I think it will be like this, going like this (*John draws graph of* $y = \sqrt{x}$). So I can see that this is just the area of this graph from zero here to infinity. This area here, this integral. I show the area from zero to infinity. That's why, when I calculate here, I say the area also tends to infinity, … it won't have a limit. The area which is bonded by the line y = 0 and the graph f, the square root of x. Ya, I think that's how I explain. Ya, can I have another page please.

3. Lines 56–58

Indicator & Formation: In answering Question 3, John uses a particular example as a core on which to build an explanation. This need to work with concrete and specific properties of a particular example rather than abstracted and generalised properties of a class of mathematical objects is indicative of complex thinking using association-with-an example. In such usage, the example (in this case, a sketch of $\int_0^{\infty} \sqrt{x}dx$) is the nucleus around which the student tries to build a concept.

EPISODE 4: LINES 59 – 66

Question 4: First pass

Summarising overview:

In Question 4, the student is asked to define $\int_{-\infty}^{b} f(x)dx$.

John starts off by correctly asserting that $\int_{-\infty}^{b} f(x)dx$ "shows me the area from negative infinity to b" [line 60]. As I discuss below, I categorise John's thinking as complex thinking with graphical associations.

John then generates a definition for $\int_{-\infty}^{b} f(x)dx$ using, by his own acknowledgement, comparison with the given definition of $\int_{a}^{\infty} f(x)dx$ [line 62]. Using analogy, he correctly argues that $\int_{-\infty}^{b} f(x)dx$ is equivalent to $\lim_{a\to-\infty}\int_{a}^{b} f(x)dx$. John's reasoning here indicates that he is using complex thinking with template–orientations (again see discussion below).

In contrast, John's argument that $\int_{-\infty}^{b} f(x)dx$ is the "opposite" of $\int_{a}^{\infty} f(x)dx$ [line 63], and so $\int_{-\infty}^{b} f(x)dx$ diverges if the limit exists, otherwise it converges [lines 63, 66], is idiosyncratic and incorrect (mathematically speaking, $\int_{-\infty}^{b} f(x)dx$ diverges if the limit *does not* exist, otherwise it converges). Thus at this stage, John exhibits little understanding of the notions of convergence and divergence. Below I explain why I categorise John's usage of signs in this sub-episode (episode 4c) as indicative of heap thinking.

***Transcript*[10]**

59. MB: Okay. Page 4, question 4.
60. J : (*John reads the question quietly to himself*). So this question asks me: how would I define the integral from negative infinity to b, f(x) dx? I can say this shows me the area from negative infinity to b. (*John looks from time to time at Questions 2 and 3*). I think maybe, that's all…. And I can say that this is not an infinite integral because b doesn't tend to infinity.
61. MB: What are you thinking?
62. J: I was just thinking of how, the thing that can define this. I don't find anything … (*John is looking at the definition of Improper Integral Type 1, while he talks*). **But what I can compare this with is the first example from Activity II** (*John is referring to the definition of improper integral, Type 1*) as there a tends to infinity, I mean b tends to infinity. It was the integral from a to infinity and here is the integral from b to negative infinity. So here I think, I get… they might have took the limit of the integral… the limit as a tends to negative infinity of the integral from a to b of f(x).
63. J: So I can say that, if the limit as a tends to negative infinity of f(x) dx exists, I can say that this is… the improper integral… this improper integral… **I can say it diverges because it**

[10] Throughout this chapter I use bold type to highlight what appear to me as particularly significant aspects of the transcript.

is the opposite of the activity two of improper integral Type 1 (*referring to definition of improper integral, Type I*).

64. MB: Do you want to write down your definition?

65. J: Yes, ja. I can write down. … (*John looks at definition of improper integral, Type 1, on page 2 as he writes down his definition. It is as if he is copying the definition on page 2, just changing relevant aspects as he goes*). So here I can say: if f is continuous, continuous on the interval, negative infinity to b, the integral from negative infinity to b, of f(x) dx, is the same as limit as a tends to negative infinity of the integral of a to b of f(x).

(*John writes "If f is continuous on the interval* $(-\infty, b]$ *then* $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx$ *. MB: This is correct.)*

66. J: So you can say that if the limit from a to infinity of the integral a to b of f(x)dx exist, and we can say that the improper… I'm just copying this, opposite….improper integral diverges or otherwise integral converges. (*John has written: If* $\lim_{a\to-\infty}\int_{a}^{b} f(x)dx$ *exist, we say that the improper integral diverges, otherwise integral converges*.) I think that is all . Example 4, Question 4. Can you give me another page please?

4a. Lines 59, 60

Indicator & Formation: Although at first glance it may seem that John has a conceptual understanding of an improper integral, his discussion about its convergence or divergence (see episode 4c below) indicates that this is not so. However John is certainly aware of a graphical representation of $\int_{-\infty}^{b} f(x)dx$ (he talks of *area*). Thus at this stage John is using complex thinking with graphical associations with regard to the notion of an improper integral.

4b. Lines 61– 62, 65

Indicator: John himself implies [line 62] that he is using the given definition of $\int_{a}^{\infty} f(x)dx$ to generate a definition for $\int_{-\infty}^{b} f(x)dx$. This indicates a focus on the template of the given definition of an improper integral with an upper infinite limit.

Formation: John's association of the template of the definition of $\int_{a}^{\infty} f(x)dx$ with the definition of $\int_{-\infty}^{b} f(x)dx$ indicates complex thinking with a template–orientation.

4c. Lines 63, 66

Indicator & Formation: John's link between the existence of a limit and convergence in the case of $\int_b^\infty f(x)dx$ and the existence of a limit and divergence in the "opposite" case of $\int_{-\infty}^b f(x)dx$ [lines 63, 66] is based on a vague but incorrect analogy between contrasting features of signifiers and contrasting features of signifieds.

In particular, John is relating the contrasting position (upper or lower) of the infinite limit on the integral signifier to a contrasting state of convergence or divergence.

Since this analogy depends on the layout of the signifiers, I categorise it as indicative of heap thinking.

EPISODE 5: LINES 67 – 69

Question 5(a): First pass

***Summarising overview*:**

John calculates $\int_1^\infty x^{-3}dx$. He uses ∞ as a real number (with no ill–effect) and computes the integral. (During these computations, he makes a small integration error using $-2x^{-2}$ as the antiderivative of x^{-3} whereas the antiderivative should be $\frac{-1}{2}x^{-2}$). This usage of ∞ in slots normally reserved for real numbers is indicative of complex thinking with a template–orientation.

John concludes that since the integral equals –2, the limit exists and so the integral diverges (initially he states that the integral converges but quickly changes his mind).

As I discuss below, it is very difficult to understand (as the researcher) why John states that the integral diverges. I suspect that John's response to Question 4, in which he established idiosyncratic links between convergence and divergence and the existence or not of the limit, is exacerbating the confusion[11]. The veracity of this suspicion is supported by what happens much later in the interview after John has corrected his answer to Question 4 (see episodes 15, 16 and 18).

Transcript

67. MB: Okay, now page 5.
68. J: Okay, thanks. So here in 5(a) I am asked to determine whether the integral from 1 to infinity of dx/x^3 converges or diverges. Okay, so this will be the same as the integral from 1 to infinity of x to the power of −3, dx (*John writes* $\int_1^\infty x^{-3}dx$). Which will be the same as −2 x to the power of −2, the integral from 1 to infinity (*John writes:* $= -2x^{-2}\Big|_1^\infty$ *: MB: It should be* $= \frac{-1}{2}x^{-2}\Big|_1^\infty$). It will be the same as integral from −2, I mean the same as $-2/x^2$. (*John writes:* $= \frac{-2}{x^2}\Big|_1^\infty$). So it is the same as −2 … I can see 0 −2 is the same as −2. (*John writes 0−2 = −2: MB Answer should be −1/2*).
69. J: So here I can say that the limit as b …let's take here, there was b… as b tends infinity. You can see that it exists, so this integral, you can say it converges. No, sorry, it diverges.

5a. Lines 67 – 68

Indicator & Formation: John's treatment of ∞ as a real number allows John to work with the improper integral signifier as if it were a definite integral signifier. He is thus able to apply the rules for a definite integral to his work with an improper integral. Accordingly John's usage of the signs in this episode is indicative of complex thinking with a template−orientation: a new signifier (the improper integral) is used according to the rules for the old signifier (the definite integral).

Although this use of ∞ as a real number is not problematic in this episode, we saw how template−orientated complex thinking may lead to conceptual errors in episode 2d (Question 2(b)).

5b. Line 69

Indicator & Formation: It is very difficult to understand (as the researcher) why John finally states that the integral diverges although I suspect that this has much to do with his confused answer to Question 4 (see Episode 4).

[11] In hindsight, I realise that I should have probed John's reasoning further.

EPISODE 6: LINES 70-106

(Connecting episode)

Seeking help

Note: Given that this episode is primarily concerned with John's attempts to get assistance and his (often silent) reflection on this assistance, I will present a broad description and interpretation of John's activities.

Broad description and interpretation:

John starts off by asking me about convergence. Initially he seems very confused and wants to relate convergence to an increasing or decreasing graph [line 70].

Given that I do not understand what he is saying (about increasing and decreasing graphs) and also that I do not wish to teach directly in the interview, I suggest that John uses the textbook [line 71]. He is most reluctant to do this, telling me that "it will just take too long" [line 72].

I thus suggest that John looks again at the definition of an improper integral Type 1, as presented in the task. He does this [lines 78–81]. Although he is still unsure about the meaning of convergence and divergence [line 80], reading the definition seems to divert his concern to the meaning and use of the expression $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$ [lines 79, 81, 85, 87].

Given John's incoherence and my unsuccessful probing, I am unfortunately unable to pinpoint exactly what is worrying him. Furthermore I am still unwilling to teach directly, so I urge John yet again to use the textbook.

At this juncture it is important for me to remind the reader that my interview plan (across all the interviews) was to encourage students to use the textbook for assistance. Only if that failed, would I assist[12]. This hopefully explains my insistence on John using the textbook [lines 71, 82, 84, 88, 90].

[12] See Chapter 5 for justification of this interview decision.

Notwithstanding this, but in hindsight, I realise that my probing could have been more sensitive and directed.

Finally John consults the textbook[13] [lines 93–105].

He looks at the textbook examples of improper integrals Type I and II, and at worked examples of converging and diverging integrals [lines 95, 99, 103] in the textbook. He also studies the definition of improper integrals with infinite limits in the textbook. (This definition presents three cases: an integral with an upper infinite limit, an integral with a lower infinite limit and an integral with both infinite limits[14]. The definition in the task presents only the first case).

In response to my asking, John tells me that this examination of the text is useful and helps answer his query [lines 97, 101]. Specifically he indicates that the worked example of an improper integral that diverges is useful: he indicates that it shows him how to use the notation when evaluating an improper integral [lines 103–105]. This is borne out in his responses to later questions (see episodes 7, 8, 12) in which he uses the correct notation.

My interpretation of what is happening here is best informed by what occurs afterwards. To wit: Although John's examination of the examples enables him to proceed correctly with regard to appropriate notation (an instance of complex thinking using association-with-an example) he is still unable to use notions of convergence and divergence adequately (see especially episode 10)). Moreover, he is only able to use notions of convergence and divergence correctly (and I later argue that this use is pseudoconceptual) after he has broken the idiosyncratic links he established between these notions and the existence or not of limits in his initial response to Question 4 (episode 4c).

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[13] As previously stated, the textbook is the textbook for the course: Calculus of a Single Variable, 5th or 6th Edition, by Larson, Hostetler and Edwards.

[14] See page 135 for a reproduction of this definition.

EPISODE 7: LINES 107-112

Question 5(a): Second pass

***Summarising overview*:**

John starts calculating the given integral, ie $\int_1^\infty \frac{dx}{x^3}$. Unlike his previous attempt (episode 5) he now uses the correct notation, writing $\int_1^\infty \frac{dx}{x^3}$ as $\lim_{b\to\infty}\int_1^b \frac{dx}{x^3}$. However he repeats the computation error he made when first dealing this integral and gives $-2x^{-2}$ as the antiderivative of x^{-3} (it should be $\frac{-1}{2}x^{-2}$). Nevertheless John correctly concludes that since his answer is –2, the limit exists and so the integral converges.

As I discuss below, I categorise John's use of signs in this episode as indicative of pseudoconceptual thinking.

In Chapter 8, I show how John's change in use of notation (from Question 2(b) to Question 5(a): First pass, to Question 5(a): Second pass) is indicative both of an evolution of his concept of the improper integral and of a process of acculturation into the community of mathematicians.

Transcript

107. J: So here I'm asked to determine whether this limit converges.
108. MB: Okay. So you are doing 5(a).
109. J: Ya, so I can….
110. MB: Will you label this. This is question 2. Won't you just put a 2. And then you can go on to the next page to do question 5. (*He was about to start writing Question 5 on same page as Question 2*). Just so that I know what's what. Okay. Perhaps use the next page … new page. Okay.
111. J: So here I'm doing 5 (a) (*John reads*): "Determine whether the integral from 1 to infinity of 1/ x^3 converges or diverges". I say this is the same as the limit as b tends to infinity of the integral from 1 to b of 1/x^3 (*John writes*: $\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^3}$). Which will be equal to the limit as b tends to infinity of –2 over x^2 (*John writes:* $\lim_{b\to\infty}\left[\frac{-2}{x^2}\right]_1^b$). So this will be the same as the limit as b tends to infinity of –2/ b^2 – 2. (*John writes:* $\lim_{b\to\infty}\left[\frac{-2}{b^2}-2\right]$) Which is the same as 0 – 2 which is the same as –2 (*John writes: (0–2) = –2*).
112. J: So I can see that the limit exists so that the integral converges (*John looks at definition on page 2 and then writes: converges*). So now I'm finished with 5(a).

7a. Lines 107 –111

Indicator & Formation: Despite the computation error, John approaches the problem in the correct way, using the correct procedure and the appropriate notation. However, given John's confusion about convergence and divergence (as manifested in Episodes 4 and 10), I categorise John's manipulations, which are directed to ascertaining the convergence or divergence of the improper integral, as pseudoconceptual.

7b. Line 112

Indicator & Formation: For the same reasons as in Episode 7a, I categorise John's reasoning (if the limit exists, the integral is convergent) as pseudoconceptual.

EPISODE 8: LINES 114-115

Question 5(b): First pass

***Summarising overview*:**

In this question John has to decide whether the integral $\int_{-\infty}^{1} x dx$ is convergent or not. He appropriately starts to calculate the integral, writing that it equals $\lim_{a \to -\infty} \left[\frac{1}{2} - \frac{a^2}{2} \right]$. His notation is fine as is his integration. John then concludes, without any justification, that the limit does not exist [line 115].

Although John's manipulations and notation are fine in Episode 8, given John's fundamental confusion about the notions of convergence and divergence (as manifested in Episode 10), I interpret John's manipulations which relate to the determination of the convergence or not of the improper integral, as indicative of pseudoconceptual thinking**.**

Transcript

114. J: So, I want to do 5(b). It asks me to determine whether the integral from minus infinity to 1 converges or diverges. So from this I can say: it equals to the limit as a tends to negative infinity of the integral from a to 1, x dx (*John writes:* $\int_{-\infty}^{1} x dx = \lim_{a \to -\infty} \int_{a}^{1} x dx$). So, which will be equal to the limit as a tends to negative infinity of ½ x^2 (*John writes:* $= \lim_{a \to -\infty} \left[\frac{1}{2} x^2 \right]_a^1$)… ½ x^2

from a to 1. Which is the same as limit as b tends to negative infinity of ½ minus a, ½ minus $a^2/2$. (*John writes:* $= \lim_{a\to-\infty}\left[\frac{1}{2}-\frac{a^2}{2}\right]$).

115. J: So here I can say: this limit does not exist.

8. Line 114, 115

Indicator & Formation: John approaches the problem in the correct way, using the correct procedure and appropriate notation. But as in Episode 7, I categorise John's activities which are directed to ascertaining the convergence or divergence of the improper integral, as pseudoconceptual. That is, John is manipulating symbols correctly but (given his confusion about the meaning of convergence and divergence as revealed in Episode 10), he is not engaging presumably with their intended meaning.

EPISODE 9: LINES 116–157

(Connecting episode)

Dealing with negative infinity

Since in this episode, John is struggling with notions of negative infinity (rather than with the convergence or divergence of an improper integral), I present a broad description of this episode rather than a microanalysis.

Broad description of activity:

John tells me that he does not know how to deal with negative infinity. His focus is on how to manipulate an expression of form $\lim_{a\to-\infty}\left[\frac{a^2}{2}\right]$ [lines 116, 118, 124]. In his words: "What would be the answer if I substitute negative infinity in $a^2/2$?" [line 118]. John argues that if he was dealing with positive infinity rather than negative infinity, $\frac{a^2}{2}$ would tend to ∞ [line 128] but that he is confused by negative infinity [lines 134, 139, 140]. "I don't know if I can substitute this negative infinity…" [line 134]. Since limits at infinity are not the focus of the interview, and I do not want John to spend hours on this problem, I tell him how to evaluate this expression.

Finally John argues correctly that $\frac{1}{2}-\frac{a^2}{2}$ will tend to minus infinity [line 156].

EPISODE 10: LINE 158

Back to Question 5(b): First pass

***Summarising overview*:**

John argues that since $\lim_{a\to-\infty}\left[\frac{1}{2}-\frac{a^2}{2}\right]$ is infinite, it does not exist and so the integral is convergent. While telling me this, he is looking at his response to Question 4 (see episode 4 above) in which he (incorrectly) argued that $\int_{-\infty}^{b} f(x)dx$ diverges if the limit exists, otherwise it converges. Of course, this answer is mathematically incorrect (although it is consistent with John's definition in Question 4). As I explain below, I classify John's usage of signs in this episode as indicative of complex thinking with template–orientation.

Transcript

158. J: So here I can conclude, according to the definition that I made here (*John turns back to the definition he wrote for Question 4*)…. because it doesn't exist, the limit as a tends to infinity doesn't exist. So I can say that the integral converges … (*John writes:* $= \infty \therefore$ *converges. He then carries on studying the definition he wrote in response to Question 4 (in which he confused converge with diverge)*).

10. Line 158

Indicator & Formation: John uses the template of his answer to Question 4 (as evidenced by him looking at this answer while answering the current question) to argue that if the limit does not exist, the improper integral converges. Accordingly I classify his argument as indicative of complex thinking with template–orientation.

EPISODE 11: LINES 159–160

Question 4: Second pass

***Summarising overview*:**

John studies his answer to Question 4 again. He looks slightly worried. Given his request to revise his answer to this question a little while later (see episode 14) he is possibly beginning to feel slightly (and appropriately) perturbed by his initial response to Question 4 (see episode 4).

Transcript

159. MB: What you're looking at there? Sorry, where were you looking?
160. J: I was looking at definition I wrote (*referring to Question 4*).

EPISODE 12: LINES 163 – 165

Question 5(c): First pass

Summarising overview:

John calculates the integral. His notation is fine as is his integration. He gets as far as writing that the integral equals $\lim_{b\to\infty}(\ln b)$. But as we see in the next episode (Episode 13), John is unsure about how to evaluate this expression and so is unable to proceed further.

Transcript

163. J: Now I can do 5(c). Here I'm asked to determine whether the limit from 1 to infinity converges or diverges. So here the integral from 1 to infinity of dx over x. It will be the same as the limit as b tends to infinity of the integral from one to b of dx over x (*John writes:* $\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x}$). Which is the same as the limit as b tend to infinity of ln x (*John writes* $\lim_{b\to\infty}(\ln x|_1^b$). Which is the same as the limit from 1 to b as b tends to infinity of ln b minus ln1. (*John writes:* $\lim_{b\to\infty}(\ln b - \ln 1)$)
164. MB: Talk louder please.
165. J: ln 1 is zero. So its the same as limit as b tend to infinity of ln b (*John writes:* $\lim_{b\to\infty}(\ln b)$) . As b tend to infinity of ln b.

12. Lines 163 – 165

Indicator & Formation: John approaches the problem in the correct way, using the correct procedure and appropriate notation. But, as with Episodes 7 and 8, I categorise his activities as indicative of pseudoconceptual thinking.

EPISODE 13: LINES 166– 191

(Connecting episode)

$$\lim_{b\to\infty}(\ln b)$$

Since this episode is concerned with John's conception of infinity and its relationship to the ln function (rather than issues of convergence and divergence), I will present a broad description rather than a microanalysis of this episode.

Broad description of activity:

John tells me that he does not know how to deal with $\lim_{b\to\infty}(\ln b)$ [line 166]. As with $\lim_{a\to-\infty}\left[\frac{1}{2}-\frac{a^2}{2}\right]$ (see episode 9), his initial concern is one of manipulation, how to "substitute" ∞ [line 166].

John remembers that this same example was done in the textbook and so locates the example in the textbook. He tells me the textbook indicates that "ln of infinity is the same as infinity" [line 172] but that he does not understand why this is [line 174]. In response I explain, using a graph of ln x, that as x tends to infinity so does ln x. John seems to be happy with this explanation [line 186].

EPISODE 14: LINES 192–213

Question 4: Third pass

***Summarising overview*:**

John tells me that he thinks that his answer to Question 4 was incorrect. He thinks that he used convergence when he should have used divergence and vice versa [lines 198, 200]. Despite my probing, John is unable to explain why he thinks his original answer was incorrect or to elaborate on his misgivings [line 202]. He asks if he may use the textbook (presumably to clarify or validate his thinking about his answer to Question 4).

John silently reads the definition in the textbook of an improper integral Type I. In this definition three cases are presented: the integral with an upper infinite limit, the integral with a lower infinite limit (the case referred to in Question 4) and an integral with two infinite limits. He then indicates (in somewhat incoherent language) that $\int_a^\infty f(x)dx$ converges if $\lim_{b\to\infty}\int_a^b f(x)dx$ exists and that since $\int_{a}^{\infty} f(x)dx$ "is the same as this one," ie as $\int_{-\infty}^{a} f(x)dx$ [line 210], the latter improper integral also converges if $\lim_{a\to-\infty}\int_{a}^{b} f(x)dx$ exists.

John then corrects what he has written in response to Question 4. His answer to Question 4 is now correct, specifically the conditions for convergence or divergence of $\int_{-\infty}^{a} f(x)dx$ are now correct. As discussed below, I categorise this episode as predominantly indicative of complex thinking with a template–orientation.

Transcript

192. J: Actually I'm not sure about the definition that I made there (*referring to Question 4*) Can I just compare it with the definition here.
193. MB: Oh, which definition are you looking at?
194. J: This (*points to definition of improper integral Type I, in which three different cases are discussed, on page 536, Larson*).
195. MB: Ja, I don't want you to just take it from there though. All right, but you… okay, as long as you explain… if you think you went wrong, where you went wrong. Okay?
196. J: I think here.
197. MB: What do you think you did wrong, if you think you did.
198. J: Here in diverges or converges.
199. MB: Right.
200. J: I think maybe I took it the other way round.
201. MB: And why should you think that?
202. J: I don't know…I just think….
203. MB: I mean is it from reading something over there, in the textbook?
204. J: No.
205. MB: Okay.
206. J: I just wanted to refer if I did …
207. MB: Okay.
208. J: (*John silently reads the definition of improper integral, Type 1, on page 536 in Larson.. He spends a while doing this.*)
209. MB: Just tell me what you read there. What's going on?
210. J: Ja, here I saw that here, I made a mistake here (*Johns points to the definition he gave for Question 4*). Because it just explains that (*mumbles as he points to definition of improper integral , Type I in Larson, page 536*)… As the limit… as b tends to infinity of the integral from a to b of f(x), exists. Therefore I can say the integral converges. **Which is the same as this one…** as a tends to negative infinity of the integral from a to b of f(x), exists. Therefore the integral converges. It also converges.
211. J: (*referring to his answer on page 4*). So I made…
212. MB: If you want you can cross it out and write it underneath, the correct thing. Don't rub it out, if you want you can cross it out… if you feel something is wrong, cross it out and then just write what you think is correct…
213. J: (*John corrects the definition he gave for Question 4; he changes converges to diverges, diverges to converges. His definition is now correct*).

14. Lines 192–213

Indicator and Formation: John seems to have suffered a perturbation regarding his use of the notions of convergence and divergence of an improper integral [192]. But he does not seem to be sure about precisely where he went wrong, hence his desire to refer to the textbook to clarify or validate his thinking [line 192].

John's usage of signs is not always evident in this episode (for example, he is silent as he studies the definition in the textbook). But his assertion [line 210] that some (unspecified) aspect of the case of an integral with an upper infinite limit "is the same as" some (unspecified) aspect of the case of an integral with a lower infinite limit, together with his usage of exactly the same verbal template, viz. "as ___ tends to ___ of the integral from a to b of f(x), exists. Therefore the integral converges" [line 210] when referring to the integral with the upper infinite limit and the integral with the lower infinite limit, indicates that John is using a form of template–orientated complex thinking. That is, his focus is on the similarity of form of the two cases.

Moreover, in several of the following episodes we see that although John is now able to use notions of convergence and divergence in deciding whether a given improper integral converges or not (see episodes 15, 16 and 18) he is unable to deal with the notions of convergence or divergence in a non–procedural context (see episodes 19 and 20). This suggests that John is not using conceptual thinking when discussing the convergence or divergence of $\int_{-\infty}^{a} f(x)dx$ and so further justifies my interpretation of John's use of complex thinking in the current episode.

EPISODE 15: LINES 218–220

Question 5(b): Second pass

***Summarising overview*:**

Having corrected his definition of $\int_{-\infty}^{a} f(x)dx$ in Episode 14, John now corrects his conclusion to Question 5(b). He states correctly the integral diverges. As I explain below, I categorise this usage of signs as pseudoconceptual.

Transcript

218. J: Yes. 5(b). (*John is looking at his answer to 5(b)).* So here I can conclude that the integral diverges. (*John changes the word converges to diverges. Q5(b) is now correct*).
219. MB: Does it make sense now, more sense?
220. J: Yes, more sense.

15. Lines 218–220

Indicator & Formation: John is now able to use the notion of divergence (and, as is seen in episode 18, convergence) effectively. This use is indicative of pseudoconceptual thinking. My classification of this use as indicative of pseudoconceptual rather than conceptual thinking is influenced by John's difficulties in Episodes 19 and 20. In those episodes we see how he is unable to use notions of convergence and divergence in non–procedural problems.

EPISODE 16: LINE 221

Question 5(c): Second pass

Summarising overview:

John completes Question 5(c), stating correctly that the improper integral diverges. Again I classify his usage of signs as pseudoconceptual.

Transcript

221.J: So I was in 5(c). So now here the limit does not exist as b tend to negative infinity therefore the integral diverges. (*John writes:* = $-\infty$ *diverges*).

Indicator & Formation: As in the previous episode, John is able to use the notion of divergence. As in that episode, I categorise this usage as indicative of pseudoconceptual thinking.

EPISODE 17: LINES 222 - 270

(Connecting episode)

In this episode, John investigates different ways of integrating $\int_1^\infty x^{-3}dx$.

Since these manipulations focus on different techniques of integration, I am not micro–analysing this episode.

Broad description of activity:

John asks if he can return to Question 5(a) to “check the limit.” [line 222].

He is unsure that he has integrated $\int_1^\infty x^{-3}dx$ correctly in his previous attempt and wishes to try different ways of integrating this expression.

He proceeds to apply different methods of integration to the integral, finally concluding that “what I did first was right” [line 267], (which it is).

EPISODE 18: LINES 271-272

Question 5(a): Third pass

***Summarising overview*:**

John decides to re-do Question 5(a) (see Episodes 5 and 7 for his first and second attempts). He thus proceeds to compute $\int_1^\infty x^{-3}dx$. His notation is fine but he repeats the computational error he made when computing Question 5(a): first and second pass (ie he uses $-2x^{-2}$ as the antiderivative of x^{-3} whereas $\frac{-1}{2}x^{-2}$ is the antiderivative).

While computing the integral John refers to his answer to the integration as “the area” of the integral [line 271]. As discussed below, this passing reference is indicative of complex thinking using graphical associations.

Soon after [line 272], John correctly concludes that since the limit exists, the improper integral converges. As with previous similar episodes, I categorise this usage as indicative of pseudoconceptual thinking.

Transcript

271. J: Yes. So here in question 5(a), I found the integral from 1 to infinity of $1/x^3$dx which is the same as the limit as b tend to infinity of 1 to b of x^{-3}dx (*John writes :* $\int_1^{\infty} x^{-3}dx = \lim_{b\to\infty}\int_1^b x^{-3}dx$). Which will be the same as the limit as b tend to infinity of $-2/x^2$. (*John writes:* $= \lim_{b\to\infty}\left|\frac{-2}{x^2}\right|_1^b$). The area of this will be… it shall be the same as the limit as b tends to infinity of minus….let's see… of zero minus 2 over b minus 2. (*John writes:* $= \lim_{b\to\infty}\left(\frac{-2}{b} - 2\right)$). So the limit as b tends to infinity of minus 2 over b, minus 2…you can say that…. it is the same as minus 2. (*John writes : =–2*).

272. J: So therefore the limit exist. Therefore the improper integral converges. (*John writes* $\therefore$ *exist* $\therefore$ *improper integral converges*). Yes. So can I have another page please?

18a. Line 271

Indicator: This is the first time in the interview that John has spoken of the answer to a specific improper integral as 'an area'. As such it is indicative of a shift in understanding which has occurred during the course of the interview.

Formation: John's reference to the improper integral as 'an area' is indicative of complex thinking using graphical associations. As I discussed in Chapter 6, such forms of complex thinking stand at the cusp of conceptual thought.

18b. Line 272

Indicator & Formation: John is able to use the notion of convergence adequately in determining whether a particular improper integral converges or not. As in episodes 15 and 16, I categorise this use as indicative of pseudoconceptual thinking.

EPISODE 19: LINES 279–292

Question 9

Summarising overview:

Question 9 is a non–procedural question which can nevertheless be solved by deduction and manipulation of the term $\int_0^{\infty} f'(x)dx$ subject to the given conditions.

But John does not attempt to manipulate the term $\int_0^\infty f'(x)dx$. Rather he tries to generate an example of a function f(x) for which the given criteria (ie f′(x) is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$) are valid.

However he is unable to find such an example (although many such examples exist). He also makes many errors while differentiating and integrating the various candidate functions.

Although John is aware that he is not expected to generate an example of f(x) which fits the given criteria [lines 290, 291], he tells me that he needs an example in order to understand the given proposition [line 291].

Transcript

279. J: (*reading*): 'If this thing, f, is continuous on zero to infinity and limit of f(x) as x tend to infinity is zero, then the limit from zero to infinity of f(x) dx is the same as minus f(0) '. And now I'm just thinking of how I can do this here.

280. MB: Sorry.

281. J: I'm just thinking of how can I do this. … (*John looks at page for a while*)

282. MB: What are you thinking about?

283. J: I am thinking of the example that I can… I just want to get the example so that I can compare it with this statement. Okay, I can take f(x), I can take an example of f(x) = $1/x^2$ (*John writes f(x) =* $\frac{1}{x^2}$). So, this thing, the derivative of f(x) … it will be… it will be…−1/x (*John writes f′(x) =* $-\frac{1}{x}$ *. MB: This is incorrect*) . And this is continuous, this is continuous from zero to positive infinity (*he is now reading question*). I'm just checking if I'm right (*MB: Derivative of* $1/x^2$ *is* $-2/x^3$*.* $\lim_{x\to\infty} f(x) = 0$ *but derivative is not continuous at x = 0*). No, this is not the right example.

284. J: I can take f(x) as maybe $2x^2$ so f(x)… the integral… I mean the derivative of f(x) will be 4x. (*John writes f′ (x) = 4x*). So the limit as x tend to infinity… you can say that it also doesn't, it also don't exist. Its also not equal to zero. Let me just check: let f(x) equals to −1/x, so derivative of f(x) will be 1, I think (*John writes* $-\frac{1}{x}$*, f′(x) = 1. MB: This is wrong*). I think….Let me see. (*mumbling*). Let me check it will be. Okay, f(x) is the same as x minus x^{-1}. (*John changes f(x) =* $-\frac{1}{x}$ *to f(x) = x −* $\frac{1}{x}$). So the integral, or the derivative of f(x) is equal to one into minus two, minus one, into x to the power of −2. It is the same as one plus one over x^2 (*John writes: f′(x) = 1 −* $(-1)x^{-2}$ *= 1 +* $\frac{1}{x^2}$). But the limit as x tends to infinity is not zero. (*MB: f′ is not continuous at x=0, and* $\lim_{x\to\infty} f(x) \neq 0$).

285. J: Let me just check another example. 1/x, let f(x) = 1/x and in this example the limit as x tend to infinity is zero (*John writes* $\frac{1}{x}$). So I just want to find…. derivative of f(x) which will be the same as $-1/x^2$. (*John writes f′(x)=−* $\frac{1}{x^2}$). So now I just want to find the integral of this, of derivative of x from zero to infinity. This will be the integral from zero to infinity $-1/x^2$dx. Which is the same as integral from zero, the limit as b tend to infinity of $-1/x^2$dx.

(*John writes on right hand side of page:* $\int_0^\infty\left(-\frac{1}{x^2}\right)dx = \lim_{b\to\infty}\int_0^b\left(-\frac{1}{x^2}\right)dx$). Which will be the same as minus x^{-2} . Let me just check here. So now I'm just trying to simplify this. I want to find the antiderivative of this (*mumbling as he scribbles on left side of page*: *[$-x^{-2}$]*, $+x^{-1}$, 1/x , $-x^{-2}$)

286. MB: Talk a little louder, sorry.

287. J: Okay. $\frac{1}{-2+1}$, x^{-2+1}, it should be $\frac{-1}{-1}$, x^{-1}, 1/x … (*John carries on scribbling on left hand side of page:* $\frac{1}{-2+1}$, x^{-2+1}, $\frac{-1}{-1}$, x^{-1}, 1/x) . So here, this will be the same as limit as b tend to infinity of 1/x … of the area from zero to b, from b to, from zero to b of 1/x. Which is the same as the limit as b tend to infinity of 1/b, of 1/b, you can say minus undefined…(*John has written on right hand side of page:* $=\lim_{b\to\infty}\left[\frac{1}{x}\right]_0^b = \lim_{b\to\infty}\frac{1}{b}-u$. *MB: This function is undefined at x = 0 and as* $x\to\infty$)....No.. (*mumbling*)…continuous.

288. MB: What's going on?

289. J: Now I just want to check if the derivative of f(x) is continuous from zero to infinity. (*MB: It clearly is not*). This is wrong. I cannot find an example that I can use.

290. MB: Do they want you to use an example?

291. J: No, but actually I don't understand this. I can't explain, because I don't even understand. I just wanted to get an example. So that I can see if it's right what I'm going to explain.

292. MB: All right. Rather move on to the next one then you can come back. I can explain all these things after the interview, if you want me to. Okay.

19. Lines 279–292

Indicator and Formation John needs an example of the mathematical object (ie a function which satisfies the given conditions) to make sense of the proposition. Support for this statement is twofold: first John makes many attempts to generate an example of f(x) which satisfies the given criteria; secondly he himself implies that he cannot understand the proposition without an example [line 291].

For these reasons, I categorise this episode as an instance of complex thinking using association-with-an example. That is, the nucleus is an example of the mathematical object (ie a function f(x) which satisfies the given criteria).

EPISODE 20: LINES 307- 312

Question 10

***Summarising overview*:**

John reads Question 10 and immediately tries to generate an example which will illustrate the given proposition. He first draws $y=x^3$ but then ignores this graph presumably realising that it is not symmetric about the y–axis [line 307].

He then sketches $y=x^2$ and states that he wants to see whether this example converges or not. Accordingly he proceeds to calculate $\int_0^\infty x^2 dx$ algebraically. His calculations are fine and he finds that the improper integral is infinite. He does not state whether this means that the integral converges or diverges, but concludes, without explanation, that the proposition in statement 10 is true.

John's need to work with a particular case of a symmetric function is an example of complex thinking using association-with-an example.

Transcript

307. J: Determine whether the following statements are true or false. Justify your answer in each case. So I can jump 9, because it was giving me some problems. I can go straight to 10 (*reading Question 10 again*): 'If the graph of f is symmetric with respect to the y-axis, then the integral from zero to infinity of f(x) converges if and only if the integral from minus infinity to infinity converges'. Yes. I can see this statement 10. But before I can conclude I can …(*John draws graph resembling* $y = x^3$ *on question paper but then ignores this graph*).

308. MB: What did you say it's… what did you say about statement 10 or didn't you say?

309. J: Yes I just wanted to conclude so… but first I just want to get an example which is symmetric about the y-axis. (*John draws graph of* $y = x^2$ *which he labels* $f(x) = x^2$ *. Looks at the question and his graph for quite some time on new page*). So here, I got the example which is symmetric about the y-axis. Which is $f(x) = x^2$. Is this right? Is it symmetric about the y-axis?

310. J: So I just want to check whether the integral converges, from zero to infinity. The limit as b tend to infinity of the integral from zero to b x^2 dx is the same as 1/3 x^3 from b to zero. So this will be same as 1/3… (*mumbling as he writes:* $\lim_{b\to\infty}\int_0^b x^2 dx = \lim_{b\to\infty}\left[\frac{1}{3}x^3\right]_0^b = \lim_{b\to\infty}\left[\frac{1}{3}b^3\right] = \infty$).

311. MB: Talk louder please.

312. J: Okay, so here I say, this statement, this statement 10 is true.

20a. Lines 307–309

Indicator & Formation: John's concern with generating an example to illustrate the given proposition indicates his need to work with a particular instance rather than the general case. As such, John's activity is indicative of complex thinking using association-with-an example; the example is the nucleus with which the general case is built.

20b. Line 310-312

Indicator & Formation. John's need to compute the integral algebraically indicates that he does not see graphically that the integral must diverge. (This does not mean that, if asked, John would not be able to see from the graph that the improper integral must diverge; it merely indicates that John is in an algebraic rather than a graphical frame of mind). Notwithstanding my bracketed comment, throughout the interview there has been no indication that John relates convergence to an area which is tending towards a finite value as $x \to \infty$, and divergence to an area which is increasing without bound.

Since John gives little indication as to why he thinks that the proposition is true, I am unable to classify his usage of signs. The most I can say is that John's need to determine *algebraically* whether the improper integral $\int_0^\infty x^2 dx$ diverges or not, implies that he is not thinking conceptually or even with complex thinking using graphical associations about convergence or divergence.

EPISODE 21: LINES 313 –317

Question 10 and the Note

***Summarising overview*:**

John now attempts to justify why the proposition in statement 10 is true. He first observes that "they just broke this into two, into two parts" (referring to the equation $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$) [line 315]. He then applies the template of this equation to the expression $\int_{-\infty}^{\infty} f(x)dx$ in Question 10 with c as zero: "Because I can say that they have split this into …minus infinity to *…* the integral from minus infinity to zero plus the integral from zero to infinity.." [line 317].

Having established that the template of the equation in the Note can be applied to the integral in Question 10 [line 317] John glosses over the rest of the Note. In particular, he argues that "if one of these integrals converges or diverges…therefore the left-hand side diverges or converges" [line 317]. This is incorrect. What is given in the Note is that "the improper integral on the left diverges **if either** of the improper integrals on the right diverges; **otherwise it converges**".

Although it is difficult to interpret this episode with great conviction (because much of John's articulation about his reasoning is incoherent), I classify (for reasons given below) John's use of signs in this episode as mostly indicative of complex thinking with a template–orientation.

Transcript:

313. J: And as compared to the note given below, the statement 10 is true.
314. MB: How do you mean: "as compared to the note"?
315. J: Okay, I mean, because in the note they just gave me an example that if the integral from minus infinity to infinity of f(x) dx. They just broke it into two, into two parts.
316. MB: Yes.
317. J: From minus infinity to c and from c to infinity. And they say that if one of these integrals converges or diverges… therefore the left-hand side diverges or converges. So here I think if this converges, this will converge, according to this note. Because I can say that they have split this into …minus infinity to … the integral from minus infinity to zero plus the integral from zero to infinity. Because I can say, ya, that statement is true.

21. Lines 313-317

Indicator & Formation: As previously stated, it is rather difficult to interpret this episode since John does not articulate his reasoning clearly. However, using clues from what he does and says, I suggest that John is using complex thinking with a template–orientation.

Specifically he seems to use the fact that he can successfully apply the template of $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$ to the expression $\int_{-\infty}^{\infty} f(x)dx$ in Question 10 (using c as zero) as a reason for asserting that the proposition in Question 10 is true [line 317]. Of course, this is incorrect.

I further speculate that John is unable to deal with the intended logical implication of the statement in the Note: "the improper integral on the left diverges if either of the improper integrals on the right diverges; otherwise it converges". He thus latches on to what he knows is correct, viz. the similarity of template of the equation in the Note and the integral in the

Question. After all he uses this similarity of template to explain his decision that the proposition in Question 10 is true.

EPISODE 22: LINES 318–323

Question 11

Summarising overview:

John's basic strategy in Question 11 is to find a counterexample to the given proposition. This is an appropriate strategy but the function John chooses for his counterexample is not apposite.

That is, John chooses f(x) to be 1/x (in episode 16 John found that $\int_1^\infty \frac{dx}{x}$ diverges). Despite drawing f(x) John does not seem to recognise how his chosen f(x) violates one of the given constraints (ie that f(x) must be continuous at x=0) and so cannot be used as a counterexample.

He then sets about evaluating $\int_0^\infty \frac{dx}{x}$. He argues that it is the same as the previous example in episode 16 (it is not: in episode 16 John evaluated an improper integral whose lower limit was 1; here the lower limit is 0) and so is divergent.

Accordingly he deduces that the proposition is false.

In this episode, John's basic argument [lines 321, 323] is correct. The problem is that his starting assumption (that f(x)=1/x is continuous at x=0) is incorrect. Furthermore, he treats $\int_0^\infty \frac{dx}{x}$ as if it were the same as $\int_1^\infty \frac{dx}{x}$ which it is clearly not.

Transcript

318. J: So in Question 11. (*Reading*): 'If f is continuous on 0 to ∞ and limit of f(x) as x tends to infinity is zero, then f(x), the integral from 0 to ∞ of f(x) converges.'

319. J: I can use the example here. Question 11. I can take this example, 1 over x. That's f(x). So the limit of this as x tends to infinity is zero (*writes* $\lim_{x\to\infty} \frac{1}{x} = 0$). Of 1/x.

320. J: So, and this f(x) is continuous from 0 to ∞. I can say 1/x is continuous from 0 to ∞. Let me just draw the graph. It will just look like this (*draws graph of y = 1/x, x >0*). (*mumbling as he draws*). So, now its continuous. (*MB: y=1/x is not continuous at x = 0*).

321. J: So, now, I just want to find if this converges. If the integral from 0 to ∞ of 1/x converges. Limit as x tends to infinity of the integral from 0 to b of 1 over x (*writes* $\lim_{x\to\infty}\int_0^b\left(\frac{1}{x}\right)dx$)…. That's b (*changes* $x \to \infty$ *to* $b\to \infty$*, to get* $\lim_{b\to\infty}\int_0^b\left(\frac{1}{x}\right)dx$). It will be the same as, the same as, its equal to infinity (*writes:* = ∞) .

322. MB: How did you know that?

323. J: According to the Example that I, we did. A previous example. It is the same as infinity. So, so here…. You can say this statement 11 is wrong. Statement 11 is wrong according to the example that I've given, f(x), where it's 1/x. Because f(x), because 1/x is continuous on zero to infinity (*MB: 1/x is not continuous at x*=0) and the limit of f(x) as x tends to infinity is zero. Then the integral of zero to infinity of f(x) does not converge. It just diverges because the limit from zero, as b tend to infinity of the integral from zero to b of 1/x diverges.

22a. Lines 318–319

Indicator and Formation: Although John adopts the correct strategy (that is he seeks a function f(x) for which the proposition is false), his choice of f(x)=1/x is inapposite.

It seems that John just latches onto an example for which he 'knows' (from episode 16) that the improper integral diverges. (The problem is that the improper integral in episode 16 was $\int_1^\infty \frac{dx}{x}$ not $\int_0^\infty \frac{dx}{x}$. The latter improper integral is problematic in that neither the integrand nor the antiderivative, ie ln x, are defined at x=0.)

This type of activity, in which the student links a previous example to his current needs while ignoring a crucial piece of information (such as the difference in lower limits in the two improper integrals), is indicative of complex thinking with a surface association.

22b. Line 320

Indicator and Formation: John claims that f(x) =1/x is continuous from 0 to ∞ (this is not so: f(x) is not continuous at x = 0). Despite drawing the graph correctly, John persists in his claim.

John's ignoring of the crucial graphical evidence (which he himself generates) is yet again indicative of complex thinking with a surface association.

22c. Lines 321– 323

Indicator and Formation: Using his (incorrect) assumptions from the previous two sub–episodes (ie that $\int_0^\infty \frac{dx}{x}$ is the same as $\int_1^\infty \frac{dx}{x}$ and that f(x) =1/x is continuous on [0, ∞)), John argues, fairly coherently, that the given proposition is false.

Although his style of argument is fine, the assumptions are incorrect. That is, John persists in ignoring two vital pieces of information. I thus classify this sub–episode as again indicative of complex thinking with a surface association.

§7.3 SUMMARY OF ANALYSIS OF JOHN'S INTERVIEW PROTOCOL

§7.3.1 SYNOPSIS

In the analysis I argue that John gains initial access to the improper integral through complex thinking with a template–orientation (Question 2(a), episode 1). In this regard he generates objectively meaningless expressions whose templates correspond in various ways to the template of an improper integral.

Similarly, for Question 2(b), I argue that John's approach is still dominated by complex thinking with a template–orientation (episode 2). In this episode John again generates sequences of objectively meaningless expressions whose templates resemble that of an improper integral to various extents.

What is interesting in these two episodes is that, despite having a seemingly ill–developed sense of an improper integral, by using complex thinking with a template–orientation, John is able to engage in activities with the mathematical signs relating to the improper integral.

In answering Question 3 (episode 3), John uses a specific example of an improper integral in his explanation. He does this without attempting a general explanation which would apply to any improper integral. I argue that this need to work with concrete and specific properties of a particular example rather than abstracted and generalised properties of a class of mathematical objects is indicative of complex thinking using association-with-an example. In such usage the example is a nucleus around which the student tries to build a concept. Although we do not see John move to a more generalised interpretation of the graphical representation of an improper integral during the interview, we see how John is able to use particular examples of the improper integral in the textbook (episode 6) to develop a more generalised notion of the improper integral (I discuss this further below).

John's response to Question 4 (episode 4) is interesting in that it is one of the very few instances of heap thinking over all the nine interviews. Although John initially seems to use complex thinking with a template–orientation in generating the correct template for an improper integral with a lower infinite limit, his use of signs in his explanation of conditions under which this integral converges or diverges, are indicative of heap thinking. My interpretation is based on John's idiosyncratic and incorrect argument that $\int_{-\infty}^{a} f(x)$ is divergent if the limit exists because "it is the opposite" of $\int_{a}^{\infty} f(x)$. In the analysis I suggest that this argument is based on a vague analogy between signifiers in contrasting positions and contrasting signifieds.

Later (and discussed below) we see how John moves away from heap thinking (episode 4) to thinking in template–orientated complexes (episode 14) when dealing with the convergence or divergence of $\int_{-\infty}^{a} f(x)$ and then to pseudoconceptual thinking when deciding whether a particular improper integral diverges or converges (episode 15 and 16). In Chapter 8, I discuss the mechanism whereby a student moves between the different phases posited by appropriation theory as he develops his concept of a convergent or divergent improper integral.

With regard to Question 5(a) (episode 5), a procedural question, I argue that John is applying the template of a definite integral to his computations with the improper integral. Although his manipulations are acceptable he appears to be confused by the notions of convergence and divergence.

Indeed in episode 6 John indicates that he is confused by these notions. Although he tries to articulate his bafflement, I am unable to understand what is troubling him specifically[15]. I thus urge John to refer to the textbook (Larson et al, 1998). This suggestion is also made in terms of the interview design whereby I require that all students first seek help in the textbook before receiving direct assistance from me.

Subsequent to John's perusing of worked examples of both convergent and divergent improper integrals in the textbook (episode 6), he is able to use the correct notation for the improper integral (episode 7 onwards), no longer using mathematically meaningless expressions as he did in episodes 1 and 2. As mentioned above, this movement, from particular examples to more general principles is indicative of complex thinking using association-with-an example.

Although John is now able to determine that the improper integral in Question 5(a) (episode 7) is convergent and he is able to use appropriate notation in his manipulations (that is, he is able to communicate his manipulations effectively), John's use of convergence and divergence is indicative of pseudoconceptual thinking. This categorisation is based on John's initial response to Question 5(b) (episodes 8 and 10). In this response, John argues (incorrectly) that the given improper integral converges because the limit is infinite. This seems to be a direct application of the template of John's heap–type answer to Question 4 (episode 4) and thus indicative of complex thinking using a template–orientation. Hence my decision not to classify John's correct usage of the notion of a convergent

[15] As I discuss in Chapter 8, I suggest that John's difficulties with the English language contribute to this poor communication.

improper integral as indicative of conceptual thought (in episode 7) is justified.

Indeed John is only able to deal consistently and correctly with notions of convergence and divergence after he revises his answer to Question 4 (episode 14). He does this by using the textbook as a reference and using complex thinking with a template–orientation to support his revision.

Subsequent to this revision of Question 4, John is able to redo Questions 5(b) and 5(c) correctly (episodes 15 and 16 respectively). But again (and as in episode 7) John is using pseudoconceptual notions of convergence and divergence in these procedural problems. As with episode 7, my interpretation of John's usage of signs is retrospectively based on his responses to the non–procedural questions 9, 10 and 11 (episodes 19, 20, 21 and 22). In these responses he is unable to work with the improper integral, let alone its properties of convergence or divergence in a useful or appropriate way.

Specifically, in answering Questions 9 and 10 (episodes 19 and 20) John, by his own admission, is unable to proceed without an illustrative example of the mathematical object presented in each of these questions. (As previously mentioned, this need to work with particular instances rather than the general case is indicative of complex thinking using association-with-an example.) However John is not able to find appropriate examples and so is unable to proceed in any obviously useful way with questions 9 and 10.

Furthermore, John's inability to appreciate graphically that a particular improper integral is divergent (Question 10, episode 20) further supports my earlier argument that John is not using a mature, well–formed notion of a convergent or divergent improper integral even when his answers to the procedural questions (in episodes 15, 16 and 18) are correct.

Because of time constraints (the interview was of maximum length of 1.5 hours) John is unable to attempt any of Questions 6, 7 and 8.

In Chapter 8 I discuss some of the important themes which are implicit in the analysis of John’s protocol and this synopsis. Hopefully that discussion will further contribute to the illuminating power of appropriation theory as a tool for the analysis of a student’s mathematical activities as he appropriates a new mathematical object.

§7.3.2 TABULAR SUMMARY OF ANALYSIS OF JOHN’S INTERVIEW PROTOCOL

In this section I present Table 3, a tabular summary of the different phases through which John moves as he goes about appropriating a mathematical object. I present this summary as an index of the analysis for reference purposes rather than as information which can be interpreted as it stands. My opposition to an interpretation of this table per se, is based partly on the fact that the different phases relate to John’s usage of *different* sets of signs.

For example, in sub–episode 1a, ‘template–orientation’ refers to John’s usage of the improper integral Type I sign; in sub–episode 5a, ‘template–orientation’ refers to John’s usage of the infinity sign. Similarly, in episode 4a, ‘association (example as nucleus)’ refers to John’s usage of a specific example of an improper integral in his graphical interpretation of an improper integral; in episode 19, ‘association (example as nucleus)’ refers to John’s attempts to generate a specific example of a function which fulfils certain criteria, and so on.

Clearly any interpretation based merely on this table would be extremely misleading and simplistic. (I remind the reader that I will discuss aspects of John’s protocol in Chapter 8 using the actual analysis rather than this simplified summary).

Table 3: A summary of the different phases through which John moves

QUESTION NUMBER	EPISODE NUMBER	PHASE
2(a): first pass	1a	template
2(a): first pass	1b	template
2(a): first pass	1c	surface
2(b): first pass	2a	template
2(b): first pass	2b	template
2(b): first pass	2c	template
2(b): first pass	2d	template
3: first pass	3	association (example as nucleus)
4: first pass	4a	graphical association
4: first pass	4b	template
4: first pass	4c	heap
5(a): first pass	5a	template
5(a): first pass	5b	not classified
Connecting	6	association (example as nucleus)
5(a): second pass	7a	pseudoconcept
5(a): second pass	7b	pseudoconcept
5(b): first pass	8	pseudoconcept
Connecting	9	not classified
5(b): first pass	10	template
4: second pass	11	not classified
5(c): first pass	12	pseudoconcept
Connecting	13	not classified
4: third pass	14	template
5(b): second pass	15	pseudoconcept
5(c): second pass	16	pseudoconcept
Connecting	17	not classified
5(a): third pass	18a	graphical
5(a): third pass	18b	pseudoconcept
9	19	association (example as nucleus)
10	20a	association (example as nucleus)
10	20b	not classified
10 and Note	21	template
11	22a	surface
11	22b	surface
11	22c	surface

§7.4 ANALYSIS: DAVID'S APPROPRIATION OF AN IMPROPER INTEGRAL

As with John, I will focus on David's appropriation of an improper integral paying especial attention to how David deals with the divergence or convergence of an improper integral. A copy of David's written responses to the task is given in Appendix C.

To avoid simplistic comparisons between John's and David's protocols (or the analyses thereof), it is instructive to note, yet again, that each interview necessarily differs on certain dimensions. In particular, since each student comes with his own particular set of mathematical experiences, and his own different resources (such as language skills), each student responds differently to the various prescribed mathematical questions. Hence my responses and interventions necessarily differ from student to student. (I discuss issues surrounding the differences in my relationships with each of John and David in §8.4). Additionally, since each student works at his own pace, each student covers a different set of questions during the set 1.5 hours of the interview. For example, David answers all questions in the interview; John's time runs out before he gets to the final three questions (Questions 6, 7 and 8). Furthermore in the first 45 minutes David attempts each of the 15 questions in the task at least once and is reasonably successful with 12 of these questions; John attempts 7 questions and is only reasonably successful with one of these questions[16].

[16] For the purposes of this comparison, I am counting each part of a question as a new question. So, for example, Question 5(a), 5(b), and 5(c) constitute three questions.

EPISODE 1: LINES 15 – 28

Definition of improper integral Type I and Question 2(a)

Summarising overview:

David reads the definition of improper integral Type 1 on page 2 aloud, underlining the words ‘exist’ and ‘converge’ in the definition as he reads. He then writes down an example of an improper integral with an infinite integration limit, looking at the definition as he does so [line 24].

David then calculates the improper integral even though this is not required. His algebraic notation is fine although his actual integration is incorrect (he stipulates that the integral of $\frac{1}{x^4}$ is $-\frac{1}{x^3}$, whereas it should be $-\frac{1}{3x^3}$).

In response to my asking what he is doing, David explains which mathematical properties he was trying to satisfy when generating an improper integral, Type I. He tells me that an improper integral with an infinite integration limit must equal a limit that exists. [line 27] He further implies that since the integral exists, it is an example of an improper integral with an infinite integration limit. [line 28]

Note: The implication of both these statements (that an improper integral must have a limit that exists, ie that it must be convergent) is incorrect; an improper integral may be convergent or divergent. In episode 2, we see how David realises that an improper integral does not have to be convergent.

As I argue below, David’s conflation of an improper integral with a necessarily convergent improper integral is indicative of complex thinking using surface association.

Transcript

15. D: Next page. (*I hand him pages 2 and 3*). (*David starts reading page 2*). “Improper Integral Type 1. If f is continuous on the integral a to infinity, then infinity to a of f(x) equals the limit as b tends to infinity of the integral from a to b of f(x). If limit as b tends to infinity of the integral exits, we say the improper integral converges. Otherwise the improper integral diverges”. (*David reads definition and Question 2(a) quietly to himself a few times. Mumbles as he reads).*

16. MB: What are you thinking?

17. D: I’m just rereading the definition of what it says over here (*referring to page 2. David underlines words, ‘ exist’ and ‘converges’.*).

18. D: Can I look at this page (*pointing to page 1 and reading it silently*).

19. MB: Yes, of course. You can look at any of them.
20. D: (reads definition on page 2 to himself again).
21. D: (*David starts writing*). Do you want me to start with this integral?
22. MB: What was the question?
23. D: It said: "Can you make up an example of an improper integral with an infinite integration limit".
24. D: (*David writes*: $\lim_{b\to\infty}\int_a^b \frac{1}{x^4}dx$
25. D: $= \lim_{b\to\infty}\left[\frac{-1}{x^3}\right]_a^b = 0 + \frac{1}{a^3}$).
26. MB: Okay, you tell me afterwards, what you did, what you are doing. What did you do?
27. D: Okay, what I did was: I had to find something, an integral which would have given a result when you substituted in b. Where if you would have taken the limit as b tends to infinity, you would have got a number that exists. That's the same as from the section of limits to infinity.
28. D: So when over here (*referring to* $\lim_{b\to\infty}\int_a^b \frac{1}{x^4}dx = \lim_{b\to\infty}\left[\frac{-1}{x^3}\right]_a^b$), I will get… I can write it here (*David now writes* $\lim_{b\to\infty}\left[-\frac{1}{b^3}+\frac{1}{a^3}\right]$). Its the limit as b tends to infinity of minus 1 over b^3 plus 1 over a^3. And then I say, as the limit tends to infinity, as b tends to infinity, this tends to zero. Therefore you're left with 1 over a^3 (*referring to* $0 + \frac{1}{a^3}$) and by the definition (*points to page 2)* **that integral exists** (*MB: If a* $\neq 0$*).* **Therefore** that is the improper integral with an infinite integration limit.

1a. Lines 15–20

Indicator & Formation: David's underlining of the words 'exist' and 'converge' presages his inappropriate attention to the existence and convergence of an improper integral (see Episode 1b). This undue focus on these two signs (or words) is retrospectively indicative of complex thinking using surface associations. See sub–episodes 1b and 1c for further justification of this categorisation.

1b. Lines 21–25

Indicator & Formation: David was not required to evaluate the improper integral which he generated. That he did so was contingent on his belief that it had to be a convergent improper integral (see sub–episode 1a and sub–episode 1c) and his desire to verify that it was. This is further evidence that David's notion of an improper integral is dominated by an inappropriate association between an improper integral and a convergent improper integral. As in sub–episodes 1a and 1c, his activities are indicative of complex thinking using surface associations.

1c. Lines 26–28

Indicator: In this sub–episode David explicates his belief that an improper integral has to be convergent. Specifically he states that he needed to generate an integral which equals a "number that exists" [line 27]. Yet again this indicates that David has isolated certain aspects of the definition of the improper integral Type I (specifically the condition for convergence) and taken this as a property of all improper integrals.

Formation: As was discussed in sub–episodes 1a and 1b and further evidenced here (by his explicit statements), David has established an association between an improper integral and a convergent improper integral. Consequently I categorise David's activities around the generation of an improper integral as complex thinking in which a surface association dominates. (In episode 2, we see how this association changes.)

EPISODE 2: LINES 29–40

Question 2(a) and 2(b)

Summarising overview:

David starts reading Question 2(b). While reading this he realises that an improper integral does not have to be convergent: "I've just realised something" [line 31]. He further acknowledges explicitly [line 36] that Question 2(a) did not require him to find a convergent improper integral even though this is what he did.

He then writes down that his answer to Question 2(b) is the same as his answer to Question 2(a).

As I justify in the analysis below, David's usage of signs in this episode as indicative of pseudoconceptual thinking, stimulated by a perturbation.

Transcript

29. D: *David now starts reading Question 2(b) silently. He also looks at Definition on page 2 and Question 2(a).*
30. MB: Which one… Also, if you can explain which one you're looking at, reading, etc.
31. D: Okay. I'm just looking at over here… I've just realised something.
32. MB: What did you realise?
33. D: (*Mumbling as he looks at Definition of Improper Integral Type I on Page 2, and then at his answer to 2(a)).* (*Then David starts reading aloud from Page 2*):"…. Definition of an improper integral with an infinite integration limit."
34. D: Okay. (*mumbling*) : "make up an example of a convergent…" This over here is a convergent… (*David is looking at Q2(a))* Is it okay to have the same answer for (a) and (b)?
35. MB: Yes.

36. D: Because over here I did an example of a convergent (pointing to Q2(a)) but over here in (a) it just says: “can you make up an example of an improper integral with a infinite integration limit”.
37. D: And its like the same thing (*David is referring to the fact that his answer to Q2(a) is an acceptable answer to Q2(b)*). Unless you want me to do a divergent one and a convergent one?
38. MB: You don’t need to.
39. D: Must I just write ’same as a’ (*David writes ‘same as (a)’*)?
40. MB: Yes.

2. Lines 29–40

Indicator and Formation: When reading Question 2(b), which specifically requires the generation of a convergent improper integral, David gets perturbed. This is evidenced by his declaration “I’ve just realised something” [line 31] and his implication [line 36] that this realisation is that an improper integral does not have to be convergent. We see how this perturbation helps David break the associative bond that he formed in the previous episode between an improper integral and a convergent improper integral [lines 36, 37].

In Episode 3a and 3c, David reveals a muddled and partial notion of the convergent improper integral. Hence, although David successfully generates an example of a convergent improper integral, I classify David’s activities with the improper integral in the current episode as commensurate with pseudoconceptual thinking.

EPISODE 3: LINES 41–54

Question 3: First pass

Summarising overview:

David states that the graph (of the function in the improper integral) has a repeating pattern, which “repeats to infinity” [line 43]. He then distinguishes between a convergent improper integral and a divergent improper integral as follows: with a convergent improper integral the gaps between the graph and the x–axis get smaller and smaller as x gets larger and larger; with a divergent improper integral the gaps between the graph and the x–axis get larger and larger as x gets larger and larger [line 43] (in talking of “gaps”, David is presumably referring to the area between the graph and the x–axis).

Since David's notion of an oscillating graph is inappropriate to an interpretation of an improper integral, I try to provoke reflection during the interview. I thus ask David if the graph of the function f(x) in the improper integral has to oscillate. He first says yes and then becomes unsure. He tries to generate a graph in which the function does not oscillate but he is unable to generate such an example. So ultimately, but without conviction (evidenced by his comment in line 53: "I don't know whether I'm right or wrong") he confirms his previous answer (that the graph must oscillate).

David's graphical interpretation of an improper integral is clearly not correct. In the analysis below, I argue that David's description of the function in the improper integral with its emphasis on the oscillating nature of the curve, derives from an inappropriate and extralingual association between the properties of alternating convergent or divergent sequences (which he is studying in lectures at the time) and the words "converges" and "diverges" in the definition of an improper integral Type I.

Notwithstanding David's apparent fixation with the oscillating nature of the function in the improper integral, his argument that the improper integral with an infinite integration limit involves the "**area under the graph** to infinity, from a to infinity as b tends to infinity" [line 51] is correct.

Transcript

41. D: (*David reading Question 3 quietly to himself*) "… Can you explain…" *(Seems to think for a time*). Okay, for 3. Do you want me to write this down or…..
42. MB: Yes. I want you to write it down.
43. D: It says: "Can you explain what an improper integral with an infinite integration limit represents graphically". I see that as a graph with a repeating pattern… (*David writes "Graph with a repeating pattern "*). Or something similar so that its just repeating (*David waves his hands as if to show me as oscillating graph*). It repeats to infinity. Like the gap between the graph and the line repeats to infinity. Like a sine graph kind of thing (*David draws oscillating graph: Graph A)* where close to infinity or where something like…. it oscillates around the x-axis and then these just gets smaller and smaller gaps. That sort of thing (*David draws oscillating graph: Graph B*) when its convergent. And like where its divergent, it would be something that is small, small, small and then just gets bigger and bigger and bigger (*David draws oscillating graph: Graph C*) and therefore you can't say whether… you can't for sure find out what the area between the graph and the axis is. That's convergent (*writes convergent next to Graph B*), divergent (*writes divergent next to Graph C*).
44. D: Is all of this work that we need to know for our… .
45. MB: We'll do it at the end of the year, yes.
46. D: So this is like stuff that we have not done yet.
47. MB: Can I, did you think that you have done it in class. Does it look familiar to you.
48. D: (*Shakes his head*) Not integrals up to infinity. I don't remember doing that in class.
49. MB: Can I just ask over there. Okay, are you happy with all of that?
50. D: Yes, I'll say so. According to what the definition says over here (*finds definition on page 2*), I'll say yes. (*David is looking at the definition on page 2)* I suppose you can have

something to the effect of …er, no. That's my thinking, understanding of what divergent and convergent means and from the definition over here (*points to definition on page 2*).

51. D: I'll say that will be my interpretation of it. Its basically something where the line continues to infinity and you are trying to find **the area under the graph** to infinity, from a to infinity as b tends to infinity?

52. MB: I told you I'm not going to say anything. I just want to ask: does it have to oscillate?

53. D: Not necessarily. Because for that same reason you could have… I'm just writing because if I have to draw a line, a wavy line like that (*David draws graph D*), you'll still have this area which as it tends to infinity, you would never be able to work out what it is because its infinity times the height. So I would say yes, but I'm not sure. There may be an example where it won't work… Maybe if you got a graph, for example, like the … graph that goes like that, it goes up like that, (*draws Graph E, a graph of an almost vertical line)* you can say as b tends to infinity…. Since b tends to infinity…I don't know. I think this sort of graph poses a problem because, ya, no (*David crosses out Graph E*). I would say it has to oscillate or something to the effect of that. Where you can see the areas getting smaller and smaller. Or I don't know if you can maybe… (*Davis draws Graph F, a slant line like* $y = -x + 1$) if you draw a graph like that and you limit the domain of it. But I… as b tends to infinity it still…. (*David crosses out Graph F*). Yes I'll say that. I don't know whether I'm right or wrong.

54. MB: I'll tell you afterward. You must tell if you want a new page.

3a. Lines 41–43, 49–50

Indicator & Formation: This sub–episode represents a good example of complex thinking with a chain association. Specifically it seems that David links the words "converges" and "diverges" in the definition of an improper integral to the notion of an alternating sequence which converges or diverges. I base this suggestion on the fact that, at the time of the interview, David's class is studying the convergence or divergence of alternating sequences; and David's discussion of the convergence or divergence of an improper integral mirrors much of the discussion which would have taken place in class around the convergence or divergence of an alternating sequence.

For further in–depth discussion of this chaining, see Example 3 in Chapter 6 in which I use this episode as an illustrative example of complex thinking with a chain association.

David's frequent reference to the gap between the function of the improper integral and the x–axis indicates an awareness that the improper integral represents the area between the function and the x–axis. David's makes this awareness explicit in episode 3b.

3b. Line 51

Indicator & Formation: David's reference to the area under the graph is probably based on David's association of the template of the improper integral with the template of the definite integral. As such it is indicative of complex thinking with template–orientation.

3c. Line 51 –53

Indicator & Formation: In asking David whether the graph has to oscillate, I am trying to focus David's attention on his assertion that it does. In episode 3a I argued that this notion has been brought into the idea of an improper integral through complex thinking with a chain association.

Although David tries to break the (inappropriate) bond between the idea of an improper integral Type I and an oscillating graph by looking for an example of an improper integral whose graph is not oscillating, he is unable to find such an example. His thinking is thus still dominated by the link he has established between an improper integral Type I and an oscillating graph. Accordingly, and as in episode 3c, David's activities are still dominated by complex thinking with a chain association.

EPISODE 4: LINES 56–59

Question 4

Summarising overview:

David writes down the definition of an improper integral with a lower infinite limit looking constantly at the definition of an improper integral with an upper infinite limit as he does so.

Transcript

56. D: (*David reads Q4 silently. He then picks up page 2 and reads the definition silently*). Do you want me to copy this out?
57. MB: I want you to write it. Whatever you would write.
58. D: (*David looks at definition of improper Integral Type I, ie page 2, while writing. It is as if he is copying the definition on page 2. He also briefly glances at page 1, the definition of an improper integral*.)
59. D: (*D writes: If f is continuous on interval (–∞;b] then* $\int_{-\infty}^{b} f(x)dx = \lim_{a\to\infty}\int_{a}^{b} f(x)dx$). Okay. Ya.

4. Lines 56–59

Indicator & Formation: David's constant looking at the given definition of the improper integral with an upper infinite, whilst writing down the definition of an improper integral with a lower infinite limit, suggests that he is using analogy with the template of the given definition to generate his definition.

Accordingly I classify his usage of signs as indicative of complex thinking with a template–orientation.

EPISODE 5: LINES 73–75

Question 5

David does Question 5(a), 5(b), 5(c) adequately. That is, he calculates each integral and then decides whether each integral converges or diverges (in terms of whether its limit exists or not). His decisions about the convergence or divergence or each integral are correct.

I argue below, via a microanalysis of David's response to Question 5(b), that David's notions of convergence and divergence are pseudoconceptual. By similar arguments I could argue that his notions of convergence and divergence as used in Questions 5(a) and 5(c) are pseudoconceptual.

My choice of using Question 5(b) as the illustrative episode is based on the fact that David's concern with the notion of divergence is the primary focus of his activities in Question 5(b) whereas in Questions 5(a) and 5(c) other prior–knowledge issues (such as the meaning of $\lim_{x\to\infty} \ln x$) are primary[17].

Question 5(b)

Summarising overview:

David is required to decide whether $\int_{-\infty}^{1} x\,dx$ is convergent or divergent. He thus calculates its value. The limit does not exist and so David decides (correctly) that the improper integral diverges.

When calculating the integral David uses x in an ambiguous fashion, using it both as a variable limit of integration (ie $\lim_{x\to-\infty} \int_{x}^{1}$) and as a function variable (ie $\int x\,dx$). Although this dual usage of the letter x is potentially confusing,

[17] Although these prior–knowledge issues are material to David's evolving concept of convergence and divergence, examining David's conceptions of these notions in this analysis would distract from the primary focus of this analysis.

David does not appear to be confused. Possibly this lack of befuddlement is indicative of David's mindful approach to the manipulation?

Notwithstanding the above remark, David makes a manipulative error: he reverses the order of terms after substitution of the limits, that is he writes

$\lim_{x\to-\infty}\left[\frac{x^2}{2}\right]_x^1 = \lim_{x\to-\infty}\left[\frac{x^2}{2}-\frac{1}{2}\right]$ whereas he should write $\lim_{x\to-\infty}\left[\frac{x^2}{2}\right]_x^1 =$

$\lim_{x\to-\infty}\left[\frac{1}{2}-\frac{x^2}{2}\right]$.

But this reversal of terms is not material to my assertion that David's computations with the improper integral are adequate.

Transcript

73. D: (*David does 5(b) silently*). (*He writes :* $\lim_{x\to-\infty}\int_x^1 x\,dx = \lim_{x\to-\infty}\left[\frac{x^2}{2}\right]_x^1 = \lim_{x\to-\infty}\left[\frac{x^2}{2}-\frac{1}{2}\right]$.).

74. D: The second one (b), I'd say, the limit does not exist (*writes lim dne*) so it diverges.

75. D: And I checked to see whether I could have used L'Hopital's Rule but $\frac{x^2-1}{2}$, you substitute in, you try to use the limit as x tends to minus infinity. Get infinity over 2 and that's not an example of an indiscriminate form….

5. Lines 73 – 75

Indicator & *Formation*: Although David is able to do Question 5(b) *as if* he has a conceptual understanding of convergence and divergence, we know from Episode 3 and Episodes 11 and 13 that his notions of convergence and divergence are actually rather muddled and incomplete (in episode 3, David's use of the notions of convergence and divergence is dominated by complex thinking with chain associations and in episodes 11 and 13 his use is dominated by complex thinking with graphical associations).

But since David can *use* notions of convergence and divergence adequately and mindfully in a procedural sense, I categorise his thinking with regard to convergence and divergence of an improper integral as pseudoconceptual.

See Example 16 in Chapter 6 for further discussion of David's usage of signs in Question 5(b).

EPISODE 6: LINES 87–94

Question 9

Summarising overview:

David reads the proposition fairly slowly and then states that it is false because “that whole thing equals nought” [line 91]. He then sets about justifying his assertion, using deduction in his argument. As a result of his deductive manipulations, David changes and mind and states (correctly) that the proposition is true.

Transcript

87. D: (*Reading*): Determine whether the following statements are true or false. Justify your answer in each case.
88. D: (*reading*) If f dash is continuous …..(*stops reading out loud and looks silently at Question 9 for a while*).
89. MB: Could you explain what you’re thinking there.
90. D: Okay. It says over here: if f dash is continuous on zero to infinity and that corresponds over here with f dash x over here (*referring to* $\int_0^\infty f'(x)dx$) which we require for an improper integral, improper infinite integral. So therefore you know it can be an improper infinite integral. And then it says: limit as x tends to infinity of f(x) equals zero. So it says over here: minus f of … equals minus f(0).
91. D: But I’d say that that whole thing equals nought. So its false. (*D writes: = 0. False*)
92. D: Because you are not sure what f(0) is. So you can’t really say that. But you can say that integral will give f(x) from infinity to zero which is the limit of f(x) as x tends to infinity minus f(0) (*writes* $[f(x)]_0^\infty$). Sorry. I’m wrong. It’s true.
93. MB: Don’t rub it out. Rather just cross it out.
94. D: Which equals limit as x tends to infinity of f(x) minus f(0) (*writing:* $= \lim_{x\to\infty} f(x) - f(0)\ \therefore$ *True*). It’s called being lazy and not finishing off the answer correctly.

6a. Lines 87–91

Indicator and Formation: David’s immediate response to the proposition is that it is false and that the improper integral is zero. The immediacy of David’s answer suggests that this response is based on complex thinking with a surface association (although I am not sure what that particular association is) in which David equates the improper integral to zero without justification.

6b. Lines 92– 94

Indicator and Formation: When trying to justify his assertion, David manipulates the improper integral using the given constraints in his determinations. That is, he uses logical deduction to decide (correctly) that the proposition is actually true.

Since David’s usage of signs in this sub–episode is both logical and abstracted from concrete examples, I classify these as indicative of conceptual thinking about the proposition.

EPISODE 7: LINES 95 – 109

Question 10

Question 10 deals with the convergence of an improper integral Type I in the context of a specific problem. I remind the reader that, unlike Questions 5 and 6, Question 10 cannot be approached purely with a computational bent. Furthermore not only does the student have to deal with the concept of convergence, but also with notions of symmetry and the logical condition, ’if and only if’.

Although my focus in this analysis is David’s evolving concept of convergence (rather than his notions of symmetry or the ‘if and only if’ condition), I am presenting a microanalysis below and so I will need to attend to all these aspects.

***Summarising overview*:**

David reads Question 10 out loud and confidently and immediately states that the proposition in Question 10 is true.

Without reading the attached note, David justifies his answer. He mentions that $\int_{-\infty}^{\infty} f(x)dx$ refers to an area and he writes: $\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$ [line 97].

This equation and David’s reference to area encapsulate the crucial notions that an improper integral represents area and that a consequence of the symmetry of the function around the y–axis is that the area between the curve f(x) and the x–axis from $-\infty$ to 0 equals the area between the curve f(x) and the x–axis from 0 to ∞.

David explains why the proposition in Question 10 is true. He argues that $\int_{-\infty}^{\infty} f(x)$ cannot be infinite since it equals $2\int_{0}^{\infty} f(x)dx$ and $2\int_{0}^{\infty} f(x)dx$ is divisible by two [line 98].

David asks me if his explanation is sufficient. Before I can indicate my approval or not, he starts looking at the Note which is part of Question 10. David reads the Note to himself and says that the Note gives the reason why the proposition is true. However his elaboration of this reason [line 108] is abbreviated (although he does correctly indicate that if the one side of the equation converges then so does the other side; if the one side of the equation diverges then so does the other side) so I am unable to interpret his usage of the sign 'if–and–only–if' (in hindsight I realise that I should have asked David to elaborate on his reasoning).

Transcript

95. D: Okay, I'm going to 10. (*reading)* "If the graph of f is symmetric with respect to the y axis then the integral from zero to infinity of f(x) converges if and only if the integral of f(x) from minus infinity to infinity converges".
96. D: I'd say that's true (*writes: True*).
97. D: For the simple reason that you cannot say.. Oh, alright. The **area** for the integral from infinity to infinity of f(x) equals 2 times **integral** from zero to infinity of f(x) dx (*writing: For* $\int_{-\infty}^{\infty} f(x) = 2\int_{0}^{\infty} f(x)dx$)
98. D: Therefore this integral over here (*referring to* $\int_{-\infty}^{\infty} f(x)$) must be a number … it must not be an infinite number so that you can divide by two to get to this number over here. So as the limit tends to infinity you won't get an infinite answer.
99. D: Would that be a sufficient answer or?....(*D looks at me*).
100. MB: If you're happy with it.
101. MB: I'm first asking…. what are you looking at?
102. D: The next definition.
103. MB: The Note?
104. D: Ya.
105. MB: Yes, that goes with it.
106. D: (*reading to himself*) This is the reason (*writes 'reason' next to Note*).
107. MB: Which part is the reason?
108. D: Its basically a definition of exactly what this is. It basically says: if this converges then that converges. If one diverges then the other one also diverges (*pointing to Statement 10*) .
109. MB: Okay.

7a. Lines 95 – 97

Indicator & Formation: Given that David speaks about area interchangeably with an integral [line 97], I suggest that David is using different signs (the improper integral and area) to talk of one mathematical object. This equivalence of concepts indicates a usage of the improper integral which is compatible with conceptual thinking.

I also suggest that a concept of symmetry around the y–axis is implicit in David's equation $\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$. But because David's argument and equation constitute an abbreviated way of dealing with the mathematical notion of symmetry, it is difficult to justify my suggestion. The most I can argue is that conceptual thinking regarding symmetry is not counter–indicated by David's usage of signs.

7b. Lines 98 –100

David's argument concerning the impossibility of an infinite number being equal to some multiple of a finite number seems to be based on a conceptual notion of infinity. David's assertion hinges on the implicit argument that there is no algebraic procedure which enables one to transform an infinite number into a finite number and an implicit acknowledgement that infinity does not behave in the same way as a real number. The flexible and logical approach implicit in such an argument is indicative of conceptual thinking. (For further discussion on this point, see Example 18 in Chapter 6.)

Although David's notions of symmetry and infinity are arguably conceptual, his notion of convergence is pseudoconceptual. I base this latter classification on David's future activities with convergence (see episodes 11 and 13) which primarily involve graphical associations.

Indeed David's use of the notion of convergence in the current episode is implicitly procedural[18], ie if an improper integral is convergent its value is necessarily finite. As such it does not necessarily require conceptual thinking[19].

7c. Lines 101–109

David's use of the Note to justify his assertion that the proposition is true, is abbreviated. Since I did not (unfortunately) probe his reasoning I am unable to comment on his use of the if–and–only–if signifiers.

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[18] A procedural approach is not necessarily pseudoconceptual, it may be conceptual. In many contexts a procedural approach is the appropriate approach; the use of such an approach may be precisely because that student has a conceptual understanding of the notion at hand and knows thus that a particular procedure is apposite to the task.

[19] This illustrates how learners may think of some mathematical notions in conceptual terms while simultaneously thinking of other mathematical notions in non-conceptual terms.

EPISODE 8: LINES 114–124

Question 11: First Pass

Summarising overview:

David's tries to find a counter–example to the given proposition [line 116]. That is, he tries to generate a function for which $\lim_{x\to\infty} f(x) = 0$ [line 116], f is continuous on [0, ∞) [line 118] and $\int_0^\infty f(x)dx$ diverges [line 122]. After a few unsuccessful attempts, he asks if he can return to this question later.

Transcript

114. D: (*reading*): 'If f is continuous from zero to infinity and limit as x tends to infinity of f of x equals zero then the integral from zero to infinity of f(x) converges'. (*D looks silently at statement for quite a while. Raises his eyebrows, scratches his head*).

115. MB: Tell me what you're thinking?

116. D: I was just trying to think of a counterexample for this. The example that I can think of is if f(x) equals 1 over x (*writing if f(x) = 1/x*). Then you can say the limit of f(x) as x tends to infinity equals zero (*writes* $\lim_{x\to\infty} f(x) = 0$) . But if you say the integral from infinity to zero of f(x) (*writes* $\int_0^\infty f(x)dx$) , you get that equals….er…. then ln x from infinity to zero (*writes* $[\ln x]_0^\infty$ …. Ooh, no, sorry. I can't use that.

117. MB: Why not?

118. D: ln x, 1 over x is not defined for zero. And the domain has to include zero.

119. MB: Why did you want to use that originally?

120. D: Because it would have been a counterexample to prove that, for this, that f(x) wouldn't have converged.

121. MB: How did you know it might be a counterexample? From doing it before, I mean, now?

122. D: Because I know as x tends to infinity for f(x) you know that there will always be a higher power in the…. lower….in the denominator. Because that's the only way how it will tend towards zero as x tends to infinity. So I was thinking of an example where x tends to infinity… but as x tends to ∞, the integral of it doesn't tend, also doesn't tend to a real number.

123. D: (*D looks for another counter example, mumbling as he does*): I suppose you could go……if f(x) equals 1 over x^n (*writing f(x) = 1/x^n ,* $\int_0^\infty f(x)dx$, $\left[\frac{-1}{(n-1)x^{n-1}}\right]_0^\infty$ *and then sitting silently and apparently thinking*).

124. D: I am stumped. May I have the next page so that I can just think about this while I'm doing something else.

8. Lines 114–124

Indicator and Formation: In this episode, David seems to be using the notion of divergence procedurally, ie $\int_0^\infty f(x)dx$ is divergent if $\int_0^\infty f(x)dx = \infty$.

Without other evidence, this would indicate a pseudoconceptual or a conceptual notion of convergence. Taken with David's other responses particularly to Questions 3 and 11 (see episodes 3, 11 and 13) in which he respectively uses complex thinking with a chain and then a graphical association in his arguments about the convergence or divergence of an improper integral, I interpret David's use of the notion of convergence or divergence of an improper integral as indicative of pseudoconceptual thinking.

EPISODE 9: LINES 140 – 148

Question 6

In Question 6 the student has to determine whether two improper integrals Type II are convergent or divergent. This is similar to Question 5 except that in Question 5 the improper integrals are Type I.

Since Question 6 is similar to Question 5, the student's responses to Question 6 as compared to Question 5, may reveal changes in the student's notions of convergence and divergence.

Certainly in David's case, we see how his use of signifiers in the improper integral sign has evolved from Question 5 (Episode 5) to Question 6 (Episodes 9 and 10): in Question 5 he idiosyncratically and ambiguously used x as both a function variable and a limit of integration in the improper integral sign; in Question 6 he restricts the use of x to refer to the function variable only.

Also David's use of the limit signifier evolves from a confused usage in Question 6(a) to an acceptable usage in Question 6(b). In Question 6(a), David talks of finding the limit as c goes to 0 whilst writing the relevant function as if it were independent of c (see details below); in Question 6(b) he corrects this notation. In the indicator and formation section below I argue that this evolution in David's usage of signs is a direct result of his mindful usage of signs. Furthermore I maintain that the latter and conventional use of the signifier x is acceptable to the community of mathematicians and so enabling of enculturation into that community (and hence communication and further conceptual evolution).

Question 6(a)

Summarising overview:

In order to decide whether a particular improper integral is convergent or divergent, the student needs to evaluate that integral. This David does by calculating $\int_0^3 \frac{dx}{x}$.

After stating that this improper integral is equal to 'ln 3 – $\lim_{c\to 0^+} \ln 0$' [line 142][20], David uses his calculator to confirm that '$\lim_{c\to 0^+} \ln 0$' is negative infinity (he uses his calculator to find ln x for x near zero and presumably extrapolates that $\lim_{c\to 0^+} \ln c = -\infty$). He tells me that he knows what the graph of ln x looks like but that he is using his calculator to confirm the value of $\lim_{c\to 0^+} \ln 0$. He also draws a quick sketch of ln x, possibly to verify his calculator answer.

David goes on to state that the limit is infinite and so the improper integral diverges. This is correct.

Throughout this episode David's calculations are fine but his use of signifiers is slightly muddled. For example he writes the expression '$\lim_{c\to 0^+} [\ln x]_0^3$' as opposed to $\lim_{c\to 0^+} [\ln x]_c^3$ [line 140] and '$\lim_{c\to 0^+} \ln 0$' as opposed to $\lim_{c\to 0^+} \ln c$ [line 142]. I argue below that this non–logical use of symbols indicates complex thinking with a template–orientation to the notion of an improper integral Type II. However in other respects David's use of signifiers has become more compatible with that of the mathematics community. That is, he no longer uses x as both a variable and a limit as he did in Question 5 (see episode 5).

[20] This is not a misprint; ie David used a 0 rather than a 'c' in the expression 'ln 0'.

I also argue that David's use of both a calculator and a sketch to evaluate $\lim_{c\to 0^+} \ln c$ is indicative of complex thinking using graphical association and that, as in Question 5, his overall conception of the convergence or divergence of an improper integral Type II is still pseudoconceptual.

Transcript

140. D: (*D writes*: $\int_0^3 \frac{dx}{x} = \lim_{c\to 0^+} \int_c^3 \frac{dx}{x} = \lim_{c\to 0^+} [\ln x]_0^3$).
141. MB: Tell me what you're doing?
142. D: I'm just… I converted the zero, I changed the zero over here to a c, over here. And I said the limit as c tends to zero plus. And I'm just working out the integral… it's from 3 to zero. So it's ln 3 minus limit as c tends to zero plus, of ln zero (*writes* $= \ln 3 - \lim_{c\to 0^+} \ln 0$).
143. D: But ln (0) …. (*uses calculator. Presumably he is calculating ln x for values of x close to zero*)… tends to negative infinity (*writes ln 3* $-(-\infty)$).
144. MB: What did you look up on the calculator?
145. D: I was just testing. I know the graph looks like that (*sketches graph of lnx*) but I was just testing to check that it continues going down and down and down.
146. D: So basically ln 3 minus minus infinity. So it would end up as being infinity. So we say the limit does not exist, therefore it converges. Therefore diverges sorry. (*writes:* $=\infty$*, lim d.n.e.* $\therefore$ *diverges*).
147. D: The reason over here.. why I said that it was the limit as x, as c tends to zero plus is because x cannot equal zero over here (*referring to expression* $\lim_{c\to 0^+}$ *in 6(a)*).
148. D: Same as over here (*referring to 6(b)*). x cannot equal zero therefore you obviously …you know that's the….what's the terminology used over here… discontinuity? You know it's the discontinuity. That's where the discontinuity happens.

9a. Line 140–142

Indicator and Formation: David's calculations are basically correct but his use of the limit notation, eg $\lim_{c\to 0^+} [\ln x]_0^3$ and $\lim_{c\to 0^+} \ln 0$, is confused (he has replaced the c which should tend to zero, with zero, in each expression). This type of replacement of one symbol with another, without reference to its intended meaning, is indicative of complex thinking with a template–orientation. In this case, the template–orientation refers to the usage of the improper integral Type II sign.

9b. Lines 143 –145

Indicator and Formation: David's use of a calculator and a graph to verify that $\lim_{c\to 0^+} \ln 0$ is $-\infty$, is an example of complex thinking using graphical association (with regard to the ln function). The use of technology or a graph means that David's reasoning regarding the value of $\lim_{c \to 0^+} \ln c$ is concrete, rather than logical or abstract.

9c. Line 146

Indicator and Formation: As with Question 5, David is able to compute the improper integral successfully and to determine that it diverges. Although his successful use of the mathematical signs in this sub–episode could be interpreted as indicative of a conceptual understanding of the notions of convergence or divergence of an improper integral, we know that David's use of these notions is actually inadequate. (This inadequacy is implied by his strange use of signifiers when dealing with the improper integral in episode 9a, and his graphical notion of convergence and divergence demonstrated in episodes 11 and 13.)

As with an improper integral Type I, I thus categorise David's use of the notions of convergence and divergence of an improper integral Type II as pseudoconceptual.

9d. Line 147–148

Indicator & Formation: The fact that David tries to explain his use of the term $\lim_{c \to 0^+}$ indicates that he is using his signifiers mindfully. That he is using the limit signifiers incorrectly, yet mindfully, strengthens my argument in episode 9a that David's use of the improper integral is indicative of template–orientated complex thinking**.**

Nonetheless I suggest that it is David's mindful use of the signifiers that quickly leads him to a more logical use of the limit notation. Indeed, in doing Question 6(b) David uses the limit signifier appropriately (see Episode 10).

EPISODE 10: LINES 149 – 151

Question 6(b)

***Summarising overview*:**

As with Question 6(a), David needs to evaluate the given integral. He does this correctly and his notation is now acceptable. That is, he uses the letter ‘c’ correctly when writing expressions of form $\lim_{c \to 0^+} f(c)$ unlike in Question 6(a).

I argue below that, despite David’s adequate usage of signs, his thinking about the convergence or divergence of an improper integral is still pseudoconceptual.

Transcript

149. D: So it will be the limit as c tends to zero …. (*mumbling as he writes:* $\lim_{c \to 0^-} \int_{-8}^{c} \frac{dx}{x^{2/3}}$). The reason over here why I said it’s c to the minus is because, you know, the graph starts at −8 and it’s moving to the other side. So therefore the graph will be defined to the left of c (*waves hands to demonstrate*). That’s why I said c from the minus…. over here. (*D writes:* $= \lim_{c \to 0^-} \left[3x^{1/3}\right]_{-8}^{c} = 6$ ∴ *Converges.*) Okay.

150. MB: So what have you written there?

151. D: I just calculated it… the limit as c tends to zero over here gives zero then you would have minused, minus your second term over here. The third root of minus 8 is minus 2. Minus 2 times 3 times minus gives you 6.

10. Lines 149–151

Indicator and Formation: As in Question 6(a) (Episode 9), David evaluates the improper integral correctly, and appropriately concludes that the improper integral converges. However, unlike in Question 6(a), he uses the limit notation correctly. Indeed, and as I put forward in episode 9, I suggest that David’s mindful albeit incorrect use of the limit signifier in that episode, contributed to this evolution of usage.

Although David uses notions of convergence and divergence as if he has a well–developed understanding of these notions, his usage of these notions is still pseudoconceptual. My primary reason for this statement is David’s graphical interpretation of convergence and divergence shortly hereafter in episodes 11 and 13.

EPISODE 11: LINES 173 –179

In Question 8 the student has to explain what an improper integral Type II represents graphically. This is a similar question to Question 3 except that in Question 3 the improper integral is Type I.

As with Questions 6 and 5 (see episodes 9 and 10), the student's responses to Question 8 as compared to Question 3, may reveal changes in the student's notions of convergence and divergence.

Indeed we see that David's use of the graphical notion of convergence and divergence in Question 8 has evolved from a use dominated by chain associations (in Question 3) to a use dominated by graphical associations (in Question 8).

Question 8: first pass

Summarising overview:

David's response to Question 8 indicates a change in David's graphical interpretation of convergence and divergence (formerly expressed in his response to Question 3). Perhaps because he is no longer dealing with a graph which extends to infinity on its domain (as with an improper integral Type I) but rather with a graph with an infinite discontinuity where "you are not finding the graph after that discontinuity" [line 178], or for some other reason, David no longer argues that the function in an improper integral needs to oscillate (as was his argument in episode 3).

Nonetheless, as in episode 3, David still explains convergence and divergence graphically in terms of increasing or decreasing amounts of added area. He argues that if the added areas decrease as x increases (presumably between the curve f(x) and the x–axis, for each equivalent change in the x–value) then the improper integral will converge; if the added areas increase as x increases (presumably between the curve f(x) and the x–axis, for each equivalent change in the x–value) then the improper integral will diverge [line 179].

As I discuss below, David's verbal reasoning is visual rather than logical (it is also incorrect). Accordingly I classify it as indicative of complex thinking using graphical associations.

Transcript

173. D: (*reading): '*Can you explain what an improper integral with an infinite discontinuity represents graphically.' This is actually what I mentioned earlier on where… remember I said that graph kind of goes up like that (*draws small graph resembling $y = \tan x$, $0 < x < \pi/2$*).

174. MB: Right.

175. D: (*D looks puzzled. He looks at definition of improper integral, Type II and at Questions 5 and 6).* I'm not sure whether this is correct. but is it… It's the integral of a graph that continues increasing but the added increments after you reach a certain x–value are so insignificant that they barely add onto to the total area under the graph? Do you want me to write it down or…?

176. MB: I want you to imagine I'm a student and explain it to me.

177. D: (*D looks at me while he speaks,. waving his hands from time to time).* Okay. First of all an improper integral….Go back to the definition over here is… (*pointing at definition of improper integral on page 1*). If one of the limits is infinite or the function has a discontinuity on [a, b], you call the integral an improper integral. So over here, from this you can say that …if the graph is discontinuous… its not continuous.. if there is a discontinuity at a point, then you say it is an improper integral.

178. D: You are finding the area between the two limits where there's, where there's a discontinuity. So you are not finding the graph after that discontinuity, the area under that graph after the discontinuity. And then you can say there's an infinity discontinuity. That is basically that the graph tends to infinity at that point.

179. D: Therefore you can say that, by saying: if the graph converges at that point … if there's convergence over there… there is the increments…. As you increase ever so slightly the increments are getting smaller, the added area is getting smaller and smaller and smaller. And if it diverges the added area is getting bigger and bigger and bigger. So therefore you can't work out…. When it converges you can work out the area because its basically a sum to infinity of those small areas. And if it diverges you can't because those areas will eventually add up to infinity.

11. Lines 173–179

Indicator & Formation: David's explanation [lines 175, 179) gives an account of convergence and divergence in terms of the addition of increasing or decreasing amounts of area between the function of the improper integral and the x–axis, for intervals of constant width, as $x \to \infty$.

This account of convergence hinges on an analogy whereby if the areas are getting smaller and smaller, then the improper integral (as the area between the curve and the x–axis) converges. But this analogy is visually–based (not abstract) and it is incorrect.

Indeed the areas of successive intervals of constant width may be getting larger and larger but the *cumulative* area may be getting smaller and smaller. For example consider the two improper integrals $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ and

$\int_0^1 \frac{1}{x^2} dx$. In each case the area between the function and the x–axis is getting larger and larger for successive intervals as $x \to 0$, but the former integral converges and the latter integral diverges

Since David's argument relies on a visual analogy rather than logic or abstract reasoning, I classify his thinking re convergence and divergence of an improper integral, Type II, as complex thinking using graphical association.

EPISODE 12: LINES 191 – 197

Question 11: Second Pass

***Summarising overview*:**

We saw in episode 8 how David was unable to find a counter–example to the proposition in Question 11.

He now adopts a different strategy. He states that the proposition in Question 11 is true and tries to prove this using spurious and irrelevant associations. Specifically, David argues that $\lim_{x \to 0} f(x)$ exists (this is true but not relevant) [line 192, 195]. He then notes that $\lim_{x \to \infty} f(x)$ also exists (this is also true) [line 192]. But David goes on to falsely imply that the existence of these two limits guarantees that one can always calculate the area between x =0 and x = ∞ [line 197].

Indeed David does not seem to engage directly with the notion of convergence in his activities in this episode. In fact, he only alludes to the idea of convergence once when concluding that "it is possible to work out the area" [line 197] (presumably implying that the improper integral must thus converge). Furthermore his focus on the existence of the limits of the function at both 0 and ∞ is irrelevant to a consideration of whether a particular improper integral is convergent or not.

Transcript

191. D: (*D is reading Question 11 again to himself*). Question 11.

192. D: I'll say that this is true (*writes "True"*). Because by simply saying that it's continuous from zero including zero, you can say that it basically implies that there is a limit as x tends to zero, of f(x). Therefore… and there's a limit as x tends to infinity. Therefore from zero to infinity you'll always have a limit.
193. D: Do you want me to write it down?
194. MB: Um…Okay.
195. D: (*D writes: True. Because 0 is included in the domain.* $\lim_{x \to 0} f(x)$ *exists. Still looks puzzled*). Ya.
196. MB: What did you write there? Sorry, I just can't read it.
197. D: Because zero is included in domain, the limit… it is implied (*writes: "it is implied" in front of* $\lim_{x \to 0} f(x)$) that the limit as x tends to zero of f(x) exists. Because that implies that there is a point on zero where it does exist… at zero… where x is equals to zero. Therefore you can say that you will be able to… it is possible to work out the area.

12. Lines 191–197

Indicator & Formation: I will repeat my interpretation which I previously gave in Example 9 in Chapter 6, of what is happening in this episode.

David is unable to find a counter–example to the proposition. He thus hones in on a part of the proposition which he is recognises. That is, he latches onto those signs of the proposition which refer to the existence of the limits of the function at both 0 and ∞ [lines 192, 195]. He then uses the existence of these limits as the nucleus of a complex to which he associates the existence of the limit of the area [line 197]. This is incorrect and not logical.

Because David's reasoning seems to be based on associating a new, unfamiliar mathematical object (the convergent improper integral) with a familiar object (the existence of limits at the endpoints of the domain of the function), I propose that David's usage of signs in this episode is indicative of complex thinking using association (of an artificial nature).

EPISODE 13: LINES 215–232

Question 3: second pass

Summarising overview:

David tells me that he is uncertain about his response to Question 3. I tell him that his answer was not correct and I ask David again (see episode 3) whether the function in the improper integral must be repeating. David replies that the function does not have to repeat. He argues that the areas that you add on (as x tends to infinity) must get smaller and smaller (he confirms in line 229 that he means for convergence of the improper integral) or bigger and bigger (presumably he means for the divergent improper integral).

This is a similar explanation to what he gave in Question 8 when he was dealing with a graphical interpretation of an improper integral Type II (see episode 11).

Since these graphical explanations of convergence and divergence are not mathematically correct, I decide[21] to show David an example of an improper integral where the size of the area between the function and the x–axis decreases for constant width as $x \to \infty$ but the improper integral diverges. Specifically I remind David of his answer to Question 5(c). There he showed that $\int_1^\infty \frac{1}{x}dx$ was divergent even though the area between the curve y = 1/x, the x–axis and an interval of constant width gets smaller and smaller[22] as x tends to ∞.

David acknowledges that his explanation was not adequate and that "you always have to mathematically show everything, not just make a huge guess and think what's happening" [line 231].

Transcript

215. D: Was this right, this oscillating thing? (*Referring to Q 3)*
216. MB: No. This one was not right. (*Reading*) "Can you explain what an improper integral with an infinite integration limit represents graphically ". It might represent that (*MB points to Graphs A, B, C*) but there is more than this. This is Question 3.
217. D: Yes, I wasn't sure.
218. MB: Why did you want it (*pointing to his statement: "graph with a repeating pattern")* to be repeating?
219. D: Well, not necessarily. I was wrong in saying that it was repeating because it was more of a sliding scale of areas. The areas decrease, get smaller and smaller and smaller or get bigger and bigger and bigger. That kind of thing (*waves hands to show graph which approaches x–axis as x tends to infinity*).
220. MB: Yes, but…

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[21] The decision to 'teach' David on this second attempt at the question is consistent with my interview guidelines. See Chapter 5 for a discussion of these.

[22] The point is that the cumulative area (as represented by $\int_1^\infty \frac{1}{x}dx$) gets infinitely large.

221. D: That's more what I really mean. Not necessarily repeating. I think that was for that graph which I was not sure about (*referring to Graph D).* Then I decided on the getting smaller and getting bigger idea over here (*referring to Graphs B and C*)
222. MB: So can you think of another example, more general, I mean. I can show you if you wish.
223. D: Yes. Its probably same as that graph that we did now (*Draws a graph resembling a graph of y = 1/x, x > 0, Graph G*) . The area that gets smaller and smaller and smaller as you continue along additional area.
224. MB: Has that got an infinite integration limit. This one that you've done here now. Do you want your x to go to infinity?
225. D: Ya. The x should go to infinity. So… a graph that would for example go like that.
226. MB: Yes
227. D: That would give…. like the area would get smaller, smaller until it just became insignificant (*referring to Graph G*)
228. MB: If its convergent?
229. D: Ya, if it's convergent. Would it not be convergent?
230. MB. Well, not necessarily actually. Well, lets see, look over here. *(MB turns to Q5(c):* $\int_1^\infty \frac{1}{x}dx$ *)* 1 over x. That one you found correctly. I don't know where you wrote it. Where did you do this (*MB looking through D's pages*)? Oh here, it diverges.
231. D: Oh ya. It does. It would create a mix up here… so it would diverge. Its not converging. I suppose all of these, you always have to mathematically show everything, not just make a huge guess and think what's happening.
232. MB: Yes. That's right. Yes, some of them… But basically as x goes to infinity it could be in the shape 1/x, the graph of f(x) = 1/x. It could be in the shape of $f(x) = 1/x^2$ which I think you show converged. Or you showed $1/x^3$ converged. That was correct.

13a. Lines 215 – 229

Indicator & Formation: Although David seems to have severed the link between the function in an improper integral and an oscillating sequence (see episode 3), his notion of convergence and divergence is still dominated by graphical associations (a form of complex thinking) rather than abstract or logical links. To wit, he explains convergence and divergence in terms of increasing or decreasing amounts of added area: areas which "decrease, get smaller and smaller and smaller or get bigger and bigger and bigger" [line 219].

Just as in episode 11, David's explanation is not logical or abstract; rather it centres on a visual analogy whereby if the areas are getting smaller and smaller, then the improper integral (that is, the area between the curve and the x–axis) converges.

I thus classify David's thinking re convergence and divergence in this episode as complex thinking using graphical association.

Note: The problem with David's notion of divergence is that the areas may be getting smaller and smaller but the cumulative area may be getting infinitely large. As stated previously, this occurs with the divergent improper integral $\int_1^\infty \frac{1}{x}dx$. Similarly the areas may be getting larger and larger but the cumulative area may be tending to a limit; this gives a convergent improper integral. Or the added areas may remain constant, thus resulting in divergence. An example of the latter case is any integral of form $\int_a^\infty c\,dx$, where a, c are constants.

13b. Lines 230 – 232

Indicator & Formation: David seems happy with an explanation in which I give him a counterexample to his conjecture [line 232] even though I do not attempt to explain why, say, $\int_1^\infty \frac{1}{x}dx$ is divergent and $\int_1^\infty \frac{1}{x^2}dx$ is convergent. His response: “you always have to mathematically show everything” [line 231] seems to imply that he is happy to treat the notions of convergence and divergence procedurally (ie if the computations show the improper integral is infinite, it is divergent; if the computations show that the improper integral is finite, then it is convergent).

Given David’s apparent lack of curiosity about why some improper integrals are convergent and other ‘similar’ improper integral are divergent, I suspect that, by the end of the interview, David still has a pseudoconceptual notion of convergence or divergence rather than a conceptual one and significantly, that he is satisfied with such a notion.

§7.5 SUMMARY OF ANALYSIS OF DAVID’S INTERVIEW PROTOCOL

§7.5.1 SYNOPSIS

In the analysis I argue that David’s initial attempt to generate an improper integral Type I in response to Question 2(a) (episode 1) is indicative of complex thinking with a surface association. Specifically, I argue that David’s implication that an improper integral must necessarily converge is consequent upon his isolation of certain words (“converge” and “exist”) when initially reading the definition of an improper integral.

In the next episode (episode 2), I suggest that this complex thinking with a surface association is perturbed when David reads that he has to generate a specifically *convergent* improper integral. My suggestion is based on the fact that after reading the question, David acknowledges that an improper integral may converge or diverge and he uses the same example he gave for Question 2(a) in response to Question 2(b). In the analysis I suggest that David’s usage of the improper integral sign in this episode is indicative of pseudoconceptual thinking.

For Question 3 (episode 3), in which the student is asked to interpret an improper integral graphically, I classify David’s usage of signs as primarily indicative of complex thinking with a chain association. Although a notion of area is implicit in David’s interpretation (ie with a convergent improper integral the “gaps” between the graph and the x–axis get smaller and smaller as x gets larger and larger whereas with a divergent improper integral the “gaps” between the graph and the x–axis get larger and larger as x gets larger and larger), David’s insistence on the oscillating or repeating pattern of the graph indicates that he is associating the convergence or divergence of an improper integral with the convergence or divergence respectively of an alternating sequence. Accordingly I classify his usage of signs in this episode as primarily indicative of complex thinking with a chain association in which a convergent improper integral is chained to a convergent alternating sequence.

In response to Question 4 (episode 4), David uses the given definition of $\int_a^{\infty} f(x)dx$ to generate a definition for $\int_{-\infty}^{b} f(x)dx$ successfully. Given that David seems to be basing his definition directly on the definition of an improper integral with an upper infinite limit, I classify David’s usage of signs in this episode as indicative of complex thinking with a template–orientation.

With regard to Questions 5(a), (b) and (c) (episode 5), David's notation for an improper integral is ambiguous (he uses the letter x to stand for both the limit of integration and the function variable). Nevertheless he is able to determine adequately whether the given improper integral converges or diverges. Despite this success, and given David's later (incorrect) interpretation of a convergent and divergent improper integral (see episode 13), I argue that David's seemingly appropriate responses to each of Questions 5(a), (b) and (c) are based on a pseudoconceptual rather than a conceptual notion of convergence and divergence.

With regard to the two non–procedural questions, Questions 9 and 10 (episodes 6 and 7 respectively), I classify certain of David's sign usages as indicative of conceptual thinking. To elaborate: David initially responds incorrectly to Question 9 stating that the proposition is false. But after careful manipulation and logical deduction he changes his mind and asserts that the proposition is true (which is correct). In the analysis I argue that this logical deduction is indicative of conceptual thinking. In Question I0, I also classify David's way of dealing with the improper integral as conceptual and I classify his flexible approach to both symmetry and infinity as probably conceptual. However, I interpret David's way of dealing with convergence and divergence in Question 10 as still pseudoconceptual.

At this juncture it is worth pointing out how David has moved from complex thinking with a surface association about the improper integral (episode 1) to a usage of signs which may be indicative of conceptual thinking (episode 6 and 7).

In a more general vein, it is important to note that movement through the phases posited by appropriation theory is not linear. That is, the student does not necessarily move in a straightforward direction towards a more coherent usage of signs as time passes. We see this clearly with regard to David's responses to Question 11; in answering this (non–procedural) question, David moves from a pseudoconceptual usage of signs (episode 8) to an incoherent usage of signs indicative of complex thinking based on a seemingly arbitrary association (episode 12).

To elaborate: Although David's initial response to Question 11 (episode 8) is acceptable (he seeks a counterexample so that he can prove that the proposition is false), he is unable to find a suitable counterexample. Accordingly he decides to move on and return to the question later. Although he does not solve the problem, his general approach to the problem is correct and I argue that it implies a pseudoconceptual way of dealing with divergence.

But when David later returns to this same question (episode 12), he adopts a different approach. That is, he asserts that the proposition is true; to prove this he makes unjustified and illogical connections between various signs so that he can argue (incorrectly) his case. Indeed this second attempt at answering Question 11 demonstrates how a "good" student like David, who is usually able to justify what he is doing, may revert to non–logical associations when he is grappling with a particularly elusive mathematical object. In Chapter 8, I discuss how these non–logical and ostensibly arbitrary associations may be the fore–conceptions of conceptual knowledge.

David's response to Question 6 (episodes 9 and 10) is also illuminating in that it shows how David's usage of signs around the improper integral evolves from his usage in Question 5[23]. In Question 5, as discussed above, David uses ambiguous and idiosyncratic notation when dealing with the improper integral; in question 6(a) (episode 9) he no longer uses the letter x ambiguously although his usage of the limit notation is idiosyncratic. However by the time he gets to Question 6(b) (episode 10), David's notation is unambiguous and correct. Indeed, and as I discuss in Chapter 8, evolution of notation may be indicative of a student's mathematical enculturation. Notwithstanding David's evident enculturation, I suggest in

[23] Question 5 and 6 are similar in that they both ask the student to determine whether certain improper integrals are convergent or not. However the improper integrals are Type I in Question 5 whereas in Question 6 they are Type II.

the analysis that David's usage of signs around convergence and divergence is still predominantly pseudoconceptual.

In Question 8 (episode 11), David has to interpret an improper integral Type II graphically. This is a similar question to Question 3 (episode 3) in which he has to interpret an improper integral Type I graphically. David's response to Question 8 is particularly significant in that it shows how David's usage of signs around the notions of convergence and divergence evolves from predominantly complex thinking with chain associations (in episode 3) to complex thinking with predominantly graphical associations (in episode 11). In particular, in episode 11 David uses a visual analogy to argue that the improper integral converges if the areas of the slices between the curve and the x–axis, for successive intervals of equal width, decreases as x tends to a limit; otherwise it diverges. This argument is actually incorrect, but given that complex thinking with a graphical association stands at the cusp of conceptual thinking (in that it is free from contradiction), I maintain that David's change in usage of signs (in his graphical explanation of convergence and divergence) is indicative of progress and intellectual growth.

In episode 13, David looks at Question 3 (episode 3) again. His argument here is very similar to his argument given in Question 8 (episode 11) and he uses the same (incorrect) visual analogy as he did in Question 8. Accordingly, and as with Question 8, I classify his usage of signs (in his graphical explanation of convergence and divergence) as dominated by complex thinking with graphical associations.

In Chapter 8 I pick up certain threads interwoven in the above synopsis and I use these to elaborate further on appropriation theory and its edifying power.

§7.5.2 TABULAR SUMMARY OF ANALYSIS OF DAVID'S INTERVIEW PROTOCOL

I present here a tabular summary of my categorisations of the different phases through which David moves as he goes about appropriating an improper integral.

As I discussed when presenting a similar summary for John (§7.3.2), the summary is intended as an index for reference purposes rather than as information for interpretation. Indeed, as previously argued, an interpretation based directly and simply on this table would be deceptive and unhelpful.

Table 4: A summary of the different phases through which David moves

QUESTION NUMBER	EPISODE NUMBER	PHASE
Question 2(a)	1	surface association
Question 2(a), 2(b)	2	pseudoconcept
Question 3	3a	chain
Question 3	3b	template
Question 3	3c	chain
Question 4	4	template
Question 5(a)	–	pseudoconcept
Question 5(b)	5	pseudoconcept
Question 5(c)	–	pseudoconcept
Question 9	6a	surface
Question 9	6b	concept
Question 10	7a	concept
Question 10	7b	concept, pseudoconcept
Question 10	7c	not classified
Question 11	8	pseudoconcept
Question 6(a)	9a	template
Question 6(a)	9b	graphical association
Question 6(a)	9c	pseudoconcept
Question 6(a)	9d	template
Question 6(b)	10	pseudoconcept
Question 7	–	graphical association
Question 7	–	template
Question 8	11	graphical association
Question 11: second pass	12	(artificial) association
Question 3: second pass	13a	graphical association
Question 3: second pass	13b	pseudoconcept

§7.6 CONCLUDING COMMENT

In this chapter I have used appropriation theory to illuminate how two different students zigzag through the different phases posited by appropriation theory as they each go about appropriating a new mathematical object in an interview situation.

Before moving on to Chapter 8, where I discuss the analyses and use them as the basis for various demonstrations and arguments, I wish to make two brief but very important points.

- I maintain that the very fact that I was able to apply my modified version of Vygotsky's stages of concept formation (ie appropriation theory) to the analyses of two very different students, each with their own mathematical profiles and personal history, indicates the applicability of appropriation theory to mathematical activity at a tertiary level.
- I maintain that these analyses, in and of themselves, demonstrate the illuminating power of appropriation theory in the mathematical domain. Specifically by presenting an up–close and detailed interpretation of each of John's and David's usage of signs, I contend that the analyses give the researcher or teacher great insight into the way these different students appropriate (or partially appropriate) a new mathematical object.

CHAPTER 8: DISCUSSION

§8.1 INTRODUCTION

My primary aim in this chapter is to highlight and expand on various themes which were interwoven into the analyses of Chapter 7.

To do this I will

- demonstrate that the mechanism for mathematical object appropriation is functional use (Vygotsky, 1986) of the mathematical sign. Related to this, I will demonstrate that a functional use of mathematical signs may lead to enculturation and the development of personal meaning.
- discuss the role of fore–conceptions in the appropriation of a mathematical object and the mechanism whereby these fore–conceptions are transformed into more acceptable forms.
- discuss the importance of the social context to the appropriation of a mathematical object.
- demonstrate how appropriation theory may illuminate the prevalence of certain usages of signs (or forms of thinking) in the activities of different students as these students go about appropriating a mathematical object.
- confirm minor modifications to appropriation theory (which I anticipated in Chapter 6).

I will use the episodic analyses of Chapter 7 as the basis for my demonstrations, elaborations and modifications.

Before commencing with these discussions, I would like to clarify the following overarching supposition: I assume that the appropriation of a mathematical object by a learner necessarily involves a production of both personal and social meaning[1] for that learner. Likewise I assume that a production of personal and social meaning of a mathematical object for a learner necessarily involves that learner's appropriation of that mathematical object.

§8.2 THE MECHANISM OF MATHEMATICAL OBJECT APPROPRIATION: AN ELABORATION AND A DEMONSTRATION

In this section I argue that it is through the functional use of particular mathematical signs that a learner appropriates a new mathematical object. Furthermore it is functional use that enables the learner to move through the different phases of appropriation theory, as he constructs a concept.

By functional use, I am referring to activities with mathematical signs such as manipulations, reflections, template–matchings, associations and imitations[2].

In terms of appropriation theory, functional use of the mathematical object can be understood in terms of an analogy with a child learning how to use new words. Vygotsky argues (1986, 1994) that the child starts to use the new word in communication before it has developed into a fully–fledged concept for that child.

Similarly I claim that a learner starts to use a new mathematical sign in mathematical pursuits such as problem–solving, applications, proofs, communication and so on before he 'knows' the mathematical object

[1] A sign has social meaning if its usage is compatible with its usage by the mathematical community.

[2] To avoid confusion, it is worth stressing that template–matching is a mathematical activity; on the other hand, complex thinking with a template–orientation is a theoretical phase in mathematical thinking. Similarly, the student may use association in his mathematical activities; associative complexes refer to a particular set of phases in mathematical thinking.

referred to by that sign. That is, the learner initially uses the sign of the new mathematical object functionally whilst having only a faint notion of the mathematical object.

Through this use of the mathematical sign, the learner is able to engage with the mathematical object and to communicate with others about his developing mathematical ideas. My claim is that on account of this functional use, regulated by social discourse (including the use of pedagogically–designed materials such as a textbook), the mathematical sign begins to acquire personal meaning for that learner and he begins to understand how it is used in mathematical discourse[3]. In other words, the learner begins to appropriate the mathematical object.

At this juncture it is apposite to remind the reader that the purposefully–designed materials in the clinical interviews consisted of the mathematical task and the textbook (Larson et al, 1998). As previously discussed (see §2.3), I regard the actual written task and the textbook as important tools for semiotic and social mediation in the ZPD. In particular, as I discussed extensively in §5.5.4, the different activities in the task were designed so as to enable rich and diverse functional usages of the mathematical signs. That is, the activities and the order of presentation of activities in the task were specifically devised so that there were ample possibilities for reflections, manipulations, associations, template–matchings, imitations and perturbations[4].

In demonstrating the learner's functional use of a mathematical sign, a very important epistemological point is the following: although it is impossible for a researcher to determine with any certainty whether a particular mathematical sign is personally meaningful to a learner, it is possible to decide with reasonable confidence that certain usages are *not* personally

[3] How the mathematical sign is used in mathematical discourse is an important aspect of its social meaning.

[4] In contrast, a task consisting of, say, procedural exercises only, would allow for a very limited functional use of the mathematical signs.

meaningful to a mathematics student at tertiary level. For example I believe that a usage of a mathematical sign that is unstable (that is the usage changes from one moment to the next) and incoherent (that is the usage contains gross inconsistencies and illogical links) is highly unlikely to be personally meaningful to a tertiary–level mathematics student.

In contrast to this, social meaningfulness is relatively easy to judge. Its measure is the extent to which the learner's usage of a particular sign is consonant with the usage of that sign by the mathematical community (of which I consider myself a member).

I will now use the episodic analyses in Chapter 7 to demonstrate how each of John and David go about appropriating a new mathematical object through a functional usage of the relevant mathematical signs. I will also show how functional use is the mechanism which enables the student to move through the different phases of the appropriation map, thereby developing personal and social meaning for this object, as they appropriate it.

Demonstrations

I will present four different examples here of how functional usage of a mathematical sign by a student leads to a change in the meaningfulness of that sign. This change in meaning may be on a personal level and/ or it may be on a social level.

Allied to this, in three of these examples, I will show how functional usage of a mathematical sign enables the student to move through different phases of the appropriation map as he appropriates an object. In the fourth example, I will demonstrate how functional usage enables enculturation of the student into the mathematical community.

All the examples derive from the analyses in Chapter 7. Accordingly all the episode numbers refer to the relevant episode numbers in Chapter 7 (obviously with due regard for whether the episode is drawn from the analysis of John's or David's activities).

John's movement from a template–orientated usage to a pseudoconceptual usage of an improper integral

Initially John does not seem to have a reasonable sense of the improper integral Type I. This is evidenced by his initial generation of (objectively meaningless and inconsistent) strings of signifiers such as $\int_0^\infty f(x)dx = \lim_{2\to\infty}\int_0^2 f(x)dx = \lim_{2\to\infty}\int_0^2 \sqrt{x}dx = \int_0^2 \sqrt{x}dx$ (episode 1) and $\lim_{2\to\infty}\int_0^2 f(x)dx = \lim_{2\to\infty}\int_0^\infty \sqrt{x}dx = \int_0^\infty \sqrt{x}dx$ (episode 2).

But despite John's idiosyncratic and mathematically unacceptable use of these signifiers, he is able to use these signifiers in mathematical activities. Specifically, in episode 1 John manipulates the template of the improper integral so that it eventually has the form of a definite integral (ie $\int_0^2 \sqrt{x}dx$), a form with which he is familiar. He then manipulates this definite integral signifier according to the rules for a definite integral.

In episode 2 John uses template–matching to generate the expression, $\lim_{2\to\infty}\int_0^\infty \sqrt{x}dx$ as an example of an improper integral. Although this is a bizarre expression, progress has been made in that the final term in the string of signifiers has the template of an improper integral, ie $\int_0^\infty \sqrt{x}dx$, rather than a definite integral. Notwithstanding this template, John calculates the term using the rules for a definite integral. That is, he uses ∞ as if it were a real number.

Thus we see how John is able to use an immature version of the improper integral in mathematical activities in both episodes 1 and 2 even while using it incorrectly. In the analyses I classified John's usage of signs in both episodes 1 and 2 as primarily indicative of complex thinking with a template–orientation. (To remind the reader, a learner using complex thinking with a template– orientation, slots new signifiers into spaces reserved for old signifiers, and then manipulates the new signifiers in terms of the rules for the old signifiers.)

The point is: by using various signs in mathematical activities (a functional usage involving template–matching and manipulations primarily) John is able to engage with the mathematical object on first contact, albeit an undeveloped version of this object. In this way, John gains a point of entry into mathematical activities with the object before he ‘knows’ that object.

Notwithstanding that functional usage gives John a way of engaging with the new object, his usage of the object in these episodes is actually incorrect (in an objective sense). I also assume that this usage is not particularly meaningful to John. For example, he does not seem to recognise that he has interpreted the peculiar expression $\lim_{2\to\infty}\int_0^2 f(x)dx$ in two entirely different ways in episodes 1 and 2.

The question now is: how does John move from this idiosyncratic (and, as I suggested, not personally meaningful) usage to a usage which is both personally satisfying and socially acceptable?

I suggest that the answer lies in John’s imitation of the improper integral sign.

Specifically, it is only after John has seen prototypes of improper integrals in the textbook (in episode 6) that he starts to use the improper integral in a way that is consonant with its definition (ie if f is continuous on the interval $[a, \infty)$, then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$). To elucidate: While looking at the worked examples in the textbook (episode 6), John states that these example are useful to him, and he indicates that his notation for the improper integral has been incorrect up till then (which it has). Subsequently (in episode 7) John’s usage of the improper integral notation becomes consistent with the definition for the first time during the interview.

My contention is: it is John's functional use of the improper integral sign (specifically imitation) that enables him to move from a usage indicative of complex thinking with a template–orientation to a pseudoconceptual use (in episode 7).

A related contention is: John's change in use of notation (from episode 1 to episode 7) is indicative both of an evolution of his concept of the improper integral and of a process of enculturation into the community of mathematicians.

John's movement from a heap–type usage to a pseudoconceptual usage of convergence and divergence

When considering John's appropriation of a convergent improper integral and a divergent improper integral, we see, from the analysis in Chapter 7, that John's functional usage of the relevant mathematical signs enables him to move from a heap–type usage of signs (which is mathematically incorrect and presumably not meaningful on a personal level[5]) to a pseudoconceptual usage (which is correct and possibly personally meaningful).

Specifically, in his initial response to Question 4 (episode 4), John uses signs in a heap–type way when arguing for a definition of an improper integral with a lower infinite limit which converges when the limit does not exist, and diverges when the limit exists. (This argument is wholly incorrect: the concept of convergence and divergence necessitates that $\int_{-\infty}^{b} f(x)dx$ diverges if the limit *does not* exist, otherwise it converges.)

My question is: how does John move from this heap–type usage of signs to a pseudoconceptual usage of signs whereby he can at least use notions of convergence and divergence in a socially acceptable and a reasonably

[5] For example, in episode 6 John tells me that he does not understand convergence and divergence.

senseful way (as he does in determining whether certain improper integrals converge or diverge in episodes 15 and 16)?

Examining the analysis in Chapter 7, we see how, despite John's (objectively) distorted usage of convergence and divergence (as articulated in episode 4), he proceeds to engage in mathematical activities with these notions. In particular, he attempts question 5 (episodes 5, 7, 10 and 12) in which he has to determine whether certain improper integrals converge or not. In these determinations John uses manipulation and template–matching with his (incorrect) answer to Question 4.

However, after several confused attempts at Question 5, John states that he thinks that his answer to Question 4, on which he is currently basing his activities, was incorrect (episode 14). He tells me that he thinks that he should have used divergence when he used convergence in his answer to Question 4, and vice versa.

My contention is that John's functional usages of convergence and divergence (particularly in episodes 5, 7, 10 and 12) enable him to develop suitable doubt about his initial usage of convergence and divergence.

I support this contention with the following argument: When John first looks at the definition of an improper integral Type I with a lower infinite limit[6] in the textbook (episode 6), it does not seem to affect his usage of convergence and divergence. For example, he does not express doubts about his initial answer to Question 4 and he goes on to answer Question 5(b) according to the template answer to Question 4 (episode 10). I suggest that this inability to appreciate how the textbook definition of an improper integral (episode 6) contradicts the definition he gave in response in Question 4 (episode 4) indicates that John was not ready at that stage (episode 6) to be perturbed by the correct definition of convergence or divergence. Indeed it is only after John has functionally used signs relating to the convergence or divergence of an improper integral (in episodes 5, 7,

[6] This is the same improper integral that he was asked to define in Question 4.

10 and 12) that he seems to be effectively perturbed by his original answer to Question 4.

Consequent to this perturbation, John revises the answer to Question 4 (episode 14) using an analogy with an improper integral with an upper infinite limit (that is, he uses template–matching). He is then able to use notions of convergence and divergence in procedural questions in which he has to decide whether a particular improper integral is convergent or not (for example, episodes 15, 16 and 18).

Thus John has achieved a way of using notions of convergence and divergence that is socially acceptable and possibly personally meaningful (being stable and consistent). In the analysis I classified this usage as pseudoconceptual.

The main point of this example, though, is the demonstration that through functional usage of the mathematical signs, John is able to communicate about the convergence or not of an improper integral and to engage in certain activities with this object no matter that he does not seem to have a fully matured or comprehensive concept[7] of a convergent or divergent improper integral.

David's movement from a chain association type usage to a graphical association type usage of convergence and divergence

Here I will look at how David's usage of mathematical signs evolves from a usage which is indicative of complex thinking with chain associations (episode 3) into a usage which is indicative of complex thinking with graphical associations (episodes 11 and 13) when he is explaining how he graphically interprets the convergence or divergence of an improper integral. This evolution is particularly significant in that, as I pointed out in Chapter 6, complex thinking with graphical associations stand at the cusp of conceptual thinking. And by definition, conceptual thinking is allied with

[7] For example, by the end of the interview John is still not able to use notions of convergence and divergence in non–procedural problems.

personally meaningful and socially acceptable usage of mathematical signs.

Specifically, on his first attempt at Question 3 (episode 3), David declares that the function in the improper integral has to oscillate (this is incorrect). He then argues that the associated improper integral converges if the gaps between the graph and the x–axis get smaller and smaller as x increases; the associated improper integral diverges if the gaps between the graph and the x–axis get larger and larger as x increases. This is an idiosyncratic interpretation of convergence or divergence and, as I proposed in the analysis, it seems to relate to the diagrams of oscillating sequences which David is studying in lectures at the time of the interview. (Oscillating sequences converge or diverge depending on whether the absolute gap between successive points and the x–axis gets smaller and smaller or larger and larger respectively.) As such I classified this usage as indicative of complex thinking with a chain association in which the words “converge” and “diverge” are chained to notions of convergence and divergence of oscillating sequences (see episode 3 in Chapter 7 and example 3 in Chapter 6 for extended discussion and justification of my classification).

However by the time David gets to do Question 8 (episode 11)[8], his explanation of convergence and divergence is based on a visual analogy (that is, he is using complex thinking with graphical associations). I suggest that it is David’s functional use of the signs of convergence and divergence (for example, his use of manipulations in episodes 5 and 9, his reflections on aspects of convergence and divergence in episodes 7 and 8 and his use of associations in episode 3) that has enabled this evolution.

Although David’s graphical explanation in response to Question 8 (episode 11) is still incorrect, it is nevertheless consistent and free of obvious

[8] I am comparing David’s usages of signs in Question 3 (episode 3) to Question 8 (episode 11) because both questions involve a graphical explanation of an improper integral. In question 3 the focus is the improper integral Type I; in Question 8 the focus is on the improper integral Type II.

contradictions. Specifically David draws an analogy between the increase or decrease in area between the function of the improper integral and the x–axis for intervals of constant width as x tends to the point of discontinuity, and the respective divergence or convergence of the improper integral.

A little later, David attempts Question 3 again (episode 13). His graphical explanation is very similar to that which he gave to Question 8 (episode 11) and he also states explicitly that the function in the improper integral does not have to oscillate. As in episode 11, David's explanation is consistent and free of obvious contradictions (albeit incorrect). Again I maintain that his explanation is based on a visual analogy and that David is using complex thinking with a graphical association (see episode 13 for justification of this classification).

In summary, my argument is that David's initial usage of the signs of convergence and divergence, which were indicative of complex thinking with chain associations (episode 3), have evolved through functional use (in particular manipulations, reflections and associations) into complex thinking with graphical associations (in episodes 11 and 13). Although these graphical associations are neither abstract nor logical (and are incorrect), they are consistent. Indeed a graphical explanation based on a visual analogy is far closer to a conceptual explanation than a chain association which is based on the isolation of an aspect of a mathematical statement (such as a particular sign) and the consequent association of this decontextualised sign with a new sign.

Functional use enables David's enculturation

First I must make clear that the enculturation of the student into the mathematical community necessarily involves a development of usage of signs that is compatible with the usage of those signs by the mathematical community. In other words, and perhaps more consistent with my previous phraseology in this chapter, enculturation involves the development by a learner of a social meaning of a mathematical sign.

In the example below I demonstrate how David moves from an initial position in which he uses the improper integral Type II notation ambiguously (see for example Episode 9) to a position in which his use of this notation is in line with that of the mathematical community (see for example, Episode 10) [9]. I argue that this change is enabled by David's functional use of the improper integral Type II sign in episode 9 (no matter that this usage is ambiguous). This functional usage takes the form of manipulations with and mindful reflections on the improper integral Type II sign (see particularly lines 147 and 148 in episode 9 where David explains his notation).

Specifically David writes expressions like ' $\lim_{c\to 0^+} \ln 0$ ' (he should be writing ' $\lim_{c\to 0^+} \ln c$ ') when doing Question 6(a) (episode 9). Although the term $\lim_{c\to 0^+} \ln 0$ is presumably meaningful to David (he is able to use expressions of this form successfully) this is not a conventionally acceptable form, containing as it does redundant terms like ' $\lim_{c\to 0^+}$ ' and terms which are ill–defined like 'ln 0'. Certainly if David persisted in this usage, his enculturation into the mathematical community would be unsuccessful and communication would be inhibited.

But by the time David gets to Question 6(b) (episode 10), his usage of the limit notation is fine and he appropriately writes (objectively meaningful) expressions such as $\lim_{c\to 0^-} \left[3x^{\frac{1}{3}}\right]_{-8}^{c}$.

In summary, my argument is that David's functional use (primarily manipulations and reflections) of the improper integral Type II sign in mathematical activities in episode 9 enables his mathematical enculturation.

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[9] I suggest in my analyses that both usages (in episodes 9 and 10) can be classified as pseudoconceptual in that they both imply an understanding of the improper integral Type II that resonates with its usage by the mathematical community.

§8.3 MATHEMATICAL OBJECT APPROPRIATION AND FORE–CONCEPTIONS

Appropriation theory implies that although the student may flounder around making what appear to be idiosyncratic, arbitrary and illogical associations based on certain attributes of the signifier or signified, these may be necessary steps on the path to object appropriation; the "fore–conceptions" (Sierpinska, 1992: 28) of personally meaningful conceptions whose use is also acceptable to the mathematical community[10].

In the analyses of both John's and David's protocols we see several examples where the student starts off using apparently random and arbitrary associations in their appropriation of a new mathematical object. Analyses of the students' usage of signs (as given in Chapter 7) shows how these apparently random and arbitrary associations (that is, fore–conceptions) evolve into pseudoconcepts through a functional use of the mathematical signs.

Demonstrations

I will illustrate how certain of David's and John's fore–conceptions evolve into a more coherent and consistent form through functional usage of the related signs. To do this I will use two examples, one of which I have already looked at in this chapter (but with a different focus).

Specifically I will examine how the fore–conceptions in John's initial response to Question 4 evolve into a pseudoconcept of convergence and divergence. And I will look at how David's fore–conception of an improper integral Type I evolves into a pseudoconcept of an improper integral.

John

[10] These fore–conceptions also relate to the prior mathematical history of the individual student.

In §8.2, I discussed how John moved from a usage of convergence and divergence in his initial response to Question 4 (episode 4) which was indicative of heap–type thinking, to a pseudoconceptual usage of these terms in Question 5(b) and Question 5(c) (episodes 15 and 16).

I will not repeat the description of events here (the reader is advised to see §8.2 above). Rather I would like to stress that John's fore–conceptions around the convergence and divergence of $\int_{-\infty}^{b} f(x)dx$ are initially incoherent and idiosyncratic. For example, John's argument that $\int_{-\infty}^{b} f(x)dx$ is the "opposite" of $\int_{a}^{\infty} f(x)dx$ and so $\int_{-\infty}^{b} f(x)dx$ diverges if the limit exists, otherwise it converges (episode 4) is bizarre and wholly incorrect.

But, as I argued in §8.2, these usages of convergence and divergence evolve through functional use (in particular template–matching and manipulation) and John is eventually able to re–define the improper integral with a lower infinite limit (episode 14) appropriately.

Consequently John is able to use notions of convergence and divergence in a socially acceptable (although limited) way; for example when deciding whether the improper integrals in Questions 5(b) and 5(c) converge or not (episodes 15, 16). In these episodes I classified his usage as pseudoconceptual.

In summary, I maintain that John's idiosyncratic fore–conceptions of convergence and divergence evolve through functional use into a pseudoconceptual and socially acceptable form. Moreover, John is ultimately able to apply these pseudoconcepts in determining whether a particular improper integral is convergent or not.

David

In answering Question 2(a) (episode 1), David argues that an improper integral Type I always has to have a limit that exists. That is, David's fore–conception of the improper integral Type I is of an improper integral that is necessarily convergent. This is, of course, incorrect. An improper integral may be convergent or divergent.

In the analysis, I suggested that David's fore–conception was indicative of complex thinking with a surface association in which he isolated (as evidenced by his underlining) the words 'converge' and 'exist' as he read the definition of an improper integral Type 1 (episode 1).

However when David reads question 2(b) (episode 2) in which he is asked to generate a convergent improper integral, this fore–conception seems to be perturbed and David acknowledges that an improper integral is not necessarily convergent (episode 2).

Thus, through a functional use of the improper integral (in particular, manipulations in episode 1 where he evaluates the integral and reflections in episode 2 when he is perturbed), I suggest that David's notion of the improper integral Type I evolves into a pseudoconceptual form (episode 2). From then on, David is clearly aware that an improper integral may diverge or converge.

§8.4 MATHEMATICAL OBJECT APPROPRIATION AS A FUNCTION OF THE SOCIAL CONTEXT

In this section I wish to argue that mathematical object appropriation is not just a function of 'pure' cognitive ability or prior mathematical knowledge or the specific design of the mathematical task; rather it is a multi–determined phenomenon of which the social relations[11] in the teaching–learning situation are an important contributing factor.

In Chapter 5, I argued that a clinical interview, like any learning–teaching situation, is a social event. As such it does not take exactly the same form

[11] In this context, I am using the term 'social relations' to refer to the way in which the student and interviewer interact. These ways are themselves determined by a plethora of factors such as the social, academic and economic standing of each person and their relative access to the academic discourse and the language of instruction.

for each student; rather the quality and mode of interaction between interviewer and student, even when constrained by broad guidelines (see Chapter 5), are unique to each interview. Indeed the particular interactions in the interview both reflect and contribute to the nuances in relationship which each student has with the interviewer. These interactions in turn point to the differences in the quality of the teaching–learning relationship between the teacher and the student. In Vygotskian terms, the zone of proximal development that is constituted in the interview necessarily differs from student to student. Consequently I maintain that the possible quality of mathematical learning (or mathematical object appropriation) is a function, inter alia, of the type of social interactions that take place during the interview (or any learning–teaching situation). As Goodnow (1993: 373) argues, albeit in a more general sense:

> It is the nature of one's positions, of one's participation in the social life of the group, that influences the extent to which one picks up, and appropriates as one's own, the skills and ways of thinking valued by the group.

In this section I hope to demonstrate how differences in John's and David's level of academic enculturation into the University of the Witwatersrand environment contributed to a difference in the quality of the modes of interaction between interviewer (myself) and student. I will do this briefly by comparing the quality of some of my interactions with each of these students.

Overall I felt that I was able to engage more successfully and richly with David than with John in the interview. With John much of my intellectual energy was spent on clarifying what he was trying to say or do; with David, on the other hand, most of my intellectual energy (both during the interview and in its analysis) was spent on interrogating or interpreting his mathematical thinking. Objective support for this interpretation can be found in terms of the number of times in which I had to ask John to clarify what he was saying (because I genuinely did not understand what he was getting at) compared to David. In the first 45 minutes of John's interview, there were seven such instances [lines 73, 75, 82, 84, 94,129, 143]; in the first 45 minutes of David's interview, there was only one such instance [line 63].

I maintain that this disparity in my interview–relationship with the students was a result not only of their apparently different mathematical abilities, but also a result of the extent to which each student was enculturated into the academic environment at the University of the Witwatersrand. Furthermore I suggest that this disparity in relationship is echoed in the various relationships which different students form with their lecturers at the University of the Witwatersrand and consequently affects the overall quality of that student's learning.

Specifically, I maintain that students like David, who come from 'good' schools and have similar cultural backgrounds to lecturers like myself, are culturally at ease with lecturers such as myself and the academic environment of the University of the Witwatersrand. For example, David speaks English fluently (it is his primary language); he has grown up around people who have studied at the University of the Witwatersrand or similar universities; all of the teachers at his school have themselves been to university. All David really needs to attend to at university is his mathematical enculturation. John, on the other hand, is an outsider at the University of the Witwatersrand. He comes from a rural background, where very few, if any, members of his community have gone to university. The vestiges of apartheid mean that most, if not all teachers at his school have gone to colleges of education rather than universities like the University of the Witwatersrand. Furthermore, John struggles with English, which is not his primary language and he has great difficulty in articulating his thinking.

Indeed, although both John and David received A symbols for their Matriculation level Higher Grade Mathematics examinations, John fails Mathematics I Major in his first attempt (in 2000) whereas David attains a very good grade of 81% for the year. I suggest that this is partly because in his first year at university John needs to attend to both his mathematical and his academic acculturation. That is, besides learning mathematics, he needs to learn ways of being in an academic environment, how to speak

English fluently and how to articulate himself, etc. Only in his second year can he devote himself fully to his mathematical growth[12].

In summary, I contend that more effective zones of proximal development are constituted in situations with academically enculturated students. In such situations the student can devote his attentions to the appropriation of mathematical objects without specifically having to attend to the language of instruction or implicit academic norms; likewise the teacher is able to engage more richly with the student, not having to devote her energies to deciphering what he is saying.

§8.5 PREDOMINANT PHASES

In this section I wish to discuss how certain forms of thinking seem to predominate in each of John's and David's protocols (as revealed by the analyses). I do this in the belief that such information would be useful for a teacher attempting to understand how a particular student appropriates new mathematical objects. Furthermore this discussion should also highlight the edifying power of appropriation theory in the mathematical domain.

To support this discussion, I have graphically represented the frequency of the occurrences of different phases in each of John's and David's protocols for questions 1 to 5 and Questions 9 to 11 (that is, questions around the improper integral Type I)[13] in Figures 5 and 6 below.

Before the reader attempts to interpret these figures, a few comments must be made:

- The data in the figures derives mainly from the episodic analyses given in Chapter 7, specifically the analyses of the student's responses to Questions 1 to 5, and Question 9 to 11 during the official interview (1.5 hours). However since certain episodes were omitted in those

[12] This argument is supported further by the fact that John received a final grade of 95% in his second attempt at Mathematics I Major in 2001. This is an excellent grade.

[13] I have used these particular questions because both students attempted all these questions at least once.

analyses[14] and David's responses to Questions 6 and 8 were also included in the analyses (but excluded from the data used for Figure 5), the correspondence between those analyses and the figures may not be immediately obvious.

- Since the particular path any individual student takes is dependent on, inter alia, the intersection of the task with that student's prior knowledge (which necessarily will differ for John and David considering that each student has gone to very different sorts of schools), the social relations in the interview context (see discussion in §8.4) and the particular student's attitudes and motivations, a simple comparison of Figure 5 and Figure 6 is misleading and highly simplistic. Accordingly I will not attempt such a comparison.

David's profile

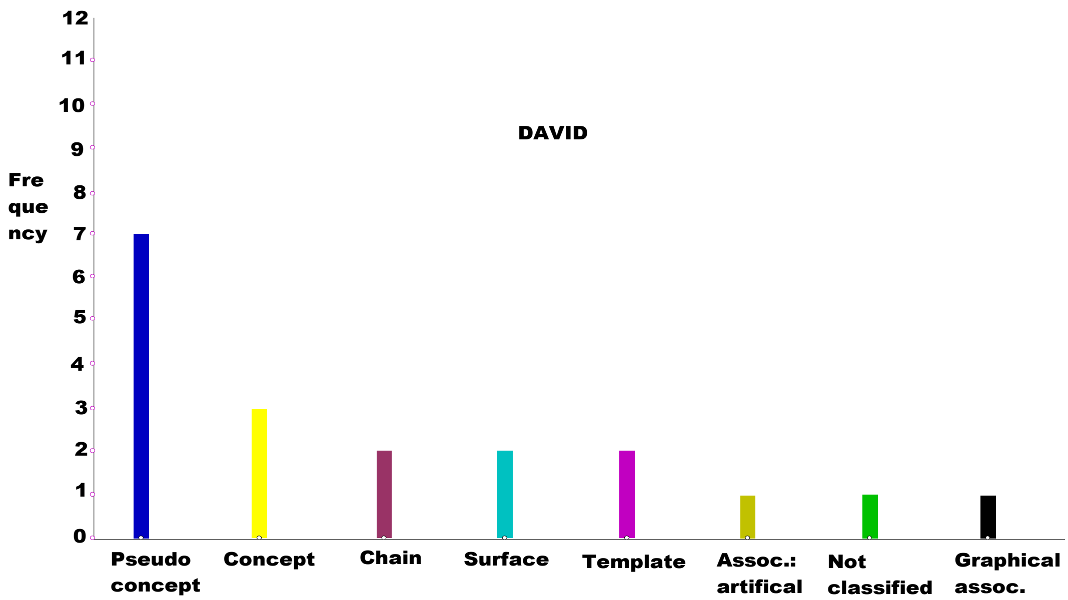

Figure 5: Frequency of occurrence of different phases in David's protocol (Questions 1 to 5, 9 to 11)

Pseudoconcept

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[14] For example, in the analysis of David's activities, I only classified his usage of signs with respect to Question 5(b) arguing that the classifications of his usage of signs in Questions 5(a) and 5(c) were very similar. In Figure 5 I have included these latter two classifications (of pseudoconceptual usage).

The prevalent form of thinking in David’s profile is pseudoconceptual (about 37% of the classifications).

Instances of pseudoconceptual thinking are very important for several reasons:

First because a pseudoconcept has the appearance of a concept (albeit the structure of a complex), it allows for effective communication between the student and various social agents (such a teacher, a peer or a textbook) about the mathematical object. Such communication is, in terms of appropriation theory, necessary for further development and maturation of the meaning of the relevant mathematical sign.

Related to this, a pseudoconcept stands at the cusp of conceptual thought. As such, I suggest that it indicates that the student, through further functional use of the relevant mathematical signs, will ultimately gain reasonable epistemological access to the mathematical object.

Thus the preponderance of pseudoconcepts in David’s protocol is particularly significant, allowing as it does for David to communicate effectively about the mathematical objects and portending future conceptual thinking.

Furthermore, pseudoconceptual usage often leads to successful performance (as in the case of David) and consequently may be motivating and confidence–building.

In contrast to the above positive features of pseudoconceptual thinking there is a negative aspect to the display of this type of thinking. Because of the dual nature of the pseudoconcept both teacher and learner may confuse pseudoconceptual thinking with conceptual thinking; indeed they may take a pseudoconceptual usage of mathematical signs as evidence that the student ‘understands’ the mathematics. In that case, opportunities for further conceptual development and mathematical exploration for that learner may be lost.

Concept

We also see three instances of conceptual thinking in David's profile. Thus more than half of David's usage of signs are conceptual or pseudoconceptual. I suggest that this partly explains why David is so performance–wise successful and confident in his Mathematics I Major course.

Surface, chains and other complexes

Finally I want to mention the presence of certain forms of complex thinking such as chain associations (two cases), surface associations (two cases), template–orientation (two cases), and association with artificial nucleus (one case) in David's profile. I mention these categories specifically because they often manifest as idiosyncratic or inexplicable usages of mathematical signs and so may appear as evidence of thoughtlessness or superficiality in the student[15].

The point I wish to make is that even very 'good' students like David use these ways of thinking when dealing with new mathematical objects (about 37% of David's responses were categorised in this way). Indeed these seemingly strange ways of using mathematical signs are often the fore–conceptions of true concepts (see §8.2.2). As such they should be treated with caution and interest by the relevant educator.

[15] I have omitted graphical association from this list since it does not usually manifest in idiosyncratic or bizarre ways.

John's profile

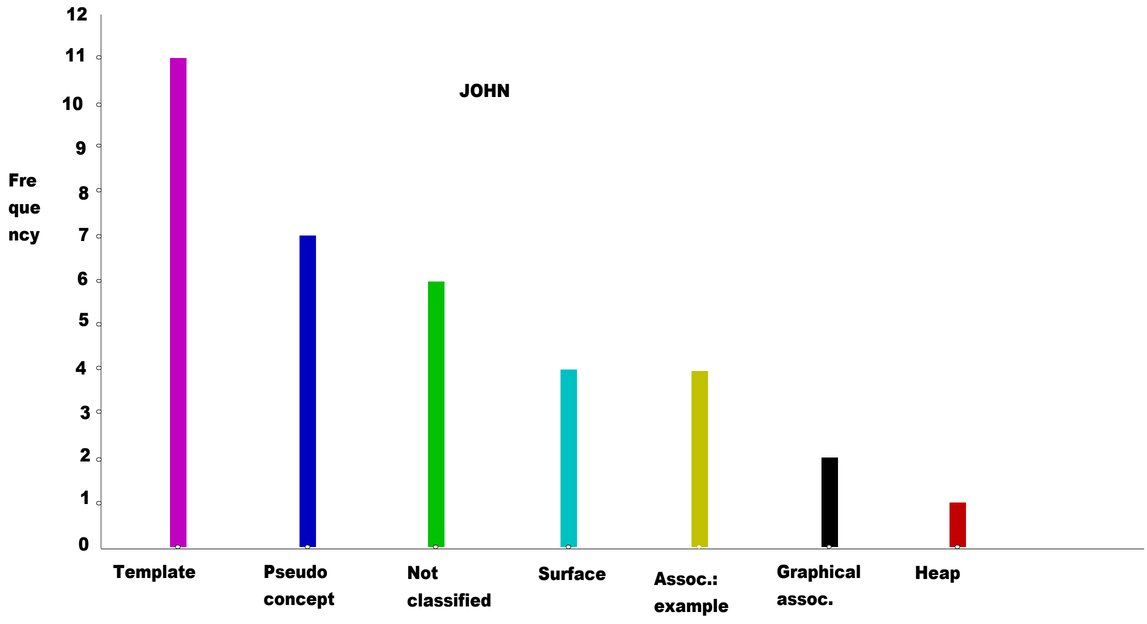

Figure 6: Frequency of occurrence of different phases in John's protocol (Questions 1 to 5, 9 to 11)

Template–orientation

Looking at Figure 6 (ie John's profile), we see how complex thinking with template–orientations dominate John's usage of signs (31% of John's responses were classified as indicative of template–orientations). Indeed in §8.2 I argued that complex thinking with template–orientations give John an entry point into activities with newly–defined mathematical objects, even before he 'knows' what those objects are.

To be sure such complex thinking with template–orientations provide the initial access to newly defined mathematical objects for many students (cf Sfard, 2000). The discerning feature of John's usage however, is the extent to which he distorts the various templates. For example, he uses the bizarre expression $\lim_{2\to\infty}\int_0^2 f(x)dx$ in both episodes 1 and 2 and he uses the equally peculiar expression $\lim_{2\to\infty}\int_0^\infty f(x)dx$ in episode 2. Nevertheless he is able to proceed with mathematical activities involving these signs.

In order to understand John's particular use of templates, it is interesting to look at the sort of distortions John brings to bear on these usages. For example, consider the template of the improper integral Type I, ie

$\lim_{b\to\infty}\int_a^b f(x)dx$, as given in the definition of an improper integral Type 1. John uses the template with seemingly little regard for whether the letters a and b represent variables or constants and which letters must be the same (for example, he seems oblivious to the fact that the letter b is used for both the upper limit and in the expression $b \to \infty$ in the definition). It is as if John has stripped the template of any detail and simplified it to its most basic form, ie

$\lim_{\square\to\infty}\int_\square^\square f(x)dx$. With this simplified template in hand he only needs to attend to the gross details.

In terms of teaching John or similar such students, the knowledge that such a student frequently uses templates to gain access to new mathematical objects (albeit frequently distorted templates), may be valuable to the teacher or the writer of curriculum materials.

Unclassified usages

Another interesting feature of John's profile is the number of cases categorised as 'not classified'. These cases mostly refer to episodes or sub–episodes in John's protocol which I was unable to classify because I was unclear, during the analysis, about what John was trying to say or do. As such it relates to my argument in §8.4 about the frequent difficulties I had with understanding what John was saying or doing during the actual interview.

Indeed a comparison with the analysis of David's profile is quite legitimate here, since this serves as further justification for why a general comparison between the analyses is not valid. As can be seen from Figures 5 and 6, there were six instances in the analysis of John's interview in which I was unable to decipher or interpret what John was doing; in David's interview there was only one such instance.

Pseudoconcepts

Another point of interest in John's profile is the relatively high number of episodes (or sub–episodes) which I classified as indicative of pseudoconceptual thinking. These constitute about 20% of the classifications.

As mentioned above, instances of pseudoconceptual thinking are very important because, although they may sometimes contribute to masking a lack of conceptual thinking, their use does allow for effective communication between learner and more knowledgeable other; this in turn enables further appropriation of the mathematical object. Related to this, their very existence serves to indicate that, with functional use of the relevant mathematical signs, the student is likely to develop conceptual thinking (that is, thinking in abstract and logical terms) about the mathematical object.

Indeed, in John's case, his relatively high number of pseudoconcepts augured his excellent performance in 2001 when he repeated the Mathematics I Major course.

§8.6 APPROPRIATION THEORY REVISITED

In the light of the analyses (Chapter 7), I would like to revisit aspects of appropriation theory which I have not referred to or discussed above. In particular I would like to examine the appropriation map in which I elaborated Vygotsky's theory of concept formation to the mathematical domain (see Chapter 6).

For ease of reference, I reproduce that map here:

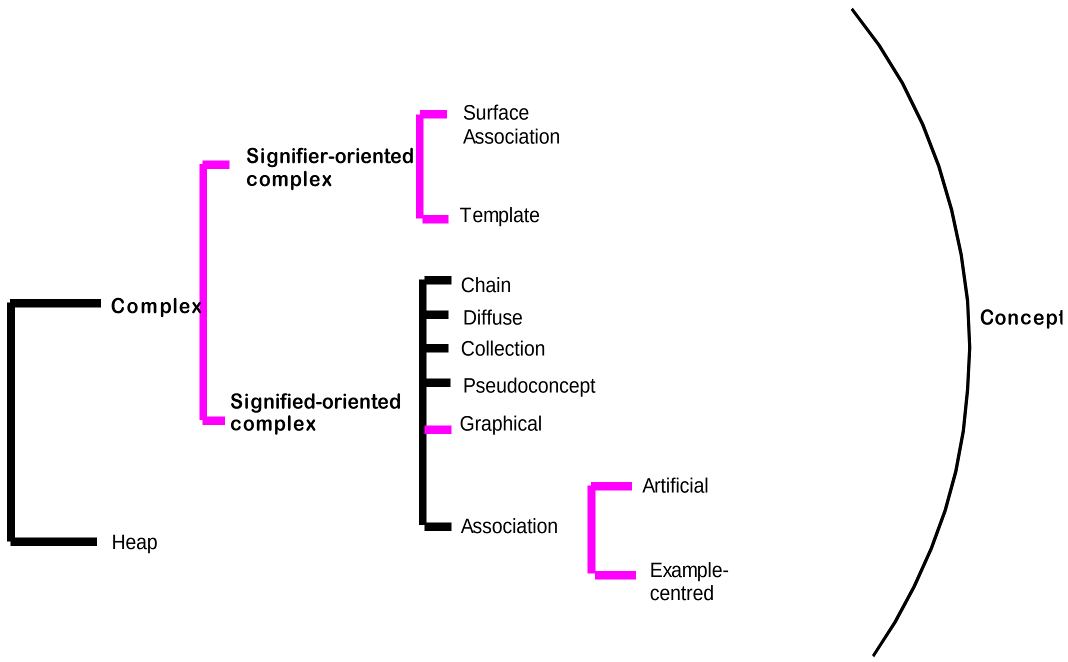

Figure 7: Phases in the appropriation of a mathematical object

Note: Pink–coloured lines represent those phases which are specific to the mathematical domain. Black–coloured lines represent those phases posited in Vygotsky's account of the different stages in concept formation

First, in my initial formulation of appropriation theory, I posited that a learner would zigzag through the different phases of the map in a non–linear fashion sometimes even returning to less coherent and idiosyncratic forms of thinking. Certainly this was evidenced in both interviews. In David's interview, for example, he moves from a pseudoconceptual usage of signs when doing Question 11 (episode 8) to complex thinking based on an apparently arbitrary association (episode 12) when doing that same question a second time. In John's interview he moves from a pseudoconceptual usage of convergence and divergence (episodes 15 and 16) to an inability to deal with convergence and divergence without an example of the described object (episode 20a), and so on.

Secondly, and as discussed extensively above, each student gained initial access to the mathematical object through a signifier–orientated complex. That is, John's initial use of the improper integral (and of convergence and divergence) was through complex thinking with a template–orientation (episodes 1 and 2); David's initial access was through complex thinking with a surface association (episode 1).

Thirdly, certain phases which I posited for appropriation theory were absent from both John's and David's protocols. In particular, I did not classify any of their activities as indicative of either the diffuse or the collection complex. Notwithstanding this absence in the two analyses, I was able to find one instance of the collection complex (in which a student confuses a mathematical operation with its inverse) in Tom's interview (see example 11 in Chapter 6) [16].

However, and as I anticipated in Chapter 6, I was unable to find any instances of the diffuse complex (in which a student associates one object with another due to a remote or vague similarity between the objects) in any of the seven interviews which were not better classified as examples of complex thinking with an artificial association (in which unfamiliar objects are associated with a familiar object even though the connection may not be logical or relevant). Indeed I contend that I have subsumed the diffuse complex into the more mathematically relevant category of complex thinking with an artificial association.

Interestingly both the diffuse and the collection complex were posited by Vygotsky in his theory of concept formation. However given that Vygotsky's theory was based on an experiment in which people had to classify concrete objects according to two different sets of signs, it is little wonder that some of his phases may be less useful in the mathematical domain. Indeed the categories I formulated specifically for mathematical activity such as complex thinking with a surface association, complex thinking with an artificial association, complex thinking with a graphical association and

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[16] Notwithstanding this claim, it must be acknowledged that my analyses of the five case study students (that is excluding John and David) were neither as thorough nor as rigorous as my analyses for John and David. That is I did not write down detailed descriptions and justifications such as I presented in Chapter 7 for John and David, when analysing the five other case study protocols.

complex thinking with a template–orientation were very useful in the analyses of the different student's protocols, and, I suggest, are far more applicable to the mathematical domain than the diffuse or the collection complex; in particular, both complex thinking with a surface association and complex thinking with a template–orientation are particularly important categories in that they provide the student with initial access to the mathematical object (episode 1 in David's protocol, episodes 1 and 2 in John's protocol, for example).

All the other phases presented in the appropriation map (Figure 7) appeared at least once in either the analysis of David's or John's protocols and were very useful in explaining how the student was using particular mathematical signs. For example, even though thinking–in–heaps occurred only once in John's interview and complex thinking with an artificial association appeared only once in David's protocol, there was no other way in which I could categorise either student's usage of signs in that particular episode. Indeed, considering the protocols of all seven students, both these phases of thinking showed up several times in the analyses (see Chapter 6 for more examples).

Certainly, and as suggested in Chapter 6, pseudoconceptual thinking was ubiquitous in both students' usage of signs. This is not surprising: one would expect that all students who managed to gain entry into the Mathematics I Major Course at the University of the Witwatersrand, would, at the least, know how to use mathematical signs *as if* they understood the concepts.

In summary, based on the analyses of the seven case study students, with particular attention to the comprehensive and detailed analyses of John's and David's protocols, the only substantial change I wish to make to my original map of the different phases in appropriation theory, is to remove diffuse complex thinking from the map.

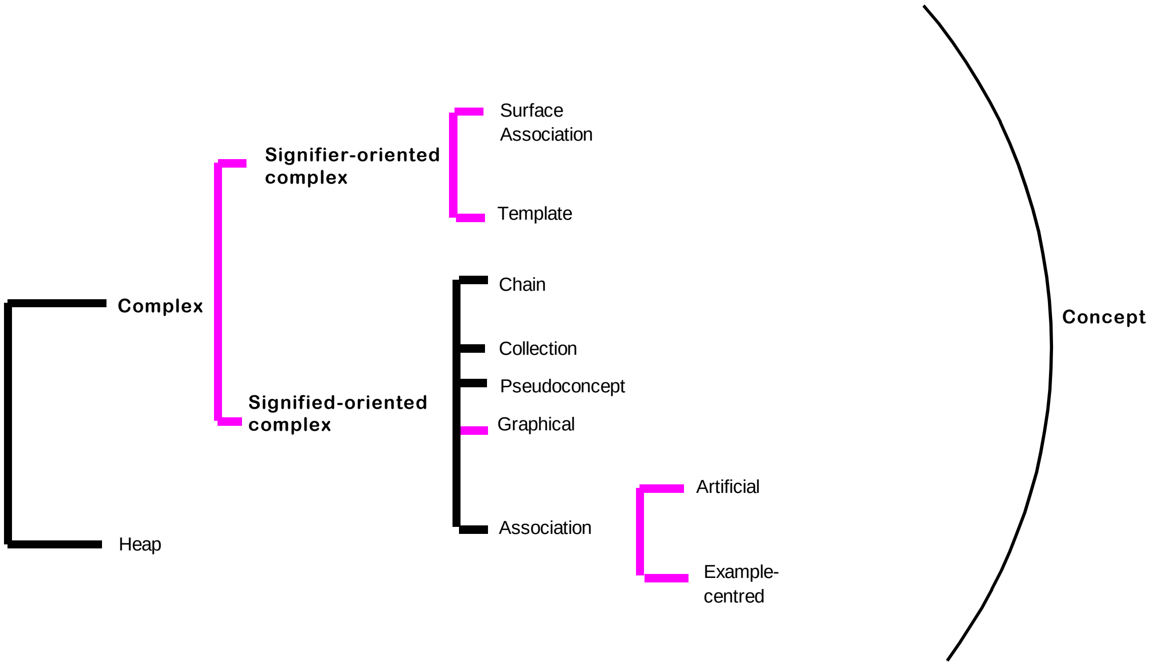

Figure 8: Revised map of phases in the appropriation of a mathematical object

§8.7 CONCLUSION

In this chapter I have used the episodic analyses (of Chapter 7) both to illustrate and elaborate various aspects of appropriation theory. In doing this, and indeed by virtue of the analyses themselves, I hope to have demonstrated how appropriation theory may illuminate the different phases (as indicated by different usages of signs) through which a student may move as he goes about appropriating a new mathematical object in an interview setting.

In summary, I have done the following in this chapter:

- I have argued that a "functional use" (Vygotsky, 1986) of a new mathematical sign enables a production of meaning that is both personally satisfying and also enabling of the learner's enculturation into the mathematical community. This functional usage takes the form of activities such as imitation, association, template–matching, manipulations and reflection. These activities in turn are afforded or constrained by the way in which the purposefully designed task encourages diverse functional usages of the mathematical sign.

Since many in the mathematics education community do not regard activities such as imitation, association, template–matching and manipulations as particularly productive of meaning–making for the student, my argument and illustration are particularly significant on a pedagogical level.

Furthermore, I have demonstrated how this functional usage enables the learner to move through the different phases of appropriation theory as he constructs a concept. That is, I have shown that functional use is the mechanism which propels the student from one phase of mathematical thinking to another. Again, this has profound implications on the pedagogical level.

- I have argued that, although the student may appear to use new mathematical signs in bizarre and unpredictable fashions, these initial usages may be the fore–runners of usages which are personally meaningful and socially apposite. Following Sierpinska (1992: 28), I have called these initial usages “fore–conceptions”.

  Indeed I suggest that many so–called misconceptions[17] are actually fore–conceptions; as such they should be regarded as necessary parts of the developmental process rather than as cognitive obstacles to be overcome. Furthermore, through functional use within social discourse, these fore–conceptions should be able to evolve into more coherent and objectively meaningful forms.

  Pedagogically, this implies that the learner should be encouraged into using such conceptions functionally.

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[17] I assume that a misconception is “ a deep and resilient idea” about specific subject matter (Disessa and Minstrel, 1998, p. 156). “Measured by a repertoire of simple qualitative problems… many students, in some cases the majority, come out of instruction essentially as they went in.” (ibid.)

- I have argued that the path through the various phases is informed not only by the actual mathematical task and the student's prior mathematical knowledge (as made manifest, for example, in the student's fore–conceptions) but also by social phenomena such as the social relations between the interviewer and the student in the interview context. (The quality of these social relations may in turn depend on wider cultural issues such as the particular student's level of academic enculturation and ease with the language of instruction.)
- I have demonstrated how different usages of signs predominate in each of the critical case students. Although the data in this project is insufficient to explain why these differences exist and indeed why different students take different paths through the appropriation map (for example, some students use predominantly chain associations between signs to develop their thinking, other students focus on the template of the signs they are using, and so on), the nature of the different paths and the prevalence of one type of sign usage over another may give the researcher or teacher useful insight into that particular student's way of mathematical thinking. Such insight can potentially inform further teaching and learning activities.
- Above all, I have argued and demonstrated that appropriation theory with some minor modifications (in particular, the subsuming of the diffuse complex into the category 'complex thinking using artificial associations') is a very useful and usable tool for understanding how different students go about appropriating a new mathematical object presented to them in the form of a mathematical definition.

CHAPTER 9: ISSUES OF VALIDITY

§9.1 INTRODUCTION

In this thesis I have elaborated Vygotsky's theory of concept formation to the mathematical domain[1] and I have demonstrated the power of the elaborated theory (and its language) as a tool with which to describe how specific students appropriate a new mathematical object presented in a mathematical definition[2]. In this penultimate chapter, I wish to address certain crucial questions surrounding the worth, the plausibility and the credibility of my elaborations and illustrations. That is, I wish to examine the validity, the reliability and the potential applicability of my elaborated theory[3].

But before doing this, I need to remind the reader of my ontological and epistemological assumptions since these assumptions necessarily inform my measure of what is worthy or plausible. I argued in Chapter 5 that order in the world does not exist independently of the human mind; rather we impose order on the world through our theoretical constructions. Accordingly, any account that I or anyone else gives, must be necessarily an interpretation of the world, an attempt to impose order and structure on the world. Allied to this, I argued that we cannot see the world unfiltered by a particular theoretical perspective. Thus the interpretation of the world necessarily takes place within a particular theoretical perspective, in my case a Vygotskian perspective.

Having restated these fundamental assumptions, I can now move on to a consideration of the extent to which my elaborations of Vygotsky's theory of concept formation are valid and reliable, and the extent to which the

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[1] See Chapters 6 and 8 especially.

[2] See Chapter 7 and 8.

[3] I have called this elaborated theory, 'appropriation theory'.

elaborated theory (ie appropriation theory) is applicable to other mathematical learners or other types of mathematical learning.

§9.2 VALIDITY

An account is valid to the extent that it is coherent and the researcher measures what she thinks she is measuring (Brown and Dowling, 1998). Indeed Brown and Dowling (1998: 137) claim that coherence is the "fundamental criterion by which educational research is to be judged". Similarly, Maxwell implies (1992) that the validity of an account should be evaluated not only on its "inner logic and coherence" (p. 285), but also "on its relationship to those things that it is intended to be an account *of* " (ibid: 281; italics in original).

Maxwell (1992: 292) usefully distinguishes between three different types of validity[4]: descriptive validity, interpretative validity and theoretical validity. These are "the ones most directly involved in assessing a qualitative account as it pertains to the actual situation on which the account is based".

I will examine my account in terms of these categories. I will also deal with Brown and Dowling's (1998) characterisation of validity as that which concerns "the relationship between the theoretical concept variables (or concepts) and empirical, indicator variables (or indicators)" (p. 26) under the heading, 'theoretical validity'.

§9.2.1 DESCRIPTIVE VALIDITY

According to Maxwell (1992), descriptive validity is concerned with the factual accuracy of the account. It is concerned with the accuracy of the reporting of events rather than with the meaning of these events (to the participants or to the observer). Maxwell (p. 286) captures this nicely in his argument that descriptive validity is concerned with "matter on which, in

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[4] He also refers to two other categories of validity, namely generalizability and evaluative validity. Neither is directly relevant to the generation of elaborated theory, which is my task in this thesis.

principle, intersubjective agreement could easily be achieved, given the appropriate data", such as a video tape of an interview.

So, in my account, descriptive validity concerns the accuracy of the transcripts. Did John really say that? Did David really write that? And so on. Certainly I believe that the transcripts I have used are very accurate records of the spoken and written events that took place.

The main reasons for this belief are the multiple sources for the transcripts and the extent to which the transcripts were revisited and revised. The multiple sources for the transcripts comprise audio and video tapes, the written work of the student in the interview and my observations of the student's activities, written during the interview. The extensive revisions of the transcripts refer to the process whereby the final transcripts were produced. The process was as follows (for each student). The audio tapes from the interviews were transcribed originally by a professional secretary. After this, I listened to the audio tapes with the transcription, and amended the transcription as required (in particular some of the students' mathematical statements were incorrectly recorded and the mathematical notation was not always satisfactory). I then used the video tapes of the interviews, together with a copy of the students' written workings, to reconstruct what and when the student wrote. As can be seen in the transcripts (see Chapter 7), this became part of the written record of the interview. I watched the video recordings several times (for each student) until I was happy that I had reconstructed the observable events of the interviews faithfully and accurately in the transcripts.

Notwithstanding the overall accuracy of the transcripts, I do acknowledge certain lapses in my reconstruction of events. For example, I have not indicated how long any of the pauses were, nor the length of time a student took over a particular activity, nor the stresses of the student's voices. Certainly such detail would contribute to the richness of the transcripts. But I do not believe that this detail would alter my analyses of the transcripts.

That is, these details would improve the descriptive validity of my account but, I believe, would not contribute significantly to the interpretative or theoretical validity of the account.

§9.2.2 INTERPRETATIVE VALIDITY

Maxwell (1992) uses this category specifically to refer to the researcher's interpretation of what the objects or events *mean to the people engaged in them*[5]. In my research, interpretative validity is a matter of inference about what the student is thinking or feeling, based on my interpretations of his or her words and actions during the interview.

So, for example, (hypothetical) statements such as "Michael understands convergence" or "Jane is confused" are statements which may or may not be interpretatively valid, since Michael may appear to understand convergence (but really may not) and Jane may not be confused (for example, she may just be distracted).

In my analyses[6] I have tried to avoid writing statements such as those quoted above. Rather I have spoken of the student's usages of signs (for example, David uses the mathematical signs in the problem correctly) or I have acknowledged that a statement about the student's experience is only what *appears* to be. So, for example, I have made statements such as "Mary *seems* to understand convergence" or "Tom *appears* confused", and so on.

Notwithstanding these conscious (and possibly cosmetic) attempts to acknowledge the interpretative nature of several of my descriptive comments, the interpretations may still be incorrect; that is, they may not mirror what the student is truly feeling or thinking. Furthermore, although I vigorously attempt to base my analyses only on the presence of specific

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[5] This is a very different sense from the way in which many authors such as Merriam (1998) use this category (Merriam's category of interpretative validity is similar to Maxwell's category of theoretical validity).

[6] I am using the term analyses to refer to my various accounts of different aspects of students' mathematical activities given in each of Chapters 6, 7 and 8.

empirical indicators[7] which are defined in terms of how the student uses the mathematical signs rather than on my interpretations of what the students are feeling, thinking, comprehending and so on, I cannot truly claim that certain of my decisions of categorisation in the analyses were not sometimes influenced by these interpretations. For example, I may have categorised some particular usage of signs as indicative of, say, heap thinking rather than, say, surface thinking, because the student *seemed* particularly confused.

The point is: I cannot claim that there are no threats to the interpretative validity of my analyses (be they in Chapters 6, 7 or 8). All I can claim is that I strove to base my analyses impartially on the empirical indicators which were defined in terms of how the student was using the mathematical signs. Thus, threats to the validity of my analyses (and the elaborated theory which flowed from it, especially the development of the concept variables) are more likely to have arisen because of issues concerning the theoretical validity of the account rather than its interpretative validity .

§9.2.3 THEORETICAL VALIDITY

Maxwell (1992) uses this category to refer to issues about the "legitimacy of the application of a given concept or theory to established facts", (p. 293) where the 'facts' derive from the descriptive and interpretative accounts. So Maxwell's category of theoretical validity corresponds to Brown and Dowling's description of validity as the measure of the quality of the relationship between the empirical indicators (the 'facts') and the concept variables. That is, an account has theoretical validity to the extent that the empirical indicators measure or indicate what they are supposed to indicate (the concept variables).

In my account, I described in great detail how each of the different theoretical concept variables, such as the pseudoconcept, heap thinking, complex thinking using chain associations, and so on, could be seen or

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[7] I discuss potential validity threats regarding inferences from these indicators in the section 'theoretical validity'.

measured empirically. I did this through the use of elaborated description (Brown and Dowling, 1998) in Chapter 6. Using such elaborated description I argued how certain usages of mathematical signs (the empirical indicators) indicated particular phases in the student's appropriation of a mathematical object (the theoretical concept variables). These theoretical concept variables were, in turn, derived from Vygotsky's stages of concept formation or from my prior experiences observing and teaching undergraduate mathematics students at the University of the Witwatersrand or from the mathematics education literature around mathematical object appropriation (or mathematical concept construction).

So, for example, I argued that a template orientation (Sfard, 2000) is indicated when the student applies properties of one sign with which he is familiar to a new or unfamiliar sign which has a similar template to the familiar sign; I argued that a student was using heap thinking (Vygotsky, 1986) when his use of signs related to the physical context of the signifiers, eg their layout on the page or the physical proximity of one signifier to another. Similarly I explicated other empirical indicators for each of the different phases of thinking (see Chapter 6).

My approach was in line with Brown and Dowling's argument that for a coherent account, the theoretical field needs to be specialised into theoretical concept variables and the empirical field needs to be localised into empirical indicators. In this way the theoretical and empirical domains are able to articulate with each other in a very specialised and localised region (see Chapter 5 for details of this methodology).

In terms of a measure of the theoretical validity of my account, the plausibility or accuracy of the link between the different phases of thinking (the concept variables) and the way in which these phases were made manifest in the interview setting (the empirical indicators) is crucial. So my account has theoretical validity to the extent that the following sorts of questions can be answered in the affirmative: Is the student truly using heap thinking[8] when his explanation of convergence and divergence relates to the order in which three different types of improper integral are defined and the order in which information about their convergence or divergence is presented[9]. Is the student truly adopting a template orientation when he states that "$\lim_{x \to c} f(x) = \infty \Rightarrow$ the limit of f(x) exists as x tends to c"?[10] And so on.

Ultimately I contend that my empirical indicators are good indicators of the phases of thinking that they purport to indicate (within the Vygotskian paradigm). This contention is based primarily on my experience of both developing and using these indicators in the analyses (in Chapter 7). Notwithstanding my conviction, my use of elaborated description allows the reader to judge the goodness of fit for her or himself. In this way, at the least, the links between the theory and the data are made visible, and, I suggest, the plausibility of my account is enhanced.

§9.3 RELIABILITY

In the analyses I used the student's different forms of activities in the mathematical interviews (the empirical indicators) to categorise the student's phase of mathematical thinking (the concept variables). The extent to which I have used these linkages in a rigorous and consistent way is a measure of the internal reliability of my study.

Although I attempted to be rigorous and impartial in my classification of the student's usages of signs, in practice the empirical indicators were not always unambiguous or complete. For example, I may have categorised some particular usage of signs as indicative of, say, surface thinking rather than say, heap thinking, because the student *seemed* to be ignoring parts of the mathematical expression, rather than because there was

[8] Heap thinking is defined by Vygotsky (1986) as the grouping together of disparate, inherently unrelated objects which are linked by chance in the person's perception.

[9] See Example 2 in Chapter 6.

[10] See Example 15 in Chapter 6.

unambiguous evidence that he was doing so. Also, the student sometimes merely wrote down an answer without articulating his reasoning for that answer. In these cases, I sometimes struggled to isolate the empirical indicator.

Because the empirical indicators were neither always unambiguous nor always explicitly delineated and because I was not always able to classify the pertinent mathematical activity easily, I safeguarded the integrity of my account by spending much time and effort in justifying my categorisations of different mathematical activities. Hence the analyses in Chapter 7 are particularly dense and detailed.

Thus, although I cannot claim that there are no inconsistencies throughout my categorisations, by forcing myself to justify and explicate each of the categorisations, the dangers of such inconsistencies are lessened. Moreover, given the extensive nature of my justifications, these inconsistencies, if they exist, should at the least be visible to the critical reader.

§9.4 APPLICABILITY

In this thesis I have elaborated a theory, based on Vygotsky's theory of concept formation, to the mathematical domain. That is, I have developed an elaborated language of description to describe how an undergraduate mathematics learner appropriates a mathematical object presented to him in the form of a definition. I have called the theory, 'appropriation theory'.

Moreover I have demonstrated in great detail that appropriation theory can be fruitfully applied to the activities of two very different Mathematics I Major students at the University of the Witwatersrand working from a mathematical definition in an interview situation . I have also indicated (although not explicitly demonstrated[11]) that appropriation theory can be applied to the activities of five other purposefully chosen Mathematics I

[11] For example I have used exemplars from their protocols in my elaborated description of the theoretical concept variables in Chapter 6.

Major students (who all differ in terms of previous type of schooling and academic grade for their matriculation Higher Grade Mathematics examination) in similar interview situations.

Important questions now are: to what extent is appropriation theory applicable to other situations? What sort of situations?

I put forward that appropriation theory is applicable to most situations in which we find an undergraduate mathematical learner, armed with a textbook and/or set of mathematical activities and a written definition of a new mathematical object. Of course, modifications and further refinements of the theory may be required; but I maintain that the theory is sufficiently coherent and complete to endure and benefit from such adaptations.

I base this proposition partly on the fact that I was able to apply the theory successfully to seven very different sorts of students, partly on the fact that appropriation theory is an elaboration of a theory posited by one of the most respected educationalists of the 20th century, Vygotsky, whose theories are widely used, a priori, in educational research, and partly on the fact that my elaborations were based on three primary sources: my own extensive experiences as a university mathematics student and teacher, my detailed observations made during the course of this study and presented in this thesis and my examination of related mathematics education literature.

Indeed I suggest that in considering the applicability and robustness of appropriation theory, due regard should be given to Merriam's (1998) category of "reader or user generalizability"[12].

> Reader or user generalizability involves leaving the extent to which a study's finding apply to other situations up to the people in those situations. (Merriam, 1998: 211)

Just like a doctor or a lawyer uses various case studies to inform his understanding of a new event, so educationalists should be able to use one person's findings (or theory) to enhance their understanding of different

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[12] According to Merriam (1998), reader or user generalizability is also called case–to case transfer by Firestone.

situations, which have certain elements in common with the original situation.

I propose that appropriation theory has the potential for “reader or user generalizability”. That is, it is possible for other researchers to apply the theoretical concept variables and their empirical indicators, possibly with some modifications, to a situation similar to that which I have dealt with here (undergraduate students appropriating mathematical objects from written definitions armed with a textbook and/or pedagogically designed mathematical activities).

It may even be that with further refinements and modifications to fit the new circumstances the theory can be applied to different situations (for example, students working in a group; students using technology to explore advanced mathematical notions; students involved in mathematical problem–solving activities; schoolchildren working on mathematical activities, and so on).

§9.5 ENHANCING CREDIBILITY

In this section I would like to highlight some ways in which the validity and reliability of my account could have been enhanced.

First, although I interviewed nine different students, I only analysed two of these in great depth, from beginning to end (see Chapter 7); with regard to the other seven students, I used the protocols of five of these students to extract exemplars of the different phases in thinking and to develop empirical indicators of these different phases (see Chapter 6). Certainly the credibility of my study would have been greatly enhanced if I had analysed all nine protocols in the same detailed fashion.

Indeed, one of the reasons why I did not comprehensively analyse all nine protocols, highlights a drawback of my methodology. Because I was using elaborated description in the analyses (so as to enhance the theoretical validity of the study), each detailed analysis took an incredible length of time – measured in months – to execute.

In some sense, thus, there was a trade–off between credibility and theoretical validity. Because my study was primarily geared to theory–generation it seemed more appropriate to focus on the theoretical validity of the account. Certainly, however, I believe that the plausibility of this entire account would be greatly improved if appropriation theory were applied to the analyses of the protocols of other students doing other mathematical tasks involving the construction of a mathematical concept.

Another possible weakness in my account is the lack of measures of inter–rater reliability. Although I have stressed that my analyses necessarily represent my own interpretations (with the desire for objectivity as a regulating force), the credibility of the analyses and hence the account, would certainly be enhanced if someone else were to analyse the protocols comprehensively using my same empirical indicators. One of the major reasons why I did not request such corroboration relates to the same 'problem' I alluded to above: each interview took so very long to analyse that I felt most uncomfortable in asking any of my very busy colleagues other than my supervisor for extensive help. Notwithstanding this lack of direct corroboration I did present and discuss the relationships between many of the theoretical concept variables and their empirical indicators in several seminars.

§9.6 SUMMARY

In summary, I believe that my account has a good measure of descriptive, interpretative and theoretical validity, although improvements in the reliability and the demonstrations of its applicability could be made.

I am also aware that, as is always the case with qualitative research, some readers may disagree with some of my interpretations and that other researchers may be able to generate different and better empirical indicators of the various phases of thinking. For this reason I acknowledge that my account is ultimately tentative and open to revisions.

Notwithstanding these and other potential revisions, I believe that I have gone a long way to developing a useful language of description for explaining the phases and mechanisms whereby different students appropriate different mathematical objects in an interview context, and that this language is potentially applicable to other scenarios involving mathematical learning.

Above all, I hope and believe that appropriation theory “offers generative research and development possibilities in the field of mathematics education”. (Adler, 2001: 143)

CHAPTER 10: CONCLUSION

§10.1 APPROPRIATION THEORY

In this thesis I have explored how mathematics students construct personal meanings and develop usages of mathematical signs which are compatible with the culturally established meanings and usages of those signs. In particular, I have focussed on the ways in which several students studying the Mathematics I Major course at the University of the Witwatersrand in South Africa appropriate mathematical objects presented to them in the form of a written definition.

These explorations and investigations have led me to posit that the appropriation of a new mathematical object by a learner takes place in phases (broadly categorised as heaps, complexes and concepts) and that these phases give a language of description for understanding this process.

This theory, which I call ‘appropriation theory’, is primarily comprised of an elaboration of Vygotsky’s (1986) theory of concept formation to the mathematical domain.

In line with Vygotsky, I have also postulated that the mechanism for moving through these phases, ie for appropriating the mathematical object, is a functional use of the mathematical sign (symbols, words, graphs, diagrams, etc.) which is regulated by social forces (for example, more knowledgeable others, textbooks and so on). In the mathematical domain, I have suggested that functional use is comprised of manipulations, reflections on, template–matchings, associations and imitations of mathematical signs.

That is, I have argued that the learner uses the mathematical signs both as objects with which to communicate (like words are used by a person) and as objects on which to focus and to organise his mathematical world (again as words are used in language) before he fully comprehends the meaning of these signs. Through this sign usage the mathematical concept evolves for that learner so that the mathematical sign eventually has personal meaning; furthermore, because the usage is socially regulated the concept evolves in a way that is consistent with the way it is used by the broader mathematical community.

Related to the above postulates, I have argued that the learner's use of signs in the construction of a formal mathematical concept[1] is mediated by the learner's knowledge of related signs or concepts which are called forth for that learner by particular attributes of the new sign, or this use is mediated by social agents such a teacher, peer or text.

10.2 SUMMARY

At this point it is apposite to backtrack and remind the reader, very briefly, of the structure and content of this thesis, all of which contributed to the development of appropriation theory.

My exploration took place along two different dimensions, the theoretical and the empirical (Brown and Dowling, 1998).

With regard to the theoretical field, I first explicated my assumptions about the nature of mathematics and my assumptions about how knowledge is constructed in Chapter 2. I argued that the appropriation of a mathematical object necessarily involves the personal construction of a mathematical concept. I also explained why I took my position with regard to the nature of mathematics from Rotman (1993). In terms of this position, mathematical objects are co-created and mutually constituted by the human mind in mathematical discourse; furthermore meaning and communication about mathematical objects are only possible because a particular use of symbols and signs in mathematical discourse is socially sanctioned.

With regard to the actual explication of how learners appropriate mathematical objects, I explained that I wished to adopt a Vygotskian perspective. I argued that several themes within this framework were apposite to such an exploration. In particular, I argued that Vygotsky's

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[1] In Vygotskian terms these are scientific concepts.

maxim that all scientific knowledge (such as mathematics) is semiotically mediated, that the individual is both biologically and socially constituted, that the meaning of word (ie a concept) evolves for a child through "functional use[2]" (1986) were relevant to my exploration. Furthermore Vygotsky's descriptions of the different stages through which an individual moves as he constructs a concept resonated with my observations over the years of undergraduate students constructing new mathematical concepts.

Since I wanted to draw on previous mathematics education research concerning the relationship between mathematical signs and meanings as well as research around mathematical concept construction at tertiary level, I interrogated this literature in Chapter 3. I argued that there are three almost–distinct bodies of literature in mathematics education which address the problem of the appropriation of mathematical objects by undergraduate learners. There is that body of research which shares an empirical field with mine and so focuses on mathematical learning at the undergraduate level (for example Tall, 1991, 1995, 1999; Tall et al, 2000a, 2000b; Dubinsky, 1991, 1997; Czarnocha et al, 1999; Sierpinska, Dreyfus and Hillel, 1999a; Sierpinska, 2000). There is that body of scholarship which shares a theoretical field with mine and focuses on the role of symbols in the appropriation of mathematical objects (for example Vile and Lerman, 1996; Cobb et al, 1997; Presmeg, 1997; Sierpinska et al, 1999, Sierpinska, 2000; Gravemeijer et al, 2000; Radford, 2000, 2001; Dörfler, 2000; Sfard, 2000; van Oers, 2000). There is also a very small body of literature which looks at the genesis of mathematical concepts for the individual learner (Sierpinska, 1993, 1994; Schmittau, 1993). I highlighted particular themes which dominate each body of literature and which I expected to use in my account

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[2] I remind the reader: "functional use" refers to Vygotsky's (1986) thesis that children use words for communication purposes and for organising their own activities, before they have a full understanding of what that word means. Through this use of words in socially–regulated discourse, the children come to develop mature understandings and usages of these words which are compatible with the socially accepted usages of these terms.

of mathematical object appropriation. I also drew attention to significant gaps in the literature. These gaps revolve largely around the dearth of research looking at the use of mathematical signs by individual learners at the undergraduate level.

Having clarified my theoretical assumptions and having highlighted significant gaps in the literature, I was ready to articulate my research question in a precise and theoretically informed manner. I did this in Chapter 4. Specifically I indicated that I planned to explore the applicability of Vygotsky's stages of concept formation to an undergraduate learner's appropriation of a new mathematical object; that I wished to elaborate on these stages in the mathematical domain and that I hoped to show that the mechanism for moving through the different phases of mathematical object appropriation is the functional use of mathematical signs. I explained that, by analogy with Vygotskian theory, a functional use of a mathematical sign involves the learner using the mathematical sign for mathematical activities and communication even before he has a fully–fledged or mature idea of the mathematical object. (In the mathematical domain, functional use is comprised of manipulations, reflections on, template–matchings, associations and imitations of the mathematical sign.) I called my intended elaboration of Vygotsky's theory of concept formation to the mathematical domain, 'appropriation theory'.

In Chapter 5, I indicated how I planned to develop appropriation theory. That is, I elaborated on the methodology of my research; this methodology derives from Brown and Dowling (1998). In line with the methodology I argued that I needed to articulate my empirical field with my theoretical field by refining and developing each of these fields. I had already discussed my theoretical assumptions, framework, problematic and research question in much detail in Chapters 2, 3 and 4. Thus in Chapter 5, I focussed on my localisations of the empirical field of my study. I explained how I had purposefully selected ten different students from the Mathematics I Major course and that I planned to conduct clinical interviews with each of these students. In these interviews each student was to be given a mathematical task consisting of several activities (which I had designed) around the improper integral. I was especially concerned to see how these students attempted to appropriate a new mathematical object (the improper integral) given to them in the form of a mathematical definition. Ultimately my purpose was to analyse these interviews, two in particular detail.

Before analysing these interviews I needed to develop appropriation theory further. To do this, I particularised each of Vygotsky's stages of concept formation to the mathematical domain and I also developed several other phases which were peculiar to object appropriation in the mathematical domain. I did this in Chapter 6. These latter phases (which do not derive from Vygotsky) in fact derive from my experience over the years as a mathematics lecturer observing different students grappling with new mathematical ideas and from the mathematics education literature around concept construction. The stages and phases of learning (such as heap thinking, the pseudoconcept, the template–orientated complex and so on) constitute the theoretical concept variables of my study. Simultaneous to this development of the theoretical concept variables, I developed the empirical indicators for each of these variables. That is, using "elaborated description" (Brown and Dowling: 1998) I discussed how each of the theoretical concept variables could be recognised in the empirical domain (ie when a student is engaging in mathematical activities). The elaborated descriptions of each of the theoretical concept variables comprise a major part of my elaboration of Vygotsky's theory (that is, appropriation theory).

Using these empirical indicators, I then analysed, in much detail, the interview protocols of two specially chosen students (who contrasted on certain crucial dimensions) in Chapter 7. The purpose of these analyses was to show how appropriation theory could be used to illuminate those processes whereby undergraduate students made meanings from new signs and how these meanings evolved through functional use of the mathematical signs (that is, that functional use of the mathematical signs is the mechanism for moving through the different phases). I also wanted to illustrate how these two students zigzagged between the various phases posited by appropriation theory as they went about appropriating a new mathematical object.

In Chapter 8 I discussed these analyses. In particular I demonstrated that the mechanism whereby a student moves from phase to phase when appropriating a mathematical object is functional use of the mathematical signs; I demonstrated how appropriation theory may highlight the prevalence of certain usages of signs in the activities of different students and I argued that the path through the various phases is informed not only by the actual mathematical task and the student's prior mathematical knowledge but also by social factors such as the quality of the social relations in the interview context. Finally I used the analyses to make minor modifications to and elaborations of appropriation theory.

I concluded that appropriation theory was a very useful and illuminating tool for understanding how a student made personal meaning of a mathematical sign which was compatible with its usage by the mathematical community. In particular I argued that appropriation theory highlights the importance of the functional use of mathematical signs via activities such as imitation, associations, manipulations, reflections on and template–matchings in the construction of a mathematical concept.

In summary, I have elaborated and applied Vygotsky's theory of concept formation to the mathematical domain, calling the elaborated theory, 'appropriation theory'. Furthermore I have illustrated the power of appropriation theory as a tool and language with which to describe how various students appropriate a new mathematical object.

§10.3 CONTRIBUTIONS TO MATHEMATICS EDUCATION SCHOLARSHIP

I contend that my thesis makes several major contributions to mathematics education research.

First most theories of mathematical object appropriation (or concept construction) at undergraduate level focus on the evolving relationship between process and object as the student constructs a concept (Tall, 1991, 1995; Tall et al, 2000a; Dubinsky, 1991, 1997; Czarnocha et al, 1999; Sfard 1991, 1994). In contrast appropriation theory focuses on the early stages of mathematical knowledge appropriation; in these early stages (which may be of extremely short or extremely long duration) the learner is not yet able to use the new mathematical signs as either object or process. Rather he seems to fumble and muddle his way through a new definition, application, procedure and/or a theorem, making apparently arbitrary connections between familiar and less familiar signs. Appropriation theory looks at how these apparently “meaningless” activities contribute to the student’s ultimate appropriation of the mathematical object.

Secondly, to the best of my knowledge, Vygotsky’s theory of concept formation has never been applied to an explication of an undergraduate learner constructing a mathematical concept. In these terms alone, my elaboration of the theory to the mathematical domain contributes to the developing body of Vygotskian scholarship, and, as I demonstrated, illuminates and informs an understanding of mathematical concept construction. In particular the notion of a pseudoconcept as a bridge between the intramental and intermental planes is an interesting theoretical premise. To remind the reader, I argued, in Chapter 2, that the pseudoconcept can be regarded as functioning as a bridge between concepts whose meaning is more or less fixed and constant in the social world (such as that body of knowledge we call mathematics) and the learner’s need to make and shape these concepts so that they become personally meaningful.

Furthermore, as van de Veer and Valsiner (1994) argue, the application of Vygotskian theory to an individual, as opposed to a group or dyad or any overtly social situation, has been long neglected in the Vygotskian research tradition in the West. My investigation of an individual appropriating a new mathematical object from a written definition, with access to a textbook and the occasional interventions of a teacher/ researcher goes some way to filling this gap in the literature.

On a methodological level, my application of Brown and Dowling's (1998) mode of interrogation to my research illustrates their maxim that credible and coherent research entails an examination of the empirical domain through the theoretical concept variables. Although I presume that other researchers have used Brown and Dowling's methodology, I believe that my thesis exemplifies this mode of interrogation with particular clarity.

§10.4 PEDAGOGICAL IMPLICATIONS OF THIS RESEARCH

Appropriation theory is particularly relevant to the various on–going debates about what constitutes desirable mathematical learning for undergraduate mathematics students.

On the one hand, the Calculus Reform Movement emphasises the value of conceptual development (Ganter, 2001) and so frequently regards activities such as imitations, manipulations, associations and template–matching as meaningless and without worth. However appropriation theory shows that such activities are necessary in the appropriation of a mathematical object (in terms of appropriation theory, the mechanism for moving through the different phases of object appropriation are the functional usages of mathematical signs). Accordingly, I suggest that a de–emphasis of these activities may inhibit and retard the development of mature concepts.

On the other hand, the Back–to–Basic Movement with its emphasis on performance seems to overvalue the role of skills and algorithmic applications at the expense of conceptual development. In this way mathematics learners in a Back–to–Basic environment (such as exists in many mathematics undergraduate courses in South Africa) may be satisfied with a pseudoconceptual notion of a mathematical object and such learners may not strive for further evolution of the pseudoconcept. Added to this, the use of skill–orientated examination questions which require pseudoconceptual rather than conceptual mathematical knowledge may further promote learning which emphasises the construction of pseudoconcepts rather than of concepts.

In contrast to some advocates of the Back–to Basics Movement, I maintain that undergraduate mathematics students, who will inevitably be using some mathematics in their careers, or going on to more advanced mathematical study, need to understand concepts as well as knowing how to use various techniques and algorithms.

Indeed, I suggest that appropriation theory shows that neither the Calculus Reform Movement's disparagement of the functional use of mathematical signs, nor the Back–to–Basic Movement's undue focus on performance and skills, is sufficient for the development of flexible and coherent mathematical concepts. What is required is a recognition that the functional uses of mathematical signs are necessary but not sufficient activities for mathematical knowledge appropriation.

§10.5 POSSIBLE EXTENSIONS TO THIS RESEARCH

In the previous chapter (Chapter 9) I indicated that certain methodological improvements would enhance the credibility and reliability of my study. Specifically I suggested that the study would be more credible if I analysed the protocols of more students engaged in appropriating a new mathematical object and I suggested that the reliability of the study would be improved if several interpreters were to analyse these protocols according to the empirical indicators. In this regard, I would like to reflect on how these methodological improvements could be made.

On a practical level, the mathematical task needs to be shorter so that the analyses are not so time–consuming and more protocols can be analysed. In this respect a lot of ingenuity and creativity will be required to design a mathematical task which has a similar pedagogic and research structure to the task in this thesis but which is much shorter (it would probably involve the appropriation of a mathematical object other than the improper integral). Furthermore such a study would benefit if a team of researchers were involved. If this were the case, the inter–rater reliability of the project could be enhanced by more researchers analysing the same protocols; also the theoretical validity of the study could be improved if more researchers contributed to the development of the relationship between the theoretical concept variables and the empirical indicators.

Moreover I believe that the quality of theory development would be enhanced and refined if the analyses were less unwieldy, repetitive and long.

In addition I hope that I, or other researchers, will demonstrate the robustness of appropriation theory by applying it (possibly with some modifications) to types and settings of mathematical activities other than the generation of mathematical objects from a definition in an interview setting.

On an educational research note, an exploration of the relationship (if any) between the prior mathematical history of the particular student and his pattern of movement through the different phases of the appropriation map as he appropriates a mathematical object would be most interesting. Questions around the relationships between the predominance of particular forms of usage of mathematical signs for a student and that student's mathematical biography could be asked. For example, do certain practices or experiences in that student's history encourage the use of surface associations; do other practices and experiences encourage the use of pseudoconceptual thinking and so on?

§10.6 SUMMARISING CONCLUSION

In my final comment I suggest that appropriation theory is a large first step in the elaboration of Vygotsky's theory of concept formation to the mathematical domain particularly at higher mathematical levels where mathematical ideas are presented through mathematical signs rather than concrete objects. Deriving as it does from such a well–respected and innovative source (namely, Vygotsky), I do hope that I and other researchers will take the opportunity to further refine and modify the theory.

Most importantly I believe that my explication of appropriation theory contributes significantly to a language of description which can be used (and further developed) to describe how different students appropriate new mathematical objects in various circumstances.

Indeed my overall hope (and belief) is that appropriation theory provides a coherent, rigorous and well–defined framework that other researchers can use and adapt in pursuit of an understanding of how people learn advanced mathematics.

APPENDIX A

Improper Integrals

Activity I

The definition of a definite integral $\int_a^b f(x)dx$ requires that the interval [a, b] is finite and that f is continuous on [a, b]

If one of the limits of integration is infinite, or the function f has an infinite discontinuity on [a, b], we call the integral an **improper integral**.

1. Can you make up an example of an improper integral?

Activity II

Improper integral Type I

If f is continuous on the interval [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$.

If $\lim_{b\to\infty}\int_a^b f(x)dx$ exists, we say that the improper integral converges. Otherwise the improper integral diverges.

The above definition is of an **improper integral with an infinite integration limit.**

2. (a) Can you make up an example of an improper integral with an infinite integration limit?

   (b) Can you make up an example of a convergent improper integral with an infinite integration limit?

3. Can you explain what an improper integral with an infinite integration limit represents graphically.

4. How would you define $\int_{-\infty}^{b} f(x)dx$?

5. (a) Determine whether $\int_1^{\infty} \frac{dx}{x^3}$ converges or diverges.

(b) Determine whether $\int_{-\infty}^{1} x\,dx$ converges or diverges.

(c) Determine whether $\int_1^{\infty} \frac{dx}{x}$ converges or diverges.

Activity IV

Determine whether the following statements are true or false. Justify your answer in each case.

9. If f ′ is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)dx = -f(0)$.

10. If the graph of f is symmetric with respect to the y-axis, then $\int_0^\infty f(x)dx$ converges if and only if $\int_{-\infty}^\infty f(x)dx$ converges.

NOTE: If f is continuous on the interval (–∞, ∞) , then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx, \qquad \text{where c is any real number.}$$

The improper integral on the left diverges if either of the improper integrals on the right diverges; otherwise it converges.

11. If f is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)dx$ converges.

Activity III

Improper integral Type II

If f is continuous on the interval (a,b], and has an infinite discontinuity at a, then

$$\int_a^b f(x)dx = \lim_{c \to a+} \int_c^b f(x)dx .$$

If $\lim_{c \to a+} \int_c^b f(x)dx$ exists, we say the improper integral converges.

Otherwise the improper integral diverges.

The above definition is of an **improper integral with an infinite discontinuity** at one of its limits.

Reminder: A function f is said to have an **infinite discontinuity** at c, if,

from the right or left, $\lim_{x \to c} f(x) = \infty$, or $\lim_{x \to c} f(x) = -\infty$.

6. (a) Determine whether $\int_0^3 \frac{dx}{x}$ converges or diverges.

(b) Determine whether $\int_{-8}^0 \frac{dx}{x^{2/3}}$ converges or diverges.

7. Can you make up an example of an improper integral with an infinite discontinuity?

8. Can you explain what an improper integral with an infinite discontinuity represents graphically.

APPENDIX B

Improper Integrals

Activity I

The definition of a definite integral $\int_a^b f(x)dx$ requires that the interval [a, b] is finite and that f is continuous on [a, b]

If one of the limits of integration is infinite, or the function f has an infinite discontinuity on [a, b], we call the integral an **improper integral**.

1. Can you make up an example of an improper integral?

Pass 1:

If student cannot give example, say: "Alright, leave this for now – we can come back to this a little later."

Activity II

Improper integral Type I

If f is continuous on the interval [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$.

If $\lim_{b\to\infty} \int_a^b f(x)dx$ exists, we say that the improper integral converges. Otherwise the improper integral diverges.

The above definition is of an **improper integral with an infinite integration limit.**

2. (a) Can you make up an example of an improper integral with an infinite integration limit?

   (b) Can you make up an example of a convergent improper integral with an infinite integration limit?

3. Can you explain what an improper integral with an infinite integration limit represents graphically.

First pass: I expect student to be able to do 2(a) easily, but 2(b), 3 with more difficulty.

If student gives example for 2(b), without verifying that it is convergent, ask: “Are you sure about that example. Explain.”

If he can’t do some of these questions, say: “Alright, leave this for now – we can come back to this a little later.” If she/he gets it wrong, say “We’ll come back to this later.” When I do, I will ask student again if he/she thinks he is right.

Second pass: After student has done examples 5. If she/he is now successful, say: “Fine, what helped you do that.”

Third Pass: Let student use Larson.

NB $\int_1^{\infty} \frac{dx}{x^p}$ converges if p>1 to 1/(p-1); else diverges.

4. How would you define $\int_{-\infty}^{b} f(x)dx$?

Q4 Ans: If f is continuous on the interval (-∞, b], then $\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)dx$.

First pass: If he can’t do this question, say: “Alright, leave this for now – we can come back to this a little later.”

Second pass: After student has done examples 5(b). If she/he is now successful, say: “Fine, what helped you do that.”

Third pass: Say: “Here is a hint: look at Question 5(b). Or say: How does what you have done relate to Q5(b)?”

Fourth pass: If they get stuck on Q4 Definition, suggest they look at definition in textbook, $7.8. Ask student to explain definition.

5. (a) Determine whether $\int_1^{\infty} \frac{dx}{x^3}$ converges or diverges.

(b) Determine whether $\int_{-\infty}^{1} x\,dx$ converges or diverges.

(c) Determine whether $\int_1^{\infty} \frac{dx}{x}$ converges or diverges.

Answer 5(a): ½

Answer 5(b): Divergent. NB. In Q. 4, student had to define limit like that in Q5(b).

If student gets stuck, let her/him use Larson.

Go back to Q2 if necessary.

If they don't use definition (i.e. if they just use ∞ as if it were a number) let them finish up to end 5(b), then say: Are these improper integrals? Why?

Activity IV

Determine whether the following statements are true or false. Justify your answer in each case.

9. If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f'(x)dx = -f(0)$.

10. If the graph of f is symmetric with respect to the y-axis, then $\int_0^{\infty} f(x)dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)dx$ converges.

NOTE: If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx, \qquad \text{where c is any real number.}$$

The improper integral on the left diverges if either of the improper integrals on the right diverges; otherwise it converges.

11. If f is continuous on [0, ∞) and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)dx$ converges.

Q9. True: Can use deductive reasoning

Q10. True in both directions – can use deductive reasoning.

NB: If whole integral converges then each part converges (Definition)

If graph symmetric wrt y-axis then $\int_{-\infty}^{0} f(x)dx = \int_0^{\infty} f(x)dx$ (if these integrals converge).

Q11. False: Counterexample: f(x) = 1/(x+1) and F(x) = ln (x+1).

NB. Can't use f(x)=1/x or f(x) =1/x^2 , etc. because f(x) is continuous on [0, ∞)

NB. f(x) = 1/$(x+1)^2$ gives convergence so not a counterexample.

Activity III

Improper integral Type II

If f is continuous on the interval (a,b], and has an infinite discontinuity at a, then

$$\int_a^b f(x)dx = \lim_{c\to a+} \int_c^b f(x)dx .$$

If $\lim_{c\to a+} \int_c^b f(x)dx$ exists, we say the improper integral converges.

Otherwise the improper integral diverges.

The above definition is of an **improper integral with an infinite discontinuity** at one of its limits.

Reminder: A function f is said to have an **infinite discontinuity** at c, if,

from the right or left, $\lim_{x\to c} f(x) = \infty$, or $\lim_{x\to c} f(x) = -\infty$.

6. (a) Determine whether $\int_0^3 \frac{dx}{x}$ converges or diverges.

(b) Determine whether $\int_{-8}^{0} \frac{dx}{x^{2/3}}$ converges or diverges.

Answer 6(a): Divergent

Answer 6(b): Converges to 6

If they don’t use definition (i.e. if they ignore definition and discontinuities), let them finish to end of 6(b), then say: Are these improper integrals? Why?

If student gets stuck, let her/him use Larson.

7. Can you make up an example of an improper integral with an infinite discontinuity?

8. Can you explain what an improper integral with an infinite discontinuity represents graphically.

Second Pass: Let student use Larson.

$\int_{0}^{1} \frac{dx}{x^p}$ converges if $p < 1$ to $1/(1-p)$; else diverges,.

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