Symmetry Reductions of Systems of Partial Differential Equations using Conservation Laws

R.M. Morris

Supervisor: Professor A.H. Kara

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DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

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R.M. Morris

This ____________ day of ________________ 2013.

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Abstract

There is a well established connection between one parameter Lie groups of transformations and conservation laws for differential equations. In this thesis, we construct conservation laws via the invariance and multiplier approach based on the well-known result that the Euler-Lagrange operator annihilates total divergences. This technique will be applied to some plasma physics models. We show that the recently developed notion of the association between Lie point symmetry generators and conservation laws lead to double reductions of the underlying equation and ultimately to exact/invariant solutions for higher-order nonlinear partial differential equations viz., some classes of Schrödinger and KdV equations.

Contents

Introduction

The study and analysis of differential equations through the realm of group theory is associated with the great mathematician Sophus Lie [59]. Some effective Lie group methods such as the classical Lie group approach [17, 42, 73], the non-classical Lie group approach [15, 57, 72] and the Clarkson and Kruskal direct method [19, 20] have been implemented successfully in finding symmetries, symmetry groups, symmetry reductions and constructions of group invariant solutions of nonlinear partial differential equations (PDEs). They have been used to construct new exact solutions for numerous physically important nonlinear PDEs arising from mathematics and physics (see [22, 31, 41, 47, 58, 60, 62, 65, 81, 84, 96] and references therein).

There are a number of reasons to compute conserved densities and fluxes of PDEs. Some conservation laws are fundamental laws of physics (e.g., conservation of momentum, mass and energy) while others facilitate the analysis of the PDE. They assist in obtaining reductions and solutions of PDEs. The existence of a large number of conservation laws is a predictor of complete integrability [38]. Without these conserved vectors (integrals of motion), an understanding of the problem would be incomplete [33]. The role of 'multipliers' has been shown to play a significant role in the construction of conservation laws and in determining hierarchies [28]. In short, knowledge of a multiplier, by formula, leads to a conserved flow.

The theory of double reduction of a PDE (and systems of PDEs) is well-known for the association of conservation laws with Noether symmetries [16, 73]. The association of conservation laws with Lie-Bäcklund symmetries [49] and non-local symmetries [85, 86] was then analysed. This lead to the expansion of the theory of double reduction for PDEs with two independent variables which do not possess a Lagrangian formulation, i.e., do not possess Noether symmetries [87]. Most of the previous analyses of nonlinear PDEs are based on the 'travelling wave' type solutions via some well-known substitutions. This method shows that the travelling wave method by the underlying symmetries of the equation is recovered and how solutions are obtained via a double reduction following an association of a Lie point symmetry with conservation laws of the equation. Such an association exists for a range of symmetries, e.g., scaling and rotational symmetries. In this thesis, we apply the fundamental theorem of double reduction for classes of higher-order nonlinear PDEs and systems of PDEs with two independent variables.

This thesis is structured as follows.

In the first chapter, we state the definitions and theorems of the fundamental concepts that will be used to perform the calculations.

In the second chapter, we perform the double reduction procedure as discussed above for a second-order system of PDEs, by analysing the Gross-Pitaevskii equation (section 2.2) and the parametrically damped-driven Schrödinger equation (section 2.3).

In the third chapter, we construct conserved vectors via the invariance and multiplier approach as discussed above and apply the double reduction procedure for a third-order scalar Hunter-Saxton type equation (section 3.2), then we apply the double reduction procedure for a version of the third-order standard Korteweg-de Vries

(KdV) equation (section 3.3) and for a third-order system of PDEs related to the Drinfeld-Sokolov-Wilson equation (section 3.4).

In the fourth chapter, we calculate conserved quantities via the invariance and multiplier approach for a second-order system of PDEs related to generalized Zakharov equations (section 4.2) and for a fourth-order wave equation related to compressional dispersive Alfvén waves (section 4.3). We also calculate conserved vectors via Noether's theorem and apply the double reduction procedure in section 4.3.

In the fifth chapter, we analyse a fourth-order system of PDEs based on a model of fluid mechanics related to unsteady hydromagnetic flows of an Oldroyd-B fluid under the influence of hall currents (section 5.2).

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we present the following definitions and theorems of the fundamental concepts that will be used throughout this thesis.

1.2 Fundamental Concepts

A function $f(x, u, u_{(1)}, \ldots, u_{(k)})$ of a finite number of variables is called a differential function of order k.

$u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denotes the collections of all first, second, $\ldots$, k^{th} order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha)$, $\ldots$respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{1.1}$$

where i represents the independent variables.

The summation convention for an index appearing twice in a term is adopted throughout this thesis.

It will be denoted that $\mathcal{A}$ is the universal vector space of differential functions, thus consider a k^{th} order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \mu = 1, \ldots, \tilde{m}. \tag{1.2}$$

The Lie-Bäcklund or generalized operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}\ , \quad \xi^i, \eta^\alpha \in \mathcal{A}. \tag{1.3}$$

The operator (1.3) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}\ , \tag{1.4}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \ldots i_s}^\alpha &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha\ , \quad s > 1. \end{aligned} \tag{1.5}$$

In (1.5), W^α is the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.6}$$

A Lie-Bäcklund operator X is said to be a Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$, if there exists a vector $B^i = (B^1, \ldots, B^n)$, $B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{1.7}$$

If $B^i = 0$, $i = 1, \ldots, n$, then X is referred to as a strict Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$.

A current $T^i = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$ is conserved if it satisfies

$$D_i T^i = 0 \tag{1.8}$$

along the solutions of (1.2).

It can be shown that every admitted conservation law arises from multipliers $Q_\mu \in \mathcal{A}$ such that

$$Q_\mu G^\mu = D_i T^i \tag{1.9}$$

holds identically (i.e., off the solution space) everywhere, not just on solutions for some current T^i.

Definition 1.1 [49] A Lie-Bäcklund symmetry generator X of the form (1.4) is associated with a conserved vector T of the system (1.2) if X and T satisfy the relations

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, \ldots, n. \tag{1.10}$$

Definition 1.2 [38] The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{1.11}$$

Theorem 1.3 [48, 50] Suppose that X is any Lie-Bäcklund symmetry of (1.2) and $T^i, i = 1, \ldots, n$ are the components of the conserved vector of (1.2). Then

$$T^{*i} = [T^i, X] = X(T^i) + T^i D_k \xi^k - T^k D_k \xi^i, \quad i = 1, \ldots, n \tag{1.12}$$

constitute the components of a conserved vector of (1.2), i.e., $D_i T^{*i} \mid_{(1.2)} = 0$.

Theorem 1.4 [18] Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (1.2). Then under a contact transformation, there exists functions $\tilde{T}^i$ such that $J\ D_i T^i = \tilde{D}_i \tilde{T}^i$, where $\tilde{T}^i$ is given as

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} \tag{1.13}$$

in which

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix} \tag{1.14}$$

and $J = \det(A)$.

Theorem 1.5 [18] **(fundamental theorem on double reduction)**
Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (1.2). Then under a similarity transformation of a symmetry X of the form (1.4) for the PDE, there exist functions $\tilde{T}^i$ such that X is still a symmetry for the PDE satisfying $\tilde{D}_i \tilde{T}^i = 0$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \tag{1.15}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x_1} & D_1 \tilde{x_2} & \cdots & D_1 \tilde{x_n} \\ D_2 \tilde{x_1} & D_2 \tilde{x_2} & \cdots & D_2 \tilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x_1} & D_n \tilde{x_2} & \cdots & D_n \tilde{x_n} \end{pmatrix} \tag{1.16}$$

and $J = \det(A)$.

The original system of PDEs (1.2) is equivalent to

$$sys_j = \begin{cases} {q_j}^1 G^1 + {q_j}^2 G^2 + {q_j}^3 G^3 + \ldots = 0, \\ {q_j}^1 G^1 - {q_j}^2 G^2 - {q_j}^3 G^3 - \ldots = 0. \end{cases} \tag{1.17}$$

The system (1.17) can be rewritten as

$$\begin{aligned} D_1 {T_j}^1 + D_2 {T_j}^2 + \ldots + D_n {T_j}^n &= 0, \\ {q_j}^1 G^1 - {q_j}^2 G^2 - {q_j}^3 G^3 - \ldots &= 0, \end{aligned} \tag{1.18}$$

where $T_j = (T_j^1, \ldots, T_j^n)$ and $Q_j = (q_j^1, q_j^2, q_j^3, \ldots)$ for $i = 1, \ldots, n$ and $j = 1, 2, \ldots$

Theorem 1.6 [73] **(Noether's theorem)**

For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there exists a corresponding vector $T^i = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = B^i - N^i(L), \quad i = 1, \ldots, n \tag{1.19}$$

which is a conserved current of the Euler-Lagrange equations $\dfrac{\delta L}{\delta u^\alpha} = 0$, $\alpha = 1, \ldots, m$ and the Noether operator associated with a Lie-Bäcklund operator X is given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{ii_1 \cdots i_s}}, \quad i = 1, \ldots, n \tag{1.20}$$

in which the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.11) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\,\frac{\partial}{\partial u_{ij_1\cdots j_s}^\alpha}\,, \quad i=1,\ldots,n, \quad \alpha=1,\ldots,m. \tag{1.21}$$

The double reduction theory results in two reductions, the first being a reduction in the number of independent variables and the second being a reduction in the order of the PDE by at least one to an ordinary differential equation (ODE) [18, 87].

When the PDE system is variational, multipliers are variational symmetries. There is a determining system for finding multipliers and hence conservation laws for any given PDE system. We resort to the invariance and multiplier approach based on the well-known result that the Euler-Lagrange operator annihilates total divergences, i.e., the defining equation is given by

$$\frac{\delta}{\delta u^\alpha}[Q_\mu G^\mu] = 0. \tag{1.22}$$

To calculate the conserved flows for each corresponding multiplier, this requires the integration (by parts) of an expression in multi-dimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. We apply the homotopy operator [5, 38, 51, 74], which is a powerful algorithmic tool (explicit formula) that originates from homological algebra and variational bi-complexes. It reduces the inversion of the total divergence operator to a standard integration of one auxiliary variable and is calculated via calculus based formulas that involve higher-order Euler-Lagrange operators [38].

Chapter 2

Reductions and Exact Solutions of some Nonlinear Schrödinger Equations

2.1 Introduction and background

Bose-Einstein condensate (BEC) [76, 77] emerged in 1995 as an example of a cold fifth state of matter called a superfluid. The particles in BEC have the coldest temperature possible, viz., zero degrees Kelvin, or absolute zero. Atoms in this state display unique characteristics. The initial idea dates back to 1924, when the physicists Bose and Einstein theorized that this other state of matter must be possible. Einstein expanded on Bose's ideas about the behaviour of light when acting as waves and particles. He applied the statistics which described how light can coalesce into a single entity (now known as a laser) and considered its impact on particles with mass. The underlying equation that describes this phenomenon is a form of a nonlinear Schrödinger equation known as the Gross-Pitaevskii (GP) equation, whose derivation is now widely available (see [9, 12, 27]).

This equation, including an external potential $V(\mathbf{r})$, is given by

$$iF_t = (k\nabla^2 + V(\mathbf{r}) + g|F|^2)F, \tag{2.1}$$

where k and g are arbitrary real constants and F is the condensate wave function of a complex order parameter. We analyse (2.1) for the one-dimensional case. Without loss of generality, we choose $k = g = 1$, so that (2.1) becomes

$$iF_t = F_{xx} + V(x)F + |F|^2 F. \tag{2.2}$$

We assume F to be of the form $F = u + iv$, so that separating (2.2) into real and imaginary parts results in the system of PDEs

$$\begin{aligned} u_t - v_{xx} - V(x)v - (u^2 + v^2)v &= 0, \\ v_t + u_{xx} + V(x)u + (u^2 + v^2)u &= 0, \end{aligned} \tag{2.3}$$

which is the version we will consider for our analysis.

We perform the double reduction procedure on (2.3) for two cases of the potential $V(x)$.

The second Schrödinger related problem we will consider is as follows.

A model that describes phenomena such as nonlinear Faraday resonance in a vertically oscillating water trough [55], propagation of magnetization waves in an easy-plane ferromagnet [21], and phase-sensitive amplification of light pulses in optical fibres [24] is governed by the parametrically damped-driven nonlinear Schrödinger equation. These types of equations arise if the dissipation coefficient and the driving strength are weak, where the driving frequency is just below the phonon band in a soliton bearing system. They exhibit localized solutions with a variety of temporal behaviours that range from stationary to periodic and chaotic [10, 11, 79].

This equation is given by [11]

$$iF_t + F_{xx} + 2|F|^2 F - F = h\bar{F} - i\gamma F, \tag{2.4}$$

where $\gamma \geq 0$ is the damping coefficient, h is the amplitude of the parametric driver (which can be assumed to be positive) and F is a wave function of a complex order parameter.

We assume F to be of the form $F = u + iv$, so that separating (2.4) into real and imaginary parts results in the system of PDEs

$$\begin{aligned} u_t + v_{xx} + 2(u^2+v^2)v + (h-1)v + \gamma u &= 0, \\ -v_t + u_{xx} + 2(u^2+v^2)u + (h-1)u - \gamma v &= 0, \end{aligned} \tag{2.5}$$

which is the version we will consider for our analysis.

We perform the double reduction procedure on (2.5) for two cases based on the relationship of the parameters γ and h.

The results for the damped-driven Schrödinger equation appear in [14].

2.2 The Gross-Pitaevskii equation

We analyse the following system of PDEs

$$\begin{aligned} G^1 &= u_t - v_{xx} - V(x)v - (u^2+v^2)v = 0, \\ G^2 &= v_t + u_{xx} + V(x)u + (u^2+v^2)u = 0, \end{aligned} \tag{2.6}$$

where G^1 and G^2 are functions satisfying (1.2).

Case 1: $V(x) = A(x)$

Equation (2.6) admits the following two Lie point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \end{aligned} \tag{2.7}$$

and the following two conserved vectors

$$\begin{aligned} T_1 &= [u^2+v^2, 2(u_x v - 2v_x u)], \\ T_2 &= [(u^2+v^2)^2 + 2(u^2+v^2)A(x) - 2(u_x^2+v_x^2), 4(u_x u_t + v_x v_t)], \end{aligned} \tag{2.8}$$

with corresponding multipliers

$$\begin{aligned} Q_1 &= [2u, 2v], \\ Q_2 &= [-4v_t, 4u_t]. \end{aligned} \tag{2.9}$$

2.2.1 A double reduction of (2.6) by $< X_1, X_2 >$

We first show that X_1 and X_2 are associated with T_2 using (1.12) for $i = 1, 2$, which is given by

$$T^* = X\binom{T^t}{T^x} - \begin{pmatrix} D_t\xi^t & D_x\xi^t \\ D_t\xi^x & D_x\xi^x \end{pmatrix}\binom{T^t}{T^x} + (D_t\xi^t + D_x\xi^x)\binom{T^t}{T^x}. \tag{2.10}$$

We have

$$\binom{T_2^{*t}}{T_2^{*x}} = X_1^{[1]}\binom{T_2^t}{T_2^x} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\binom{T_2^t}{T_2^x} + (0)\binom{T_2^t}{T_2^x} = \binom{U_1}{U_2}$$

where

$$U_1 = \frac{\partial}{\partial t}[(u^2 + v^2)^2 + 2(u^2 + v^2)A(x) - 2(u_x^2 + v_x^2)]$$

and

$$U_2 = \frac{\partial}{\partial t}[4(u_x u_t + v_x v_t)].$$

This computation shows that

$$U_1 = 0 = U_2,$$

where the prolongation of X_1 from (1.4) and (1.5) is given by

$$X_1^{[1]} = \frac{\partial}{\partial t}.$$

Therefore X_1 is associated with T_2.

Similarly for X_2,

$$\binom{T_2^{*t}}{T_2^{*x}} = X_2^{[1]}\binom{T_2^t}{T_2^x} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\binom{T_3^t}{T_3^x} + (0)\binom{T_3^t}{T_3^x} = \binom{U_1}{U_2}$$

where

$$U_1 = 4uv(u^2+v^2) + 4uvA(x) - 4uv(u^2+v^2) - 4uvA(x) - 4u_xv_x + 4u_xv_x$$

and

$$U_2 = 4u_xv_t + 4u_tv_x - 4u_tv_x - 4u_xv_t.$$

This shows that

$$U_1 = 0 = U_2,$$

where the prolongation of X_2 from (1.4) and (1.5) is given by

$$X_2^{[1]} = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} + v_t\frac{\partial}{\partial u_t} + v_x\frac{\partial}{\partial u_x} - u_t\frac{\partial}{\partial v_t} - u_x\frac{\partial}{\partial v_x}.$$

Therefore X_2 is also associated with T_2.

We can get a reduced conserved form for the first equation of (1.17) for $j = 2$, since X_1 and X_2 are both associated symmetries of T_2.

We now consider a linear combination of X_1 and X_2, i.e., of the form $X = X_1 + cX_2$ (c is an arbitrary constant) and transform this generator to its canonical form $Y = \frac{\partial}{\partial s}$, where this generator is of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial p}$.

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$ and $X(p) = 0$, we have

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{cv} = \frac{dv}{-cu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \tag{2.11}$$

Equation (2.11) is solved using the well-known method of invariance.

The invariants of X from (2.11) are given by

$$
\begin{aligned}
b_1 &= x, \\
b_2 &= u^2 + v^2, \\
b_3 &= \arctan\left(\frac{u}{v}\right) - ct, \\
b_4 &= r, \\
b_5 &= s - t, \\
b_6 &= w, \\
b_7 &= p, \qquad (2.12)
\end{aligned}
$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$ and $b_7 = b_3$, we obtain the canonical coordinates

$$
\begin{aligned}
r &= x, \\
s &= t, \\
w &= \sqrt{u^2 + v^2}, \\
p &= \arctan\left(\frac{u}{v}\right) - ct. \qquad (2.13)
\end{aligned}
$$

We note that $w = w(r)$ and $p = p(r)$.

The inverse canonical coordinates from (2.13) are given by

$$
\begin{aligned}
t &= s, \\
x &= r, \\
u &= w \sin(p + cs), \\
v &= w \cos(p + cs). \qquad (2.14)
\end{aligned}
$$

The computation of A and $(A^{-1})^T$ from (1.14) and (2.14) is given by

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} D_t r & D_t s \\ D_x r & D_x s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (A^{-1})^T,$$

where $J = \det(A) = -1$.

The partial derivatives of u and v from (2.14) are given by

$$\begin{aligned}
u_t &= cw\cos(p + cs), \\
u_x &= w_r \sin(p + cs) + wp_r \cos(p + cs), \\
v_t &= -cw \sin(p + cs), \\
v_x &= w_r \cos(p + cs) - wp_r \sin(p + cs), \\
u_{xx} &= (2w_r p_r + wp_{rr}) \cos(p + cs) + (w_{rr} - wp_r^2) \sin(p + cs), \\
v_{xx} &= (-2w_r p_r - wp_{rr}) \sin(p + cs) + (w_{rr} - wp_r^2) \cos(p + cs). \qquad (2.15)
\end{aligned}$$

We now apply the formula (1.13) for $i = 1, 2$, which is given by

$$\begin{pmatrix} T_j^r \\ T_j^s \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_j^t \\ T_j^x \end{pmatrix}. \qquad (2.16)$$

Equation (2.16) also satisfies

$$D_r T_j^r = 0. \qquad (2.17)$$

We note that (2.17) is independent of s, since $Y = \frac{\partial}{\partial s}$.

By substituting (2.14) and (2.15) into (2.16) for $j = 2$, we obtain

$$\begin{aligned} T_2^r &= -4cw^2 p_r, \\ T_2^s &= -[w^4 + 2(Aw^2 - w_r^2 - w^2 p_r^2)]. \end{aligned} \tag{2.18}$$

The next step of this procedure is to combine (2.17) with (2.18), which results in

$$w^2 p_r = k, \tag{2.19}$$

or equivalently

$$p = k \int \frac{1}{w^2} dx + m, \tag{2.20}$$

where k and m are integration constants.

Differentiating (2.19) implicitly with respect to r and then taking out a common factor of w results in

$$2w_r p_r + w p_{rr} = 0. \tag{2.21}$$

The second equation of (1.17) for $j = 2$ in simplified form is given by

$$-2u_t v_t - u_t u_{xx} + v_t v_{xx} + A(v v_t - u u_t) + (u^2 + v^2)(v v_t - u u_t) = 0. \tag{2.22}$$

After transforming (2.22) using (2.14) and (2.15), and then taking out a common factor of cw, we obtain

$$\begin{aligned} &\left[2(c - A)w + 2wp_r^2 - 2(w_{rr} + w^3)\right] \cos(p + cs) \sin(p + cs) \\ &\qquad - 2(2w_r p_r + w p_{rr}) \cos 2(p + cs) = 0. \end{aligned} \tag{2.23}$$

Then substituting (2.19) and (2.21) into (2.23), and dividing both sides by 2, this results in the nonlinear ODE

$$k^2 = w^3 w_{rr} + (c - A)w^4 - w^6. \tag{2.24}$$

Combining (2.14) and (2.20), we obtain the final solution to our original equation (2.6) as

$$
\begin{aligned}
u &= f(x)\sin\left(ct + k\int \frac{1}{f(x)^2}dx + m\right), \\
v &= f(x)\cos\left(ct + k\int \frac{1}{f(x)^2}dx + m\right),
\end{aligned}
\tag{2.25}
$$

where $w = f(x)$ is a solution of the nonlinear ODE

$$
k^2 = f(x)^3\left(\frac{d^2}{dx^2}f(x)\right) + (c - A(x))f(x)^4 - f(x)^6. \tag{2.26}
$$

Case 2: $V(x) = x^2$

In this case, (2.6) admits the following four Lie point symmetries

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \\
X_3 &= e^{2t}\frac{\partial}{\partial x} + xe^{2t}v\frac{\partial}{\partial u} - xe^{2t}u\frac{\partial}{\partial v}, \\
X_4 &= e^{-2t}\frac{\partial}{\partial x} - xe^{-2t}v\frac{\partial}{\partial u} + xe^{-2t}u\frac{\partial}{\partial v},
\end{aligned}
\tag{2.27}
$$

and the following four conserved vectors

$$
\begin{aligned}
T_1 &= \left[u^2 + v^2, 2(u_x v - v_x u)\right], \\
T_2 &= \left[(u^2 + v^2)^2 + 2(u^2 + v^2)A(x) - 2(u_x^2 + v_x^2), 4(u_x u_t + v_x v_t)\right], \\
T_3 &= \Big[4e^{2t}(xv^2 + xu^2 - u_x v + v_x u), -4e^{2t}\Big(\frac{u^4 + v^4}{2} + (x^2 + v^2)u^2 + (2xv_x + v_t)u \\
&\quad + x^2v^2 - (2xu_x + u_t)v + u_x^2 + v_x^2\Big)\Big],
\end{aligned}
$$

$$T_4 = \Big[4e^{-2t}(xv^2 + xu^2 + u_x v - v_x u), 4e^{-2t}\Big(\frac{u^4+v^4}{2} + (x^2+v^2)u^2 + (-2xv_x + v_t)u + x^2v^2 + (2xu_x - u_t)v + u_x^2 + v_x^2\Big)\Big], \qquad (2.28)$$

with corresponding multipliers

$$\begin{aligned} Q_1 &= [2u, 2v], \\ Q_2 &= [-4v_t, 4u_t], \\ Q_3 &= \Big[8e^{2t}(xu + v_x), 8e^{2t}(xv - u_x)\Big], \\ Q_4 &= \Big[8e^{-2t}(xu - v_x), 8e^{2t}(xv + u_x)\Big]. \end{aligned} \qquad (2.29)$$

2.2.2 A reduction of (2.6) by $< X_4 >$

We show that X_4 is associated with T_1 using (2.10).

We have

$$\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} = X_4^{[1]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -2e^{-2t} & 0 \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (0)\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = -2xe^{-2t}uv + 2xe^{-2t}uv$$

and

$$U_2 = 2xe^{-2t}vv_x + 2xe^{-2t}uu_x - 2e^{-2t}v(v+xv_x) - 2e^{-2t}u(u+xu_x) + 2e^{-2t}u^2 + 2e^{-2t}v^2.$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_4^{[1]} = e^{-2t}\frac{\partial}{\partial x} - xe^{-2t}v\frac{\partial}{\partial u} + xe^{-2t}u\frac{\partial}{\partial v} - e^{-2t}(v + xv_x)\frac{\partial}{\partial u_x} + e^{-2t}(u + xu_x)\frac{\partial}{\partial v_x}.$$

Therefore X_4 is associated with T_1.

We can get a reduced conserved form for the first equation of (1.17) for $j = 1$, since X_4 is an associated symmetry of T_1.

We transform the generator X_4 to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X_4(r) = 0$, $X_4(s) = 1$, $X_4(w) = 0$ and $X_4(p) = 0$, we have

$$\frac{dt}{0} = \frac{dx}{e^{-2t}} = \frac{du}{-xe^{-2t}v} = \frac{dv}{xe^{-2t}u} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \qquad (2.30)$$

The invariants of X_4 from (2.30) are given by

$$\begin{aligned} b_1 &= t, \\ b_2 &= u^2 + v^2, \\ b_3 &= \arctan\left(\frac{v}{u}\right) - \frac{x^2}{2}, \\ b_4 &= r, \\ b_5 &= s - xe^{2t}, \\ b_6 &= w, \\ b_7 &= p, \end{aligned} \qquad (2.31)$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$ and $b_7 = b_3$, we obtain the canonical coordinates

$$\begin{aligned} r &= t, \\ s &= xe^{2t}, \\ w &= \sqrt{u^2 + v^2}, \end{aligned}$$

$$p = \arctan\left(\frac{v}{u}\right) - \frac{x^2}{2}. \tag{2.32}$$

The inverse canonical coordinates from (2.32) are given by

$$\begin{aligned} t &= r, \\ x &= se^{-2r}, \\ u &= w\cos\left(p + \frac{s^2e^{-4r}}{2}\right), \\ v &= w\sin\left(p + \frac{s^2e^{-4r}}{2}\right). \end{aligned} \tag{2.33}$$

The computation of A and $(A^{-1})^T$ is given by

$$A = \begin{pmatrix} 1 & -2se^{-2r} \\ 0 & e^{-2r} \end{pmatrix}$$

and

$$(A^{-1})^T = \begin{pmatrix} 1 & 0 \\ 2s & e^{2r} \end{pmatrix}$$

where $J = e^{-2r}$.

The partial derivatives of u and v from (2.33) are given by

$$\begin{aligned} u_t &= w_r\cos\left(p + \frac{s^2e^{-4r}}{2}\right) - wp_r\sin\left(p + \frac{s^2e^{-4r}}{2}\right), \\ u_x &= -se^{-2r}w\sin\left(p + \frac{s^2e^{-4r}}{2}\right), \\ v_t &= w_r\sin\left(p + \frac{s^2e^{-4r}}{2}\right) + wp_r\cos\left(p + \frac{s^2e^{-4r}}{2}\right), \\ v_x &= se^{-2r}w\cos\left(p + \frac{s^2e^{-4r}}{2}\right), \\ u_{xx} &= -w\sin\left(p + \frac{s^2e^{-4r}}{2}\right) - s^2e^{-4r}w\cos\left(p + \frac{s^2e^{-4r}}{2}\right), \\ v_{xx} &= w\cos\left(p + \frac{s^2e^{-4r}}{2}\right) - s^2e^{-4r}w\sin\left(p + \frac{s^2e^{-4r}}{2}\right). \end{aligned} \tag{2.34}$$

By substituting (2.33) and (2.34) into (2.16) for $j = 1$, we obtain

$$\begin{aligned} T_1^r &= e^{-2r} w^2, \\ T_1^s &= 0. \end{aligned} \tag{2.35}$$

Solving (2.17) and (2.35) simultaneously results in

$$e^{-2r} w^2 = k, \tag{2.36}$$

or equivalently

$$w = \sqrt{k} e^r, \tag{2.37}$$

where k is an integration constant.

Differentiating (2.36) implicitly with respect to r and then taking out a common factor of $-2e^{-2r}w$ results in

$$w - w_r = 0. \tag{2.38}$$

The second equation of (1.17) for $j = 1$ in simplified form is given by

$$uu_t - vv_t - uv_{xx} + vu_{xx} - 2uv(x^2 + u^2 + v^2) = 0. \tag{2.39}$$

After transforming (2.39) using (2.33) and (2.34), and then taking out a common factor of w, we obtain

$$-2w(p_r + w^2) \sin\left(p + \frac{s^2 e^{-4r}}{2}\right) \cos\left(p + \frac{s^2 e^{-4r}}{2}\right) + w - w_r = 0. \tag{2.40}$$

We now substitute (2.36) and (2.38) into (2.40).

This results in the ODE

$$p_r + ke^{2r} = 0. \tag{2.41}$$

After integrating (2.41) with respect to r, we obtain

$$p = -\frac{k}{2}e^{2r} + m, \tag{2.42}$$

where m is an integration constant.

Combining (2.33), (2.37) and (2.42), we obtain the final solution to our original equation (2.6) as

$$\begin{aligned} u &= \sqrt{k}e^t \cos\left(-\frac{k}{2}e^{2t} + m + \frac{x^2}{2}\right), \\ v &= \sqrt{k}e^t \sin\left(-\frac{k}{2}e^{2t} + m + \frac{x^2}{2}\right). \end{aligned} \tag{2.43}$$

2.3 The parametrically damped-driven Schrödinger equation

In this section, we analyse the following system of PDEs

$$\begin{aligned} G^1 &= u_t + v_{xx} + 2(u^2+v^2)v + (h-1)v + \gamma u \ = \ 0, \\ G^2 &= -v_t + u_{xx} + 2(u^2+v^2)u + (h-1)u - \gamma v \ = \ 0. \end{aligned} \tag{2.44}$$

Case 1: $\gamma \neq 0, h \neq 0$

Equation (2.44) admits a four-dimensional Lie point symmetry algebra spanned by

$$\begin{aligned} X_1 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \\ X_2 &= \frac{\partial}{\partial t} - (h-1)v\frac{\partial}{\partial u} + (h-1)u\frac{\partial}{\partial v}, \end{aligned}$$

$$
\begin{aligned}
X_3 &= \frac{\partial}{\partial x}, \\
X_4 &= 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v},
\end{aligned} \tag{2.45}
$$

and only one conserved vector

$$
\begin{aligned}
T_1 &= \left[\frac{1}{2}e^{2\gamma t}(uv_x - vu_x), \frac{1}{2}e^{2\gamma t}((u^2+v^2)^2 + (h-1)(u^2+v^2)\right. \\
&\left. + vu_t - uv_t + u_x^2 + v_x^2)\right],
\end{aligned} \tag{2.46}
$$

with corresponding multiplier

$$
Q_1 = [e^{2\gamma t}v_x, e^{2\gamma t}u_x]. \tag{2.47}
$$

2.3.1 A reduction of (2.44) by $< X_1, X_3 >$

We show that X_1 and X_3 are associated with T_1.

We have

$$
\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} = X_1^{[1]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (0)\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}
$$

where

$$
U_1 = \frac{1}{2}e^{2\gamma t}(-vv_x - uu_x + vv_x + uu_x)
$$

and

$$
\begin{aligned}
U_2 &= \frac{1}{2}e^{2\gamma t}[-4uv(u^2+v^2) - 2(h-1)uv + vv_t + 4uv(u^2+v^2) + 2(h-1)uv \\
&+ uu_t - vv_t - 2u_xv_x - uu_t + 2u_xv_x].
\end{aligned}
$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_1^{[1]} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} - v_t\frac{\partial}{\partial u_t} - v_x\frac{\partial}{\partial u_x} + u_t\frac{\partial}{\partial v_t} + u_x\frac{\partial}{\partial v_x}.$$

Therefore X_1 is associated with T_1.

Similarly for X_3,

$$\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} = X_3^{[1]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (0)\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{\partial}{\partial x}\left[\frac{1}{2}e^{2\gamma t}(uv_x - vu_x)\right]$$

and

$$U_2 = \frac{\partial}{\partial x}\left[\frac{1}{2}e^{2\gamma t}((u^2+v^2)^2 + (h-1)(u^2+v^2) + vu_t - uv_t + u_x^2 + v_x^2)\right].$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_3^{[1]} = \frac{\partial}{\partial x}.$$

Therefore X_3 is also associated with T_1.

We consider a linear combination of X_1 and X_3, i.e., of the form $X = X_3 + cX_1$ (c is an arbitrary constant) and transform this generator to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$ and $X(p) = 0$, we have

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{-cv} = \frac{dv}{cu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \qquad (2.48)$$

The invariants of X from (2.48) are given by

$$
\begin{aligned}
b_1 &= t, \\
b_2 &= u^2 + v^2, \\
b_3 &= \arctan\left(\frac{v}{u}\right) - cx, \\
b_4 &= r, \\
b_5 &= s - x, \\
b_6 &= w, \\
b_7 &= p,
\end{aligned}
\tag{2.49}
$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$ and $b_7 = b_3$, we obtain the canonical coordinates

$$
\begin{aligned}
r &= t, \\
s &= x, \\
w &= \sqrt{u^2 + v^2}, \\
p &= \arctan\left(\frac{v}{u}\right) - cs.
\end{aligned}
\tag{2.50}
$$

The inverse canonical coordinates from (2.50) are given by

$$
\begin{aligned}
t &= r, \\
x &= s, \\
u &= w\cos(p + cs), \\
v &= w\sin(p + cs).
\end{aligned}
\tag{2.51}
$$

The computation of A and $(A^{-1})^T$ is given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (A^{-1})^T,$$

where $J = 1$.

The partial derivatives of u and v from (2.51) are given by

$$\begin{aligned} u_t &= w_r \cos(p+cs) - wp_r \sin(p+cs), \\ u_x &= -cw \sin(p+cs), \\ v_t &= w_r \sin(p+cs) + wp_r \cos(p+cs), \\ v_x &= cw \cos(p+cs), \\ u_{xx} &= -c^2 w \cos(p+cs), \\ v_{xx} &= -c^2 w \sin(p+cs). \end{aligned} \tag{2.52}$$

By substituting (2.51) and (2.52) into (2.16) for $j = 1$, we obtain

$$\begin{aligned} T_1^r &= \frac{1}{2} c e^{2\gamma r} w^2, \\ T_1^s &= \frac{1}{2} e^{2\gamma r} [w^4 + (h-1)w^2 - p_r w^2 + c^2 w^2]. \end{aligned} \tag{2.53}$$

Solving (2.17) and (2.53) simultaneously results in

$$e^{2\gamma r} w^2 = k, \tag{2.54}$$

or equivalently

$$w = \sqrt{k} e^{-\gamma r}, \tag{2.55}$$

where k is an integration constant.

Differentiating (2.54) implicitly with respect to r and then taking out a common factor of $2e^{2\gamma r}w$ results in

$$\gamma w + w_r = 0. \tag{2.56}$$

The second equation of (1.17) for $j = 1$ is given by

$$\begin{aligned} & e^{2\gamma t}v_x\Big[u_t + v_{xx} + 2(u^2+v^2)v + (h-1)v + \gamma u\Big] \\ & \quad - e^{2\gamma t}u_x\Big[-v_t + u_{xx} + 2(u^2+v^2)u + (h-1)u - \gamma v\Big] = 0. \end{aligned} \tag{2.57}$$

After transforming (2.57) using (2.51) and (2.52), we obtain

$$\begin{aligned} & -2ce^{2\gamma r}w^2\Big[p_r + c^2 - 2w^2 - (h-1)\Big]\cos(p+cs)\sin(p+cs) \\ & \quad + ce^{2\gamma r}w(\gamma w + w_r)\cos 2(p+cs) = 0. \end{aligned} \tag{2.58}$$

We now substitute (2.54) and (2.56) into (2.58), then dividing both sides by $-2ck$, this results in the ODE

$$p_r = 2w^2 + h - 1 - c^2. \tag{2.59}$$

Substituting (2.55) into (2.59), and then integrating with respect to r results in

$$p = \frac{-k}{\gamma}e^{-2\gamma r} + (h-1-c^2)r + m, \tag{2.60}$$

where m is an integration constant.

Combining (2.51), (2.55) and (2.60), we obtain the final solution to our original equation (2.44) as

$$\begin{aligned} u &= \sqrt{k}e^{-\gamma t}\cos\left(\frac{-k}{\gamma}e^{-2\gamma t} + (h-1-c^2)t + m + cx\right), \\ v &= \sqrt{k}e^{-\gamma t}\sin\left(\frac{-k}{\gamma}e^{-2\gamma t} + (h-1-c^2)t + m + cx\right). \end{aligned} \tag{2.61}$$

Case 2: $\gamma \neq 0, h = 0$

In this case, (2.44) admits a four-dimensional Lie point symmetry algebra spanned by

$$
\begin{aligned}
X_1 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},\\
X_2 &= \frac{\partial}{\partial t} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},\\
X_3 &= \frac{\partial}{\partial x},\\
X_4 &= 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v},
\end{aligned} \tag{2.62}
$$

and the following two conserved vectors

$$
\begin{aligned}
T_1 &= \Big[-\frac{1}{2}e^{2\gamma t}(x(u^2+v^2) + 2t(vu_x - uv_x)),\\
&\quad e^{2\gamma t}(tu^4 - tv^2 + tv^4 + tu^2(-1+2v^2) + v(tu_t + xu_x))\\
&\quad - u(tv_t + xv_x) + t(u_x^2 + v_x^2)\Big],\\
T_2 &= \left[\frac{1}{2}e^{2\gamma t}(u^2+v^2), e^{2\gamma t}(-vu_x + uv_x)\right],
\end{aligned} \tag{2.63}
$$

with corresponding multipliers

$$
\begin{aligned}
Q_1 &= \left[-xue^{2\gamma t}v_x + 2te^{2\gamma t}v_x, xve^{2\gamma t} + 2te^{2\gamma t}u_x\right],\\
Q_2 &= \left[e^{2\gamma t}u, -e^{2\gamma t}v\right].
\end{aligned} \tag{2.64}
$$

2.3.2 A reduction of (2.44) by $< X_4 >$

We show that X_4 is associated with T_2.

We have

$$\begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} = X_4^{[1]}\begin{pmatrix} T_2^{t} \\ T_2^{x} \end{pmatrix} - \left(\begin{array}{cc} 0 & 0 \\ 2 & 0 \end{array} \right) \begin{pmatrix} T_2^{t} \\ T_2^{x} \end{pmatrix} + (0)\begin{pmatrix} T_2^{t} \\ T_2^{x} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{1}{2}e^{2\gamma t}(-2xuv + 2xuv)$$

and

$$U_2 = e^{2\gamma t}\big(- xvv_x - xuu_x + xvv_x + v^2 + xuu_x + u^2 - u^2 - v^2\big).$$

Thus

$$U_1 = 0 = U_2,$$

where

$$\begin{aligned} X_4^{[1]} &= 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v} - (xv_t + 2u_x)\frac{\partial}{\partial u_t} - (xv_x + v)\frac{\partial}{\partial u_x} \\ &\quad + (xu_t - 2v_x)\frac{\partial}{\partial v_t} + (xu_x + u)\frac{\partial}{\partial v_x}. \end{aligned}$$

Therefore X_4 is associated with T_2.

We transform the generator X_4 to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X_4(r) = 0$, $X_4(s) = 1$, $X_4(w) = 0$ and $X_4(p) = 0$, we have

$$\frac{dt}{0} = \frac{dx}{2t} = \frac{du}{-xv} = \frac{dv}{xu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \qquad (2.65)$$

The invariants of X_4 from (2.65) are given by

$$\begin{aligned} b_1 &= t, \\ b_2 &= u^2 + v^2, \\ b_3 &= \arctan\left(\frac{v}{u}\right) - \frac{x^2}{4t}, \end{aligned}$$

$$\begin{aligned}
b_4 &= r, \\
b_5 &= s - \frac{x}{2t}, \\
b_6 &= w, \\
b_7 &= p, \qquad (2.66)
\end{aligned}$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$ and $b_7 = b_3$, we obtain the canonical coordinates

$$\begin{aligned}
r &= t, \\
s &= \frac{x}{2t}, \\
w &= \sqrt{u^2 + v^2}, \\
p &= \arctan\left(\frac{v}{u}\right) - \frac{x^2}{4t}. \qquad (2.67)
\end{aligned}$$

The inverse canonical coordinates from (2.67) are given by

$$\begin{aligned}
t &= r, \\
x &= 2rs, \\
u &= w\cos(p + rs^2), \\
v &= w\sin(p + rs^2). \qquad (2.68)
\end{aligned}$$

The computation of A and $(A^{-1})^T$ is given by

$$A = \begin{pmatrix} 1 & 2s \\ 0 & 2r \end{pmatrix}$$

and

$$(A^{-1})^T = \begin{pmatrix} 1 & 0 \\ -\frac{s}{r} & \frac{1}{2r} \end{pmatrix}$$

where $J = 2r$.

The partial derivatives of u and v from (2.68) are given by

$$\begin{aligned}
u_t &= w_r \cos(p + rs^2) - wp_r \sin(p + rs^2) + s^2 w \sin(p + rs^2),\\
u_x &= -swr \sin(p + rs^2),\\
v_t &= w_r \sin(p + rs^2) + wp_r \cos(p + rs^2) - s^2 w \cos(p + rs^2),\\
v_x &= sw \cos(p + rs^2),\\
u_{xx} &= -\frac{w \sin(p + rs^2)}{2r} - s^2 w \cos(p + rs^2),\\
v_{xx} &= \frac{w \cos(p + rs^2)}{2r} - s^2 w \sin(p + rs^2).
\end{aligned} \tag{2.69}$$

By substituting (2.68) and (2.69) into (2.16) for $j = 2$, we obtain

$$\begin{aligned}
T_2^r &= re^{2\gamma r} w^2,\\
T_2^s &= 0.
\end{aligned} \tag{2.70}$$

Solving (2.17) and (2.70) simultaneously results in

$$re^{2\gamma r} w^2 = k, \tag{2.71}$$

or equivalently

$$w = \sqrt{\frac{k}{r}} e^{-\gamma r}, \tag{2.72}$$

where k is an integration constant.

Differentiating (2.71) implicitly with respect to r and then taking out a common factor of $e^{2\gamma r} w$ results in

$$2rw_r + (2\gamma r + 1)w = 0, \tag{2.73}$$

or equivalently after dividing both sides by $2r$

$$w_r + \frac{w}{2r} + \gamma w = 0. \tag{2.74}$$

The second equation of (1.17) for $j = 2$ is given by

$$e^{2\gamma t}\Big[uu_t - vv_t + uv_{xx} + vu_{xx} + 2uv(2(u^2+v^2)-1) + \gamma(u^2 - v^2)\Big] = 0. \tag{2.75}$$

After transforming (2.75) using (2.68) and (2.69), we obtain

$$\begin{aligned} &e^{2\gamma r}(-2wp_r + 4w^3 - 2w)\cos(p+rs^2)\sin(p+rs^2) \\ &\quad + e^{2\gamma r}\left(w_r + \frac{w}{2r} + \gamma w\right)\cos 2(p+rs^2) = 0. \end{aligned} \tag{2.76}$$

After multiplying both sides of (2.76) by $-\frac{w}{2}$, and then substituting (2.72) and (2.74) into (2.76), this results in the ODE

$$p_r = 2w^2 - 1. \tag{2.77}$$

Substituting (2.72) into (2.77), and then integrating with respect to r results in

$$p = 2k\int \frac{e^{-2\gamma r}}{r}dr - r. \tag{2.78}$$

We note that $\int \frac{e^{-2\gamma r}}{r}dr = \ln r + \sum_{j=1}^{\infty}\frac{(-1)^j(2\gamma r)^j}{jj!} + m$, where m is an integration constant.

Combining (2.68), (2.72) and (2.78), we obtain the final solution to our original equation (2.44) as

$$\begin{aligned} u &= \sqrt{\frac{k}{t}}e^{-\gamma t}\cos\left(p + \frac{x^2}{4t}\right), \\ v &= \sqrt{\frac{k}{t}}e^{-\gamma t}\sin\left(p + \frac{x^2}{4t}\right), \end{aligned} \tag{2.79}$$

where $p = 2k\left(\ln t + \sum_{j=1}^{\infty} \frac{(-1)^j (2\gamma t)^j}{j j!} + m\right) - t$.

2.4 Discussion and conclusion

We applied the double reduction procedure to the Gross-Pitaevskii equation for two cases of the potential $V(x)$. In the first case, we obtained a new exact solution that can be given as a solution of the nonlinear ODE (2.26) for an arbitrary function $f(x)$. In the second case, we obtained a new exact solution which approximates to e^t.

The same procedure was also applied to the parametrically damped-driven Schrödinger equation for two cases on the relationship of the parameters γ and h. In the first case, we obtained a new exact solution which approximates to $e^{-\gamma t}$ and in the second case, we obtained a new exact solution which approximates to $\frac{e^{-\gamma t}}{\sqrt{t}}$.

Chapter 3

Some Classes of Third-order PDEs Related to the KdV Equation

3.1 Introduction and background

The well-known KdV equation and its variations have been extensively studied and analysed in many texts through different numerical and analytical approaches (see [38, 74, 89, 93] and references therein). In order to study the dynamics of shallow water waves, the general improved KdV equation models this phenomenon in detail.

This equation is given by [6, 35, 54]

$$u_t + au^n u_x + bu_{xxt} + cu_{xxx} = 0, \tag{3.1}$$

for $n \neq 0, -1, -2$.

The coefficients b and c of (3.1) relate to the dispersion terms, where b accounts for the improved KdV equation and a represents the power law nonlinearity. If $b = 0$, then (3.1) reduces to the regular KdV equation. We analyse another class of nonlinear wave equations related to (3.1) but with greater generality that studies shallow water waves in lake or ocean shores [53].

The version we will consider is given by

$$au_t - 2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx} = 0, \tag{3.2}$$

where $m(u), a, k \neq 0$.

In [53], the authors study various cases of (3.2) construing it as one that is "lying 'mid-way' between the periodic Hunter-Saxton and Camassa-Holm equations, and which describes evolution of rotators in liquid crystals with external magnetic field and self-interaction."

We calculate the Lie point symmetries and apply the invariance and multiplier approach on (3.2) for two cases of the parameter a. When calculating the conservation laws, we will also consider two choices of the function $m(u)$. One case of the double reduction procedure will be performed on (3.2) using a specific choice of $m(u)$.

The results for the Hunter-Saxton type equation appear in [67].

We will consider a version of the well-known standard KdV equation given by

$$u_t - u_{xxx} - uu_x = 0. \tag{3.3}$$

Since the travelling wave solution is well-known for (3.3), we do not perform the double reduction for this case. Instead, we consider a reduction via a scaling symmetry with some conserved vector.

A system of KdV type equations that we will consider is as follows.

Drinfeld and Sokolov, followed by Wilson, constructed an equation involving affine Lie algebras [25] and the affine Kac-Moody Lie algebra $C_2^{(1)}$ [95] called the Drinfeld-Sokolov-Wilson (DSW) equation [4, 38, 74, 91]. The DSW equation is an extension of the KdV equation and it is a member of the Kadomtsev-Petviashvili hierarchy, which confirms its integrability [46]. The soliton structure and Painlevé analysis was analysed for this equation in [39], while its recursive operator and bi-Hamiltonian formulation was given in [34]. The solutions to this equation are very unusual which are called static solitons; these are static solutions that interact with moving solitons without deformations. The generalized DSW equation was recently analysed in [88].

We consider the version given by the system of PDEs [71]

$$\begin{aligned} u_t + 2vv_x &= 0, \\ v_t - av_{xxx} + 3bu_xv + 3kuv_x &= 0, \end{aligned} \tag{3.4}$$

where a, b and k are arbitrary real constants.

In [52], the relationship between the Lie point symmetries and the multipliers of (3.4) was investigated. In [71], the invariance and multiplier approach was used to obtain additional conserved forms of (3.4) for special cases of the parameters, and therefore possibilities for additional solutions may exist. We perform the double reduction procedure based on this property of the special cases.

The results for the standard KdV and DSW equations appear in [68].

3.2 The Hunter-Saxton type equation

We analyse the following scalar PDE

$$G = au_t - 2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx} = 0. \qquad (3.5)$$

Case 1: $a \neq 0$

By (1.3), we define $X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u}$ to be the Lie-Bäcklund operator that leaves invariant (3.5), i.e.,

$$X^{[3]}(au_t - 2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx}) = 0, \qquad (3.6)$$

where $\tau = \tau(t, x, u)$, $\xi = \xi(t, x, u)$ and $\phi = \phi(t, x, u)$.

The governing equations of (3.6) are obtained by using the computer algebra system (CAS) package Mathematica for the separation of monomials and solving the over determined system of PDEs.

The calculations reveal that the principal Lie algebra of Lie point symmetries of (3.5) is given by $\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right\rangle$.

In the process of separating the monomials of (3.6), it turns out that the case $m = u$ admits an additional generator given by

$$Z = \frac{(2+a)}{k}t\frac{\partial}{\partial t} + 2t\frac{\partial}{\partial x} - \frac{[ak + (2+a)u]}{k}\frac{\partial}{\partial u}. \qquad (3.7)$$

We determine the possible existence of higher-order multipliers and corresponding conserved vectors via the invariance and multiplier approach.

By (1.9), we require

$$q_j G = D_t T^t + D_x T^x$$

so that

$$\frac{\delta}{\delta u}[q_j(au_t - 2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx})] = 0, \tag{3.8}$$

since the Euler-Lagrange operator annihilates total divergences.

We assume the multiplier q_j to be of second-order derivative dependence, i.e.,

$$q_j = g(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}).$$

Equation (3.8) has to be satisfied for all functions $u(x,t)$, not only the solutions of (3.5).

The expansion of the left hand side of (3.8) is extensive and requires the use of the CAS package Maple to enumerate, particularly in the separation of the monomials and solving the over determined system of PDEs.

The calculations after expansion and separation by monomials of (3.8) reveals that

$$q_j = a_1 + a_2 u + a_3\left[\frac{1}{2}(2u_{xt} + 2uu_{xx} + 2ku_{xx} + u_x^2) - 2\int m(u)\mathrm{d}u\right]. \tag{3.9}$$

To calculate the conservation laws, the conserved densities and fluxes are calculated by using the homotopy operator [38].

The choice functions $m = u$ and $m = \cos u$ are merely used for illustrative purposes to demonstrate cases for the general function $m(u)$.

(a) $q_1 = 1$

$m = \cos u$

$$
\begin{aligned}
T_1^t &= au + \frac{1}{3}u_{xx}, \\
T_1^x &= -2\sin u + \frac{1}{2}{u_x}^2 + \frac{2}{3}u_{xt} + ku_{xx} + uu_{xx},
\end{aligned}
$$

$m = u$

$$
\begin{aligned}
T_1^t &= au + \frac{1}{3}u_{xx}, \\
T_1^x &= -u^2 + \frac{1}{2}{u_x}^2 + \frac{2}{3}u_{xt} + ku_{xx} + uu_{xx},
\end{aligned}
$$

(b) $q_2 = u$

$m = \cos u$

$$
\begin{aligned}
T_2^t &= \frac{1}{6}(3au^2 - {u_x}^2 + 2uu_{xx}), \\
T_2^x &= 2 - 2\cos u - \frac{1}{3}u_t u_x - \frac{1}{2}k{u_x}^2 + u^2 u_{xx} + u\left(-2\sin u + \frac{2}{3}u_{xt} + ku_{xx}\right),
\end{aligned}
$$

$m = u$

$$
\begin{aligned}
T_2^t &= \frac{1}{6}(3au^2 - {u_x}^2 + 2uu_{xx}), \\
T_2^x &= \frac{1}{6}[-4u^3 - u_x(2u_t + 3ku_x) + 6u^2 u_{xx} + u(4u_{xt} + 6ku_{xx})],
\end{aligned}
$$

(c) $q_3 = \frac{1}{2}(2u_{xt} + 2uu_{xx} + 2ku_{xx} + u_x^2) - 2\int m(u)\mathrm{d}u$

$m = \cos u$

$$
\begin{aligned}
T_3^t \;=\; & \frac{1}{36u^2}\Big[-36(-1+\cos u){u_x}^2 - 36u(\sin u{u_x}^2 - (-1+\cos u)u_{xx}) + 3u^2(3au_tu_x \\
& + 2{u_x}^2(2\cos u + u_{xx}) + 2(12a(-1+\cos u) + (2\sin u + u_{xt})u_{xx} + k{u_{xx}}^2) \\
& + u_x(u_{xxt} + ku_{xxx})) + 2u^4(6au_{xx} - u_{xxxx}) + u^3(6a{u_x}^2 + 9au_{xt} + 18aku_{xx} \\
& - 4u_xu_{xxx} - 3u_{xxxt} - 3ku_{xxxx})\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_3^x \;=\; & \frac{1}{72u^2}\Big[72(-1+\cos u)u_tu_x + 72u(\sin u u_tu_x - (-1+\cos u)u_{xt}) \\
& + 3u^2(6a{u_t}^2 + 3{u_x}^4 + {u_x}^2(-24\sin u + 8u_{xt} + 12ku_{xx}) \\
& + 4(2{u_{xt}}^2 + 3(-2\sin u + ku_{xx})^2 + u_{xt}(-14\sin u + 5ku_{xx})) - 4u_x(u_{xtt} \\
& + ku_{xxt}) + 2u_t((6ak - 4\cos u)u_x + u_{xxt} + ku_{xxx})) + 4u^4(-6au_{xt} + 9{u_{xx}}^2 + u_{xxxt}) \\
& + u^3(-18au_{tt} + 8u_t(3au_x + u_{xxx}) + 6(-6u_{xt}(ak - 2u_{xx}) + 6(-4\sin u + {u_x}^2)u_{xx} \\
& + 12k{u_{xx}}^2 + u_{xxtt} + ku_{xxxt})\Big],
\end{aligned}
$$

$m = u$

$$
\begin{aligned}
T_3^t \;=\; & \frac{1}{36}\Big[-12au^3 + 3(3au_tu_x + 2{u_x}^2u_{xx} + 2u_{xx}(u_{xt} + ku_{xx}) + u_x(u_{xxt} + ku_{xxx})) \\
& + 2u^2(6au_{xx} - u_{xxxx}) + u(6a{u_x}^2 + 9au_{xt} + 18aku_{xx} - 4u_xu_{xxx} \\
& - 3u_{xxxt} - 3ku_{xxxx})\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_3^x \;=\; & \frac{1}{72}\Big[36u^4 - 72u^3u_{xx} + 3(6a{u_t}^2 + 3{u_x}^4 + 4{u_x}^2(2u_{xt} + 3ku_{xx}) + 4(2{u_{xt}}^2 \\
& + 5ku_{xt}u_{xx} + 3k^2{u_{xx}}^2) - 4u_x(u_{xtt} + ku_{xxt}) + 2u_t(6aku_x + u_{xxt} + ku_{xxx})) \\
& - 4u^2(9{u_x}^2 + 6(3+a)u_{xt} + 18ku_{xx} - 9{u_{xx}}^2 - u_{xxxt}) + u(-18au_{tt} \\
& + 8u_t(3au_x + u_{xxx}) + 6(-6u_{xt}(ak - 2u_{xx}) + 6{u_x}^2u_{xx} + 12k{u_{xx}}^2 + u_{xxtt} + ku_{xxxt})\Big].
\end{aligned}
$$

Case 2: $a = 0$

Equation (3.5) is reduced to

$$G = -2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx} = 0. \qquad (3.10)$$

We now solve

$$X^{[3]}(-2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx}) = 0. \qquad (3.11)$$

The calculations reveal that the principal Lie algebra of Lie point symmetries of (3.10) is given by $\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right\rangle$.

In the process of separating the monomials of (3.11), it turns out that the general function $m(u)$ admits additional Lie point symmetries for the following choices of $m(u)$

(i) $m = u$: $X_1 = -kt\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$,

(ii) $m = u^\beta$: $X_2 = \left(\frac{2kt}{\beta - 1} + x\right)\frac{\partial}{\partial x} + \frac{(1+\beta)}{(\beta - 1)}t\frac{\partial}{\partial t} - \frac{2}{\beta - 1}u\frac{\partial}{\partial u}$,

(iii) $m = e^u$: $X_3 = (-2t + x)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}$,

where in (ii), $\beta \neq 0, 1$.

We now solve

$$\frac{\delta}{\delta u}\left[q_j(-2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx})\right] = 0, \qquad (3.12)$$

where we also assume q_j to be of second-order derivative dependence.

Equation (3.12) has to be satisfied for all functions $u(x,t)$, not only the solutions of (3.10).

The calculations after expansion and separation by monomials of (3.12) reveals that

$$q_j = a_1 u + F_1(t) + F_2\Big(t, -\int 2m(u)\mathrm{d}u + u_{xt} + uu_{xx} + ku_{xx} + \frac{1}{2}u_x^2\Big). \tag{3.13}$$

We use the choice functions $m = u$ and $m = e^u$ for illustrative purposes to demonstrate cases for the general function $m(u)$.

In (c), we just state the conserved densities.

(a) $q_1 = g(t)$

$m = u$

$$\begin{aligned} T_1^t &= \frac{1}{3}g(t)u_{xx}, \\ T_1^x &= \frac{1}{6}\Big[-2g'u_x + g(t)(-6u^2 + 3{u_x}^2 + 4u_{xt} + 6ku_{xx} + 6uu_{xx})\Big], \end{aligned}$$

$m = e^u$

$$\begin{aligned} T_1^t &= \frac{1}{3}g(t)u_{xx}, \\ T_1^x &= -\frac{1}{3}g'u_x + g(t)\Big(2 - 2e^u + \frac{1}{2}{u_x}^2 + \frac{2}{3}u_{xt} + ku_{xx} + uu_{xx}\Big), \end{aligned}$$

(b) $q_2 = u$

$m = u$

$$\begin{aligned} T_2^t &= \frac{1}{6}(-{u_x}^2 + 2uu_{xx}), \\ T_2^x &= \frac{1}{6}\Big[-4u^3 - u_x(2u_t + 3ku_x) + 6u^2u_{xx} + u(4u_{xt} + 6ku_{xx})\Big], \end{aligned}$$

$m = e^u$

$$\begin{aligned} T_2^t &= \frac{1}{6}(-{u_x}^2 + 2uu_{xx}), \\ T_2^x &= -2 + 2e^u - \frac{1}{3}u_tu_x - \frac{1}{2}k{u_x}^2 + u^2u_{xx} + u\Big(- 2e^u + \frac{2}{3}u_{xt} + ku_{xx}\Big), \end{aligned}$$

(c) $q_3 = -\int 2m(u)\mathrm{d}u + u_{xt} + uu_{xx} + ku_{xx} + \frac{1}{2}u_x^2$

$m = u$

$$\begin{aligned} T_3^t &= \frac{1}{36}\Big[6{u_x}^2u_{xx} + 6u_{xt}u_{xx} + 6k{u_{xx}}^2 + u_x(3u_{xxt} + (3k - 4u)u_{xxx}) \\ &\quad - 3uu_{xxxt} - 3kuu_{xxxx} - 2u^2u_{xxxx}\Big], \end{aligned}$$

$m = e^u$

$$\begin{aligned} T_3^t &= \frac{1}{36u^2}\Big[6{u_x}^2(6(-1 + e^u) - 6e^uu + u^2(2e^u + u_{xx})) + u^2u_x(3u_{xxt} \\ &\quad + (3k - 4u)u_{xxx}) - u(- 6(6 - 6e^u + u(2e^u + u_{xt}))u_{xx} \\ &\quad - 6kuu_{xx}{}^2 + u^2(3u_{xxxt} + (3k + 2u)u_{xxxx}))\Big]. \end{aligned}$$

3.2.1 A reduction of (3.10) by $< X_1 >$

We perform the double reduction procedure for case (a) where $m = u$ using X_1.

Without loss of generality, we choose $g(t) = t$.

We show that X_1 is associated with T_1.

We have

$$\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} = X_1^{[2]} \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \left(\begin{array}{cc} -1 & 0 \\ -k & 0 \end{array} \right) \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = -\frac{1}{3}tu_{xx} + \frac{1}{3}tu_{xx}$$

and

$$\begin{aligned} U_2 &= tu^2 - \frac{1}{2}tu_x^2 - \frac{2}{3}tu_{xt} - ktu_{xx} - tuu_{xx} - 2tu^2 + tuu_{xx} - \frac{1}{3}u_x + tu_x^2 + ktu_{xx} + tuu_{xx} \\ &\quad + \frac{4}{3}tu_{xt} + \frac{2}{3}ktu_{xx} + \frac{1}{3}ktu_{xx} + \frac{1}{3}u_x + tu^2 - \frac{1}{2}tu_x^2 - \frac{2}{3}tu_{xt} - ktu_{xx} - tuu_{xx}. \end{aligned}$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_1^{[2]} = -t\frac{\partial}{\partial t} - kt\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_{xx}\frac{\partial}{\partial u_{xx}} + (2u_{xt} + ku_{xx})\frac{\partial}{\partial u_{xt}}.$$

Therefore X_1 is associated with T_1.

As in the second chapter, we transform the generator X_1 to its canonical form $Y = \frac{\partial}{\partial s}$, where this generator is of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}$.

From $X_1(r) = 0$, $X_1(s) = 1$ and $X_1(w) = 0$, we have

$$\frac{dt}{-t} = \frac{dx}{-kt} = \frac{du}{u} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}. \tag{3.14}$$

The invariants of X_1 from (3.14) are given by

$$\begin{aligned} b_1 &= kt - x, \\ b_2 &= tu, \\ b_3 &= r, \\ b_4 &= s + \ln t, \\ b_5 &= w, \end{aligned} \tag{3.15}$$

where b_3, b_4 and b_5 are arbitrary functions all dependent on b_1 and b_2.

By choosing $b_3 = b_1$, $b_4 = 0$ and $b_5 = b_2$, we obtain the canonical coordinates

$$\begin{aligned} r &= kt - x, \\ s &= -\ln t, \\ w &= tu. \end{aligned} \tag{3.16}$$

The inverse canonical coordinates from (3.16) are given by

$$\begin{aligned} t &= e^{-s}, \\ x &= ke^{-s} - r, \\ u &= we^{s}. \end{aligned} \tag{3.17}$$

The computation of A and $(A^{-1})^T$ is given by

$$A = \begin{pmatrix} 0 & -1 \\ -e^{-s} & -ke^{-s} \end{pmatrix}$$

and

$$(A^{-1})^T = \begin{pmatrix} k & -1 \\ -e^{s} & 0 \end{pmatrix}$$

where $J = -e^{-s}$.

The partial derivatives of u from (3.17) are given by

$$\begin{aligned} u_x &= -w_r e^s, \\ u_{xt} &= e^s(-kw_{rr} + w_r e^s), \\ u_{xx} &= w_{rr} e^s, \\ u_{xxt} &= e^s(kw_{rrr} - w_{rr} e^s), \\ u_{xxx} &= -w_{rrr} e^s. \end{aligned} \tag{3.18}$$

By substituting (3.17) and (3.18) into (2.16) for $j = 1$, we obtain

$$\begin{aligned} T_1^r &= w_r - w^2 + \frac{1}{2}w_r^2 + ww_{rr}, \\ T_1^s &= \frac{1}{3}w_{rr}. \end{aligned} \tag{3.19}$$

Solving (2.17) and (3.19) simultaneously results in

$$w_r - w^2 + \frac{1}{2}w_r^2 + ww_{rr} = n, \tag{3.20}$$

where n is an integration constant.

We note that for scalar PDEs, when a multiplier is multiplied with the equation and substituted in the differential consequence of the reduced conserved form, it tends to zero.

We now analyse (3.20) for $n = 0$, i.e.,

$$w_r - w^2 + \frac{1}{2}w_r^2 + ww_{rr} = 0. \tag{3.21}$$

Since $\frac{\partial}{\partial r}$ is a Lie point symmetry of (3.21), we have the zero, first-order and second-order invariants given by

$$\begin{aligned} \alpha &= w, \\ \beta &= w_r, \\ \frac{d\beta}{d\alpha} &= \frac{w_{rr}}{w_r}. \end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21) results in the first-order ODE

$$\frac{d\beta}{d\alpha} = \frac{\alpha}{\beta} - \frac{1}{\alpha} + \frac{\beta}{2\alpha}. \tag{3.23}$$

Equation (3.23) can be solved using classical integration methods.

3.3 The standard KdV equation

We analyse the following scalar PDE

$$G = u_t - u_{xxx} - uu_x = 0. \tag{3.24}$$

Equation (3.24) admits the following four Lie point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= t\frac{\partial}{\partial t} + \frac{1}{3}x\frac{\partial}{\partial x} - \frac{2}{3}u\frac{\partial}{\partial u}, \\ X_3 &= -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ X_4 &= \frac{\partial}{\partial x}, \end{aligned} \tag{3.25}$$

and the following three conserved vectors

$$\begin{aligned}
T_1 &= \left[\frac{1}{2}u^2, -uu_{xx} + \frac{1}{2}u_x^2 - \frac{1}{3}u^3\right], \\
T_2 &= \left[\frac{1}{2}tu^2 + xu, -tuu_{xx} - xu_{xx} + \frac{1}{2}tu_x^2 + u_x - \frac{1}{3}tu^3 - \frac{1}{2}xu^2\right], \\
T_3 &= \left[u, -u_{xx} - \frac{1}{2}u^2\right], \qquad (3.26)
\end{aligned}$$

with corresponding multipliers

$$\begin{aligned}
q_1 &= u, \\
q_2 &= x + tu, \\
q_3 &= 1. \qquad (3.27)
\end{aligned}$$

3.3.1 A reduction of (3.24) by $< X_2 >$

We show that X_2 is associated with T_2.

We have

$$\begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} = X_2^{[2]} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} + \left(\frac{4}{3}\right) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{1}{2}tu^2 + \frac{1}{3}xu - \frac{2}{3}tu^2 - \frac{2}{3}xu + \frac{1}{6}tu^2 + \frac{1}{3}xu$$

and

$$\begin{aligned}
U_2 = {} & -tuu_{xx} + \frac{1}{2}tu_x^2 - \frac{1}{3}tu^3 - \frac{1}{3}xu_{xx} - \frac{1}{6}xu^2 + \frac{2}{3}tuu_{xx} + \frac{2}{3}tu^3 + \frac{2}{3}xu^2 - tu_x^2 \\
& - u_x + \frac{4}{3}tuu_{xx} + \frac{4}{3}xu_{xx} - tuu_{xx} - xu_{xx} + \frac{1}{2}tu_x^2 + u_x - \frac{1}{3}tu^3 - \frac{1}{2}xu^2.
\end{aligned}$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_2^{[2]} = t\frac{\partial}{\partial t} + \frac{1}{3}x\frac{\partial}{\partial x} - \frac{2}{3}u\frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_x} - \frac{4}{3}\frac{\partial}{\partial u_{xx}}.$$

Therefore X_2 is associated with T_2.

We transform the generator X_2 to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X_2(r) = 0$, $X_2(s) = 1$ and $X_2(w) = 0$, we have

$$\frac{dt}{t} = \frac{3dx}{x} = \frac{3du}{-2u} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}. \tag{3.28}$$

The invariants of X_2 from (3.28) are given by

$$\begin{aligned} b_1 &= \frac{x^3}{t}, \\ b_2 &= x^2 u, \\ b_3 &= r, \\ b_4 &= s - \ln t, \\ b_5 &= w, \end{aligned} \tag{3.29}$$

where b_3, b_4 and b_5 are arbitrary functions all dependent on b_1 and b_2.

By choosing $b_3 = b_1$, $b_4 = 0$ and $b_5 = b_2$, we obtain the canonical coordinates

$$\begin{aligned} r &= \frac{x^3}{t}, \\ s &= \ln t, \\ w &= x^2 u. \end{aligned} \tag{3.30}$$

The inverse canonical coordinates from (3.30) are given by

$$
\begin{aligned}
t &= e^s, \\
x &= (re^s)^{\frac{1}{3}}, \\
u &= w(re^s)^{-\frac{2}{3}}.
\end{aligned} \tag{3.31}
$$

The computation of A and $(A^{-1})^T$ is given by

$$
A = \frac{1}{3}e^{\frac{s}{3}} \begin{pmatrix} 0 & r^{-\frac{2}{3}} \\ 3e^{\frac{2s}{3}} & r^{\frac{1}{3}} \end{pmatrix}
$$

and

$$
(A^{-1})^T = e^{-\frac{s}{3}} \begin{pmatrix} -re^{-\frac{2s}{3}} & 3r^{\frac{2}{3}} \\ e^{-\frac{2s}{3}} & 0 \end{pmatrix}
$$

where $J = -\frac{1}{3}\left(\frac{e^{4s}}{r^2}\right)^{\frac{1}{3}}$.

The partial derivatives of u from (3.31) are given by

$$
\begin{aligned}
u_t &= -(re^{-5s})^{\frac{1}{3}} w_r, \\
u_x &= e^{-s}\left(3w_r - \frac{2w}{r}\right), \\
u_{xx} &= 3\,(re^s)^{\frac{-4}{3}}\,(3r^2 w_{rr} - 2rw_r + 6w), \\
u_{xxx} &= 3\,(re^s)^{\frac{-5}{3}}\,(9r^3 w_{rrr} + 8rw_r - 8w).
\end{aligned} \tag{3.32}
$$

By substituting (3.31) and (3.32) into (2.16) for $j = 2$, we obtain

$$
\begin{aligned}
T_2^r &= \frac{2w^2}{3r} + \frac{w}{3} + 9ww_{rr} + \frac{4w^2}{r^2} + 9rw_{rr} - 9w_r + \frac{8w}{r} - \frac{9w_r^2}{2} + \frac{w^3}{3r^2}, \\
T_2^s &= -\frac{w^2}{6r^2} - \frac{w}{3r}.
\end{aligned} \tag{3.33}
$$

Solving (2.17) and (3.33) simultaneously results in

$$
\frac{2w^2}{3r} + \frac{w}{3} + 9ww_{rr} + \frac{4w^2}{r^2} + 9rw_{rr} - 9w_r + \frac{8w}{r} - \frac{9w_r^2}{2} + \frac{w^3}{3r^2} = k, \tag{3.34}
$$

or equivalently after multiplying both sides by $6r^2$

$$4rw^2+2r^2w+54r^2ww_{rr}+24w^2+54r^3w_{rr}-54r^2w_r+48rw-27r^2w_r^2+2w^3-6kr^2=0, \tag{3.35}$$

where k is an integration constant.

Equation (3.35) is the second Painlevé transcendent. There are numerous and alternative analytical or numerical approaches that can be adopted in solving (3.35). We refer the reader to [43] for an extensive discussion.

3.4 The Drinfeld-Sokolov-Wilson equation

In this section, we analyse the following system of PDEs

$$\begin{aligned} G^1 &= u_t+2vv_x = 0, \\ G^2 &= v_t-av_{xxx}+3bu_xv+3kuv_x = 0. \end{aligned} \tag{3.36}$$

Equation (3.36) admits a three-dimensional Lie point symmetry algebra spanned by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-2u\frac{\partial}{\partial u}-2v\frac{\partial}{\partial v}. \end{aligned} \tag{3.37}$$

Case 1: $b = k$

In this case, (3.36) admits the following three conserved vectors

$$\begin{aligned} T_1 &= [u, v^2], \\ T_2 &= [v, -av_{xx} + 3buv], \\ T_3 &= \left[\frac{3b}{4}u^2 + \frac{v^2}{2}, 3buv^2 - avv_{xx} + \frac{a}{2}v_x^2\right], \end{aligned} \tag{3.38}$$

with corresponding multipliers

$$\begin{aligned} Q_1 &= [1, 0], \\ Q_2 &= [0, 1], \\ Q_3 &= \left[\frac{3b}{2}u, v\right]. \end{aligned} \tag{3.39}$$

3.4.1 A reduction of (3.36) by $< X_1, X_2 >$

We show that X_1 and X_2 are associated with T_3.

We have

$$\begin{pmatrix} T_3^{*t} \\ T_3^{*x} \end{pmatrix} = X_1^{[2]} \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} + (0) \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{\partial}{\partial t}\left(\frac{3bu^2}{4} + \frac{v^2}{2}\right)$$

and

$$U_2 = \frac{\partial}{\partial t}\left(3buv^2 - avv_{xx} + \frac{av_x^2}{2}\right).$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_1^{[2]} = \frac{\partial}{\partial t}.$$

Therefore X_1 is associated with T_3.

Similarly for X_2,

$$\begin{pmatrix} T_3^{*t} \\ T_3^{*x} \end{pmatrix} = X_2^{[2]} \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} - \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} + (0) \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{\partial}{\partial x}\left(\frac{3bu^2}{4} + \frac{v^2}{2}\right)$$

and

$$U_2 = \frac{\partial}{\partial x}\left(3buv^2 - avv_{xx} + \frac{av_x^2}{2}\right).$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_2^{[2]} = \frac{\partial}{\partial x}.$$

Therefore X_2 is also associated with T_3.

We consider a linear combination of X_1 and X_2, i.e., of the form $X = X_1 + cX_2$ (c is an arbitrary constant) and transform this generator to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$ and $X(p) = 0$, we have

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} = \frac{dv}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \tag{3.40}$$

The invariants of X from (3.40) are given by

$$
\begin{aligned}
b_1 &= x - ct, \\
b_2 &= u, \\
b_3 &= v, \\
b_4 &= r, \\
b_5 &= s - t, \\
b_6 &= w, \\
b_7 &= p, \qquad (3.41)
\end{aligned}
$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = b_2$ and $b_7 = b_3$, we obtain the canonical coordinates

$$
\begin{aligned}
r &= x - ct, \\
s &= t, \\
w &= u, \\
p &= v. \qquad (3.42)
\end{aligned}
$$

The inverse canonical coordinates from (3.42) are given by

$$
\begin{aligned}
t &= s, \\
x &= r + cs, \\
u &= w, \\
v &= p. \qquad (3.43)
\end{aligned}
$$

The computation of A and $(A^{-1})^T$ is given by

$$
A = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}
$$

and

$$A^{-1} = \begin{pmatrix} -c & 1 \\ 1 & 0 \end{pmatrix} = (A^{-1})^T,$$

where $J = -1$.

The partial derivatives of u and v from (3.43) are given by

$$\begin{aligned} u_t &= -cw_r, \\ u_x &= w_r, \\ v_t &= -cp_r, \\ v_x &= p_r, \\ v_{xx} &= p_{rr}, \\ v_{xxx} &= p_{rrr}. \end{aligned} \tag{3.44}$$

By substituting (3.43) and (3.44) into (2.16) for $j = 3$, we obtain

$$\begin{aligned} T_3^r &= \frac{3bcw^2}{4} + \frac{cp^2}{2} - 3bwp^2 + app_{rr} - \frac{ap_r^2}{2}, \\ T_3^s &= -\frac{3bw^2}{4} - \frac{p^2}{2}. \end{aligned} \tag{3.45}$$

Solving (2.17) and (3.45) simultaneously results in

$$\frac{3bcw^2}{4} + \frac{cp^2}{2} - 3bwp^2 + app_{rr} - \frac{ap_r^2}{2} = m, \tag{3.46}$$

where m is an integration constant.

Differentiating (3.46) implicitly with respect to r results in

$$\frac{3bcww_r}{2} + cpp_r - 3bw_rp^2 - 6bwpp_r + app_{rrr} = 0. \tag{3.47}$$

The second equation of (1.17) for $j = 3$ is given by

$$\frac{3bu}{2}(u_t + 2vv_x) - v[v_t - av_{xxx} + 3b(u_xv + uv_x)]. \tag{3.48}$$

After transforming (3.48) using (3.43) and (3.44), we obtain

$$-\frac{3bcww_r}{2} + cpp_r + app_{rrr} - 3bw_rp^2 = 0. \tag{3.49}$$

Substituting (3.47) into (3.49) and then taking out a common factor of $-3bw$ yields the first-order ODE

$$cw_r - 2pp_r = 0. \tag{3.50}$$

Integrating (3.50) with respect to r results in

$$w = \frac{1}{c}(p^2 + n), \tag{3.51}$$

where n is an integration constant.

Substituting (3.51) into (3.46) and then multiplying both sides by $4c$ results in the second-order ODE

$$2ac(2pp_{rr} - p_r^2) - 9bp^4 - 2(3bn - c^2)p^2 = 4cm - 3bn^2. \tag{3.52}$$

We present a numerical simulation for (3.52) in the figure below, using Mathematica, where the parameter values were chosen as $a = c = m = n = 1$ and $b = -1$, for $r \in [0, 20]$. The initial conditions were given as $p(0) = 1$ and $p_r(0) = 0$.

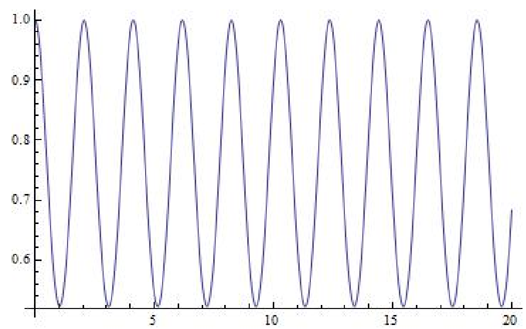

Figure 3.1: Profile of solution for $p(r)$

This in turn imposes (3.51) with a travelling wave form for $w(r)$. Other approaches such as homotopy analysis and alternative numerical approaches can also be adopted to extract the solutions of (3.51) and (3.52).

Case 2: $2b = k$

In this case, (3.36) admits the following three conserved vectors

$$\begin{aligned} T_4 &= \left[\frac{1}{2}v^2, -avv_{xx} + \frac{1}{2}av_x^2 + 3buv^2\right], \\ T_5 &= \left[\frac{1}{2}(tv^2 - xu), -\frac{1}{2}xv^2 - at\Big(vv_{xx} - \frac{1}{2}v_x^2\Big) + 3btuv^2\right], \end{aligned} \tag{3.53}$$

and T_1 from (3.38).

The corresponding multipliers are

$$\begin{aligned} Q_4 &= [0, 1], \\ Q_5 &= \Big[-\frac{1}{2}x, tv\Big], \end{aligned} \tag{3.54}$$

and Q_1 from (3.39).

3.4.2 A reduction of (3.36) by $< X_3 >$

We show that X_3 is associated with T_5.

We have

$$\begin{pmatrix} T_5^{*t} \\ T_5^{*x} \end{pmatrix} = X_3^{[2]} \begin{pmatrix} T_5^{t} \\ T_5^{x} \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_5^{t} \\ T_5^{x} \end{pmatrix} + (4) \begin{pmatrix} T_5^{t} \\ T_5^{x} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{3}{2}tv^2 - \frac{1}{2}xu + xu - 2tv^2 + \frac{1}{2}tv^2 - \frac{1}{2}xu$$

and

$$\begin{aligned} U_2 &= -3atvv_{xx} + \frac{3}{2}atv_x^2 + 9btuv^2 - \frac{1}{2}xv^2 - 6btuv^2 + 2xv^2 + 2atvv_{xx} - 12btuv^2 \\ &\quad - 3atv_x^2 + 4atvv_{xx} - \frac{3}{2}xv^2 - 3atvv_{xx} + \frac{3}{2}atv_x^2 + 9btuv^2. \end{aligned}$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_3^{[2]} = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 3v_x\frac{\partial}{\partial v_x} - 4v_{xx}\frac{\partial}{\partial v_{xx}}.$$

Therefore X_3 is associated with T_5.

We transform the generator X_3 to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X_3(r) = 0$, $X_3(s) = 1$, $X_3(w) = 0$ and $X_3(p) = 0$, we have

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{du}{-2u} = \frac{dv}{-2v} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \tag{3.55}$$

The invariants of X_3 from (3.55) are given by

$$
\begin{aligned}
b_1 &= \frac{x^3}{t}, \\
b_2 &= \frac{v}{u}, \\
b_3 &= x^2 u, \\
b_4 &= r, \\
b_5 &= s - \ln x, \\
b_6 &= w, \\
b_7 &= p,
\end{aligned} \tag{3.56}
$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = b_2$ and $b_7 = b_3$, we obtain the canonical coordinates

$$
\begin{aligned}
r &= \frac{x^3}{t}, \\
s &= \ln x, \\
w &= \frac{v}{u}, \\
p &= x^2 u.
\end{aligned} \tag{3.57}
$$

The inverse canonical coordinates from (3.57) are given by

$$
\begin{aligned}
t &= \frac{e^{3s}}{r}, \\
x &= e^{s}, \\
u &= \frac{p}{e^{2s}}, \\
v &= \frac{pw}{e^{2s}}.
\end{aligned} \tag{3.58}
$$

The computation of A and $(A^{-1})^T$ is given by

$$A = e^s \begin{pmatrix} -\frac{e^{2s}}{r^2} & 0 \\ \frac{3e^{2s}}{r} & 1 \end{pmatrix}$$

and

$$(A^{-1})^T = \frac{1}{e^s} \begin{pmatrix} -\dfrac{r^2}{e^{2s}} & 3r \\ 0 & 1 \end{pmatrix}$$

where $J = -\dfrac{e^{4s}}{r^2}$.

The partial derivatives of u and v from (3.58) are given by

$$\begin{aligned}
u_t &= -\frac{r^2 p_r}{e^{5s}}, \\
u_x &= \frac{1}{e^{3s}}(3rp_r - 2p), \\
v_t &= -\frac{r^2}{e^{5s}}(pw_r + wp_r), \\
v_x &= \frac{1}{e^{3s}}[3r(pw_r + wp_r) - 2pw], \\
v_{xx} &= -\frac{3}{e^{4s}}[2r(pw_r + wp_r) - 3r^2(2p_r w_r + pw_{rr} + wp_{rr}) - 2pw], \\
v_{xxx} &= \frac{3}{e^{5s}}[8r(pw_r + wp_r) + 27r^3(p_{rr}w_r + p_r w_{rr}) \\
&\quad + 9r^3(pw_{rrr} + wp_{rrr}) - 8pw]. \qquad (3.59)
\end{aligned}$$

By substituting (3.58) and (3.59) into (2.16) for $j = 5$, we obtain

$$\begin{aligned}
T_5^r &= \frac{2p^2w^2}{r} - \frac{p}{2} + 27apw(p_r w_r + pw_{rr} + wp_{rr}) + \frac{3p^2w^2}{r^2}(4a - 3bp) \\
&\quad - \frac{27a}{2}(p^2w_r^2 + w^2p_r^2), \\
T_5^s &= \frac{p^2w^2}{2r^2} + \frac{9apw}{r}(p_r w_r + pw_{rr} + wp_{rr}) + \frac{4ap^2w^2}{r^3} - \frac{9a}{2r}(p^2w_r^2 + p_r^2w^2) \\
&\quad - \frac{3bp^3w^2}{r^3}. \qquad (3.60)
\end{aligned}$$

Solving (2.17) and (3.60) simultaneously results in

$$\begin{aligned}\frac{2p^2w^2}{r} - \frac{p}{2} + 27apw(p_rw_r + pw_{rr} + wp_{rr}) + \frac{3p^2w^2}{r^2}(4a-3bp) \\ - \frac{27a}{2}(p^2w_r^2 + w^2p_r^2) = m,\end{aligned} \tag{3.61}$$

where m is an integration constant.

Differentiating (3.61) implicitly with respect to r results in

$$\begin{aligned}&\frac{4pw}{r}(wp_r + pw_r) - \frac{2p^2w^2}{r^2} - \frac{p_r}{2} \\ &\quad + 27apw(p_{rr}w_r + 2p_rw_{rr} + pw_{rrr} + w_rp_{rr} + wp_{rrr}) \\ &\quad + 27a(wp_r + pw_r)(p_rw_r + pw_{rr} + wp_{rr}) \\ &\quad + \frac{6pw}{r^3}[r(wp_r + pw_r) - pw](4a-3bp) - \frac{9bp^2w^2p_r}{r^2} \\ &\quad - 27a(pp_rw_r^2 + p^2w_rw_{rr} + ww_rp_r^2 + w^2p_rp_{rr}) = 0.\end{aligned} \tag{3.62}$$

or equivalently after multiplying both sides by $2r^3$

$$\begin{aligned}&8r^2pw(wp_r + pw_r) - 4rp^2w^2 - r^3p_r + 162ar^3pw(w_rp_{rr} + p_rw_{rr}) \\ &\quad + 54ar^3pw(pw_{rrr} + wp_{rrr}) + 48arpw(wp_r + pw_r) - 48ap^2w^2 \\ &\quad - 54brp^2w^2p_r - 36brp^3ww_r + 36bp^3w^2 = 0.\end{aligned} \tag{3.63}$$

The second equation of (1.17) for $j = 5$ is given by

$$-\frac{x}{2}(u_t + 2vv_x) - tv[v_t - av_{xxx} + 3b(u_xv + uv_x)]. \tag{3.64}$$

After transforming (3.64) using (3.58) and (3.59), taking out a common factor of $\frac{1}{e^{4s}}$ and multiplying both sides by $2r$, we obtain

$$r^3 p_r - 4r^2 pw(wp_r + pw_r) + 4rp^2 w^2 + 48arpw(wp_r + pw_r)$$
$$+ 162ar^3 pw(w_r p_{rr} + p_r w_{rr}) + 54ar^3 pw(pw_{rrr} + wp_{rrr}) - 48ap^2 w^2$$
$$- 54brp^2 w^2 p_r + 36bp^3 w^2 - 36brp^3 ww_r \;=\; 0. \tag{3.65}$$

Substituting (3.63) into (3.65) and taking out a common factor of $2r$ results in the first-order ODE

$$r^2 p_r - 6rpw(pw_r + wp_r) + 4p^2 w^2 = 0. \tag{3.66}$$

Solving (3.66) and (3.61) simultaneously for w and p leads to a solution for u and v to our original equation (3.36).

Case 3: $2b \neq k$

In this case, (3.36) admits the following two conserved vectors

$$T_6 \;=\; \left[\frac{3}{4}(2b-k)u^2 + \frac{1}{2}v^2, 3buv^2 - avv_{xx} + \frac{1}{2}av_x^2\right] \tag{3.67}$$

and T_1 from (3.38).

The corresponding multipliers are

$$Q_6 \;=\; \left[\frac{3}{2}(2b-k)u, v\right] \tag{3.68}$$

and Q_1 from (3.39).

3.4.3 A reduction of (3.36) by $< X_1, X_2 >$

We show that X_1 and X_2 are associated with T_6.

We have

$$\begin{pmatrix} T_6^{*t} \\ T_6^{*x} \end{pmatrix} = X_1^{[2]} \begin{pmatrix} T_6^t \\ T_6^x \end{pmatrix} - \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \begin{pmatrix} T_6^t \\ T_6^x \end{pmatrix} + (0) \begin{pmatrix} T_6^t \\ T_6^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{\partial}{\partial t} \left(\frac{3}{4}(2b - k)u^2 + \frac{v^2}{2} \right)$$

and

$$U_2 = \frac{\partial}{\partial t} \left(3buv^2 - avv_{xx} + \frac{av_x^2}{2} \right).$$

This shows that

$$U_1 = 0 = U_2,$$

where

$$X_1^{[2]} = \frac{\partial}{\partial t}.$$

Therefore X_1 is associated with T_6.

Similarly for X_2,

$$\begin{pmatrix} T_6^{*t} \\ T_6^{*x} \end{pmatrix} = X_2^{[2]} \begin{pmatrix} T_6^t \\ T_6^x \end{pmatrix} - \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \begin{pmatrix} T_6^t \\ T_6^x \end{pmatrix} + (0) \begin{pmatrix} T_6^t \\ T_6^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$U_1 = \frac{\partial}{\partial x} \left(\frac{3}{4}(2b - k)u^2 + \frac{v^2}{2} \right)$$

and

$$U_2 = \frac{\partial}{\partial x} \left(3buv^2 - avv_{xx} + \frac{av_x^2}{2} \right).$$

Thus

$$U_1 = 0 = U_2,$$

where

$$X_2^{[2]} = \frac{\partial}{\partial x}.$$

Therefore X_2 is also associated with T_6.

We consider a linear combination of X_1 and X_2, i.e., of the form $X = X_1 + cX_2$ (c is an arbitrary constant) and transform this generator to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$ and $X(p) = 0$, we have

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} = \frac{dv}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \tag{3.69}$$

The invariants of X from (3.69) are given by

$$\begin{aligned} b_1 &= x - ct, \\ b_2 &= u, \\ b_3 &= v, \\ b_4 &= r, \\ b_5 &= s - t, \\ b_6 &= w, \\ b_7 &= p, \end{aligned} \tag{3.70}$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = b_2$ and $b_7 = b_3$, we obtain the canonical coordinates

$$\begin{aligned} r &= x - ct, \\ s &= t, \\ w &= u, \\ p &= v. \end{aligned} \tag{3.71}$$

The inverse canonical coordinates from (3.71) are given by

$$\begin{aligned} t &= s, \\ x &= r + cs, \\ u &= w, \\ v &= p. \end{aligned} \tag{3.72}$$

The computation of A and $(A^{-1})^T$ is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} -c & 1 \\ 1 & 0 \end{pmatrix} = (A^{-1})^T,$$

where $J = -1$.

The partial derivatives of u and v from (3.72) are given by

$$\begin{aligned} u_t &= -cw_r, \\ u_x &= w_r, \\ v_t &= -cp_r, \\ v_x &= p_r, \\ v_{xx} &= p_{rr}, \\ v_{xxx} &= p_{rrr}. \end{aligned} \tag{3.73}$$

By substituting (3.72) and (3.73) into (2.16) for $j = 6$, we obtain

$$
\begin{aligned}
T_6^r &= \frac{3c}{4}(2b-k)w^2 + \frac{cp^2}{2} - 3bwp^2 + app_{rr} - \frac{a}{2}p_r^2, \\
T_6^s &= -\frac{3}{4}(2b-k)w^2 - \frac{p^2}{2}.
\end{aligned} \tag{3.74}
$$

Solving (2.17) and (3.74) simultaneously results in

$$
\frac{3c}{4}(2b-k)w^2 + \frac{cp^2}{2} - 3bwp^2 + app_{rr} - \frac{a}{2}p_r^2 = m, \tag{3.75}
$$

where m is an integration constant.

Differentiating (3.75) implicitly with respect to r results in

$$
\frac{3c}{2}(2b-k)ww_r + cpp_r - 3bw_rp^2 - 6bwpp_r + app_{rrr} = 0. \tag{3.76}
$$

The second equation of (1.17) for $j = 6$ is given by

$$
\frac{3}{2}(2b-k)u(u_t + 2vv_x) - v[v_t - av_{xxx} + 3b(u_xv + uv_x)]. \tag{3.77}
$$

After transforming (3.77) using (3.72) and (3.73), we obtain

$$
-\frac{3c}{2}(2b-k)ww_r + 6bwpp_r - 6kwpp_r + cpp_r + app_{rrr} - 3bw_rp^2 = 0. \tag{3.78}
$$

After substituting (3.76) into (3.78) and then taking out a common factor of $-3(2b-k)w$, we obtain (3.50) and consequently (3.51).

Substituting (3.51) into (3.75) and then multiplying both sides by $4c$ results in the second-order ODE

$$
2ac(2pp_{rr} - p_r^2) - 3(2b+k)p^4 - 2(3kn - c^2)p^2 = 4cm - 3(2b-k)n^2. \tag{3.79}
$$

Similarly as in case 1, solving (3.79) for p leads to a solution for w in (3.51) and hence a solution for u and v to our original equation (3.36).

3.5 Discussion and conclusion

We obtained new conservation laws via the invariance and multiplier approach for a class of KdV equations, specifically relating to a Hunter-Saxton type equation. Second-order multipliers were calculated and thus new conserved quantities were then obtained. One case of the double reduction procedure was applied to this equation without the evolution term and this resulted in reductions to a first-order ODE.

We showed how the interplay between underlying symmetries and conservation laws lead to double reductions for a class of Drinfeld-Sokolov-Wilson equations. In all the cases on the specific relationship of the parameters b and k, we obtained a reduction to an ODE of order, at most, two. After performing the double reduction procedure for one of the cases, we adopted a numerical approach via Mathematica to illustrate the profile of the solution for one of the reduced ODEs.

Chapter 4

Multipliers, Conservation Laws and Reductions of Higher-order PDEs Related to Plasma Physics

4.1 Introduction and background

One of the most fundamental and fascinating phenomena in plasma physics that was analysed in the 1920's is Langmuir turbulence. This turbulence consists only of high frequency electron oscillations in a low amplitude range. However, the presence of larger amplitude waves induces nonlinearities which couple the high frequency electron oscillations to low frequency ion oscillations. These nonlinearities lead to parametric instabilities. The strongly nonlinear state leads to the formation of solitons, where these structures are stable in one dimension and can collapse catastrophically in higher dimensions. Zakharov derived a set of equations to describe all of these physical phenomena. These equations are commonly referred to as the Zakharov equations [99]. Generalized Zakharov equations (GZEs) are a universal model of interaction between high and low frequency waves [13, 98, 100].

The dimensionless form of the GZE with power law nonlinearity is given by

$$\begin{aligned} iF_t + aF_{xx} + b|F|^{2n}F &= Fw, \\ w_{tt} - k^2 w_{xx} &= \left(|F|^{2n}\right)_{xx}, \end{aligned} \tag{4.1}$$

where F is a complex order parameter that represents a high frequency wave, w represents a real low frequency field, the coefficient of a is the group velocity dispersion and b represents the power law nonlinearity. In the second equation of (4.1), the left hand side represents the wave operator, where k is an arbitrary real constant. When $b = 0$, (4.1) is reduced to the classical Zakharov equations.

Taking F to be of the form $F = u + iv$ and separating the first equation of (4.1) into real and imaginary parts results in the system of PDEs

$$\begin{aligned} u_t + av_{xx} + b(u^2+v^2)^n v - vw &= 0, \\ -v_t + au_{xx} + b(u^2+v^2)^n u - uw &= 0, \\ w_{tt} - k^2 w_{xx} - [(u^2+v^2)^n]_{xx} &= 0. \end{aligned} \tag{4.2}$$

The invariance and multiplier approach will be applied on (4.2) to extract conservation laws for $n = 1$.

The results for the GZE appear in [69].

The second plasma physics model we will consider is based on Alfvén waves.

Alfvén suggested the existence of electromagnetic-hydromagnetic waves [2]. These waves have been mainly investigated in the fields of astrophysics and plasma physics

[23, 63, 90, 101]. The study of the amplitude modulation of compressional dispersive Alfvén (CDA) waves against quasi-stationary magnetic field pertubations in a low-β plasma [82] and the study of a theory for large amplitude compressional electromagnetic solitary pulses in a magnetized electron-positron plasma [83] was conducted. It was shown in both of these articles how a system of three PDEs relating to the nonlinear propagation of the waves, governed by the ion continuity equation, the ion momentum equation (which used Ampere's law) and Faraday's law of electromagnetic induction were linearized and combined.

This resulted in the fourth-order wave equation

$$u_{tt} - (3a^2 + c^2)u_{xx} - \delta^2 u_{xxxx} - \delta^2 u_{xxtt} = 0, \tag{4.3}$$

where a, c and δ are arbitrary real constants.

Since (4.3) admits a Lagrangian, we will determine conservation laws via the well-known Noethers theorem [73]. We also apply the invariance and multiplier approach, and the double reduction procedure on (4.3).

The results for the CDA wave equation appear in [70].

4.2 Generalized Zakharov equations

In this section, we analyse the system of PDEs given by

$$\begin{aligned} G^1 &= u_t + av_{xx} + b(u^2+v^2)v - vw = 0, \\ G^2 &= -v_t + au_{xx} + b(u^2+v^2)u - uw = 0, \\ G^3 &= w_{tt} - k^2 w_{xx} - [(u^2+v^2)]_{xx} = 0. \end{aligned} \tag{4.4}$$

The multiplier $Q_j = (q_j^1, q_j^2, q_j^3)$ satisfies the 'joint' Euler-Lagrange operator

$$\frac{\delta}{\delta(u,v,w)}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3) = 0. \tag{4.5}$$

Equation (4.5) is a consequence of three dependent variables and is equivalent to the action of the Euler-Lagrange operator on each dependent variable u, v and w, given by

$$\begin{aligned} \frac{\delta}{\delta u}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3) = 0, \\ \frac{\delta}{\delta v}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3) = 0, \\ \frac{\delta}{\delta w}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3) = 0. \end{aligned} \tag{4.6}$$

Thus by (1.9), we require

$$q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 = D_x \Phi^x + D_t \Phi^t.$$

We assume the multiplier $Q_j = (q_j^1, q_j^2, q_j^3)$ to be of first-order derivative dependence, i.e.,

$$q_j^1 = g^1(x, t, u, v, u_x, v_x, u_t, v_t),$$

$$q_j^2 = g^2(x, t, u, v, u_x, v_x, u_t, v_t),$$

and

$$q_j^3 = g^3(x, t, u, v, u_x, v_x, u_t, v_t).$$

Equation (4.6) has to be satisfied for all functions $u(x,t)$, $v(x,t)$ and $w(x,t)$, not only the solutions of (4.4).

The calculations after expansion and separation by monomials of (4.6) reveals the following multipliers and corresponding components of conserved vectors

(a) $Q_1 = (u, -v, 1)$

$$
\begin{aligned}
T_1^t &= \frac{1}{2}(u^2 + v^2 + 2w_t), \\
T_1^x &= -2uu_x - avu_x + auv_x - 2vv_x - k^2 w_x,
\end{aligned}
$$

(b) $Q_2 = (u, -v, t)$

$$
\begin{aligned}
T_2^t &= \frac{1}{2}(u^2 + v^2 - 2w + 2tw_t), \\
T_2^x &= -2tuu_x - avu_x + auv_x - 2tvv_x - tk^2 w_x,
\end{aligned}
$$

(c) $Q_3 = \left(-2tu, 2tv, \frac{1}{2}k^2t^2 + \frac{1}{2}x^2\right)$

$$
\begin{aligned}
T_3^t &= \frac{1}{2}(-2tu^2 - 2tv^2 - 2tk^2 w + x^2 w_t + t^2k^2 w_t), \\
T_3^x &= \frac{1}{2}\Big[2xu^2 + 2xv^2 - 2u((x^2 + t^2k^2)u_x + 2atv_x) - v(-4atu_x + 2(x^2 + t^2k^2)v_x) \\
&\quad - k^2(-2xw + (x^2 + t^2k^2)w_x)\Big],
\end{aligned}
$$

(d) $Q_4 = \left(-t^2u, t^2v, \frac{1}{6}k^2t^3 + \frac{1}{2}tx^2\right)$

$$
\begin{aligned}
T_4^t &= \frac{1}{6}(-3t^2u^2 - 3t^2v^2 - 3x^2w - 3t^2k^2w + 3tx^2w_t + t^3k^2w_t), \\
T_4^x &= -\frac{1}{6}t\Big[-6xu^2 - 6xv^2 + 2u((3x^2 + t^2k^2)u_x + 3atv_x) + v(-6atu_x + 2(3x^2 + t^2k^2)v_x) \\
&\quad + k^2(-6xw + (3x^2 + t^2k^2)w_x)\Big].
\end{aligned}
$$

4.3 Compressional dispersive Alfvén waves

In this section, we analyse the scalar PDE given by

$$
u_{tt} - (3a^2 + c^2)u_{xx} - \delta^2 u_{xxxx} - \delta^2 u_{xxtt} = 0. \tag{4.7}
$$

Equation (4.7) admits the Lagrangian

$$L = -\frac{1}{2}u_t^2 + \frac{1}{2}(3a^2 + c^2)u_x^2 - \frac{1}{2}\delta^2 u_{xx}^2 - \frac{1}{2}\delta^2 u_{xt}^2. \tag{4.8}$$

The Noether symmetries from (1.7) for $(B_j^x, B_j^t) = (0, 0)$, where $j = 1, 2, 3$ are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial u}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \frac{\partial}{\partial x}. \end{aligned} \tag{4.9}$$

The corresponding components of conserved vectors from (1.19) are given by

$$\begin{aligned} T_1^t &= u_t - \frac{1}{2}\delta^2 u_{xxt}, \\ T_1^x &= -\frac{1}{2}\delta^2 u_{xtt} - \delta^2 u_{xxx} - (3a^2 + c^2)u_x, \end{aligned}$$

$$\begin{aligned} T_2^t &= -\frac{1}{2}u_t^2 - \frac{3}{2}a^2 u_x^2 - \frac{1}{2}c^2 u_x^2 + \frac{1}{2}\delta^2 u_{xx}^2 + \frac{1}{2}\delta^2 u_t u_{xxt}, \\ T_2^x &= -\frac{1}{2}\delta^2 u_{tt} u_{xt} - \delta^2 u_{xt} u_{xx} + \frac{1}{2}\delta^2 u_t u_{xtt} + \delta^2 u_t u_{xxx} + (3a^2 + c^2)u_x u_t, \end{aligned}$$

and

$$\begin{aligned} T_3^t &= -\frac{1}{2}\delta^2 u_{xx} u_{xt} + \frac{1}{2}\delta^2 u_x u_{xxt} - u_x u_t, \\ T_3^x &= \frac{1}{2}u_t^2 + \frac{3}{2}a^2 u_x^2 + \frac{1}{2}c^2 u_x^2 - \frac{1}{2}\delta^2 u_{xx}^2 + \frac{1}{2}\delta^2 u_x u_{xtt} + \delta^2 u_x u_{xxx}. \end{aligned}$$

For non-zero gauge terms, the Noether symmetries and components of the gauge vectors are

$$\begin{aligned} X_4 &= t\frac{\partial}{\partial u}, \quad B_4^t = -u, \quad B_4^x = 0, \\ X_5 &= x\frac{\partial}{\partial u}, \quad B_5^t = 0, \quad B_5^x = (c^2 + 3a^2)u, \end{aligned} \tag{4.10}$$

from which the corresponding components of conserved vectors are given by

$$T_4^t = \frac{1}{6}\left[-6u + 6tu_t + \delta^2(u_{xx} - 3tu_{xxt})\right],$$
$$T_4^x = \frac{1}{6}\left[-6t(c^2+3a^2)u_x + \delta^2(2u_{xt} - 3t(u_{xtt} + 2u_{xxx}))\right],$$

and

$$T_5^t = xu_t + \frac{1}{6}\delta^2(2u_{xt} - 3xu_{xxt}),$$
$$T_5^x = (c^2+3a^2)u + \frac{1}{6}\delta^2 u_{tt} - c^2xu_x - 3a^2xu_x - \frac{1}{2}x\delta^2u_{xtt} + \delta^2u_{xx} - x\delta^2u_{xxx}.$$

We now solve

$$\frac{\delta}{\delta u}\left[q_j(u_{tt} - (3a^2+c^2)u_{xx} - \delta^2u_{xxxx} - \delta^2u_{xxtt})\right] = 0, \qquad (4.11)$$

where we assume q_j to be up to first-order in derivatives.

Equation (4.11) has to be satisfied for all functions $u(x,t)$, not only the solutions of (4.7).

The calculations after expansion and separation by monomials of (4.11) reveals the following multipliers and corresponding components of conserved vectors

(a) $q_1 = \cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)$

$$T_6^t = \frac{1}{6}\left[(6+c^2+3a^2)\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_t - \delta^2\left(2\sqrt{\frac{c^2+3a^2}{\delta^2}}\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xt} + 3\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xxt}\right)\right],$$

$$T_6^x = -\frac{1}{6}\delta^2\left[\sqrt{\frac{c^2+3a^2}{\delta^2}}\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{tt} + 3\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xtt} + 6\sqrt{\frac{c^2+3a^2}{\delta^2}}\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xx} + 6\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xxx}\right].$$

(b) $q_2 = \sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)$

$$
\begin{aligned}
T_7^t &= \frac{1}{6}\left[(6+c^2+3a^2)\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_t\right.\\
&\left.+\delta^2\left(2\sqrt{\frac{c^2+3a^2}{\delta^2}}\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xt}-3\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xxt}\right)\right],\\
T_7^x &= \frac{1}{6}\delta^2\left[\sqrt{\frac{c^2+3a^2}{\delta^2}}\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{tt}-3\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xtt}\right.\\
&\left.+6\sqrt{\frac{c^2+3a^2}{\delta^2}}\cos\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xx}-6\sin\left(\sqrt{\frac{c^2+3a^2}{\delta^2}}x\right)u_{xxx}\right].
\end{aligned}
$$

4.3.1 A reduction of (4.7) by $< X_1, X_2, X_3 >$

We note that X_1, X_2 and X_3 are associated with their corresponding conserved vectors T_1, T_2 and T_3.

We consider a linear combination of X_1, X_2 and X_3, i.e., of the form $X = k\frac{\partial}{\partial t} + m\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$ (k and m are arbitrary constants) and transform this generator to its canonical form $Y = \frac{\partial}{\partial s}$.

From $X(r) = 0$, $X(s) = 1$ and $X(w) = 0$, we have

$$\frac{dt}{k} = \frac{dx}{m} = \frac{du}{1} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}. \tag{4.12}$$

The invariants of X from (4.12) are given by

$$
\begin{aligned}
b_1 &= x - \frac{m}{k}t,\\
b_2 &= u - \frac{x}{m},
\end{aligned}
$$

$$\begin{aligned} b_3 &= r, \\ b_4 &= s - \frac{t}{k}, \\ b_5 &= w, \end{aligned} \tag{4.13}$$

where b_3, b_4 and b_5 are arbitrary functions all dependent on b_1 and b_2.

By choosing $b_3 = b_1$, $b_4 = 0$ and $b_5 = b_2$, we obtain the canonical coordinates

$$\begin{aligned} r &= x - \frac{m}{k}t, \\ s &= \frac{t}{k}, \\ w &= u - \frac{x}{m}. \end{aligned} \tag{4.14}$$

The inverse canonical coordinates from (4.14) are given by

$$\begin{aligned} t &= ks, \\ x &= r + ms, \\ u &= w + s + \frac{r}{m}. \end{aligned} \tag{4.15}$$

The computation of A and $(A^{-1})^T$ is given by

$$A = \begin{pmatrix} 0 & 1 \\ k & m \end{pmatrix}$$

and

$$(A^{-1})^T = \begin{pmatrix} -\frac{m}{k} & 1 \\ \frac{1}{k} & 0 \end{pmatrix}$$

where $J = -k$.

The partial derivatives of u from (4.15) are given by

$$\begin{aligned}
u_t &= -\frac{m}{k}w_r, \\
u_x &= w_r + \frac{1}{m}, \\
u_{tt} &= \frac{m^2}{k^2}w_{rr}, \\
u_{xx} &= w_{rr}, \\
u_{xt} &= -\frac{m}{k}w_{rr}, \\
u_{xtt} &= \frac{m^2}{k^2}w_{rrr}, \\
u_{xxt} &= -\frac{m}{k}w_{rrr}, \\
u_{xxx} &= w_{rrr}, \\
u_{xxtt} &= \frac{m^2}{k^2}w_{rrrr}, \\
u_{xxxx} &= w_{rrrr}.
\end{aligned} \tag{4.16}$$

By substituting (4.15) and (4.16) into (2.16) for $j = 2$, we obtain

$$\begin{aligned}
T_2^r &= -\frac{(3a^2+c^2)}{2m} + \frac{(mk^2(3a^2+c^2) - m^3)}{2k^2}w_r^2 - \frac{m\delta^2(k^2+m^2)}{2k^2}w_{rr}^2 \\
&\quad + \frac{m\delta^2(m^2+k^2)}{k^2}w_r w_{rrr}, \\
T_2^s &= \frac{3a^2+c^2}{2m^2} + \frac{3a^2+c^2}{m}w_r + \frac{(k^2(3a^2+c^2)+m^2)}{2k^2}w_r^2 - \frac{1}{2}\delta^2 w_{rr}^2 \\
&\quad - \frac{\delta^2 m^2}{2k^2}w_r w_{rrr}.
\end{aligned} \tag{4.17}$$

Solving (2.17) and (4.17) simultaneously results in

$$\begin{aligned}
&- \frac{(3a^2+c^2)}{2m} + \frac{(mk^2(3a^2+c^2) - m^3)}{2k^2}w_r^2 - \frac{m\delta^2(k^2+m^2)}{2k^2}w_{rr}^2 \\
&\quad + \frac{m\delta^2(m^2+k^2)}{k^2}w_r w_{rrr} = n_1,
\end{aligned} \tag{4.18}$$

where n_1 is an integration constant.

Differentiating (4.18) implicitly with respect to r results in

$$(m^2 - k^2(3a^2 + c^2))w_{rr} - \delta^2(m^2 + k^2)w_{rrrr} = 0. \tag{4.19}$$

Integrating (4.19) with respect to r by applying D-operator methods results in

$$\begin{aligned} w &= n_2 \cos\left(\sqrt{\frac{(m^2 - k^2(3a^2 + c^2))}{-\delta^2(m^2 + k^2)}}r\right) + n_3 \sin\left(\sqrt{\frac{(m^2 - k^2(3a^2 + c^2))}{-\delta^2(m^2 + k^2)}}r\right) \\ &+ \frac{1}{(m^2 - k^2(3a^2 + c^2))}(n_4 r + n_5), \end{aligned} \tag{4.20}$$

where n_2, n_3, n_4 and n_5 are integration constants.

Combining (4.15) and (4.20), we obtain the final solution to our original equation (4.7) as

$$\begin{aligned} u &= n_2 \cos\left(\sqrt{\frac{(m^2 - k^2(3a^2 + c^2))}{-\delta^2(m^2 + k^2)}}\left(x - \frac{m}{k}t\right)\right) \\ &+ n_3 \sin\left(\sqrt{\frac{(m^2 - k^2(3a^2 + c^2))}{-\delta^2(m^2 + k^2)}}\left(x - \frac{m}{k}t\right)\right) \\ &+ \frac{1}{(m^2 - k^2(3a^2 + c^2))}\left(n_4\left(x - \frac{m}{k}t\right) + n_5\right) + \frac{x}{m}. \end{aligned} \tag{4.21}$$

4.4 Discussion and conclusion

We applied the Euler-Lagrange operator to extract multipliers and conserved quantities for a generalized Zakharov equation with power law nonlinearity. Four nontrivial multipliers were determined and they were all derivative independent, from which additional conserved vectors were obtained.

The invariance and multiplier approach was adopted to an Alfvén wave equation, and this generated two multipliers in the form of triangular periodic functions. Noether symmetries were calculated from which conservation laws were extracted by Noether's theorem. The double reduction procedure was carried out via the association of conserved vectors with a linear combination of Noether symmetries, in which an exact/invariant solution was obtained.

Chapter 5

Analysis of a Fourth-order System of PDEs

5.1 Introduction and background

In recent years, the effectiveness of Lie group analysis has attracted several authors working in fluid mechanics, particularly non-Newtonian fluids [3, 8, 26, 29, 30]. A problem of unsteady hydromagnetic flows of an Oldroyd-B fluid under the influence of Hall currents is not only helpful in establishing a relationship among the different solutions, but it also has its own significance in various ways. The constitutive relations of non-Newtonian fluids involve a number of complex parameters that give rise to systems of higher-order PDEs which are more complicated to analyse as compared to viscous fluids. Consequently, the additional terms due to rheological parameters in the differential systems pose various interesting challenges. Due to the complexity of the magnetohydrodynamic (MHD) rotating flows of non-Newtonian fluids, limited research is available [7, 36, 37, 56, 78]. To date, there has been no symmetry analysis for the MHD rotating flow of an Oldroyd-B fluid, as well as for the hydrodynamic situation.

In view of the constitutive equations used in the derivation of the governing equation in [7] and after re-defining some of the constants, we present the equation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 F}{\partial z\partial t}+2i\omega\frac{\partial F}{\partial z}\right)+\frac{\mu}{1-im}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial F}{\partial z}=\nu\frac{\partial^3 F}{\partial z^3}+\nu\gamma\frac{\partial^4 F}{\partial z^3\partial t}, \quad (5.1)$$

in which F is a complex order parameter of the form $F = u + iv$, where u and v are the velocity components in the x and y-directions, ω is the constant angular velocity, λ and γ are the material time constants referred to as relaxation and retardation times respectively, with the condition $\lambda \geq \gamma \geq 0$, $\nu = \frac{\beta}{\rho}$ (where ρ is the density and β is the dynamic viscosity) is the kinematic viscosity, $\mu = \frac{\sigma B_0^2}{\rho}$ (where B_0 is the applied magnetic field parallel to the z-axis and σ is the electrical conductivity) and m is the Hall parameter.

Separating (5.1) into real and imaginary parts results in the system of PDEs

$$\begin{aligned} & u_{zt}-2\omega v_z+\lambda(u_{ztt}-2\omega v_{zt})+\frac{\mu}{1+m^2}(u_z+\lambda u_{zt})-\frac{m\mu}{1+m^2}(v_z+\lambda v_{zt}) \\ &= \nu u_{zzz}+\nu\gamma u_{zzzt}, \\ & v_{zt}+2\omega u_z+\lambda(v_{ztt}+2\omega u_{zt})+\frac{\mu}{1+m^2}(v_z+\lambda v_{zt})+\frac{m\mu}{1+m^2}(u_z+\lambda u_{zt}) \\ &= \nu v_{zzz}+\nu\gamma v_{zzzt}. \end{aligned} \quad (5.2)$$

The invariance and multiplier approach will be applied on (5.2) to extract conservation laws.

The results of this work and an analysis of similarity solutions obtained via reductions through translation and rotational symmetries of (5.2) appear in [32].

5.2 Conservation laws of the underlying model

In this section, we analyse the system of PDEs

$$
\begin{aligned}
G^1 &= u_{zt} - 2\omega v_z + \lambda(u_{ztt} - 2\omega v_{zt}) + \frac{\mu}{1+m^2}(u_z + \lambda u_{zt}) - \frac{m\mu}{1+m^2}(v_z + \lambda v_{zt}) \\
&\quad - \nu u_{zzz} - \nu\gamma u_{zzzt} = 0, \\
G^2 &= v_{zt} + 2\omega u_z + \lambda(v_{ztt} + 2\omega u_{zt}) + \frac{\mu}{1+m^2}(v_z + \lambda v_{zt}) + \frac{m\mu}{1+m^2}(u_z + \lambda u_{zt}) \\
&\quad - \nu v_{zzz} - \nu\gamma v_{zzzt} = 0. \qquad (5.3)
\end{aligned}
$$

We require

$$q_j^1 G^1 + q_j^2 G^2 = D_t T^t + D_z T^z,$$

so that

$$\frac{\delta}{\delta(u,v)}(q_j^1 G^1 + q_j^2 G^2) = 0. \qquad (5.4)$$

We assume the multiplier $Q_j = (q_j^1, q_j^2)$ to be of second-order derivative dependence, i.e.,

$$q_j^1 = g^1(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt}, u_{tt}, v_{tt})$$

and

$$q_j^2 = g^2(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt}, u_{tt}, v_{tt}).$$

Equation (5.4) has to be satisfied for all functions $u(x,t)$ and $v(x,t)$, not only the solutions of (5.3).

The calculations after expansion and separation by monomials of (5.4) results in

$$q_j^1 = f^1(z,t),$$

$$q_j^2 = e^{t/\lambda} f^2(z) + f^3(t) + \int \frac{1}{\lambda}\left(\left(\int \frac{1}{2\omega + 2\omega m^2 + m\mu}\left(e^{\frac{-t}{\lambda}}\left(\nu\gamma(1+m^2) f^1_{zzzt} + \lambda(1+m^2) f^1_{ztt} + (-\nu m^2 - \nu) f^1_{zzz} + (-1 - \mu\lambda - m^2) f^1_{zt} + f^1_z \mu\right)\right) dt\right) e^{\frac{t}{\lambda}}\right) dz.$$

We note that not all of Q_j lead to zero when we check (5.4).

A sample of Q_j that do satisfy (5.4) are, with the corresponding conserved densities

(a) $Q_1 = (t,k)$, where k is an arbitrary constant

$$T_1^t = \frac{1}{12(1+m^2)}\Big[2\left(3t(1+m^2+\lambda\mu) - \lambda(2+3km\mu+6k\omega+m^2(2+6k\omega))\right) u_z - 6(k(1+m^2+\lambda\mu) + t\lambda(m\mu+2\omega+2m^2\omega)) v_z + (1+m^2)(8t\lambda u_{zt} - 8k\lambda v_{zt} + 3\gamma\nu(-tu_{zzz} + kv_{zzz}))\Big].$$

(b) $Q_2 = (t, ze^{t/\lambda})$

$$T_2^t = \frac{-1}{12(1+m^2)}\Big[-6e^{t/\lambda}\lambda(m\mu+2\omega+2m^2\omega)u + 2e^{t/\lambda}(1+m^2-3\lambda\mu)v - 4e^{t/\lambda}\lambda v_t - 4e^{t/\lambda}m^2\lambda v_t - 6tu_z - 6m^2tu_z + 4\lambda u_z + 4m^2\lambda u_z - 6t\lambda\mu u_z + 6e^{t/\lambda}mz\lambda\mu u_z + 12e^{t/\lambda}z\lambda\omega u_z + 12e^{t/\lambda}m^2z\lambda\omega u_z + 2e^{t/\lambda}zv_z + 2e^{t/\lambda}m^2zv_z + 6mt\lambda\mu v_z + 6e^{t/\lambda}z\lambda\mu v_z + 12t\lambda\omega v_z + 12m^2t\lambda\omega v_z - 8t\lambda u_{zt} - 8m^2t\lambda u_{zt} + 8e^{t/\lambda}z\lambda v_{zt} + 8e^{t/\lambda}m^2z\lambda v_{zt} + 3e^{t/\lambda}\gamma\nu v_{zz} + 3e^{t/\lambda}m^2\gamma\nu v_{zz} + 3t\gamma\nu u_{zzz} + 3m^2t\gamma\nu u_{zzz} - 3e^{t/\lambda}z\gamma\nu v_{zzz} - 3e^{t/\lambda}m^2z\gamma\nu v_{zzz}\Big].$$

5.3 Discussion and conclusion

We obtained new conserved densities for an Oldroyd-B fluid by assuming the possible existence of higher-order multipliers. We listed a sample of two multipliers determined by the Euler-Lagrange operator and they were derivative independent.

Conclusion

In this thesis, our main objective was to analyse the relationship between symmetries and conservation laws for higher-order nonlinear scalar PDEs and systems of PDEs with two independent variables. This entailed performing the generalized fundamental theorem of double reduction via the recently developed notion of an association between Lie point symmetries and conservation laws. In this procedure, conservation laws and their corresponding multipliers were used to construct equivalent systems relating to the original ones. This lead to resulting equations in which new exact/invariant solutions were obtained. In the case of scalar PDEs, it was unnecessary to consider the multiplier multiplied by the PDE under consideration because when this is substituted into the differential consequence of the reduced conserved form, it tends to zero.

We note that when applying the method of invariance, the dependent invariants were chosen conveniently in such a way that made the calculations of the transformed variables easier to manage.

To calculate the conservation laws, we first determined the possible existence of higher-order multipliers via the Euler-Lagrange operator acting on a total divergence. The corresponding conserved quantities were then determined via the homotopy operator. It was necessary to exclude the tedious calculations that gave rise to the multipliers and corresponding conservation laws via the use of CAS packages Maple and Mathematica, as they were very involved.

In the second chapter, we analysed two Schrödinger systems of PDEs. After performing the double reduction procedure, we obtained new non-trivial exact/invariant solutions.

In the third chapter, we applied the same approach to various classes of KdV equations. This resulted in reduced ODEs that can be solved via numerous numerical and analytical approaches.

In the fourth chapter, we considered classes of higher-order PDEs related to plasma physics. This chapter mainly focussed on the construction of multipliers and conservation laws.

In the fifth and final chapter, we extended the invariance and multiplier approach for a fluid mechanics model that inherits a fourth-order system of PDEs.

The double reduction theory is a move away from the classical and standard approaches in which nonlinear PDEs are analysed. The generalization of this theory to PDEs of higher dimensions and an increase in independent variables is still an open problem for further investigations, for example, the Ito equation [94], the Sawada-Kotera equation [80], the Benney-Luke equation [75] and other fluid mechanics models.

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