UNIVERSITY OF THE WITWATERSRAND

DOCTORATE’S THESIS

THE k–RAMSEY NUMBER FOR CYCLES

Student: Ronald John Maartens (1779477)

Supervisor: Prof. Elizabeth Jonck

Co-Supervisor: Prof. Johannes Hattingh

A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy.

27 June 2022

DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signed

Ronald John Maartens

27th day of June 2022 at the University of the Witwatersrand, Johannesburg.

UNIVERSITY OF THE WITWATERSRAND

ABSTRACT

Let F and H be two graphs. The *Ramsey number* $R(F, H)$ is defined as the smallest positive integer n such that for any red-blue coloring of the edges of K_n there is a subgraph of K_n isomorphic to F whose edges are all colored red, or a subgraph of K_n isomorphic to H whose edges are all colored blue.

Let F and H now be two bipartite graphs with Ramsey number $R(F, H)$. Further, let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order $n = \sum_{i=1}^{k} n_i$ with $n_i \in \{\lceil n/k \rceil, \lfloor n/k \rfloor\}$ for $i = 1, \ldots, k$ and $k = 2, \ldots, R(F, H)$. The k-Ramsey number $R_k(F, H)$ is then defined as the smallest positive integer n such that for any red-blue coloring of the edges of G there is a subgraph of G isomorphic to F whose edges are all colored red, or a subgraph of G isomorphic to H whose edges are all colored blue. The k-Ramsey number $R_k(F, H)$ is defined in [2] for two bipartite graphs F and H only.

In the thesis we investigate the k-Ramsey number of two cycles which are not both bipartite. Amongst other results, we determine $R_k(C_3, C_4)$, $R_k(C_3, C_5)$, $R_k(C_4, C_5)$ and $R_k(C_5, C_5)$ for all the possible values of k in each case. From these results and others, we conclude with a conjecture regarding the formula for $R_k(C_{2n+1}, C_{2m+1})$ where $n \geq m \geq 1$, $(n, m) \neq (1,1)$ and $k = 5, \ldots, 4n + 1$.

We show that $R_2(F, H)$ does not exist when F is nonbipartite and H is a nonempty graph. We also show that $R_k(K_n, H)$ does not exist when H is a nonempty graph and $2 \leq k < n$.

ACKNOWLEDGMENTS

I wish to express my deepest gratitude to both my supervisors, Professor Elizabeth Jonck and Professor Johannes Hattingh. Thank you for your motivation, encouragement, passion, and knowledge so that I could pursue my research dreams. Thank you for the reading and correcting of my work that has led to an improved thesis.

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF SYMBOLS

Symbol	**Meaning**	**Page**
$BR(F,H)$	Bipartite Ramsey number	6
C_n	Cycle	3
$\deg_G(v)$	Degree of a vertex	2
$E(G)$	Edge set	2
$ex(n,G)$	Largest number of edges possible that does not contain G.	10
G	Graph	2
$\bar{G}$	Complement	3
G_B	Blue subgraph	4
G_R	Red subgraph	4
K_n	Complete graph	2
$K_{n_1,n_2,\ldots,n_k}$	Complete k-partite graph	3
ℓ	Length of a longest cycle	13
m	Size	2
$N_G(v)$	Open neighborhood	2
n	Order	2
P_n	Path	3
(R,B)	Red-blue edge coloring	4

$R(F,H)$	Ramsey number	4
$R_k(F,H)$	k-Ramsey number	7
$V(G)$	Vertex set	2
$\Delta(G)$	Highest degree	2
$G_1 \cong G_2$	G_1 isomorphic to G_2.	3
$H \subseteq G$	H is a subgraph of G	2
$[X,Y]$	Set of edges joining a vertex in X and a vertex in Y.	2
$\langle X \rangle$	Subgraph induced by set X.	2
$\langle [X,Y] \rangle$	Subgraph induced by set $[X,Y]$.	2

CHAPTER 1

INTRODUCTION

1.1 Overview of Thesis

We want to investigate the question,

What is the k-Ramsey number for two cycles which are not both bipartite?

We begin this chapter with an overview of the work done in the thesis. We then define the relevant notations and terminologies in Graph Theory that we make use of throughout the thesis. We define and state some results of the Ramsey number, bipartite Ramsey number and the k-Ramsey number specifically relating to two cycles. The k-Ramsey number $R_k(F, H)$ is defined in [2] for two bipartite graphs F and H only.

In Chapter 2 we present results that appear in the manuscript [12]. We begin by providing an alternative proof to the existence of $R_k(F, H)$ for two bipartite graphs F and H. Thereafter we determine the k-Ramsey number $R_k(C_5, C_5)$ for $k = 5, \ldots, 9$ where $R(C_5, C_5) = 9$. Note that a 5-cycle is nonbipartite.

In Chapter 3 we investigate the k-Ramsey number of a 4-cycle and a 5-cycle which is bipartite and nonbipartite, respectively. We first show that the 2-Ramsey number of a nonbipartite graph F and a nonempty graph H does not exist. Thereafter we show that the k-Ramsey number of a n-clique F and a nonempty graph H does not exist if $2 \leq k < n$. Lastly, we determine $R_k(C_4, C_5)$ for $k = 2, \ldots, 7$ where $R(C_4, C_5) = 7$.

In Chapter 4 we continue our investigation by determining $R_k(C_3, C_4)$ for $k = 2, \ldots, 7$ where $R(C_3, C_4) = 7$, and $R_k(C_3, C_5)$ for $k = 5, \ldots, 9$ where $R(C_3, C_5) = 9$. From these two results, as well as the other results mentioned in the thesis, we conclude Chapter 4 with a conjecture regarding the formula for $R_k(C_{2n+1}, C_{2m+1})$ where $n \geq m \geq 1$, $(n, m) \neq (1,1)$ and $k = 5, \dots, 4n + 1$. Note that $R(C_{2n+1}, C_{2m+1}) = 4n + 1$.

Finally, in Chapter 5 we conclude by presenting some suggestions for possible future work we plan to pursue relating to the thesis.

1.2 Graph Theory

The graphs we consider in the thesis are simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G, denoted by n, is the number of vertices in G. The *size* of G, denoted by m, is the number of edges in G.

Two vertices u and v are *adjacent* in G if $uv \in E(G)$; otherwise, the two vertices are *nonadjacent*. The edges uv and vx are *adjacent* in G, while the edges vx and wy are *nonadjacent*. A *matching* is a set of edges which are pairwise nonadjacent.

The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G. The *degree* of a vertex v, denoted by $\deg_G(v)$, is given by $|N_G(v)|$. The highest degree of the vertices in G is denoted by $\Delta(G)$. A graph G is r*-regular* if every vertex in G has degree r. An *isolated vertex* in G is a vertex with degree zero. A *pendant edge* is the edge incident to a vertex with degree one.

A graph H is a *subgraph* of graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For two sets X and Y of vertices of a graph G, the set of edges in G joining a vertex in X and a vertex in Y is denoted by $[X, Y]$. The subgraph induced by $[X, Y]$ is denoted by $G\langle[X, Y]\rangle$, while the subgraph induced by X is denoted by $G\langle X\rangle$.

An *empty graph* is a graph with no edges, while a *complete graph* is a graph with an edge between every two vertices in the graph. We denote a complete graph of order n by K_n. Note that K_n has $\binom{n}{2}$ edges. If $H = K_n$ is a subgraph of G for some $n \leq |V(G)|$, then H is called a *n-clique*.

A graph G is *k-partite* if $V(G)$ can be partitioned into the sets $V_1, \dots, V_k$ such that $V_i \cap V_j = \emptyset$ for distinct $i, j \in \{1, \dots, k\}$. A graph G is *bipartite* when G can be partitioned into a 2-partite graph; otherwise G is *nonbipartite*. The *complete k-partite graph*, denoted by $K_{n_1, n_2, \dots, n_k}$, is the k-partite graph with an edge between every two vertices that are not in the same set V_i where $n_i = |V_i|$ for $1 \leq i \leq k$.

A $u - v$ *walk* in G is a sequence of vertices where consecutive vertices in the sequence are adjacent in G, starting at vertex u and ending at vertex v. A $u - v$ *path* in G is a $u - v$ walk in G which does not repeat vertices. The *length* of a path is given by the number of edges on the path, and we denote a path of order n by P_n. A *cycle* in G is a sequence of vertices where consecutive vertices in the sequence are adjacent in G, which starts and ends at the same vertex. The *length* of a cycle is given by the number of edges on the cycle. We denote a cycle of length n, in short, an *n-cycle*, by C_n. A cycle C_n is called *odd* or *even* depending on whether n is odd or even. A graph G is *C_n-free* if G does not have C_n as an induced subgraph.

A graph G is called *connected* if for every two vertices u and v in G there is a $u - v$ path in G; otherwise G is called *disconnected*. Graph H is a *component* of G if $H \subseteq G$ and there does not exist a connected graph F such that $H \subset F \subset G$. A *tree* (*forest*) is a connected (disconnected) graph with no cycles. Note that a tree of order n has $n - 1$ edges.

Let G be a graph of order n. The graph $\bar{G}$ is the *complement* of G if $V(\bar{G}) = V(G)$ and $E(\bar{G}) = E(K_n) - E(G)$. A graph G_1 is *isomorphic* to graph G_2, denoted by $G_1 \cong G_2$, if there exists a one-to-one mapping ϕ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$.

In a *red-blue coloring*, denoted by (R, B), (of the edges) of a graph G, every edge of G is colored red or blue. We let G_R (G_B) denote the subgraph of G induced by the set of edges all colored red (blue) in G.

We omit G in the notation if the context is clear. For simplicity in our notation, we, for example, denote the open neighborhood of vertex v in G_R by $N_R(v)$, and the size of G_B by m_B, if the context is clear.

The reader is referred to [7] for any undefined terms and concepts.

1.3 Ramsey Theory

Ramsey Theory originated from [17] by Ramsey which was submitted in 1928 and published in 1930. Sadly, Ramsey passed away at age 26 in 1930.

Photo 1.1: *Frank Plumpton Ramsey (1903-1930).* [22]

Let F and H be two graphs. The *Ramsey number* $R(F, H)$ is defined as the smallest positive integer n such that for any (R, B)-coloring of the edges of K_n there is a subgraph of K_n isomorphic to F whose edges are all colored red (called a *red* F), or a subgraph of K_n isomorphic to H whose edges are all colored blue (called a *blue* H). The Ramsey number $R(F, H)$ is known to exist for each pair F, H of graphs.

Note that the Ramsey number $R(F, H) = n$ is an extremal result as every (R, B)-coloring of the edges of K_n contains a red F or a blue H, while there exists a (R, B)-coloring of the edges of K_{n-1} that contains neither a red F nor a blue H. Note also that the subgraph induced by the set of edges all colored red is the complement of the subgraph induced by the set of edges all colored blue in some (R, B) coloring of K_n.

Over the years, the Ramsey number for various graphs F and H have been determined. Regarding the Ramsey number of two cycles, Greenwood and Gleason showed in [11] that $R(C_3, C_3) = 6$. The following result is by Chartrand and Schuster [8].

Theorem 1.1

i. *If* $n \geq 3$, *then*

$$R(C_3, C_n) = \begin{cases} 6 & if \quad n = 3, \\ 2n - 1 & if \quad n \geq 4. \end{cases}$$

ii. *If* $n \geq 4$, *then*

$$R(C_4, C_n) = \begin{cases} 6 & if \quad n = 4, \\ 7 & if \quad n = 5, \\ n + 1 & if \quad n \geq 6. \end{cases}$$

iii. *If* $n \geq 5$, *then* $R(C_5, C_n) = 2n - 1$.

iv. $R(C_6, C_6) = 8$.

Later, Bondy and Erdös found partial results for $R(C_n, C_m)$ where n and m are restricted to certain conditions in [6]. All these results were part of the building blocks that lead eventually to the formula for $R(C_n, C_m)$ where $n \geq 3$ and $m \geq 3$. Finally, the formula for $R(C_n, C_m)$ was independently proved by Faudree and Schelp [10], and Rosta [19].

Theorem 1.2

i. *For* $n \geq m \geq 3$, m *is odd, and* $(n, m) \neq (3,3)$, $R(C_n, C_m) = 2n - 1$.

ii. *For* $n \geq m \geq 4$, m *and* n *are even, and* $(n, m) \neq (4,4)$, $R(C_n, C_m) = n - 1 + m/2$.

iii. *For* $n > m \geq 4$, m *is even and* n *is odd,* $R(C_n, C_m) = max\{n - 1 + m/2, 2m - 1\}$.

A unified, self-contained, and simplified proof of Theorem 1.2 was given in [15] by Károlyi and Rosta.

1.4 Bipartite Ramsey Theory

Over the years many different versions of the Ramsey number have been defined. The bipartite Ramsey number was defined by Beineke and Schwenk in [3].

Given any two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F,H)$ is defined as the smallest positive integer n such that for every (R,B)-coloring of the edges of $K_{n,n}$ there will either be a subgraph in $K_{n,n}$ isomorophic to F whose edges are all colored red (called a *red* F), or a subgraph in $K_{n,n}$ isomorophic to H whose edges are all colored blue (called a *blue* H). The bipartite Ramsey number $BR(F,H)$ is known to exist for each pair F,H of bipartite graphs (see [13]).

The bipartite Ramsey number for various bipartite graphs F and H have been determined over the years. Regarding the bipartite Ramsey number of two cycles, Beineke and Schwenk proved that $BR(C_4,C_4)=5$ in [3]. A few years later, Zhang and Sun proved in [20] that $BR(C_{2m},C_{2n})\geq m+n-1$ and

$$BR(C_{2m},C_4)=\begin{cases} 5 & \text{if} \quad m=2 \text{ or } 3, \\ m+1 & \text{if} \quad m\geq 4. \end{cases}$$

Thereafter, Zhang, Sun and Wu showed in [21] that

$$BR(C_{2m},C_{2n})\geq\begin{cases} m+n-1 & \text{if} \quad m\neq n, \\ 2m & \text{if} \quad m=n, \end{cases}$$

and

$$BR(C_{2m},C_6)=\begin{cases} 6 & \text{if} \quad m=3, \\ m+2 & \text{if} \quad m\geq 4. \end{cases}$$

In [21] the authors conjectured the following.

Conjecture 1.3

For $m>n$, $BR(C_{2m},C_{2n})=m+n-1$.

To date this conjecture has not yet been proved.

1.5 k-Ramsey Theory

A relatively new version of Ramsey numbers, called *k-Ramsey numbers*, was introduced in [2] by Andrews, Chartrand, Lumduanhom and Zhang. Further work was done in a dissertation by Johnston [14]. See also [1], [4], [5] and [9].

For an integer $k \geq 2$, a *balanced complete k-partite graph G* of order $n \geq k$ is the complete k-partite graph in which every partite set has either $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ vertices. Suppose $n = kq + r$ where $0 \leq r \leq k-1$. Then $\lfloor n/k \rfloor = q$ and $\lceil n/k \rceil = q+1$. Let ℓ be the number of partite sets in G of order $\lfloor n/k \rfloor = q$. Then the number of partite sets of order $\lceil n/k \rceil = q+1$ equals $k - \ell$. Hence, $n = \ell q + (k-\ell)(q+1) = kq + k - \ell$, and $kq + r = kq + k - \ell$; whence $\ell = k - r$. Thus, we have $k - r$ partite sets of order $\lfloor n/k \rfloor$ and r partite sets of order $\lceil n/k \rceil$, and so G is isomorphic to

$$K_{\underbrace{\lceil n/k \rceil, \ldots, \lceil n/k \rceil}_{r}, \underbrace{\lfloor n/k \rfloor, \ldots, \lfloor n/k \rfloor}_{k-r}} = K_{\underbrace{q+1, \ldots, q+1}_{r}, \underbrace{q, \ldots, q}_{k-r}}.$$

If $r = 0$, then G is a $(k-1)q$-regular graph, which we denote by $K_{k(q)}$. If $1 \leq r \leq k-1$, then we denote G by $K_{r(q+1),(k-r)q}$.

For bipartite graphs F and H, and an integer k with $2 \leq k \leq R(F,H)$, the *k-Ramsey number* $R_k(F,H)$ is defined as the smallest positive integer n such that every (R,B)-coloring of the balanced complete k-partite graph of order n produces a red F or a blue H.

Note that the k-Ramsey number $R_k(F,H) = n$ is an extremal result as every (R,B)-coloring of the edges of the balanced complete k-partite graph of order n contains a red F or a blue H, while there exists a (R,B)-coloring of the edges of the balanced complete k-partite graph of order $n-1$ that contains neither a red F nor a blue H.

Note also that the red subgraph and the blue subgraph of the balanced complete k-partite graph G above are not complements of each other as in the case for Ramsey numbers.

We say that the k-Ramsey number *does not exist* if for any positive integer n, there exists a (R, B)-coloring of the edges of the balanced complete k-partite graph of order n that contains neither a red F nor a blue H. Trivially it follows that the 1-Ramsey number $R_1(F, H)$ does not exist for any graphs F and H.

In [2], a formula for $R_k(K_{1,s}, K_{1,t})$ is presented for each pair $K_{1,s}, K_{1,t}$ of stars of sizes $s \geq 2$ and $t \geq 2$, respectively, and each integer k with $2 \leq k \leq n-1$ where $R(K_{1,s}, K_{1,t}) = n$. The authors also proved the following theorem.

Theorem 1.4

$R_k(C_4, C_4) = 12 - k$ *where* $2 \leq k \leq 5 = R(C_4, C_4) - 1$.

Johnston made the following observation in [14].

Observation 1.5

$R_k(F, H)$ *does not exist for any two nonbipartite graphs* F *and* H *when* $k = 2$, 3 *and* 4.

The author further showed that $R_5(C_3, C_3)$ does not exist.

* * * * * * *

CHAPTER 2

THE k-RAMSEY NUMBER OF TWO FIVE CYCLES

We present results in this chapter from [12] which has been submitted to *Graphs and Combinatorics*. Our objective in this chapter is to investigate $R_k(C_5, C_5)$ for $k = 5, \dots, 9$ where $R(C_5, C_5) = 9$.

2.1 Introduction

Let F and H be bipartite graphs with Ramsey number $R(F, H)$. The k-Ramsey number $R_k(F, H)$ then exists for $k = 1, \dots, R(F, H)$. To see this, let $BR(F, H) = p$ and consider the complete k-partite graph $K_{k(p)}$. Let $V_1, \dots, V_k$ denote the partite sets of G. Then $G' = \langle [V_1, V_2] \rangle \cong K_{p,p}$. Let (R, B) be any two-coloring of G. Then, as we have a 2-coloring of the edges of G' and $BR(F, H) = p$, either a blue F or a red H is produced in G' and therefore in G. Thus, $R_k(F, H) \leq kp = kBR(F, H)$ and so $R_k(F, H)$ exists.

It is not necessarily the case that $R_k(F, H)$ exists when F and H are nonbipartite, which was pointed out by Johnston in his Ph.D. dissertation [14]. To see this, let G be any balanced complete bipartite graph. No matter how we color the edges of G, there is no subgraph isomorphic to F all whose edges are colored red nor is there a subgraph isomorphic to H all whose edges are colored blue.

Next, consider the case when $k = 3$. Let G be any balanced complete 3-partite graph with partite sets V_1, V_2 and V_3. Assigning the color red to every edge of $[V_1, V_2]$ and blue to all other edges of G results in G_R and G_B both being bipartite. Thus, $R_3(F, H)$ does not exist if both F and H are nonbipartite.

For $k = 4$, let G be a balanced complete 4-partite graph with partite sets V_1, V_2, V_3 and V_4 and color each edge of $[V_1, V_2]$, $[V_2, V_3]$ and $[V_3, V_4]$ red and assign blue to all other edges of G. Then both G_R and G_B are bipartite. Thus, $R_4(F, H)$ does not exist if both F and H are nonbipartite.

We note next that $R_5(C_3, C_3)$ does not exist. To see this, let G be a balanced complete 5-partite graph with partite sets V_i for $1 \leq i \leq 5$. If the edges in $[V_1, V_2]$, $[V_2, V_3]$, $[V_3, V_4]$, $[V_4, V_5]$, $[V_5, V_1]$ are colored red and all other edges are colored blue, then G does not contain a monochromatic K_3. Consequently, $R_k(C_3, C_3)$ only exists when $k = R(C_3, C_3) = 6$ (by Theorem 1.2), since then $R_6(C_3, C_3) = R(C_3, C_3)$.

The following result is due to Johnston.

Theorem 2.1 [14]

Let $p_1 = BR(K_{k,k}, K_{k,k})$, *and let* $p_{i+1} = BR(K_{p_i}, K_{p_i})$ *for* $i = 1, \ldots, 5$. *For integers* $k \geq \ell \geq 2$, *the Ramsey number* $R_5(C_{2\ell+1}, C_{2k+1}) \leq 5p_5$.

Recall that $R_k(C_5, C_5)$ does not exist for $2 \leq k \leq 4$, and note that $R(C_5, C_5) = 9$ by Theorem 1.2.

2.2 Preliminary Remarks

We begin this section by stating results that will be used in the remainder of this chapter.

Let n be a positive integer and G a graph. We define $ex(n, G)$ to be the largest number of edges possible in a graph on n vertices that does not contain G as a subgraph. Reiman determined the following upper bound on $ex(n, C_4)$.

Theorem 2.2 [18]

Let n *be a positive integer. Then* $ex(n, C_4) \leq (n/4)\left(1 + \sqrt{4n - 3}\right)$.

Lemma 2.3

Let H *be a graph of order five. If* $m(H) \geq 8$, *then* H *has a* 5*-cycle.*

Proof

The result is obviously true for $m(H) = 9$ or $m(H) = 10$.

Next, consider the case when $m(H) = 8$. Let $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$. If two adjacent edges, say v_1v_3 and v_1v_4, are omitted, then $v_1, v_2, v_3, v_4, v_5, v_1$ is one 5-cycle of H. If two nonadjacent edges, say v_1v_2 and v_3v_5 are omitted, then $v_1, v_3, v_2, v_5, v_4, v_1$ is a 5-cycle of H. ■

Lemma 2.4

Let H be a graph of order five obtained from K_5 by deleting exactly one edge. Delete two nonadjacent edges from H to form the graph H'. Then H' has a 5-cycle.

Proof

Let $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ and suppose v_1v_2 is the deleted edge. Consider any two nonadjacent edges e and f of H. Suppose first that e and f are both incident with v_1 and v_2, say $e = v_2v_3$ and $f = v_1v_5$. Then $v_1, v_3, v_5, v_2, v_4, v_1$ is a 5-cycle of H'. Next, suppose that exactly one of e or f is incident with a vertex in $\{v_1, v_2\}$, say $e = v_2v_3$. But then $f = v_4v_5$, and $v_1, v_3, v_5, v_2, v_4, v_1$ is a 5-cycle of H'. ■

Next, we show that determining $R_k(C_5, C_5) = 9$ for $5 \leq k \leq 9$ reduces to showing that $R_5(C_5, C_5) \leq 9$. Consider the following complete multipartite graphs G_k for $5 \leq k \leq 9$:

$$G_5 = K_{4(2),1} \quad \text{if } k = 5,$$

$$G_6 = K_{3(2),3(1)} \quad \text{if } k = 6,$$

$$G_7 = K_{2(2),5(1)} \quad \text{if } k = 7,$$

$$G_8 = K_{2,7(1)} \quad \text{if } k = 8,$$

$$G_9 = K_9 \qquad \text{if } k = 9.$$

Note that $G_k \subseteq G_{k+1}$ for $5 \le k \le 8$.

Observation 2.5

If $R_k(C_5, C_5) \le 9$, *then* $R_{k+1}(C_5, C_5) \le 9$ *for* $5 \le k \le 8$.

Proof

Let $5 \le k \le 8$ and consider any 2-coloring $\mathcal{C} = (R, B)$ of the edges of G_{k+1}. Then, as $G_k \subseteq G_{k+1}$, the coloring $\mathcal{C}$ induces a 2-coloring on the edges of G_k. Moreover, as $G_k(C_5, C_5) \le 9$, the red subgraph of this induced coloring contains a C_5 or the blue subgraph of this induced coloring contains a C_5. As $G_k \subseteq G_{k+1}$, it now follows that the red subgraph associated with $\mathcal{C}$ contains a C_5 or the blue subgraph associated with $\mathcal{C}$ contains a C_5; whence $R_{k+1}(C_5, C_5) \le 9$. ■

Consider the following complete multipartite graphs H_k for $5 \le k \le 8$:

$$H_5 = K_{3(2),2(1)} \quad \text{if } k = 5,$$

$$H_6 = K_{2(2),4(1)} \quad \text{if } k = 6,$$

$$H_7 = K_{2,6(1)} \quad \text{if } k = 7,$$

$$H_8 = K_{8(1)} \quad \text{if } k = 8.$$

As $R(C_5, C_5) = 9$, there exists a 2-coloring $\mathcal{C} = (R, B)$ of the edges of K_8 such that neither the red graph nor the blue graph has a 5-cycle. Note that $H_k \subseteq K_8$, and that the coloring $\mathcal{C}$ induces a coloring on the edges of H_k containing no monochromatic 5-cycle for $5 \le k \le 8$; whence $R_k(C_5, C_5) \ge 9$ for $5 \le k \le 8$.

Thus, if we can show that $R_5(C_5, C_5) \le 9$, then, by Observation 2.5 and the latter remark, it follows that $R_k(C_5, C_5) = 9$ for $5 \le k \le 9$.

2.3 Main Result

We now show that $R_5(C_5, C_5) \leq 9$.

Theorem 2.6

$R_5(C_5, C_5) \leq 9$.

Proof

Let graph $G = K_{4(2),1}$ and assume to the contrary that there exists a (R, B)-coloring of the edges of G that contains neither a red C_5 nor a blue C_5. Note that $m := m(G) = \binom{9}{2} - 4 = 32$. Further, let G_R and G_B denote the subgraphs of G induced by the red edges and the blue edges of G respectively. Lastly, let m_R and m_B denote the size of G_R and G_B respectively. Then,

$$m = m_R + m_B = 32.$$

Without loss of generality, suppose $m_R \geq 16$. The graph G_R is not a forest as $m_R > 8 = 9 - 1$. By Theorem 2.2, we have

$$ex(|V(G_R)|, C_4) \leq \frac{9}{4}\left(1 + \sqrt{4(9) - 3}\right) \approx 15.2,$$

which implies that G_R has a 4-cycle, as $m_R \geq 16$. Let ℓ denote the length of the longest cycle of G_R. Then, as G_R does not contain a 5-cycle, we have $\ell \in \{4,6,7,8,9\}$. We now consider each of these possibilities for ℓ.

Let $C := v_1, v_2, \ldots, v_\ell, v_1$ be a longest cycle in G_R and let $V(G_R) - V(C) = \{v_{\ell+1}, \ldots, v_9\}$. Let G_1 (G_2, respectively) be the subgraph induced by $V(C)$ ($V(G_R) - V(C)$, respectively) in G_R, and let G_3 be the subgraph of G_R induced by the edges $E(G_R) - E(G_1) - E(G_2)$. Lastly, let m_i denote the size of the graph G_i for $i = 1,2,3$. Note that $m_R = m_1 + m_2 + m_3$.

Case 1: $\ell = 4$.

As $C_4 \subseteq G_1$, we see that $4 \leq m_1 \leq 6$.

We first establish a few useful facts.

Fact 2.7

If $v' \in V(G_R)$, then G_2 does not contain a 4-cycle C' such that $|N(C') \cap \{v'\}| \geq 3$.

Proof

Suppose G_2 contains a 4-cycle $C' := u_1', u_2', u_3', u_4', u_1'$ such that $|N(C') \cap \{v'\}| \geq 3$. As $|N(C') \cap \{v'\}| \geq 3$, we have two consecutive vertices on C' which are adjacent to v'. Without loss of generality, assume $\{u_1', u_2'\} \subseteq N(v')$. Then $v', u_1', u_4', u_3', u_2', v'$ is a 5-cycle in G_R, which is a contradiction. The result now follows. □

Fact 2.8

Suppose $u \in V(G_2)$. Then $|N(u) \cap V(C)| \leq 2$. Moreover, if $|N(u) \cap V(C)| = 2$, then u is adjacent to two nonconsecutive vertices of C, and $m_1 \leq 5$.

Proof

By Fact 2.7, we have $|N(u) \cap V(C)| \leq 2$. If $N(u) = \{v_1, v_3\}$, say, then v_2 is nonadjacent to v_4, since otherwise u, v_1, v_2, v_4, v_3, u is a 5-cycle of G_R, which is a contradiction. Thus, if $N(u) = \{v_1, v_3\}$, then $m_1 \leq 5$. □

Fact 2.9

Let G_{21} be a non-trivial component of G_2 and let $u \in V(G_{21})$. If $|N(u) \cap V(C)| = 2$, then $N(v) \cap V(C) = \emptyset$ for all $v \in V(G_{21}) - \{u\}$.

Proof

Let $u \in V(G_{21})$ and suppose $|N(u) \cap V(C)| = 2$. Without loss of generality, assume $N(u) \cap V(C) = \{v_1, v_3\}$. Let $v \in V(G_{21}) - \{u\}$. As G_{21} is connected, there is a nontrivial path P connecting u and v. If v is adjacent to v_2, then P followed by the vertices v_2, v_3, v_4, v_1, u is a t-cycle of G_R where $t \geq 6$, which is a contradiction. Thus, v is nonadjacent to v_2, and, by symmetry, neither is v adjacent to v_4. If v is adjacent to v_3, then P followed by the vertices v_3, v_4, v_1, u is a t-cycle of G_R, where $t \geq 5$, which is a contradiction. If v is adjacent to v_1, then P followed by the vertices v_1, v_2, v_3, u is a t-cycle of G_R, where $t \geq 5$, which is a contradiction. Thus, $N(v) \cap V(C) = \emptyset$. □

Fact 2.10

Let G_{21} be a non-trivial component of G_2. Then $\left|N\big(V(G_{21})\big) \cap V(C)\right| \leq |V(G_{21})|$. Moreover, if $|N(u) \cap V(C)| = 2$ for some $u \in G_{21}$, then $\left|N\big(V(G_{21})\big) \cap V(C)\right| = 2$.

Proof

If $|N(u) \cap V(C)| = 2$ for some $u \in V(G_{21})$, then, by Fact 2.9, we have $N(v) \cap V(C) = \emptyset$ for all $V(G_{21}) - \{u\}$, and so $\left|N\big(V(G_{21})\big) \cap V(C)\right| = 2 \leq |V(G_{21})|$. Thus, $|N(u) \cap V(C)| \leq 1$ for all $u \in V(G_{21})$, and so $\left|N\big(V(G_{21})\big) \cap V(C)\right| \leq |V(G_{21})|$. □

Fact 2.11

Let G_{21} be a non-trivial component of G_2. If $|N(u) \cap V(C)| = 1$ for some $u \in V(G_{21})$, then $N\big(V(G_{21})\big) \cap V(C) = N(u) \cap V(C)$.

Proof

Let G_{21} be a non-trivial component of G_2 and suppose $|N(u) \cap V(C)| = 1$ for some $u \in V(G_{21})$. Without loss of generality, assume $N(u) \cap V(C) = \{v_1\}$. Note that $N(u) \cap V(C) \subseteq N\big(V(G_{21})\big) \cap V(C)$.

We show that $N(V(G_{21})) \cap V(C) \subseteq N(u) \cap V(C)$. Let $v \in N(V(G_{21})) \cap V(C)$. If $v = v_1$, then $v \in N(u) \cap V(C)$. Thus, $v \neq v_1$. Let $u' \in V(G_{21})$ be adjacent to v. If $u = u'$, then $N(u) \cap V(C) = \{v_1, v\}$, which is a contradiction. Hence, $u \neq u'$. As G_{21} is connected, there exists a path P joining u and u'. If $v \in \{v_2, v_4\}$, say $v = v_2$, then P followed by v, v_3, v_4, v_1, u is a t-cycle of G_R for $t \geq 6$, which is a contradiction. If $v = v_3$, then P followed by v, v_2, v_1, u is a t-cycle of G_R for $t \geq 5$, which is a contradiction.

Thus, $N(V(G_{21})) \cap V(C) = N(u) \cap V(C)$. □

Fact 2.12

Let G_{2i} be the nontrivial components of G_2 for $i = 1, \dots, s$, let $n_i = |V(G_{2i})|$ for $i = 1, \dots, s$ and let $c = \sum_{i=1}^{s} n_i$. Then $m_R \leq max\{m_2 + (15 - c), m_2 + 11\}$.

Proof

For simplicity, let $V(G_2) = \{u_1, \dots, u_5\}$. Without loss of generality, assume the vertices u_i for $i = c+1, \dots, 5$ are isolated in G_2. Then, by Fact 2.8 and Fact 2.10, $m_3 = \sum_{i=1}^{c} |N(u_i) \cap V(C)| + \sum_{i=c+1}^{5} |N(u_i) \cap V(C)| \leq \sum_{i=1}^{s} n_i + 2(5 - c) = c + 10 - 2c = 10 - c$. Thus, $m_R \leq m_1 + m_2 + 10 - c$. If $|N(u_i) \cap V(C)| = 2$ for some $i \in \{c+1, \dots, 5\}$, then, by Fact 2.8, we have $m_1 = 5$, and so $m_R \leq m_1 + m_2 + 10 - c = 15 + m_2 - c$. Otherwise, $|N(u_i) \cap V(C)| \leq 1$ for $i = c+1, \dots, 5$, and so $m_3 = \sum_{i=1}^{c} |N(v_i) \cap V(C)| + \sum_{i=c+1}^{5} |N(u_i) \cap V(C)| \leq \sum_{i=1}^{s} n_i + 5 - c = c + 5 - c = 5$; whence $m_R \leq 6 + m_2 + 5 = m_2 + 11$. It now follows that $m_R \leq \max\{m_2 + (15 - c), m_2 + 11\}$. □

Note that $m_2 \leq 7$, since otherwise G_R has a 5-cycle (cf. Lemma 2.3).

Case 1.1: $m_2 \leq 4$.

As $m_2 + 11 \leq 15$ and $m_R \geq 16$, we have $16 \leq m_R \leq \max\{m_2 + (15 - c), m_2 + 11\} = m_2 + (15 - c)$, and so $c \leq m_2 - 1$. First consider the case when $m_2 \leq 3$. As $c \geq 2$, we have that $m_2 \geq 3$, and so $m_2 = 3$. Then $c = 2$ and so $G_2 \cong K_2 \cup 3P_1$; whence $m_2 = 1$, which is a contradiction. Next, consider the case when $m_2 = 4$. In this case, $2 \leq c \leq 3$. If $c = 2$, then $G_2 \cong K_2 \cup 3P_1$; whence $m_2 = 1$, which is a contradiction. If $c = 3$, then $G_2 \cong P_3 \cup 2P_1$ or $G_2 \cong K_3 \cup 3P_1$; whence $m_2 \leq 3$, which is a contradiction.

Case 1.2: $m_2 = 5$.

Note that G_2 has at most two nontrivial components, since otherwise $|V(G_2)| \geq 6$. If G_2 has two nontrivial components, then $G_2 \cong 2K_2 \cup K_1$, $G_2 \cong K_2 \cup P_3$ or $G_2 \cong K_3 \cup K_2$; whence $m_2 \leq 4$, which is a contradiction. Thus, G_2 has one nontrivial component. Let D denote the graph obtained from K_4 by deleting an edge. Then $G_2 \cong K_2 + 3K_1$ (having size 1), $G_2 \cong P_3 \cup 2K_1$ (having size 2), $G_2 \cong K_3 + 2K_1$ (having size 3), $G_2 \cong H \cup K_1$ where H is a connected subgraph of order four or G_2 is connected. As $m_2 = 5$, we must have that $G_2 \cong D \cup K_1$ or G_2 is connected.

First consider the case when $G_2 \cong D \cup K_1$. Let v_5 be the isolated vertex of G_2 and let $V(D) = \{v_6, v_7, v_8, v_9\}$. If $|N(v_5) \cap V(C)| = 2$, then, by Fact 2.8, we have $m_1 \leq 5$, and so $m(\langle\{v_1, v_2, v_3, v_4, v_5\}\rangle) \leq 7$. If $|N(v_5) \cap V(C)| \leq 1$, then, as $m_1 \leq 6$, we have $m(\langle\{v_1, v_2, v_3, v_4, v_5\}\rangle) \leq 7$. As $N\big(V(D)\big) \cap V(C) = m_R - m_2 - m(\langle\{v_1, v_2, v_3, v_4, v_5\}\rangle)$, we see that $\big|N\big(V(D)\big) \cap V(C)\big| \geq 16 - 5 - 7 = 4$. If $|N(u) \cap V(C)| = 2$ for some $u \in V(D)$, then, by Fact 2.10, we have $\big|N\big(V(D)\big) \cap V(C)\big| = 2$, which is a contradiction. Thus, every vertex of D is adjacent to at most one vertex of C. As $\big|N\big(V(D)\big) \cap V(C)\big| \geq 4$ and D has order four, we see that every vertex of D is adjacent to exactly one vertex of C. By Fact 2.11, every vertex of D is adjacent to the same vertex of C, say v_1. But then D contains a 4-cycle C' such that $|N(C') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7.

Next, consider the case when G_2 is connected. If $|N(v_i) \cap V(C)| = 2$ for some $i \in \{5, \ldots, 9\}$, then $\big|N\big(V(G_2)\big) \cap V(C)\big| = 2$ (cf. Fact 2.10), and it follows that $m_R = m_1 + m_2 + m_3 \leq 6 + 5 + 2 = 13$, which is a contradiction. Thus, $|N(v_i) \cap V(C)| \leq 1$ for $i = 5, \ldots, 9$. If $N(v_i) \cap V(C) = \emptyset$ for some $i \in \{5, \ldots, 9\}$, then $m_R = m_1 + m_2 + m_3 \leq 6 + 5 + 4 = 15$, which is a contradiction. Thus, every vertex of G_2 is adjacent to exactly one vertex of C. By Fact 2.11, every vertex of G_2 is adjacent to the same vertex of C, say v_1.

Before proceeding further, we prove the following useful result.

Claim 2.13

G_2 contains P_4 as a subgraph.

Proof

If $\Delta(G_2) = 1$, then $m_2 \leq 5/2$, and so $m_2 \leq 2$, which is a contradiction.

For simplicity, assume $V(G_2) = \{u_1, \dots, u_5\}$. Assume, without loss of generality, that u_1 is the vertex of maximum degree in G_2 and that $N_{G_2}(u_1) = \{u_2, u_3, \dots, u_{\Delta(G_2)}\}$.

First consider the case when $\Delta(G_2) = 2$. Then $\deg_{G_2}(u_2) = \deg_{G_2}(u_3) = 2$, since otherwise $m_2 \leq 4$, which is a contradiction. Note that u_2 and u_3 are nonadjacent in G_2, since otherwise $\langle\{u_1, u_2, u_3\}\rangle$ is a component of G_2, which is a contradiction. Thus, u_2 is adjacent to u_4 (say), and u_4, u_2, u_1, u_3 is a path of order four in G_2.

Next, consider the case when $\Delta(G_2) = 3$. As G_2 is connected, u_5 must be adjacent to u_i in G_2 for some $i \in \{2, \dots, 4\}$, say u_2. But then u_5, u_2, u_1, u_3 is path of order four in G_2.

Lastly, consider the case when $\Delta(G_2) = 4$. Then, as $m_2 = 5$, two vertices of $\{u_2, \dots, u_5\}$ must be adjacent in G_2, say u_2 and u_3. But then u_3, u_2, u_1, u_4 is a path of order four in G_2. □

Let P be a path of order 4 of G_2. As v_1 is adjacent to every vertex of G_2, v_1 is adjacent to the endpoints of P which forms a 5-cycle in G_R, which is a contradiction.

Case 1.3: $m_2 = 6$.

Reasoning as previously, we see that $G_2 \cong K_4 \cup K_1$ or G_2 is connected.

First consider the case when $G_2 \cong K_4 \cup K_1$. Let v_5 be the isolated vertex of G_2 and let $V(K_4) = \{v_6, v_7, v_8, v_9\}$. If $|N(v_5) \cap V(C)| = 2$, then, by Fact 2.8, we have $m_1 \leq 5$, and so $m(\{v_1, v_2, v_3, v_4, v_5\}\rangle \leq 7$. If $|N(v_5) \cap V(C)| \leq 1$, then, as $m_1 \leq 6$, we have $m(\langle\{v_1, v_2, v_3, v_4, v_5\}\rangle) \leq 7$. As $N\big(V(K_4)\big) \cap V(C) = m_R - m_2 - m\langle\{v_1, v_2, v_3, v_4, v_5\}\rangle$, we see that $\big|N\big(V(K_4)\big) \cap V(C)\big| \geq 16 - 6 - 7 = 3$. If $|N(u) \cap V(C)| = 2$ for some $u \in V(K_4)$, then, by Fact 2.10, we have $\big|N\big(V(K_4)\big) \cap V(C)\big| = 2$, which is a contradiction. Thus, every vertex of K_4 is adjacent to at most one vertex of C. As $\big|N\big(V(K_4)\big) \cap V(C)\big| \geq 3$ and $|V(K_4)| = 4$, we see that at least three vertices of K_4 are adjacent to exactly one vertex of C. By Fact 2.11, at least three vertices of K_4 are adjacent to the same vertex of C, say v_1. As $C_4 \subseteq K_4$, the graph G_2 contains a 4-cycle C' such that $|N(C') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7.

Next, consider the case when G_2 is connected. If $|N(v_i) \cap V(C)| = 2$ for some $i \in \{5, \ldots, 9\}$, then $\big|N\big(V(G_2)\big) \cap V(C)\big| = 2$, and so $m_R = m_1 + m_2 + m_3 \leq 6 + 6 + 2 = 14$, which is a contradiction. Thus, $|N(v_i) \cap V(C)| \leq 1$ for $i = 5, \ldots, 9$. As $m_R \geq 16$, $m_1 \leq 6$ and $m_2 = 6$, we see that $m_3 \geq 4$. Thus, we see that at least four vertices of G_2 are adjacent to exactly one vertex of C. By Fact 2.11, at least four vertices of G_2 are adjacent to the same vertex of C, say v_1.

Claim 2.14

G_2 contains a P_4 with both its endpoints adjacent to v_1.

Proof

For simplicity, let $V(G_2) = \{u_1, \ldots, u_5\}$. If $\Delta(G_2) \leq 2$, then $m_2 \leq 5$, which is a contradiction. Thus, $\Delta(G_2) \geq 3$.

First consider the case when $\Delta(G_2) = 3$. Suppose u_1 is a vertex of maximum degree in G_2, and suppose, without loss of generality, that $N_{G_2}(u_1) = \{u_2, u_3, u_4\}$. If $|N(v_5) \cap N(u_1)| \geq 2$, then we have a 4-cycle C' in G_2 such that $|N(C') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7. Thus, v_5 is adjacent to exactly one vertex of $N(u_1)$ and nonadjacent to u_1. As $m_2 = 6$, the graph $\langle\{v_1, v_2, v_3, v_4\}\rangle \cong K_4 - e$.

As $C_4 \subseteq K_4 - e$, the graph G_2 contains a 4-cycle C' such that $|N(C') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7.

Next, consider the case when $\Delta(G_2) = 4$. Suppose u_1 is a vertex of maximum degree in G_2, and suppose, without loss of generality, that $N_{G_2}(u_1) = \{u_2, u_3, u_4, u_5\}$. As $m_2 = 6$, the graph $\langle N(u_1) \rangle$ contains exactly two edges.

First consider the case when these two edges are adjacent, i.e., suppose u_i is adjacent to u_j and u_k where i, j and k are distinct integers in the set $\{2,3,4,5\}$. Then $C' := u_1, u_j, u_i, u_k, u_1$ is a 4-cycle in G_2 such that $|N(C') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7.

Next, consider the case when these two edges are nonadjacent, i.e., suppose $u_i u_j$ and $u_k u_\ell$ are edges with $\{i, j, k, \ell\} = \{2,3,4,5\}$. If u_i is nonadjacent to v_1, then u_j, u_1, u_k, u_ℓ is a P_4 with both endpoints adjacent to v_1. If u_j is nonadjacent to v_1, then u_i, u_1, u_k, u_ℓ is a P_4 with both endpoints adjacent to v_1. If u_k is nonadjacent to v_1, then u_i, u_j, u_1, u_ℓ is a P_4 with both endpoints adjacent to v_1. If u_ℓ is nonadjacent to v_1, then u_i, u_j, u_1, u_k is a P_4 with both endpoints adjacent to v_1. Lastly, if u_1 is nonadjacent to v_1, then u_i, u_j, u_1, u_k is a P_4 with both endpoints adjacent to v_1. $\square$

As G_2 contains a P_4 with both its endpoints adjacent to v_1, we see that v_1 followed by the path of order four followed by v_1 is a 5-cycle of G_R, which is a contradiction.

Case 1.4: $m_2 = 7$.

Reasoning as before, we see that G_2 is connected. If $|N(v_i) \cap V(C)| = 2$ for some $i \in \{5, \dots, 9\}$, then $\left|N\left(V(G_2)\right) \cap V(C)\right| = 2$, and so $m_R = m_1 + m_2 + m_3 \leq 6 + 7 + 2 = 15$, which is a contradiction. Thus, $|N(v_i) \cap V(C)| \leq 1$ for $i = 5, \dots, 9$. As $m_R \geq 16$, $m_1 \leq 6$ and $m_2 = 7$, we see that $m_3 \geq 3$. Thus, we see that at least three vertices of G_2 are adjacent to exactly one vertex of C. By Fact 2.11, at least three vertices of G_2 are adjacent to the same vertex of C, say v_1.

If G_2 is C_4-free, then, by Theorem 2.2,

$$ex(|V(G_2)|, C_4) \leq (5/4)\left(1 + \sqrt{4(5) - 3}\right) \approx 6.4.$$

As $m_2 \geq 7$, we may assume G_2 has a 4-cycle. For simplicity, assume $V(G_2) = \{u_1, u_2, u_3, u_4, u_5\}$ where u_1, u_2, u_3, u_4, u_1 (say) forms a 4-cycle C'.

As G_2 is connected, vertex u_5 is adjacent to some vertex of C', say u_3. To avoid a 5-cycle, u_5 is nonadjacent to u_2 or u_4. So, possibly u_5 is adjacent to u_1.

First consider the case when u_5 is adjacent to only u_3. Then, as $m_2 = 7$, we see that $\langle\{u_1, \ldots, u_4\}\rangle \cong K_4$. Moreover, as $m_3 \geq 3$, vertex v_1 is adjacent to at least two vertices of the clique of order 4, resulting in a 5-cycle of G_R, which is a contradiction.

Next, consider the case when u_5 is adjacent to both u_1 and u_3. To avoid the 5-cycle $u_5, u_1, u_4, u_2, u_3, u_5$, we see that u_2 is nonadjacent to u_4. As $m_2 = 7$, it then follows that u_1 is adjacent to u_3.

If u_5 is nonadjacent to v_1, then $|N(C') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7. Thus, v_1 is adjacent to u_5. If v_1 is nonadjacent to u_2, then the 4-cycle $C'' = u_1, u_4, u_3, u_5, u_1$ has $|N(C'') \cap \{v_1\}| \geq 3$, which contradicts Fact 2.7. Thus, v_1 is adjacent to u_2. But then $v_1, u_2, u_1, u_3, u_5, v_1$ is a 5-cycle of G_R, which is a contradiction.

Case 2: $\ell = 6$.

As G_R does not contain any 5-cycles, the pairs of vertices v_1v_3, v_1v_5, v_2v_4, v_2v_6, v_3v_5, v_4v_6 are not in G_R, and so $m(G_1) \leq 9$.

If $|N(v_i) \cap V(C)| \leq 1$ for all $i = 7,8,9$, then $m_R \leq 9 + 3 + 3 = 15$, which is a contradiction. Assume, without loss of generality, that $|N(v_7) \cap V(C)| \geq 2$ and that v_7 is adjacent to v_1. Note that v_7 cannot be adjacent to either v_2 or v_6, since otherwise we obtain a cycle of length 7. To avoid the 5-cycle $v_7, v_1, v_6, v_5, v_4, v_7$, we see that v_7 is nonadjacent to v_4. Thus, $N(v_7) \cap \{v_2, v_4, v_6\} = \emptyset$, and so $N(v_7) \cap V(C) \subseteq \{v_1, v_3, v_5\}$.

If $N(v_i) \cap V(C) = \emptyset$ for $j = 8{,}9$, then $m_R \leq 9 + 3 + 3 = 15$, which is a contradiction. Thus $N(v_i) \cap V(C) \neq \emptyset$ for some $i \in \{8{,}9\}$.

Case 2.1: $N(v_i) \cap \{v_1, v_3, v_5\} \neq \emptyset$ *for some* $i \in \{8{,}9\}$.

As before, $N(v_i) \cap \{v_2, v_4, v_6\} = \emptyset$. But then $H = \langle\{v_2, v_4, v_6, v_7, v_i\}\rangle - \{v_7, v_i\}$ is a 5-clique with an edge removed in $\overline{G_R}$. At most two nonadjacent edges of H are not in G_B. Thus, by Lemma 2.4, the subgraph induced by $\{v_2, v_4, v_6, v_7, v_i\}$ in G_B contains a 5-cycle, which is a contradiction.

Case 2.2: $N(v_i) \cap \{v_1, v_3, v_5\} = \emptyset$ *for* $i \in \{8{,}9\}$.

In this case, $H = \langle\{v_1, v_3, v_5, v_8, v_9\}\rangle - \{v_8, v_9\}$ is a 5-clique with an edge removed in $\overline{G_R}$. Again, at most two nonadjacent edges of H are not in G_B. Thus, by Lemma 2.4, the subgraph induced by $\{v_1, v_3, v_5, v_8, v_9\}$ in G_B contains a 5-cycle, which is a contradiction.

Before proceeding further, we establish the following useful result.

Claim 2.15

If $S = v_1, v_2, \ldots, v_7, v_1$ *is a* 7*-cycle of* G_R*, then* $m(\langle S\rangle) \leq 9$.

Proof

To avoid a 5-cycle in G_R, none of the vertex pairs v_1v_4, v_1v_5, v_2v_5, v_2v_6, v_3v_6, v_3v_7, v_4v_7 occur as edges of G_R.

Suppose v_1 and v_3 are adjacent in G_R. To avoid the 5-cycle $v_1, v_6, v_5, v_4, v_3, v_1$, we see that v_1 is nonadjacent to v_6. To avoid the 5-cycle $v_1, v_3, v_4, v_6, v_7, v_1$, we see that v_4 is nonadjacent to v_6. To avoid the 5-cycle $v_1, v_3, v_4, v_5, v_7, v_1$, we see that v_5 is nonadjacent to v_7. To avoid the 5-cycle $v_1, v_3, v_5, v_6, v_7, v_1$, we see that v_3 is nonadjacent to v_5. If v_2 is adjacent to both v_4 and v_7, then

$v_2, v_4, v_5, v_6, v_7, v_2$ is a 5-cycle, which is a contradiction. Hence, at most one of the pairs v_2v_7 or v_2v_4 is an edge of G_R. Thus, if v_1 and v_3 are adjacent, then at most one of v_2v_7 or v_2v_4 is an edge of G_R; whence $m(\langle S \rangle) \leq 7 + 2 = 9$.

We may therefore assume that v_1 is nonadjacent to v_3. Similarly, we may assume that none of the pairs v_2v_4, v_3v_5, v_4v_6, v_5v_7, v_1v_6, v_2v_7 are edges of G_R. In this case, $m(\langle S \rangle) \leq 7$. □

Case 3: $\ell = 7$.

By Claim 2.15, we have $m_1 \leq 9$. As $m_2 \leq 1$ and $m_R \geq 16$, we see that $m_3 \geq 6$. Thus, either v_8 or v_9 is adjacent to at least three vertices of C. Without loss of generality, assume v_8 is adjacent to at least three vertices of C. Moreover, suppose without loss of generality, that v_8 is adjacent to v_1. To avoid an 8-cycle, we see that v_8 is nonadjacent to both v_2 and v_7. To avoid the 5-cycle $v_8, v_1, v_7, v_6, v_5, v_8$, we see that v_5 is nonadjacent to v_8. To avoid the 5-cycle $v_8, v_1, v_2, v_3, v_4, v_8$, we see that v_4 is nonadjacent to v_8. Thus, v_8 must be adjacent to both v_3 and v_6, and we obtain the 5-cycle $v_8, v_3, v_4, v_5, v_6, v_8$, which is a contradiction. Thus, this case cannot occur.

Claim 2.16

Let $T = v_1, v_2, \dots, v_8, v_1$ be an 8-cycle of G_R where u, v, w are three consecutive vertices of T. If u and w adjacent, then $m(\langle T \rangle) \leq 12$.

Proof

Assume $u = v_1$ and $w = v_3$ are adjacent. To avoid the 5-cycle $v_2, v_4, v_3, v_1, v_8, v_2$, we see that at most one of the pairs v_2v_4 or v_2v_8 occurs as an edge in G_R. To avoid the 5-cycle $v_2, v_3, v_1, v_8, v_7, v_2$, we see that v_2 is nonadjacent to v_7. To avoid the 5-cycle $v_2, v_3, v_4, v_5, v_6, v_2$, we see that v_2 is nonadjacent to v_6. To avoid the 5-cycle $v_2, v_5, v_4, v_3, v_1, v_2$, we see that v_2 is nonadjacent to v_5. By Claim 2.15, we have that $m(\langle V(T) - \{v_2\} \rangle) \leq 9$; whence $m(\langle T \rangle) = m(\langle V(T) - \{v_2\} \rangle) + \deg_{G_R}(v_2) \leq 9 + 3 = 12$. □

Case 4: $\ell = 8$.

As G_R does not contain any 5-cycles, the pairs v_1v_5, v_2v_6, v_3v_7 and v_4v_8 are not edges of G_R.

Claim 2.17

If u, v, w are three consecutive vertices of C, then u is nonadjacent to w.

Proof

Suppose u, v, w are three consecutive vertices of C with u adjacent to w. Without loss of generality, assume $u = v_1$ and that $w = v_3$. By Claim 2.16, we have $m_1 \leq 12$, and so v_9 is adjacent to at least four vertices of C.

Suppose v_9 is adjacent to vertex v_1. To avoid a 9-cycle, v_9 is nonadjacent to both v_2 and v_8. Note that vertex v_i ($i = 4,5,6$) is the endpoint of a path P_4 having v_1 as its other endpoint. Thus, to avoid a 5-cycle, vertex v_9 is nonadjacent to any of the vertices v_4, v_5 or v_6. But then vertex v_9 is only possibly adjacent to the additional vertices v_3 and v_7, which implies that v_9 is adjacent to at most three vertices of C, which is a contradiction. Thus, v_9 is nonadjacent to v_1. By symmetry, v_9 is nonadjacent to v_3.

Suppose v_9 is adjacent v_7. To avoid 9-cycles, v_9 is nonadjacent to both v_6 and v_8. But then v_9 is adjacent to v_2, v_4 and v_5, resulting in a 9-cycle, which is a contradiction. Thus, v_9 is nonadjacent to v_7.

If v_9 is adjacent to v_5, then, to avoid a 9-cycle, vertices v_4 and v_6 are nonadjacent to v_9. The only additional vertices which v_9 can be adjacent to are v_2 and v_8, which implies that v_9 is adjacent to at most three vertices of C, which is a contradiction. Thus, v_9 is nonadjacent to v_5.

It now follows that v_9 is adjacent to the vertices v_2, v_4, v_6 and v_8, which form the 5-cycle $v_9, v_4, v_3, v_1, v_8, v_9$, which is a contradiction.

Thus, v_1 is nonadjacent to v_3. The result now follows. ◻

Suppose v_9 is adjacent to a vertex of C, say v_1. To avoid a 9-cycle, v_9 is nonadjacent to both v_2 and v_8. Note that vertex v_i ($i = 4{,}6$) is the endpoint of a path P_4 having v_1 as its other endpoint. Thus, to avoid a 5-cycle, vertex v_9 is nonadjacent to both v_4 and v_6. By Claim 2.17, we see that $\{v_4, v_6, v_7, v_8, v_9\}$ induces a 5-clique in $\overline{G_R}$. Thus, v_9 is nonadjacent to any vertices of C, and again $\{v_4, v_6, v_7, v_8, v_9\}$ induces a 5-clique in $\overline{G_R}$. Thus, by Lemma 2.3, the subgraph induced by $\{v_1, v_3, v_5, v_8, v_9\}$ in G_B contains a 5-cycle, which is a contradiction.

Case 5: $\ell = 9$.

First consider the case when we have three consecutive vertices u, v, w on the cycle C with u adjacent to w. Without loss of generality, assume $u = v_1$ and $w = v_3$. Let $C' = v_1, v_3, \ldots, v_9, v_1$. To avoid the 5-cycle $v_2, v_3, v_1, v_9, v_8, v_2$, we see that v_2 is nonadjacent to v_8. By symmetry, we see that v_2 is nonadjacent to vertex v_5 either. To avoid the 5-cycle $v_2, v_7, v_8, v_9, v_1, v_2$, we see that v_2 is nonadjacent to v_7. By symmetry, v_2 is nonadjacent to v_6 either. To avoid the 5-cycle $v_3, v_1, v_9, v_2, v_4, v_3$, we see that at most one of the pairs v_2v_9 and v_2v_4 is an edge of G_R. Thus, $\deg_{G_R}(v_2) \leq 3$.

Suppose u', v', w' are three consecutive vertices of C' such that u' and w' are adjacent. Then, by Claim 2.16, $m(\langle C' \rangle) \leq 12$, and so $m_R = m(\langle C' \rangle) + \deg_{G_R}(v_2) \leq 12 + 3 = 15$, which is a contradiction.

Thus, if u', v', w' are three consecutive vertices of C', then u' is nonadjacent to w'. So, the pairs v_1v_4, v_3v_5, v_4v_6, v_5v_7, v_6v_8, v_7v_9, v_1v_8, v_3v_9 are not edges of G_R. Recall that v_2 is nonadjacent to any of the vertices in the set $\{v_5, v_6, v_7, v_8\}$. To avoid the 5-cycle $v_1, v_2, v_3, v_4, v_5, v_1$, the pair v_1v_5 is not an edge. To avoid the 5-cycle $v_1, v_3, v_4, v_5, v_6, v_1$, the pair v_1v_6 is not an edge. To avoid the 5-cycle $v_3, v_4, v_5, v_6, v_7, v_3$, vertex v_3 is nonadjacent to v_7. To avoid the 5-cycle $v_3, v_2, v_1, v_9, v_8, v_3$, vertex v_3 is nonadjacent to v_8. To avoid the 5-cycle, $v_4, v_5, v_6, v_7, v_8, v_4$, vertex v_4 is nonadjacent to v_8. To avoid the 5-cycle $v_4, v_3, v_2, v_1, v_9, v_4$, vertex v_4 is nonadjacent to v_9. To avoid the 5-cycle $v_5, v_6, v_7, v_8, v_9, v_5$, vertex v_5 is nonadjacent to v_9. To avoid the 5-cycle

$v_6, v_9, v_1, v_2, v_3, v_6$, only one of the pairs v_6v_9 and v_3v_6 may be an edge of G_R. To avoid the 5-cycle $v_7, v_1, v_2, v_3, v_4, v_7$, only one of the pairs v_1v_7 and v_4v_7 may be an edge of G_R. Recall that at most one of the pairs v_2v_9 and v_2v_4 may be an edge of G_R. The pair v_5v_8 may be an edge of G_R, and so $m(G_R) \leq 10 + 3 + 1 = 14$, which is a contradiction.

We therefore assume that if we have three consecutive vertices u, v, w on the cycle C, then u is nonadjacent to w. Thus, $v_1, v_3, v_5, v_7, v_9, v_2, v_4, v_6, v_8, v_1$ is a cycle in $\overline{G_R}$.

To avoid the 5-cycle $v_1, v_2, v_3, v_4, v_5, v_1$, vertex v_1 is nonadjacent to v_5. To avoid the 5-cycle $v_1, v_9, v_8, v_7, v_6, v_1$, vertex v_1 is nonadjacent to v_6.

Similarly, pairs v_2v_6, v_2v_7, v_3v_7, v_3v_8, v_4v_8, v_4v_9, v_5v_9 are not edges of G_R. Only an additional 9 edges may be in G_R.

We next show that v_5 and v_8 are nonadjacent. Suppose, to the contrary, that v_5 and v_8 are adjacent. To avoid the 5-cycle $v_8, v_5, v_4, v_3, v_2, v_8$, we see that v_2 is nonadjacent to v_8. To avoid the 5-cycle $v_3, v_9, v_8, v_5, v_4, v_3$, we see that vertex v_3 is nonadjacent to v_9. To avoid the 5-cycle $v_8, v_5, v_2, v_1, v_9, v_8$, we see that vertex v_2 is nonadjacent to v_5. Thus, we have ten edges in G_R so far, with only five additional edges which may be in G_R. Thus, $m_R \leq 15$, which is a contradiction.

We conclude that v_5 and v_8 are nonadjacent, and, by symmetry, that v_3 and v_6 are nonadjacent. It now follows that $\{v_1, v_3, v_8, v_5, v_6\}$ induces a $K_5 - e$ in $\overline{G_R}$. By Lemma 2.4, the subgraph induced by $\{v_1, v_3, v_8, v_5, v_6\}$ in G_B has a 5-cycle, which is a contradiction. ■

2.4 Conclusion

We began this chapter by providing an alternative proof to the existence of $R_k(F, H)$ for two bipartite graphs F and H. Thereafter we proved that $R_k(C_5, C_5) = 9 = R(C_5, C_5)$ for $k = 5, \dots, 9$.

* * * * * * *

CHAPTER 3

THE k-RAMSEY NUMBER OF A FOUR CYCLE AND A FIVE CYCLE

Our objective in this chapter is to investigate $R_k(C_4, C_5)$ for $k = 2, \ldots, 7$ where $R(C_4, C_5) = 7$.

3.1 Introduction

The k-Ramsey number $R_k(F, H)$ is defined in [2] for two bipartite graphs F and H. As in Section 2.2, it is not necessarily the case that $R_k(F, H)$ exists when F and H are not both bipartite. In Chapter 2 we investigated the case where $F = H = C_5$ which is nonbipartite. In this chapter we investigate the case where $F = C_4$ is bipartite and $H = C_5$ is nonbipartite.

3.2 Preliminary Remarks

We begin this section by stating results that will be used in the remainder of this chapter.

Proposition 3.1

For a nonbipartite graph F and a nonempty graph H, $R_2(F, H)$ does not exist.

Proof

Let G be any balanced complete 2-partite graph. Color all the edges of G red. Then, $G_R = G$ is a bipartite graph which does not contain a nonbipartite subgraph F. Also, G_B is then the empty graph of order $|V(G)|$ which does not contain a nonempty graph H as subgraph. The result then follows. ■

Proposition 3.2

For $n \geq 3$ and a nonempty graph H, $R_k(K_n, H)$ does not exist if $2 \leq k < n$.

Proof

Let G be a balanced complete k-partite graph. Color all the edges of G red so that $G_R = G$ and G_B is the empty graph of order $|V(G)|$. Then, the clique of largest order in G_R is K_k as each vertex of K_k is in a different partite set of $V(G)$. Thus, G contains neither a red K_n nor a blue H. The result then follows. ■

The next corollary follows directly from Proposition 3.2.

Corollary 3.3

For $n \geq m \geq 3$, $R_k(K_n, K_m)$ does not exist if $k < n$.

To investigate $R_k(C_4, C_5)$ we need to consider $2 \leq k \leq 6$ as $R(C_4, C_5) = 7$ by Theorem 1.2. Recall that $R_2(C_4, C_5)$ does not exist by Proposition 3.1. We consider each case separately for $k = 3, \dots, 6$.

3.3 $\boldsymbol{R_3(C_4, C_5)}$

The case for $k = 3$ is considered in the following result.

Theorem 3.4

$R_3(C_4, C_5) = 10.$

Proof

Let graph $H = K_{3,3,3}$ with vertex partite sets $V = \{v_1, v_2, v_3\}$, $W = \{w_1, w_2, w_3\}$ and $X = \{x_1, x_2, x_3\}$. Color the edges of H as described in Table 3.1 where R indicates that the edge is colored red, and B the edge is colored blue. Then H_R does not contain a 4-cycle and H_B does not contain a 5-cycle. Hence, $R_3(C_4, C_5) > 9$.

	w_1	w_2	w_3	x_1	x_2	x_3
v_1	R	R	B	R	B	B
v_2	B	R	R	B	B	R
v_3	R	B	R	B	R	B
w_1				B	B	R
w_2				B	R	B
w_3				R	B	B

Table 3.1: *Edge coloring of graph H.*

To show that $R_3(C_4, C_5) \leq 10$, let graph $G = K_{4,3,3}$ and assume to the contrary that there exists a red-blue coloring of the edges of G that contains neither a red C_4 nor a blue C_5. Further, let the vertex partite sets of G be $V = \{v_1, v_2, v_3, v_4\}$, $W = \{w_1, w_2, w_3\}$ and $X = \{x_1, x_2, x_3\}$. Without loss of generality, assume that v_1 is the vertex in V such that $\deg_{G_R}(v_1) \geq \deg_{G_R}(v_i)$ for $i = 2,3,4$. We consider each possibility for $\deg_{\mathrm{G_R}}(v_1)$.

Before proceeding further, we prove the following useful result.

Fact 3.5

Let $N_R(v_1) = A$, let $W' = W \cap A$ and let $X' = X \cap A$. Then $|N_R(v_i) \cap (W' \cup X')| \leq 1$ where $i = 2,3,4$. Suppose $|W'| \leq |X'| \leq 2$. Then the set of edges colored red in $[W', X']$ form a matching from some subset of W' to some subset of X' and has cardinality at most $|W'|$.

Proof

We first show that $|N_R(v_i) \cap (W' \cup X')| \leq 1$, where $i = 2,3,4$. To see this, let $i \in \{2,3,4\}$ and suppose edges $u_1 v_i$ and $u_2 v_i$ are colored red for $u_1, u_2 \in W' \cup X'$ with $u_1 \neq u_2$. Then G_R has the 4-cycle v_1, u_1, v_i, u_2, v_1, which is a contradiction.

Furthermore, $|N_R(w_i) \cap X'| \leq 1$ and $\left|N_R(x_j) \cap W'\right| \leq 1$ for all $w_i \in W'$ and for all $x_j \in X'$. To see this, suppose, without loss of generality, that $\{x_1, x_2\} \subseteq N_R(w_1) \cap X'$ where $w_1 \in W'$. Then G_R has the 4-cycle w_1, x_1, v_1, x_2, w_1, a contradiction. If, without loss of generality, $\{w_1, w_2\} \subseteq N_R(x_1) \cap W'$ where $x_1 \in X'$, then G_R has the 4-cycle w_1, x_1, w_2, v_1, w_1, which is a contradiction.

As $|N_R(w_i) \cap X'| \leq 1$ and $\left|N_R(x_j) \cap W'\right| \leq 1$ for all $w_i \in W'$ and for all $x_j \in X'$, we see that edges colored red in $[W', X']$ form a matching from some subset of W' to some subset of X'. The cardinality of this matching is at most $|W'|$. □

We now continue with the proof of Theorem 3.4.

Case 1: $\deg_{G_R}(v_1) = 6$.

Here $A = N_R(v_1) = W \cup X$. By Fact 3.5, the set of edges colored red in $[W, X]$ form a matching from some subset of W to some subset of X and has cardinality at most $|W|$. Without loss of generality, assume that the three mutually nonadjacent edges in $[W, X]$ which are possibly colored red are w_1x_2, w_2x_3 and w_3x_1.

Consider the vertex v_2. By Fact 3.5, we have that $|N_R(v_2) \cap A| \leq 1$, and so $|N_B(v_2) \cap A| \geq 5$. We may, without loss of generality, assume that $w_1, w_2, x_1, x_2 \in N_B(v_2) \cap A$. But then G_B has the 5-cycle $v_2, w_1, x_1, w_2, x_2, v_2$, which is a contradiction.

Case 2: $\deg_{G_R}(v_1) = 5$.

Let $A = \{w_1, w_2, x_1, x_2, x_3\}$. Assume, without loss of generality, that $N_R(v_1) = A$. In this case $W' = \{w_1, w_2\}$ and $X' = X$. By Fact 3.5, the set of edges colored red in $[W', X']$ form a matching from some subset of W' to some subset of X'. This means that there are possibly two mutually nonadjacent edges which are colored red in $[W', X']$, say x_1w_1 and x_2w_2.

Consider the vertex v_2. Suppose v_2x_2 is colored blue. Then, to avoid the 5-cycle $v_2, x_2, w_1, x_3, w_2, v_2$, v_2w_2 is colored red. By Fact 3.5, we see that $|N_R(v_2) \cap$

$A| \leq 1$, and so v_2w_1 and v_2x_1 are colored blue. But then $v_2, x_1, w_2, x_3, w_1, v_2$ is a 5-cycle in G_B, which is a contradiction.

Thus, v_2x_2 is colored red. By Fact 3.5, we have that $|N_R(v_2) \cap A| \leq 1$, and so v_2w_1 and v_2x_1 are colored blue. But then $v_2, x_1, w_2, x_3, w_1, v_2$ is a 5-cycle in G_B, which is a contradiction.

Case 3: $\deg_{\mathrm{G_R}}(v_1) = 4$.

Case 3.1: *Let $A = \{w_1, w_2, x_1, x_2\}$ and suppose that $N_R(v_1) = A$.*

Then $W' = \{w_1, w_2\}$ and $X' = \{x_1, x_2\}$. By Fact 3.5, the set of edges colored red in $[W', X']$ form a matching from some subset of W' to some subset of X'. This means that there are possibly two mutually nonadjacent edges which are colored red in $[W', X']$, say x_1w_2 and x_2w_1. Then x_1w_1 and x_2w_2 are colored blue.

Consider the vertex v_2. By Fact 3.5, we see that $|N_R(v_2) \cap A| \leq 1$, and so $|N_B(v_2) \cap A| \geq 3$. Without loss of generality, assume $\{w_1, w_2, x_1\} \subseteq N_B(v_2)$. If $\{w_1, w_2, x_1\} \subseteq N_B(v_3)$, then G_B contains the 5-cycle $x_1, w_1, v_2, w_2, v_3, x_1$, which is a contradiction. This means that v_3x_2 is colored blue. If v_3x_1 is colored blue, then G_B contains the 5-cycle $v_2, x_1, v_3, x_2, w_2, v_2$, which is a contradiction. Thus, $\{w_1, w_2, x_2\} \subseteq N_B(v_3)$, and G_B then contains the 5-cycle $x_2, w_2, v_2, w_1, v_3, x_2$, a contradiction.

Case 3.2: *Let $A = \{w_1, w_2, w_3, x_1\}$ and suppose that $N_R(v_1) = A$.*

By Fact 3.5, $|N_B(v_i) \cap A| \geq 3$. If, for $i = 1,2,3$, $|N_R(x_i) \cap W| \geq 2$, say $w_1, w_2 \in N_R(x_i) \cap W$, then G_R has the 4-cycle x_i, w_1, v_1, w_2, x_i. Thus, $|N_R(x_i) \cap W| \leq 1$, and it follows that at most three edges in $[W, X]$ are colored red.

Note that $F = \langle [W, X] \rangle \cong K_{3,3}$ in G. As F is bipartite, F does not contain odd cycles. Since $|E(F_R)| \leq 3$, $|E(F_B)| = |E(F)| - |E(F_R)| \geq 6$. Thus, $|V(F_B)| \leq |E(F_B)|$ and so F_B is not a forest. Thus, F_B has a 4-cycle or a 6-cycle. We now consider each possibility.

Case 3.2.1: *F_B contains a* 4*-cycle, say C.*

As F is bipartite, $|V(C) \cap W| = |V(C) \cap X| = 2$ and $2 \leq |A \cap V(C)| \leq 3$.

Case 3.2.1.1: $|A \cap V(C)| = 2$.

Without loss of generality, assume $C = w_1, x_2, w_2, x_3, w_1$ in F_B. For $i = 2,3,4$, $|N_B(v_i) \cap A| \geq 3$ and so $N_B(v_i) \cap V(C) \neq \emptyset$. Without loss of generality, assume $w_1 \in N_B(v_2) \cup V(C)$. Edges v_2x_2 and v_2x_3 are colored red; otherwise G_B contains the 5-cycles $v_2, w_1, x_3, w_2, x_2, v_2$ and $v_2, w_1, x_2, w_2, x_3, v_2$.

Consider the vertex v_3. Since $|N_B(v_3) \cap V(C)| \geq 1$, $w_1 \in N_B(v_3) \cap V(C)$ or $w_2 \in N_B(v_3) \cap V(C)$.

If $w_1 \in N_B(v_3) \cap V(C)$, then G_R contains a 4-cycle. To see this, let $w_1 \in N_B(v_3) \cap V(C)$. Edges v_3x_2 and v_3x_3 are colored red; otherwise G_B contains the 5-cycles $v_3, w_1, x_3, w_2, x_2, v_3$ and $v_3, w_1, x_2, w_2, x_3, v_3$. However, G_R then contains the 4-cycle v_2, x_2, v_3, x_3, v_2, a contradiction. Hence, $w_2 \in N_B(v_3) \cap V(C)$. Now, v_3x_2 and v_3x_3 are colored red; otherwise G_B contains the 5-cycles $v_3, w_2, x_3, w_1, x_2, v_3$ and $v_3, w_2, x_2, w_1, x_3, v_3$. However, G_R then contains the 4-cycle v_2, x_2, v_3, x_3, v_2, a contradiction.

Case 3.2.1.2: $|A \cap V(C)| = 3$.

Without loss of generality, assume $C = w_1, x_1, w_2, x_2, w_1$. As $|N_B(v_i) \cap A| \geq 3$, we have $|N_B(v_i) \cap V(C)| \geq 2$ for $i = 2,3,4$. If $\{x_1, w_1\} \subseteq N_B(v_2)$, then G_B contains the 5-cycle $v_2, w_1, x_2, w_2, x_1, v_2$. If $\{x_1, w_2\} \subseteq N_B(v_2)$, then G_B contains the 5-cycle $v_2, w_2, x_2, w_1, x_1, v_2$. Thus, $\{w_1, w_2\} \subseteq N_B(v_2)$. Similarly, $\{w_1, w_2\} \subseteq N_B(v_3)$.

For $i = 2,3$ we have that $x_1 \in N_R(v_i)$ and $x_2 \in N_R(v_i)$; otherwise G_B has the 5-cycles $v_i, x_1, w_2, x_2, w_1, v_i$ and $v_i, x_2, w_2, x_1, w_1, v_i$. However, this forms the 4-cycle v_2, x_1, v_3, x_2, v_2 in G_R, a contradiction.

Case 3.2.2: F_B *contains a* 6-*cycle, say* C.

Without loss of generality, assume $C = w_1, x_1, w_2, x_2, w_3, x_3, w_1$.

Consider vertices v_2 and v_3. Let $i \in \{2,3\}$. Note that if $\{w_3, x_1\} \subseteq N_B(v_i)$, then G_B has the 5-cycle $v_i, w_3, x_2, w_2, x_1, v_i$. Thus, $\{w_1, w_2\} \subseteq N_B(v_i)$.

If $x_1 v_2$ is blue, then G_B has the 5-cycle $v_3, w_1, x_1, v_2, w_2, v_3$. Thus, $x_1 v_2$ is red. It now follows that $v_2 w_3$ is colored blue. To avoid the 5-cycle $v_2, x_2, w_3, x_3, w_1, v_2$ in G_B, $x_2 v_2$ is colored red. Similarly, $x_1 v_3$ and $x_2 v_3$ are colored red.

It now follows that x_1, v_2, x_2, v_3, x_1 is a 4-cycle in G_R, which is a contradiction.

Case 4: $\deg_{G_R}(v_1) = 3$.

Case 4.1: *Suppose that* $N_R(v_1) = W$.

Then $N_B(v_1) = X$. Note that at least one of the edges $v_i x_j$ for $i, j = 2{,}3$ is colored blue; otherwise G_R has the 4-cycle v_2, x_2, v_3, x_3, v_2. Without loss of generality, say $v_2 x_2$ is colored blue.

Let $A = X \cup \{v_2, v_3, v_4\}$. For $u \in A$ and $w_1, w_2 \in N_R(u) \cap W$, G_R has the 4-cycle u, w_1, v_1, w_2, u. Thus, $|N_R(u) \cap W| \leq 1$ and so $|N_B(u) \cap W| \geq 2$ for $u \in A$. As $|N_B(v_2) \cap W| \geq 2$ and $|N_B(x_1) \cap W| \geq 2$, we have $\left|\left(N_B(v_2) \cap N_B(x_1)\right) \cap W\right| \geq 1$. Suppose $w_i \in N_B(v_2) \cap N_B(x_1)$. Then G_B contains the 5-cycle $v_2, w_i, x_1, v_1, x_2, v_2$, a contradiction.

Case 4.2: *Suppose* $N_R(v_1) = \{w_1, w_2, x_1\}$.

Then $N_B(v_1) = \{w_3, x_2, x_3\}$. Note that at least one of the edges $v_i x_j$ for $i, j = 2{,}3$ is colored blue; otherwise G_R has the 4-cycle v_2, x_2, v_3, x_3, v_2. Without loss of generality, say $v_2 x_2$ is colored blue.

Let $A = \{w_1, w_2, x_1\}$. By Fact 3.5, we have $|N_R(v_i) \cap A| \leq 1$ and so $|N_B(v_i) \cap A| \geq 2$ where $i = 2{,}3{,}4$.

Suppose w_1v_2 and w_2v_2 are both colored blue. For $i = 1,2$, edge w_ix_3 is colored red; otherwise G_B has the 5-cycle $w_i, x_3, v_1, x_2, v_2, w_i$. However, then w_2, x_3, w_1, v_1, w_2 is a 4-cycle in G_R, a contradiction. Hence, one of the edges in $\{w_1v_2, w_2v_2\}$ is colored red, and as $|N_R(v_i) \cap A| \leq 1$, the other edge is colored blue, while v_2x_1 is also colored blue. Similarly, we see that exactly one of the edges in $\{w_1v_i, w_2v_i\}$ is colored blue, while v_ix_1 is also colored blue for $i = 3,4$.

Without loss of generality, assume w_2v_2 is colored red. Then $w_1, x_1 \in N_B(v_2)$. Edges w_1x_3 and w_3x_1 are colored red; otherwise G_B contains the 5-cycles $w_1, x_3, v_1, x_2, v_2, w_1$ and $w_3, x_1, v_2, x_2, v_1, w_3$. Hence, w_2x_3 is colored blue; otherwise G_R contains the 4-cycle w_2, x_3, w_1, v_1, w_2.

Suppose w_1v_3 is colored blue. Edge w_1x_2 is colored red; otherwise G_B contains the 5-cycle $w_1, x_2, v_2, x_1, v_3, w_1$. To avoid the 4-cycle x_2, w_2, v_1, w_1, x_2 in G_R, w_2x_2 is colored blue. Edges w_3x_2 and w_3x_3 are colored red; otherwise G_B contains the 5-cycles $w_3, x_2, w_2, x_3, v_1, w_3$ and $w_3, x_3, w_2, x_2, v_1, w_3$. However, w_3, x_3, w_1, x_2, w_3 is a 4-cycle in G_R, a contradiction.

Hence, $\{w_2, x_1\} \subseteq N_B(v_3)$ with w_1v_3 colored red. Edges v_3x_2 and v_3w_3 are colored red; otherwise G_B contains the 5-cycles $v_3, x_2, v_1, x_3, w_2, v_3$ and $v_3, w_3, v_1, x_3, w_2, v_3$. Further, w_2x_2 is colored red; otherwise G_B contains the 5-cycle $w_2, x_2, v_2, x_1, v_3, w_2$. Then, w_1x_2 is colored blue; otherwise G_R contains the 4-cycle w_1, x_2, w_2, v_1, w_1.

Edges v_4x_2 and v_4x_3 are colored red; otherwise G_B contains the 5-cycles $v_4, x_2, w_1, v_2, x_1, v_4$ and $v_4, x_3, w_2, v_3, x_1, v_4$. In turn, v_4w_3 is colored blue; otherwise G_R contains the 4-cycle v_4, w_3, v_3, x_2, v_4. If w_1v_4 is colored blue, then $v_4, w_1, x_2, v_2, x_1, v_4$ is a 5-cycle in G_B. Thus, w_1v_4 is colored red and so,v_4w_2 is colored blue as $|N_R(v_4) \cap A| \leq 1$. However, G_B then contains the 5-cycle $v_4, w_2, x_3, v_1, w_3, v_4$, a contradiction.

Case 5: $\deg_{G_R}(v_1) = 2$.

For $i = 2,3,4$, recall that $\deg_{G_R}(v_i) \leq \deg_{G_R}(v_1)$. As $\deg_{G_B}(v_1) = 4$, we have that $\deg_{G_B}(v_i) \geq 4$. Further, for $i, j = 1,2,3,4$ with $i \neq j$, we have that $\left|N_B(v_i) \cap N_B(v_j)\right| \geq 2$.

Case 5.1: *Suppose* $w_1, w_2 \in N_R(v_1)$.

First consider the case when $\{x_1, x_2\} \subseteq N_B(v_1) \cap N_B(v_2)$.

Then, w_3x_1 and w_3x_2 are colored red; otherwise G_B contains the 5-cycles $w_3, x_1, v_2, x_2, v_1, w_3$ and $w_3, x_2, v_2, x_1, v_1, w_3$. Edges v_2w_1 or v_2w_2 are colored blue; otherwise G_R contains the 4-cycle v_2, w_1, v_1, w_2, v_2. Without loss of generality, say v_2w_1 is colored blue. Then, w_1x_1 is colored red; otherwise G_B contains the 5-cycle $w_1, x_1, v_1, x_2, v_2, w_1$. Similarly, w_1x_2 is colored red. However, G_R then contains the 4-cycle w_1, x_2, w_3, x_1, w_1, a contradiction.

Next, consider the case when $\{w_3, x_1\} \subseteq N_B(v_1) \cap N_B(v_2)$.

Then, w_3x_2 is colored red; otherwise G_B contains the 5-cycle $w_3, x_2, v_1, x_1, v_2, w_3$. Similarly, w_3x_3 is colored red. For $i = 2,3$, if x_iv_2 is colored blue, then $\{x_1, x_i\} \subseteq N_B(v_1) \cap N_B(v_2)$, which was considered above. But then v_2, x_2, w_3, x_3, v_2 is a 4-cycle in G_R, which is a contradiction.

Case 5.2: *Suppose* $w_1, x_1 \in N_R(v_1)$.

First consider the case when $\{w_2, w_3\} \subseteq N_B(v_1) \cap N_B(v_2)$.

Edge w_2x_2 is colored red; otherwise G_B contains the 5-cycle $w_2, x_2, v_1, w_3, v_2, w_2$. Similarly, w_2x_3, w_3x_2 and w_3x_3 are colored red. But then G_R then contains the 4-cycle w_3, x_3, w_2, x_2, w_3, a contradiction.

Next, consider the case when $\{w_2, x_2\} \subseteq N_B(v_1) \cap N_B(v_2)$.

Edges w_2x_3 and w_3x_2 are colored red; otherwise G_B contains the 5-cycles $w_2, x_3, v_1, x_2, v_2, w_2$ and $w_3, x_2, v_2, w_2, v_1, w_3$. If v_2w_3 is colored blue, then

$\{w_2, w_3\} \subseteq N_B(v_1) \cap N_B(v_2)$, a case already considered. We may therefore assume that $v_2 w_3$ is colored red. If $v_2 x_3$ is colored blue, then $\{x_2, x_3\} \subseteq N_B(v_1) \cap N_B(v_2)$, which is similar to the case $\{w_2, w_3\} \subseteq N_B(v_1) \cap N_B(v_2)$ already considered. We may therefore also assume that $v_2 x_3$ is colored red.

Further, either $w_2 x_2$ or $w_3 x_3$ is colored blue; otherwise G_R contains the 4-cycle w_2, x_2, w_3, x_3, w_2.

Case 5.2.1: *Suppose $w_2 x_2$ is colored blue.*

Suppose $w_2 v_3$ is colored blue. Then, $v_3 w_3$ and $v_3 x_3$ are colored red; otherwise G_B contains the 5-cycles $v_3, w_3, v_1, x_2, w_2, v_3$ and $v_3, x_3, v_1, x_2, w_2, v_3$. However, G_R then contains the 4-cycle v_3, x_3, v_2, w_3, v_3, a contradiction.

Therefore, $w_2 v_3$ is colored red, and similarly $w_2 v_4$ is colored red. Suppose $w_3 v_3$ is colored red. To avoid the 4-cycle v_3, x_3, v_2, w_3, v_3 in G_R, $v_3 x_3$ is colored blue. To avoid the 5-cycle $v_3, x_2, w_2, v_1, x_3, v_3$ in G_B, $v_3 x_2$ is colored red. To avoid the 4-cycles v_4, w_3, v_3, w_2, v_4 and v_4, x_2, v_3, w_2, v_4 in G_R, $v_4 w_3$ and $v_4 x_2$ are colored blue. However, G_B then contains the 5-cycle $v_4, x_2, w_2, v_1, w_3, v_4$, which is a contradiction. Therefore, $w_3 v_3$ is colored blue. Similarly, $w_3 v_4$ is colored blue.

Edge $v_3 x_2$ is colored red; otherwise G_B contains the 5-cycle $v_3, x_2, w_2, v_1, w_3, v_3$. Similarly, $v_4 x_2$ is colored red. But then G_R then contains the 4-cycle v_4, x_2, v_3, w_2, v_4, a contradiction.

Case 5.2.2: *Suppose $w_3 x_3$ is colored blue.*

Suppose $w_2 v_3$ is colored blue. Then, $v_3 w_3$ and $v_3 x_3$ are colored red; otherwise G_B contains the 5-cycles $v_3, w_3, x_3, v_1, w_2, v_3$ and $v_3, x_3, w_3, v_1, w_2, v_3$. However, G_R then contains the 4-cycle v_3, x_3, v_2, w_3, v_3, a contradiction.

Therefore, $w_2 v_3$ is colored red, and similarly $w_2 v_4$ is colored red. Suppose $w_3 v_3$ is colored red. To avoid the 4-cycle v_3, x_3, v_2, w_3, v_3 in G_R, $v_3 x_3$ is colored blue. To avoid the 5-cycle $v_3, x_2, v_1, w_3, x_3, v_3$ in G_B, $v_3 x_2$ is colored red. To avoid the 4-cycles v_4, w_3, v_3, w_2, v_4 and v_4, x_2, v_3, w_2, v_4 in G_R, $v_4 w_3$ and $v_4 x_2$ are colored blue. However, G_B then contains the 5-cycle $v_4, x_2, v_1, x_3, w_3, v_4$, which is a contradiction. Therefore, the w_3v_3 is colored blue. Similarly, w_3v_4 is colored blue.

Edge v_3x_2 is colored red; otherwise G_B contains the 5-cycle $v_3, x_2, v_1, x_3, w_3, v_3$. Similarly, v_4x_2 is colored red. But then G_R then contains the 4-cycle v_4, x_2, v_3, w_2, v_4, a contradiction.

Case 6: $\deg_{G_R}(v_1) = 1$.

Without loss of generality, assume that v_1w_1 is colored red. Then, $N_B(v_1) = \{w_2, w_3, x_1, x_2, x_3\}$. For $i = 2,3,4$, recall that $\deg_{G_R}(v_i) \leq \deg_{G_R}(v_1)$. As $\deg_{G_B}(v_1) = 5$, we have that $\deg_{G_B}(v_i) \geq 5$. Consider the vertex v_2. As $|N_B(v_1) \cap N_B(v_2)| \geq 4$, we may, without loss of generality, assume that $w_2, x_1, x_2 \in N_B(v_2)$.

For $i = 2,3$ and $j = 1,2$, if the edges w_ix_j are colored red, then G_R contains the 4-cycle w_2, x_1, w_3, x_2, w_2. Thus, at least one of the edges w_ix_j is colored blue. If w_2x_1 is colored blue, then G_B contains the 5-cycle $w_2, x_1, v_1, x_2, v_2, w_2$. If w_2x_2 is colored blue, then G_B contains the 5-cycle $w_2, x_2, v_1, x_1, v_2, w_2$. If w_3x_1 is colored blue, then G_B contains the 5-cycle $w_3, x_1, v_2, x_2, v_1, w_3$. Therefore, w_3x_2 is colored blue. However, G_B then contains the 5-cycle $w_3, x_2, v_2, x_1, v_1, w_3$, a contradiction.

Case 7: $\deg_{G_R}(v_1) = 0$.

Here $N_B(v_i) = W \cup X$ for $i = 1,2,3,4$. For $i = 1,2$ and $j = 1,2$, if the edges w_ix_j are colored red, then G_R contains the 4-cycle w_1, x_1, w_2, x_2, w_1. Thus, at least one of the edges w_ix_j is colored blue. Without loss of generality, assume w_1x_1 is colored blue. Then, G_B contains the 5-cycle $w_1, x_1, v_1, x_2, v_2, w_1$, a contradiction.

…….■

3.4 $R_4(C_4, C_5)$

We consider the case for $k = 4$ next.

Theorem 3.6

$R_4(C_4, C_5) = 8$.

Proof

Let graph $H = K_{3(2),1}$ with vertex partite sets $V = \{v\}$, $W = \{w_1, w_2\}$, $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Color the edges of H as described in Table 3.2 where R indicates the edge is colored red, and B the edge is colored blue. Then H_R does not contain a 4-cycle and H_B does not contain a 5-cycle. Hence, $R_4(C_4, C_5) > 7$.

	w_1	w_2	x_1	x_2	y_1	y_2
v	R	B	R	B	R	B
w_1			R	B	B	B
w_2			B	R	R	R
x_1					B	R
x_2					R	B

Table 3.2: *Edge coloring of graph* H.

For $R_4(C_4, C_5) \leq 8$, let graph $G = K_{4(2)}$ and assume to the contrary that there exists a red-blue coloring of the edges of G that contains neither a red C_4 nor a blue C_5. Further, let the vertex partite sets of G be $V = \{v_1, v_2\}$, $W = \{w_1, w_2\}$, $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Without loss of generality, assume that v_1 is the vertex of highest degree in G_R. We now consider each possibility for $\deg_{\mathrm{G_R}}(v_1)$.

Case 1: $\deg_{G_R}(v_1) = 6$.

Let $A = W \cup X \cup Y$. Then, $N_R(v_1) = A$. For $u_1, u_2 \in N_R(v_2) \cap A$, G_R has the 4-cycle v_2, u_1, v_1, u_2, v_2. Thus, $|N_R(v_2) \cap A| \leq 1$ and so, $|N_B(v_2) \cap A| \geq 5$. Without loss of generality, say $N_B(v_2) \supseteq A - \{y_2\}$. Further, for $u \in A$ and $u_1, u_2 \in N_R(u) \cap A$, G_R has the 4-cycle u, u_1, v_1, u_2, u. Thus, $|N_R(u) \cap A| \leq 1$ and so, $|N_B(u) \cap A| \geq 3$. Consider the vertex y_1. As $|N_B(y_1) \cap A| \geq 3$, assume without loss of generality that $\{w_1, w_2, x_1\} \subseteq N_B(y_1)$. Now, w_1x_2 and w_2x_2 are colored red; otherwise $w_1, x_2, v_2, x_1, y_1, w_1$ and $w_2, x_2, v_2, w_1, y_1, w_2$ are 5-cycles in G_B. Hence, $|N_R(x_2) \cap A| \geq 2$, a contradiction.

Case 2: $\deg_{G_R}(v_1) = 5$.

Let $A = \{w_1, w_2, x_1, x_2, y_1\}$. Without loss of generality, assume $N_R(v_1) = A$. As before, $|N_R(v_2) \cap A| \leq 1$ and so $|N_B(v_2) \cap A| \geq 4$. Without loss of generality, either $\{w_1, w_2, x_1, x_2\} \subseteq N_B(v_2)$ or $\{w_1, w_2, x_1, y_1\} \subseteq N_B(v_2)$. Also, $|N_R(y_2) \cap A| \leq 1$ and so $|N_B(y_2) \cap A| \geq 3$. Lastly, for $u \in A$, we have $|N_R(u) \cap A| \leq 1$.

Suppose $\{w_1, w_2, x_1, x_2\} \subseteq N_B(v_2)$. As $|N_B(y_2) \cap A| \geq 3$, assume without loss of generality that $\{w_1, w_2, x_1\} \subseteq N_B(y_2)$. Edge w_1x_2 is colored red; otherwise $w_1, x_2, v_2, x_1, y_2, w_1$ is a 5-cycle in G_B. Similarly, w_2x_2 is colored red. Hence, $|N_R(x_2) \cap A| \geq 2$, a contradiction.

Suppose next that $\{w_1, w_2, x_1, y_1\} \subseteq N_B(v_2)$. Suppose $\{w_1, w_2\} \subseteq N_B(y_2)$. Then w_1x_1 is colored red; otherwise $w_1, x_1, v_2, w_2, y_2, w_1$ is a 5-cycle in G_B. Similarly, w_2x_1 is colored red; whence $|N_R(x_1) \cap A| \geq 2$, a contradiction.

Thus, $\{w_1, w_2\} \nsubseteq N_B(y_2)$. Without loss of generality, assume w_1y_2 is red. As $|N_R(y_2) \cap A| \leq 1$, we have that $\{w_2, x_1, x_2\} \subseteq N_B(y_2)$. To avoid the 5-cycle $y_2, x_2, y_1, v_2, w_2, y_2$, edge x_2y_1 is colored red. As $|N_R(y_1) \cap A| \leq 1$, we have $\{w_1, w_2, x_1\} \subseteq N_B(y_1)$. But then $w_2, y_2, x_1, y_1, v_2, w_2$ is a 5-cycle in G_B, which is a contradiction.

Case 3: $\deg_{G_R}(v_1) = 4$.

Case 3.1: *Suppose* $\{w_1, w_2, x_1, x_2\} = N_R(v_1)$.

Then $y_1, y_2 \in N_B(v_1)$. Let $A_1 = \{w_1, w_2, x_1, x_2\}$ and $A_2 = \{v_2, y_1, y_2\}$. As before, for $u \in A_2$, we have $|N_R(u) \cap A_1| \leq 1$ and so $|N_B(u) \cap A_1| \geq 3$. Without loss of generality, we may assume that $w_1, w_2, x_1 \in N_B(v_2)$.

If $w_1, w_2 \in N_B(y_2)$, then G_R contains a 4-cycle. To see this, suppose $w_1, w_2 \in N_B(y_2)$. Then, x_1w_1 and x_1w_2 are colored red; otherwise G_B contains the 5-cycles $x_1, w_1, y_2, w_2, v_2, x_1$ and $x_1, w_2, y_2, w_1, v_2, x_1$. However, G_R then contains the 4-cycle x_1, w_1, v_1, w_2, x_1, a contradiction. Thus,$\{w_1, w_2\} \nsubseteq N_B(y_2)$.

Therefore, let $w_1, x_1 \in N_B(y_2)$. Edge w_2x_1 is colored red; otherwise G_B contains the 5-cycle $w_2, x_1, y_2, w_1, v_2, w_2$. Edge w_1x_1 is colored blue; otherwise G_R contains the 4-cycle w_1, x_1, w_2, v_1, w_1. Edges w_1y_1 and x_1y_1 are colored red; otherwise G_B contains the 5-cycles $w_1, y_1, v_1, y_2, x_1, w_1$ and $x_1, y_1, v_1, y_2, w_1, x_1$. However, $|N_R(y_1) \cap A_1| \geq 2$ which is a contradiction.

Case 3.2: *Suppose* $\{w_1, w_2, x_1, y_1\} = N_R(v_1)$.

Then $x_2, y_2 \in N_B(v_1)$. Let $A = \{w_1, w_2, x_1, y_1\}$. As before, $|N_B(v_2) \cap A| \geq 3$, while, for $i = 1,2$ we have $|N_B(y_i) \cap \{w_1, w_2, x_1\}| \geq 2$.

First consider the case when $\{w_1, w_2, x_1\} \subseteq N_B(v_2)$.

If $w_1, w_2 \in N_B(y_1)$, then G_R contains a 4-cycle. To see this, let $w_1, w_2 \in N_B(y_1)$. Edge x_1w_1 is colored red; otherwise G_B contains the 5-cycle $x_1, w_1, y_1, w_2, v_2, x_1$. Similarly, x_1w_2 is colored red. However, G_R contains the 4-cycle x_1, w_1, v_1, w_2, x_1, a contradiction.

Hence, $\{w_1, w_2\} \nsubseteq N_B(y_1)$, and so, without loss of generality, assume $\{w_1, x_1\} \subseteq N_B(y_1)$. Edge w_2x_1 is colored red; otherwise G_B contains the 5-cycle $w_2, x_1, y_1, w_1, v_2, w_2$. Edge w_1x_1 is colored blue; otherwise G_R contains the 4-cycle w_1, x_1, w_2, v_1, w_1. Edge w_2y_1 is colored red; otherwise G_B contains the 5-

cycle $w_2, y_1, w_1, x_1, v_2, w_2$. However, G_R contains the 4-cycle w_2, y_1, v_1, x_1, w_2, a contradiction.

Next, consider the case when $\{w_1, x_1, y_1\} \subseteq N_B(v_2)$. Note that $v_2 w_2$ is colored red.

Assume that $v_2 y_2$ is colored blue. Then to avoid the 5-cycle $v_2, y_2, v_1, x_2, y_1, v_2$ in G_B, $x_2 y_1$ is colored red. To avoid the 5-cycle $v_2, y_2, v_1, x_2, w_1, v_2$ in G_B, $x_2 w_1$ is colored red. But then x_2, y_1, v_1, w_1, x_2 is a 4-cycle in G_R, which is a contradiction.

Thus, $v_2 y_2$ is colored red, and, similarly, $v_2 x_2$ is colored red. To avoid the 4-cycle w_2, x_2, v_2, y_2, w_2, we may, without loss of generality, assume that $w_2 x_2$ is colored blue.

Assume $w_2 y_2$ is colored red. Then, as $|N_B(y_2) \cap \{w_1, w_2, x_1\}| \geq 2$, it follows that $y_2 x_1$ and $y_2 w_1$ are colored blue. To avoid the 5-cycle $v_2, x_1, y_2, w_1, y_1, v_2$, the $w_1 y_1$ is colored red. As $|N_B(y_1) \cap \{w_1, w_2, x_1\}| \geq 2$, it follows that $y_1 x_1$ and $y_1 w_2$ are colored blue. But then $v_2, w_1, y_2, x_1, y_1, v_2$ is 5-cycle in G_B, which is a contradiction.

We may therefore assume that $w_2 y_2$ is colored blue. To avoid the 4-cycle w_2, y_1, v_1, x_1, w_2 in G_R, we may assume, without loss of generality, that $w_2 x_1$ is colored blue. Then, to avoid the 5-cycle $v_1, x_2, w_2, x_1, y_2, v_1$ in G_B, $x_1 y_2$ is colored red. As $|N_B(y_2) \cap \{w_1, w_2, x_1\}| \geq 2$, edge $w_1 y_2$ must be colored blue. But then $v_2, w_1, y_2, w_2, x_1, v_2$ is a 5-cycle in G_B, which is a contradiction.

Case 4: $\deg_{G_R}(v_1) = 3$.

Case 4.1: *Suppose* $w_1, w_2, x_1 \in N_R(v_1)$.

Then, $x_2, y_1, y_2 \in N_B(v_1)$. At least one of the edges in $[W, Y]$ is colored blue; otherwise w_1, y_1, w_2, y_2, w_1 is a 4-cycle in G_R. Say $w_1 y_1$ is colored blue. Let $A_1 = \{w_1, w_2, x_1\}$ and $A_2 = \{v_2, y_1, y_2\}$. As before, for $u \in A_2$, we have $|N_B(u) \cap A_1| \geq 2$. We now consider vertex v_2.

Case 4.1.1: *Suppose* $w_1, w_2 \in N_B(v_2)$.

Suppose w_1x_1 is colored red. Edge w_2x_1 is colored blue; otherwise G_R contains the 4-cycle w_2, x_1, w_1, v_1, w_2. Edge x_1y_1 is colored red; otherwise G_B contains the 5-cycle $x_1, y_1, w_1, v_2, w_2, x_1$. Edge w_2y_1 is colored blue; otherwise G_R contains the 4-cycle w_2, y_1, x_1, v_1, w_2. As $\deg_{G_R}(x_1) \leq \deg_{G_R}(v_1)$, we have that x_1y_2 is colored blue. However, G_B then contains the 5-cycle $x_1, y_2, v_1, y_1, w_2, x_1$, a contradiction. Thus, w_1x_1 is colored blue.

Now, v_2x_2 and v_2y_2 are colored red; otherwise G_B contains the 5-cycles $v_2, x_2, v_1, y_1, w_1, v_2$ and $v_2, y_2, v_1, y_1, w_1, v_2$. Also, x_1y_2 is colored red; otherwise G_B contains the 5-cycle $x_1, y_2, v_1, y_1, w_1, x_1$. As $|N_R(y_2) \cap A_1| \leq 1$, edges w_1y_2 and w_2y_2 are colored blue. Then, v_2x_1 and v_2y_1 are colored red; otherwise G_B contains the 5-cycles $v_2, x_1, w_1, y_2, w_2, v_2$ and $v_2, y_1, w_1, y_2, w_2, v_2$. Hence, $|N_R(v_2)| \geq 4$ which contradicts the fact that $\deg_{G_R}(v_2) \leq \deg_{G_R}(v_1)$. Therefore, $\{w_1, w_2\} \nsubseteq N_B(v_2)$.

Case 4.1.2: *Suppose* $w_1, x_1 \in N_B(v_2)$.

Note that v_2w_2 is colored red. Also, v_2x_2 and v_2y_2 are colored red; otherwise G_B contains the 5-cycles $v_2, x_2, v_1, y_1, w_1, v_2$ and $v_2, y_2, v_1, y_1, w_1, v_2$.

Suppose w_1x_1 is colored red. Edge w_2x_1 is colored blue; otherwise G_R contains the 4-cycle w_2, x_1, w_1, v_1, w_2. Edge w_2y_1 is colored red; otherwise G_B contains the 5-cycle $w_2, y_1, w_1, v_2, x_1, w_2$. Edge x_1y_1 is colored blue; otherwise G_R contains the 4-cycle x_1, y_1, w_2, v_1, x_1. Edge w_2x_2 and w_2y_2 are colored red; otherwise G_B contains the 5-cycles $w_2, x_2, v_1, y_1, x_1, w_2$ and $w_2, y_2, v_1, y_1, x_1, w_2$. However, G_R now has the 4-cycle w_2, y_2, v_2, x_2, w_2, a contradiction. Thus, w_1x_1 is colored blue.

Edge x_1y_2 is colored red; otherwise G_B contains the 5-cycle $x_1, y_2, v_1, y_1, w_1, x_1$. Edges w_1y_2 and w_2y_2 are colored blue; otherwise G_R contains the 4-cycles w_1, y_2, x_1, v_1, w_1 and w_2, y_2, x_1, v_1, w_2. Edge w_2x_1 is colored red; otherwise G_B contains the 5-cycle $w_2, x_1, v_2, w_1, y_2, w_2$. However, G_R now has the 4-cycle w_2, x_1, y_2, v_2, w_2, a contradiction. Hence, $\{w_1, x_1\} \nsubseteq N_B(v_2)$.

Case 4.1.3: *Suppose* $w_2, x_1 \in N_B(v_2)$.

As $\{w_1, w_2\} \nsubseteq N_B(v_2)$, we see that w_1v_2 is colored red.

Suppose w_1x_1 is colored blue. Then w_2y_1 and x_1y_2 are colored red; otherwise G_B contains the 5-cycles $w_2, y_1, w_1, x_1, v_2, w_2$ and $x_1, y_2, v_1, y_1, w_1, x_1$. Further, w_1y_2 and x_1y_1 are colored blue; otherwise G_R has the 4-cycles w_1, y_2, x_1, v_1, w_1 and x_1, y_1, w_2, v_1, x_1. Hence, G_B contains the 5-cycle $x_1, y_1, v_1, y_2, w_1, x_1$, a contradiction. Therefore, w_1x_1 is colored red. Also, w_2x_1 is colored blue; otherwise G_R contains the 4-cycle w_2, x_1, w_1, v_1, w_2.

Suppose w_2y_2 is colored blue. Then v_2y_1 and x_1y_1 are colored red; otherwise G_B contains the 5-cycles $v_2, y_1, v_1, y_2, w_2, v_2$ and $x_1, y_1, v_1, y_2, w_2, x_1$. Hence, G_R contains the 4-cycle x_1, y_1, v_2, w_1, x_1, a contradiction. Hence, w_2y_2 is colored red.

Suppose w_1y_2 is colored blue. Then x_2y_1 and x_2y_2 are colored red; otherwise G_B contains the 5-cycles $x_2, y_1, w_1, y_2, v_1, x_2$ and $x_2, y_2, w_1, y_1, v_1, x_2$. The edge w_2y_1 is colored blue; otherwise G_R contains the 4-cycle w_2, y_1, x_2, y_2, w_2. In turn, x_1y_2 is colored red; otherwise G_B contains the 5-cycle $x_1, y_2, v_1, y_1, w_2, x_1$. However, G_R then contains the 4-cycle x_1, y_2, w_2, v_1, x_1, a contradiction.

Hence, w_1y_2 is colored red and so G_R contains the 4-cycle w_1, y_2, w_2, v_1, w_1, a contradiction. Thus, $\{w_2, x_1\} \nsubseteq N_B(v_2)$ and so, $|N_B(v_2) \cap A_1| < 2$ which is a contradiction.

Case 4.2: *Suppose* $w_1, x_1, y_1 \in N_R(v_1)$.

Then, $w_2, x_2, y_2 \in N_B(v_1)$. For every $u \in V(G) - \{v_1\}$ we have $\deg_{G_R}(u) \leq 3$ and $\deg_{G_B}(u) \geq 3$ as $\deg_{G_R}(u) \leq \deg_{G_R}(v_1)$. Let $A = \{w_1, x_1, y_1\}$. As before, $|N_R(v_2) \cap A| \leq 1$ and so $|N_B(v_2) \cap A| \geq 2$. Without loss of generality, we may assume that $w_1, x_1 \in N_B(v_2)$.

Suppose v_2w_2 is colored blue. Then w_1y_2 is colored red; otherwise G_B contains the 5-cycle $w_1, y_2, v_1, w_2, v_2, w_1$. By symmetry, x_1y_2 is colored red. However, G_R then contains the 4-cycle x_1, y_2, w_1, v_1, x_1, a contradiction.

Thus, v_2w_2 is colored red. By symmetry, v_2x_2 is also colored red. Suppose y_2v_2 is colored blue. Edges w_1x_2 and w_2x_1 are colored red; otherwise G_B contains the 5-cycles $w_1, x_2, v_1, y_2, v_2, w_1$ and $w_2, x_1, v_2, y_2, v_1, w_2$. Edges w_2y_1 and x_2y_1 are colored blue; otherwise G_R contains the 4-cycles w_2, y_1, v_1, x_1, w_2 and x_2, y_1, v_1, w_1, x_2. Edges w_2y_2 and x_2y_2 are colored red; otherwise G_B contains the 5-cycles $w_2, y_2, v_1, x_2, y_1, w_2$ and $x_2, y_2, v_1, w_2, y_1, x_2$. However, G_R then contains the 4-cycle x_2, y_2, w_2, v_2, x_2, a contradiction. Hence, y_2v_2 is colored red. As $\deg_{\mathrm{G_R}}(v_2) \leq 3$, edge y_1v_2 is colored blue.

To avoid the 4-cycle x_2, w_1, v_1, y_1, x_2 in G_R, either x_2w_1 or x_2y_1 is colored blue. Without loss of generality, assume x_2w_1 is colored blue. To avoid the 4-cycle x_2, w_2, v_2, y_2, x_2 in G_R, either x_2w_2 or x_2y_2 is colored blue.

Suppose x_2w_2 is colored blue. Then w_2x_1 and w_2y_1 are colored red; otherwise G_B contains the 5-cycles $w_2, x_1, v_2, w_1, x_2, w_2$ and $w_2, y_1, v_2, w_1, x_2, w_2$. However, G_R then contains the 4-cycle w_2, y_1, v_1, x_1, w_2, a contradiction.

Thus, x_2w_2 is colored red, while x_2y_2 is colored blue.

Suppose x_2y_1 is colored blue. Then x_1y_2 is colored red, since otherwise $y_2, x_1, v_2, y_1, x_2, y_2$ is a 5-cycle in G_B. Edge w_1y_2 is also colored red, since otherwise $y_2, w_1, v_2, y_1, x_2, y_2$ is a 5-cycle in G_B. However, G_R then contains the 4-cycle y_2, w_1, v_1, x_1, y_2, a contradiction.

Thus, we may assume that x_2y_1 is colored red. To avoid the 4-cycle y_1, x_2, v_2, w_2, y_1 in G_R, y_1w_2 is colored blue. To avoid the 5-cycle $w_1, y_1, w_2, v_1, x_2, w_1$ in G_B, w_1y_1 is colored red. To avoid the 4-cycle w_2, x_2, v_2, y_2, w_2 in G_R, w_2y_2 is colored blue. To avoid the 5-cycle $w_1, y_2, w_2, v_1, x_2, w_1$ in G_B, w_1y_2 is colored red. As $\deg_{G_R}(w_1) = 3$, w_1x_1 is colored blue. To avoid the 5-cycle $y_2, x_1, w_1, x_2, v_1, y_2$ in G_B, the x_1y_2 must be colored red. But then x_1, y_2, w_1, v_1, x_1 is a 4-cycle in G_R, which is a contradiction.

Case 5: $\deg_{G_R}(v_1) = 2$.

Case 5.1: *Suppose* $w_1, w_2 \in N_R(v_1)$.

Then $X \cup Y = N_B(v_1)$. As before, $|N_R(v_2) \cap W| \leq 1$ and so $|N_B(v_2) \cap W| \geq 1$. Assume that w_1v_2 is colored blue.

Further, for every $u \in V(G) - \{v_1\}$ we have $\deg_{G_R}(u) \leq 2$ or, equivalently, $\deg_{G_B}(u) \geq 4$ as $\deg_{G_R}(u) \leq \deg_{G_R}(v_1)$. Thus, $|N_B(v_1) \cap N_B(v_2)| \geq 2$ as $\deg_{G_B}(v_2) \geq 4$.

Suppose $x_1, x_2 \in N_B(v_2)$. Then w_1x_1 and w_1x_2 are colored red; otherwise G_B contains the 5-cycles $w_1, x_1, v_1, x_2, v_2, w_1$ and $w_1, x_2, v_1, x_1, v_2, w_1$. But then $\deg_{G_R}(w_1) \geq 3$, which is a contradiction. Hence, $\{x_1, x_2\} \not\subseteq N_B(v_2)$.

Therefore, without loss of generality, $x_1, y_1 \in N_B(v_2)$. Then w_1x_1 and w_1y_1 are colored red; otherwise G_B contains the 5-cycles $w_1, x_1, v_1, y_1, v_2, w_1$ and $w_1, y_1, v_1, x_1, v_2, w_1$. But then $\deg_{G_R}(w_1) \geq 3$, which is a contradiction.

Case 5.2: *Suppose* $w_1, x_1 \in N_R(v_1)$.

Then $\{w_2, x_2\} \cup Y = N_B(v_1)$. As before, $|N_R(v_2) \cap \{w_1, x_1\}| \leq 1$ and so $|N_B(v_2) \cap \{w_1, x_1\}| \geq 1$. Assume that w_1v_2 is colored blue.

Further, for every $u \in V(G) - \{v_1\}$ we have $\deg_{G_R}(u) \leq 2$ or, equivalently, $\deg_{G_B}(u) \geq 4$ as $\deg_{G_R}(u) \leq \deg_{G_R}(v_1)$. Thus, $|N_B(v_1) \cap N_B(v_2)| \geq 2$ as $\deg_{G_B}(v_2) \geq 4$.

Suppose $w_2, x_2 \in N_B(v_2)$. Then w_1x_2 and w_1y_1 are colored red; otherwise G_B contains the 5-cycles $w_1, x_2, v_1, w_2, v_2, w_1$ and $w_1, y_1, v_1, w_2, v_2, w_1$. But then $\deg_{G_R}(w_1) \geq 3$, which is a contradiction.

Suppose $w_2, y_1 \in N_B(v_2)$. Then, w_1x_2 and w_1y_2 are colored red; otherwise G_B contains the 5-cycles $w_1, x_2, v_1, y_1, v_2, w_1$ and $w_1, y_2, v_1, y_1, v_2, w_1$. But then $\deg_{G_R}(w_1) \geq 3$, which is a contradiction.

Suppose $x_2, y_1 \in N_B(v_2)$. Then w_1x_2 and w_1y_1 are colored red; otherwise G_B contains the 5-cycles $w_1, x_2, v_1, y_1, v_2, w_1$ and $w_1, y_1, v_1, x_2, v_2, w_1$. But then $\deg_{G_R}(w_1) \geq 3$, which is a contradiction.

Therefore, $y_1, y_2 \in N_B(v_2)$. Then w_1y_1 and w_1y_2 are colored red; otherwise G_B contains the 5-cycles $w_1, y_1, v_1, y_2, v_2, w_1$ and $w_1, y_2, v_1, y_1, v_2, w_1$. But then $\deg_{G_R}(w_1) \geq 3$, which is a contradiction.

Case 6: $\deg_{G_R}(v_1) = 1$.

Note that $\deg_{G_B}(v_1) = 5$. Thus, for every $u \in V(G) - \{v_1\}$ we have $\deg_{G_R}(u) \leq 1$ and $5 \leq \deg_{G_B}(u) \leq 6$ as $\deg_{G_R}(u) \leq \deg_{G_R}(v_1)$. Thus, G_R is a spanning subgraph of $4K_2$. We claim that G_B has a 5-cycle.

To see this, consider the subgraph H induced by $\{v_1, v_2, w_1, x_1, y_1\}$ in G. Then $H \cong K_5 - \{v_1v_2\}$. The subgraph H' of G_B induced by $\{v_1, v_2, w_1, x_1, y_1\}$ is obtained by removing at most two nonadjacent edges from H. By Lemma 2.4, $H' \subseteq G_B$ has a 5-cycle, which is a contradiction.

Case 7: $\deg_{G_R}(v_1) = 0$.

Here the subgraph H induced by $\{v_1, v_2, w_1, x_1, y_1\}$ in G_B is isomorphic to $K_5 - \{v_1v_2\}$, which has the 5-cycle $v_1, w_1, x_1, v_2, y_1, v_1$, which is a contradiction. ■

3.5 $R_5(C_4, C_5)$

We now consider $k = 5$.

Theorem 3.7

$R_5(C_4, C_5) = 7$.

Proof

To show that $R_5(C_4, C_5) > 6$ we need to consider the graph $H = K_{1(2),4(1)}$. As $R(C_4, C_5) = 7$, there exists a (R, B)-coloring of the edges of K_6 such that K_6 has neither a red C_4 nor a blue C_5. As $H \subset K_6$, we have an induced (R, B)-coloring of the edges of H such that H has neither a red C_4 nor a blue C_5.

To show that $R_5(C_4, C_5) \leq 7$, let graph $G = K_{2(2),3(1)}$ and assume to the contrary that there exists a red-blue coloring of the edges of G that contains neither a red C_4 nor a blue C_5. Further, let the vertex partite sets of G be $V = \{v_1, v_2\}$, $W = \{w_1, w_2\}$, $X = \{x\}$, $Y = \{y\}$ and $Z = \{z\}$. Without loss of generality, assume that $\deg_{G_R}(v_1) \geq \deg_{G_R}(v_2)$. We now consider each possibility for $\deg_{G_R}(v_1)$.

Case 1: $\deg_{G_R}(v_1) = 5$.

Let $A = \{w_1, w_2, x, y, z\}$. Then $N_R(v_1) = A$. As before, we have $|N_R(v_2) \cap A| \leq 1$ and so $|N_B(v_2) \cap A| \geq 4$.

Suppose $w_1, w_2, x, y \in N_B(v_2)$. At least one of the edges in $[W, (X \cup Y)]$ is colored blue; otherwise w_1, x, w_2, y, w_1 is a 4-cycle in G_R. Without loss of generality, assume that w_1x is colored blue.

Suppose yz is colored red. Then xz and w_2z are colored blue; otherwise G_R contains the 4-cycles x, z, y, v_1, x and w_2, z, y, v_1, w_2. However, G_B then contains the 5-cycle w_2, z, x, w_1, v_2, w_2, a contradiction.

Thus, yz is colored blue. Edge w_1z and xz are colored red; otherwise G_B contains the 5-cycles w_1, z, y, v_2, x, w_1 and x, z, y, v_2, w_1, x. However, G_R then contains the 4-cycle x, z, w_1, v_1, x, a contradiction.

Therefore, $w_1, x, y, z \in N_B(v_2)$. We have that xy or yz is colored blue; otherwise x, y, z, v_1, x is a 4-cycle in G_R. Without loss of generality, assume that xy is colored blue.

Suppose w_2z is colored red. Then w_2x and w_2y are colored blue; otherwise G_R contains the 4-cycles w_2, x, v_1, z, w_2 and w_2, y, v_1, z, w_2. Also, xz is colored blue; otherwise G_R contains the 4-cycle x, z, w_2, v_1, x. However, G_B then contains the 5-cycle x, z, v_2, y, w_2, x, a contradiction.

Thus, w_2z is colored blue. Now w_2x is colored red; otherwise G_B contains the 5-cycle w_2, x, y, v_2, z, w_2. By symmetry, w_2y is also colored red. However, G_R then contains the 4-cycle w_2, y, v_1, x, w_2, a contradiction.

Case 2: $\deg_{G_R}(v_1) = 4$.

Case 2.1: *Let* $A = \{w_1, x, y, z\}$ *and* $N_R(v_1) = A$.

As before, $|N_R(v_2) \cap A| \leq 1$ and so, $|N_B(v_2) \cap A| \geq 3$.

Suppose $x, y, z \in N_B(v_2)$. We have that xy or yz is colored blue; otherwise x, y, z, v_1, x is a 4-cycle in G_R. Without loss of generality, assume that xy is colored blue.

Suppose w_1z is colored red. Then w_1x and w_1y are colored blue; otherwise G_R contains the 4-cycles w_1, x, v_1, z, w_1 and w_1, y, v_1, z, w_1. Also, yz is colored blue; otherwise G_R contains the 4-cycle y, z, w_1, v_1, y. However, G_B then contains the 5-cycle y, z, v_2, x, w_1, y, a contradiction.

Thus, assume w_1z is colored blue. Edge w_1x is colored red, since otherwise G_B contains the 5-cycle w_1, x, y, v_2, z, w_1. By symmetry, w_1y is colored red. However, G_R then contains the 4-cycle w_1, y, v_1, x, w_1, a contradiction.

We may therefore assume that $w_1, x, y \in N_B(v_2)$. We have that w_1x or w_1y is colored blue; otherwise w_1, x, v_1, y, w_1 is a 4-cycle in G_R. Without loss of generality, assume w_1x is colored blue.

Assume that yz is colored red. Then xz, w_1z and xy are colored blue; otherwise G_R contains the 4-cycles x, z, y, v_1, x; w_1, z, y, v_1, w_1 or x, y, z, v_1, x. However, G_B then contains the 5-cycle w_1, z, x, y, v_2, w_1, a contradiction.

Thus, yz is colored blue. Edge w_1z is colored red; otherwise G_B contains the 5-cycle w_1, z, y, v_2, x, w_1. By symmetry, xz is also colored red. However, G_R then contains the 4-cycle x, z, w_1, v_1, x, a contradiction.

Case 2.2: *Let* $A = \{w_1, w_2, x, y\}$ *and* $N_R(v_1) = A$.

As before, $|N_R(v_2) \cap A| \leq 1$ and so, $|N_B(v_2) \cap A| \geq 3$.

Suppose $w_1, x, y \in N_B(v_2)$.

We have that w_1x or w_1y is colored blue; otherwise w_1, x, v_1, y, w_1 is a 4-cycle in G_R. Without loss of generality, assume that w_1x is colored blue.

Assume that yz is colored red. Then, w_1z and xz are colored blue; otherwise G_R contains the 4-cycles w_1, z, y, v_1, w_1 and x, z, y, v_1, x. If xy is colored blue, then G_B then contains the 5-cycle w_1, z, x, y, v_2, w_1, a contradiction. Thus, xy is colored red. To avoid the 4-cycle w_2, z, y, v_1, w_2 in G_R, w_2z is colored blue. To avoid the 4-cycle y, w_2, v_1, x, y in G_R, w_2y is colored blue. But then v_2, w_1, z, w_2, y, v_2 is a 5-cycle in G_B, which is a contradiction.

Thus yz is colored blue. Edge w_1z is colored red; otherwise G_B contains the 5-cycle w_1, z, y, v_2, x, w_1. Edge xz is colored red, since otherwise G_B contains the 5-cycle x, z, y, v_2, w_1, x. However, G_R then contains the 4-cycle x, z, w_1, v_1, x, a contradiction. Thus, $\{w_1, x, y\} \not\subseteq N_B(v_2)$.

Therefore, $w_1, w_2, x \in N_B(v_2)$.

We have that w_1x or w_2x is colored blue; otherwise w_1, x, w_2, v_1, w_1 is a 4-cycle in G_R. Without loss of generality, assume w_1x is colored blue.

Suppose yz is colored red. Then to avoid the 4-cycle z, y, v_1, p, z in G_R, where $p \in \{w_1, w_2, x\}$, edges w_1z, w_2z and xz are all colored blue. But then v_2, w_1, x, z, w_2, v_2 is a 5-cycle in G_B, which is a contradiction.

Thus, yz is colored blue.

We first show that w_1z is colored blue. Suppose, to the contrary, that w_1z is colored red. Then to avoid the 4-cycle z, w_1, v_1, p, z in G_R, where $p \in \{w_2, x\}$, edges w_2z and xz are both colored blue. But then v_2, w_1, x, z, w_2, v_2 is a 5-cycle in G_B, which is a contradiction. Thus, w_1z is colored blue.

Next, we show that w_1y is colored blue. Suppose, to the contrary, that w_1y is colored red. To avoid the 4-cycle y, w_1, v_1, x, y in G_R, we see that xy is colored blue. But then v_2, w_1, z, y, x, v_2 is a 5-cycle in G_B, which is a contradiction.

Thus, w_1y is colored blue. If zx and zw_2 are both colored red, then we have the 4-cycle z, w_2, v_1, x, z in G_R, which is a contradiction. Thus, zp is colored blue where $p \in \{w_2, x\}$. But then v_2, w_1, y, z, p, v_2 is a 5-cycle in G_B, which is a contradiction.

Case 3: $\deg_{\mathrm{G_R}}(v_1) = 3$.

Case 3.1: *Suppose* $x, y, z \in N_R(v_1)$.

Then $w_1, w_2 \in N_B(v_1)$. If at most one of the edges in $\{xy, yz, xz\}$ is colored blue, then G_R contains a 4-cycle. Without loss of generality, assume that xy and yz are colored blue. Let $A = \{x, y, z\}$. As before, for $i = 1,2$, we have$|N_B(w_i) \cap A| \geq 2$. Therefore, $|N_B(w_1) \cap N_B(w_2) \cap A| \geq 1$.

Suppose $y \in N_B(w_1) \cap N_B(w_2)$. Then w_1x is colored red; otherwise G_B contain the 5-cycle w_1, x, y, w_2, v_1, w_1. By symmetry, w_1z is also colored red. However, G_R then contains the 4-cycle w_1, z, v_1, x, w_1, a contradiction.

Therefore, $x \in N_B(w_1) \cap N_B(w_2)$. Edges w_1y and w_2y are colored red; otherwise G_B contains the 5-cycles w_1, y, x, w_2, v_1, w_1 and w_2, y, x, w_1, v_1, w_2. Edge w_1z is colored blue; otherwise G_R contains the 4-cycle w_1, z, v_1, y, w_1. To avoid the 4-cycle z, w_2, y, v_1, z, the edge w_2z must be colored blue. Note that either v_2w_1 or v_2w_2 is colored blue; otherwise v_2, w_1, y, w_2, v_2 is a 4-cycle in G_R. Without loss of generality, assume that v_2w_1 is colored blue.

Then, v_2x and v_2z are colored red; otherwise G_B contains the 5-cycles v_2, x, y, z, w_1, v_2 and v_2, z, y, x, w_1, v_2. However, G_R then contains the 4-cycle v_2, z, v_1, x, v_2, a contradiction. Hence, $\{x, y, z\} \not\subseteq N_R(v_1)$.

Case 3.2: *Suppose* $w_1, x, y \in N_R(v_1)$.

Then $w_2, z \in N_B(v_1)$. As before, at least two of the edges of $\{w_1x, xy, w_1y\}$ are colored blue. Thus, $x, y \in N_B(w_1)$ or $w_1, y \in N_B(x)$.

Suppose $x, y \in N_B(w_1)$. Then, xz or yz is colored blue; otherwise z, x, v_1, y, z is a 4-cycle in G_R. Without loss of generality, assume that xz is colored blue.

Assume v_2w_1 is colored red. Then v_2x and v_2y are colored blue; otherwise G_R contains the 4-cycles v_2, x, v_1, w_1, v_2 and v_2, y, v_1, w_1, v_2. Edges v_2w_2 and v_2z are colored red; otherwise G_B contains the 5-cycles v_2, w_2, v_1, z, x, v_2 and v_2, z, x, w_1, y, v_2. Also, w_1z is colored red; otherwise G_B contains the 5-cycle w_1, z, x, v_2, y, w_1. Edges w_2z and yz are colored blue; otherwise G_R contains the 4-cycles w_2, z, w_1, v_2, w_2 and y, z, w_1, v_1, y. Edge w_2y is colored red; otherwise G_B contains the 5-cycle w_2, y, w_1, x, z, w_2. Lastly, w_2x is colored blue; otherwise G_R contains the 4-cycle w_2, x, v_1, y, w_2. However, G_B then contains the 5-cycle w_2, x, v_2, y, z, w_2, a contradiction.

Hence, v_2w_1 is colored blue.

Suppose yz is colored red. Then w_1z is colored blue; otherwise G_R contains the 4-cycle w_1, z, y, v_1, w_1. Moreover, w_2x and w_2y are colored red; otherwise G_B contains the 5-cycles w_2, x, w_1, z, v_1, w_2 and w_2, y, w_1, z, v_1, w_2. However, G_R then contains the 4-cycle w_2, y, v_1, x, w_2, a contradiction.

Hence, yz is colored blue. Now, v_2x and v_2y are colored red; otherwise G_B contains the 5-cycles v_2, x, z, y, w_1, v_2 and v_2, y, z, x, w_1, v_2. However, G_R then contains the 4-cycle v_2, y, v_1, x, v_2, a contradiction. Therefore, $\{x, y\} \not\subseteq N_B(w_1)$.

Let $w_1, y \in N_B(x)$. We have that xz or yz is colored blue; otherwise x, z, y, v_1, x is a 4-cycle in G_R.

Suppose xz is colored blue. Edge w_2y is colored red; otherwise G_B contains the 5-cycle w_2, y, x, z, v_1, w_2. Edge w_2x is colored blue; otherwise G_R contains the 4-cycle w_2, x, v_1, y, w_2. Edges w_1z and yz are colored red; otherwise G_B contains the 5-cycles w_1, z, v_1, w_2, x, w_1 and y, z, v_1, w_2, x, y. However, G_R then contains the 4-cycle y, z, w_1, v_1, y, a contradiction.

Hence, xz is colored red and yz is colored blue.

Edge $w_1 z$ is colored blue; otherwise G_R contains the 4-cycle w_1, z, x, v_1, w_1. Edge $w_2 x$ is colored red; otherwise G_B contains the 5-cycle w_2, x, y, z, v_1, w_2. Edge $w_2 y$ is colored blue; otherwise G_R contains the 4-cycle w_2, y, v_1, x, w_2. Edges $w_1 y$ and $w_2 z$ are colored red; otherwise G_B contains the 5-cycles w_1, y, w_2, v_1, z, w_1 and w_2, z, w_1, x, y, w_2.

Suppose $v_2 z$ is colored red. Then $v_2 w_2$ and $v_2 x$ are colored blue; otherwise G_R contains the 4-cycles v_2, w_2, x, z, v_2 and v_2, x, w_2, z, v_2. Moreover, $v_2 w_1$ and $v_2 y$ are colored red; otherwise G_B contains the 5-cycles v_2, w_1, z, y, x, v_2 and v_2, y, z, v_1, w_2, v_2. However, G_R then contains the 4-cycle v_2, w_1, v_1, y, v_2, a contradiction.

Hence, $v_2 z$ is colored blue.

Now, $v_2 w_1$ and $v_2 y$ are colored red; otherwise G_B contains the 5-cycles v_2, w_1, x, y, z, v_2 and v_2, y, x, w_1, z, v_2. However, G_R then contains the 4-cycle v_2, y, v_1, w_1, v_2, a contradiction.

Case 3.3: *Suppose* $w_1, w_2, x \in N_R(v_1)$.

Then $y, z \in N_B(v_1)$. It follows that $w_1 y$ or $w_2 y$ is colored blue; otherwise w_1, y, w_2, v_1, w_1 is a 4-cycle in G_R. Without loss of generality, assume that $w_1 y$ is colored blue. Let $A = \{w_1, w_2, x\}$. As before, $|N_B(v_2) \cap A| \geq 2$.

Suppose $w_1 v_2$ is colored blue. First consider the case when $w_1 z$ is colored red. Edges $w_2 z$ and xz are colored blue; otherwise G_R contains the 4-cycles w_2, z, w_1, v_1, w_2 and x, z, w_1, v_1, x. Edge $w_1 x$ is colored red; otherwise G_B contains the 5-cycle w_1, x, z, v_1, y, w_1. Edge $w_2 x$ is colored blue; otherwise G_R has the 4-cycle w_2, x, w_1, v_1, w_2. Edges $w_2 y$ and xy are colored red; otherwise G_B

contains the 5-cycles w_2, y, v_1, z, x, w_2 and x, y, v_1, z, w_2, x. However, G_R then contains the 4-cycle x, y, w_2, v_1, x, a contradiction.

Thus, we may assume that $w_1 z$ is colored blue. Edge $v_2 y$ is colored red; otherwise G_B contains the 5-cycle v_2, y, v_1, z, w_1, v_2. By symmetry, $v_2 z$ is also colored red.

Suppose xy is colored red. Then $w_2 y$ and xz are colored blue; otherwise G_R contains the 4-cycles w_2, y, x, v_1, w_2 and x, z, v_2, y, x. Also, $w_1 x$ and $w_2 x$ are colored red; otherwise G_B contains the 5-cycles w_1, x, z, v_1, y, w_1 and w_2, x, z, v_1, y, w_2. However, G_R then contains the 4-cycle w_2, x, w_1, v_1, w_2, a contradiction.

Thus, xy is colored blue. Edge $w_1 x$ is colored red; otherwise G_B contains the 5-cycle w_1, x, y, v_1, z, w_1. Edge $w_2 x$ is colored blue; otherwise G_R has the 4-cycle w_2, x, w_1, v_1, w_2. Edge $v_2 w_2$ is colored red; otherwise G_B contains the 5-cycle v_2, w_2, x, y, w_1, v_2.

Recall that $\deg_{G_R}(v_2) \leq \deg_{G_R}(v_1)$. As $|N_R(v_2)| = 3$, this implies that $v_2 x$ is colored blue. Edges $w_2 y$ and $w_2 z$ are colored red; otherwise G_B contains the 5-cycles w_2, y, w_1, v_2, x, w_2 and w_2, z, w_1, y, x, w_2. However, G_R then contains the 4-cycle w_2, y, v_2, z, w_2, a contradiction.

Hence, $w_1 v_2$ is colored red, and so $w_2, x \in N_B(v_2)$ as $|N_B(v_2) \cap A| \geq 2$.

Suppose xz is colored blue. Edges $v_2 y$ and $w_1 x$ are colored red; otherwise G_B contains the 5-cycles v_2, y, v_1, z, x, v_2 and w_1, x, z, v_1, y, w_1. Edges $w_2 x$ and xy are colored blue; otherwise G_R contains the 4-cycles w_2, x, w_1, v_1, w_2 and x, y, v_2, w_1, x. Edges $v_2 z$ and $w_2 z$ are colored red; otherwise G_B contains the 5-cycles v_2, z, v_1, y, x, v_2 and w_2, z, v_1, y, x, w_2. Also, $w_2 y$ is colored red; otherwise G_B contains the 5-cycle w_2, y, v_1, z, x, w_2. However, G_R then contains the 4-cycle w_2, y, v_2, z, w_2, a contradiction.

Thus, xz is colored red.

First, w_1z and w_2z are colored blue; otherwise G_R contains the 4-cycles w_1, z, x, v_1, w_1 and w_2, z, x, v_1, w_2. Edges v_2y and w_1x are colored red; otherwise G_B contains the 5-cycles v_2, y, w_1, z, w_2, v_2 and w_1, x, v_2, w_2, z, w_1. Edges v_2z and xy are colored blue; otherwise G_R contains the 4-cycles v_2, z, x, w_1, v_2 and x, y, v_2, w_1, x. However, G_B then contains the 5-cycle x, y, w_1, z, v_2, x, a contradiction.

Case 4: $\deg_{G_R}(v_1) = 2$.

Case 4.1: *Suppose* $w_1, w_2 \in N_R(v_1)$.

Then $N_B(v_1) = \{x, y, z\}$. Let $A = \{x, y, z\}$. As before, for $u \in A$, we have $|N_R(u) \cap W| \leq 1$; whence $|N_B(u) \cap W| \geq 1$. As $|A| = 3$, there are at least two vertices $u_1, u_2 \in A$ such that $|N_B(u_1) \cap N_B(u_2) \cap W| \geq 1$. Without loss of generality, say $w_1 \in N_B(x) \cap N_B(y)$. Then, xz and yz are colored red; otherwise G_B contains the 5-cycles x, z, v_1, y, w_1, x and y, z, v_1, x, w_1, y, a contradiction.

We have that v_2x or v_2y is colored blue; otherwise v_2, x, z, y, v_2 is a 4-cycle in G_R. Suppose, without loss of generality, that v_2x is colored blue. Then v_2w_1 is colored red; otherwise G_B contains the 5-cycle v_2, w_1, y, v_1, x, v_2. Edge v_2w_2 is colored blue; otherwise G_R has the 4-cycle v_2, w_2, v_1, w_1, v_2. Edges w_2y and w_2z are colored red; otherwise G_B contains the 5-cycles w_2, y, w_1, x, v_2, w_2 and w_2, z, v_1, x, v_2, w_2. Edges w_1z and xy are colored blue; otherwise G_R contains the 4-cycles w_1, z, w_2, v_1, w_1 and x, y, w_2, z, x. However, G_B then contains the 5-cycle x, y, w_1, z, v_1, x, a contradiction.

Case 4.2: *Suppose* $x, y \in N_R(v_1)$.

Then $N_B(v_1) = \{w_1, w_2, z\}$. To avoid the 4-cycle w_1, x, v_1, y, w_1 in G_R, we may assume, without loss of generality, that w_1x is colored blue.

Assume v_2x is colored blue. To avoid the 5-cycles $v_1, w_1, x, v_2, w_2, v_1$ and v_1, w_1, x, v_2, z, v_1 in G_B, both of the v_2w_2 and v_2z are colored red. As $\deg_{G_R}(v_2) = \deg_{G_R}(v_1)$, we see that v_2w_1 and v_2y are both colored blue. To avoid the 5-cycle $v_2, w_1, v_1, w_2, x, v_2$ and $v_2, w_1, v_1, w_2, y, v_2$, both w_2x and w_2y

are colored red. But then w_2, x, v_1, y, w_2 is a 4-cycle in G_R, which is a contradiction.

Thus, v_2x is colored red. To avoid to 4-cycle v_2, x, v_1, y, v_2, edge v_2y is colored blue. If w_1y is colored blue, then, following the reasoning in the previous paragraph, we see that v_2w_2 and v_2z are both colored red; whence $\deg_{G_R}(v_2) = 3$, which is a contradiction. Thus, w_1y is colored red. Similarly, w_2y is colored red. To avoid the 4-cycle w_2, x, v_1, y, w_2 in G_R, we see that w_2x is colored blue. To avoid the 5-cycles z, v_1, w_2, x, w_1, z and z, v_1, w_1, x, w_2, z, edges zw_1 and zw_2 are both colored red. But then z, w_1, y, w_2, z is a 4-cycle in G_R, which is a contradiction.

Case 4.3: *Suppose* $w_1, x \in N_R(v_1)$.

Then $N_B(v_1) = \{w_2, y, z\}$. As before $|N_B(u) \cap \{w_1, x\}| \geq 1$ for $u \in \{y, z\}$.

Suppose $x \in N_B(y) \cap N_B(z)$. Edges w_2y and w_2z are colored red; otherwise G_B contains the 5-cycles w_2, y, x, z, v_1, w_2 and w_2, z, x, y, v_1, w_2. We have that v_2y or v_2z is colored blue; otherwise v_2, y, w_2, z, v_2 is a 4-cycle in G_R. Without loss of generality, assume that v_2y is colored blue. Edge v_2x is colored red; otherwise G_B contains the 5-cycle v_2, x, z, v_1, y, v_2. Edge v_2w_1 is colored blue; otherwise G_R contains the 4-cycle v_2, w_1, v_1, x, v_2. Edge w_1z is colored red; otherwise G_B contains the 5-cycle w_1, z, x, y, v_2, w_1. Edge w_1y is colored blue; otherwise G_R contains the 4-cycle w_1, y, w_2, z, w_1. Edge w_1x is colored red; otherwise G_B contains the 5-cycle w_1, x, z, v_1, y, w_1. Lastly, v_2z is colored blue; otherwise G_R contains the 4-cycle v_2, z, w_1, x, v_2. However, G_B then contains the 5-cycle v_2, z, x, y, w_1, v_2, a contradiction. Hence, $x \notin N_B(y) \cap N_B(z)$.

Suppose $w_1 \in N_B(y) \cap N_B(z)$. Edges w_2y and w_2z are colored red; otherwise G_B contains the 5-cycles w_2, y, w_1, z, v_1, w_2 and w_2, z, w_1, y, v_1, w_2. We have that v_2y or v_2z is colored blue; otherwise v_2, y, w_2, z, v_2 is a 4-cycle in G_R. Without loss of generality, assume that v_2y is colored blue. Edge v_2w_1 is colored red; otherwise G_B contains the 5-cycle v_2, w_1, z, v_1, y, v_2. Edge v_2x is colored blue; otherwise G_R contains the 4-cycle v_2, x, v_1, w_1, v_2. Edges xw_2 and xz are colored red; otherwise

G_B contains the 5-cycles x, w_2, v_1, y, v_2, x and x, z, w_1, y, v_2, x. Edge xy is colored blue; otherwise G_R contains the 4-cycle x, y, w_2, z, x. Edge w_1x is colored red; otherwise G_B contains the 5-cycle w_1, x, y, v_1, z, w_1. Lastly, v_2w_2 is colored blue; otherwise G_R contains the 4-cycle v_2, w_2, x, w_1, v_2. However, G_B then contains the 5-cycle v_2, w_2, v_1, y, x, v_2, a contradiction. Hence, $w_1 \notin N_B(y) \cap N_B(z)$.

Therefore, $w_1 \in N_B(y)$ and $x \in N_B(z)$. Since $w_1 \notin N_B(y) \cap N_B(z)$, we have that $w_1 \in N_R(z)$. Also, $x \in N_R(y)$ as $x \notin N_B(y) \cap N_B(z)$. Edge w_1x is colored red; otherwise G_B contains the 5-cycle w_1, x, z, v_1, y, w_1. Edge yz is colored blue; otherwise G_R contains the 4-cycle y, z, w_1, x, y. Edge xw_2 is colored red; otherwise G_B contains the 5-cycle x, w_2, v_1, y, z, x. Edge w_2z is colored blue; otherwise G_R contains the 4-cycle w_2, z, w_1, x, w_2.

If $w_1 \in N_R(v_2)$, then G_B contains a 5-cycle. To motivate this, let $w_1 \in N_R(v_2)$. Then, v_2x and v_2y are colored blue; otherwise G_R contains the 4-cycles v_2, x, v_1, w_1, v_2 and v_2, y, x, w_1, v_2. However, G_B then contains the 5-cycle v_2, y, v_1, z, x, v_2, a contradiction.

Hence, $w_1 \in N_B(v_2)$. Now, v_2x and v_2z are colored red; otherwise G_B contains the 5-cycles v_2, x, z, y, w_1, v_2 and v_2, z, v_1, y, w_1, v_2. However, G_R then contains the 4-cycle v_2, z, w_1, x, v_2, a contradiction.

Case 5: $\deg_{G_R}(v_1) = 1$.

Recall that $\deg_{G_R}(v_2) \leq \deg_{G_R}(v_1)$. As $\deg_{G_B}(v_1) = 4$, $\deg_{G_B}(v_2) \geq 4$ and $|N_B(v_1) \cap N_B(v_2)| \geq 3$. Suppose v_1w_1 is colored red. Then $N_B(v_1) = \{w_2, x, y, z\}$. As $|N_B(v_1) \cap N_B(v_2)| \geq 3$, assume without loss of generality that $x, y \in N_B(v_2)$. Edges w_2x and w_2y are colored red; otherwise G_B contains the 5-cycles w_2, x, v_2, y, v_1, w_2 and w_2, y, v_2, x, v_1, w_2. Also, xz and yz are colored red; otherwise G_B contains the 5-cycles x, z, v_1, y, v_2, x and y, z, v_1, x, v_2, y. However, G_R then contains the 4-cycle y, z, x, w_2, y, a contradiction. Hence, we may assume that v_1w_1 is colored blue. Thus, the subgraph of G_B induced by $V \cup W$ is isomorphic to a 4-cycle.

Without loss of generality, assume that v_1x is colored red. Then, $N_B(v_1) = \{w_1, w_2, y, z\}$. As $|N_B(v_1) \cap N_B(v_2)| \geq 3$, assume without loss of generality that v_2y is colored blue. Then w_1z and yz are colored red; otherwise G_B contains the 5-cycles w_1, z, v_1, y, v_2, w_1 and y, z, v_1, w_1, v_2, y.

Suppose xv_2 is colored blue. Then w_1x and xy are colored red; otherwise G_B contains the 5-cycles w_1, x, v_2, y, v_1, w_1 and x, y, v_1, w_1, v_2, x. However, G_R then contains the 4-cycle x, y, z, w_1, x, a contradiction.

Thus, xv_2 is colored red. Now v_2z is colored blue as $\deg_{\mathrm{G_R}}(v_2) = 1$. Edges w_2y and w_2z are colored red; otherwise G_B contains the 5-cycles $w_2, y, v_1, w_1, v_2, w_2$ and $w_2, z, v_1, w_1, v_2, w_2$. Also, w_1y is colored red; otherwise G_B contains the 5-cycle w_1, y, v_1, z, v_2, w_1. However, G_R then contains the 4-cycle w_1, y, w_2, z, w_1, a contradiction.

Case 6: $\deg_{\mathrm{G_R}}(v_1) = 0$.

Therefore $\deg_{G_R}(v_1) = \deg_{G_R}(v_2) = 0$. Interchanging the roles of the sets V and W, we see we may also assume that $\deg_{G_R}(w_1) = \deg_{G_R}(w_2) = 0$. We now see that the subgraph of G_B induced by the set $V \cup W \cup \{x\}$ is isomorphic to $C_4 + K_1$ which has 5-cycles. An example of one such 5-cycle in G_B is x, v_1, w_2, v_2, w_1, x.■

3.6 $\boldsymbol{R_6(C_4, C_5)}$

Finally, we consider the case for $k = 6$.

Theorem 3.8

$R_6(C_4, C_5) = 7$.

Proof

For $R_6(C_4, C_5) > 6$ we need to consider the graph $H = K_6$. As $R(C_4, C_5) = 7$, there exists a (R, B)-coloring of the edges of K_6 such that K_6 has neither a red C_4 nor a blue C_5.

For $R_6(C_4, C_5) \leq 7$ we need to consider the graph $G = K_{1(2),5(1)}$. Consider any (R, B)-coloring of G. As $K_{2(2),3(1)} \subset G$, we have an induced (R, B)-coloring of the edges of $K_{2(2),3(1)}$, and, by Theorem 3.7, this two coloring now contains a red C_4 or a blue C_5. ■

As $R_6(C_4, C_5) = 7$ by Theorem 3.8, and $K_{2(2),3(1)} \subset K_7$, note that an argument like the one used in the proof of Theorem 3.8 gives us the following result.

Corollary 3.9

$R_7(C_4, C_5) = R(C_4, C_5) = 7$.

3.7 Conclusion

In this chapter we proved that $R_3(C_4, C_5) = 10$, $R_4(C_4, C_5) = 8$, and $R_k(C_4, C_5) = 7$ for $k = 5$, 6 and 7. Note that $R_2(C_4, C_5)$ does not exist by Proposition 3.1 whereby we proved that $R_2(F, H)$ does not exist for a nonbipartite graph F and a nonempty graph H. Lastly, we proved in Proposition 3.2 that $R_k(K_n, H)$ does not exist if $2 \leq k < n$ and H is a nonempty graph. From this result, Corollary 3.3 was deduced which states that $R_k(K_n, K_m)$ does not exist if $n \geq m \geq 3$ and $k < n$.

* * * * * * *

CHAPTER 4

MORE $\boldsymbol{k}$-RAMSEY NUMBERS:

$\boldsymbol{R_k(C_3, C_4)}$ AND $\boldsymbol{R_k(C_3, C_5)}$.

Our objective in this chapter is to investigate $R_k(C_3, C_4)$ for $k = 2, \dots, 7$ where $R(C_3, C_4) = 7$, and $R_k(C_3, C_5)$ for $k = 5, \dots, 9$ where $R(C_3, C_5) = 9$.

4.1 Introduction

The k-Ramsey number $R_k(F, H)$ is defined in [2] for two bipartite graphs F and H. As in Section 2.2, it is not necessarily the case that $R_k(F, H)$ exists when F and H are nonbipartite. In Chapter 2 we investigated the case where $F = H = C_5$, which is nonbipartite. In Chapter 3 we investigated the case where $F = C_4$ is bipartite and $H = C_5$ is nonbipartite. In this chapter we continue our investigation by considering the pair F, H for pairs C_3, C_4 and C_3, C_5.

4.2 Preliminary Remarks

In this section we state results that are used in the remainder of the chapter.

Lemma 4.1

Let $G \cong K_5 - \{e, f\}$ where $e, f \in E(K_5)$. Then G has a 5-cycle.

Proof

Let $V(G) = \{v_i : i = 1, \dots, 5\}$. If two adjacent edges are omitted from K_5 to form G, say v_1v_2 and v_2v_3, then G has the 5-cycle $v_1, v_3, v_5, v_2, v_4, v_1$. Alternatively, if two nonadjacent edges are omitted from K_5 to form G, say v_1v_5 and v_2v_4, then G has the 5-cycle $v_1, v_2, v_5, v_3, v_4, v_1$. ∎

Proposition 4.2

Let k, n and m be integers with $n \geq m \geq 1$ and $(n, m) \neq (1,1)$. For $5 \leq k \leq 4n$, $R_k(C_{2n+1}, C_{2m+1}) > 4n$.

Proof

Let G_k be the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order $4n = \sum_{i=1}^{k} n_i$ with $n_i \in \{\lceil 4n/k \rceil, \lfloor 4n/k \rfloor\}$ for $i = 1, \ldots, k$. Note that $G_k \subseteq K_{4n}$ for all $k = 5, \ldots, 4n$. From Theorem 1.2 we have $R(C_{2n+1}, C_{2m+1}) = 4n + 1$, and so, there exists a (R, B)-coloring $\mathcal{C}$ of the edges of K_{4n} such that K_{4n} neither has a red C_{2n+1} nor a blue C_{2m+1}. Thus, coloring $\mathcal{C}$ induces a coloring on the edges of G_k so that G_k has neither a red C_{2n+1} nor a blue C_{2m+1}. The result then follows. ■

4.3 $R_k(C_3, C_4)$

Note that $R(C_3, C_4) = 7$ by Theorem 1.2 and recall that $R_2(C_3, C_4)$ does not exist by Proposition 3.1. We proceed by considering each value of $k \in \{3,4,5,6\}$ for $R_k(C_3, C_4)$ separately.

Theorem 4.3

$R_3(C_3, C_4) = 9$.

Proof

To show that $R_3(C_3, C_4) > 8$, consider the graph $H = K_{3,3,2}$ with partite sets V_1, V_2 and $V_3 = \{v_1, v_2\}$. Color each edge of $[V_1 \cup \{v_1\}, V_2 \cup \{v_2\}]$ red and assign blue to all other edges of G. Then H_R is bipartite and $H_B = 2K_{1,3}$.

To show that $R_3(C_3, C_4) \leq 9$, consider the graph $G = K_{3,3,3}$ and assume to the contrary that there exists a (R, B)-coloring of the edges of G that contains neither a red C_3 nor a blue C_4. Further, let the partite sets of G be $V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1, v_2, v_3\}$ and $V_3 = \{w_1, w_2, w_3\}$. Assume without loss of generality that v_1 has the highest degree in G_R. We now consider each possibility for $\deg_{G_R}(v_1)$.

Case 1: $\deg_{G_R}(v_1) \in \{5,6\}$.

Without loss of generality, assume that the edges v_1u_i for $i = 1,2,3$ (v_1w_i for $i = 1,2$, respectively) are colored red. Then, as we have no red 3-cycles, u_1, w_1, u_2, w_2, u_1 forms a 4-cycle in G_B, a contradiction.

Case 2: $\deg_{G_R}(v_1) = 4$.

Consider first the case when edges v_1u_i and v_1w_i are colored red for $i = 1,2$. Then, as G_R contains no 3-cycles, u_iw_j is colored blue for $i,j = 1,2$. But then u_1, w_1, u_2, w_2, u_1 forms a 4-cycle in G_B, a contradiction.

Next, consider the case when the edges in $\{v_1w_1\} \cup \{v_1u_i : i = 1,2,3\}$ are colored red. Then the edges in $\{w_1u_i : i = 1,2,3\}$ are colored blue as G_R does not have any 3-cycles. Necessarily then, each of the vertices in $\{v_2, v_3, w_2, w_3\}$ has at most one neighbor in the set $\{u_1, u_2, u_3\}$ in G_B; otherwise, a 4-cycle is formed. Thus, each of the vertices in $\{v_2, v_3, w_2, w_3\}$ has at least two neighbors in the set $\{u_1, u_2, u_3\}$ in G_R. Each pair (v_i, w_j) then has $|N_R(v_i) \cap N_R(w_j)| \geq 1$ for $i,j = 2,3$. This implies that the edges v_iw_j for $i,j = 2,3$ are colored blue as G_R does not have a 3-cycle. However, then, a 4-cycle forms in G_B, a contradiction.

Case 3: $\deg_{G_R}(v_1) = 3$.

First consider the case when the edges in $\{v_1u_i : i = 1,2,3\}$ are colored red and the edges in $\{v_1w_i : i = 1,2,3\}$ blue. Then, each of the vertices in $\{u_1, u_2, u_3, v_2, v_3\}$ has at most one neighbor in the set $\{w_1, w_2, w_3\}$ in G_B; otherwise, a 4-cycle is formed. Thus, each of the vertices in $\{u_1, u_2, u_3, v_2, v_3\}$ has at least two neighbors in the set $\{w_1, w_2, w_3\}$ in G_R. Each pair (u_i, v_j) has $|N_R(u_i) \cap N_R(v_j)| \geq 1$ for $i = 1,2,3$ and $j = 2,3$. This implies that the edges u_iv_j for $i = 1,2,3$ and $j = 2,3$ are colored blue as G_R does not have a 3-cycle. But then a 4-cycle forms in G_B, a contradiction.

Next, assume that v_1u_1, v_1u_2 and v_1w_1 are colored red, and v_1u_3, v_1w_2 and v_1w_3 are colored blue. Then, w_1u_1 and w_1u_2 are blue as G_R has no 3-cycles.

Since $\deg_{G_R}(v_2) \leq \deg_{G_R}(v_1)$, it follows that $\deg_{G_R}(v_2) \leq 3$. To avoid the 4-cycle v_2, u_2, w_1, u_1, v_2, one of the edges u_1v_2 or u_2v_2 is colored red. Without loss of generality, assume u_1v_2 is colored red.

To avoid the blue 4-cycle v_1, w_2, v_2, w_3, v_1, one of the edges v_2w_2 or v_2w_3 is red. Again, without loss of generality, assume v_2w_2 is red. To avoid the red 3-cycle v_2, w_2, u_1, v_2, the edge u_1w_2 is blue. To avoid the blue 4-cycle w_2, u_2, w_1, u_1, w_2, the edge u_2w_2 is colored red. To avoid the red 3-cycle v_2, w_2, u_2, v_2, the edge u_2v_2 is colored blue.

If v_2w_1 is red, then, as $\deg_{G_R}(v_2) \leq 3$, the edges v_2u_3 and v_2w_3 must be colored blue, and we obtain the blue 4-cycle v_2, u_3, v_1, w_3, v_2, which is a contradiction. Thus, v_2w_1 is blue.

To avoid the blue 4-cycle w_3, u_1, w_2, v_1, w_3, we see that u_1w_3 is colored red. To avoid the red 3-cycle w_3, u_1, v_2, the edge v_2w_3 is colored blue. To avoid the blue 4-cycle u_3, v_1, w_3, v_2, u_3, the edge u_3v_2 is colored red. To avoid the red 3-cycle u_3, v_2, w_2, u_3, the edge u_3w_2 is colored blue. Note that every two distinct vertices of V_3 are the endvertices of a blue P_3.

As $\deg_{G_R}(v_3) \leq 3$, we must have that $\deg_{G_B}(v_3) \geq 3$. At most one of the edges v_3w_i for $i = 1,2,3$ is colored blue, since otherwise a blue 4-cycle is formed. We conclude that at least two of the edges v_3u_i for $i = 1,2,3$ are colored blue. As u_1 and u_2 (u_1 and u_3, respectively) are endvertices of a blue P_3, we see that v_3u_2 and v_3u_3 are colored blue, while v_3u_1 is colored red. Thus, exactly one of the edges v_3w_i where $i \in \{1,2,3\}$ is colored blue. Note that u_2 and w_3 are the endvertices of a blue P_3, while u_3 and w_2 (u_3 and w_3, respectively) are also endvertices of a blue P_3. We now have a blue 4-cycle, which is a contradiction.

Case 4: $\deg_{G_R}(v_1) \in \{0,1,2\}$.

Note that $4 \leq \deg_{G_B}(v_1) \leq 6$. Since $\deg_{G_R}(v_2) \leq \deg_{G_R}(v_1)$, it follows that $4 \leq \deg_{G_B}(v_2) \leq 6$ also. If $|N_B(v_1) \cap N_B(v_2)| \leq 1$, then G has order at least ten, which is a contradiction. Thus, $|N_B(v_1) \cap N_B(v_2)| \geq 2$, and so we have a blue 4-cycle, which is a contradiction. ■

Theorem 4.4

$R_4(C_3, C_4) = 8$.

Proof

To show that $R_4(C_3, C_4) > 7$, consider graph $H = K_{2,2,2,1}$ with $V(H) = \{v_i : 1 \leq i \leq 7\}$ and $E(H) = E(K_7) - \{v_1v_4, v_2v_5, v_3v_6\}$. Color the edges of the 7-cycle $v_1, v_2, \dots, v_7, v_1$ red and the edges of the 7-cycle $v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_1$ blue. Further, color v_2v_6 and v_3v_7 red, and the remaining edges v_1v_5 and v_4v_7 blue. Then H has neither a red C_3 nor a blue C_4.

To show that $R_4(C_3, C_4) \leq 8$, let graph $G = K_{2,2,2,2}$ and assume to the contrary that there exists a (R, B)-coloring of the edges of G that contains neither a red C_3 nor a blue C_4. Further, let the partite sets of G be $V_1 = \{v_1, v_2\}$, $V_2 = \{w_1, w_2\}$, $V_3 = \{x_1, x_2\}$ and $V_4 = \{y_1, y_2\}$. Assume without loss of generality that v_1 has the highest degree in G_R. We now consider each possibility for $\deg_{G_R}(v_1)$.

Case 1: $\deg_{G_R}(v_1) \in \{4,5,6\}$.

As G_R does not contain any 3-cycles, it follows that every vertex in $N_R(v_1) \cap V_i$ is joined with every vertex in $N_R(v_1) \cap V_j$ with a blue edge for distinct $i, j \in \{2,3,4\}$.

Suppose $N_R(v_1) \cap V_i \neq \emptyset$ for $i = 2,3,4$. As $\deg_{G_R}(v_1) \geq 4$, assume, without loss of generality, that $V_3 \subseteq N_R(v_1)$ and that $\{w_1, y_1\} \subseteq N_R(v_1)$. But then w_1, x_1, y_1, x_2, w_1 is a 4-cycle of G_B, which is a contradiction.

Next, suppose $N_R(v_1) \cap V_i = \emptyset$ for some $i = 2,3,4$. Without loss of generality, assume $N_R(v_1) \cap V_2 = \emptyset$, and so $N_R(v_1) = V_3 \cup V_4$. But then x_1, y_1, x_2, y_2, x_1 is a 4-cycle of G_B, which is a contradiction.

Case 2: $\deg_{G_R}(v_1) = 3$.

First, suppose v_1w_1, v_1x_1 and v_1y_1 are colored red, and that the remaining edges incident to v_1 are colored blue. As $G_R[N_R(v_1)]$ contains no 3-cycles, we see that $C \coloneqq w_1, x_1, y_1, w_1$ forms a 3-cycle in G_B. If v_2 is adjacent to two vertices of C in G_B, then we have a 4-cycle in G_B, which is a contradiction. Thus, $|N_R(v_2) \cap V(C)| \geq 2$. Without loss of generality, assume $\{w_1, y_1\} \subseteq N_R(v_2)$. If v_2y_2 is colored red, then, as $\deg_{G_R}(v_1) = 3$, all remaining edges incident with v_1 must be colored blue. But then v_1, w_2, v_2, x_2, v_1 forms a 4-cycle in G_B, which is a contradiction. Thus, v_2y_2 is colored blue, and, similarly, we see that w_2v_2 is colored blue. It now follows that v_2, w_2, v_1, y_2, v_2 is a 4-cycle of G_B, which is a contradiction.

Next, suppose v_1w_1, v_1w_2 and v_1x_1 are colored red, while all other edges incident with v_1 are colored blue. Then w_1, x_1, w_2 is a path in G_B, as G_R does not contain any 3-cycles. If $\{w_1, w_2\} \subseteq N_B(y_i)$ for $i = 1{,}2$, then y_i, w_1, x_1, w_2, y_i is a 4-cycle of G_B, which is a contradiction. Thus, either w_1y_i or w_2y_i is colored red for $i = 1{,}2$. Similarly, if x_2 is adjacent to both the vertices w_1 and w_2 in G_B, then x_2, w_1, x_1, w_2, x_2 is a 4-cycle in G_B, which is a contradiction. Thus, either w_1x_2 or w_2x_2 is colored red. Without loss of generality, assume that w_1x_2 is red. If w_1 is adjacent to either y_1 or y_2 in G_R, then, as $\Delta(G_R) = 3$, we have that w_1 is adjacent to exactly one of the vertices y_i and we have the case considered in the previous paragraph. Thus, w_1y_1 and w_1y_2 are colored blue. But then w_1, y_1, v_1, y_2, w_1 is a 4-cycle of G_B, which is a contradiction.

Case 3: $\deg_{G_R}(v_1) \in \{0,1,2\}$.

Note that $4 \leq \deg_{G_B}(v_1) \leq 6$. Since $\deg_{G_R}(v_2) \leq \deg_{G_R}(v_1)$, it follows that $4 \leq \deg_{G_B}(v_2) \leq 6$ also. If $|N_B(v_1) \cap N_B(v_2)| \leq 1$, then G has order at least nine, which is a contradiction. Thus, $|N_B(v_1) \cap N_B(v_2)| \geq 2$, and so we have a blue 4-cycle, which is a contradiction. ■

Theorem 4.5

$R_5(C_3, C_4) = 8$.

Proof

For $R_5(C_3, C_4) > 7$, consider graph $H = K_{2,2,1,1,1}$ with $V(H) = \{v_i : 1 \leq i \leq 7\}$ and $E(H) = E(K_7) - \{v_2v_5, v_3v_6\}$. Color the edges of the 7-cycle $v_1, v_2, \ldots, v_7, v_1$ red and the edges of the 7-cycle $v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_1$ blue. Further, color v_1v_4, v_2v_6 and v_3v_7 red, and the remaining edges v_1v_5 and v_4v_7 blue. Then H has neither a red C_3 nor a blue C_4.

For $R_5(C_3, C_4) \leq 8$ we need to consider the graph $G = K_{2,2,2,1,1}$. Consider any (R, B)-coloring of G. As $K_{2,2,2,2} \subset K_{2,2,2,1,1}$, we have an induced (R, B)-coloring of the edges of $K_{2,2,2,2}$, and, by Theorem 4.4, this two coloring now either contains a red C_3 or blue C_4. ■

Theorem 4.6

$R_6(C_3, C_4) = 7$.

Proof

For $R_6(C_3, C_4) > 6$, we need to consider the graph $H = K_6$. As $R(C_3, C_4) = 7$, there exists a (R, B)-coloring of the edges of K_6 such that H_R does not have a 3-cycle and H_B does not have a 4-cycle.

For $R_6(C_3, C_4) \leq 7$, let graph $G = K_{2,5(1)}$ and assume to the contrary that there exists a (R, B)-coloring of the edges of G that contains neither a red C_3 nor a blue C_4. Further, let the partite sets of G be $V_i = \{v_i\}$ for $i = 1, \ldots, 5$ and $V_6 = \{v_6, v_7\}$. Assume without loss of generality that $\deg_{G_R}(v_6) \geq \deg_{G_R}(v_7)$. We now consider each possibility for $\deg_{G_R}(v_6)$.

Case 1: $\deg_{G_R}(v_6) \in \{4,5\}$.

Since G_R contains no 3-cycles, the set $N_R(v_6)$ induces a clique of order at least four in G_B, which contains a 4-cycle, and we obtain a contradiction.

Case 2: $\deg_{G_R}(v_6) = 3$.

Let $N_R(v_6) = \{v_1, v_2, v_3\}$ and $N_B(v_6) = \{v_4, v_5\}$. Then v_1, v_2, v_3, v_1 is a 3-cycle in G_B, as G_R contains no 3-cycles. If vertex v_7 is adjacent in G_B to two vertices in $\{v_1, v_2, v_3\}$, then we have a 4-cycle in G_B, which is a contradiction. Thus, without loss of generality, we may assume that both v_1 and v_3 are adjacent to v_7 in G_R.

First consider the case when $v_2 v_7$ is colored red. Then $\deg_{G_R}(v_7) = 3$, and so, $v_4 v_7$ and $v_5 v_7$ are both colored blue. But then v_6, v_4, v_7, v_5, v_6 is a 4-cycle in G_B, which is a contradiction.

Now consider the case when $v_2 v_7$ is colored blue. If v_4 is adjacent to both v_1 and v_3 in G_B, then we obtain the 4-cycle v_4, v_1, v_2, v_3, v_4 in G_B, which is a contradiction. Thus, without loss of generality, assume that $v_3 v_4$ is colored red. But then $v_4 v_7$ is colored blue, since otherwise we obtain the 3-cycle v_3, v_4, v_7, v_3 in G_R, which is a contradiction. To avoid the 4-cycle v_4, v_6, v_5, v_7, v_4 in G_B, $v_5 v_7$ is colored red. To avoid the 3-cycles v_1, v_5, v_7, v_1 and v_3, v_5, v_7, v_3, both $v_1 v_5$ and $v_3 v_5$ are colored blue. But then we obtain the 4-cycle v_1, v_2, v_3, v_5, v_1 in G_B, which is a contradiction.

Case 3: $\deg_{G_R}(v_6) \in \{0,1,2\}$.

Note that $3 \leq \deg_{G_B}(v_6) \leq 5$. Since $\deg_{G_R}(v_7) \leq \deg_{G_R}(v_6)$, it follows that $3 \leq \deg_{G_B}(v_2) \leq 5$ also. If $N_B(v_6) \cap N_B(v_7) = \emptyset$, then G has order at least eight, which is a contradiction. If $|N_B(v_6) \cap N_B(v_7)| \geq 2$, then we have a 4-cycle in G_B, which is a contradiction. So, assume $|N_B(v_6) \cap N_B(v_7)| = 1$. Without loss of generality, assume $N_B(v_6) = \{v_1, v_2, v_3\}$ while $N_B(v_7) = \{v_3, v_4, v_5\}$. Then $N_R(v_6) = \{v_4, v_5\}$, while $N_R(v_7) = \{v_1, v_2\}$. To avoid the 3-cycle v_1, v_2, v_7, v_1 in G_R, we see that that $v_1 v_2$ is colored blue. Similarly, we see that $v_4 v_5$ is colored blue. To avoid the 4-cycle v_3, v_i, v_j, v_6, v_3 where $\{i,j\} = \{1,2\}$ in G_B, we see that both v_1v_3 and v_2v_3 are colored red. Similarly, both v_3v_4 and v_3v_5 are colored red. To avoid the 3-cycles v_2, v_3, v_4, v_2 and v_1, v_3, v_5, v_1, we see that v_2v_4 and v_1v_5 are both colored blue. But then v_1, v_2, v_4, v_5, v_1 is a 4-cycle in G_B, which is a contradiction. ■

As $R_6(C_3, C_4) = 7$ by Theorem 4.6, and $K_{2,5(1)} \subset K_7$, note that an argument like the one used in the proof of Theorem 4.5 gives us the following result.

Corollary 4.7

$R_7(C_3, C_4) = R(C_3, C_4) = 7.$

4.4 $R_k(C_3, C_5)$

Note that $R(C_3, C_5) = 9$ by Theorem 1.2 and recall that $R_k(C_3, C_5)$ does not exist for $k = 2$, 3 and 4 by Observation 1.5. In this section we prove the following result.

Theorem 4.8

$R_k(C_3, C_5) = 9$ *for* $k = 5, \ldots, 9.$

Proof

For $k = 5, \ldots, 9$, $R_k(C_3, C_5) > 8$ follows from Proposition 4.2. We show now that determining $R_k(C_3, C_5) \leq 9$ for $k = 5, \ldots, 9$ reduces to showing that $R_5(C_3, C_5) \leq 9$. In this regard, let G_k be the complete k-partite graph $K_{n_1, n_2, \ldots, n_k}$ of order $9 = \sum_{i=1}^{k} n_i$ with $n_i \in \{\lceil 9/k \rceil, \lfloor 9/k \rfloor\}$ for $i = 1, \ldots, k$. Note that

$$G_5 = K_{4(2),1} \quad \text{if } k = 5,$$

$$G_6 = K_{3(2),3(1)} \quad \text{if } k = 6,$$

$$G_7 = K_{2,5(1)} \quad \text{if } k = 7,$$

$$G_8 = K_{2,7(1)} \qquad \text{if } k = 8,$$

$$G_9 = K_9 \qquad \text{if } k = 9,$$

and so, $G_k \subseteq G_{k+1}$ for $k \in \{5,6,7,8\}$.

Observation 4.9

If $R_k(C_3, C_5) \leq 9$, *then* $R_{k+1}(C_3, C_5) \leq 9$ *for* $k \in \{5,6,7,8\}$.

Proof

For $5 \leq k \leq 8$, let $\mathcal{C}$ be a (R, B)-coloring of the edges of G_{k+1}. As $G_k \subseteq G_{k+1}$, $\mathcal{C}$ induces a (R, B)-coloring on the edges of G_k. Moreover, as $R_k(C_3, C_5) \leq 9$, either the red subgraph associated with G_k has a C_3 or the blue subgraph associated with G_k has a C_5. As $G_k \subseteq G_{k+1}$, it now follows that the red subgraph associated with G_{k+1} has a C_3 or the blue subgraph associated with G_{k+1} has a C_5. Hence, $R_{k+1}(C_3, C_5) \leq 9$. □

We proceed to show that $R_5(C_3, C_5) \leq 9$ which completes the proof of Theorem 4.8.

Theorem 4.10

$R_5(C_3, C_5) \leq 9$.

Proof

Let graph $G = K_{4(2),1}$ and assume to the contrary that there exists a (R, B)-coloring of the edges of G that contains neither a red C_3 nor a blue C_5. Further, let the partite sets of G be $V = \{v_1\}$, $W = \{w_1, w_2\}$, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$. We now consider each possibility for $\deg_{G_R}(v_1)$.

Case 1: $\deg_{G_R}(v_1) \in \{5,6,7,8\}$.

Suppose $|N_R(v_1)| = 5$. Without loss of generality, either $N_R(v_1) = \{w_1, w_2, x_1, x_2, y_1\}$ or $N_R(v_1) = \{w_1, w_2, x_1, y_1, z_1\}$. Let $A_1 = \{w_1, w_2, x_1, x_2, y_1\}$ and $A_2 = \{w_1, w_2, x_1, y_1, z_1\}$.

First consider the case $N_R(v_1) = A_1$. For $u_1, u_2 \in A_1$ such that $u_1 \in N_R(u_2)$, G_R has the 3-cycle u_1, u_2, v_1, u_1. Thus, $N_R(u_2) \cap A_1 = \emptyset$ and so, graph $H_1 = \langle A_1 \rangle \cong K_{2,2,1}$ in G_B. By Lemma 4.1, H_1 contains a 5-cycle, a contradiction.

Therefore, $N_R(v_1) = A_2$. For $u_1, u_2 \in A_2$ such that $u_1 \in N_R(u_2)$, G_R has the 3-cycle u_1, u_2, v_1, u_1. Thus, $N_R(u_2) \cap A_2 = \emptyset$ and so, graph $H_2 = \langle A_2 \rangle \cong K_{1(2),3(1)}$ in G_B. By Lemma 4.1, H_2 contains a 5-cycle, a contradiction.

For $|N_R(v_1)| = 6$, 7 and 8 with $N_R(v_1) = D$, either $A_1 \subseteq D$ or $A_2 \subseteq D$, without loss of generality. Thus, $H_1 \subseteq \langle D \rangle$ or $H_2 \subseteq \langle D \rangle$ in G_B. As H_1 and H_2 contain a 5-cycle, G_B contains a 5-cycle which is a contradiction.

Case 2: $\deg_{G_R}(v_1) = 4$.

Case 2.1: Let $A_1 = \{w_1, x_1, y_1, z_1\}$ and $A_2 = \{w_2, x_2, y_2, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. For $u_1, u_2 \in A_1$ such that $u_1 \in N_R(u_2)$, G_R has the 3-cycle u_1, u_2, v_1, u_1. Thus, $N_R(u_2) \cap A_1 = \emptyset$ and so, all the edges in $\langle A_1 \rangle \cong K_4$ are colored blue. Let $u_1 \in A_1$ and $u_2 \in A_2$.

We next show that if $u_2 \in N_B(u_1)$, then G_B contains a 5-cycle. Without loss of generality, assume that $w_2 \in N_B(x_1)$. Edge $w_2 z_1$ is colored red, since otherwise G_B contains the 5-cycle $w_2, z_1, w_1, y_1, x_1, w_2$. Similarly, $w_2 y_1$ is red.

Suppose $x_2 z_1$ is colored blue. Then, to avoid the blue 5-cycle $x_2, z_1, y_1, x_1, w_2, x_2$, edge $x_2 w_2$ is red. To avoid the red 3-cycle x_2, y_1, w_2, x_2, we see that $x_2 y_1$ is blue. But then G_B contains the 5-cycle $x_2, z_1, x_1, w_1, y_1, x_2$, a contradiction.

Thus, we may assume that $x_2 z_1$ is red. Then, $w_2 x_2$ is colored blue; otherwise G_R contains the 3-cycle w_2, x_2, z_1, w_2. Furthermore, $w_1 x_2$ and $w_1 y_2$ are colored red;

otherwise G_B contains the 5-cycles $w_1, x_2, w_2, x_1, y_1, w_1$ and $w_1, y_2, v_1, w_2, x_1, w_1$. Also, x_2y_2 is colored blue; otherwise G_R contains the 3-cycle x_2, y_2, w_1, x_2. Third, w_1z_2 is colored red; otherwise G_B contains the 5-cycle $w_1, z_2, v_1, w_2, x_1, w_1$. Then, y_2z_2 is colored blue; otherwise G_R contains the 3-cycle y_2, z_2, w_1, y_2. However, G_B then contains the 5-cycle $y_2, z_2, v_1, w_2, x_2, y_2$, a contradiction. Thus, $w_2 \in N_R(x_1)$ and so, $u_2 \in N_R(u_1)$.

Next, we show that for $u_3, u_4 \in A_2$, if $u_3 \in N_R(u_4)$, then G_R contains a 3-cycle. Without loss of generality, assume that $w_2 \in N_R(x_2)$. Then, G_R contains the 3-cycle w_2, x_2, y_1, w_2 which is a contradiction. Thus, $w_2 \in N_B(x_2)$ and so, $u_3 \in N_B(u_4)$. Hence, $\langle\{v_1, w_2, x_2, y_2, z_2\}\rangle \cong K_5$ in G_B which contains a 5-cycle.

Case 2.2: Let $A_1 = \{w_1, w_2, x_1, y_1\}$ and $A_2 = \{x_2, y_2, z_1, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. To avoid 3-cycles, we see that all the edges of $\langle A_1\rangle \cong K_4 - \{w_1w_2\}$ in G are colored blue.

We now show that x_2y_1 is colored red. Suppose, to the contrary, that x_2y_1 is colored blue. Edges w_2x_2 and w_2y_2 are colored red; otherwise G_B contains the 5-cycles $w_2, x_2, y_1, w_1, x_1, w_2$ and $w_2, y_2, v_1, x_2, y_1, w_2$. Then x_2y_2 is colored blue; otherwise G_R contains the 3-cycle x_2, y_2, w_2, x_2. Furthermore, w_2z_1 and w_2z_2 are colored red; otherwise G_B contains the 5-cycles $w_2, z_1, v_1, x_2, y_1, w_2$ and $w_2, z_2, v_1, x_2, y_1, w_2$. But then x_2z_1 and y_2z_2 are colored blue; otherwise G_R contains the 3-cycles x_2, z_1, w_2, x_2 and y_2, z_2, w_2, y_2. However, G_B then contains the 5-cycle $y_2, z_2, v_1, z_1, x_2, y_2$, a contradiction. We conclude that x_2y_1 is colored red. Similarly, x_1y_2 is red.

Next, we show that x_2w_1 is colored red. Suppose, to the contrary, that x_2w_1 is colored blue. Edges y_1z_1 and y_1z_2 are colored red; otherwise G_B contains the 5-cycles $y_1, z_1, v_1, x_2, w_1, y_1$ and $y_1, z_2, v_1, x_2, w_1, y_1$. Edges x_2z_1 and x_2z_2 are colored blue; otherwise G_R contains the 3-cycles x_2, z_1, y_1, x_2 and x_2, z_2, y_1, x_2. Edge x_1z_1 is colored red; otherwise G_B contains the 5-cycle $x_1, z_1, v_1, x_2, w_1, x_1$. But then y_2z_1 is colored blue; otherwise G_R contains the 3-cycle y_2, z_1, x_1, y_2.

However, G_B then contains the 5-cycle $y_2, z_1, x_2, z_2, v_1, y_2$, a contradiction. Thus, x_2w_1 is colored red. Similarly, x_2w_2, y_2w_1 and y_2w_2 are all colored red.

Edge x_2y_2 is colored blue, since otherwise G_R contains the 3-cycle x_2, y_2, w_2, x_2 which is a contradiction.

Suppose x_2z_1 is colored red. To avoid the red 3-cycle x_2, z_1, w_1, x_2, the edge w_1z_1 is colored blue. To avoid the red 3-cycle w_2, x_2, z_1, w_2, the edge w_2z_1 is colored blue. But then we have the 5-cycle $z_1, w_1, y_1, x_1, w_2, z_1$ in G_B, which is a contradiction. Thus, x_2z_1 is colored blue, and, similarly, x_2z_2 is colored blue. Moreover, in a similar way, we see that y_2z_1 and y_2z_2 are colored blue. Hence, $\langle\{v_1, x_2, y_2, z_1, z_2\}\rangle \cong K_5 - \{z_1z_2\}$ in G_B which contains a 5-cycle, a contradiction.

Case 2.3. Let $A_1 = \{w_1, w_2, x_1, x_2\}$ and $A_2 = \{y_1, y_2, z_1, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. To avoid 3-cycles, we see that all the edges of $\langle A_1\rangle \cong C_4$ in G are colored blue.

We show that w_1y_1 is colored red. Suppose, to the contrary, that w_1y_1 is colored blue. Edges x_1y_1 and x_1z_1 are colored red; otherwise G_B contains the 5-cycles $x_1, y_1, w_1, x_2, w_2, x_1$ and $x_1, z_1, v_1, y_1, w_1, x_1$. Then, y_1z_1 is colored blue; otherwise G_R contains the 3-cycle y_1, z_1, x_1, y_1. Edges x_1y_2 and x_1z_2 are colored red; otherwise G_B contains the 5-cycles $x_1, y_2, v_1, y_1, w_1, x_1$ and $x_1, z_2, v_1, y_1, w_1, x_1$. Then, y_2z_1 and y_2z_2 are colored blue; otherwise G_R contains the 3-cycles y_2, z_1, x_1, y_2 and y_2, z_2, x_1, y_2. However, G_B then contains the 5-cycle $y_2, z_2, v_1, y_1, z_1, y_2$, a contradiction. Thus, w_1y_1 is colored red. Similarly, if $u \in A_2$, then w_1u is colored red.

Let u', u'' be two distinct vertices of A_2. If $u'u''$ is colored red, then we have the 3-cycle w_1, u', u'', w_1 in G_R, which is a contradiction. Hence, all the edges of $\langle\{v_1, y_1, y_2, z_1, z_2\}\rangle \cong K_5 - \{y_1y_2, z_1z_2\}$ of G are colored blue, and so G_B contains a 5-cycle by Lemma 4.1.

Case 3: $\deg_{G_R}(v_1) = 3$.

Case 3.1. Let $A_1 = \{w_1, x_1, y_1\}$ and $A_2 = \{w_2, x_2, y_2, z_1, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. To avoid 3-cycles which are red, all the edges in $\langle A_1 \rangle \cong C_3$ of G are colored blue.

We first show that x_1w_2 is colored red. Suppose, to the contrary, that x_1w_2 is colored blue. The edge w_1x_2 is colored red; otherwise G_B contains the 5-cycle $w_1, x_2, v_1, w_2, x_1, w_1$. Similarly, w_1y_2 is colored red. Edge x_2y_2 is colored blue; otherwise G_R contains the 3-cycle x_2, y_2, w_1, x_2. Edge w_1z_1 is colored red; otherwise G_B contains the 5-cycle $w_1, z_1, v_1, w_2, x_1, w_1$. Similarly, w_1z_2 is colored red. But then x_2z_2 and y_2z_1 are colored blue; otherwise G_R contains the 3-cycles x_2, z_2, w_1, x_2 and y_2, z_1, w_1, y_2. However, G_B then contains the 5-cycle $y_2, z_1, v_1, z_2, x_2, y_2$, a contradiction. Thus, x_1w_2 is colored red. Similarly, y_1w_2, x_2y_1, x_2w_1, y_2w_1 and y_2x_1 are all colored red.

To avoid the red 3-cycle w_2, x_2, y_1, w_2, the edge w_2x_2 is colored blue. To avoid the red 3-cycle x_2, y_2, w_1, x_2, the edge x_2y_2 is colored blue. To avoid the red 3-cycle y_2, w_2, x_1, y_2, the edge w_2y_2 is colored blue. It now follows that the edges of $\langle \{v_1, w_2, x_2, y_2\} \rangle \cong K_4$ in G are all colored blue.

Note that both z_1 and z_2 are joined with a blue edge to vertex v_1. If either z_1 or z_2 is joined with a blue edge to one of the vertices of $\{w_2, x_2, y_2\}$, then we have a 5-cycle in G_B, which is a contradiction. Thus, all the edges z_iw_2, z_ix_2 and z_iy_2 are colored red for $i = 1,2$. To avoid the red 3-cycle z_i, x_2, y_1, z_i, the edge z_iy_1 is colored blue where $i = 1,2$. To avoid the red 3-cycle z_i, x_1, w_2, z_i, the edge z_ix_1 is colored blue where $i = 1,2$. To avoid the red 3-cycle z_i, w_1, x_2, z_i, the edge z_iw_1 is colored blue where $i = 1,2$. Hence, all the edges $\langle \{w_1, x_1, y_1, z_1, z_2\} \rangle \cong K_5 - z_1z_2$ of G are colored blue, and so G_B contains a 5-cycle.

Case 3.2. Let $A_1 = \{w_1, w_2, x_1\}$ and $A_2 = \{x_2, y_1, y_2, z_1, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. To avoid 3-cycles which are red, all the edges in $\langle A_1 \rangle \cong P_3$ of G are colored blue.

We show that x_1y_1 is colored red. Suppose, to the contrary, that x_1y_1 is colored blue. Edges w_1x_2 and w_1y_2 are colored red; otherwise G_B contains the 5-cycles $w_1, x_2, v_1, y_1, x_1, w_1$ and $w_1, y_2, v_1, y_1, x_1, w_1$. Then, x_2y_2 is colored blue; otherwise G_R contains the 3-cycle x_2, y_2, w_1, x_2. Edge w_1z_i is colored red; otherwise G_B contains the 5-cycle $w_1, z_i, v_1, y_1, x_1, w_1$ where $i = 1,2$. Then, x_2z_1 and y_2z_2 are colored blue; otherwise G_R contains the 3-cycles x_2, z_1, w_1, x_2 and y_2, z_2, w_1, y_2. However, G_B then contains the 5-cycle $y_2, z_2, v_1, z_1, x_2, y_2$, a contradiction. Hence, x_1y_1 is colored red. Similarly, x_1y_2, x_1z_1 and x_1z_2 are all colored red.

To avoid red 3-cycles, the edges of $\langle\{y_1, y_2, z_1, z_2\}\rangle \cong K_4 - \{y_1y_2, z_1z_2\}$ in G are all colored blue. But then the edges of $\langle\{v_1, y_1, y_2, z_1, z_2\}\rangle \cong K_5 - \{y_1y_2, z_1z_2\}$ in G are all colored blue, and so G_B contains a 5-cycle by Lemma 4.1.

Case 4: $\deg_{G_R}(v_1) = 2$.

Case 4.1. Let $A_1 = \{w_1, x_1\}$ and $A_2 = \{w_2, x_2, y_1, y_2, z_1, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. Note that w_1x_1 is colored blue; otherwise G_R has the 3-cycle w_1, x_1, v_1, w_1.

Suppose x_1y_1 is colored blue. Note that w_1, x_1, y_1, v_1, t is a path of order 5 in G_B for $t \in \{x_2, y_2, z_1, z_2\}$. To avoid a 5-cycle in G_B, it follows that w_1x_2, w_1y_2, w_1z_1 and w_1z_2 are all colored red. To avoid red 3-cycles, the edges of $\langle\{x_2, y_2, z_1, z_2\}\rangle \cong K_4 - \{z_1z_2\}$ in G are all colored blue. But then the edges of $\langle\{v_1, x_2, y_2, z_1, z_2\}\rangle \cong K_5 - \{z_1z_2\}$ in G are all colored blue, and so G_B contains a 5-cycle which is a contradiction.

It now follows that all the edges x_1y_1, x_1y_2, x_1z_1, x_1z_2 are colored red. To avoid red 3-cycles, the edges of $\langle\{y_1, y_2, z_1, z_2\}\rangle \cong K_4 - \{y_1y_2, z_1z_2\}$ in G are all colored blue. But then the edges of $\langle\{v_1, y_1, y_2, z_1, z_2\}\rangle \cong K_5 - \{y_1y_2, z_1z_2\}$ in G are all colored blue, and so G_B contains a 5-cycle by Lemma 4.1.

Case 4.2. Let $A_1 = \{w_1, w_2\}$ and $A_2 = \{x_1, x_2, y_1, y_2, z_1, z_2\}$. Suppose $N_R(v_1) = A_1$. Then, $N_B(v_1) = A_2$. We first show that if $|N_B(w_1) \cap A_2| \geq 2$, then G_B

contains a 5-cycle. To see this, we consider, without loss of generality, the two cases $x_1, y_1 \in N_B(w_1)$ and $x_1, x_2 \in N_B(w_1)$.

Suppose that $x_1, y_1 \in N_B(w_1)$. Edges x_1y_2 and x_2y_1 are colored red; otherwise G_B contains the 5-cycles $x_1, y_2, v_1, y_1, w_1, x_1$ and $x_2, y_1, w_1, x_1, v_1, x_2$. Edge y_1z_i is colored red; otherwise G_B contains the 5-cycle $y_1, z_i, v_1, x_1, w_1, y_1$ where $i = 1,2$. Then, x_2z_i is colored blue; otherwise G_R contains the 3-cycle x_2, z_i, y_1, x_2 where $i = 1,2$. Edge x_1z_2 is colored red; otherwise G_B contains the 5-cycle $x_1, z_2, v_1, y_1, w_1, x_1$. Then, y_2z_2 is colored blue; otherwise G_R contains the 3-cycle y_2, z_2, x_1, y_2. However, G_B then contains the 5-cycle $y_2, z_2, x_2, z_1, v_1, y_2$, a contradiction.

Therefore, let $x_1, x_2 \in N_B(w_1)$. Edge x_1y_i is colored red; otherwise G_B contains the 5-cycle $x_1, y_i, v_1, x_2, w_1, x_1$ where $i = 1,2$. Similarly, x_1z_1 and x_1z_2 are colored red. To avoid red 3-cycles, the edges of $\langle\{y_1, y_2, z_1, z_2\}\rangle \cong K_4 - \{y_1y_2, z_1z_2\}$ in G are all colored blue. But then the edges of $\langle\{v_1, y_1, y_2, z_1, z_2\}\rangle \cong K_5 - \{y_1y_2, z_1z_2\}$ in G are all colored blue, and so G_B contains a 5-cycle by Lemma 4.1.

Hence, $|N_B(w_1) \cap A_2| \leq 1$ and so, $|N_R(w_1) \cap A_2| \geq 5$. Without loss of generality, say $\{y_1, y_2, z_1, z_2\} \subseteq N_R(w_1)$. As before, the edges of $\langle\{v_1, y_1, y_2, z_1, z_2\}\rangle \cong K_5 - \{y_1y_2, z_1z_2\}$ in G are all colored blue, and so G_B contains a 5-cycle.

Case 5: $\deg_{G_R}(v_1) = 1$.

Let $w_1 \in N_R(v_1)$ and so, $N_B(v_1) = V(G) - \{w_1\}$. Let $A = V(G) - \{v_1, w_1\}$. Using the same reasoning is in Case 4.2, we see that if $|N_B(w_1) \cap A| \geq 2$, then G_B contains a 5-cycle. Hence, $|N_B(w_1) \cap A| \leq 1$ and so, $|N_R(w_1) \cap A| \geq 5$. Without loss of generality, say $\{y_1, y_2, z_1, z_2\} \subseteq N_R(w_1)$. As before, the edges of $\langle\{v_1, y_1, y_2, z_1, z_2\}\rangle \cong K_5 - \{y_1y_2, z_1z_2\}$ in G are all colored blue, and so G_B contains a 5-cycle.

Case 6: $\deg_{G_R}(v_1) = 0$.

Again, the reasoning is the same as in Case 4.2. ■

4.5 Closing Remarks

In this chapter we proved that $R_3(C_3, C_4) = 9$, $R_4(C_3, C_4) = R_5(C_3, C_4) = 8$, $R_6(C_3, C_4) = R_7(C_3, C_4) = 7$, and $R_k(C_3, C_5) = 9$ for $5 \leq k \leq 9 = R(C_3, C_5)$.

By Theorem 1.1 we have that $R(C_3, C_n) = 2n - 1$ for $n \geq 4$. Considering the results obtained in Section 4.3 for $R_k(C_3, C_4)$, $2 \leq k \leq 7$, and Theorem 4.8 for $R_k(C_3, C_5)$, $2 \leq k \leq 9$, we propose that $R_k(C_3, C_n)$ does not have the same formula for when $n \geq 4$ is odd or even.

Let $n \geq m \geq 1$ with $(n, m) \neq (1,1)$. Recall that $R(C_{2n+1}, C_{2m+1}) = 4n + 1$ by Theorem 1.2. Also recall that $R_k(C_{2n+1}, C_{2m+1})$ does not exist for $k = 2$, 3 and 4 by Observation 1.5, and that $R_k(C_{2n+1}, C_{2m+1}) > 4n$ for $5 \leq k \leq 4n$ by Proposition 4.2. Following the results obtained for $R_k(C_5, C_5)$, $5 \leq k \leq 9$ (see Section 2.2 and Theorem 2.6), as well as the results obtained for $R_k(C_3, C_5)$, $5 \leq k \leq 9$ (see Theorem 4.8), we propose the following conjecture.

Conjecture 4.11

$R_k(C_{2n+1}, C_{2m+1}) \leq 4n + 1$ *for* $k = 5,6, \ldots, 4n$.

That is, $R_k(C_{2n+1}, C_{2m+1}) = R(C_{2n+1}, C_{2m+1})$ for $k = 5,6, \ldots, 4n + 1$.

* * * * * * *

CHAPTER 5

IN CLOSING

The aim of this thesis is to investigate the question,

What is the k-Ramsey number for two cycles which are not both bipartite?

5.1 The Existence of the k-Ramsey Number

The k-Ramsey number $R_k(F, H)$ is defined in [2] for two bipartite graphs F and H only. In Section 2.1 we gave an alternative proof to confirm that $R_k(F, H)$ exists for two bipartite graphs F and H. From the definition of a k-Ramsey number we asked the question,

Does the k-Ramsey number exist when graphs F and H are not both bipartite?

To investigate this question, we considered the following two cases:

(i) F and H are both nonbipartite, and
(ii) F is bipartite and H is nonbipartite (or *vice versa*).

Note that an even cycle graph is bipartite, and an odd cycle graph is nonbipartite.

Suppose first that F and H are any two nonbipartite graphs. Johnston observed in [14] that $R_k(F, H)$ does not exist when $k = 2$, 3 and 4. The author further showed that $R_5(C_3, C_3)$ does not exist so that $R_k(C_3, C_3)$ only exists when $k = 6$, that is, $R_6(C_3, C_3) = R(C_3, C_3)$. Johnston also determined an upper bound for $R_5(C_{2\ell+1}, C_{2k+1})$, proving the existence of this number (see Theorem 2.1).

In Section 2.2 and Theorem 2.6 we showed that $R_k(C_5, C_5)$ exists for $5 \leq k \leq 9$ by determining the actual value thereof. Similarly, we have that $R_k(C_3, C_5)$ exists for $5 \leq k \leq 9$ by Theorem 4.8. For integers k, n and m with $n \geq m \geq 1$ and $(n, m) \neq (1,1)$, we conjecture that $R_k(C_{2n+1}, C_{2m+1})$ exists for $5 \leq k \leq 4n$ (see Conjecture 4.11).

For $(F, H) \neq (C_3, C_3)$, $R_k(F, H)$ does not necessarily exist when $k \geq 5$. For $n \geq 3$, we showed in Proposition 3.2 that $R_k(K_n, H)$ does not exist if $2 \leq k < n$.

Suppose now that graph F is bipartite and graph H is nonbipartite. In Proposition 3.1 we showed that $R_2(F, H)$ does not exist. By Proposition 3.2 we have that $R_k(F, K_n)$ does not exist for $n \geq 3$ and $2 \leq k < n$.

From the results in Chapter 3 we have that $R_k(C_4, C_5)$ exists for $3 \leq k \leq 7$ by determining the actual value thereof. Similarly, we have that $R_k(C_3, C_4)$ exists for $3 \leq k \leq 7$ from the results in Section 4.3.

From the results we obtained in this thesis, we propose that the existence of $R_k(F, H)$, for graphs F and H not both bipartite, depends on the specific graphs F and H under consideration. Thus, we pose the following open problem.

Open Problem 5.1

When does $R_k(F, H)$ exist for graphs F and H not both bipartite?

5.2 The k -Ramsey Number for Two Cycles

We started our investigation of the k-Ramsey number for two cycles by first analyzing the historic results that led up to the formula for the Ramsey number of two cycles. A short overview thereof is given in Section 1.3, and for the bipartite Ramsey number for two cycles in Section 1.4. Our objective in this thesis was to duplicate (to some extent) a portion of this process to find the k-Ramsey number for some pairs of cycles.

To investigate $R_k(C_n, C_m)$ for $n \geq 3$, $m \geq 3$ and $k = 2, \ldots, R(F, H)$, the following five cases need to be considered:

(i) $R_k(C_{2n}, C_{2n})$ for $n \geq 2$,

(ii) $R_k(C_{2n}, C_{2m})$ for $n \geq 2$ and $m \geq 2$ with $n \neq m$,

(iii) $R_k(C_{2n+1}, C_{2n+1})$ for $n \geq 1$,

(iv) $R_k(C_{2n+1}, C_{2m+1})$ for $n \geq 1$ and $m \geq 1$ with $n \neq m$, and

(v) $R_k(C_{2n+1}, C_{2m})$ for $n \geq 1$ and $m \geq 2$.

The k-Ramsey numbers in Cases (i) and (ii) exists as mentioned in Section 5.1. The authors of [1] showed that $R_k(C_4, C_4) = 12 - k$ for $2 \leq k \leq 6 = R(C_4, C_4)$. To our knowledge this is the only result known with regards to Case (i), and no results have been found regarding Case (ii). Thus, we pose the following open problem where $R(C_{2n}, C_{2m}) = 2n + m - 1$ from Theorem 1.2.

Open Problem 5.2

(i) *Let* $n \geq 3$. *For* $2 \leq k \leq 3n - 2$, *what is* $R_k(C_{2n}, C_{2n})$*?*

(ii) *Let* $n \geq m \geq 2$ *with* $(n, m) \neq (2,2)$. *For* $2 \leq k \leq 2n + m - 2$, *what is* $R_k(C_{2n}, C_{2m})$*?*

Regarding Open Problem 5.2, we are currently investigating $R_k(C_4, C_6)$ and $R_k(C_6, C_6)$ for the relevant values of k.

As mentioned in Section 5.1, the k-Ramsey numbers as in Cases (iii) and (iv) do not exist when $k = 2$, 3 and 4. Also, $R_k(C_3, C_3)$ only exists when $k = 6$, that is, $R_6(C_3, C_3) = R(C_3, C_3)$.

In Section 2.2 and Theorem 2.6 we proved that $R_k(C_5, C_5) = 9$ for $5 \leq k \leq 9 = R(C_5, C_5)$. We further proved that $R_k(C_3, C_5) = 9$ for $5 \leq k \leq 9 = R(C_3, C_5)$ in Theorem 4.8. In the general case, we observed that $R_k(C_{2n+1}, C_{2m+1}) > 4n$ for $n \geq m \geq 1$, $(n, m) \neq (1,1)$ and $5 \leq k \leq 4n = R_k(C_{2n+1}, C_{2m+1}) - 1$ in Proposition 4.2. We posed the following open problem in Conjecture 4.11 where $R(C_{2n+1}, C_{2m+1}) = 4n + 1$ from Theorem 1.2.

Open Problem 5.3

Let k, n *and* m *be integers with* $n \geq m \geq 1$ *and* $(n, m) \neq (1,1)$. *For* $5 \leq k \leq 4n$, $R_k(C_{2n+1}, C_{2m+1}) \leq 4n + 1$.

For Case (v), we showed that $R_2(C_{2n+1}, C_{2m})$ does not exist for $n \geq 1$ and $m \geq 1$ in Proposition 3.1. For $R_k(C_4, C_5)$ we proved that $R_3(C_4, C_5) = 10$ (see Theorem

3.4), $R_4(C_4, C_5) = 8$ (see Theorem 3.6), and $R_k(C_4, C_5) = 7$ for $k = 5$, 6 and 7 (see Theorem 3.7, Theorem 3.8 and Corollary 3.9, respectively).

For $R_k(C_3, C_4)$ we proved that $R_3(C_3, C_4) = 9$ (see Theorem 4.3), $R_4(C_3, C_4) = R_5(C_3, C_4) = 8$ (see Theorem 4.4 and Theorem 4.5, respectively), and $R_6(C_3, C_4) = R_7(C_3, C_4) = 7$ (see Theorem 4.6 and Corollary 4.7, respectively).

For Case (v), we pose the following open problem where $R(C_3, C_{2m}) = 4m - 1$ and $R(C_{2n+1}, C_{2m}) = \max\{2n + m, 4m - 1\}$ from Theorem 1.2.

Open Problem 5.4

Let k, n and m be integers with $m \geq 2$.

- *(i)* *For $3 \leq k \leq 4m - 1$, what is $R_k(C_3, C_{2m})$?*
- *(ii)* *For $n \geq 2$ and $3 \leq k \leq \max\{2n + m, 4m - 1\}$, what is $R_k(C_{2n+1}, C_{2m})$?*

5.3 More Open Problems

The Ramsey number $R(F, H)$ is known to exist for each pair F, H of graphs. The k-Ramsey number $R_k(F, H)$, for two bipartite graphs F and H, is defined with the constraint $2 \leq k \leq R(F, H)$. Thus, knowing the value of $R(F, H)$, we have an upper bound for k and we can proceed to determine $R_k(F, H)$ if it exists.

We propose the following approach to determine unknown values of $R(F, H)$.

Remark 5.5

For $k \geq 2$, let G_k be the balanced complete k-partite graph $K_{n_1,\dots,n_k}$. Let F and H be graphs and $k \geq 2$ be the smallest integer such that $R_k(F, H)$ exists. First determine $R_k(F, H)$. Determine now $R_{k+1}(F, H), \dots, R_{k'}(F, H)$, in that order, where k' is the value such that $G_{k'} \cong K_{k'}$. Then, $R_{k'}(F, H) = R(F, H)$.

Note that Remark 5.5 was used in this thesis to provide an alternative proof for $R(C_5, C_5) = R_9(C_5, C_5)$, $R(C_4, C_5) = R_7(C_4, C_5)$, $R(C_3, C_4) = R_7(C_3, C_4)$, and

$R(C_3, C_5) = R_9(C_3, C_5)$. For these results, see Observation 2.5, Corollary 3.9, Corollary 4.7, and Theorem 4.8, respectively.

One of the well-known outstanding Ramsey numbers is the formula for $R(K_n, K_m)$. For specific values of n and m, see [16] for a review of some of the known values or bounds for $R(K_n, K_m)$. From Corollary 3.3 we have that $R_k(K_n, K_m)$ does not exist if $k < n$, and Johnston observed in [14] that $R_k(C_3, C_3) = R_k(K_3, K_3)$ does not exist for $2 \leq k \leq 5$. We end this thesis with the following open problem.

Open Problem 5.6

Using Remark 5.5, can we find a formula for $R(K_n, K_m)$ if such a formula exists?

* * * * * * *

BIBLIOGRAPHY

[1] E. Andrews, Z. Bi, D. Johnston, C. Lumduanhom and P. Zhang, On k-Ramsey numbers of stripes. *Utilitas Mathematica*. **106** (2018), 233-249.

[2] E. Andrews, G. Chartrand, C. Lumduanhom and P. Zhang, Stars and their k-Ramsey numbers. *Graphs and Combinatorics*. **33** (2017) 257-274.

[3] L.W. Beineke and A.J. Schwenk, On a bipartite form of the Ramsey problem. *Proceedings of the Fifth British Combinatorial Conference*. *1975*. **15** (1975) 17-22.

[4] Z. Bi, G. Chartrand and P. Zhang, A new view of bipartite Ramsey numbers. *Journal of Combinatorial Mathematics and Combinatorial Computing*. **108** (2019), 193-203.

[5] Z. Bi, G. Chartrand and P. Zhang, Another view of bipartite Ramsey numbers. *Discussiones Mathematicae Graph Theory*. **38** (2018), 587-605.

[6] J.A. Bondy and P. Erdös, Ramsey numbers for cycles in graphs. *Journal of Combinatorial. Theory*. **11** (1973) 46-54.

[7] G. Chartrand, L. Lesniak and P. Zhang, Graphs & Digraphs, Sixth Edition. CRC Press/Taylor & Francis Group, Boca Raton, FL (2016).

[8] G. Chartrand and S. Schuster, On the existence of specified cycles in complementary graphs. *Bulletin of the American Mathematical Society*. **77** (1971) 995-998.

[9] G. Chartrand and P. Zhang, New directions in Ramsey theory, *Discrete Mathematics Letters*. **6** (2021),84-96.

[10] R.J. Faudree and R.H. Schelp, All Ramsey numbers for cycles in graphs. *Discrete Mathematics.* **8** (1974) 313-329.

[11] R.E. Greenwood, A.M. Gleason, Combinatorial relations and chromatic graphs. *Canadian Journal of Mathematics.* **7** (1955) 1-7.

[12] J.H. Hattingh, E. Jonck and R.J. Maartens, The k-Ramsey number of two five cycles. **Manuscript**.

[13] J.H. Hattingh and M.A. Henning, Bipartite Ramsey theory. *Utilitas Mathematica.* **53** (1998) 217-230.

[14] D. Johnston, Edge colorings of graphs and their applications (2015). Dissertations. 586. http://scholarworks.wmich.edu/dissertati ons/586.

[15] G. Károlyi and V. Rosta, Generalized and geometric Ramsey numbers for cycles. *Theoretical Computer Science.* **263** (2001) 87-98.

[16] S. Radziszowski, Small Ramsey numbers. *The Electronic Journal of Combinatorics*. (2011) 1000-1115.

[17] F. Ramsey, On a problem of formal logic. *Proceedings of London Mathematical Society.* **30** (1930) 264-286.

[18] I. Reiman, Über ein problem von K. Zarankiewicz. *Acta Mathematica Academiae Scientiarum Hungarica.* **9** (1958) 269-273.

[19] V. Rosta, On a Ramsey type problem of J.A. Bondy and P. Erdös I & II. *Journal of Combinatorial Theory Series B.* **15** (1973) 94-120.

[20] R. Zhang and Y. Sun, The bipartite Ramsey numbers $b(C_{2m}, K_{2,2})$. *Electronic Journal of Combinatorics.* **18** (2011) 51-60.

[21] R. Zhang, Y. Sun and Y. Wu, The bipartite Ramsey numbers $b(C_{2m}, C_{2n})$. *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering.* **1** (2013) 80-83.

[22] Biography of Frank Ramsey, https://ideasofeconomists.com/2020/09/05/frank-ramsey-1903-1930/.