Symmetry Classifications on a Curved Geometry

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DECLARATION

I declare that the contents of this thesis are original, except where due references have been made. It is submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

AS Mathebula

Signed at Johannesburg on the 19^{th} day of June 2020.

Abstract

In this thesis, we consider one-parameter point transformations that leave a differential equation invariant. In particular, we show that Noether symmetry classifications of any diagonal metric may be simplified by geometric criteria. We describe the Klein-Gordon equation for some general spaces and deal with the corresponding Killing algebra. Moreover, our investigation consists of several metrics, their Lie algebras, the point generators of the Klein-Gordon equation and their associated potential functions. Finally, we study a class of ecological diffusive equations and determine higher-order symmetries of non-linear diffusion equations.

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Last but not least, I am proud of myself for having kept on going even though there were many obstacles in my life. I never gave up because I knew that one day everything will work out. May this lesson motivate me to excel in everything I do!

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Chapter 1

Introduction

During an academic career, the majority of students first come across Differential Equations (DEs) in the form of exact, separable and homogeneous equations. It was in the nineteenth century, when a well known mathematician, Sophus Lie discovered that, the special solving techniques used was in fact all special cases of a general integration method depended on the invariance of the DEs under a continuous group of symmetries. This discovery expanded on the availability of integration methods and was the beginning of the development and application of the continuous groups, also known as Lie groups. The application of Lie groups has an impact in mathematics, engineering, physics and other related mathematical sciences areas.

There was a significant explosion of research activity in the field of Lie groups over the last few decades. The research ranged from application to concrete physical systems to extensions of the scope and depth of the theory. Today, there is still extensive research in the field of Lie groups. A group which transforms solutions of a system of DEs to other solutions is a symmetry group of a system of DEs. There are common examples of symmetry groups, which are rotations, translations and groups of scaling symmetries. The continuous symmetry groups can be determined using explicit computational methods. The method is composed of mechanical computations - several symbolic manipulation computer programs have also been developed for this task.

After the symmetry group of system of DEs has been determined, a variety of applications become available. The defining property of a symmetry group can be used to construct new solutions to the system of DEs from the known symmetry groups. A symmetry may be used to reduce the order of a given equation by one. Additionally, generalized symmetries have been found to be of paramount importance in the recent study of nonlinear DEs after many years of being neglected.

Noether's theorems relating to symmetry groups of a variational integral to properties of its associated Euler-Lagrange equations were discovered by E. Noether in 1918. The first theorem revealed that each one - parameter variational symmetry group produces a conservation law of the Euler-Lagrange equations. According to [1], the general result necessitates the introduction of " generalized symmetries " which are groups whose infinitesimal generators depend not only on the independent and dependent variables of the system, but also the derivatives of the dependent variables. In the second Noether's theorem, there is a nontrivial relation among the ensuing Euler-Lagrange equations, and, consequently, symmetries giving rise to conservation laws.

1.1 Outline of Chapters

Chapter 2 provides the notation and theory that we utilize to determine Noether point symmetries and potential functions. In Chapter 3, the general form of Noether symmetries admitted by Lagrangians corresponding to a diagonal metric are determined. We apply this general result in order to classify different metric functions for the determination of Noether generators for the equations of motion. For the two broad cases considered, we identify symmetry algebras up to dimension thirteen. The contents of Chapter 3 have been published in [2]. In Chapter 4, we consider a general set of Einstein-Maxwell fields in 2+1-dimensional space. Two broad categories of solutions are discussed, namely solutions of vanishing covariant derivatives (uniform electromagnetic fields) and stationary cyclic symmetric spaces. Subsequently, several major subclasses of solutions arise that may be classified according to the conformal algebra they possess. A key feature of these algebras is the presence of the $SO(2) \times R$ Killing group. It is shown that this group and other elements of the conformal algebra of each solution satisfies a special contingency relation with the potential function of the Klein-Gordon equation. Chapter 4 has been accepted for publication [3]. In Chapter 5, we investigate the potential functions $V(x^i)$ that appear in the Lagrangian $L(x^i, \Phi, \Phi_i) = \frac{1}{2}\sqrt{g}\, g^{ij}\Phi_i\Phi_j - \frac{1}{2}\sqrt{g}\, V(x^i)\,\Phi^2$, where g is the metric of an arbitrary space. Our approach is based on a connection between the conformal Killing group associated with g and the Noether symmetries of $L(x^i, \Phi, \Phi_i)$. To this effect, we select certain spaces that are characterized by a nonzero Weyl tensor. Chapter 5 has been published [4].

In Chapter 6, spacetimes which are 2+2 decomposable are investigated according to the Klein-Gordon equations of the space. Nine classes of solutions are presented, including their conformal algebras. Some of the spaces represent perfect fluids or vacuum spaces. For each isometry, explicit expressions are obtained to define the potential function embedded into the Klein-Gordon equation. Chapter 6 has been accepted for publication [5]. In Chapter 7, we consider different routes to generalized symmetries for some ecological equations that arise in spatial theory. Two primary methods for the derivation of generalized symmetries are the standard Lie invariance condition with vector fields dependent on derivatives and secondly, a recursive operator. The former is less efficient especially if it includes derivatives that become increasingly higher in order, and this necessarily complicates the nature of the computations. The latter involves a nontrivial analysis to define a recursion operator, if one exists, but is successful in providing higher-order analogs of the equation or equivalently, higher-order symmetries. A linear Kierstead-Slobodkin and Skellam model is shown to possess a recursion operator that renders the equation completely integrable, by verifying the presence of infinitely many higher-order symmetries. Moreover, we apply the scheme of the multiplier approach to establish nontrivial conserved vectors from multipliers $\Lambda(t, x, u, u_x, u_t)$, that are analogous to integrating factors. Chapter 7 has been published in [6].

Chapter 2

Notation and Theory

2.1 Differential Functions

For the Lie algebraic treatment of differential equations, consider the universal space $\mathcal{A}$ [1]. A locally analytic function $h(x, u, u_{(1)}, \ldots, u_{(k)})$ of a finite number of variables is called a differential function of order k. The space $\mathcal{A}$ is the vector space of all differential functions of all finite orders and forms an algebra. A total derivative converts any differential function of order k to a differential function of order $k+1$. Hence, the space $\mathcal{A}$ is closed under total derivations D_i. There are also other operators on $\mathcal{A}$ which we recall below, see the books [7] , [8], [1] and [9].

2.2 Symmetry Methods

In this section, we outline the general procedure for determining point symmetries for an arbitrary system of equations. Consider a nonlinear system with q unknown functions u^a which depends on p independent variables x^i. We denote u and x as $u = (u^1, \ldots, u^q)$ and $x = (x^1, \ldots, x^p)$, respectively. Let

$$G_\alpha\left(x, u^{(k)}\right) = 0, \quad \alpha = 1, \ldots, q, \tag{1}$$

be a system of m nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x. A one-parameter Lie group of transformations (ϵ is the group parameter) that is invariant under (1) is given by

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \tag{2}$$

Invariance of (1) under the transformation (2) implies that any solution $u = \Theta(x)$ of (1) maps to another solution $v = \Psi(x; \epsilon)$ of (1). Expanding (2) around the identity $\epsilon = 0$, we can generate the following infinitesimal transformations:

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon\xi^i(x, u) + \mathcal{O}(\epsilon^2), \quad i = 1, \ldots, p, \\ \bar{u}^\alpha &= u^\alpha + \epsilon\eta^\alpha(x, u) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following infinitesimal vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{4}$$

The action of X is extended to all derivatives appearing in the equation in question through the appropriate prolongation. The infinitesimal criterion for invariance is given by

$$X\left[\text{LHS Eq.(1)}\right] |_{Eq.(1)} = 0. \tag{5}$$

Eq. (5) yields an overdetermined system of linear homogeneous equations which can be solved algorithmically, more details can be found in [1] among other texts.

2.3 Fundamental Operators

Definition 1. The *Euler* operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\, \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\ , \quad \alpha = 1,\ldots,m\ . \tag{6}$$

Definition 2. The Lie-Bäcklund or generalised operator can be expressed by the following infinite formal sum

$$X = \xi^i\, \frac{\partial}{\partial x^i} + \eta^\alpha\, \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1\ldots i_s}\, \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{7}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}\ , \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s>1. \end{aligned} \tag{8}$$

The total differentiation operator D_i with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij}\frac{\partial}{\partial u^\alpha_j} + \ldots, \quad i = 1,\ldots,n. \tag{9}$$

In (8), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j\ . \tag{10}$$

Definition 3. Lie-Bäcklund or generalised operators $\tilde{X}$ and X are said to be *equivalent* if

$$X - \tilde{X} = \lambda^i D_i, \quad \lambda^i \in \mathcal{A}\,.$$

In particular, a generalized operator of the form $\tilde{X} = \eta^\alpha \partial/\partial u^\alpha + \cdots$ is called a *canonical* or *evolutionary* representation of X.

Definition 4. The Noether operator associated with a Lie-Bäcklund operator X is given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}}\ , \quad i=1,\ldots,n, \tag{11}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (6) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}} \quad i=1,\ldots,n, \quad \alpha=1,\ldots,m\ . \tag{12}$$

The Euler, Lie-Bäcklund or generalised, and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i\ . \tag{13}$$

Definition 7. A Lie-Bäcklund operator X is called a Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$, if there exists a vector $B^i = (B^1,\ldots,B^n), B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{14}$$

If $B^i = 0\,(i=1,\ldots,n)$, then X is called a strict Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$.

2.4 Noether's Theorem

Noether [10] discovered the interesting link between symmetries and conservation laws, showing that for every infinitesimal transformation admitted by the action integral of a system, there exists a conservation law. That is, for any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there exists a current $\Phi^i = (\Phi^1,\ldots,\Phi^n), \quad \Phi^i \in \mathcal{A}$, defined by

$$\Phi^i = B^i - N^i(L), \qquad i=1,\ldots,n, \tag{15}$$

A current $\mathbf{T} = (T^1, \dots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{16}$$

along the solutions of (1),

2.5 Multipliers

The multipliers Λ^α of (1) satisfy the relation

$$D_i T^i = \Lambda^\alpha G_\alpha \tag{17}$$

for the function u^α [1, 11]. The overdetermined equations for Λ^α are

$$\frac{\delta}{\delta u^\alpha}[\Lambda^\alpha G_\alpha] = 0. \tag{18}$$

Eq.(17) is satisfied for the functions u^α and not only for the solutions of (1). Conserved quantities may be derived using (17) as the determining equation, or by elementary manipulation once the multiplier has been obtained. Alternatively, the conserved quantities are determined by a homotopy operator, see [12] for details.

2.6 Recursion Operators

Polynomial systems arising from evolution equations are given by

$$u_t = G(u), \tag{19}$$

where G is simply expressed as $G(u)$ even though G depends on u and it's x-derivatives up to order n. Assuming that all parameters in the system are non-zero.

A higher-order symmetry, $H(u)$, leaves the partial differential equation invariant under the substitution $u \to u + \epsilon H$ within order ϵ [13]. A linear integro-differential operator which is a recursion operator, $\mathcal{R}$, links higher-order symmetries [13]

$$H^{(p+q)} = \mathcal{R}H^{(p)}, p = 1, 2, 3, \ldots \tag{20}$$

where $q = 1$ in most cases and $H^{(p)}$ is the $p - th$ higher-order symmetry.

2.7 Collineation Criterion

Suppose we have the coordinates $\{x^i\}$ for an arbitrary space K^n of dimension n, and a one parameter point transformation

$$\tilde{x}^i = x^i + \epsilon\zeta^i\left(x^k\right), \tag{21}$$

where the ζ^i are vector field components called the infinitesimal generator of Eq. (21). If Υ is a geometric object in K^n, under the action of (21), Υ transforms to $\Theta\left(\Upsilon^i, x^i, x^{i'}\right)$ and the transformation law is $\Upsilon^{j'} = \Theta^{j'}\left(\Upsilon^i, x^i, x^{i'}\right)$. The Lie derivative $\mathcal{L}_\zeta$ of Υ is then [14]

$$\mathcal{L}_\zeta\Upsilon = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Theta\left(\Upsilon^i, x^i, x^{i'}\right) - \Upsilon\Big). \tag{22}$$

For functions $\Psi\left(x^i\right)$, the transformation law is $\Psi'\left(\tilde{x}^i\right) = \Psi\left(x^i\right)$, so that under the action of Eq. (21), one has

$$\tilde{\Psi}\left(\tilde{x}^i\right) = \Psi\left(x^i + \epsilon\zeta^i\left(x^k\right)\right) = \Psi\left(x^i\right) + \epsilon\Psi_{,i}\zeta^i + O\left(\epsilon^2\right).$$

Thus, by Eq. (22) we have

$$\mathcal{L}_\zeta\Psi = \Psi_{,i}\zeta^i. \tag{23}$$

If the RHS of Eq. (23) vanishes, then $\Psi\left(x^i\right)$ is invariant under the point transformation (21), and ζ is known as a symmetry of $\Psi\left(x^i\right)$. If $\mathcal{L}_\zeta\Upsilon = A$, where A is a tensor having the identical number and symmetries of the indices with Υ, the vector field ζ is called a collineation of Υ whose type depends on A.

In the context of general relativity, our geometric objects Υ are defined in terms of a metric. That is, let $\Upsilon = g_{ab}$ and $A = 2\psi g_{ab}$, then we have

$$\mathcal{L}_{\zeta} g_{ab} = 2\psi g_{ab}, \tag{24}$$

and ζ is a conformal Killing vector (CKV) [15]. Here, $\psi = \psi(x^a)$ is a conformal factor. If $\psi_{;ab} \neq 0$, the CKV is said to be proper. We have

$$\begin{aligned} \psi_{;ab} = 0 \quad &\Longleftrightarrow \quad X \text{ is a special CKV (sCKV)}, \\ \psi_{,a} = 0 \quad &\Longleftrightarrow \quad X \text{ is a homothetic vector (HV)}, \\ \psi = 0 \quad &\Longleftrightarrow \quad X \text{ is a Killing vector (KV)}. \end{aligned} \tag{25}$$

These vector fields form a Lie algebra and the dimension of the conformal algebra is less than or equal to $\frac{1}{2}(n+1)(n+2)$.

Chapter 3

Noether Symmetry Classifications of Generalized Diagonal Spaces

Diagonal metrics are frequently used in literature and have featured in numerous Noether symmetry classifications, for example, [16, 17, 18, 19] and many others. A Noether symmetry of a geodesic Lagrangian is also a Lie symmetry of the equations of motion; however, the inverse of this result is not always true. In particular, Noether symmetries form a Lie algebra called a Noether algebra. It is also well known that the following holds

$$KV \subseteq HV \subset \text{ Noether symmetry},$$

where KV and HV denote a Killing and homothety vector, respectively.

The existence of a Noether symmetry supplies us with a Noether integral by Noether's theorem [10]. Such Noether integrals are invariant functions of the Noether symmetry, and thus, the number of Noether symmetries are important for a dynamical system. Dynamical systems of n degrees of freedom that admit a minimum of n linear independent first integrals I_k $(k = 1, \ldots n)$, with the Poisson bracket $\{I_k, I_m\} = 0$, can be solved by quadrature. These important consequences have driven most investigations involving Noether symmetries, see for example [20, 21, 22, 23, 24], and inter alia; the investigations into minisuperspace quantum cosmology [25], hybrid gravity [26], $f(T)$ cosmology [27], $f(R, \mathcal{G})$ [28] and Horndeski theories [29], Noether symmetries are used as model selection criteria.

In this investigation, we explicitly state the general symmetry determining equations and symmetry vector forms of geodesic Lagrangians that arise from diagonal metrics. As an application, we then consider a general nonstatic space whereby the unknown metric functions provide various Noether symmetries of the associated geodesic Lagrangian.

3.1 Noether Symmetries and Diagonal Metrics

We now turn our attention to a Riemannian space with metric $g_{ii}(x^k)$. Then the classical Lagrangian describing the motion of the particle is

$$L = \tfrac{1}{2} g_{ii} \left(\dot{x}^i\right)^2 , \quad \dot{x} = \tfrac{dx}{ds} \tag{26}$$

(s is the arclength parameter) with the action

$$\bar{S} = \int ds \left(L \left(x^k, \dot{x}^k \right) \right) .$$

The vector

$$X = \xi(s, x^k) \partial_s + \eta^i(s, x^k) \partial_{x^i} \tag{27}$$

is a Noether symmetry of the Lagrangian (26) if the Noether condition

$$X^{[1]} L + \frac{d\xi}{ds} L = \frac{dh}{ds} \tag{28}$$

where $h(s,x^k)$ is a gauge term, is satisfied and, $X^{[1]}$ is the first prolongation of X. Condition (28) is equivalently expressed as [30]:

$$\begin{aligned} X^{[1]}L &= \left(\xi\partial_s + \eta^k\partial_{x^k} + \left(\dot{\eta}^k - \dot{x}^k\dot{\xi}\right)\partial_{\dot{x}^k}\right)\left(\frac{1}{2}g_{ii}(\dot{x}^i)^2\right) \\ &= \frac{1}{2}g_{ii,k}\eta^k\left(\dot{x}^i\right)^2 + g_{ii}\left(\eta^i_{,k} - \xi_{,s}\right)\dot{x}^i\dot{x}^k + g_{ii}\eta^i_{,s}\dot{x}^i - g_{ii}\xi_{,k}\left(\dot{x}^i\right)^2\dot{x}^k, \end{aligned}$$

$$\dot{\xi}L = \left(\xi_{,s} + \xi_{,k}\dot{x}^k\right)\left(\frac{1}{2}g_{ii}\left(\dot{x}^i\right)^2\right) = \frac{1}{2}\xi_{,s}g_{ii}\left(\dot{x}^i\right)^2 + \frac{1}{2}\xi_{,k}g_{ii}\left(\dot{x}^i\right)^2\dot{x}^k,$$

$$\dot{h} = h_{,s} + h_{,k}\dot{x}^k.$$

Hence, we find the following system of determining equations:

$$\xi_{,k} = 0, \quad h_{,s} = 0, \tag{29}$$

$$\eta^i_{,s}g_{ii} = h_{,i} \tag{30}$$

$$g_{ii}\left(\eta^k_{,i} - \tfrac{1}{2}\xi_{,s}\right) + \tfrac{1}{2}\eta^i g_{ii,k} = 0. \tag{31}$$

Eq. (29) implies that $\xi\left(s\right)$ and $h\left(x^i\right)$, while Eq. (30) solves to give $\eta^i = g^{ii}\left(h_{,i}s + J_i\left(x^i\right)\right)$. Thus the general Noether symmetry vector is of the form

$$X = \xi(s)\partial_s + g^{ii}\left(h_{,i}s + J_i\left(x^i\right)\right)\partial_{x^i}. \tag{32}$$

Moreover, Eq. (31) becomes

$$\left(g_{ii}g^{ii}_{,i} + \frac{1}{2}g_{ii,k}\right)\left(h_{,i}s + J_i\right) - \frac{1}{2}\xi_{,s}g_{ii} + h_{,ii}s + J_{i,i}. \tag{33}$$

Note that, if one wanted symmetries independent of s, we need $\xi = 0$ and $h =$ *constant*. In the next section, we show the usefulness of the application of the formulae (32) and (33).

3.2 Conservation laws

A study about variational symmetries is incomplete without the corresponding conservation laws guaranteed by Noether's theorem. Relative to each Noether symmetry (27) of the Lagrangian (26), is an integral of motion given by

$$I = \xi \left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L \right) - \eta^i \frac{\partial L}{\partial \dot{x}^i} + h,$$

which is a first integral or Noether integral of the equations of motion, i.e. $\frac{dI}{ds} = 0$. Of course conservation laws can be derived without Noether's theorem. Nevertheless, Noether's work provides a simplistic and elegant route to quantities left dynamically invariant.

3.3 A Genaralized Nonstatic space

Consider a nonstatic line element given by

$$ds^2 = -dt^2 + e^{P(t,r)} dr^2 + e^{Q(t,r)} d\theta^2 + e^{K(t,r)} d\phi^2, \tag{34}$$

in the coordinate system (t, r, θ, ϕ), which admits the geodesic Lagrangian

$$L = \frac{1}{2} \left(-\dot{t}^2 + e^{P(t,r)} \dot{r}^2 + e^{Q(t,r)} \dot{\theta}^2 + e^{K(t,r)} \dot{\phi}^2 \right). \tag{35}$$

The Euler-Lagrange equations are

$$\begin{aligned}
\ddot{t} &= -\tfrac{1}{2} P_t e^P \dot{r}^2 - 1/2\, Q_t e^Q \dot{\theta}^2 - 1/2\, K_t e^K \dot{\phi}^2, \\
\ddot{r} &= -\tfrac{1}{2e^P} \left(2\, P_t e^P \dot{r} \dot{t} + P_r e^P \dot{r}^2 - Q_r e^Q \dot{\theta}^2 - K_r e^K \dot{\phi}^2 \right), \\
\ddot{\theta} &= -\dot{\theta} \left(Q_r \dot{r} + Q_t \dot{t} \right), \\
\ddot{\phi} &= -\dot{\phi} \left(K_r \dot{r} + K_t \dot{t} \right).
\end{aligned} \tag{36}$$

We assume that the Noether symmetry for the above Lagrangian takes the form of (27), that is

$$X = \xi\partial_s + \eta^1\partial_t + \eta^2\partial_r + \eta^3\partial_\theta + \eta^4\partial_\phi,$$

with gauge function h, where ξ, η^i, h are functions of (s,t,r,θ,ϕ). Using (29) - (31) the determining equations are explicitly

$$\begin{aligned}
&\xi_\theta = 0, \quad \xi_\phi = 0, \quad \xi_r = 0, \quad \xi_s = 0, \quad h_t = 0, \quad \xi_t - 2\,\eta^1_s = 0,\\
&e^P\left(-\xi_t + P_s\eta^1 + P_r\eta^2 + 2\,\eta^2_r\right) = 0,\\
&e^K\left(2\eta^4_\phi - \xi_t + K_s\eta^1 + K_r\eta^2\right) = 0,\\
&e^Q(-\xi_t + Q_s\eta^1 e^Q Q_r\eta^2 + 2\,\eta^3_\theta) = 0,\\
&2\,\eta^4_\theta e^C + 2\,\eta^3_\phi e^Q = 0, \quad -2\,\eta^1_r + 2\,\eta^2_s e^P = 0,\\
&-2\,\eta^1_\phi + 2\,\eta^4_s e^K = 0, \quad -2\,\eta^1_\theta + 2\,\eta^3_s e^Q = 0,\\
&2\,\eta^2_\theta e^P + 2\,\eta^3_r e^Q = 0, \quad 2\,\eta^2_\phi e^P + 2\,\eta^4_r e^K = 0,\\
&2\,\eta^3_t e^Q - h_\theta = 0, \quad 2\,\eta^4_t e^K - h_\phi = 0, \quad 2\,\eta^2_t e^P - h_r = 0, \quad -2\,\eta^1_t - h_s = 0.
\end{aligned} \tag{37}$$

The Noether symmetries of this space may be classified according to the functional forms of the metric functions P, Q and K- a task facilitated by Eq. (32) - (33). In fact, according to Olver [1], in situations containing generic functions, say f, one must search for symmetries with arbitrary f, and thereafter determine particular forms of f that will expand the symmetry group. To this end, one may consider the functions P, Q and K as defined above, or specify that they are functions of single variables, i.e. t or r. Hence, we split our classification into these two considerations in which $a, b, c, d, e, f \neq 0$ are arbitrary constants. Nonzero gauge terms are listed where applicable.

3.4 Two-variable Cases

This case can be further divided into many cases if P, Q and K are explicitly defined. Here we will discuss five general cases:

$K(t,r) = P(t,r) = Q(t,r)$:

In this case, metric functions are equivalent and we find four Noether symmetries:

$$X_1 = \partial_s,\ X_2 = \theta\partial_\phi - \phi\partial_\theta,\ X_3 = \partial_\phi,\ X_4 = \partial_\theta.$$

Similarly other combinations of $K(t,r) = P(t,r) = Q(t,r)$ together with their admitted Noether symmetries are listed below.

$K(t,r) = P(t,r) \neq Q(t,r)$:

$$X_1,\ X_3,\ X_4.$$

$K(t,r) = Q(t,r) \neq P(t,r)$:

$$X_1,\ X_2,\ X_3,\ X_4.$$

$P(t,r) = Q(t,r) \neq K(t,r)$:

$$X_1,\ X_3,\ X_4.$$

$P(t,r) \neq Q(t,r) \neq K(t,r):$

In general, for arbitrary $P(t,r), Q(t,r), K(t,r)$ we have the Noether generators

$$X_1,\ X_3,\ X_4,$$

but in the special case of $P(r) = \frac{1}{r}$, $Q(t,r) = ar^2 + br + crt + dt^2 + et + f$ and $K(t) = a\mathrm{e}^{bt}$ we have the additional symmetries

$$X_5 = \frac{s\mathrm{e}^{\frac{-1}{2r}}}{2}\partial_r \text{ with gauge } h_5 = r\mathrm{e}^{\frac{1}{2r}} + \frac{1}{2}Ei\left(1, \frac{-1}{2r}\right), \text{ and } X_6 = \mathrm{e}^{\frac{-1}{2r}}\partial_r,$$

where Ei refers to the exponential integral.

3.5 One-variable Cases

In this section, we examine single variable metric functions in both t and r. Some specific choices of the metric functions admit additional generators. For example, we choose to consider quadratic, linear, logarithmic, exponential, reciprocal functions etc.

$K(r) = P(r) = Q(r) = ar^2 + br + c:$

$$X_1,\ X_3,\ X_4,\ X_7 = \frac{-s}{2}\partial_t \text{ with gauge } h_7 = t,\ X_8 = \partial_t.$$

$K(t) = P(t) = Q(t) = at^2 + bt + c:$

$$X_1,\ X_2,\ X_3,\ X_9 = \partial_r,\ X_{10} = r\partial_\theta - \theta\partial_r,\ X_{11} = r\partial_\phi - \phi\partial_r.$$

$K(r) = P(r) = Q(r) = a\ln(r) + b:$

$$X_1,\, X_2,\, X_3,\, X_4,\, X_7,\, X_8,\, X_{12} = \frac{\phi}{a+2}\partial_\phi + \frac{\theta}{2}\partial_\theta + \frac{r}{a+2}\partial_r + \frac{t}{2}\partial_t + s\partial_s.$$

$K(t) = P(t) = Q(t) = a\ln(t) + b:$

$$X_1,\, X_2,\, X_3,\, X_9,\, X_{10},\, X_{11},\, X_{13} = \frac{t}{2}\partial_t - \frac{\phi(a-2)}{4}\partial_\phi + \frac{\theta}{2}\partial_\theta + s\partial_s - \frac{r(a-2)}{4}\partial_r.$$

$K(r) = P(r) = Q(r) = a\mathrm{e}^{br}:$

$$X_1,\, X_2,\, X_3,\, X_4,\, X_7,\, X_8.$$

$K(t) = P(t) = Q(t) = a\mathrm{e}^{bt}:$

$$X_1,\, X_2,\, X_3,\, X_4,\, X_9,\, X_{10},\, X_{11}.$$

$K(r) = P(r) = Q(r) = ar + b:$

This is the case with the most additional Noether symmetries.

$$X_1,\, X_2,\, X_3,\, X_4,\, X_7,\, X_8,\, X_{14} = \frac{\theta}{2}\partial_\theta + \frac{1}{a}\partial_r + \frac{t}{2}\partial_t + s\partial_s,$$

$$X_{15} = \frac{as}{4}\cos\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_\phi + \frac{as}{4}\sin\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_r \text{ with gauge } h_{15} = \sin\left(\frac{1}{2}a\phi\right)\mathrm{e}^{\frac{1}{2}ar},$$

$$X_{16} = \frac{as}{4}\cos\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_r - \frac{as}{4}\sin\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_\phi \text{ with gauge } h_{16} = \cos\left(\frac{1}{2}a\phi\right)\mathrm{e}^{\frac{1}{2}ar},$$

$$X_{17} = \frac{at}{2}\sin\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_r + \sin\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{ar}{2}}\partial_t + \frac{at}{2}\cos\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_\phi,$$

$$X_{18} = \frac{au}{2}\cos\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_r + \cos\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{ar}{2}}\partial_t - \frac{at}{2}\sin\left(\frac{a\phi}{2}\right)\mathrm{e}^{\frac{-ar}{2}-b}\partial_\phi,$$

$$X_{19} = \mathrm{e}^{\frac{-ar}{2}}\sin\left(\frac{a\phi}{2}\right)\partial_r + \mathrm{e}^{\frac{-ar}{2}}\cos\left(\frac{a\phi}{2}\right)\partial_\phi,$$

$$X_{20} = \mathrm{e}^{\frac{-ar}{2}}\cos\left(\frac{a\phi}{2}\right)\partial_r - \mathrm{e}^{\frac{-ar}{2}}\sin\left(\frac{a\phi}{2}\right)\partial_\phi.$$

$K(t) = P(t) = Q(t) = at + b$:

$$X_1,\ X_2,\ X_3,\ X_4,\ X_9,\ X_{10},\ X_{11},$$
$$X_{21} = \frac{a^2\theta\phi}{2}\mathrm{e}^b\partial_\phi - a\theta\mathrm{e}^b\partial_t + \frac{a^2\theta r}{2}\mathrm{e}^b\partial_r + \left(\mathrm{e}^{-au} - \frac{a^2}{4}(r^2+\phi^2)\mathrm{e}^b\right)\partial_\theta,$$

$$X_{22} = \frac{a^2(\phi^2 - r^2) - 4\mathrm{e}^{-at-b}}{4a}\partial_r - \frac{ar\phi}{2}\partial_\phi + r\partial_t,$$

$$X_{23} = \frac{a^2(r^2 - \phi^2) - 4\mathrm{e}^{-at-b}}{4a}\partial_\phi - \frac{ar\phi}{2}\partial_r + \phi\partial_t,$$

$$X_{24} = \partial_t - \frac{ar}{2}\partial_r - \frac{a\phi}{2}\partial_\phi.$$

$K(r) = P(r) = Q(r) = \frac{1}{r}$:

$$X_1,\, X_2,\, X_3,\, X_4,\, X_7,\, X_8.$$

$K(t) = P(t) = Q(t) = \frac{1}{t}$:

$$X_1,\, X_2,\, X_3,\, X_4,\, X_9,\, X_{10},\, X_{11}.$$

$K(r) = P(r) = Q(r) = ar^3 + br^2 + cr + d$:

$$X_1,\, X_2,\, X_3,\, X_4,\, X_7,\, X_8.$$

$K(t) = P(t) = Q(t) = at^3 + bt^2 + ct + d$:

$$X_1,\, X_2,\, X_3,\, X_4,\, X_9,\, X_{10},\, X_{11}.$$

$K(r) = P(r) = ar^2 + br + c$, $Q(r) = a\ln(r) + b$:

$$X_1,\, X_3,\, X_4,\, X_7,\, X_8.$$

$K(t) = P(t) = at^2 + bt + c$, $Q(t) = a\ln(t) + b$.

$$X_1,\ X_3,\ X_4,\ X_9,\ X_{11}.$$

$K(r) = P(r) = a\ln(r) + b$, $Q(r) = ar^2 + br + c:$

$$X_1,\ X_3,\ X_4,\ X_7,\ X_8,\ X_{12}.$$

$K(t) = P(t) = a\ln(t) + b$, $Q(u) = at^2 + bt + c:$

$$X_1,\ X_3,\ X_4,\ X_9,\ X_{11},\ X_{13}.$$

$K(r) = P(r) = a\mathrm{e}^{br}$, $Q(r) = ar + b:$

$$X_1,\ X_3,\ X_4,\ X_7,\ X_8.$$

$K(t) = P(t) = a\mathrm{e}^{bt}$, $Q(t) = au + b$.

$$X_1,\ X_3,\ X_4,\ X_9,\ X_{11}.$$

$K(r) = P(r) = ar + b$, $Q(r) = a\mathrm{e}^{br}$.

$$X_1,\ X_3,\ X_4,\ X_7,\ X_8,\ X_{14},\ X_{15},\ X_{16},\ X_{17},\ X_{18},\ X_{19},\ X_{20}.$$

$K(t) = P(t) = at + b$, $Q(t) = a\mathrm{e}^{bt}$.

$$X_1,\ X_3,\ X_4,\ X_9,\ X_{11},\ X_{22},\ X_{23},\ X_{24}.$$

$K(r) = Q(r) = a\mathrm{e}^{br}$, $P(r) = ar + b$:

$$X_1,\ X_2,\ X_3,\ X_4,\ X_7,\ X_8.$$

$K(t) = Q(t) = a\mathrm{e}^{bt}$, $P(t) = at + b$:

$$X_1,\ X_2,\ X_3,\ X_4,\ X_9.$$

$P(r) = Q(r) = a\ln(r) + b$, $K(r) = \frac{1}{r}$:

$$X_1,\ X_3,\ X_4,\ X_7,\ X_8.$$

$P(t) = Q(t) = a\ln(t) + b$, $K(t) = \frac{1}{t}$:

$$X_1,\ X_3,\ X_4,\ X_9,\ X_{10}.$$

$P(r) = ar + b$, $Q(r) = ar^2 + br + c$, $K(r) = a\ln(r) + b$:

$$X_1,\ X_3,\ X_4,\ X_7,\ X_8.$$

$P(t) = at^3 + bt^2 + ct + d$, $Q(t) = \frac{1}{t}$, $K(t) = a\mathrm{e}^{bt}$:

$$X_1,\ X_3,\ X_4,\ X_9.$$

$P(r) = ar^3 + br^2 + cr + d$, $Q(r) = \frac{1}{r}$, $K(t) = a\mathrm{e}^{bt}$:

$$X_1,\ X_3,\ X_4,$$

$$X_{25} = \frac{s\mathrm{e}^{-d}}{2}\mathrm{e}^{\frac{-r(ar^2+br+c)}{2}}\partial_r \text{ with gauge } h_{25} = \int \mathrm{e}^{r(ar^2+br+c)}\mathrm{e}^{\frac{-r(ar^2+br+c)}{2}}dr,$$

$$X_{26} = \left(\int \mathrm{e}^{\frac{ar^4+br^3+cr^2-1}{r}}\mathrm{e}^{\frac{-r(ar^2+br+c)}{2}}dr\right)\partial_\theta,$$

$$X_{27} = -\theta\mathrm{e}^{-d}\mathrm{e}^{\frac{-r(ar^2+br+c)}{2}}\partial_r,\ X_{28} = \mathrm{e}^{-\frac{-r(ar^2+br+c)}{2}}\partial_r.$$

$P(r) = ar^3 + br^2 + cr + d$, $Q(t) = \frac{1}{t}$, $K(t) = a\mathrm{e}^{bt}$:

$$X_1,\ X_3,\ X_4,\ X_{25},\ X_{28}.$$

$P(t) = at + b$, $Q(t) = at^2 + bt + c$, $K(r) = a\ln(r) + b$:

$$X_1,\ X_3,\ X_4,\ X_{29} = \partial_t - \frac{ar}{2}\partial_r + \frac{a^2\phi}{4}\partial_\phi.$$

$P(t) = \frac{1}{t}$, $Q(r) = ar + b$, $K(r) = a\ln(r) + b$:

$$X_1,\ X_3,\ X_4.$$

3.6 Corresponding First Integrals

We display the corresponding Noether integrals in Tables 3.1 and 3.2.

Table 3.1: The First Integrals $I_1 - I_{14}$.

Noether Symmetry	**First Integral** I_k
X_1	$I_1 = \frac{1}{2}\bigg(- \dot{t}^2 + \mathrm{e}^{P(t,r)}\dot{r}^2 + \mathrm{e}^{Q(t,r)}\dot{\theta}^2 + \mathrm{e}^{K(t,r)}\dot{\phi}^2 \bigg),$
X_2	$I_2 = -\mathrm{e}^{K(t,r)}\dot{\phi}\theta + \mathrm{e}^{Q(t,r)}\dot{\theta}\phi,$
X_3	$I_3 = -\mathrm{e}^{K(t,r)}\dot{\phi},$
X_4	$I_4 = -\mathrm{e}^{Q(t,r)}\dot{\theta},$
X_5	$I_5 = -1/2\, s\dot{r}\mathrm{e}^{1/2\, r^{-1}},$
X_6	$I_6 = -\dot{r}\mathrm{e}^{1/2\, r^{-1}},$
X_7	$I_7 = -1/2\, \dot{t}s,$
X_8	$I_8 = \dot{t},$
X_9	$I_9 = -\mathrm{e}^{at^2+bt+c}\dot{r},$
X_{10}	$I_{10} = -\mathrm{e}^{Q(t)}\dot{\theta}r + \mathrm{e}^{P(t)}\dot{r}\theta,$
X_{11}	$I_{11} = -\mathrm{e}^{K(t)}\dot{\phi}r + \mathrm{e}^{P(t)}\dot{r}\phi,$
X_{12}	$I_{12} = \frac{s}{2}\bigg(- \dot{t}^2 + \mathrm{e}^{P(t,r)}\dot{r}^2 + \mathrm{e}^{Q(t,r)}\dot{\theta}^2 + \mathrm{e}^{K(t,r)}\dot{\phi}^2 \bigg) + \frac{t\dot{t}}{2} - \frac{\mathrm{e}^{P(t,r)}r\dot{r}}{a+2} - \frac{\mathrm{e}^{Q(t,r)}\theta\dot{\theta}}{2} - \frac{\mathrm{e}^{K(t,r)}\phi\dot{\phi}}{a+2},$
X_{13}	$I_{13} = \frac{s}{2}\bigg(- \dot{t}^2 + \mathrm{e}^{P(t,r)}\dot{r}^2 + \mathrm{e}^{Q(t,r)}\dot{\theta}^2 + \mathrm{e}^{K(t,r)}\dot{\phi}^2 \bigg) + \frac{t\dot{t}}{2} - \frac{\mathrm{e}^{P(t,r)}(a-2)r\dot{r}}{4} - \frac{\mathrm{e}^{Q(t,r)}\theta\dot{\theta}}{2} - \frac{\mathrm{e}^{K(t,r)}(a-2)\phi\dot{\phi}}{4},$
X_{14}	$I_{14} = \frac{s}{2}\bigg(- \dot{t}^2 + \mathrm{e}^{P(t,r)}\dot{r}^2 + \mathrm{e}^{Q(t,r)}\dot{\theta}^2 + \mathrm{e}^{K(t,r)}\dot{\phi}^2 \bigg) + \frac{t\dot{t}}{2} - \frac{\mathrm{e}^{P(t,r)}\dot{r}}{a} - \frac{\mathrm{e}^{Q(t,r)}\theta\dot{\theta}}{2}.$

Table 3.2: The First Integrals $I_{15} - I_{29}$.

Noether Symmetry	**First Integral** I_k
X_{15}	$I_{15} = -1/4\, as \left(\dot{r} \sin(1/2\, a\phi) + \dot{\phi} \cos(1/2\, a\phi)\right) \mathrm{e}^{-1/2\, a(-2\,r+r)}$,
X_{16}	$I_{16} = 1/4\, as \left(\dot{\phi} \sin(1/2\, a\phi) - \dot{r} \cos(1/2\, a\phi)\right) \mathrm{e}^{-1/2\, a(-2\,r+r)}$,
X_{17}	$I_{17} = -1/2\, a\dot{\phi}t \cos(1/2\, a\phi)\, \mathrm{e}^{-1/2\, a(-2\,r+r)} + \dot{t} \sin(1/2\, a\phi)\, \mathrm{e}^{1/2\, ar} - 1/2\, \dot{r}at \sin(1/2\, a\phi)\, \mathrm{e}^{-1/2\, a(-2\,r+r)}$,
X_{18}	$I_{18} = 1/2\, a\dot{\phi}t \sin(1/2\, a\phi)\, \mathrm{e}^{-1/2\, a(-2\,r+r)} + \dot{t} \cos(1/2\, a\phi)\, \mathrm{e}^{1/2\, ar} - 1/2\, \dot{r}at \cos(1/2\, a\phi)\, \mathrm{e}^{-1/2\, a(-2\,r+r)}$,
X_{19}	$I_{19} = -\left(\cos(1/2\, a\phi)\, \dot{\phi} + \sin(1/2\, a\phi)\, \dot{r}\right) \mathrm{e}^{-1/2\, ar+ar+b}$,
X_{20}	$I_{20} = \left(\sin(1/2\, a\phi)\, \dot{\phi} - \cos(1/2\, a\phi)\, \dot{r}\right) \mathrm{e}^{-1/2\, ar+ar+b}$,
X_{21}	$I_{21} = -1/2\, \mathrm{e}^{at+2\,b} \dot{\phi} a^2 \theta \phi - \mathrm{e}^{-au+at+b} \dot{\theta} + 1/4\, \mathrm{e}^{at+2\,b} \dot{\theta} a^2 r^2 + 1/4\, \mathrm{e}^{at+2\,b} \dot{\theta} a^2 \phi^2 - 1/2\, \mathrm{e}^{at+b} \dot{r} a^2 \theta r - \dot{t} a \theta \mathrm{e}^b$,
X_{22}	$I_{22} = 1/4\, \frac{\mathrm{e}^{at+b} a^2 r^2 \dot{r} + 2\,\mathrm{e}^{at+b} \dot{\phi} a^2 r \phi - \mathrm{e}^{at+b} a^2 \phi^2 \dot{r} + 4\,\mathrm{e}^{-at+at} \dot{r} + 4\, \dot{t} r a}{a}$,
X_{23}	$I_{23} = 1/4\, \frac{-\mathrm{e}^{at+b} a^2 r^2 \dot{\phi} + 2\,\mathrm{e}^{at+b} \dot{r} a^2 r \phi + \mathrm{e}^{at+b} a^2 \phi^2 \dot{\phi} + 4\,\mathrm{e}^{-at+at} \dot{\phi} + 4\, \dot{t} \phi a}{a}$,
X_{24}	$I_{24} = 1/2\, \mathrm{e}^{at+b} \dot{\phi} a \phi + 1/2\, \mathrm{e}^{at+b} \dot{r} a r + \dot{t}$,
X_{25}	$I_{25} = -1/2\, \mathrm{e}^{at+b-d-1/2\, r\left(ar^2+br+c\right)} r_1 s$,
X_{26}	$I_{26} = -\mathrm{e}^{at+b} \dot{\theta} \int \mathrm{e}^{-1/2\, \frac{-2\, ar^4 - 2\, br^3 - 2\, cr^2 + r\left(ar^2+br+c\right) r + 2}{r}}\, \mathrm{d}r$,
X_{27}	$I_{27} = \mathrm{e}^{at+b-d-1/2\, r\left(ar^2+br+c\right)} \dot{r} \theta$,
X_{28}	$I_{28} = -\mathrm{e}^{at+b+1/2\, r\left(ar^2+br+c\right)} \dot{r}$,
X_{29}	$I_{29} = -1/4\, \mathrm{e}^{at+b} \dot{\phi} a^2 \phi + 1/2\, \mathrm{e}^{at+b} \dot{r} a r + \dot{t}$.

3.7 Conclusion

In this study, we derived general Noether symmetry determining equations for Lagrangians that originate from a diagonal metric. As an example, we performed Noether symmetry classifications of a nonstatic space containing unknown metric functions. Two main cases were identified where more interesting and additional Noether symmetries were obtained for simpler single variable functions. As a final remark we highlight the notable applications of Noether symmetries to cosmology and relativistic astrophysics, as seen above and in the references below for applications involving the selection of viable models.

Chapter 4

Contingent Relations for Klein-Gordon Equations

The last three decades have seen the importance of 2+1-dimensional gravity. This, in part is due to its relevance in black hole studies, quantization of fields coupled to gravity and their solutions, and many other contexts. In fact the discovery of the 2+1 -dimensional Banados-Teitelboim-Zanelli (BTZ) black hole [31] that admits features associated with 3+1-dimensional black hole physics, fuelled many 2+1 gravity research endeavours.

The purpose of this contribution is to provide a classification of electromagnetic and static cyclic gravitational solutions to the Einstein-Maxwell fields in 2+1 gravity, together with their symmetry properties and potential functions of the underlying Klein-Gordon equation. Specifically, we present some uniform electromagnetic solutions, i.e. solutions that possess vanishing covariant derivatives, as well as stationary cyclic symmetric solutions. Additionally, the dynamics of black holes are inherent to some subclasses. The role of symmetries in physics is ever growing. Recent advances can be seen in [23, 32, 28], for example.

In 2+1 gravity with a cosmological constant, solutions are considered that possess stationary and cyclic symmetries. We begin with a stationary cyclic symmetric space-time with signature $(-,+,+)$. Among the classes of solutions with a cosmological constant, we investigate a general metric for static cyclic symmetric gravitational fields coupled to Maxwell electromagnetic fields. A number of solutions exist in this context. Naturally, one encounters well known solutions such as the Peldan [33] and Clement [34] fields. In this regard, we refer the reader to [35] for an insightful look into electromagnetic solutions and black hole physics, as well as some extra solutions. Where possible we relate our study to the solutions in the literature, and essentially complement the existing results.

4.1 The Potential Functions

A related study is the investigation of Klein-Gordon equations for general spaces. Specifically, CKVs of a space coincide with the Lie point symmetries $X = \xi^i(x^k)\partial_i + \eta\left(x^k,\Phi\right)\partial_\Phi$ of the Klein-Gordon equation constructed on the space. That is, for the metric g, the equation is

$$\tfrac{1}{\sqrt{|-g|}}\tfrac{\partial}{\partial x^i}\left(\sqrt{|-g|}g^{ij}\tfrac{\partial\Phi}{\partial x^j}\right) + V(x^i)\Phi = 0, \tag{39}$$

and is derived from the Lagrangian

$$L\left(x^i,\Phi,\Phi_i\right) = \frac{1}{2}\sqrt{g}\,g^{ij}\Phi_i\Phi_j - \frac{1}{2}\sqrt{g}\,V(x^i)\,\Phi^2, \tag{40}$$

where the potential $V(x^i)$ is [36, 37, 18]

$$\xi^k V_k + 2\psi V - \frac{2-n}{2}\triangle\psi = 0. \tag{41}$$

Also, the linearity of the Klein-Gordon equation implies that it will always admit the Lie point symmetry $P_{(\Phi)} = \Phi\partial_\Phi$ and the infinite dimensional abelian subalgebra of solutions $P_{(\infty)} = F(x^i)\partial_\Phi$, where $F(x^i)$ is a solution of the Klein-Gordon equation.

4.2 Gravitational Field Heuristics

Consider the general 2+1- dimensional gravity line element

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2, \tag{42}$$

for stationary cyclic symmetric. Coordinates may be chosen to express (42) as

$$ds^2 = -\frac{F(r)}{H(r)}dt^2 + \frac{dr^2}{F(r)} + H(r)\left(d\phi + W(r)dt\right)^2, \tag{43}$$

without loss of generality. In the presence of an electromagnetic field, the field contravariant tensor is given by

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & \beta & -\frac{\gamma}{F(r)} \\ -\beta & 0 & \alpha \\ \frac{\gamma}{F(r)} & -\alpha & 0 \end{pmatrix}$$

where the constants α, β, γ characterise the field. The energy-momentum tensor is

$$\begin{pmatrix} \frac{\left(\beta^2H^2W^2 - F\beta^2 - \alpha^2H^2 - \gamma^2H\right)}{8\pi FH} & \frac{\alpha\gamma}{4\pi} & -\frac{\alpha\left(\beta H^2W^2 - F\beta - \alpha H^2W\right)}{4\phi FH} \\ \frac{\gamma HW\beta - \gamma H\alpha}{4\phi F^2} & \frac{-\beta^2F + H^2\left(W^2\beta^2 - 2W\beta\alpha + \alpha^2\right) + \gamma^2H}{8\pi FH} & -\frac{\gamma\left(\beta H^2W^2 - F\beta - \alpha H^2W\right)}{4\phi F^2H} \\ \frac{HW\beta^2 - \beta H\alpha}{4\phi F} & \frac{\beta\gamma}{4\pi} & -\frac{\left(\beta^2H^2W^2 - F\beta^2 - \alpha^2H^2 + \gamma^2H\right)}{8\pi FH} \end{pmatrix}. \tag{44}$$

The corresponding Einstein-Maxwell equations with cosmological constant $\Lambda = -\frac{1}{l^2}$ are

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} - \frac{1}{l^2}g_{\mu\nu} - 8\pi T_{\mu\nu} = 0,$$

where $g_{\mu\nu}, R_{\mu\nu}$, and $T_{\mu\nu}$ represent the metric, Ricci and energy-momentum tensors, respectively; and R is called the Ricci scalar.

Explicitly, they are

$$
\begin{aligned}
&\frac{1}{2}\left(H^2 W W_{rr} + \frac{F H_{rr}}{H}\right) + \frac{1}{4}\left(H^2 W_r^2 - \frac{F H_r^2}{H^2} + \frac{H_r F_r}{H}\right) \\
&\qquad + \frac{\beta\left(F - W^2 H^2\right)}{FH} + \frac{\gamma^2}{F} \\
&\qquad + H_r H W W_r + \Lambda + \frac{\alpha^2 H}{F}, \\
&\qquad -2\alpha\gamma, \\
&\frac{1}{2}\left(F_{rr} W - F W_{rr} - H^2 W^2 W_{rr}\right) - 2\left(\alpha\beta\frac{F - H^2 W^2}{FH} - \frac{\alpha^2 H W}{F}\right) \\
&\qquad - \frac{F W_r H_r}{H} - H H_r W^2 W_r - \frac{W F_r H_r}{H} - H^2 W W_r \\
&\qquad - \frac{F W H_{rr}}{H} + \frac{F W H_r^2}{H^2}, \\
&\qquad -2\gamma\frac{H\left(W\beta - \alpha\right)}{F^2}, \\
&\frac{1}{4}\left(\frac{F H_r^2}{H^2} - \frac{H_r F_r}{H} + H^2 W_r^2\right) - \frac{H\left(\beta W - \alpha\right)^2}{F} \\
&\qquad + \Lambda + \frac{\beta^2}{H} - \frac{\gamma^2}{F}, \\
&\qquad - \frac{2\gamma\alpha H W}{F^2} - \frac{2\gamma\beta\left(F - H^2 W^2\right)}{F^2 H}, \\
&\frac{1}{2}\left(2 H W_r H_r + H^2 W_{rr} - \frac{4\beta\left(W\beta - \alpha\right) H}{F}\right), \\
&\qquad -2\beta\gamma, \\
&\frac{1}{4}\left(-\frac{3 H_r F_r}{H} + \frac{3 F H_r^2}{H^2} + 2F_{rr} - 2\frac{F H_{rr}}{H}\right) \\
&\qquad - \frac{3}{4} H^2 W_r^2 - \frac{1}{2} H^2 W W_{rr} \\
&\qquad - H W H_r W_r + \frac{\gamma^2}{F} + \frac{\alpha^2 H}{F} + \Lambda - \frac{\beta^2\left(F - H^2 W^2\right)}{FH}.
\end{aligned}
\tag{45}
$$

The geometrical significance of the solutions of Einstein's equations lies in their prediction of singularities, such as black holes.

Solutions may be found by integrating the field equations and by characterising the constants α, β, γ. In the text below, we focus on two primary categories of solutions.

4.3 Static Cyclic Symmetric Solutions

In the static solution of the Einstein-Maxwell system, $W(r) = 0$ and this results in three classes based on the constants α, β, γ: an electric field ($\alpha = \gamma = 0, \beta \neq 0$), a magnetic field ($\beta = \gamma = 0, \alpha \neq 0$) and a hybrid field ($\alpha = \beta = 0, \gamma \neq 0$).

In Table 4.1, we present some spaces that showcase these three classes. Cases I and II are electrostatic solutions, where II is equivalent to a Peldan electrostatic solution [33] with $\Lambda = -\frac{1}{l^2}$ under some coordinate and parameter changes. Furthermore, the solution for case I endowed with mass, electric charge and radial parameters, allows for a charged black hole interpretation [35]. The case III, IV and V fall under magnetostatic solutions. Case III allows for a hydrodynamics interpretation in terms of a perfect fluid energy-momentum tensor for a stiff fluid and Case IV is equivalent to a Peldan magnetostatic solution with $\Lambda = -\frac{1}{l^2}$. Case V is that of the magnetostatic solutions for vanishing cosmological constant, comparable to the solutions of Melvin [38] and Barrow et al. [39]. The Case VI is a static hybrid solution. The last case, case VII corresponds to a vanishing Λ and is equivalent to the findings of Gott et al. [40]. The parameters $C_{0-1}, \kappa_0, m, r_{0-2}, \propto$ are constants. As for KVs, cases I-VI admit the symmetries

$$X_1 = \partial_\phi,\ X_2 = \partial_t, \tag{46}$$

and generate the group $SO(2) \times R$. The Lie bracket is

$$[X_1, X_2] = 0.$$

Table 4.1: Static Cyclic Symmetric Solutions

Case	**Metric**	**Structural Functions & Field Vector**
I	$ds^2 = -\frac{F(r)}{h(r)}dt^2 + \frac{dr^2}{F(r)} + h(r)d\phi^2$	$F(r) = 2r\left[\frac{2r}{l^2} - \beta^2 \ln \frac{r}{r_0}\right]$, $h(r) = 2r$, $A = \frac{\beta}{2} \ln \frac{r}{r_0}\ dt$,
II	$ds^2 = -F(\rho)\, dt^2 + \frac{d\rho^2}{F(\rho)} + \rho^2\, d\phi^2$,	$F(\rho) = \frac{\rho^2}{l^2} - m - 2\beta^2 \ln \rho$, $A = b \ln \rho\ dt$,
III	$ds^2 = -h(r)\, dt^2 + \frac{dr^2}{H(r)h(r)} + H(r)\, d\phi^2$	$H(r) = \frac{4}{C_1^2 l^2}\left[\kappa_0 + h(r) + \alpha^2 l^2 \ln h(r)\right]$, $F(r) = H(r)h(r)$, $h(r) = C_1 r + C_0$, $A = \frac{\alpha}{C_1} \ln h\ d\phi$,
IV	$ds^2 = -\rho^2\, dt^2 + \frac{d\rho^2}{F(\rho)} + F(\rho)\, d\phi^2$	$F(\rho) = \kappa_0 + \frac{\rho^2}{l^2} + 2\alpha^2 \ln \rho$ $A = \alpha \ln \rho\ d\phi$,
V	$ds^2 = -\rho^2\, dt^2 + \frac{d\rho^2}{F(\rho)} + F(\rho)\, d\phi^2$	$F(\rho) = \kappa_0 + 2\alpha^2 \ln \rho$ $A = \alpha \ln \rho\ d\phi$,
VI	$ds^2 = -\frac{F(r)}{H(r)}\, dt^2 + \frac{dr^2}{F(r)} + H(r)\, d\phi^2$	$F(r) = \frac{4}{l^2}(r - r_1)(r - r_2)$, $H(r) = (r - r_1)^{\frac{(1+\sqrt{\propto})}{2}}(r - r_2)^{\frac{1-\sqrt{\propto}}{2}}$ $A = \frac{\gamma}{2}\left(t\ d\phi - \phi\ dt\right)$,
VII	$ds^2 = -F(\rho)\, dt^2 + \frac{d\rho^2}{F(\rho)} + \rho^2\, d\phi^2$	$F(\rho) = \kappa_0 - 2\beta^2 \ln \rho$, $A = \beta \ln \rho\ dt$.

With the application of (41), the KVs X_1 and X_2 are potential functions of the Klein-Gordon equation (39) contingent upon the potential function being of the form

$$V_1\left(t,r\right),\ V_2\left(\phi,r\right) \tag{47}$$

respectively, for cases I, III and VI. The subalgebra $Y = \{X_1, X_2\}$ and the field created by the linear combination $Z = aX_1 + bX_2$ are potential functions of the Klein-Gordon equation (39) if and only if

$$V_Y\left(r\right), \tag{48}$$

$$V_Z\left(\frac{b\phi - at}{b}, r\right), \tag{49}$$

respectively. For cases II, IV, V and VII the potentials are equivalent except r is replaced with ρ.

4.4 Uniform Electromagnetic Solutions

In this section, we turn our attention to uniform electromagnetic solutions that occur for zero covariant derivatives $F_{\mu\nu;\gamma}$. To this end, families of solutions can be found for $\alpha \neq 0$ and/or $\beta \neq 0$, but not for the hybrid situation $\gamma \neq 0$.

4.4.1 Uniform Stationary Solutions

The metric (43) has the equivalent formulation

$$ds^2 = -\frac{F(r)}{h(r)}\left(dt - \omega(r)d\phi\right)^2 + h(r)d\phi^2 + \frac{dr^2}{F(r)}, \tag{50}$$

with structural function relations for (43) and (50) as

$$F(r) = F(r),\ H(r) = h(r) - \frac{F(r)\omega^2(r)}{h(r)},\ W(r) = \frac{\omega(r)F(r)}{H(r)h(r)}. \tag{51}$$

Suppose that $\alpha = \gamma = 0$ and the covariant derivatives $F_{\phi r;r} = F_{tr,r} = 0$. Then $h(r) = h;\ \omega(r) = \omega$ where h, ω are required to be constants. Furthermore if $F(r) = F$ =constant, we find the maximal dimensional symmetry algebra. In particular we redefine the structural function

$$F(r) = \frac{2}{l^2}\mathrm{e}^{\left(\frac{2\sqrt{2}\propto}{l}\right)}, \quad \propto \ \text{constant} \tag{52}$$

and

$$H(r) = \beta^2 l^2,$$

the isometries expand to:

$$X_1,\ X_2,\ X_3 = \partial_r,\ X_4 = \phi\partial_\phi,$$

and

$$
\begin{aligned}
X_5 &= \frac{\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)t}{\beta^4 l^6}\partial_\phi, \\
X_6 &= -\frac{r}{\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)\beta^2}\partial_\phi, \\
X_7 &= \frac{\left(\mathrm{e}^{\frac{4\sqrt{2}\propto}{l}} l^4+4\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+4\right)t}{\beta^2 l^2}\partial_r, \\
X_8 &= t\phi\partial_t+\phi r\partial_r+\left[\frac{\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)t^2}{2\beta^4 l^6}+\frac{\phi^2}{2}-\frac{r^2}{2\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)\beta^2}\right]\partial_\phi, \\
X_9 &= rt\partial_t+\phi r\partial_\phi+ \\
&\left[\frac{r^2}{2}+\frac{\left(\mathrm{e}^{\frac{4\sqrt{2}\propto}{l}} l^4+4\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+4\right)t^2}{2\beta^2 l^2}-\frac{\beta^2\left(\mathrm{e}^{\frac{4\sqrt{2}\propto}{l}} l^4+4\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+4\right)\phi^2}{\left(2\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+4\right)}\right]\partial_r, \\
X_{10} &= \left[\frac{\phi^2}{2}+\frac{\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)t^2}{2\beta^4 l^6}+\frac{\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)r^2}{2\beta^2\left(\mathrm{e}^{\frac{4\sqrt{2}\propto}{l}} l^4+4\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+4\right)}\right]\partial_t \\
&+\frac{rt\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)t}{\beta^4 l^6}\partial_r+\frac{t\phi\left(\mathrm{e}^{\frac{2\sqrt{2}\propto}{l}} l^2+2\right)t}{\beta^4 l^6}\partial_\phi.
\end{aligned}
\tag{53}
$$

In this case, the contingency relations for the above vector fields to be Lie symmetries of Eq. (39), are

$$
\begin{aligned}
X_3 \ \&\ X_7 &: V_3(t,\phi), \\
X_5 &: V_4(t,\phi,r)=\frac{1}{\phi^{2/3}}P(t,r), \\
X_5 \ \&\ X_6 &: V_5(t,r),
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
X_8 \quad : \quad V_8\left(t,\phi,r\right) &= \tfrac{1}{t^2} Q\Bigg(\tfrac{r}{t}, \tfrac{1}{l^6\beta^4 t}\Bigg(\mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}}\beta^4 l^8\phi^2 + 2\,\phi^2\beta^4 l^6 \\
&\quad + r^2\beta^2 l^6 - \mathrm{e}^{4\,\frac{\sqrt{2}\alpha}{l}} l^4 t^2 - 4\,\mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}} l^2 t^2 - 4\,t^2\Bigg) \\
&\quad \left(2 + \mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}} l^2\right)^{-1}\Bigg), \qquad (55)\\
X_9 \quad : \quad V_9\left(t,\phi,r\right) &= \tfrac{1}{t^2} S\Bigg(\tfrac{\phi}{t}, \tfrac{1}{\beta^2 l^2 t}\Bigg(\mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}}\beta^4 l^4\phi^2 + 2\,\beta^4\phi^2 l^2 \\
&\quad - \mathrm{e}^{4\,\frac{\sqrt{2}\alpha}{l}} l^4 t^2 + r^2\beta^2 l^2 - 4\,\mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}} l^2 t^2 - 4\,t^2\Bigg)\Bigg).
\end{aligned}
$$

The potential corresponding to X_{10} is voluminous and therefore omitted. In addition, some subalgebras of the maximal algebra (53) admit potentials of the form:

$$
\begin{aligned}
\{X_3, X_4\} \quad &: \quad \tfrac{1}{\phi^{2/3}} V\left(t\right), \\
\{X_3, X_5\}\ \&\ \{X_3, X_6\} \quad &: \quad V\left(t\right).
\end{aligned} \qquad (56)
$$

Moreover, the following linear combinations of vector fields are Lie point symmetries of Eq. (39) contingent upon V being of the form:

$$
\begin{aligned}
jX_3 + X_4 \quad &: \quad \tfrac{1}{\phi^{2/3}} V\left(t, r - j\ln\phi\right), \\
jX_3 + X_6 \quad &: \quad V\left(t, 2\mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}} l^2\beta^2\phi + r^2 + 4b^2 j\phi\right), \\
jX_3 + X_5 \quad &: \quad V\Bigg(t, \tfrac{1}{t}\left(-jb^4 l^6\phi + \mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}} r t l^2 + 2\,tr\right) \\
&\qquad \left(2 + \mathrm{e}^{2\,\frac{\sqrt{2}\alpha}{l}} l^2\right)^{-1}\Bigg), \qquad (57)\\
kX_4 + X_7 \quad &: \quad V\left(t,\phi,r\right) = \tfrac{1}{\phi^{2/3}} T\Bigg(t, \tfrac{1}{b^2 l^2 k}\Bigg(- t l^4 \ln\left(\phi\right)\mathrm{e}^{4\,\frac{\sqrt{2}m}{l}} \\
&\qquad + r b^2 l^2 k - 4\,t\ln\left(\phi\right)\mathrm{e}^{2\,\frac{\sqrt{2}m}{l}} l^2 - 4\,t\ln\left(\phi\right)\Bigg)\Bigg).
\end{aligned}
$$

On the other hand, from the Einstein-Maxwell field equations (45) we have that

$$
F(r) = \frac{2r^2}{l^2} + c_1 r + c_0, \;\; h(r) = \beta^2 l^2, \quad c_{0-1} \text{ constant.} \qquad (58)
$$

The corresponding field vector is

$$A = \frac{r}{\beta l^2}(dt - \omega d\phi).$$

This uniform stationary electromagnetic metric only admits the $SO(2) \times R$ algebra as KVs

$$X_1, X_2.$$

Moreover, [34] reported a uniform stationary solution equivalent to

$$F(r) = \frac{2r^2}{l^2} + c_1 r + c_0,\ h(r) = \frac{2\kappa\pi_0 l^2}{4}, \quad k, \pi_0 \text{ constant.} \tag{59}$$

The field vector here is:

$$A = \frac{r}{l\sqrt{h(r)}}(dt - \omega d\phi).$$

This case also admits the $SO(2) \times R$ algebra as KVs

$$X_1, X_2.$$

The potential functions associated with the cases of (58) and (59) are identical to those derived in (47)- (49).

4.4.2 Uniform Electrostatic Solutions

Consider once again the metric (43). A case of a uniform electrostatic solutions arises when $\alpha = 0,\ \beta \neq 0$ and $W(r) = 0,\ H(r) = H_0 = constant$. As such the field equations mandate that $H_0 = \beta^2 l^2$, together with the other structural function

$$F(r) = \frac{2r^2}{l^2} + 4c_1 r + c_0,\ c_{0-1} \text{ constant.} \tag{60}$$

The metric and field vector are:

$$
\begin{aligned}
ds^2 &= -\frac{F(r)}{\beta^2 l^2}\,dt^2 + \frac{dr^2}{F(r)} + \beta^2 l^2\,d\phi^2, \\
A &= \frac{r}{\beta l^2} dt.
\end{aligned}
\tag{61}
$$

This solution is equivalent to the one presented in [41] in terms of hyperbolic structural functions and is comparable to the Bertotti [42] and Robinson [43] uniform electromagnetic solution (for a constant slice of a spatial coordinate).

The conformal vectors of this space are:

$$X_1,\ X_2,$$

and the vector fields

$$
\begin{aligned}
X_{11} &= \frac{(l^2c_1+r)\cos\left(\frac{\sqrt{-4l^2c_1^2+2c_0}\,t}{\beta l^2}\right)}{\sqrt{(4rc_1+c_0)l^2+2r^2}}\partial_t \\
&\quad + \frac{\sqrt{(4rc_1+c_0)l^2+2r^2}\sqrt{-4l^2c_1^2+2c_0}\sin\left(\frac{\sqrt{-4l^2c_1^2+2c_0}\,t}{\beta l^2}\right)}{\beta l^2}\partial_r, \\
X_{12} &= \frac{(l^2c_1+r)\sin\left(\frac{\sqrt{-4l^2c_1^2+2c_0}\,t}{\beta l^2}\right)}{\sqrt{(4rc_1+c_0)l^2+2r^2}}\partial_t \\
&\quad - \frac{\sqrt{(4rc_1+c_0)l^2+2r^2}(4l^2rc_1+l^2c_0+2r^2)\cos\left(\frac{\sqrt{-4l^2c_1^2+2c_0}\,t}{\beta l^2}\right)}{\sqrt{(4rc_1+c_0)l^2+2r^2}\,\beta l^2}\partial_r.
\end{aligned}
\tag{62}
$$

As before, the above symmetries are Lie point symmetries of Eq. (39) subject to the potential functions for this case being (47)- (49), and:

$$
\begin{aligned}
V_{11}\Bigg(\phi, \frac{1}{\sqrt{-4\,c_1^2l^2+2\,c_0}}\bigg(\tfrac{1}{4}\,\beta l^2\ln\left(4\,c_1l^2r+c_0l^2+2\,r^2\right) \\
-\int\cot\left(\frac{\sqrt{-4\,c_1^2l^2+2\,c_0t}}{\beta l^2}\right)\mathrm{d}t \\
\sqrt{-4\,c_1^2l^2+2\,c_0}\bigg)\Bigg)\sigma_0,
\end{aligned}
\tag{63}
$$

where

$$\sigma_0 = \exp\left(\int -\frac{2}{3} \frac{2\sqrt{-4\,c_1^2 l^2 + 2\,c_0}\sqrt{-4\,c_1^2 l^2 + 2\,c_0 t} - c_0}{\sqrt{-4\,c_1^2 l^2 + 2\,c_0 t}\,\beta l^2} \cot\left(\frac{\sqrt{-4\,c_1^2 l^2 + 2\,c_0 t}}{\beta l^2} \right) \mathrm{d}t \right), \tag{64}$$

and for $c_0 = 0$, X_{12} is a Lie symmetry of (39) contingent upon the potential being

$$V_{12} = \frac{1}{\left(r\left(-2\,c_1 l^2 + r\right)\right)^{2/3}} \, T\left(-\frac{1}{16}\frac{1}{c_1^2 l^2 r}\left(3\,\sin\left(2\,\frac{\sqrt{-c_1^2 l^2}}{\beta l^2}\right) \beta l^2 \sigma_1 c_1 r - 2\,\sin\left(2\,\frac{\sqrt{-c_1^2 l^2}}{\beta l^2}\right)\beta\sqrt{c_1^2 l^4}\sqrt{2}\sqrt{r\left(2\,c_1 l^2 + r\right)} - 16\,t\cos\left(2\,\frac{\sqrt{-c_1^2 l^2}}{\beta l^2}\right)\sqrt{c_1^2 l^4} c_1 r\right) \left(\cos\left(2\,\frac{\sqrt{-c_1^2 l^2}}{\beta l^2}\right)\right)^{-1}, \phi \right), \tag{65}$$

where

$$\sigma_1 = \ln\left(-2\,\frac{2\,c_1^2 l^4 + 3\,c_1 l^2 r + 2\,\sqrt{2}\sqrt{c_1^2 l^4}\sqrt{r\left(2\,c_1 l^2 + r\right)}}{2\,c_1 l^2 - r} \right).$$

4.4.3 Uniform Stationary Magnetic Solutions

In the case of positive cosmological constant $\Lambda = \frac{1}{l^2}$ there exists a uniform stationary magnetic solution with $W(r) = W_0 =$ constant. This solution is equivalent to Clement's solution [34] with $\Lambda = \frac{1}{l^2}$. The metric functions are

$$F(r) = -\frac{2r^2}{l^2} + 4c_1 r + c_0, \quad c_{0-1} \text{ constant}, \tag{66}$$

$$H(r) = \frac{F(r)}{\alpha^2 l^2}. \tag{67}$$

The metric and field vector are:

$$ds^2 = -\frac{F(r)}{H(r)}\,dt^2 + \frac{dr^2}{F(r)} + H(r)\,\left(d\phi + W_0 dt\right)^2, \tag{68}$$

$$A = \frac{r\,(d\phi + W_0)}{\alpha l^2}.$$

The conformal symmetries in this case are

$$X_1,\ X_2,\ X_{13} = \left(-t - \frac{\phi}{W_0}\right)\partial_t + \left(\phi + \frac{(-\alpha^4 l^4 + W_0^2)}{W_0}\right)\partial_\phi. \tag{69}$$

As before, X_1 and X_2 are Lie point symmetries if and only if the potentials are (47)-(49). For X_{13} to be a Lie point symmetry, the necessary form of the potential is

$$V_{13}\left(\alpha^4 l^4 t - \phi t W_0 - t W_0^2 - \frac{1}{2}\phi^2, r\right).$$

On the other hand, the contingency relation of the field $aX_1 + bX_2 + X_{13}$ to be a Lie point symmetry is

$$V\left(\alpha^4 l^4 t - \beta\phi W_0 - \phi t W_0 - a t W_0 - t W_0^2 - \frac{1}{2}\phi^2, r\right).$$

4.5 Conclusion

Point symmetries are powerful instruments in physics - Lie and Noether vectors are often applied in several different scenarios across the literature. The recurring theme in this text is the usage of geometric tools to derive physical potential functions. We have exploited a distinctive geometric connection between generators of the Klein-Gordon equation and the collineations of the spacetime metric. Hence we have documented the potential functions, for some appropriate choice of the KVs describing the underlying geometry. Interestingly, linear combinations and real subalgebras of the conformal algebra, led to an expansion in the number of results. Finally, we point out that the results contained here may be used to construct conservation laws for the equation of motions of a particle in the classical or semi-classical approach.

Chapter 5

Noether Generators and the Klein-Gordon Potential on Spaces with Nonzero Weyl Tensor

Spacetimes with nonzero Weyl tensor (i.e. non conformally flat spaces), can admit a maximum of seven conformal Killing vectors - an example of such a space is the homogeneous type N plane wave spacetimes [44]. In this work we put forward a classification of the potential functions inherent to Klein-Gordon equations based on some conformal-reducible spaces with nonzero Weyl tensor. Such conformal-reducible spacetimes may include all non conformally flat spherical symmetric, plane symmetric and hyperbolic symmetric spacetimes [45]. Recall that the general space-time admitting a group G_3 on spacelike orbits V_2 is [46]

$$ds^2 = -e^{2\nu}dt^t + e^{2\omega}dx^2 + \mu^2 \left(dy^2 + \sigma(y,k)\ dz^2\right)$$

where ν, ω, μ are functions of t and x, and

- $\sigma(y,k) = \sin y$ if $k = 1$,
- $\sigma(y,k) = y$ if $k = 0$, and
- $\sigma(y,k) = \sinh y$ if $k = -1$.

In particular, $k = 1$ corresponds to spherical symmetry for which the spatial coordinates (x, y, z) are usually recast as (r, θ, ϕ); $k = 0$ corresponds to plane symmetry for which it is conventional to use Cartesian coordinates in the orbits so that $\sigma \equiv 1$; and $k = -1$ corresponds to hyperbolic symmetry.

At this junction, let us turn to the partial differential equation under study. Based on the previous chapter, a clear connection can be seen between the Klein-Gordon equation and the geometry described by the space g. With this in mind, the primary purpose in this paper is to construct various functional forms of the potential corresponding to elements of the conformal algebra once again. Here, we relabel the linear symmetries as $\chi_{(\Phi)} = \Phi\partial_\Phi$ and the infinite dimensional abelian subalgebra of solutions $\chi_{(\infty)} = M(x^i)\partial_\Phi$, where $M(x^i)$ is a solution of the Klein-Gordon equation.

We shall explore a vacuum Taub space, Kasner vacuum space, a space admitting a sCKV and a plane symmetric Bianchi I spacetime, and show that the conformal groups of these spaces lead to a detailed discovery of the potential functions of the Klein-Gordon equation.

5.1 Conformal-reducible spaces

We now turn our attention to some examples of conformal-reducible spaces which are not conformally flat. To start, we first mention two vacuum spaces, the Taub and Kasner solutions, followed by some plane symmetric cases below. In particular, we have considered spaces that are plentiful in symmetry to obtain a classification of more depth. The spaces we consider admit between three and four KVs - some also admit a HV, CKV or sCKV. We shall obtain the potential functions admitted by the Klein-Gordon equation (39) within the aforementioned spaces.

5.1.1 CKV Class I:

Suppose we take the static vacuum solution [47], which has the spacetime metric, in Cartesian coordinates,

$$ds^2 = -\frac{dt^2}{\sqrt{x}} + \frac{dx^2}{\sqrt{x}} + xdy^2 + xdz^2. \tag{70}$$

This case admits a homothetic algebra comprising of the four KVs

$$X_1^I = \partial_t,\ X_2^I = \partial_y,\ X_3^I = \partial_z,\ X_4^I = z\partial_y - y\partial_z, \tag{71}$$

and the HV

$$H^I = 4t\partial_t + 4x\partial_x + y\partial_y + z\partial_z, \tag{72}$$

with homothetic constant $\psi^I = 3$. The commutators of these vector fields are provided in Table 5.1.

From condition (41), we have that X_1^I to X_4^I are Lie and Noether point symmetries of the Klein-Gordon equation (39), if and only if the potential $V(t,x,y,z)$ is of the respective forms

$$\begin{aligned} X_1^I &: V_1^I = V(x,y,z), \\ X_2^I &: V_2^I = V(t,x,z), \\ X_3^I &: V_3^I = V(t,x,y), \\ X_4^I &: V_4^I = V(t,x,y^2+z^2). \end{aligned} \tag{73}$$

Table 5.1: Lie brackets of the conformal algebra.

	${X^I}_1$	${X^I}_2$	${X^I}_3$	${X^I}_4$	H^I
X_1^I	0	0	0	0	$4X_1^I$
X_2^I	0	0	0	$-X_3^I$	X_2^I
X_3^I	0	0	0	X_2^I	X_3^I
X_4^I	0	X_3^I	$-X_2^I$	0	0
H^I	$-4X_1^I$	$-X_2^I$	$-X_3^I$	0	0

The HV H^I is a Lie point symmetry of Eq. (39) when

$$V_H^I = \frac{1}{t^{\frac{3}{2}}} V\left(\frac{x}{t}, \frac{y}{\sqrt[4]{t}}, \frac{z}{\sqrt[4]{t}}\right).$$

Additionally, more can be said about the potential, if one considers certain linear combinations of the vectors. That is, from linear combinations of the KVs

$$\begin{aligned} X_{(1)}^I &= X_1^I + X_2^I, \\ X_{(2)}^I &= X_1^I + X_3^I, \\ X_{(3)}^I &= X_1^I + X_4^I, \\ X_{(4)}^I &= X_2^I + X_3^I, \\ X_{(5)}^I &= X_2^I + X_4^I, \\ X_{(6)}^I &= X_3^I + X_4^I, \end{aligned} \tag{74}$$

we have that the vector fields (96) are Lie/Noether point symmetries of (39), when

$$\begin{aligned} X_{(1)}^I &: V_{(1)}^I = V\left(x, -t+y, z\right), \\ X_{(2)}^I &: V_{(2)}^I = V\left(x, y, -t+z\right), \\ X_{(3)}^I &: V_{(3)}^I = V\left(x, y^2+z^2, -\arctan\left(\tfrac{y}{z}\right)+t\right), \\ X_{(4)}^I &: V_{(4)}^I = V\left(t, x, -y+z\right) \\ X_{(5)}^I &: V_{(5)}^I V\left(t, x, -\tfrac{z^2}{2} - \tfrac{y^2}{2} - z\right), \\ X_{(6)}^I &: V_{(6)}^I = V\left(t, x, y^2+z^2-2\,y\right). \end{aligned} \tag{75}$$

Regarding the HV, the construction of linear combinations with the four KVs of the space, are the following

$$H_{(1)}^{I} = X_{1}^{I} + H^{I},\ H_{(2)}^{I} = X_{2}^{I} + H^{I}, \tag{76}$$

and

$$H_{(3)}^{I} = X_{3}^{I} + H^{I},\ H_{(4)}^{I} = X_{4}^{I} + H^{I}. \tag{77}$$

Here, we find that the vector fields (98) and (99) are Lie/Noether point symmetries, if and only if the corresponding potential functions of (39) are

$$\begin{aligned} H_{(1)}^{I} &: V_{(H1)}^{I} = \frac{1}{\sqrt{(1+4t)^{3}}} V\left(\frac{x}{1+4t}, \frac{y}{\sqrt[4]{1+4t}}, \frac{z}{\sqrt[4]{1+4t}}\right), \\ H_{(2)}^{I} &: V_{(H2)}^{I} = \frac{1}{t^{\frac{3}{2}}} V\left(\frac{x}{t}, \frac{1+y}{\sqrt[4]{t}}, \frac{z}{\sqrt[4]{t}}\right), \\ H_{(3)}^{I} &: V_{(H3)}^{I} = \frac{1}{t^{\frac{15}{16}}} V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{1+z}{\sqrt[4]{t}}\right). \end{aligned} \tag{78}$$

We find no real solution for $H_{(4)}^{I}$. We also note that the plane symmetric non null Einstein-Maxwell solution [48]

$$ds^{2} = -\left(mx^{-1} + e^{2}x^{-2}\right) dt^{2} + \left(mx^{-1} + e^{2}x^{-2}\right)^{-1} dx^{2} + x^{2}\left(dy^{2} + dz^{2}\right), \tag{79}$$

possesses a four-dimensional Killing algebra, i.e. its KVs are (93). Hence, the Lie/Noether symmetries and potentials in this case all follow from the above.

5.1.2 CKV Class II

For this case, consider the Kasner vacuum solution [49]

$$ds^{2} = -\frac{dt^{2}}{\sqrt{t}} + \frac{dx^{2}}{\sqrt{t}} + tdy^{2} + tdz^{2}, \tag{80}$$

with a five dimensional homothetic algebra. This algebra is spanned by the four KVs of the space:

$$X_1^{II} = \partial_x,\ X_2^I,\ X_3^I,\ X_4^I, \tag{81}$$

and the HV H^I. The Lie brackets of this homothetic algebra can be obtained by reading the entries of Table 5.1 and

$$\left[X_1^{II}, X_2^I\right] = 0,\ \ \left[X_1^{II}, X_3^I\right] = 0,\ \ \left[X_1^{II}, X_4^I\right] = 0,\ \ \left[X_1^{II}, H^I\right] = 4X_1^{II}. \tag{82}$$

Proceeding as before, we construct the linear combinations of the KVs. Note that many of the results of Class I are inherited by this Class, while those different from Class I are listed as

$$\begin{aligned} X_{(1)}^{II} &= X_1^{II} + X_2^I, \\ X_{(2)}^{II} &= X_1^{II} + X_3^I, \\ X_{(3)}^{II} &= X_1^{II} + X_4^I. \end{aligned} \tag{83}$$

For the HV linear combination, unique to this class, we have the generator

$$H_{(1)}^{II} = H^I + X_1^{II}. \tag{84}$$

In this way, X_1^{II} is a Lie point symmetries of Eq. (39) when

$$V_1^{II} = \ V\left(t, y, z\right).$$

The linear combinations of vector fields (106) are Lie/Noether point symmetries of (39) if the potentials are

$$\begin{aligned} X_{(1)}^{II} &\ :\ V_{(1)}^{II} = \ V\left(t, -x+y, z\right), \\ X_{(2)}^{II} &\ :\ V_{(2)}^{II} = \ V\left(t, y, -x+z\right), \\ X_{(3)}^{II} &\ :\ V_{(3)}^{II} = \ V\left(t, y^2+z^2, -\arctan\left(\tfrac{y}{z}\right)+x\right), \end{aligned} \tag{85}$$

and the HV linear combination (107) is admitted by the potential function

$$H_{(1)}^{II}\ :\ V_{H1}^{II} = \frac{1}{t^{\frac{1}{4}}} V\left(\frac{4x+1}{4(1+4t)}, \frac{y}{\sqrt[4]{1+4t}}, \frac{z}{1+4t}\right)\left(1+4t\right)^{-\frac{5}{4}}. \tag{86}$$

5.1.3 CKV Class III:

Suppose we investigate the non conformally flat spacetime [50]

$$ds^2 = -dt^2 + dx^2 + x^2(dy^2 + f^2(y,z)dz^2), \tag{87}$$

which is interesting because it possesses a sCKV. Indeed, it admits a three-dimensional Lie algebra, spanned by a KV X_1^I, a HV

$$H^{III} = t\partial_t + x\partial_x, \quad \psi^{III} = 1,$$

and a sCKV

$$C^{III} = \left(t^2 + x^2\right)\partial_t + 2tx\partial_x, \quad \psi_C^{III} = 2t.$$

The Lie brackets of the KV, HV and sCKV are

$$\left[X_1^I, H^{III}\right] = X_1^I, \;\; \left[X_1^I, C^{III}\right] = 2H^{III}, \;\; \left[H^{III}, C^{III}\right] = C^{III}.$$

The HV H^{III} is a Lie/Noether point symmetry of Eq. (39) when

$$V_H^{III} = \frac{1}{t^2}V\left(\frac{x}{t}, y, z\right),$$

while on the other hand, $C^{III} - \frac{1}{2}\psi^{III}\Phi X_\Phi$ is a Lie/Noether point symmetry if and only if

$$V_C^{III} = \frac{1}{x^2}V\left(\frac{x}{-t^2 + x^2}, y, z\right).$$

The generator $H_{(1)}^{III} = X_1^I + H^{III}$ is a Lie point symmetries of Eq. (39) if and only if the potential is

$$V_1^{III} = \frac{1}{(1+t)^2}V\left(\frac{x}{1+t}, y, z\right).$$

5.1.4 CKV Class IV:

The plane symmetric Bianchi I spacetime has line element

$$ds^2 = A^2(t)(-dt^2 + dx^2) + B^2(t)(dy^2 + dz^2), \tag{88}$$

where $A(t)$ and $B(t)$ are arbitrary functions. In general, this space has four KVs (105) and an extra two vector fields in some cases (see [45]). For example, the specialization $B = A\sinh(at)$ where a is a constant, gives rise to a further two symmetries which are CKVs of the space, viz.

$$\begin{aligned} C_1^{IV} &= \cosh(ax)\sinh(at)\partial_t + \sinh(ax)\cosh(at)\partial_x, \\ C_2^{IV} &= \sinh(ax)\sinh(at)\partial_t + \cosh(ax)\cosh(at)\partial_x. \end{aligned} \tag{89}$$

The respective conformal factors are

$$\begin{aligned} \psi_1^{IV} &= \cosh(ax)\left(a\cosh(at)\partial_t + \sinh(at)A^{-1}A_t\right), \\ \psi_2^{IV} &= \tfrac{\sinh(ax)}{\cosh(ax)}\psi_1^{IV}. \end{aligned} \tag{90}$$

In another special case, $A = \operatorname{cosech}(at)$, the CKVs (90) become KVs instead. The complete set of commutators of the Lie algebra can be seen in Table 5.1, the Lie brackets (82) and

$$\left[C_1^{IV}, X_1^{II}\right] = -aC_2^{IV},\ \ \left[X_2^I, C_1^{IV}\right] = 0,\ \ \left[X_3^I, C_1^{IV}\right] = 0,$$

$$\left[C_2^{IV}, X_1^{II}\right] = -aC_1^{IV},\ \ \left[X_4^I, C_1^{IV}\right] = 0,\ \ \left[X_2^I, C_2^{IV}\right] = 0,$$

$$\left[X_3^I, C_2^{IV}\right] = 0,\ \ \left[X_4^I, C_2^{IV}\right] = 0,$$

$$\left[C_1^{IV}, C_2^{IV}\right] = -aX_1^{II}.$$

Now in terms of the potential of the Klein-Gordon equation, all results involving the vectors (105) and their linear combinations exclusively, coincide with the corresponding calculations from Class II. However, in the case of the generator

$X^{IV}_{(1)} = X^{II}_1 + C^{IV}_2 - \frac{1}{2}\psi^{IV}_2 \Phi X_\Phi$, it is only a Lie/Noether point symmetry if and only if,

$$V^{IV}_{(1)} = V\left(-\frac{\cosh(at)+\sinh(ax)}{\sinh(at)a}, y, z\right). \tag{91}$$

5.2 Conclusion

Klein-Gordon equations have significant applications in many physical spheres, such as quantum field theory and nonlinear optics. Moreover, it possesses variational properties, including the existence of Noether symmetries and conservation laws. Every symmetry defined above is also a Lie/Noether symmetry of the field equations defining every class. As such, the Klein-Gordon equation has been the subject of numerous investigations in different contexts. We, in lieu of exploring the potential function of the Klein-Gordon equation, considered its geometric character. This study led to a collection of the functional forms of the potential based on two factors: a) the conformal-reducible spaces with non zero Weyl tensor and, b) the corresponding CKVs of these spaces.

Chapter 6

Classifications of the Klein-Gordon Symmetries for Conformally Decomposable 2+2 Spacetimes

The solution of differential equations is vastly simplified by the existence of point symmetries [51, 52, 28, 53, 54, 55, 23, 56]. One particular set of equations that has showcased this idea is that of the field equations in General Relativity. Knowledge of symmetries in this area is useful in classifying spacetimes by the structure of the Lie algebra spanned by these symmetries. The study of isometries and homotheties in particular is of considerable interest. In a related study, the geometric nature of the symmetries of Klein-Gordon equations have been recently explored [37, 18]. Klein-Gordon equations feature in numerous physical applications, including solid state physics, quantum field theory and nonlinear optics.

The main purpose of this chapter is to investigate the conformal symmetries in relation to a Klein-Gordon equation in a special, but important, class of spacetimes, namely conformally decomposable 2+2 spaces. Decomposable or reducible spacetimes are characterized by the existence of certain covariantly constant tensor fields or, if its holonomy group is non-degenerately reducible [57]. A 2+2 conformally decomposable space admits a rank-2 symmetric, covariantly constant tensor field and its manifold is the product of two two-dimensional manifolds [46]. More concisely, a spacetime (M, g) is said to be 2+2 conformally decomposable if there exists a coordinate chart x^a such that the line element can be expressed as [58]

$$ds^2 = \exp\left(2\mu\left(x^a\right)\right)\left(d\sigma_1^2 + d\sigma_2^2\right),$$

where

$$d\sigma_1^2 = g_{\alpha\beta}\left(x^\gamma\right) dx^\alpha dx^\beta, \quad \alpha, \beta, \ldots = 0, 1,$$

and

$$d\sigma_2^2 = g_{\bar{\alpha}\bar{\beta}}\left(x^{\bar{\gamma}}\right) dx^{\bar{\alpha}} dx^{\bar{\beta}}, \quad \bar{\alpha}, \bar{\beta}, \ldots = 2, 3.$$

The respective signatures of the above are zero and +2. In other words, (M, g) is conformally related to a $2 + 2$ locally decomposable spacetime, say (M,$\tilde{g}$), with line element

$$d\Sigma^2 = d\sigma_1^2 + d\sigma_2^2.$$

Structurally, the 2+2 reducible spacetimes (M, $\tilde{g}$) is of the Petrov type D or O, and therefore any conformally related spacetime will also be of one of those types. Its Ricci tensor is of the Segre type (1, 1)(11).

From a geometrical perspective, such 2+2 reducible spaces may not always be physically interesting, but they are conformally related to important and relevant spaces. Some notable conformally decomposable 2+2 spacetimes are the spherical, plane and hyperbolic symmetric spacetimes [45].

In this study our classification has been put forward based on some selected 2+2 decomposable spaces and their conformal algebras that exist in the literature. We have concentrated on some 2+2 reducible spacetimes that are perfect fluids or vacuum spaces. Taking some of these explicit classes of 2+2 decomposable spaces, a thorough study of the potential functions of the Klein-Gordon equation is performed.

NB: In Chapter 6, we use the same potential function relation as Chapter 4.

6.1 The 2+2 Decomposable Spaces

We now turn our attention to some examples of 2+2 spaces that are reducible. To start, we first mention the conformal algebra of each 2+2 decomposable metric space in the nine cases A-I below. In particular, we have considered examples of spaces that are rich in symmetry. This ensures a classification that is more interesting and has more depth. The spaces we consider, admit between two and four KVs - some also admit a HV or a CKV. We shall obtain all the potential functions admitted by the Klein-Gordon equation (39) in the context of the aforementioned spaces.

6.1.1 Class A:

Although there are many examples of geometries that admit four KVs, we would like to consider the plane symmetric $2+2$ reducible space

$$ds^2 = -\frac{dt^2}{x^6} + dx^2 + dy^2 + dz^2. \tag{92}$$

This case reveals a homothetic algebra comprising of the four KVs

$$X_1^A = \partial_t,\ X_2^A = \partial_y,\ X_3^A = \partial_z,\ X_4^A = z\partial_y - y\partial_z, \tag{93}$$

and the HV

$$H^{A}=4t\partial_{t}+x\partial_{x}+y\partial_{y}+z\partial_{z}. \tag{94}$$

with homothetic constant $\psi^{A}=1$. The commutators of these vector fields are provided in Table 6.1.

Table 6.1: Lie brackets of the conformal algebra.

	${X^A}_1$	${X^A}_2$	${X^A}_3$	${X^A}_4$	H^A
${X^A}_1$	0	0	0	0	$4{X^A}_1$
${X^A}_2$	0	0	0	$-{X^A}_3$	${X^A}_2$
${X^A}_3$	0	0	0	${X^A}_2$	${X^A}_3$
${X^A}_4$	0	${X^A}_3$	$-{X^A}_2$	0	0
H^A	$-4{X^A}_1$	$-{X^A}_2$	$-{X^A}_3$	0	0

From condition (41) we have that $X_1^A - X_4^A$ are Lie point symmetries of the Klein-Gordon equation (39), if and only if,

$$\begin{aligned} X_1^A &: V_1^A = V\left(x,y,z\right), \\ X_2^A &: V_2^A = V\left(t,x,z\right), \\ X_3^A &: V_3^A = V\left(t,x,y\right), \\ X_4^A &: V_4^A = V\left(t,x,y^2+z^2\right). \end{aligned} \tag{95}$$

The HV H^A is a Lie point symmetry of Eq. (39) when

$$V_H^A = \frac{1}{t^{\frac{3}{2}}}V\left(\frac{x}{\sqrt[4]{t}},\frac{y}{\sqrt[4]{t}},\frac{z}{\sqrt[4]{t}}\right).$$

Additionally, from linear combinations of the KVs, viz.

$$
\begin{aligned}
X^A_{(1)} &= X^A_1 + X^A_2, \\
X^A_{(2)} &= X^A_1 + X^A_3, \\
X^A_{(3)} &= X^A_1 + X^A_4, \\
X^A_{(4)} &= X^A_2 + X^A_3, \\
X^A_{(5)} &= X^A_2 + X^A_4, \\
X^A_{(6)} &= X^A_3 + X^A_4,
\end{aligned} \tag{96}
$$

we have that the vector fields (96) are Lie point symmetries of (39), when

$$
\begin{aligned}
X^A_{(1)} &: V^A_{(1)} = V\left(x, -t + y, z\right), \\
X^A_{(2)} &: V^A_{(2)} = V\left(x, y, -t + z\right), \\
X^A_{(3)} &: V^A_{(3)} = V\left(x, y^2 + z^2, -\arctan\left(\tfrac{y}{z}\right) + t\right), \\
X^A_{(4)} &: V^A_{(4)} = V\left(t, x, -y + z\right) \\
X^A_{(5)} &: V^A_{(5)} V\left(t, x, -\tfrac{z^2}{2} - \tfrac{y^2}{2} - z\right), \\
X^A_{(6)} &: V^A_{(6)} = V\left(t, x, y^2 + z^2 - 2\,y\right).
\end{aligned} \tag{97}
$$

Concerning the HV, one may easily construct linear combinations with the four KVs of the space,

$$
H^A_{(1)} = X^A_1 + H^A,\ H^A_{(2)} = X^A_2 + H^A, \tag{98}
$$

and

$$
H^A_{(3)} = X^A_3 + H^A,\ H^A_{(4)} = X^A_4 + H^A. \tag{99}
$$

In this way, we find that the vector fields (98) and (99) are Lie point symmetries, if and only if the corresponding potential functions of (39) are

$$
\begin{aligned}
H^A_{(1)} &: V^A_{(H1)} = \tfrac{1}{\sqrt{1+4\,t}} V\left(\tfrac{x}{\sqrt[4]{1+4\,t}}, \tfrac{y}{\sqrt[4]{1+4\,t}}, \tfrac{z}{\sqrt[4]{1+4\,t}}\right), \\
H^A_{(2)} &: V^A_{(H2)} = \tfrac{1}{\sqrt{t}} V\left(\tfrac{x}{\sqrt[4]{t}}, \tfrac{1+y}{\sqrt[4]{t}}, \tfrac{z}{\sqrt[4]{t}}\right), \\
H^A_{(3)} &: V^A_{(H3)} = \tfrac{1}{\sqrt{t}} V\left(\tfrac{x}{\sqrt[4]{t}}, \tfrac{y}{\sqrt[4]{t}}, \tfrac{1+z}{\sqrt[4]{t}}\right).
\end{aligned} \tag{100}
$$

6.1.2 Class B:

Similarly, suppose we take the 2+2 decomposable space with line element

$$ds^2 = -\frac{dt^2}{x^2} + x^4 dx^2 + x^4 dy^2 + x^4 dz^2, \tag{101}$$

which shares the four KVs (93) and the HV H^A, but instead with homothetic constant $\psi^B = 3$. The results of the previous subclass apply directly to this subclass.

6.1.3 Class C:

The metric of a (comoving) perfect fluid (and $2+2$ decomposable) spacetime

$$\begin{aligned} ds^2 &= -\left(a^2x^3 + \tfrac{b^2}{x^2}\right)^{-2} \tfrac{dt^2}{x^6} + \left(a^2x^3 + \tfrac{b^2}{x^2}\right)^{-2} dx^2 + \left(a^2x^3 + \tfrac{b^2}{x^2}\right)^{-2} dy^2 \\ &\quad + \left(a^2x^3 + \tfrac{b^2}{x^2}\right)^{-2} dz^2, \end{aligned}$$

has its density and pressure given by

$$\mu = 15a^2x^{-1}\left(4b^2 - a^2x^5\right), \quad p = 15a^2x^{-1}\left(-2b^2 + 3a^2x^5\right),$$

respectively. The conformal algebra is five dimensional, spanned by the KVs (93), and where the HV H^A becomes a CKV with conformal factor

$$\psi^C = \left(b^2 + a^2x^5\right)^{-1}\left(3b^2 - 2a^2x^5\right).$$

We relabel the CKV to be $\mathcal{C}^C = H^A$. Thus, the vector field $\mathcal{C}^C - \psi^C \Phi \partial_\Phi$ is a Lie point symmetry of (39) if

$$V_{\mathcal{C}^C} = \frac{V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{z}{\sqrt[4]{t}}\right)}{t^{\frac{3}{2}}}\left(a^4x^{10} + 2\ a^2b^2x^5 + b^4\right). \tag{102}$$

The linear combinations of vector fields (96) and its associated potentials (97) all follow directly from Class A. However, linear combinations involving the CKV can be read off from (98) and (99), and these symmetries are only Lie point symmetries of Eq. (39) if we have the potentials

$$\begin{aligned} H^A_{(1)} \;\; &: \;\; V^C_{(H1)} = V\left(\frac{x}{\sqrt[4]{1+4t}}, \frac{y}{\sqrt[4]{1+4t}}, \frac{z}{\sqrt[4]{1+4t}}\right)\left(\frac{4a^4x^{10}}{(1+4t)^{\frac{5}{2}}} + \frac{8a^2b^2x^5}{(1+4t)^{\frac{5}{2}}} + \frac{4b^4}{(1+4t)^{\frac{5}{2}}}t + \frac{a^4x^{10}}{(1+4t)^{\frac{5}{2}}}\right. \\ &\qquad\qquad \left. + \frac{2a^2b^2x^5}{(1+4t)^{\frac{5}{2}}} + \frac{b^4}{(1+4t)^{\frac{5}{2}}}\right), \\ H^A_{(2)} \;\; &: \;\; V^C_{(H2)} = \;\; V\left(\frac{x}{\sqrt[4]{t}}, \frac{1+y}{\sqrt[4]{t}}, \frac{z}{\sqrt[4]{t}}\right)\left(\frac{1}{t^{\frac{3}{2}}}\left(a^4x^{10} + 2\,a^2b^2x^5 + b^4\right)\right), \\ H^A_{(3)} \;\; &: \;\; V^C_{(H3)} = \;\; V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{1+z}{\sqrt[4]{t}}\right)\left(\frac{1}{t^{\frac{3}{2}}}\left(a^4x^{10} + 2\,a^2b^2x^5 + b^4\right)\right). \end{aligned} \tag{103}$$

6.1.4 Class D:

This time, suppose we take a nonstatic plane-symmetric $2+2$ reducible spacetime

$$ds^2 = -\frac{dt^2}{t} + \frac{dx^2}{t} + dy^2 + dz^2, \tag{104}$$

with a five dimensional conformal algebra. The algebra is spanned by the four KVs of the space:

$$X^D_1 = \partial_x,\; X^A_2,\; X^A_3,\; X^A_4, \tag{105}$$

and a HV, we label as $H^D = 2t\partial_t + 2x\partial_x + y\partial_y + z\partial_z$ with homothetic constant $\psi^D = 1$. The Lie brackets of the KVs and HV are given in Table 6.2.

Table 6.2: Lie brackets of the conformal algebra.

	${X^D}_1$	${X^A}_2$	${X^A}_3$	${X^A}_4$	H^D
${X^D}_1$	0	0	0	0	$2{X^D}_1$
${X^A}_2$	0	0	0	$-{X^A}_3$	${X^A}_2$
${X^A}_3$	0	0	0	${X^A}_2$	${X^A}_3$
${X^A}_4$	0	${X^A}_3$	$-{X^A}_2$	0	0
H^D	$-2{X^D}_1$	$-{X^A}_2$	$-{X^A}_3$	0	0

Proceeding as before, we construct the linear combinations of the KVs as

$$\begin{aligned}
X^D_{(1)} &= X^D_1 + X^A_2, \\
X^D_{(2)} &= X^D_1 + X^A_3, \\
X^D_{(3)} &= X^D_1 + X^A_4, \\
X^D_{(4)} &= X^A_{(4)}, \\
X^D_{(5)} &= X^A_{(5)}, \\
X^D_{(6)} &= X^A_{(6)},
\end{aligned} \tag{106}$$

while the linear combinations comprising of the HV are listed as

$$\begin{aligned}
H^D_{(1)} &= H^D + X^D_1, \\
H^D_{(2)} &= H^D + X^A_2, \\
H^D_{(3)} &= H^D + X^A_3, \\
H^D_{(4)} &= H^D + X^A_4.
\end{aligned} \tag{107}$$

The generator X_1^D is a Lie point symmetries of Eq. (39) when

$$V_1^D = \ V(t, y, z),$$

while the HV H^D is a Lie point symmetry if the potential is

$$V_H^D = \frac{1}{t} V\left(\frac{x}{t}, \frac{y}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right).$$

In a similar analysis as the previous classes, the linear combinations of vector fields (106) are Lie point symmetries of (39) if the potentials are

$$\begin{aligned} X^D_{(1)} &: V^D_{(1)} = V\left(t, -x+y, z\right), \\ X^D_{(2)} &: V^D_{(2)} = V\left(t, y, -x+z\right), \\ X^D_{(3)} &: V^D_{(3)} = V\left(t, y^2+z^2, -\arctan\left(\frac{y}{z}\right)+x\right), \end{aligned} \tag{108}$$

and identically from Class A

$$\begin{aligned} X^D_{(4)} &: V^A_{(4)}, \\ X^D_{(5)} &: V^A_{(5)}, \\ X^D_{(6)} &: V^A_{(6)}. \end{aligned} \tag{109}$$

The HV combinations (107) are admitted by the potential functions

$$\begin{aligned} H^D_{(1)} &: V^D_{H1} = \frac{1}{t}V\left(\frac{1+2\,x}{2t}, \frac{y}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right), \\ H^D_{(2)} &: V^D_{H2} = \frac{1}{t}V\left(\frac{x}{t}, \frac{1+y}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right), \\ H^D_{(3)} &: V^D_{H3} = \frac{1}{t}V\left(\frac{x}{t}, \frac{y}{\sqrt{t}}, \frac{1+z}{\sqrt{t}}\right). \end{aligned} \tag{110}$$

6.1.5 Class E:

Consider a metric

$$ds^2 = -\frac{tdt^2}{(-2ct^2+cx^2+b)^2x^6} + \frac{tdx^2}{(-2ct^2+cx^2+b)^2} + \frac{tdy^2}{(-2ct^2+cx^2+b)^2} + \frac{tdz^2}{(-2ct^2+cx^2+b)^2}, \tag{111}$$

where b, c are nonzero constants. This space represents a perfect fluid solution and is spanned by a five-dimensional conformal algebra. The associated KVs are X^A_2, X^A_3 and X^A_4. The symmetries X^D_1 and H^D become CKVs, that is, let $\mathcal{C}^E_1 = X^D_1$ and $\mathcal{C}^E_2 = H^D$. The respective conformal factors are $\psi^E_1 = -2cx\left(cx^2-2ct^2+b\right)^{-1}$ and $\psi^E_2 = -2+4b\left(cx^2-2ct^2+b\right)^{-1}$.

The potentials corresponding to X^A_2, X^A_3 and X^A_4 are the same as those listed in (95).

Now, $\mathcal{C}_1^E - \psi_1^E \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_1^E = \ V\left(t,y,z\right)\left(4\,x^{\frac{3}{2}}c^2t^4 - 4\,x^{\frac{7}{2}}c^2t^2 + x^{\frac{11}{2}}c^2 - 4\,x^{\frac{3}{2}}bct^2 + 2\,x^{\frac{7}{2}}bc + x^{\frac{3}{2}}b^2\right),$$

and $\mathcal{C}_2^E - \psi_2^E \Phi \partial_\Phi$ is a Lie point symmetry if

$$\begin{aligned} V_H^E = V\left(\tfrac{x}{t}, \tfrac{y}{\sqrt{t}}, \tfrac{z}{\sqrt{t}}\right)\Bigg(& 4\,c^2t^3 + \left(-4\,c^2x^2 - 4\,bc\right)t \\ & + \tfrac{1}{t}\left(c^2x^4 + 2\,bcx^2 + b^2\right)\Bigg). \end{aligned} \tag{112}$$

There are major differences to some of the potentials of the linear combinations, which we mention below. The vector field $\mathcal{C}_1^E + X_2^A - \psi_1^E \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_{(1)}^E = \ V\left(t,-x+y,z\right)\left(4\,x^{\frac{3}{2}}c^2t^4 + \left(-4\,x^{\frac{7}{2}}c^2 - 4\,x^{\frac{3}{2}}bc\right)t^2 + x^{\frac{11}{2}}c^2 + 2\,x^{\frac{7}{2}}bc + x^{\frac{3}{2}}b^2\right),$$

and $\mathcal{C}_1^E + X_3^A - \psi_1^E \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_{(2)}^E = V\left(t,y,-x+z\right)\left(4\,x^{\frac{3}{2}}c^2t^4 + \left(-4\,x^{\frac{7}{2}}c^2 - 4\,x^{\frac{3}{2}}bc\right)t^2 + x^{\frac{11}{2}}c^2 + 2\,x^{\frac{7}{2}}bc + x^{\frac{3}{2}}b^2\right).$$

Other linear combinations remain unchanged, namely (109).

The corresponding potential function to the linear combinations of the CKVs $\mathcal{C}_1^E + \mathcal{C}_2^E - \left(\psi_1^E + \psi_2^E\right)\Phi \partial_\Phi$ is

$$\begin{aligned} V_{C^E} = V\left(\tfrac{1+2x}{2t}, \tfrac{y}{\sqrt{t}}, \tfrac{z}{\sqrt{t}}\right)\Bigg(& 128\sqrt{2}x^{\frac{3}{2}}c^2t^{\frac{3}{2}} + \tfrac{1}{\sqrt{t}}\left(-128\,x^{\frac{7}{2}}\sqrt{2}c^2 - 128\,x^{\frac{3}{2}}\sqrt{2}bc\right) \\ & + \tfrac{1}{t^{\frac{5}{2}}}\left(32\,x^{\frac{11}{2}}\sqrt{2}c^2 + 64\,x^{\frac{7}{2}}\sqrt{2}bc + 32\,x^{\frac{3}{2}}\sqrt{2}b^2\right)\Bigg). \end{aligned} \tag{113}$$

The vector field $\mathcal{C}_2^E + X_2^A - \psi_1^E \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_{E_1} = V\left(\tfrac{x}{t}, \tfrac{1+y}{\sqrt{t}}, \tfrac{z}{\sqrt{t}}\right)\Bigg(4c^2t^3 + \left(-4\,c^2x^2 - 4\,cb\right)t + \tfrac{1}{t}\left(c^2x^4 + 2\,cbx^2 + b^2\right)\Bigg). \tag{114}$$

Moreover, the vector field $\mathcal{C}_2^E + X_3^A - \psi_1^E \Phi \partial_\Phi$ is also a Lie point symmetry, if and only if

$$V_{E_2} = V\left(\tfrac{x}{t}, \tfrac{y}{\sqrt{t}}, \tfrac{1+z}{\sqrt{t}}\right)\Bigg(4\,c^2t^3 + \left(-4\,c^2x^2 - 4\,cb\right)t + \tfrac{1}{t}\left(c^2x^4 + 2\,cbx^2 + b^2\right)\Bigg). \tag{115}$$

6.1.6 Class F:

As another case of a perfect fluid solution, consider the 2+2 reducible space

$$ds^2 = -\tfrac{t(\sec(kx))^2 dt^2}{x^6} + t(\sec(kx))^2 dx^2 + t(\sec(kx))^2 dy^2 + t(\sec(kx))^2 dz^2, \tag{116}$$

where k is a nonzero constant. Analogous to the previous class, this space admits a five dimensional conformal algebra, spanned by the KVs X_2^A, X_3^A and X_4^A, and as CKVs $\mathcal{C}_1^E$ and $\mathcal{C}_2^E$. The conformal factors are $\psi_1^F = k\tan(kx)$ and $\psi_2^F = 2 + 2kx\tan(kx)$, respectively.

Once again, the potentials corresponding to X_2^A, X_3^A and X_4^A are the same as those listed in (95). The generator $\mathcal{C}_1^E - \psi_1^F \Phi \partial_\Phi$ corresponds to $V_1^F = V(t,y,z)x^{\frac{3}{2}}$, while $\mathcal{C}_2^E - \psi_1^F \Phi \partial_\Phi$ corresponds to V_H^D.

Differences in some of the potentials are as follows. Here the vector field $\mathcal{C}_1^E + X_2^A - \psi_1^F \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_{F_1} = \ V\left(t, -x+y, z\right) x^{\frac{3}{2}},$$

$\mathcal{C}_1^E + X_3^A - \psi_1^F \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_{F_2} = \ V\left(t, y, -x+z\right) x^{\frac{3}{2}},$$

and $\mathcal{C}_1^E + X_4^A - \psi_1^F \Phi \partial_\Phi$ is a Lie point symmetry if

$$V_{F_3} = \ V\left(t, y^2+z^2, -\arctan\left(\frac{y}{z}\right)+x\right) x^{\frac{3}{2}}.$$

The equivalent results from Class D are found for the symmetries and potentials in Eq. (109). The corresponding potential function to the linear combinations of the CKVs $\mathcal{C}_1^E + \mathcal{C}_2^E - \left(\psi_1^F + \psi_2^F\right) \Phi \partial_\Phi$, is

$$V_{C^F} = \frac{2\sqrt{2}x^{\frac{3}{2}}}{t^{\frac{5}{2}}} \ V\left(\frac{1+2\,x}{2t}, \frac{y}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right),$$

$\mathcal{C}_2^E + X_2^A - \psi_2^F \Phi \partial_\Phi$ is a Lie point symmetry if the potential is V_{H2}^D, and $\mathcal{C}_2^E + X_3^A - \psi_2^F \Phi \partial_\Phi$ is a Lie point symmetry if the potential is V_{H3}^D.

6.1.7 Class G:

For this class, let us take the line element

$$ds^2 = -\tfrac{dt^2}{x^6} + dx^2 + dy^2 + \tfrac{dz^2}{y^6}. \qquad (117)$$

The conformal algebra is three dimensional, spanned by the KVs

$$X_1^A,\ X_3^A, \qquad (118)$$

Table 6.3: Lie brackets of the conformal algebra.

	${X^A}_1$	${X^A}_2$	H^G
${X^A}_1$	0	0	$4{X^A}_1$
${X^A}_2$	0	0	$4{X^A}_2$
H^G	$-4{X^A}_1$	$-4{X^A}_2$	0

and the HV $H^G = 4t\partial_t + x\partial_x + y\partial_y + 4z\partial_z$, with homothetic constant $\psi^G = 1$. The corresponding Lie brackets are depicted in Table 6.3. The potentials of the KVs can be taken from (95) and the linear combination symmetry $X^A_{(2)}$ is a Lie point symmetries of (39) when the poetntial is $V^A_{(2)}$. The HV H^G is a Lie point symmetry of (39) if

$$V^G_H = \frac{1}{\sqrt{t}}\ V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{z}{t}\right). \tag{119}$$

Other linear combinations of the symmetries are $X^G_{(1)} = X^A_1 + H^G$ and $X^G_{(2)} = X^A_3 + H^G$. Hence, the corresponding potential functions to the linear combinations of the symmetries are:

$$\begin{aligned} X^G_{(1)} &: \frac{1}{\sqrt{t}}\ V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{1+4z}{4t}\right), \\ X^G_{(2)} &: \frac{1}{\sqrt{1+4t}}\, V\left(\frac{x}{\sqrt[4]{1+4t}}, \frac{y}{\sqrt[4]{1+4t}}, \frac{z}{1+4t}\right). \end{aligned} \tag{120}$$

6.1.8 Class H:

The space

$$ds^2 = -\frac{dt^2}{(x^{-2}+y^{-2})^2 x^6} + \frac{dx^2}{(x^{-2}+y^{-2})^2} + \frac{dy^2}{(x^{-2}+y^{-2})^2} + \frac{dz^2}{(x^{-2}+y^{-2})^2}, \tag{121}$$

is a vacuum spacetime. The space shares the three-dimensional conformal algebra of Class G, with the KVs X^A_1, X^A_3 but the H^G has homothetic constant $\psi^H = 3$.

The corresponding potential functions to the two symmetries $X_1^A,\ X_3^A$ are $V_1^A,\ V_3^A$, respectively. The vector H^G is a Lie point symmetry if

$$V_H^H = V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{z}{t}\right) t^{-\frac{15}{8}}. \tag{122}$$

The corresponding potential functions to the linear combinations $X_{(2)}^A,\ X_{(1)}^G$ and $X_{(2)}^G$ of the symmetries are:

$$\begin{aligned} X_{(2)}^A &: V_{(2)}^A, \\ X_{(1)}^G &: V_{(1)}^H = V\left(\frac{x}{\sqrt[4]{1+4t}}, \frac{y}{\sqrt[4]{1+4t}}, \frac{z}{1+4t}\right)(1+4t)^{-\frac{15}{8}}, \\ X_{(2)}^G &: V_{(2)}^H = V\left(\frac{x}{\sqrt[4]{t}}, \frac{y}{\sqrt[4]{t}}, \frac{1+4z}{4t}\right) t^{-\frac{15}{8}}. \end{aligned} \tag{123}$$

6.1.9 Class I:

This case is an example of a $2+2$ reducible space (also a vacuum spacetime) that admits two KVs and one HV. The underlying $2+2$ metric is:

$$ds^2 = -\frac{dt^2}{x^6} + \frac{dx^2}{t^6} + \frac{dy^2}{z^6} + \frac{dz^2}{y^6}, \tag{124}$$

which admits the two KVs

$$\begin{aligned} X_1^I &= (t^9 + 3x^8 t)\partial_t + (x^9 + 3t^8 x)\partial_x, \\ X_2^I &= (y^9 - 3z^8 y)\partial_y + (-z^9 + 3y^8 z)\partial_z, \end{aligned} \tag{125}$$

and the HV $H^I = t\partial_t + x\partial_x + y\partial_y + z\partial_z$, with homothetic constant $\psi^I = -2$.

The Lie brackets can be found in Table 6.4.

Based on condition (41) we have that $X_1^I,\ X_2^I$ and H^I are Lie point symmetries

Table 6.4: Lie brackets of the conformal algebra.

	${X^I}_1$	${X^I}_2$	H^I
${X^I}_1$	0	0	-8${X^I}_1$
${X^I}_2$	0	0	-8${X^I}_2$
H^I	8 ${X^I}_1$	8 ${X^I}_2$	0

when

$$\begin{aligned} X_1^I \;&:\; V_1^I = V\left(\tfrac{t^2x^2}{-t^8+x^8}, y, z\right), \\ X_2^I \;&:\; V_2^I = V\left(t, x, \tfrac{yz}{\sqrt{y^8+z^8}}\right), \\ H^I \;&:\; V_H^I = V\left(\tfrac{x}{t}, \tfrac{y}{t}, \tfrac{z}{t}\right) t^4. \end{aligned} \tag{126}$$

In this instance, we do not find potential functions that give rise to linear combinations of the vector fields.

6.2 Conclusion

We have considered the problem of determining the Lie point symmetries of the Klein Gordon equation in several 2+2 conformally reducible spaces. Within the cases studied, a detailed analysis of the Klein-Gordon's potential function was performed. To determine all the potential functions, it was necessary to apply the conformal algebra and linear combinations of its basis vectors. In Table 6.5, we list the Noether symmetry vectors corresponding to each Class A-I. One may use Noether's theorem to obtain first integrals of the field equations of each space. Alternatively, using the same theorem, it is easy to obtain conservation laws for the Klein-Gordon equation itself. A classification of this nature may also facilitate the reduction and analytical solution process of the Klein-Gordon equation in these spaces via the Lie invariant method.

Table 6.5: Noether Symmetries of Eq. (39).

2+2 Case	**Dimension**	**Noether Symmetry**
A	5	$X_1^A,\ X_2^A,\ X_3^A,\ X_4^A,\ H^A,$
B	5	$X_1^A,\ X_2^A,\ X_3^A,\ X_4^A,\ H^A,$
C	5	$X_1^A,\ X_2^A,\ X_3^A,\ X_4^A,\ \mathcal{C}^C - \psi^C \Phi \partial_\Phi,$
D	5	$X_1^D,\ X_2^A,\ X_3^A,\ X_4^A,\ H^D,$
E	5	$\mathcal{C}_1^E - \psi_1^E \Phi \partial_\Phi,\ X_2^A,\ X_3^A,\ X_4^A,\ \mathcal{C}_2^E - \psi_2^E \Phi \partial_\Phi,$
F	5	$\mathcal{C}_1^E - \psi_1^F \Phi \partial_\Phi,\ X_2^A,\ X_3^A,\ X_4^A,\ \mathcal{C}_2^E - \psi_2^F \Phi \partial_\Phi,$
G	3	$X_1^A,\ X_3^A,\ H^G,$
H	3	$X_1^A,\ X_3^A,\ H^G,$
I	3	$X_1^I,\ X_2^I,\ H^I.$

Chapter 7

Generalized Symmetries and Recursive Operators of some Diffusive Equations

In 1977, Olver [13] pioneered a study of recursion operators for several evolution equations that possess infinitely many symmetries. One such discussion revolved around the higher-order analogs of the KdV equation

$$u_t = u_{xxx} + uu_x,$$

which could be reinterpreted as "higher-order symmetries". The operator itself was due to Lenard [59]

$$\mathcal{R} = D + \frac{1}{2}u + \frac{1}{2}u_x D^{-1},$$

where D denotes the total derivative with respect to x unless stated otherwise, namely $D = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xxx}\partial_{u_{xx}} + \ldots$.Inspired by these works, we obtain higher-order analogs of a special diffusion equation by formulating a recursion operator.

Symmetry methods feature in the analysis of phenomena that arise in physical and mathematical fields. For example, several studies were devoted to fundamental equations in physics, general relativity, biology and financial mathematics, see [60, 61, 62, 63] and references therein. These days the determination of point symmetries can be done mechanically by a number of symbolic computer programs, e.g. to list just a few: Macsyma [64], Mathematica [65, 66] and Maple has an automated routine as well. We point out here that point symmetries depend only on the independent variables ($\underline{x^k} = (x^1, \ldots x^p)$) of a system and its associated dependent variables only ($\underline{u^l} = (u^1, \ldots u^q)$), and excludes derivatives of the dependent variable, that is $X = \xi^k(\underline{x}, \underline{u}) + \eta^l(\underline{x}, \underline{u})$. Whilst generalized symmetries, also commonly referred to as Lie Bäcklund symmetries, include derivatives of the dependent variable. The aim of this paper is also to study some properties surrounding generalized symmetries that arise from techniques apart from recursions.

At this point it is appropriate to summarize several strategies to construct higher-order symmetries: a) The standard way of computing Lie point symmetries can be extended to include derivative dependent terms thereby resulting in higher-order Lie Bäcklund symmetries. b) Implement the multiplier or Characteristic approach [11] to construct multipliers (integrating factors) of an equation. In this case the multiplier itself is a symmetry of the underlying equation and can be defined in such a way as to consist of higher-order terms. The advantage of this method is that every multiplier provides a conserved quantity. Obviously such a conservation law is not Noetherian and does not stem from a variational principle. For further discussion of this method we refer the reader to recent literature on the subject [67, 68, 69, 70, 71]. Last but not least, (c) A recursion operator, if it exits, generates infinite hierarchies of higher-order symmetries. Historically, recursive operators were found by guesswork [13], although it can debated that this is simply a practice of searching for a pattern in a perceptive fashion. By far, (c) is exceedingly sought after, more useful and increasingly more efficient.

An important problem in analytical theories is the integrability of partial differential equations (PDEs). From the mathematical point of view their importance is due to the following circumstances. Complete integrability provides important information about the nature of the solutions of PDEs. Once ascertained that a PDE is completely integrable, numerous methods then exist for solving, for instance the method of inverse scattering transform. These reasons have motivated many investigations regarding integrability, including this work.

7.1 Some Diffusive Equations

We now mention some important PDEs that have received much attention in the literature and also the equations that underpin this study. Reaction-diffusion problems have been studied extensively in fundamental areas of engineering, mathematics, biology, population ecology and many others. Numerous models exist to explain spatial theory and population dynamics alone. The expansion of muskrat populations in Europe instigated mathematical attempts to model the problem which lead to reaction-diffusion equations [72]. A typical reaction-diffusion model, with u as the population density at spatial coordinates and time t, is

$$u_t = (\mathcal{C}(u)u_x)_x + (\mathcal{C}(u)u_y)_y + f(u), \tag{127}$$

where $\mathcal{C}(u)$ describes the diffusive movement, while reaction and population dynamics is represented by $f(u)$. Emphasis is also placed on reaction-diffusion models in which a combination of population dynamics such as movement and multi-species interactions is considered [73]. The notion of a correlated random walk [73] by species gave rise to a telegraph PDE model [74]:

$$u_t = \frac{s^2}{2\lambda}(u_{xx} + u_{yy}) - \frac{1}{2\lambda}u_{tt}, \tag{128}$$

where 2λ is a measure of the correlation between directions of travel from one step to the next and s is the velocity. The Fisher model [75] is arguably the most important reaction-diffusion model, which represents Brownian random dispersal and logistic population growth:

$$u_t = ru\left(1 - \frac{u}{K}\right) + B(u_{xx} + u_{yy}), \tag{129}$$

where r is the population's growth rate, B is the diffusion constant that measures the rate of dispersal and the carrying capacity is represented by K. Prior to these models, early PDE models of population ecology such as

$$u_t = B(u_{xx} + u_{yy}), \tag{130}$$

were used to analyze the dispersion of numerous organisms in mark-recapture studies (e.g. [76]), whereby the simplistic conjecture is assumed that expects organisms to have Brownian movement, the rate of which is invariant in space and time [73, 77]. To build on this model, one considers when organisms adapt to external stimuli or are moved by rainwater or air-stream currents, and therefore convection or drift terms are added to (130), which leads to the model [78]

$$u_t = B(u_{xx} + u_{yy}) - w_x u_x - w_y u_y, \tag{131}$$

where w_x and w_y are convection velocities. The lie point symmetries of (131) are easily obtainable and are surprisingly many, viz. we find a 10-dimensional algebra:

$$
\begin{aligned}
&\Gamma_1 = \partial_y,\ \Gamma_2 = \partial_x,\ \Gamma_3 = \partial_t,\ \Gamma_4 = u\partial_u,\\
&\Gamma_5 = t\partial_x + \frac{1}{2}\frac{u\,(tw_y - x)}{B}\partial_u, X_6 = t\partial_y + \frac{1}{2}\frac{u\,(tw_x - y)}{B}\partial_u,\\
&\Gamma_7 = y\partial_x - x\partial_y - \frac{1}{2}\frac{u\,(w_x x - w_y y)}{B}\partial_u,\\
&\Gamma_8 = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + t\partial_t - \frac{1}{4}\frac{u\left(\left(w_x^2 + w_y^2\right)t - w_x y - w_y x\right)}{B}\partial_u,\\
&\Gamma_9 = \frac{1}{2}tx\partial_x + \frac{1}{2}ty\partial_y + \frac{1}{2}t^2\partial_t\\
&\quad -\frac{1}{8}\frac{u\left(t^2w_x^2 + t^2w_y^2 - 2tw_x y - 2tw_y x + 4Bt + x^2 + y^2\right)}{B}\partial_u,\\
&\Gamma_\infty = F(t,x,y)\partial_u,
\end{aligned}
\tag{132}
$$

where $F(t, x, y)$ is the infinite symmetry that is the infinite-dimensional abelian subalgebra of solutions which is a solution of the equation (131). We remark that since Eq. (131) is linear, it naturally admits the linear symmetry Γ_4 and the infinite symmetry Γ_∞ [79]. Several PDEs have been designed to model interactions between conspecifics whereby attraction or repellence between species leads to a simple diffusion equation being replaced by a biased nonlinear diffusive equation [73, 80]:

$$u_t = Bu_{xx} + (kuu_x)_x, \tag{133}$$

where again $u(t, x)$ is the density of population, and k is a measure of the tendency to travel away from conspecifics ($k > 0$) and is a measure of the tendency to travel near conspecifics ($k < 0$). Such a model admits a four dimensional Lie algebra of point symmetries,

$$\Sigma_1 = \partial_t,\ \Sigma_2 = \partial_x,\ \Sigma_3 = \tfrac{1}{2}x\partial_x + t\partial_t,\ \Sigma_4 = \tfrac{1}{2}kx\partial_x + (ku + B)\,\partial_u. \tag{134}$$

7.2 Generalized Symmetries of a Nonlinear Diffusion Equation

If a vector field with components ξ^j, η^i relies on the following derivatives, where $u^{(s)}$ represents the s^{th} derivative of u with respect to x

$$\begin{aligned}(\bar{x})^j &= x^j + \epsilon\xi^j\left(x, u^{(s)}\right) + \mathcal{O}(\epsilon^2),\ j = 1, \ldots, n,\\ (\bar{x})^i &= u^i + \epsilon\eta^i\left(x, u^{(s)}\right) + \mathcal{O}(\epsilon^2),\ i = 1, \ldots, m\end{aligned} \tag{135}$$

then the resulting symmetries are said to be of higher-order. Without loss of generality, the symmetry generators in a higher-order context are usually expressed in the evolutionary or characteristic form, videlicet

$$\bar{X} = \phi^i(x, u^{(s)})\partial_{u^i}, \text{ assuming that } \xi^j = 0. \tag{136}$$

Hence, suppose we restrict our vector field to the form

$$\bar{X} = \phi(x, t, u, u_x, u_{xx}), \tag{137}$$

which is to be the higher-order symmetry generator of Eq.(133). The infinitesimal criterion of invariance is given by

$$\bar{X}\,[\text{Eq.(133)}]\,|_{Eq.(133)} = 0, \tag{138}$$

and therefore higher-order symmetries of our equation are given by the determining equations, for the simplest scenario $B = k = 1$,

$$\begin{aligned}&\phi_{u_{xx},u_{xx}} + \phi_{u_{xx},u_{xx}u} = 0,\\ &2\phi_{u_{xx},u_{u_x}}u_{xx} + 2\phi_{u_{xx},u}uu_x + 2\phi_{u_{xx},u_x}uu_{xx} - 2\phi_{u_{xx}}u_x + 2\phi_{x,u_{xx}}u\\ &+2\phi_{x,uxx} + 2\phi_{u_{xx},u}u_x = 0,\\ &2\phi_{u_x,u}uu_xu_{xx} - \phi_{u_x}u_xu_{xx} + \phi_{u,u}u_x^2 - 3\phi_{u_{xx}}u_{xx}^2 + \phi_{x,x} + \phi u_{xx}\\ &+2\phi_{x,u}uu_x + \phi_{u,u}uu_x^2 + \phi_{u_x,u_x}uu_{xx}^2\\ &+2\phi_{u_x,u}u_xu_{xx} + 2\phi_xu_x + 2\phi_{x,u_x}uu_{xx} + \phi_{x,x}u + \phi_uu_x^2 - \phi_t + 2\phi_{x,u}u_x+\\ &\phi_{u_x,u_x}u_{xx}^2 + 2\phi_{x,u_x}u_{xx} = 0.\end{aligned} \tag{139}$$

The solution of the system (139) gives the following four higher-order symmetries:

$$\begin{aligned}
\bar{X}_1 &= u_x \partial_u, \\
\bar{X}_2 &= (u_{xx} - 2u - 2)\partial_u, \\
\bar{X}_3 &= (uu_{xx} + u_x^2 + u_{xx})\partial_u, \\
\bar{X}_4 &= (uu_{xx}t + u_x^2 t + u + 1)\partial_u.
\end{aligned} \tag{140}$$

Note that the evolutionary symmetries $\bar{X}_{1-4}$ do not normally yield conservation laws.

7.2.1 The Multiplier Approach

Information about conservation laws is important to any symmetry study of a PDE. These conservation laws are of paramount importance and it is well known that they show a vital part in mathematical physics as they define critical physical properties of the modelled process. Conservation laws are also applicable when eliminating numerical errors of PDEs [81]. As mentioned previously, once a multiplier is found, conserved vectors may be derived systematically by using a Homotopy operator (see details and references in [12, 82]), however in some cases it is simple to construct the conserved vectors by elementary manipulations. The explicit relation between multipliers and conserved densities is summarized by Anco and Bluman [83]. To apply this approach, consider a multiplier that contains the dependent variable, the independent variables and derivatives of dependent variables up to some fixed order, i.e., let $\Lambda = \Lambda(t, x, u, u_x, u_t)$ of Eq.(133) have the property that

$$\Lambda\,[\text{Eqs.}(133)] = D_x T^x + D_t T^t, \tag{141}$$

for all functions $u(t, x)$. The right-hand side of (141) is a divergence expression and the conserved vector $T = (T^x, T^t)$ has components T^j $(j = x, t)$. The determining equations for the multipliers Λ are obtained from the expressions

$$\frac{\delta}{\delta u}\left[\Lambda(\text{Eq.133})\right] = 0, \tag{142}$$

where $\dfrac{\delta}{\delta u}$ are the Euler-Lagrange operators which annihilate divergence expressions. Solving Eq. (142) yields,

$$\begin{aligned} &u_x^2(ku+B)\Lambda_{uu} + 2u_x(ku+B)\Lambda_{ux} + (ku+B)\Lambda_{xx} \\ &+ \left(ku_x^2 + 2u_{xx}(ku+B)\right)\Lambda_u + \Lambda_t = 0. \end{aligned} \tag{143}$$

Solving for Λ, we find the solution (C_1, C_2 are arbitrary constants):

$$\Lambda(t, x, u, u_x, u_t) = C_1 x + C_2. \tag{144}$$

Hence the solutions of the determining system are the multipliers,

$$\Lambda_1 = 1 \quad \text{and} \quad \Lambda_2 = x,$$

which we note are not higher-order in the end, but each yields a nontrivial generalized biased diffusion conservation law

$$T_1^t = -u \quad \text{and} \quad T_1^x = (B + ku)u_x,$$

and

$$T_2^t = -xu \quad \text{and} \quad T_2^x = -\frac{1}{2}ku^2 + Bxu_x + u(kxu_x - B),$$

respectively. Next, we shift our focus to a special evolution equation for which we prove that it possesses an infinite hierarchy of symmetries.

7.3 The Kierstead-Slobodkin and Skellam Problem

In this section, we consider the linear Kierstead-Slobodkin [84] and Skellam [72] problem, commonly known as the KiSS model, and extend our study to the nonlinear model. Such a model is also described as the basic critical patch equation. Determination of critical patch size to guarantee the sustenance of the population is an important study. The rate at which a population exits the area, the population dynamics in the patch, the spatial area and the region surrounding the patch are some of the factors that influence the critical patch size.

A generalized $(1+2)$ KiSS model is expressed as

$$u_t = B\left(u_{xx} + u_{yy}\right) + rF(u)^{\rho}. \tag{145}$$

Here, $\rho > 0$ is the critical exponent parameter that determines whether the model is linear ($\rho = 1$) or nonlinear ($\rho \geq 2$), and r is the growth rate. An investigation of several special cases that produce interesting symmetries is presented in Table 7.1.

As mentioned before, the linear symmetry and the infinite symmetry will be added to the Table 7.1 whenever Eq.(145) is a linear model. Our interest lies in the higher-order symmetries and it turns out that we are able to find a infinite sequence of symmetries for this particular model.

Table 7.1: Classification of Lie point symmetries of model (145)

Case	Lie Symmetry	ρ	$\mathbf{F(u)}$
I	$X_1 = \partial_t, X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = u\partial_u,$ $X_5 = y\partial_x - x\partial_y,$ $X_6 = t\partial_x - \frac{1}{2}\frac{ux}{B}\partial_u,\ X_7 = t\partial_y - \frac{1}{2}\frac{uy}{B}\partial_u,$ $X_8 = t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + urt\partial_u,$ $X_9 = \dfrac{1}{2}t^2\partial_t + \dfrac{1}{2}xt\partial_x + \dfrac{1}{2}yt\partial_y +$ $\dfrac{1}{8}\dfrac{u\left(Brt^2 4 - 4Bt - x^2 - y^2\right)}{B}\partial_u.$	1	u
II	$X_1, X_2, X_3, X_5,$ $X_{10} = t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y - \frac{u}{n-1}\partial_u.$	n $(n \neq 1)$	u
III	X_1, X_2	1	arbitrary
IV	$X_1,\ X_2,$ $X_{11} = t\partial_t + \frac{x}{2}\partial_x - \frac{1}{b}\partial_u.$	1	e^{bu} ($b \neq 0$ is const.)
V	$X_1,\ X_2,\ X_4,$ $X_{12} = t\partial_t + \frac{x}{2}\partial_x - 2Bt\partial_x + xu\partial_u,$ $X_{13} = \frac{1}{2}t^2\partial_t + \frac{1}{2}tx\partial_x + \left(\frac{t}{4} - \frac{x^2}{8B}\right)u\partial_u.$	$-$	0^1

7.3.1 Higher-order Symmetries via Recursion Operators

In this subsection, we are concerned about determining an infinite series of higher-order symmetries by defining a recursion operator. A study by [85] studied higher-order symmetries as the fundamental feature of completely integrable equations, and proposed that an equation is completely integrable if and only if it admits infinitely many time-independent Lie Bäcklund symmetries. Motivated by this consideration, we define a recursion operator for the diffusive KiSS model to prove its complete integrability. In practice, and for the sake of simplicity, we study the PDE in $(1+1)$ dimensions. We stipulate the form of the model by the selection of the free function $F(u) = u$ and assume that all parameters are nonzero, specifically $B = r = \rho = 1$.

For the convenience of the reader we present the basic theoretical framework of recursive operators. For a polynomial system that arises from evolution equations

$$u_t = A(u^{(s)}), \tag{146}$$

a higher-order symmetry $Y(u)$, leaves the above PDE invariant under the substitution $u \to u + \epsilon Y$ up to order ϵ, and must satisfy the relation [13]

$$D_t Y(u) = A'(u)[Y(u)],$$

where the right hand side is equivalent to the adjoint Fréchet derivative

$$\frac{\partial}{\partial \epsilon} A\left(u + \epsilon Y\right) |_{\epsilon=0} \, .$$

A recursion operator, $\mathcal{R}$, links higher-order symmetries [13]

$$Y^{(p+q)} = \mathcal{R} Y^{(p)}, \quad p = 1, 2, \ldots \tag{147}$$

where $q = 1$ and $Y^{(p)}$ is the $p-$th higher-order symmetry.

To return to our model, we let

$$u_t = Y(u) = u_{xx} + u. \tag{148}$$

We define the recursion operator to be $\mathcal{R} = D$, therefore the infinite series of generalized symmetries

$$Y^{(p)}(u) = \mathcal{R}^p Y,$$

can be written in evolution form $u_t = Y^{(p)}(u)$. The first few of these are

$$\begin{aligned} u_t &= Y^{(0)}(u) = u_{xx} + u, \\ u_t &= Y^{(1)}(u) = u_{xxx} + u_x, \\ u_t &= Y^{(2)}(u) = u_{xxxx} + u_{xx}, \text{ etc.,} \end{aligned} \tag{149}$$

which preserve the flows of the KiSS equation, and we conclude that we have infinite symmetries of the equation.

7.4 Conclusion

A recursion operator not only provides a connection between the generalized symmetries of an equation but is also an important tool to prove the existence of an infinite series of flows - a strong indicator of complete integrability. In fact, any equation that passes the Painlevé test or possesses a recursion operator is a candidate for being solvable by the inverse scattering transformation [85]. However, it is worth mentioning that recursive operators do not yield an exhaustive list of all possible higher-order symmetries. In this work, we verified the presence of infinitely many generalized symmetries, all of which preserve the linear KiSS equation and thus proved that it belongs to a class of evolution equations that are completely integrable.

Furthermore, multipliers and Lie Bäcklund transformations were obtained for a nonlinear diffusion equation. The multipliers were defined to contain terms up to first order in derivatives and we found two multipliers that lead to two nontrivial conservation laws.

Conclusion

In our investigations, we have provided a general form of Noether symmetries in connection with diagonal metrics. Illustratively, a nonstatic space consisting of generic metric functions was considered. The study continued to classify the Noether symmetries according to two cases, the first case was determining the Noether symmetries when the metric functions are in the form two-variable functions. The second case investigated was when the metric functions were in a single-variable form.

In chapters 4-6 were devoted to the Klein-Gordon equation and its potential. We have demonstrated potential functions related to the equation constructed on Einstein-Maxwell fields in 2+1-dimensional space, specific cases with spaces having a nonzero Weyl tensor and several 2+2 conformally reducible spaces. The potential functions were obtained through applying conformal algebra and linear combinations of it's basis vectors. Now that the nature of the potential was explicitly found, one may use Lie's invariant method to construct exact solutions of the Klein-Gordon equation. Moreover, conservation laws may be determined for every Noether symmetry.

Furthermore, in the last chapter, we have analyzed a class of evolution equations. Both the multiplier approach and higher-order symmetries via recursion operators were explored.

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