Learner errors in algebraic equations:

Seeking sense in the chaos

Yvonne Laurain Sanders

A thesis submitted to the Wits School of Education,

Faculty of Humanities, University of the Witwatersrand

In fulfilment of the requirements for the degree of Doctor of Philosophy,

Johannesburg

Abstract

The fundamental aim of this mixed-methods research was to investigate learner errors in solving linear equations, and the possible relationships between errors. Solving linear equations was used as the context in which to compare errors. This aim was chosen because of the persistent poor standard of mathematics attainment in the majority of schools in South Africa. Conducting an error analysis provides one with the tools that will then allow for the diagnosis and remediation of learning deficits. To solve a linear equation a learner needs to be competent in other mathematical topics such as equality, integers and adding like terms.

Quantitative research methods were used on a large sample of Grade 9 and 10 learners (n =2135) as well as on a subset consisting of 150 Grade 9 learners and 150 Grade 10 learners. T-tests and correlations were used to analyse pre- and post-tests results. Taking a social constructivist (Vygotsky, 1978) approach to learning and learner errors, the subset of data was also analysed qualitatively. Both typological and inductive data analysis was employed in the topics of equality, integers, expressions and equations. Having found common errors, a change in errors was explored between the pre- and post-tests, and possible relationships between errors were investigated. In general learners performed poorly in the topics mentioned above. However, Grade 10 learners made more gains than Grade 9s, suggesting that the extra exposure to mathematical content was necessary to improve test scores and reduce errors.

I term this theoretical idea of more exposure and time needed for internalisation to take place, 'synk time'. I argue that learners need time for content to settle in their minds and built this idea from three theoretical constructs: Vygotsky's (1978) notion of *internalisation*; Tomasello's (2003) notions of *intention reading* and *pattern finding*, and Gopnik and Meltzoff's (1997) notion of *The Theory Theory*. The integration of these constructs and the notion of 'synk time' is a theoretical contribution of this thesis.

The qualitative analysis revealed a surprising finding: that the majority of Grade 9 learners did not use inverses to solve equations but rather used arithmetic. In Grade 10, more learners treated the equal sign as a relational symbol. In the related topics, learners who made integer errors and errors with letters did so more in the individual topics rather than when solving equations.

The two major contributions of the thesis are theoretical and methodological. Theoretical in that I integrate three theoretical constructs to explain why Grade 10 learners make more gains than Grade 9 learners. The methodological contribution is that I show how a randomly selected sample of 150 learners can be generalised to a larger group.

Recommendations for future research point towards an investigation into the amount of time given for backlogs in mathematics as well as what revision is covered in the curriculum. Other recommendations include conducting this research with the lens of determining whether there is a hierarchy of errors. This would enable us to see improvement amidst poor performance.

Declaration

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Yvonne Laurain Sanders

November 2021

Acknowledgements

Every PhD student and every supervisor will say how taxing doing a PhD is. They will also be upfront and say how, throughout the journey, you will have many ups and downs. What they do not tell you is the effect size, standard deviation or even the range of the up or down!

I have never felt so alone, broken, lost or desperate as I have during the past four years. I have not only been challenged academically but also physically, spiritually and emotionally. I have learnt more than simply what research has to say about learner errors in integers, equations and expressions; and I have learnt more than just how to perform and interpret T-test results. What I learnt, in addition to the academic expectation of a PhD, is the importance of keeping mentally, physically, emotionally and spiritually healthy.

Many times, more often than I can count, I did not want to continue with my PhD. I am therefore grateful for the people God placed in my life who helped me to continue.

Prof Craig Pournara, one of my supervisors, you have been a support and rock over the past four years. You were the calm in my storm and the voice of reason. You gently reminded me of who I am and what my strengths are. You never told me what to do but rather encouraged me to make the decisions. Although I would have preferred being told what to do, I have learnt more about myself in the process, and have grown in so many more ways (and more important ways) than my title. Your knowledge in the field; your experience; how you manage life; how you treat others and your gentle nature in dealing with people, have always had me captivated. You are an exemplary human being! Thank you for playing such an important part in my life. I have so much to thank you for.

Dr Graham Dampier, my second supervisor, thank you for all your kind words, and your belief in me. I enjoyed all of our stats sessions and talk-and-walk sessions. Even though it took a while for me to get on top of the quantitative side of my study and fix my language errors, you persisted and never gave up or altered your view on my capability. You constantly reminded me of my worth and contribution and for that I am eternally grateful. You are a fountain of knowledge, and yet so humble and real. Thank you for sharing some of that knowledge with me, and thank you for being so real, encouraging and open with me.

Marie, no words can express my gratitude to you. You were there for me in my darkest moments, and you loved me no matter what. Thank you for holding me accountable, and for being honest and direct with me. Thank you too for all the emotional and spiritual input you have had in my life.

Ian, my husband and best friend. Your love, patience and acceptance of me and my flaws have been overwhelming. It is more than I will ever deserve. Thank you for all the long hugs, for holding me and letting me cry, for sitting with me and just listening. Thank you for trying to help me think more deeply about my study and for reading some of my work. Thank you also for being a single parent, the main caregiver, lunch-maker, supper-giver, taxi-driver, puppy-patroller and homework-signer. I know it has been hard and that you have taken strain. No words can express how grateful I am and how much I love you.

Thank you too to Prof Jill Adler for not only the opportunity of being a PhD fellow but for all the various inputs you gave me over the past four years and for the opportunity to use your project's rich data.

A big thank you to the National Research Foundation for funding my studies for the last four years.

Thank you to the people who helped with capturing my data (Karabo and Michael) and who read and gave feedback on my work (the whole WMCS team). A special thanks to Yvonne Fontyn for editing my work, it could not have been an easy job.

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Chapter 1 : Introduction

"The algebraic skills of the learners are poor… because they cannot do the basic mathematics of Grade 8 and 9" (Department of Basic Education [DBE], 2014, p.126)

This thesis is located in the context of the persistent poor mathematics attainment in the majority of schools in South Africa, and in particular the poor performance in algebra and solving linear equations. The above quote is from a Grade 12 diagnostic report of learner errors and alludes to the fact that the algebraic skills of many learners are poor. Learners' poor performance is then often accompanied by responses that make no sense or are chaotic. This study is about creating order from the chaos of learner errors in mathematics in South Africa. With so many chaotic responses, I developed a methodology that makes sense of the chaos. I investigate learners' errors, how they change, and how they relate to each other in different mathematical topics. This chapter foregrounds the problem and rationale for this study, and provides the reasons that led to pursuing this topic.

1.1 Background and motivation

The background of this study rests on three main pillars: my personal interest in mathematics teaching and learning; how knowledge of learner errors can be useful in teaching and learning; and wanting to contribute to the body of knowledge of mathematics education locally and internationally. I elaborate on each of these below.

1.1.1. Experience as a student and teacher

My motivation for this study (and for originally wanting to be a mathematics teacher) stems from the deep personal desire to help learners improve the level of their mathematical achievement. As a teacher, I can assert that knowing what errors learners make informed my teaching in that I have been able to anticipate learner responses and talk to, but not eradicate, misconceptions before they develop. The teacher in me wants to help the learners who struggle; aid them in understanding more of the mathematical problems they face; and ultimately to see an improvement in their performance. I am both sympathetic and empathetic towards such learners because I was one of them. As a primary school learner I was reprimanded, punished and ridiculed for incorrect answers in mathematics. Making mistakes and not understanding certain topics was frowned on. It was only when I reached high school and had a mathematics teacher who took the time to explain my errors to me that I started to improve and develop a love for the subject.

1.1.2. Poor learner attainment and error analysis

There is consensus that learners entering secondary school struggle with mathematics because they enter the system with a rocky conceptual foundation, which, in turn, hinders the rest of their mathematics education. Spaull (2013, p8.) states that "the learning deficits that children acquire in their primary school career grow over time to the extent that they become insurmountable and preclude pupils from following the curriculum at higher grades, especially in subjects that are vertically demarcated like mathematics and science". One way to address these learning deficits is for teachers to address the errors. But first, teachers need to be aware of the errors. Conducting an error analysis provides one with the tools that will then allow for the diagnosis and remediation of these learning deficits.

1.1.3. Contribution to the body of knowledge

As a researcher, this study was motivated by my experiences in, and involvement with, the Wits Maths Connect Secondary Project (WMCS). In addition, this study was motivated by the awareness that there is a gap in research regarding the relationships between learner errors. Working on the Learning Gains (LG) Project, an off-shoot of the WMCS project, has given me the opportunity of working with large data sets that comprise learner responses to items on equality, integers, algebraic expressions and equations. The process of initial analysis of these data sets served as a reality check to me of the performance of possibly the majority of South African Grade 9 and 10 mathematics learners. Literature has shown that learner difficulties are not unique to South Africa. In fact, certain errors are made despite the teacher, school, curriculum and country (Godden, Mbekwa, & Julie, 2013; Smith, Disessa, & Roschelle, 1994). Literature does not, however, elaborate on the vast variety of learner errors, especially from South African learners. Nor is there evidence of a relationship between errors made in different topics. So, although errors are normal, natural and necessary (Brodie, 2014), this study shows that they are also vast in quantity and possibly interconnected.

1.2 Research problem

The crisis in secondary school mathematics learning in South Africa is evident in the poor learner attainment in local (DBE, 2016) and international assessments (Reddy et al., 2012). Embedded in the rationale for this study is the importance of algebra, and in particular, linear equations. Since algebra is a prominent topic in secondary schools, the poor performance of learners in mathematics in general could be linked to their poor performance in algebra. Solving linear equations is an important algebraic skill and literature shows that one area of difficulty for learners regarding this is in developing symbolic understanding (Arcavi, 2005; Stacey & MacGregor, 1999). This would include treating the equal sign as a relational symbol (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005) rather than an operator; dealing with the multiple uses of the minus symbol (Vlassis, 2004) and the multiple uses of letters (Küchemann, 1981), and in particular the simplification of algebraic terms. The South African curriculum document confirms that "recognising and using the properties of operations for different numbers provides a critical foundation for work in algebra when learners work with variables in place of numbers and manipulate algebraic expressions and solve algebraic equations" (DBE, 2011, p. 21). Learners' difficulties in understanding negative numbers, simplifying algebraic expressions and dealing with the equal sign as a relational symbol are evident in the plethora of literature and research conducted all over the world (these are discussed in more detail in Chapter 3).

Research shows that transitioning from concrete mathematics (arithmetic) to abstract mathematical concepts (algebra) is partially responsible for these difficulties (Kilpatrick & Izsák, 2008). Equality, integers, expressions and equations have been identified as key concepts needed for a smoother transition from arithmetic to algebra (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Some attempts to alleviate this problem have been to introduce algebra as a strand that appears in every grade rather than just a Grade 8 or 9 mathematics course (Carraher et al., 2017). Some advocate the use of manipulatives (Sherman & Bisanz, 2009) using different representations (Font, Godino, & D'amore, 2007) and exposing learners to multiple strategies (Durkin, Star, & Rittle-Johnson, 2017) to bridge the gap. Interestingly, the South African curriculum document does not mention the different ways one could teach, for example balance, sameness or equality, only that it can be done through inspection, trial and error, and the use of inverses.

1.3 Focus of the study

My study is grounded in the use of three symbols: the equality symbol; the minus symbol, and letters. It also has two foci: changes in errors and the relationships between the errors made when solving linear equations and errors made in other topics. Figure 1.1 shows four learner responses to the same equation and highlights the errors made. The errors are highlighted and colour-coded to match the different topics I focus on. The blue highlight is to show an error within equations, the green an error relating to negative numbers, the orange a letter error, and the yellow an equality error. How errors were coded is discussed in detail in Chapter 5, while chapters 7, 8, 9 and 10 discuss the errors made in the different topics. Learner 1 has made an inverse error (-2 instead of $+2$) and then also incorrectly added $-2+4$ and got -6 instead of 2. Learner 2 made a letter error when 4 was added to x to obtain $4x$. This error was repeated with $4x+2 \rightarrow 6x.$ The learner then made an additional error and simplified $\frac{6x}{3} \rightarrow 2$. Learner 3's response shows a letter error and an equality error where the equation is converted into an expression and then back to an equation. In Learner 4's response we see the learner substitute $x=2$ but does so only on the right hand side. By doing this the learner gets a value of 4 and so ignores the fact that there is an addition $+x$ on the left hand side.

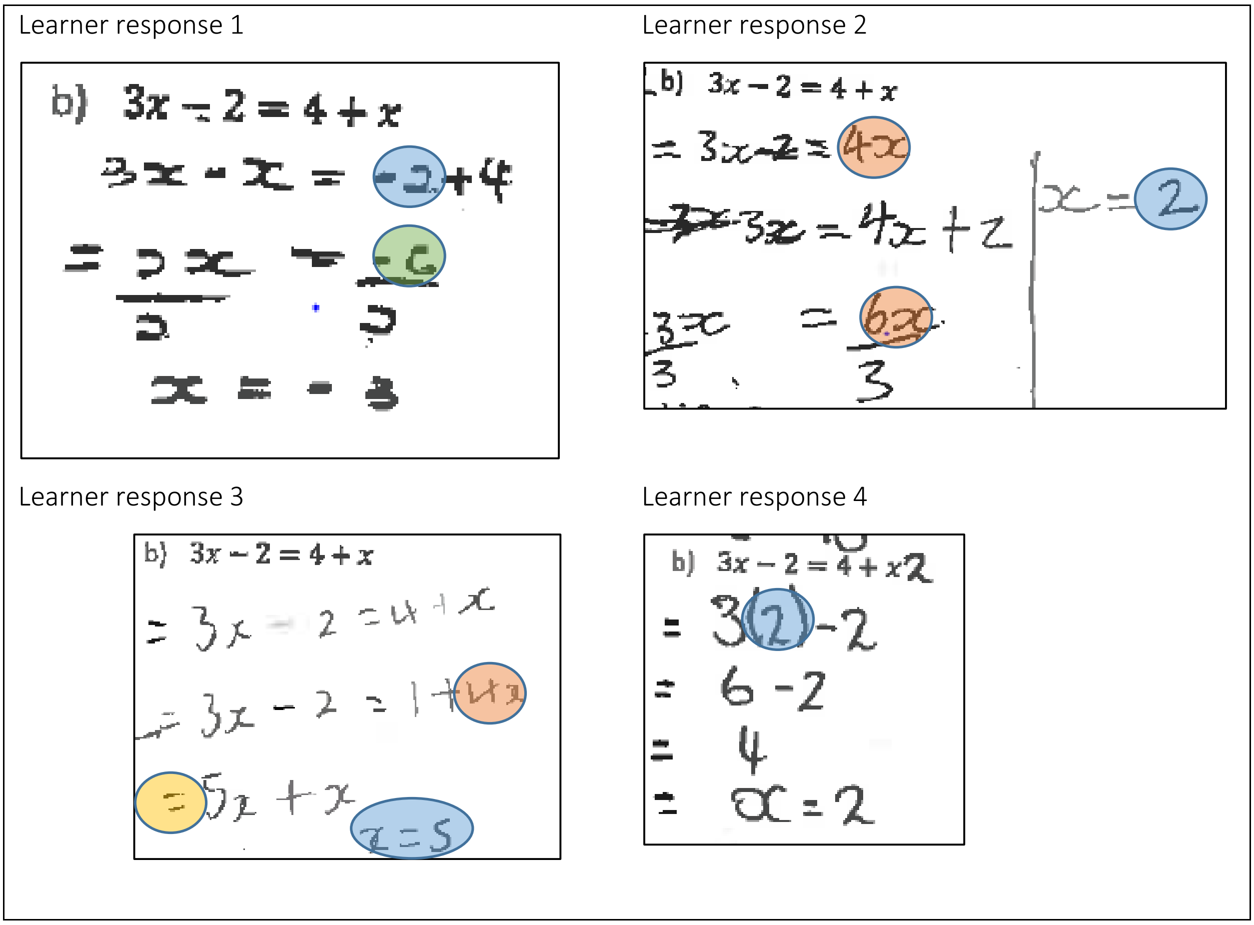


Figure 1.1: Highlighting errors made in learner responses

In order to investigate the change and interrelatedness of learners' errors, an error analysis on pre- and post-tests was conducted. These foci point towards to a mixed-methods approach where the error analysis forms the qualitative component and the relationships between errors form the quantitative component.

1.4 Research questions

My dual focus on changes in learner error and the interrelatedness of errors leads me to three main research questions:

1) What is the general performance of a cohort of Grade 9 and 10 learners in terms of simplifying integers, simplifying expressions, solving equations and understanding the dual role of the equal sign?
2) To what extent do learners' errors with integers, equality, algebraic expressions and linear equations change over time?

   This question relates to the qualitative component of the mixed-methods design and involves analysing learner responses and coding the errors made. At a simplistic level, it is about listing, counting and comparing the type, and number of errors made. There is scope to compare the results within each item, topic and grade as well as across items, topics and grades. In order to answer this question, the following sub-questions need to be asked:

a) What errors are made when dealing with integers, expressions, equality and equations in Grade 9 and Grade 10?
b) Which errors disappear, are persistent or change over time?

3) To what extent are there relationships between the errors?

1.5 Significance of the study

Although this study has similarities to other research regarding solving equations and learner error (Falle, 2009; Hall, 2002), it has differences that begin to address gaps in research. This study draws on learner errors, a topic that has been researched and reported on for many years, yet it contributes to the field in at least three ways: methodologically, practically and theoretically.

1.5.1. Methodological contribution

There is a gap in quantitative research on learners' errors in mathematics education. In general, educational research is predominantly qualitative, but in mathematics education in particular, mixed-methods research is used infrequently as an approach (Hart, Smith, Swars, & Smith, 2009; Lopez-Fernandez & Molina-Azorín, 2011; Ross & Onwuegbuzie, 2012; Truscott et al., 2010). This mixed-methods study is about the changes in learners' errors as well as the relationships between the errors, and as such goes beyond merely presenting descriptive statistics. The rationale for adopting a mixed-methods approach is that in quantitative studies, where coding consists mainly of incorrect, correct and missing responses, we learn very little about what learners can or cannot actually do. However, when coding is more detailed, the sample is typically smaller. Hence, quantitative work on its own is limited. On the other hand, conducting a large qualitative error analysis is not feasible. Conducting a small qualitative error analysis, although useful, is limited because it cannot be generalised to a larger population. Hence, the need for a mixed-methods approach to address the limitations of conventional quantitative and qualitative research. Throughout the analysis of my data I move between qualitative and quantitative methods. I qualitatively code learner errors and then count the number of occurrences of that theme and present the qualitative data quantitatively. In Chapter 11, I compare the qualitative findings of the full data set with the sub-sample to show that my results are generalisable. This is another example of how I have created order from chaos. Having a large data set is chaotic and being able to code and generalise results from the sub-sample is a way of creating order.

1.5.2. Theoretical contribution

My theoretical contribution is an output of this study. I did not start this research with these ideas in mind but rather, it came as a way to explain the data. I therefore end my thesis with the theoretical contribution rather than have it in the theoretical framework section.

A common thread that runs through all my analyses points towards Grade 10 learners making more gains than Grade 9 learners on Grades 8 and 9 mathematical content. This implies that there is a delay in the learning of concepts and skills which should have been mastered in Grade 8 or 9 as these are only starting to be mastered in Grade 10. This delayed learning points towards learners needing more 'synk-time' for content and skills to settle in their minds. 'Synk-time' is a theoretical construct I propose and define in full in Chapter 12. It is a combination of 'sink' and 'sync', implying that it is a combination of time for content to 'sink-in' and the *sync*hronisation of concepts from, by drawing on Vygotskian concepts, the external social world to the internal individual world.

In order to explain Grade 10s making more gains than Grade 9 learners, I argue that learners need time for content to settle in their minds and built this idea by integrating three theoretical constructs -- one from Vygotsky and two from language acquisition: Vygotsky's (1978) notion of *internalisation*; Tomasello's (2003) notions of *intention reading* and *pattern finding*, and Gopnik and Meltzoff's (1997) notion of *The Theory Theory*. The integration of these constructs and the notion of 'synk time' is a theoretical contribution of this thesis. This integration is a contribution and outcome of my study and is therefore discussed in Chapter 12 rather than in Chapter 2, which is the theoretical framework.

1.5.3. Relationships between errors

Although there is a vast body of literature on error analysis in mathematics education (see for example Brodie (2014); Hatano (1996); Makonye and Luneta (2014); Nesher (1987); Ryan and Williams (2007) and Smith et al. (1994)), and although research already exists on the change of errors over time (e.g. Rittle-Johnson, Matthews, Taylor, and McEldoon (2011)), errors tend to be discussed as discrete and isolated problems of learning. I have analysed errors in relation to one another, meaning, within and between mathematical topics to determine whether certain errors are carried through into other topics. I also investigated whether the same mathematical content produces the same or different errors depending on the topic. For example, do learners add unlike terms by operating on the numbers and then attaching the letter (for example $2x + 5 \rightarrow 7x$) when simplifying an expression as well as when solving an equation? Identifying relationships between errors and items takes an error analysis a step beyond reporting what the common errors are, hence being a contribution to both research and teaching. In all the analysis chapters

(Chapters 7-11) I show the correlation of each topic's scores against the other topics to show the relationships between the topics.

1.6. Organisation of the thesis

My dual focus on change in, and the interrelatedness of, errors as well as the mixing of research methods has led me to conduct this study in two phases, where phase 1 results are presented in Chapter 6 and phase 2 results in Chapters 7-10. Chapter 11 is a synthesis of the quantitative results from all the analysis chapters and Chapter 12 discusses my theoretical contribution. Every PhD journey is different. In my study I started with analysing the data, initially using codes from literature. Findings from the analysis led me to search for theoretical constructs that could explain what I was seeing. Three theories, one related to Vygotskian concepts and two related to language acquisition, were helpful in understanding my findings. The interrelatedness of these three theories is a contribution and an outcome of this study and is therefore discussed after my analysis chapters.

I begin with this introductory chapter and what follows is the theoretical framework and literature review (Chapters 2 and 3). I then present two methods chapters: Chapter 4, the research design, and Chapter 5 on the codes used on learner responses. Chapters 6 through 11 are all analysis chapters, with Chapter 11 pulling together all the quantitative analysis done in this study. Chapters 12 and 13 are two concluding chapters, with Chapter 12 discussing the theoretical constructs that emerged from the data analysis. Chapter 13 is the final concluding chapter. Figure 1.2 is a diagrammatic outline of the thesis.

To re-emphasize, theoretical constructs, such as synk-time, were an output of this thesis and not an input. It was not an idea I had when I started the analysis and therefore it appears at the end of the thesis and not in the literature review.

Figure 1.2: Organisation of the thesis

Chapter 2 : Literature review, theoretical framework

2.1. Introduction

The focus of this study is on learners' errors when solving equations, the relationship between these errors and the errors made in other topics such as integers, expressions and equality. This study is located in the context of the persistent poor mathematics attainment in the majority of schools in South Africa. There are five basic assumptions that guide this study and hence guide the structure of this chapter: a) mathematics is a human activity; b) learning is mediated; c) algebra is generalised arithmetic; d) errors made are normal and useful in learning and teaching and e) poor performance is due to a lack of basic skills. I discuss the above by referring to *views* on: mathematics, learning, algebra; errors and poor performance. In Chapter 3, I provide a literature review on symbolism in the transition from arithmetic to algebra.

2.2. View of mathematics

One's view of the nature of mathematics has important implications for the development of school mathematics curriculum, instruction, and research (Dossey, 1992). Researchers agree that there are two overarching competing views of the nature of mathematics. Lakatos (1976) distinguishes between the Euclidean and quasi-empirical views; Lerman (1990) between absolutism and fallibilism; and Ernest (1985) discusses two dichotomies in the nature of mathematics: a prescriptive-descriptive and the process-product distinction. Viewing mathematics as prescriptive or as a product means to view mathematics as logic, ideal structures or formal structures (Ernest, 1985). This is where all of mathematics is reduced to rules and formulas without reference to the meaning behind the symbols. In addition, a prescriptive view of mathematics legitimises how mathematics should be understood "rather than providing accurately descriptive accounts of the nature of mathematics" (Ernest, 1992, p. 89). Viewing maths as a product, therefore, overshadows the idea of mathematics as coming to know which has social dimensions (Ernest, 1992). A process or descriptive view of mathematics would be to see mathematics as a construction or a human activity. This means that the "meaning of mathematical objects consists of the processes by which they are constructed" (Ernest, 1985, p. 607). Within this fallibilistic view of mathematics, Lerman (1990) states that teaching mathematics is about "processes rather than content" (p.53) and that uncertainty and incorrect answers should be embraced and used for learning. I adopt a process view of mathematics and this is evident in the social constructivist view of learning and errors that I take in this thesis

2.3. View of learning

2.3.1. Broad view of learning

Social constructivism is a variant of constructivism that emphasises the social nature of learning. Lev Vygotsky is the founder of social constructivism and although he was a cognitivist, he rejected Piaget's assumption that it was possible to separate learning from its environment, and that learning was based on maturation stages (Vygotsky, 1986). Piaget (1964) argued that in order to be able to reach full development we need to pass through four stages (sensori-motor, pre-operational, concrete operational and formal operational). Although he believed that cognitive development is influenced by social transmission, Piaget asserted that development precedes learning, meaning that one needs to mature first before learning can take place. Vygotsky (1986) agrees that learning happens through our environment but claims that learning precedes development, ultimately believing that age does not determine what you can learn. Vygotsky claims that our current stage of development is enhanced when confronted with tasks that aim to bridge the gap between our pre-existing development stage and the next stage. This gap is called the *zone of proximal development* (ZPD). He focused on how instruction or social engagement enhances one's journey through the ZPD as one internalises the new information or skill. This explains how learning precedes development, in opposition to Piaget. Another difference between Piaget and Vygotsky, and hence between constructivism and social constructivism, is the role of the more knowledgeable other. For Piaget, their role is to support while for Vygotsky their role is to provide opportunities for the child to learn.

2.3.1.1. Vygotsky's theory of learning

According to Vygotsky (1978), one of the central elements of teaching and learning is semiotic mediation. As an example, in mathematics education, learners need to be able to understand that $2x$ means two multiplied by x, but also $x + x$. My study deals with mathematical signs and symbols such as the equal sign, operation signs, letters and numbers, and in mathematics, and in particular, algebra, these symbols are important. This is because success in mathematics in general requires learners to use these symbols appropriately to ensure a smoother transition from arithmetic into algebra as well as to ensure success in solving linear equations (Stacey & MacGregor, 1999).

Vygotsky's theory of learning asserts that social interaction is fundamental to cognitive development (Vygotsky, 1978), which implies that learning is a social process where society and culture play an important role in the development of higher mental functions. This means that learning is mediated through people as well as with the use of signs. Due to the type of data I collected, I limit my discussion to the notion of semiotic mediation and do not draw on, for example, the role of language, a more knowledgeable other or the zone of proximal development.

2.3.1.2. Semiotic mediation

Semiotic mediation is the use of signs and tools to mediate learning. Vygotsky offers examples of semiotic mediators such as language, everyday signs (for example, a yield sign, the Twitter icon and emoticons), esoteric signs (for example, a mathematical graph, the equal sign or subtraction sign), or mnemonic aids (for example, in trigonometry, SohCahToa). I focus on the esoteric signs aspect of semiotic mediation. "In the context of mathematics education, a sign may be a mathematical symbol, a mathematical statement, a mathematical expression, (or) the name of a mathematical object" (Berger, 2004, p.83). In order not to confuse the Vygotskian use of signs with the mathematics operation signs, I will refer to *signs* (italicised for the Vygotskian perspective) and signs (un-italicised for the mathematics use of signs).

Semiotic mediation occurs when learners attach meaning to a *sign* (or symbol) that was not previously relevant to their lives. Vygotsky (1978) states that the role of *signs* is that they are oriented inward, meaning that they are "a means of internal activity aimed at mastering oneself" or mastering one's behaviour (Vygotsky, 1978, p. 55). In other words, *signs* are used to mediate the mental world of an individual. They are symbols that cause an intended reaction but do not interfere directly with the process. In relation to mathematics, a graph with two functions is a *sign* that sometimes enables the intended reaction of solving simultaneous equations. The learner needs to learn to read the intention of the question and symbols to know whether, for example, they are required to solve an equation or simplify an expression. The *sign*, however, does not interfere with the procedure of solving an equation. Berger (2005) explains that using a word or *sign* to refer to an object before a learner has come to know something resonated with her sense of how undergraduate students made a new mathematical object meaningful to themselves. She continues by saying that in practice, the student would communicate with other students using the *signs* of the new mathematical object before they have fully understood the mathematical sign. It is then this communication, and this functional use of *signs* that gives students initial access to the new object. Vygotsky elaborates and says that "a functional use of a word, or any other sign, as a means of focusing one's attention, selecting distinctive features and analysing and synthesizing them, that plays a central role in concept formation" (Vygotsky, 1986, p. 106). This means that it is only through the use of a mathematical sign that one's attention is drawn to the emerging structures, and that synthesising the signs, or internalising them, is what helps a learner come to know something.

2.4. View on the nature of Algebra

Purposes for algebra are determined by, or are related to, different conceptions of algebra, which correlate with the different relative importance given to various uses of variables (Usiskin, 1988, p. 9).

There are different schools of thought about how algebra can be approached. Some authors have proposed three approaches to algebra (e.g. Thornton (2001), while others have proposed four (e.g. Usiskin (1988)). Some describe what algebra is rather than how it can be approached (Watson, 2009), and others have described algebraic activities (Kieran, 2004) and forms of algebraic reasoning (Kaput, 2000). Algebra can be viewed as different activities and can be approached and defined in different ways. Having reviewed the literature, key characteristics of algebraic activities can be identified as being able to generalise, formalise and symbolise, and these characteristics are related to the different approaches to school algebra (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Kieran, 1990; Sfard, 1991). How algebra is approached will influence how it is reasoned. For example, if solving $2x - 1 = 4x + 3$ is approached structurally by solving the equation, a learner is less likely to reason functionally by drawing $y = 2x - 1$ and $y = 4x + 3$ and finding the intersection.

Kieran (2004) proposed a model of algebraic activity that involves three activities: generational (for example, generating expressions and equations); transformational (working to produce equivalent expressions); and global meta-level activities (generalisation, justifying, and proving). Kilpatrick, Swafford and Findell (2001) offer a similar model, with three elements aligning with strands of mathematical proficiency: representational activity (aligning with conceptual understanding of operations and symbols and strategic competence: formulate expressions or equations); transformational activities: employing the rules and aligning with procedural fluency, conceptual understanding and strategic competence; and generalisation/justification, which aligns with adaptive reasoning and productive disposition. What both these models suggest is that algebra is about symbolising (transformational activities); formalising (generational and representational); and generalising. Kilpatrick et al. (2001) state that the strands of mathematical proficiency are interwoven and should not exist in isolation. So too are the approaches to algebra. I discuss below different conceptions of algebra.

2.4.1. Algebra as generalised arithmetic (and patterns)

With arithmetic one deals with concrete numbers and operates on these numbers using operators such as addition, subtraction, multiplication and division. When we substitute letters for numbers we generalise arithmetic operations. Usiskin (1988) states that the conception of algebra that is adopted correlates to how a variable is used, he argues that in this conception a variable is a pattern generaliser. For example, the commutative property of addition $5 + 3 = 3 + 5$ could be generalised to $x + y = y + x$ in algebra. This is similar to how Watson (2009) describes algebra: as generalisations of laws about numbers and patterns. Where Watson (2009) includes generalising about patterns in the generalised arithmetic approach, Thornton (2001) specifically proposes a patterns approach rather than a generalised arithmetic approach. He argues that taking a patterns approach is about alternative representations and about visualising the problem in different ways producing different yet equivalent expressions. Some researchers believe that a patterns approach is not helpful in learning algebra and that it relies on "thinking processes not generally believed to be needed in early algebra" (Warren, 2003, p. 123). Caspi and Sfard (2012) explain the generalised arithmetic approach as formalised meta-arithmetic, where algebra is defined as a form of communication. Algebra as generalised arithmetic is the traditional approach to algebra (Kieran, 1992), as arithmetic is widely believed to precede algebra. This is mainly because arithmetic is easy and algebra difficult; arithmetic is about operations involving particular numbers where algebra is about generalizing numbers (Schliemann, 2013). Adopting this conception, algebra can be defined as a branch of mathematics where the principles of arithmetic are generalised by using letter symbols to represent numbers. Even though algebra can be viewed as a generalised form of arithmetic, there are fundamental differences between them that make transitioning from arithmetic to algebra difficult. This is discussed in greater detail in Chapter 3.

2.4.2. Algebra as the study of relationships or functions

This functional approach to algebra is one that accepts algebra as the study of relationships. Usiskin (1988) includes modelling and functions in the approach. With this conception of algebra, he argues that the variable is an argument or parameter. Watson (2009) describes this part of algebra as learning about variables, functions and expressing change and relationships. Function is a key algebraic topic in secondary school and should include both a pointwise and global approach (Kieran, 2007). Researchers have made efforts to study the learning of functions through software (Schwartz & Yerushalmy, 1992) and the difficulties experienced by connecting multiple representations, for example with the slope of a linear function. (Zaslavsky, Sela, & Leron, 2002). Thornton (2001) argues that this approach emphasises the application of graphs and encourages learners to represent situations in words. This however seldom happens in secondary schools.

2.4.3. Algebra as problem solving and modelling

Algebra can also be viewed as a language and a tool to study the types of the relationship between specific variables in certain situations. This makes algebra powerful in that it can provide models to describe and analyse situations as well as to obtain additional, unknown information about the situation (Watson, 2009). Watson (2009) describes this view of algebra as being able to model mathematical structures of situations within and outside mathematics. A problem solving approach requires learners to think beyond routine procedures and hence is an amalgamation of concepts and representations and requires mastery of basic mathematics skills. This view of algebra is linked to real world mathematics (Stillman & Galbraith, 1998) where ‘problem solving’ has a very specific context and is not about, for example, solving an equation problem such as $2x - 5 = 10$.

2.4.4. Algebra as procedures and structures

This approach to algebra can be distinguished into two conceptions, one about procedures and one about structures. For a procedural conception Usiskin (1988) views algebra as procedures used for solving specific problems and a variable is viewed as an unknown, which is closely related to solving linear equations. An equation has expressions combined by an equal sign. To solve an equation correctly, the student must know the rules or procedures of simplifying algebraic expressions. Watson (2009) describes this part of algebra as manipulation and transformation of symbolic statements as well as the rules for transforming and solving equations. Thornton (2001) would include this approach to algebra in what he calls formal symbolic algebra and Kaput (2000) terms this approach ‘algebra as syntactically guided manipulation’. For the conception of algebra as the study of structures, Usiskin (1988) relates this category to understanding a variable as an arbitrary object. The variable can take on any value, hence being an arbitrary object. Watson (2009) describes this part of algebra as the study of structures and systems abstracted from computations and relations. Kieran (1992) refers to a procedural structural approach, where procedural refers to arithmetic operations that get carried out on numbers to get numbers as the result, and structural refers to operations that get carried out on algebraic expressions. Arithmetic operations involve a process conception while operations carried out on algebraic expressions involve an object conception. As an example, replacing x with 2 and y with 4, $3x + 2y$ becomes 14. Here the object is no longer the expression but rather a numeric process. This forms part of the procedural aspect of algebra. Simplifying $2a + 5b + 4a$ leads to $6a + 5b$ where what was operated on were objects and not numbers. This forms part of the structural aspect of algebra.

It is important to acknowledge that from a teaching perspective, any approach in isolation would be inadequate because it would not promote connections between representations. Bednarz, Kieran and Lee (1996) state that the separation into four approaches is artificial as all the conceptions are needed in any algebra programme. Although I concur with Bednarz et al. (1996), my study focuses on learners’ responses to items involving integers, algebraic expressions, equality and solving equations and hence I adopt a generalised arithmetic view because the topics in focus lean on a smooth transition from arithmetic to algebra.

2.5. View on the nature of mathematical errors

"One way of trying to find out what makes algebra difficult is to identify the kinds of errors students commonly make in algebra and then to investigate the reasons for these errors" (L. Booth, 1988, p. 20).

There is a plethora of literature on analysing learners' mathematical errors, specifically on the nature of learners' errors and possible reasons for their underlying misconceptions (Booth, Barbieri, Eyer, and Paré-Blagoev, 2014; Borasi, 1987; Erlwanger, 1973; Gardee & Brodie, 2015; Godden et al., 2013; Herholdt & Sapire, 2014; Hodgen, Brown, Coe, & Küchemann, 2012; Olivier, 1989; Radatz, 1980; Ryan & Williams, 2007; Smith et al., 1994). The extent of literature is evidence that the analysis of learner error has been of great interest to the mathematical community for many years. It is justified as being important from a theoretical point of view as well as for its pedagogical implications (Corder, 1982): mathematical errors can provide valuable insights for a teacher into their learners' thinking (Brodie, 2014). Willis and colleagues (2007) suggest that "understanding the mathematics that children understand helps us make better professional judgements" (p. 24) and as a result makes us better mathematics educators. Investigating errors informs teachers on how instruction could be altered and how tasks could be designed to address the problems that the error analysis helped diagnose (Hiebert & Carpenter, 1992).

As mentioned earlier, I hold a social constructivist view of learning. From this perspective, errors are embraced and seen as opportunities for learning (Nesher, 1987) and should therefore be celebrated (Falle, 2009) - although within reason. Drawing on Nesher's (1987) theory of errors, I view errors as rational and rooted in some (mis)understanding of prior knowledge. I also adopt the view that errors are "normal and necessary" (Brodie, 2014, p. 221) and that "each error has the potential to become a significant milestone in learning" (Nesher, 1987, p. 39). Students need to learn how algebra is mediated and therefore errors are natural and useful in coming to know how it is mediated. A person or tool can mediate an action or a process and so when I speak of mediating algebra I am referring to the process of doing algebra. To be clear, mediation is the process where the social and the individual worlds mutually shape each other. The social (the teacher, worksheet, exercise, etc.) shapes the individual (current understanding held by the student) and vice versa. For example, when learning about like terms in mathematics a student may assume that $3x$ and 3 are like terms because they both have the numeral 3. However, when the learner encounters other examples, worksheets or questions (the external or social world) it will cause the student to re-evaluate what they previously assumed and hence helps shape the individual world. This is how the social world mediates the process of algebra. Errors help us understand where in the process learners are struggling. All the authors mentioned above agree that learners' errors should be viewed as an opportunity for teaching and learning, and not as a problem that should be avoided. Errors provide insight into learner thinking and as Falle (2009) states, they are a window into the mind of learners. In addition, Olivier (1989) states that errors play a central role in the mathematics classroom and reflect the way in which learners reason.

Some errors that are made appear illogical to the teacher or researcher. However, there is often an underlying cause, which makes that the error is not random but a logical and rational attempt (Ben-Zeev, 1998) that “originated in a consistent conceptual framework based on earlier acquired knowledge” (Olivier, 1989, p. 8). In general, some errors that appear illogical are left unexplained by the researcher and hence are classified as an unknown error.

The categories of errors used in an error analysis are dependent on the mathematical content as well as the focus of the study. Some errors are only disaggregated into very broad categories. Taylor (1997) distinguished between random and systematic errors and Olivier (1989) distinguished between three types of incorrect responses: slips, errors and misconceptions, where misconceptions generate specific errors. In contrast to errors being systematic with some underlying conceptual cause, slips are defined as a lapse of concentration while solving a task (Nesher, 1987). These are easily rectified when pointed out (Olivier, 1989), whereas systematic errors are more complex (Luneta & Makonye, 2013). Literature provides three reasons for this:

1) Errors are persistent and resistant to instruction (Godden et al., 2013; Smith et al., 1994);
2) Errors arise even though they are not explicitly taught (Hatano, 1996); and
3) Similar errors occur amongst learners in different classes, schools and even countries (Hodgen, Küchemann, Brown, & Coe, 2009).

In contrast to Taylor (1997) and Olivier (1989), Bell (1994) offers narrower categories of errors and describes them in terms of erroneous notation and manipulation. Godden et al. (2013) distinguish between application, procedural, calculation and careless errors, while Booth, et al. (2014) provide domain-specific categorisations consisting of mathematical properties, equality/inequality, variable errors, operation errors, negative sign errors and fraction errors. Similarly, Hall (2002) categorised errors into domain-specific categories that relate to solving linear equations. Brodie (2015) investigated errors from a teacher’s point of view to identify how a teacher uses the errors learners bring. She found four types of errors: language related; slips; misconceptions and errors from incorrect calculator usage. Movshovitz-Hadar, Zaslavsky, and Inbar (1987) classified learners’ errors into five categories: misused data; misinterpreted language; logically invalid inference; distorted theorem or definition; unverified solution and technical error. The content in focus for their study was logarithms, probability, geometry, quadratic equations and trigonometry. The focus of my study is the different types of systematic errors within domain-specific categories, as was done with Hall (2002) and Booth et al. (2014). The content in focus is equality, integers, expressions and solving equations. In relation to my view of learning, the errors learners make are indicative of either reading the intention of the question incorrectly or having internalised the incorrect pattern and having the incorrect theory. When an error changes it reflects a change in theory or rather, a change in the patterns established. A correct answer shows the correct established patterns and correct internalisation that have occurred.

2.6. View on poor mathematical performance

Clearly there are many factors that have been found on school, class and student levels to have positive and negative effects on mathematics achievement (Howie, 2003, p. 3).

Despite the myriad research into learner understanding of mathematics, it continues to be portrayed as a difficult and intimidating school subject (Arcavi, 2008). It is also a subject that is poorly understood both on an international and national scale (DBE (2019); Kieran (2007)). The 2015 Trends in International Mathematics and Science Study (TIMSS) of the worldwide trends in learner performance in Mathematics and Science (Zuze, Reddy, Visser, Winnaar, & Govender, 2017) revealed that internationally, learners' performance in algebra is very weak. Although South African mathematics achievement scores in TIMSS have improved from a 'very low' (1995, 1999, 2003) to a 'low' (2011, 2015) national average, we remain one of the lowest-performing countries in mathematics in comparison with other TIMSS participating countries. Even with the improvement, most South African Grade 9 learners are not yet able to achieve a minimum level of competency in mathematics.

It is widely accepted that mathematics is difficult to learn, but it is it important to remember that the difficulties associated with the teaching and learning of mathematics are multidimensional (Ball, 1960). There are many interrelated 'problems' that influence the teaching and learning of mathematics in South Africa (Setati & Adler, 2000). Some of the contextual factors that make learning (mathematics in particular) difficult are related to language, poverty, teacher knowledge, reading ability, curriculum and lack of basic numeracy. Some factors are international problems, for example curriculum, but some are more pertinent to South Africa, for example language and poverty. Some of these are briefly discussed below.

2.6.1. Lack of basic mathematics skills

South African learners perform poorly in tests that measure knowledge of basic mathematical skills (Spaull, 2013). This is particularly concerning because learning algebra relies on the knowledge and application of basic mathematics skills such as numbers (Cox, 1975), fractions (Booth, Newton, & Twiss-Garrity, 2014), integers (Vlassis, 2008), and adding like terms (Kieran, 1992) and the view of the equal sign (Knuth et al., 2005). A report on the TIMSS study (Zuze et al., 2017, p. 13) states quite explicitly that “South African learners struggle with displaying knowledge and understanding of basic mathematical concepts.” Zuze et al (2017) continue to state that the lack of basic mathematics concepts could point to teaching challenges and that senior phase teachers often have to reteach basic concepts before progressing to abstract ideas. One reason that a lack of basic skills hinders future learning is because learners have not yet made the transition from arithmetic to algebra. They have not progressed from concrete to abstract ideas. Research has made suggestions on how to bridge this gap, for example the introduction of early algebra in primary school (Carraher et al., 2017); focusing on the meaning of symbols (Stephens et al., 2013); and using concrete manipulatives (Moyer, 2001).

2.6.2. Curriculum

Research has also shown that curriculum pace, variation and coverage are issues that can hinder learner performance (see for example Cai & Knuth (2005); Cai & Moyer (2006) and Reeves & Muller (2005)). In the TIMMS 2015 report, Zuze et al. (2017) state that a difficulty in learning mathematics is due to a lack of progression and thread in the curriculum. The CAPS curriculum appears too packed, with few links to the different representations of, for example, functions. Teachers struggle to get through the content because they often have to reteach basic concepts, or even worse, they move on to new concepts with previous ones not mastered, all for the sake of coverage. This affects the depth of teaching and the amount of time dedicated to mastering procedures (Zuze et al., 2017).

2.6.3. Teacher knowledge

Although the curriculum provides learners with opportunities to develop their mathematical skills, teachers arguably have the most impact on what learners actually learn. Therefore, a learner’s success in developing algebraic thinking largely rests on the ability of a teacher to foster such thinking. It is a debated topic as to whether it is ethical to measure and expose teachers’ level of mathematical knowledge as it affects their confidence (Sanders & Morris, 2000). Some research, however, has attributed learners’ poor performance to teachers’ lack of pedagogical content knowledge (Mji & Makgato, 2006). Without exposing teachers’ levels of mathematical proficiency, Pournara, Hodgen, Adler, and Pillay (2015) were able to show that improving teacher knowledge (no matter their starting point) can improve learner performance.

2.6.4. Language

Various studies have indicated that language is one of the major factors contributing to the poor performance of many students in mathematics (see for example Bell (1998); Howie (2003) and Setati & Adler (2000)). Bell (1998) states that the language of a student influences how the student will interpret and build understandings. Hence, without sufficient language to communicate the ideas being developed, students will have their mathematical development seriously restricted. One suggested approach to this problem is that one must promote and encourage English Second Language learners to use their home language as a resource for sense making and understanding of mathematical concepts (Setati & Adler, 2000). Language and mathematics education will always be interrelated and mathematics will always depend on language (Gorgorió & Planas, 2001). Language will therefore always provide tensions in the teaching and learning of mathematics.

2.6.5. Reading ability

The TIMSS 2015 study (Zuze et al., 2017) showed that many South African learners are unable to read with understanding, express their thoughts in words and structure ideas in a logical manner. They also found that with mathematics being a language in itself, learners who struggled with sentence structure in language also struggled with mathematics. Isphording, Piopiunik and Rodríguez-Planas (2016) conducted quantitative research on the relationship between reading and mathematical performance and found that raising a learner's reading performance by one standard deviation improved their mathematics performance by 0.57 standard deviations. A second quantitative study examined the relationship between working memory, reading ability, and a learner's ability to solve algebraic word problems and found that the literacy measures predictor provided a unique contribution to the prediction of mathematical performance (Lee, Ng, Ng, & Lim, 2004). When all predictor variables were used, they accounted for 49% of the variation in mathematical performance. This further shows how learning mathematics is indeed an amalgamation of different aspects.

2.6.6. Poverty

Research shows strong and consistent links between poverty and poor academic competence (see for example Fan (2012); Reardon (2011) and Spaull (2015)). Studies such as these indicate that poverty-stricken children are at high risk for performing worse academically. Reasons for worse performance include malnutrition and not being able to afford the required text books, equipment and other necessary resources that encourage learning at home and school (Fan, 2012). Although many interventions exist and have been found to be effective in reducing the gap between the 'haves' and 'have nots' (see Sirin (2005) for a review of the literature), these interventions deal with, for example, class size and extra sessions which address the poor performance rather than attempting to address the poverty. There is little research on how for example a food scheme could improve learner performance. Although not written for a paper, a principal in a school in Cape Town distributed lunch to learners so that they ate at least three times in a day. This small intervention produced learners who could concentrate for longer, which improved their overall academic performance (personal email communication with Roberts (2020)).

It is clear that mathematics teaching is an amalgamation of social, cultural, political and linguistic factors (Gorgorio & Planas, 2001) and that many interrelated factors affect learning. I have only briefly discussed five factors, but many more are at play. Although they are all important to consider, my focus on errors made in elementary algebra with Grade 9 and 10 learners lends itself to framing my study around the lack of basic numeracy and mathematical skills being a large contributing factor to poor performance.

2.7. Conclusion

In this chapter, I addressed the five assumptions that guide my study. My view of mathematics affected the approach I took in this study, which in turn influenced how I view errors. I started this chapter by looking at mathematics as a human construct in contrast to mathematics as a product. I then discussed social constructivism, a broad view of learning. I further discussed my view of errors and explained that learners need to learn how algebra is mediated and therefore that errors are normal and useful in coming to know how they are mediated. I also discussed how errors are compounded by other issues such as basics skills, which points towards a misunderstanding of how mathematics is mediated more broadly.

An outcome of this study and a contribution of this thesis is integrating three theoretical components in order to explain my findings. This is discussed in Chapter 12, after the data analysis chapters.

Chapter 3 : Literature review

A "literature review helps to determine whether the topic is worth studying, and it provides insight into ways in which the researcher can limit the scope to a needed area of inquiry".

(Creswell & Creswell, 2017)

3.1. Introduction

This literature review is a comprehensive summary of articles that relate to my topic, which is an investigation of the performance of secondary school learners: the relationships between errors made when solving linear equations. The purpose of this chapter is to familiarise the reader with both prior and current knowledge of the topic and to identify the gaps in research where I make a contribution to the mathematics education community. I not only present key findings from other research but use some of these to form the foundation of my investigation.

My study is grounded in the use of three symbols: the equality symbol; the minus symbol and algebraic letters. The equality symbol relates to literature on the equal sign, equality and equations. The minus symbol relates to literature on negatives, and letters relate to literature on expressions. In this chapter I not only present a discussion on these content areas but also discuss why they are problematic for learners. I therefore begin with talking about how symbols present a problem to learners and then discuss the transition from arithmetic to algebra. This is followed by a literature review on each of the content areas: the equal sign; expressions; negatives; and linear equations.

3.2. Symbolism and symbol sense

Symbolism in the context of mathematics describes the use of symbols to represent a mathematical relationship or concept. Cobb (2002, p. 17) defines symbols in a mathematics context as "a concrete entity that stands for or signifies a mathematical idea or object or concept or process". Similarly, Hiebert (1988) defines symbols as entities that represent mathematical ideas or processes. Pimm (2002) provides three attributes of a mathematical symbol:

1) Materiality, which refers to what the symbol looks like;
2) Syntax, which deals with how the symbol is combined with other symbols; and
3) Meaning.

For more than 20 years, much progress has been made in understanding some of the challenges learners have in understanding the meaning of symbols (Stacey & MacGregor, 1999). Symbol sense has been described generally (Arcavi, 1994; Kinzel, 2001; Zorn, 2002) and others have provided a set of characteristics that describe specific elements of symbol sense (Arcavi, 1994; Fey, 1990). Symbol sense is the awareness (Kinzel, 2001) of symbols that enable a learner to relate symbolic expressions and operational properties. It is making sense of the symbols presented (Arcavi, 2005). In other words, a learner's ability to differentiate between the different meanings of symbols is what can be described as "symbol sense." The interpretation of the symbols and their meanings is a challenge for learners (Rubenstein & Thompson, 2001).

There is consensus that algebraic symbolisation is a generalisation of arithmetic, and that it is an important conceptual leap for students to make in order to move from the concrete world of arithmetic to the more abstract world of algebra. With the view of algebra as generalised arithmetic, many studies have investigated the components that make the transition difficult for learners. One of the components needed for the transition from arithmetic to algebraic thinking requires learners to make sense of symbolic notation.

Mathematics derives its power from the use of symbols, however, its abstractness can serve as a barrier to learning (Arcavi, 2005). If learners are unable to see abstract ideas beneath the symbols, they develop a limited understanding of algebraic concepts (MacGregor & Stacey, 1997). Being competent in algebra demands an interaction with symbols. Hiebert (2013) conducted a study that revealed the use of mathematical symbols as one of the reasons why learners experience difficulties. Chirume (2012) and Vlassis (2008) argue that the multiple meanings of a mathematical symbol confuse most learners: for example, the relational and operational meaning of the equal sign (Knuth, Stephens, McNeil, & Alibali, 2006) or the triple nature of the minus symbol (Gallardo, 2002). Even the learners themselves express that symbols were the start of their problems with mathematics and that they understood the subject before symbols were introduced (Christou & Vosniadou, 2005). Not only does this reinforce the difficulties of dealing with symbols but it also re-emphasises the problem of transitioning from arithmetic to algebra. Another common difficulty that learners experience when using symbols is when they deal with expressions that cannot be simplified further. Learners require closure and are unable to accept that in algebra some answers remain as the process and the object (Herscovics & Linchevski, 1994). This occurs when a learner does not accept a symbolic expression as a final answer. The error associated with this difficulty is called conjoining. The section below elaborates on the difficult transition from arithmetic to algebra.

From a review of literature and for the purposes of this study, I identify three important concepts that make the transition for learners difficult: view of the equal sign, the use of letters and the minus symbol. These are discussed in the next section.

3.3. Transition from arithmetic to algebra

Even though algebra can be viewed as a generalised form of arithmetic, there are fundamental differences between algebra and arithmetic, and the procedural-structural perspective of algebra mentioned in Chapter 2 gives an indication of how great the difference is. It is in these differences that learners struggle. Both arithmetic and algebra involve the use and understanding of abstract symbols and operations such as the equal sign (Sfard, 1991), the minus symbol (Vlassis, 2004) and letters (Küchemann, 1981) but they operate on different levels of abstraction: arithmetic is limited to numbers and numerical computations (Norton and Cooper, 2001; Sfard and Linchevski, 1994) whereas algebra incorporates generalisations of numeric structures as well as the manipulation of these structures (Kieran, 1992). These differences require a shift in thinking and focusing on key elements that help bridge the gap between arithmetic and algebra, making the transition smoother. I briefly discuss three key elements, which I refer to as three dualities that make learning algebra difficult. In sections 3.4 to 3.7 I discuss the literature pertaining to the four content areas. However, embedded in the discussion is a reference to the dualities that make learning algebra difficult.

3.4. Equal sign duality and literature pertaining to the equal sign

One of the core algebraic ideas needed for the successful transition from arithmetic to algebra is related to the view of the equal sign (Essien & Setati, 2006; Knuth et al., 2008). It is vitally important that the equal sign is viewed as a symbol of equivalence when solving equations. Literature has suggested two main views of the equal sign: a relational view and an operational view. An operational view is where learners see the symbol '=' as an operation meaning 'the result is'. Throughout primary school, learners are confronted with the equal sign operating as a 'to do something' symbol. When they reach high school, however, they are expected to include a relational view of the equal sign where the symbol means 'the same as'. Typically in arithmetic, learners are confronted with examples such as $3 + 4 = 7; 2 \times 9 = 18$; the difference between 10 and 8 is 2. In all these examples, the right-hand side represents the answer. In algebra, learners are expected to deal with equations where there are multiple terms on the right-hand side, for example $2 + x = 2x + 5.$ Again, in terms of the equal sign and solving equations, learners have not made the transition, and to do so they need to be comfortable with the equal sign meaning 'the same as'. Behr et al (1976) are the authors who coined the phrase "to do something" symbol (p. 13) to explain the use of the symbol in its operational sense rather than as a relational symbol.

Research to measure students' understanding of the equal sign dates back as far as the early 1930s. My review of literature states that the first piece of research on the dichotomy of operational and relational meanings was done by Renwick (1932) (as cited in Jones & Pratt (2012, p. 3). Renwick (1932) found that young children treated the equal sign as an operator rather than as a relational symbol. In 1976, 40 years later, Behr et al (1976) found that not only did students (Grades 1-6) treat the equal sign operationally but that as they got older, their view of the equal sign did not extend to include a relational view. In 2003, Carpenter, Franke and Levi (2003) confirmed Behr's (1976) results and stated that there was no evidence to suggest learners improved their understanding of equality as they got older. Saenz-Ludlow and Walgamuth (1998), however, suggest that learners' knowledge of the equal sign changes as a function of their experience in mathematics as well as their exposure to the equal sign in a variety of contexts (McNeil & Alibali, 2005).

Behr's (1976) findings show that when learners encounter number sentences such as $3 + 4 = \square$ or $3 + 4 = \square + 5$ they "perceive it as a stimulus for an answer to be placed in the box" (pg. 13). There is also evidence that viewing the equal sign as an operating symbol is not confined to young children. Fyfe, Matthews, and Amsel (2020) found that an operational interpretation of the equal sign can persist into college and is a key indicator of being able to reason mathematically and doing algebra.

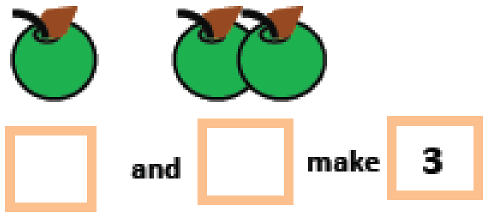
For the past 90 years, researching students' understanding and use of the equal sign has focused on the dichotomy of operational and relational interpretations. The research has spanned across pre-school to tertiary level. (See for example Behr, Erlwanger, & Nichols (1976); Carpenter, Levi, Franke, & Zeringue (2005); Falkner, Levi, & Carpenter (1999); Kieran (1981); Knuth et al. (2005); Vincent, Bardini, Pierce, & Pearn (2015)). The plethora of research done on the equal sign and all the findings suggest that not only is there little difference in the way students understand and use the symbol but also that the idea that being successful in algebra (and specifically solving equations), requires learners to interpret the equal sign as a relational symbol (Knuth et al., 2006). Moreover, Falkner et al. (1999) suggest that maintaining equality becomes meaningless without a relational view of the equal sign, and hence learners are reduced to memorising many rules to transform equations.

3.2.1.1. Contributors to a persistent operational view of the equal sign

Literature suggests that there are three main contributors to why the operational view becomes so firmly entrenched in young learners (Falkner et al., 1999) and persists through instruction (Blanton et al., 2018).

Textbooks and curriculum

Textbooks and curriculum are closely related since textbooks operationalise the curriculum, and hence if the textbook introduces the equal sign as an operational symbol the curriculum probably does the same (Essien, 2009). In the South African curriculum, the equal sign is assigned the words *makes* or *gives*. Learners are introduced to these words in Grade 1 mathematics:



(DBE, 2011c, p. 111)

Throughout primary school, the equal sign as an operator is emphasised, for example when solving equations by inspection (at the end of Grade 7), the examples in the curriculum emphasise the operational view of the equal sign:

a) Solve x if $x + 4 = 7$, where x is a natural number. (What must be added to 4 to give 7?)
b) Solve x if $x + 4 = -7$, where x is an integer. (What must be added to 4 to give -7?)
c) Solve x if $2x = 30$, where x is a natural number. (What must be multiplied by 2 to give 30?)

(DBE, 2011, p. 64)

Even when learners enter Grade 8 and are introduced to integers, the equal sign is seen as a 'to do something' symbol. For example $7 - 4 = 7 + (-4) = 3$ (DBE, 2011, p. 67).

Instructional approaches

Hattikudur and Alibali (2010); Franke and Levi (2003) and McNeil (2007) found that the instructional approaches used in the United States of America cannot help learners develop a relational understanding of the equal sign. Hiebert and Carpenter (1992) recommend using multiple representations to help students build a better conceptual understanding of equivalence. Behr et al (1980, p. 15) attribute learners' lack of understanding of the equality symbol to how they are taught. They state that it is "an expected outcome of the instruction the children receive". In addition, teachers should challenge learners' existing conceptions (Carpenter et al., 2003) by exposing them to number sentences of the form $a = a$; $a = b + c$; and $a + b = c + d$ (McNeil et al., 2006).

Calculators

Vincent et al. (2015) suggest that calculators interfere with extending learners' knowledge to incorporate a relational view. When one types a problem into the calculator, the equal sign button is pressed to reveal the answer. When learners present a string of false equalities, they are mimicking what they would do on a calculator.

3.2.1.2. Going beyond an operational-relational distinction of the equal sign

As mentioned earlier, research on the different view of the equal sign has focused on the dual use as an operational and relational symbol. However, there are some authors that have attempted to go beyond an operational and relational view.

Jones and Pratt (2012) studied the conception of substitution as an additional use of the equal sign. Where there is a consensus that a relational view means that the equal sign is viewed as meaning the same as, Jones and Pratt (2012) suggest that substitution and sameness are two components of a relational view. The idea of substitution as a relational component was also suggested by Cortes, Vergnaud and Kavafian (1990) but was termed *specification* (as cited by Prediger (2010)). Simsek, Xenidou-Dervou, Karadeniz and Jones (2019) found that substitution is a unique predictor of algebraic performance.

Carpenter et al. (2003) disaggregated the operational view of the equal sign and had three sub-categories: 1) answer comes next; 2) use all numbers and 3) extend the problem. What they mean by 'the answer comes next' is the typical view of an operational view, for example $10 + 3 = \_ + 5$ and the learners respond with 13, because the equal sign is a cue to give an answer. When a learner uses all the numbers it is referring to when the learner would add, for example the 10, 3 and 5 and place 18 in the blank space. One learner said that this must be done because the question is asking him to add the 3 and add the 5. Extending the problem is synonymous with what Kieran calls a string of operations; it means that, using the same example as above, the learner would put 13 in the blank space and then finish the sentence with 18 ($10 + 3 = 13 + 5 = 18$) producing a string of untrue equalities. In my analysis of equality items, I have seen these three types of operational views and classified them all as operational rather than by their separate categories.

Stephens et al. (2013) disaggregated the relational approach to be a relational-computational approach and a relational-structural approach, making three main categories rather than two: 1) an operational approach; 2) a relational-computational approach; and 3) a relational-structural approach. The latter approach requires a deeper and more flexible understanding of the equal sign because it draws on a learner's "ability to 'see' abstract ideas hidden behind symbols" (p.174). Using a relational-computational or a relational-structural approach requires relational thinking where learners "attend to relations and fundamental properties of arithmetic operations rather than focusing exclusively on procedures for calculating answers" (Carpenter et al., 2005, p. 53). Stephens et al. (2013) also argued that when a calculation involves small positive numbers, the calculations are straightforward, and distinguishing between a relational-computational and relational-structural view is difficult. When large numbers are used, it is easier to observe whether learners need to calculate (having a relational-computational view) or whether they are able to determine the missing number by observing the structure (having a relational-structural view). Hence, using large numbers may encourage learners to think relationally about how the two sides of the equal sign relate rather than where learners just do a procedural calculation. Knuth et al. (2006) investigated the relationship between one's view of the equal sign and solving equations and found that it is a good predictor of equation-solving success.

Rittle-Johnson et al. (2011) developed a framework that involved four levels to how one views the equal sign, hence disaggregating both the operational and relational categories:

Level	Description	Core equation structures
Level 4: Comparative relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognising that performing the same operations on both sides maintains equivalence. Recognise relational definition of equal sign as the best definition.	Operations on both sides with multi-digit numbers or multiple instances of a variable
Level 3: Basic relational	Successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign. Recognise and generate a relational definition of the equal sign.	Operations on both sides, e.g.: $a + b = c + d$ $a + b - c = d + e$
Level 2: Flexible operational	Successfully solve, evaluate and encode atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right: $c = a + b$ or No operations: $a = a$
Level 1: Rigid operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally.	Operations on left: $a + b = c$ (including when blank is before the equal sign)

Construct Map for Mathematical Equivalence Knowledge (Rittle-Johnson et al., 2011, p. 87)

It is well known that the equal sign is typically viewed as an operational symbol that means *makes* or *gives* or is a *to do something symbol* rather than *the same as.* There is much research on this dichotomy.

Literature has shown how learners' understanding does not extend over time to include a relational view but that the operational view is persistent throughout school mathematics and even into college, and becomes entrenched. It also seems as though relational and operational views are not mutually exclusive;

learners do not just transition from an operational view to a relational view; rather, it is context-dependant. McNeil et al. (2006) stated that although the two mathematical concepts appear to be incompatible, they are in fact complementary. This is evident in the additional categories of relational and operational views of the equal sign. Much research has shown that a relational view of the equal sign is needed for success in algebra and solving equations (Knuth et al., 2005). Learners should have a flexible understanding of the equal sign for arithmetic competence and algebra (Jones, Inglis, Gilmore, & Dowens, 2012) and hence it is important to continue to do research on learners' views, especially on higher grades, where there are fewer studies on this age group.

3.5. Minus symbol duality and literature pertaining to integers

Negatives are "one of the principal breeding grounds for student maladies in algebra".
(Haner, 1947, p. 656)

Another key element that makes the transition from arithmetic to algebra easier is being flexible with the minus symbol and understanding that it has two main functions: as a sign and as an operation. In arithmetic, learners only encounter the minus symbol as an operator being used in questions about difference, subtraction or taking away. In algebra, however, learners experience signed numbers and need to adjust their thinking to include the minus symbol as a sign. The fact that so many learners struggle with integers and signed numbers is because they are so used to working in the whole-number system.

Working in algebra requires the use of previously learnt arithmetic skills but also requires extending their knowledge to include variables, signed numbers and a relation view of the equal sign. Overcoming all three dilemmas requires teacher intervention. This study is however not about teachers, teaching or pedagogy, but rather about the relationship between the errors learners make when dealing with these dilemmas in a mathematics test. Sections 3.4-3.8 are discussions related to the content areas and discuss the literature pertaining to the equal sign, negatives, expressions and equations.

The aim of my PhD is to investigate the extent to which relationships exist between errors made when solving linear equations and errors made in other topics, such as integers. The motivation for this focus lies in the uncontested claim that the extension of whole-number understanding to negative numbers plays an important part in mathematical competence in school mathematics. Unfortunately, this extension happens just as algebra is taught. Vlassis (2002) found that many errors made when solving linear equations were caused by the presence of negative numbers. Gaining understanding of negative numbers has been difficult for early mathematicians to comprehend (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014), as well as for teachers to teach and for learners to learn (Ball, 1993). An outline of the historical struggle to overcome the obstacles related to integers can be found in Hefendehl-Hebeker (1991).

My focus on negative numbers is based on the premise that there are two types of understanding needed to understand negative numbers (Vlassis, 2004). The first relates to students' understandings about natural numbers and the rules needed to operate with negatives. For example, learners need to adjust their thinking that one always subtracts a smaller number from a larger, and that not all solutions are positive. The second type of conceptual change is about the minus symbol and that it has multiple functions. This change develops through an extended understanding and a flexible use of the symbol. This is what Vlassis (2004) calls 'negativity'. Literature suggests that there are four main factors that explain why negative numbers are a complex topic to learn.

3.5.1. Why are negatives so difficult to learn?

a) Cardinality, ordinality and whole-number extension

Literature has shown that in understanding numbers in general one needs to have a cardinal and ordinal understanding of number (Bishop et al., 2014). A cardinal understanding refers to magnitude and counting, which, for positive numbers, is not only familiar to learners but is concrete. A cardinal understanding also refers to the concept of deficiency with respect to zero. The magnitude of a negative number is one of the first abstract ideas a learner will encounter before algebra, for example, knowing what -2 means or stands for. Bishop et al. (2014) argue that one difficulty in understanding negative numbers is that they represent both a cardinal and an ordinal context. It is important to accept that negative numbers exist in order to move beyond the cardinality of a negative number. Bishop et al. (2014) conducted a case study on a student, Violet. They found that although she was able to solve $5 + \_ = 3$ correctly, she did not believe negative numbers were real: "Negative numbers aren't really numbers… They're just acting like other numbers except there is a minus in front of them… We don't really count them in school, and there's no negative 1 cube and stuff" (Bishop et al., 2014, p. 49). An ordinal understanding is about direction and order in relation to zero. Learners need to know which number is smaller and which is bigger. Being able to order integers is what Peled (1991) says is the first of four levels of understanding negative numbers. Level 2 is when a learner is able to add positives to any integer. Level 3 is where a learner can add or subtract two positives or two negatives and at level 4, learners can add or subtract any two integers. In addition, learners need to expand (and challenge) their current understanding of whole numbers and accept that it is indeed possible to subtract a bigger number from a smaller number and that a solution can be negative. Linked to cardinality and ordinality are the two levels of mental models offered by Bofferding (2014). The initial level contains two different ways of thinking: whole number and absolute value. Whole-number thinking is where learners lack the cardinality of negative numbers. This is operationalised by students who ignore the negative signs and operate with the numbers as though they were positive. For example, by ordering their absolute value correctly. These learners do not comprehend negative numbers as numbers below zero. In contrast, absolute-value thinking is where learners comprehend negative numbers on the number line but in the wrong order, relating to a lack of ordinality of negative numbers. The problem is that the learners treat all integers as their absolute value when comparing them and hence order them in the reverse order. The second mental model is the synthetic model, which is about magnitude. This involves understanding negative numbers as numbers below zero and differentiating negative from positive numbers. However, they consider negative numbers with a larger absolute value as greater than negative numbers with less of an absolute value. For example, $-7 > -2$. These mental models are of importance for my data when I analyse learners' responses to ordering integers because they informed my coding.

b) Abstract numbers: lack of concrete association

Another difficulty in the conceptualization of negative numbers is the inability to represent negative numbers with concrete objects. Piaget (1964) argued that in the concrete operational stage of development children first develop number sense as a one-to-one correspondence, then as an ordinal relation, and then the cardinality of set. However, when encountering negative numbers, learners are unable to make these tactile one-to-one correspondences or model a set with concrete manipulatives (Herscovics & Linchevski, 1994). Negative numbers not being concrete created a focus in finding different ways to try and explain the concept of 'below zero'. Contexts such as money and debt, temperature, and the number of floors in a building are used to try and convey the message of negative numbers; models such as integer tiles or the number line are also used. Each strategy has its downfall: for example, using the number line and trying to subtract a negative or using the idea of money and debt but needing to explain multiplication. Küchemann (1981) states that difficulties in dealing with negative numbers are not easily overcome by any particular model. With a variety of contexts and models, it is understandable that teachers and learners resort to the teaching and learning of rules. These models are sometimes demonstrated in conjunction with, or sometimes following, the introduction of the rules used to simplify negative numbers. Students are expected to then memorise and apply these rules in calculations.

Various studies have shown that learners rely on a different mental model, context and rules in attempting to understand negative numbers. Bofferding (2014) conducted a study with 61 Grade 1s to investigate learners' mental models before and after instruction. She found that learners relied on whole-number principles when trying to understand negative numbers. In a different study, Bofferding (2010) found that learners with a binary understanding of the minus symbol ignored negatives or treated them as a subtraction sign. She also found that students with a unary understanding often begin counting from a negative number. For example, students would solve $-5 + 2$ by counting "negative six, negative seven" or "negative four, negative three".

Gallardo and Rojano (1990) found that learners relied on multiple models and contexts such as the number line, temperature, money and debt, or they rely on rules. Peled, Mukhopadhyay and Resnick (1989) found that learners who were able to order integers typically referred to integers abstractly by counting up or down or using the number line rather than using, for example, money or temperature.

c) Multiple meanings of the symbol $'-'$

In addition to needing both a cardinal and ordinal view of numbers, and the lack of concrete association, literature suggests that a challenge learners encounter with regards to negative numbers is that they need to be "flexible" (Vlassis, 2004, p. 469) with the different ways in which the symbol, '$-$', is used. In mathematics it can signify subtraction, the sign of a numeral or the inverse operation. It is unfortunate that the sign remains the same; however, there is research where different signs are used for the different function. Three examples come from Ball (1993); Küchemann (1981); Shiu (1978), who used a circumflex, for example $\hat{4}$ above the numeral; a smaller elevated sign, for example, 4^{-} and a horizontal line on top of the numeral, for example $\bar{4}$ respectively, to denote a negative number and focus learners' attention on the negative number rather than as an operation. Gallardo and Rojano (1994, p. 121) and Vlassis (2004) have done extensive research in learners' understandings of negative numbers and the minus symbol. Gallardo and Rojano (1994) identified three main functions of the minus symbol: the unary, binary and symmetric functions. The unary function of the minus symbol is about the formal definition and understanding of negative numbers. For example, considering the number -5, the '$-$' in front of the 5 is a sign that identifies 5 as a negative number rather than having subtracted 5 from another value. The binary function is where the symbol '$-$' signifies an operation, for example in $13 - 8$ where 8 is subtracted or taken away from 13. The symmetric function is about taking the opposite sign and is particularly important when dealing with inverses. Vlassis (2004) coined the term 'negativity' to show the multidimensionality of the minus symbol.

In South African primary schools, learners' only experience with the minus symbol is as an operation. Subtraction in number sentences such as $4 - 1 = \_$ are introduced in Grade 1 (DBE, 2011c) and learners do not appear to encounter the minus symbol performing a different function until Grade 7. In Grade 7, learners' previous understandings are challenged when a) a negative number is to be objectified and seen in its unary function having its own cardinality and position on the number line; and b) when the binary function of the minus symbol can produce a negative number, hence including the unary function. This challenges the previously learnt strategy of taking the difference with a smaller number subtracted from a larger number. Thompson and Dreyfus (1988) conducted a teaching experiment on how learners understand integers in their binary function. They used a computer program with a turtle that moved left or right depending on the commands given. They found that the learners could determine the sign and magnitude of a sum by the end of the six-week sessions. Vlassis (2002) found that for most of her sample of Grade 8 learners, the minus symbol had its meaning in relation to the procedure which is subtraction, otherwise known as the binary function. She particularly noted that no student explicitly considered that the minus symbol could have a double status.

In addition to a unary and binary function, learners are also confronted with number sentences that include the third function of the minus symbol: the symmetric function (Gallardo & Rojano, 1994). This is where the minus symbol includes "taking the opposite of a number", which is vital in applying inverse operations when solving equations. Moreover, in algebra the unary function of the minus symbol in $'-x'$ causes much confusion as $-x$ could be positive (Lamb et al., 2012). When solving linear equations it is vital that learners recognise that $(-x) + x = 0$ and that to solve an algebraic equation of the type $Ax + B = Cx + D$, the additive inverse of Cx, which is $-Cx$, needs to be applied. It is therefore not surprising that when secondary school learners are faced with the symbol '$-$', they produce many different errors. In contrast to the South African curriculum, in the United States, the National Council of Teachers of Mathematics (NCTM) states in the Number and Operations standard that children in grades three to five should be expected to "explore numbers less than zero by extending the number line and through familiar applications" (2000, p. 148).

Similar to the triple meaning of the minus symbol is Vergnaud's (1982) distinction between a state (unary), transformation (binary) and static (symmetric) relationship of negative numbers. Not only are there three functions of the minus symbol but there are four functions of subtraction (Fuson, 1984): firstly, subtraction as take away, where items are removed; for example, I have 8 marbles, and I give 3 away, how many do I have left? The second is subtraction as a difference or comparison, which compares two or more sets and the answer is always positive, for example, I am 12 and my sister is 5, what is the difference in age? The third function of subtraction is 'join-missing-addend', which is an addition situation where one of the addends is known; for example, I have 4 toys, how many more do I need to have 9 toys? Similarly, the fourth function is subtraction as 'combine-missing addend', which is also an addition situation but the context is a part-whole situation; for example, I have 8 cakes, 3 are chocolate, how many are vanilla (Fuson, 1984, p. 221)? What all four functions have in common is that they deal with subtraction and the problem can be written symbolically as, for example, $8 - 3$. Some researchers argue that engaging learners with the symbolic notation of subtraction too early could be counterproductive for them (Fuson, 1984; Matthews, 1983; Selter, Prediger, Nührenbörger, & Hußmann, 2012). This raises the question, when is it best to learn about negative numbers?

3.5.2. When is best to learn about negative numbers?

There are discussions in literature around when is the best time to start learning about negative numbers. Some literature suggests that learning it later (as is done in South Africa) only exacerbates the problems as learners' understanding of the minus symbol is entrenched in it meaning take away or difference. Research on negative numbers is typically conducted with learners in lower secondary school (aged between 12 and 16 years). For example, see Bruno & Martinon (1999); Chiu (2001); Gallardo (2002); Fuadiah & Suryadi (2017); Küchemann (1981); Makonye & Hantibi (2014); Stephan & Akyuz (2012); Vlassis (2004) and Whitacre et al. (2015). However, some research deals with primary school children in Grades 1 to 3 prior to formal instruction. See for example Bishop et al. (2014) and Bofferding (2014). Streefland (1996) states that negative numbers should begin with 9- to 10-year-olds before they master natural numbers, and Hativa and Cohen (1995) suggest that some, not all, negative number concepts could be taught to Grade 4 learners. For example, concepts such as representing numbers on the number line and determining the distance between integers. However, they assert that learners are not ready to operate with integers. Similarly, Schliemann et al. (2003) assert that although the learners (Grades 2 to 4) in their study were able to "accept the idea of negative numbers" (2003, p. 128), they were unable to operate with negative numbers. In contrast to Hativa and Cohen (1995) and Schliemann et al. (2003), Murray (1985) suggests that learners should be introduced to negative numbers early on in their schooling and argues that extending the number system to include negative numbers is no more difficult than extending their number system to include rational numbers. However, they do state that it is on condition that negative numbers are carefully introduced, with only the addition of signed numbers and not with rules or algorithms. Goldin and Shteingold (2001) interviewed Grade 1 and 2 learners who had no prior instruction on negative numbers. Their findings suggest that learners begin to develop representations of negative numbers earlier than one thinks. They also argue that curricula should consider early introduction of negative numbers before they become obstacles in learning. When negative numbers become obstacles in learning, learners make errors.

3.5.3. Errors when operating with negative numbers

Although there is much research in integers, there is little on the errors made when dealing with integers. Tatsuoka (1984) found over 200 "bugs" regarding "erroneous rules of operation" (p.120). Of these errors, 89 related to the addition and subtraction of negative numbers. The errors found were very specific, for example: always taking a positive sign to answers; take the sign of the first number; or take the sign of the larger number. They were not classified into broader categories. Tatsuoka (1984) was able to find a large set of errors as she had 64 items regarding the addition and subtraction of negative numbers. Since I have five items, her list of errors was not helpful in my analysis. Vlassis (2004), however, identified different types of reasoning about operations with integers. These types of reasoning give birth to errors and hence informed the error categories I used when analysing my data. The different types of reasoning discussed are: right-to-left reasoning; bracket reasoning; signs-rule reasoning (Vlassis, 2004); and too many signs (Gallardo & Rojano, 1990).

a) Right-to-left reasoning

Right-to-left reasoning is used when an expression is simplified by operating from right to left instead of left to right (Vlassis, 2004). For example, if $3-8$ is read from right to left, it is transformed into $8-3$, which yields 5. Learners reverse the order to make operating with the minus symbol more "comfortable" (Vlassis, 2004, p. 477). I agree with this interpretation, however, if there is a leading negative, for example $-8+3 \rightarrow 5$, then the same interpretation cannot be used. A possible reason for $-8+3 \rightarrow 5$ is that an operation is used to yield a positive number. Ryan and Williams (2007) provide an explanation for the error $3-8 \rightarrow 5$ in that learners seem to be overgeneralising the commutative property of addition and applying it to subtraction, learners overgeneralise $a+b=b+a$ to also mean $a-b=b-a$. This interpretation covers both $3-8 \rightarrow 5$ as well as $-8+3 \rightarrow 5$. Although Hall (2002) categorised these two types of errors as *number line errors,* it does not describe the underlying problem. From my experience, reasoning from right to left or overgeneralising the commutative property of addition is done because learners require a positive solution and not because they went in the wrong direction on a number line. It is important to note that in arithmetic, learners only work with positive numbers, hence subtracting a larger value from a smaller one is not familiar to them and often does not make sense.

b) Bracket reasoning

Bracket reasoning is characterised by inserting brackets explicitly or mentally (Vlassis, 2004), for example $-8+3 \rightarrow -(8+3) \rightarrow -11$. The minus symbol is separated from 8, the numbers 8 and 3 are operated on and then the minus symbol is placed with the answer. This is where learners view the signs and numbers as separate objects (Ryan & Williams, 2007). Originally, bracket reasoning was found in algebra when multiple terms in an expression were present, for example, $3x-4+x-2$. Bracket reasoning was seen as re-grouping the like terms but not focusing on the operations (Vlassis, 2004). Using the example above, $3x-4+x-2$ is reduced to $(3x-x)+(4-2)$. Overgeneralising the commutative property for addition could explain the error but overgeneralising the associative property for addition and ignoring the operation could be used too. The associative property states that different groupings of addition do not affect the result, for example: $a+(b+c)=(a+b)+c$. Learners however overgeneralise this to apply to subtraction. In the numeric example: $-8+3 \rightarrow -(8+3) \rightarrow -11$ as well as the algebraic example: $3x-4+x-2 \rightarrow (3x-x)+(4-2)$, bracket reasoning is applied and learners do not appear to have the signs in focus. A similar error, the *detachment of the minus symbol*, is proposed by Herscovics and Linchevski (1994). This is where the minus symbol is detached from the numeral and reattached after the learner has operated on the numerals. For example, if given $-8+3$, the minus symbol is detached, 8 and 3 are operated on yielding 11 and the minus symbol is reattached giving -11. This is different to bracket reasoning because it can apply to, for example, $8-11 \rightarrow -19$. Since detachment of the minus symbol is more versatile, I use this term instead of bracket reasoning in the analysis of my data.

c) Signs-rule reasoning

The signs rule is supposed to be used when multiplying integers: *'a negative and a negative make a positive'*. In dealing with the addition and subtraction of negative numbers, a common error is to apply the signs rule. Reasoning with the signs rules shows an explicit focus on the operations. The error made with this form of reasoning is characterised by overgeneralising the *signs rule* (Gallardo & Rojano, 1994; Vlassis, 2004). The signs rule is the multiplication rule for integers where '*a negative and a negative make a positive'* and '*a negative and a positive make a negative'*. Using the same example as above, $-8+3$ could be simplified to -11 because a *negative and a positive make a negative* and 8 and 3 make 11. This error was also seen in the discussion of bracket reasoning and the detachment of the minus symbol. This shows how a single error can be due to multiple forms of reasoning. A criticism of the signs rule reasoning is that it is unclear as to whether $-8-3 \rightarrow +24$, where the two integers are multiplied together, is also considered the signs rule. Another example is $-3+8 \rightarrow -5$, where the signs rule is used as the sign as well as the operation. I therefore build on this form of reasoning to include different types of sign rule errors.

d) Too many signs reasoning

This reasoning is where learners deliberately leave out (or ignore) one of the minus symbols (Gallardo & Rojano, 1994). They provide interview evidence where a learner reduces $-a-(-b)$ to $-a-b$ because the minus symbol in the bracket, attached to b, "is not required" (p. 162). Similarly, Murray (1985) found that learners remove the duplicate minus symbol, for example, $-a-(-b) \rightarrow -a-b$ (as cited in Bofferding (2010)). Other research (Schwarz, Kohn, & Resnick, 1994) found that learners ignore the first minus symbol, reduce, and add the sign at the end, for example, using $a-(-b)$, the learners would ignore the subtraction sign operate and get $a-b$ and then add the sign at the end, $-(a-b)$.

3.6. Literature pertaining to algebraic expressions

An algebraic expression is a statement made up of integers, variables, constants and operations (for example addition, subtraction, division, multiplication and exponentiation). There are different types of expressions, for example:

a) Numeric expression: $3 + 8$

b) Linear polynomial expression: $3x + 5 + 2x$

c) Quadratic polynomial expression: $x^2 + 5x - 6$

d) Rational expression: $\frac{2x+4}{6x-2}$

e) Trigonometric expression: $(1 + \tan^2 x)(1 - \sin^2 x)$

f) Complex expression: $\int 6\cos(x) + \frac{4}{\sqrt{1-x^2}} dx$

For the purpose of this study, I only refer to numeric and linear polynomial expressions, which are categorised as part of elementary algebra. For the remainder of the thesis I refer to a linear polynomial expression as an expression. Since an expression consists of symbols, numbers and letters, there are various places learners could struggle when simplifying an expression. Before discussing the difficulties learners experience, I provide a short description of what the South African Mathematics Curriculum expects from learners in Grades 7, 8 and 9.

3.6.1. Curriculum: simplifying expressions

Letters are at the heart of simplifying expressions. Besides negative numbers, the introduction of letters is the first abstract idea a learner encounters when transitioning from arithmetic to algebra, and this begins in Grade 7 of the South African Mathematics curriculum. The South African Mathematics Curriculum states that in Grade 7, "recognising properties of operations for different numbers provides a critical foundation for working with algebra… and manipulating algebraic expressions…" (DBE, 2011, p. 12). Besides working with numeric examples, learners are expected to identify variables and constants. Interestingly they are also expected to determine the numeric value of an expression by substituting. This is interesting because it is an outcome written under the outcomes for algebraic equations rather than expressions. In Grade 8, learners revise the Grade 7 work and then extend it by:

a) recognising conventions for writing expressions

b) identifying like and unlike terms

c) recognising and identifying exponents and coefficients in an algebraic expression

d) using the commutative, associative and distributive laws to add and subtract like terms

e) multiplying and dividing integers or monomials by monomials, binomials or trinomials. They are also expected to simplify squares, cubes and square roots, and determine the numerical value of an expression by substitution.

In Grade 9 the only extension to dealing with expressions is that learners need to simplify fractions and encounter the product of two binomials as well as the square of two binomials and factorising expressions. This suggests that, in light of the items used in this study, all the expression items are based on Grade 8 work. In particular, the expression items only deal with the addition and subtraction of like terms, including the use of the distribution law.

3.6.2. Difficulties and errors when simplifying expressions

There is much research on the difficulties learners encounter with introductory algebra. However, the scope of the difficulties is limited. Literature suggests that there are five main difficulties learners experience when simplifying expressions. These are discussed below.

3.6.2.1. The interpretation of letters

The concept of variable, or letters, is fundamental to mathematics in high school. Understanding this concept provides the foundation for the transition from arithmetic to algebra and is necessary for the meaningful use of all advanced mathematics (Schoenfeld & Arcavi, 1988). Algebraic expressions consist of letters, which means there is a close relationship between understanding the different meaning of letters in the context of an expression. Kieran (1990) gives an example where, in arithmetic, $12m$ can mean 12 metres but in algebra means 12 multiplied by some unknown number. Therefore, the letter m carries two different meanings depending on the context. Küchemann (1981) distinguished between six different interpretations of letters: letter evaluated, where the letter is assigned a numerical value, for example, determine $x + 5$ if $x = 7$. Secondly, letter not used, where the letter is ignored, or not used. It is where the letter's existence is acknowledged but without meaning. For example, if $a + b = 9$ then $a + b + 2 =$? The third interpretation is a letter used as an object. This is where the letter is used as a shorthand for an object, for example: $5a$ = 5 apples. Although this interpretation can lead learners to the correct answer, for example $5a + 2a = 7a$, it does not help them to understand that a letter stands for a number. It would be better to interpret $5a + 2a = 7a$ and 5 boxes of apples + 2 boxes of apples = 7 boxes of apples. The fourth interpretation is a letter used as a specific unknown; this is where a letter is regarded as a specific but unknown number and can be operated upon directly. For example, $2x + 4 = 10$. The specific unknown in this case is 3. The fifth interpretation is a letter used as a generalised number. This is where the letter is seen as representing several values rather than just one, for example $2x + 4$, where x can be any real number. The last interpretation of letters is a letter used as a variable: this is where the letter is seen as representing a range of unspecified values and a relationship is seen to exist between the values. For example, $y = 2x + 4$. Ten years after Küchemann's (1981) research, Philipp (1992) extended Küchemann's (1981) categories and distinguished between seven interpretations of letters. These are letters as: labels (for example 12cm); constants (for example, $2 + m$); unknowns (for example, $2x + 4$); generalised numbers (for example, $a + b = b + a$); varying quantities ($y = 2x + 4$); parameters (for example, a and c in $y = ax + c$) and abstract symbols (for example, $e^x = x$). A further 16 years later Ryan and Williams (2007) provided evidence of errors that resemble the different interpretations of letters, proving that understanding that letters stand for numbers in algebra is a complex issue and results in persistent errors. Watson (2009, p. 19) argues that this "is evidence that these (errors) are due to students' normal sense-making of algebra, given their previous experiences with arithmetic and the inherent non-obviousness of algebraic notation".

3.6.2.2. Conjoining terms

One of the prominent and persistent errors that learners make when simplifying expressions is related to learners adding unlike terms. When there are like terms, expressions can be simplified (also known as a reduction (Matz, 1980)). For example: $2x + 4 - x + 2 = 3x + 6$. A difficulty in dealing with expressions is when the expression cannot be simplified. For example $3x + 7$. Many learners are unable to accept that the expression remains unclosed, and instead need to close it to get a single term: $10x$. This error is referred to as conjoining. Sfard (1991) and Tall (1999) have pointed to this difficulty as understanding the process-product duality of algebraic expressions. This is where the operational instruction and the product of the operations are the same. Other reasons for this error are linked to the previous experiences of the learners, for example in arithmetic $2 + \frac{1}{2} = 2\frac{1}{2}$ or in chemistry $C + O + O = CO_2$. More recent research (Landy, Brookes, & Smout, 2014) has offered a third reason for conjoining, that is, that it is visually salient to add unlike terms. For example, in $4x + 3$, visually, learners are drawn to the $4 + 3$ even though this goes against the rule about adding unlike terms. In arithmetic, when learners are presented with addition or subtraction, they obtain a single number, e.g. $3 + 4 = 7$. This has been their normal way of working up until Grade 8, when letters are introduced and hence the visual pattern of 'something plus something equals a number' is not only ingrained in the learners but interferes with the rule of not adding unlike terms. Conjoining is also described as the *inability to accept the lack of closure* (Collis, 1978); *premature closure* (Küchemann, 1981); *concatenation* (Matz, 1980); *tag-along approach* (Matz, 1980); and *parsing error* (Tall & Thomas, 1991). Although this error is well known and often referred to in literature, there are in fact very few empirical studies in reputable journal articles that report the frequency of conjoining in high school. In addition, many studies used the CSMS test items and hence we don't really get a sense of the extent of the problem when there is little variety of items to compare with. This further suggests that future research should develop more items that will test for the prominence of conjoining.

Stacey and MacGregor (1994) found that only 2% of learners between Grades 7 and 10 conjoined and added 12 to x and got $12x$. This is noticeably fewer learners than those in the Küchemann (1981) study, where for example, 31% of Grade 10 learners added 4 onto $3n$ and got $7n$. Judah Makonye (2015) did a study with Grade 10 learners and found that in the pre-test, 41% conjoined but in the post-test, after an intervention only 3% conjoined. These results suggest that where conjoining is concerned, intervention studies may help alleviate the problem. In the study conducted by Pournara et al. (2015), where there was no intervention, we see that conjoining is persistent with 53% of Grade 9; 50% of Grade 10, and 42% of Grade 11 learners conjoining $2a + 5b + a$. A longitudinal intervention study was done on younger learners, following Grade 5 learners into Grade 6, and it was found that as learners aged, conjoining became less of an issue (Banerjee & Subramaniam, 2012). At the end of Grade 5, 31% conjoined and in the middle of Grade 6, 26% conjoined but by the end of Grade 6, nobody conjoined. These results do point toward early algebra and deliberately intervening to lessen the gap between arithmetic and algebra.

The literature suggests three main reasons for the prevalence of the conjoining error: a) prior learning of combining symbols and numbers; b) the process-object duality; and c) visual saliency.

a) Prior learning of combining symbols and numbers

There are instances in mathematics but also in other subjects where we do not differentiate between conjoining and adding (Stacey and MacGregor, 1994). For example, in chemistry, adding two parts of oxygen to one part of nitrogen $(O + O + N)$ produces NO_2, and in fractions $5 + \frac{1}{2} = 5\frac{1}{2}$. In both cases, symbols are placed alongside each other. While Matz (1980) refers to this as concatenation, Booth (1981) refers to it as conjoining.

b) Process/object duality as a reason for conjoining:

One of the key elements to bridging the gap is to focus on the object-process duality. While arithmetic is primarily procedural, in algebra, the procedure is part of the object (for example, $2x + y$ represents both the procedure of adding $2x$ to y as well as the object $2x + y$). Linchevski and Herscovics (1996) argue that, unlike with algebra, arithmetic is unique in separating the procedures from the objects obtained from those procedures. This difficulty in separating the procedures from the objects has caused a common error called conjoining (Booth, 1988), also known as concatenation (Matz, 1980), or as a lack of closure (Biggs, & Collis, 1982). In arithmetic, concatenation is used in place value notation and mixed fractions, for example, $40 +$

$\frac{1}{2} = 40\frac{1}{2}$ but in algebra $a + b \neq ab$. Many secondary students are not ready for algebra, and to become ready learners must come to understand that an expression can be manipulated without closure (Norton & Cooper, 2001). Herscovics and Linchevski (1994) proposed the existence of a cognitive gap between thinking arithmetically and thinking algebraically that is characterised by learners' inability to operate on unknowns.

Several authors of mathematics education differentiate between processes and objects (Gray, 1994; Sfard, 1994). These authors use this distinction to account for errors made in algebra, including conjoining. In algebra, $5 + 2n$ describes the process of adding 5 to $2n$, and the outcome of that process is the same object. Whereas in arithmetic, adding $5 + 2$ describes the process and 7 describes the new object. In making the transition from arithmetic to algebra, learners need to see the expression as both a process and an object.

c) Visual saliency

More recent research offers a third reason for conjoining: that it is using visual saliency to do so. Li and Gao (2014) argue that while the retina can receive up to 10 billion bits of information, this is too much for the brain to cope with. Therefore, the human vision system identifies important visual subsets, called salient subsets, which it foregrounds in our brains and which thus receive attention first. The remaining subsets are "often inhibited or even ignored to increase the processing efficiency" (p.1). The visual alignment of numbers and symbols, or of unlike terms next to each other, e.g. $3x - 3$, creates a tension between formal rules and visual patterns from prior learning (Landy et al., 2014). This tension then creates visual structures that go against the underlying mathematical ideas intended to be learnt, e.g. in $3x - 3$, visually learners are drawn to the $3 - 3$, but this goes against the notion that one cannot add or subtract unlike terms. Typically, when learners are presented with an addition or subtraction symbol, they operate on it and obtain a single number, e.g. $3 + 4 = 7$. This visual pattern of 'something plus something equals a number', interferes with the rule of not adding unlike terms. Visually it appears to make sense to learners to add unlike terms.

3.6.2.3. Dealing with brackets

In mathematics there are three types of brackets: curly, square and round brackets. Round brackets, which are relevant in this thesis, are used for a variety of functions in mathematics. They are used to highlight what operation needs to be performed first, for example $2 \times (3 + 5)$. They are also used as part of notation in a function, for example $f(x) = 2x + 5$. They also help learners see the difference between the minus symbol as a sign and an operation (Vlassis, 2004), for example $2x - (-3x)$. Both anecdotal evidence and logic tell us that expressions with brackets are more complex than similar expressions without brackets. However, sometimes, depending on the structure of an item, an expression is more complex when brackets are implicit, for example $5 + 3 \times 2 - 1$ instead of $5 + (3 \times 2) - 1$. Sometimes brackets are required, for example $6 \times (3+2)$ and sometimes although not needed, they are inserted to make the order of operations explicit, for example $6 + (3 \times 2) - 1$. Hoch and Dreyfus (2005) found that the presence of brackets helped learners see the structure of the expression. Because learners work from left to right, they do not always see the need for brackets (Kieran, 1992), and Stacey and MacGregor (1999) found that learners do not always realise the importance of brackets in algebraic expressions and therefore may not use them at all. This means that a learner who simplifies $5 + 3 \times x$ will not see the need for imaginary brackets around the 3 and x. The results from the study by (MacGregor & Stacey, 1997) show that between 11% and 22% of learners in Grades 7 to 10 omit brackets in their working out, concurring with (Kieran, 1992). Sometimes there are explicit brackets such as $2 \times (5 + 3) \times x$ and sometimes the brackets are implicit, which we call the order of operations such as $2 \times 5 + 3 \times x$, which tell us to do the multiplications first before doing the addition. Very little empirical research has been done on the effect of brackets in an expression (or even an equation), suggesting that future research can focus on this.

3.6.2.4. Interpreting an algebraic expression as an equation

When a learner interprets an expression as an equation they respond to, for example, $2x + 3 - 4x + 6$ with $2x + 3 - 4x + 6 = 0$ and then proceed to solve the equation. This error has been reported in Hall (2002) and Kieran (1992). Interpreting an expression as an equation has links to dealing with the equal sign incorrectly, which is a topic dealt with in depth in the equality section.

3.6.2.5. The misuse of exponential laws

This section is about learners using the exponential laws when adding and subtracting terms, not when multiplying powers. Hence, it falls under the literature pertaining to letters and conjoining. There is little literature on secondary students' use and misuse of the exponential laws. A study by MacGregor & Stacey (1997) showed that learners thought of exponents as an instruction to multiply while not knowing what was being multiplied. For example, if asked what x^3 means, learners would say "x times 3". In their study the percentage of learners that misused exponential laws was small, between 1% and 2% for Grades 7, 8, 9 and 10. This error was however found to occur only with older students who had been taught the exponential laws.

Other studies looked at teaching strategies (Barnes, 2006) but as noted by Confrey and Smith (1995), who looked at the exponential function, there is little research on students' conceptions of exponents. Pitta-Pantazi, Christou and Zachariades' (2007) research focused on numerical exponents such as $0{,}7^3$ but also noted that research on exponents as a mathematical object was very limited.

3.7. Literature pertaining to linear equations

This section of the literature review is dedicated to literature on solving linear equations. Solving equations is an important skill for learners to master in school as it appears in almost all areas of mathematics. Solving linear equations is the first real opportunity for learners to link their knowledge to arithmetic and the symbolic nature of mathematics (Andrews & Sayers, 2012). However, despite its importance, learners still struggle and this is not unique to South Africa.

I begin by discussing two types of equations as well as reductions and deductions. This is followed by a short discussion on equations in the South African Curriculum. I then provide a lengthy discussion on the different studies that investigated linear equations. By reviewing these studies, I identified areas of research that have been focused on and use this to show the gap in research regarding solving equations. I begin, however, with a section on constraint equations, tautologies, reductions and deductions, and then give a brief discussion on equations in the curriculum.

3.7.1. Constraint equations, tautologies, reductions and deductions

When shifting from arithmetic to algebraic thought, and solving a linear equation, learners are required to extend their operational view of the equal sign to incorporate two algebraic uses of the equal sign (Matz, 1980). The first use is in a tautology and the second in a constraint equation. Tautologies are statements of equivalent expressions, for example, $5(x+2)=5x+10.$ Matz (1980) states that simplifying expressions "produces a single chained sequence of tautologies where each link in the chain is some simplification or change in form of its predecessor" (p. 137). She refers to the transformations that produced each link as a reduction. Transformations such as adding like terms, dealing with numbers or the minus symbol, or even applying the distributive rule, would count as a reduction.

We read tautologies across from left to right, for example:

$2(x+1)-3x+2=5x+4$

Or we read them from top to bottom, for example:

$2(x+1)-3x+2$

$=2x+2-3x+2$

$=5x+4$

A constraint equation on the other hand is an equation that is to be solved, for example, $5\,(x+2)=x+18$. The processes involved in obtaining a solution go beyond mere reductions. In order to isolate x, one needs to perform certain operations to both the left- and right-hand side. The term *deduction* is used to describe the transformation that produces a constraint equation in a different form and requires the understanding of inverses and balance.

Solving linear equations requires the use of both reductions and deductions, which causes confusion. For example: $2(x+5)=x+12$ requires learners to reduce $2(x+5)$ to $2x+10$, such that $2x+10=x+12$, but also requires learners to deduce that 10 (and x) must be subtracted from both sides, resulting in: $2x+10-10-x=x-x+12-10$. Using reductions, we obtain the solution $x=2$. Sometimes one needs to perform actions on the left *and* the right side of an equal sign but other times it is only on one side. This creates confusion and hence poor performance in solving linear equations.

In a constraint equation, learners not only perform reductions but need to start performing deductions. A deduction is the process of applying at least one of the properties of equations:

1) Addition property: For all real numbers x, y and z, if $x=y$, then $x+z=y+z$
2) Subtraction property: For all real numbers x, y and z, if $x=y$, then $x-z=y-z$
3) Multiplication property: For all real numbers x, y and z, if $x=y$, then $xz=yz$
4) Division property: For all real numbers x, y and z, $z \neq 0$, if $x=y$, then $\frac{x}{z}=\frac{y}{z}$
5) Substitutive property: For all real numbers x and y, if $x=y$ then y can be substituted for x in any expression

For example:

$2(x+3)=3x+7$	
$2x+6=3x+7$	Reduction $2(x+3)=2x+6$, a tautology
$2x-2x+6=3x-2x+7$	Deduction adding $-2x$ to both sides
$6=x+7$	Reduction
$6-7=x+7-7$	Deduction adding -7 to each side
$-1=x$	Reduction

A problem noted in literature about the transition from tautologies to constraint equations is that, when first being introduced to a constraint equation, the equations are simple and typically in the form $Ax+B=C$, which strongly resembles the familiar process-result tautology. Even though they are constraint equations, they fail to extract the underlying idea of a constraint. Students therefore solve these equations by undoing or by inspection. Filloy and Rojano (1989) refer to this problem as the didactic cut, where, as learners transition from an equation of the form $Ax+B=C$ to $Ax+B=Cx+D$, their previous methods fail them. Herscovics and Linchevski (1994) refer to this jump as a cognitive gap, where learners are unable to operate with or on the unknown. The visual comparison between an algebraic tautology and a constraint equation is not easy for learners to read. For example, knowing that $2(x+3)=2x+6$ and $2(x+3)=3x+6$ are semantically different even though syntactically similar, presents itself as troublesome knowledge for the students as they are encouraged to read the statement from left to right rather than see the equation as a whole.

Another difference between an algebraic tautology and a constraint equation is that constraint equations are not universally true statements. Solving an equation means determining the restrictions of the unknown, for example $2(x+3)=3x+9$ is only a true statement if $x=-3$, whereas in an algebraic tautology, for example $2(x+3)=2x+6$, there are no restrictions on x, it can take on any value. This is what Küchemann (1981) would call a letter as a variable rather than a letter as an unknown. These two types of equations, tautologies and constraint equations, differ in terms of the domain in which they are true.

3.7.2. Equations in the curriculum

Equations are important for the progression into higher grades. In Grade 8, learners are introduced to solving one- and two-step equations. The curriculum does not explicitly state that learners are to encounter multi-step equations where there is a letter on both sides of the equation but it appears that in Grade 9 it is assumed that learners have encountered such equations. In Grade 9, learners are introduced to quadratic equations in algebra (for example $x^2-5x=-6$) and geometric equations in geometry (for example $4(x+10°)=180°-3x$). In Grade 10 they then encounter exponential equations (for example $2^{x+1}=64$); hyperbolic equations (for example $\frac{14}{x+1}=\frac{2}{x}$) and financial equations (for example $2000000=p\left(1+\frac{0.095}{4}\right)^{4(45)}$. In Grade 11 learners are then exposed to trigonometric equations (for example $\sin\theta-2sec\theta=2\ sec\theta$). All these examples draw on being able to solve a simple linear equation such as $3x-2=10$ or $3x-2=4+x$. If one cannot correctly solve these, how can we expect them to solve any of the abovementioned equations?

3.7.3. Studies that investigated solving equations

There has been much research done on solving linear equations. These research outputs have focused on five main areas: young learners solving equations; solving equations in light of one's view of the equal sign; the didactic cut; strategies used to solve equations; and errors made when solving linear equations.

3.7.3.1. Early algebra and young learners solving equations

As mentioned earlier, some researchers argue that equations can be taught to younger learners. They argue for early algebra, as opposed to algebra early. Early algebra is different to algebra early in that the former builds on background contexts and only gradually exposes learners to formal notation (Carraher et al., 2017). It is also connected to other mathematics topics in primary school such as addition, subtraction, multiplication, division, ratio and proportion and measurement (Carraher et al., 2017).

3.7.3.2. The didactic cut

The problematic realm of moving from arithmetic to algebra was discussed in depth in section 3.3. However, in terms of equation solving, studies have focused on what's called the *didactic cut* (Filloy & Rojano, 1989), or *cognitive gap* (Linchevski & Herscovics, 1996) when moving from arithmetic equations (equations with a letter on one side) to non-arithmetic equations (equations with a letter on both sides of the equal sign). Filloy and Rojano (1989) suggested that when solving linear equations, learners respond arithmetically to equations that contain a letter on one side and then are unable to operate on the letters when faced with equations with a letter on both sides of the equal sign because their previous arithmetic methods fail them. This was termed the *didactic cut.* Linchevski and Herscovics (1996) did not find evidence of the didactic cut in their study but rather found that learners struggled with an equation in the form $Ax + B = Cx + D$ just as much as they struggled with $Ax + B = C$. They did however agree that there was a gap in going from the one to the other but that it was about operating on the unknowns rather than where the unknowns are in an equation. This was corroborated by Vlassis (2002), who stated that the didactic cut relates more to the degree of abstraction of the equation and whether the equation is associated with a concrete model. Lima and Healy (2010) also didn't find evidence of the didactic cut, but agreed that there was a cut. However, this suggests a different reason for it. Filloy and Rojano (1989)'s reason for the cut was the unknown being on both sides and not just one side. Herscovics and Linchevski (1994)'s reason was that learners are not able to operate on or with an unknown no matter how many unknowns there are. Lima and Healy (2010) suggest yet another reason for the cut; their evidence pointed towards moving mathematical objects around in an embodied world (Lima & Tall, 2008) rather than the symbolic one.

3.7.3.3. The different strategies learners use to solve equations

Another area related to solving equations that has received attention is the different strategies that learners use. I have identified seven different strategies and discuss them below.

Guess and check

This method is also referred to as "trial and error substitution" (Kieran, 1992) and requires students to recognise letters as representing numeric values and then requires them to have some basic arithmetic skills. Ideally, learners should substitute a numeric value and then check that it indeed leads to an equality; however, students rarely check their solutions. The strategy is arithmetically based and so works best with arithmetic equations where there is a letter on one side and where the solution is a whole number. The breakdown of this strategy occurs when learners are faced with an equation with letters on both sides and a fraction and/or negative solution.

Counting techniques

Counting techniques or number facts are also arithmetically based and just as Linsell (2009) states that an equation of the form $x + 5 = 11$ can be solved using counters or through guess and check methods, it can equally be solved using number facts, or as a "part whole" (p. 332) where the equation can visualise 11 and $6 + 5$.

Undoing

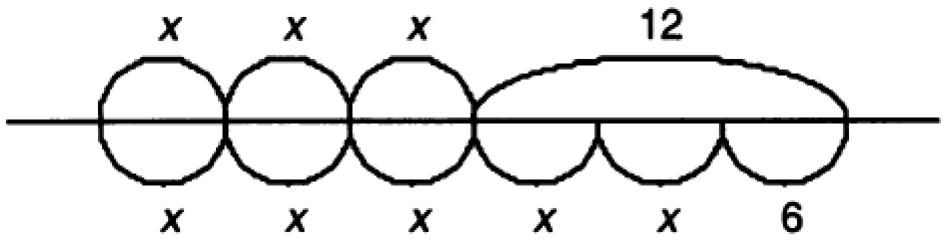
When learners undo an equation, they identify the operations that hold the equation together and then reverse the operations until the solution is reached. For example, in solving $3x - 2 = 10$, the learner would first identify that x is multiplied by 3 and 2 is then subtracted from it. The learner would secondly apply the opposite operations. The solution process would therefore be to add 2 and divide by $x = \frac{10+2}{3}$. Besides knowing the opposite operations for addition, subtraction, multiplication and division, learners would need to know about the order of operations in relation to x. This strategy also requires the ability to capture the structure of the relationships between the numbers and letters in the equation (Kieran, 1992; Watson, 2009). Kieran (1992) states that once learners learn the formal methods for solving equations, they stop using the undoing, working backwards method. However, more importantly, learners also stop using it to check their solutions (Lewis, 1980).

Transposing

Transposing is often seen as a formal method of solving equations (Kieran, 1992), but it is also known as 'change the side-change the sign' (Tall, de Lima, & Healy, 2014). They showed that some learners solve linear equations by relying on procedural embodiments. They see a procedural embodiment as a procedure that involves actions on the symbols such as picking them up and moving them to the other side of an equation.

Number line

Dickinson and Eade (2004) offered a less known method for solving equations: the number line. This method uses a double number line and jumps that represent x and/or constants. For example, in their article they show how to solve an equation with a letter on both sides of the equation $3x + 12 = 5x + 6$. The first number line is above the line, which represents $3x + 12$. We see this with three jumps of x and then a jump of length of 12. The second number line is below the line and represents 5x+6. The two sets of jumps need to start and end at the same place because they are equal. This visual form of solving an equation is helpful to see that the three jumps of x on top "cancel with the three jumps of x at the bottom, leaving $12 = 2x + 6$. We can then say the jump of six can cancel with half the jumps of 12, leaving $6 = 2x$; two jumps that make six would be two jumps of three.

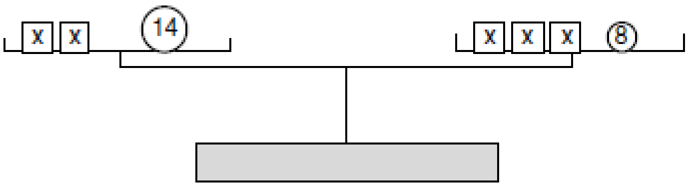


(Dickinson & Eade, 2004, p. 42)

This method allows for what seems like a natural progression when dealing with different types of equations, for example $2x + 1 = 4$ and $2x + 1 = x + 4$. This contrasts with other methods where moving to a letter on both sides causes difficulty for students (Filloy & Rojano, 1989). The limitation with this method is in dealing with subtraction or when x has a negative solution. For example, $3x - 5 = 4x + 2$ or $4x + 10 = -2$.

Balance method/ Inverse operations

The balance method is based on the concrete model of using balance scales. Vlassis (2002) uses the analogy of scales. In order to keep the scales balanced one needs to add or remove from both scales. The 'rule' we use that represents this model is that what is done to the left is done to the right. For example, the figure below comes from Vlassis (2002). The scales are balanced and to solve for the unknown we can take off two of the x boxes from both sides, leaving 14 on the left and an eight and an x on the right. We can then remove 8 from the right and left scale, yielding $x = 6$



(Vlassis, 2002, p. 344)

The formal method that represents these scales would be:

$$2x + 14 = 3x + 8$$
$$2x - 2x + 14 = 3x - 2x + 8$$
$$14 - 8 = x + 8 - 8$$
$$6 = x$$

Graphing method

The last and most sophisticated method for solving equations is the graphing method. This is used by creating two functions and determining their intersection. For example, to solve the equation $3x - 2 = 4 + x$, one would create two functions: $f(x) = 3x - 2$ and $g(x) = 4 + x$, on drawing these two functions, the place of intersection where $f(x) = g(x)$ would be the solution to $3x - 2 = 4 + x$. This method is usually only introduced when solving simultaneous equations, and hence is not popular amongst Grade 9 learners. In addition, Yerushalmy (2006) found that learners did not use the graphing software because it did not "support symbolic formulations and manipulations".

Although some debate exists on the levels of difficulty of each strategy or approach (Linsell, 2009), each breaks down at some point. Despite the proposed level of difficulty of each strategy, in the study conducted by Roberts and le Roux (2019), they found that that learners who achieve above-average assessment scores when solving linear equations have difficulty explaining why or when to use a particular strategy.

This emphasises the importance of learners being exposed to multiple strategies so that they can decide which is best for them in a given context. Although Linsell (2009) found evidence of approaches being hierarchical and an indication of learners' level of performance in algebra, teachers restrict the number of methods they teach and this therefore proves to be an ineffective way to build up students' understandings of the concept of algebra (Kieran, 1992).

Discussed here were all correct strategies: perhaps some are less sophisticated but when done correctly they produce the correct solution. There is a gap in research in that nobody talks about the 'wrong' approaches, the approaches that are illogical or made up, and approaches that will never lead to a correct response. In the study by Linsell (2009), she had a large category "no successful strategy". This category was the most common code given. It is well known that many learners do not know how to solve equations, but what we don't know is what they actually do when they use these unsuccessful categories, what pieces of equality and equations they do know. Looking deeper into the unsuccessful categories to see what a learner can do is an outcome of my study.

The above discussion was about the strategies learners use. This discussion is about the concrete models that are available for learners to use. One of the problems with the transition from arithmetic to algebra is that learners move from working concretely and then in algebra need to start working symbolically. Therefore, by using concrete models to help them solve equations we are not only adding to their conceptual understanding but are providing a smoother transition to algebra.

3.7.3.4. Learners' errors in solving linear equations

Knowing how to perform a deduction when solving a linear equation essentially requires two vitally important pieces of knowledge: that of inverses and that of balance. The majority of research in solving linear equations does not focus on the errors made when solving equations but rather on, for example, the strategies used, one's view of the equal sign, how teachers perceive equations or how equations are treated in textbooks (see for example Booth & Davenport (2013); Huntley & Terrell (2014); Vermeulen & Meyer (2017)). There is a paucity of research reporting on the errors made when solving equations that are directly related to deductions rather than reductions. Even when a research study focuses on errors when solving linear equations, it is the errors within the reduction that are mainly reported on (see for example Vermeulen and Meyer (2017)). The only errors that are directly related to deductions that I have found reported in literature are redistribution and switching addends (Kieran, 1992). Redistribution describes the situation when a learner explicitly adds different values to the left and right and hence does not maintain balance. Switching addends could be described as a confused variation of the rule: *do to the left as you do to the right.* Hall (2002) termed this error *the other inverse* because when learners need to use the multiplicative inverse (for example), they instead use *the other inverse*, the additive inverse. One researcher who has done extensive work with errors when solving equations is Merilyn Matz (Matz, 1980), who listed six different errors when solving equations, four of which were related to fractions within equations, one to quadratic equations, and one to linear equations. The error she listed was for applying an inverse without following the order of operations. Hence there is little research on errors related to deductions when solving equations.

Other research on linear equations relates to the teaching of linear equations (Andrews & Sayers, 2012); teachers' views about equations (Chazan & Yerushalmy, 2003; Pirie & Martin, 1997); linear equations as represented in textbooks (Huntley & Terrell, 2014); and equivalent equations (Steinberg, Sleeman, & Ktorza, 1991).

A literature review on the different studies that deal with linear equations has led me to identified gaps pertaining to equations: errors made related to deductions and research that looks at the different factors/ topics that can predict learner scores. There is little literature on combining equations with another concept in mathematics, for example variables or negative numbers. There is also not much literature on the movement from algebraic tautologies to constraint equations and also little on the relationship between errors when solving equations and errors in other isolated topics, for example, negativity. I intend to address all three gaps by investigating the relationships between errors made in number sentences, integer items and expressions, and the errors made in equations. I also investigate whether there are certain items that can predict learners' equation scores.

3.8. Conclusion

In this chapter I discussed symbol sense and the transition from arithmetic to algebra. This transition is said to be one of the obstacles to equation solving and that focusing on content such as the equal sign, letters and negatives would improve learners' performance in equation solving. These three dualities explain why so many learners struggle with algebra, and more specifically, solving equations. This led to a lengthy discussion on the four content areas that are in focus for this study: the equal sign; expressions; negatives and equations. The literature review on the different studies that involve equations led me to identify gaps in the literature and I explained how I was addressing these gaps.

Chapter 4 : Methodology

This chapter describes the methodology used for my mixed-methods, sequential triangulation design study. I started with a large data set which could only be analysed quantitatively but a random subset of that data allowed me to analyse both quantitatively and qualitatively. This is a way for me to make sense of much data. The general purpose of this study is to research the errors made when solving linear equations and investigate any changes in the errors made. A sequential triangulation design study is one that has a quantitative and a qualitative phase that generally is conducted after the quantitative phase. In my analysis and in the presentation of my data I first address the quantitative side and then the qualitative analysis. In Chapter 11, I pull together and compare the quantitative results of the large data set and the sub-sample. The quantitative part demonstrates differences in performance between different groups and provides evidence of the strength of the relationships between different variables. The qualitative part describes these differences and provides specific examples that support claims. The quantitative and qualitative analysis therefore complement each other. This chapter is separated into five parts:

4.1 Research design: The mixed-methods research

4.2 Background of the data

4.3 Description and contents of each phase

4.4 Item analysis

4.5 Validity, reliability and ethical considerations.

In Chapter 5 I discuss the codes used for the analysis.

4.1. Research design: Mixed-methods research

This study follows a sequential triangulation, mixed-methods research design (Teddlie & Tashakkori, 2009). Mixed-methods research (MMR) can be viewed as a third paradigm alongside qualitative and quantitative research paradigms, or as a mixture of the two paradigms (Johnson & Onwuegbuzie, 2004). A research paradigm is an established model of thinking that is widely accepted by a research community (Bertram & Christiansen, 2013). It is a combination of one's ontological, epistemological and methodological beliefs (Hatch, 2002). In other words, it is the overlap of one's view of reality (ontological belief), how that reality is investigated (epistemological belief) and what instruments are used for the investigation (methodological beliefs). A key tenet of MMR is its methodological pluralism, and Johnson and Onwuegbuzie (2004, p. 15) argue that "epistemological and methodological pluralism should be promoted in educational research so that researchers are informed about epistemological and methodological possibilities and, ultimately, so that we are able to conduct more effective research".

4.1.1. Defining MMR

Although mixed-methods research is only recently gaining acceptance as an alternative research design (Caruth, 2013), it is, in fact, not a new methodology (Maxwell, 2016). For this reason, it is surprising that a single, universally accepted definition of MMR does not exist (Johnson, Onwuegbuzie, & Turner, 2007). I adopt the definition offered by Johnson and Onwuegbuzie (2004, p. 17) as it includes very broad criteria that describes it as "the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study or set of related studies." Johnson et al. (2007, p. 120) elaborate on the last part of the definition by stating that MMR is "the mixing of qualitative (kind, type) and quantitative (amount) language or discourse (e.g., in one's methodological worldview, in forming interpretations, and in writing and communicating research findings)". I have deliberatly chosen Johnson and Onwuegbuzie (2004)'s definition over, for example, Creswell (2014)'s definition. Creswell (2014)'s definition states that MMR is when the researcher collects, analyses, *and* mixes (integrates or connects) both quantitative *and* qualitative data in a single study. His use of the word "and" versus Johnson and Onwuegbuzie (2004)'s use of the word "or" was more suited to the design of my study.

4.1.2. Justification for MMR

MMR offers refined insights and a greater depth of understanding into that which is being studied (Caruth, 2013). Although qualitative research is generally accepted to be the paradigm that gains a greater depth of understanding (Lund, 2012), it is difficult to generalise and duplicate (Onwuegbuzie & Johnson, 2006). Quantitative research, on the other hand, offers better objectivity and generalisability, but is criticised for having "utilized immaterial hypotheses and shallow descriptions" (Caruth, 2013, p. 112). The intention behind using MMR is to maintain the strengths and limit the weaknesses of each design (Truscott et al., 2010). Conducting a purely quantitative study would have meant compromising on knowing what learners are doing when dealing with transformational activities in algebra. For example, conducting a cluster analysis on a large data set would group learners' responses that appear to be similar. However, without a qualitative component to the study the mathematical underpinning of why the clusters were possibly grouped together would not be known. On the other hand, a purely qualitative study is not possible as interviews with the learners or in-depth coding of all the errors would not be feasible on a large data set. I have therefore designed a study that draws on both quantitative and qualitative methods.

4.2. Background of the data

I draw on data collected by the Wits Maths Connect Secondary project for a study called *Learning Gains (LG)*. The sections that follow give a background to the data and explain my study in the context of the Learning Gains data.

4.2.1. The Wits Maths Connect Secondary Project

The Wits Maths Connect Secondary (WMCS) project is a research and development project my research is located in. It is funded by a public-private partnership and started in 2010. The aim of WMCS could be summarised as: enabling better mathematics teaching to improve learner performance, where an underlying assumption is that focusing on teachers' mathematical and pedagogic knowledge is not only an important first step to improved teaching, but will lead to improved learner attainment (Ntow and Adler (2019); Pournara et al. (2015)). The project has therefore offered a professional development course (Transition Mathematics 1 (TM1)) to Grade 8-10 mathematics teachers, and then researched the impact of the teachers' participation in the professional development (PD) on their learners. I have been a member of the WMCS project since 2015 and hence have not only been a part of the PD, but have also assisted in terms of collecting and analysing test scripts. Details about the PD course; TM 1; its design; focus; scope; participants; and assessment can be found in Pournara and Barmby (2019). The part of the WMCS project focused on learner attainment is called the Learning Gains Study (LG) and is the data on which my research is based.

4.2.2. Learning Gains study

In the context of the WMCS project, the term *Learning Gains* refers to an increase in learners' test scores in a pre-test, post-test design. It is important to note that there was no intervention between the two tests. The pre- and post-tests were identical and learners wrote in the first and third quarter of the academic school year (February and September).

The LG study was first conducted in 2013 and 2014, with the findings published in 2015 (see Pournara et al. (2015)). Pournara et al. (2015) reported on the learning gains made by Grade 10 learners taught by teachers who participated in the TM1 course. Findings revealed that although the learning gains were small, they were statistically significant and hence provided empirical evidence that improving teachers' mathematical knowledge can have a positive effect on learning. In 2016 the WMCS project began a new study focusing on Grade 9 learning gains. The year 2016 was devoted to developing and piloting a new test instrument informed by the SOLO taxonomy (Biggs & Collis, 2014) and a Rasch Analysis was conducted to test for its reliability and validity. The background for the necessity of developing our own assessment can be found in Pournara, Hodgen, Sanders, and Adler (2016). I played a part in the development of the test items, administering the test and the initial coding of the responses. I was also involved in the interviews that took place in 2016. The test consisted of 45 items, 18 of which I am interested in because they relate to integers, expressions, equality and solving equations. Although I am basing my research on these 18 questions, it is important to note that the items were not developed with my research in mind.

In February 2017, WMCS administered the test to 2518 Grade 9 learners from 10 secondary schools in Gauteng. In September 2017 we administered the test to 2416 Grade 9 learners in the same schools. There were 2279 learners who were paired, meaning they wrote both tests. The 10 schools can be described as “schools for the poor” (Shalem & Hoadley, 2009). These are low-performing schools and the learners are taught in English although it is not their first language. The schools were chosen because they had at least one teacher who had participated in TM1 in the previous year (2016), and both the principal and teacher agreed to participate in the study. Again, I played a major role in the organisation, administrating of the test, invigilating, coding and capturing learner responses as well as in anonymising and archiving the test scripts.

In 2018, WMCS extended the LG study to include Grade 10 learners since it allowed for the inclusion of more teachers who had completed TM1 in 2016 and 2017. The test was administered in February 2018 to 3025 Grade 9 and 1860 Grade 10 learners (PD-learners) from 25 schools (PD-schools), and then to 2 949 Grade 9 learners and 1643 Grade 10 learners in September (See table 4.1 for a summary). For the WMCS project, the aim of their research is to determine the impact of their PD and so it was important to have a comparison group of learners who were not taught by teachers who participated in the PD. The WMCS team worked in collaboration with the Department of Education to identify matched comparison schools for the 25 “PD schools”. In total, over the two years, 2017-2018, the WMCS project collected 14 095 scripts from 7 943 learners in 40 low-performing secondary schools in Gauteng. In both 2017 and 2018, learners wrote the pre-test and/or the post-test. Some learners only wrote the pre-test, some only the post-test and some wrote both. In addition, three PD schools were part of the sample in both 2017 and 2018, which meant there were some learners who wrote the two tests in 2017 when they were in Grade 9 and then again in 2018, when they were in Grade 10. This yielded four data points for each of these learners. Table 4.1 shows the breakdown of the population and the number of test scripts collected.

2017 and 2018 data in the WMCS project	Grade 9		Grade 10		Total no. scripts
	Feb	Sept	Feb	Sept	
1 data point 2017	239	137	0	0	376
1 data point 2018	282	447	229	136	1094
1 data point 2018 comparison	341	100	159	35	635

2 data points 2017	2121	2121	0	0	5578
2 data points 2018	1358	1358	668	668	2716
2 data points 2018 comparison	886	886	646	646	3064
4 data points 2017-2018	158	158	158	158	632
Total	5385	5207	1860	1643	14095

Table 4.1: Table of the Learning Gains sample

This large data set inspired me to pursue a mixed-methods study on Grades 9 and 10 learner attainment and the errors they make. Due to the scope of my research, I chose to focus on the PD learners. I am therefore not investigating the impact of the PD on learners and do not analyse the data collected from the comparison group. The focus of this research is on the relationships and changes in errors made in the context of solving linear equations. Therefore, I selected data from learners a) whose scripts can be paired (i.e. who have two or four data points); b) who were taught by teachers who participated in the PD and c) who attempted the three equation questions. These criteria have meant that I have a sample of 1671 Grade 9, paired, PD learners and 464 Grade 10, paired, PD learners.

4.2.3. My sample

The quantitative phase (phase 1) of my study incorporates the full sample of paired PD learners who attempted the equation items ($n = 1671$ Grade 9; $n = 464$ Grade 10). For the qualitative phase, phase 2, a random sample of 150 Grade 9 and 150 Grade 10 learners were used from 23 different schools. This means that I analysed 9% of the Grade 9 paired data and 32% of the Grade 10 paired data. This number was chosen for two reasons: I wanted to have the same number of Grade 9 and Grade 10 learners for statistical reasons, and I needed to have a number that was practically feasible to analyse within the constraints of a doctoral study. To be clear, the Grade 9 and 10 learners are not the same learners but are compatible in the sense that they are from similar schools and have similar backgrounds. The learners were also chosen randomly which is an important consideration for quantitative analysis. The Grade 10 learners have had one year extra of learning and so it stands to reason that they have more correct responses in the test however the focus is on the amount of gains each grade makes on the same test.

4.2.4. Instrument design

As mentioned earlier in this chapter, the test was developed by the WMCS project and informed by the Solo taxonomy ((Biggs & Collis, 2014). Some of the items were taken from the CSMS test, for example 2b) $2a + 5b + a$ and others were informed by them, for example, $2a(a-4)-8.$ A Rasch analysis was conducted to test for its reliability and validity. Since I draw on the LG data, a result is that this methodology section does not have an instrument design or data collection component. For the WMCS project, all learner responses to all items of the test were coded using a three-part scale: missing, correct and incorrect responses, which were assigned codes 0, 1 and 2 respectively. I played a part in the coding and capturing of the responses. Following from the initial three-part scale coding, my study begins with learner scripts and a spreadsheet. In order to answer my research questions, I have designed two phases of analysis, where the sample for phase 1 is different to that of phase 2.

The original test contained 45 items but only 18 of these are relevant to my study (see table 4.2) as they relate to integers, expressions, equality and linear equations. For the purpose and remainder of this thesis, I refer to the 18 items as 'the test'. Sub-totals for the topics mentioned above were calculated (integer scores; expression scores, etc.). All scores were converted to percentages to make comparisons easier.

Item No.	Item
	Integer items
1	Write these numbers in order from smallest to largest: $30 \quad -35 \quad -2 \quad -500 \quad -10 \quad 4$
3a	$5 - 7 =$
3b	$5 + (-7) =$
3c	$-5 + 7 =$
3d	$6 - (-10) =$
3e	$-7 - 5 =$
	Equality items
4a	$7 + 5 = \_\_\_\_\_ + 2$
4b	$4747 + 3945 = \_\_\_\_\_\_\_ + 3943$
	Expression items
5a	$2a + 5a =$
5b	$2a + 5b + a =$
5c	$(a + b)b =$
5d	$a + 4 + a - 4 =$
5e	$3a - (b + a) =$
5f	$a + b + a - b =$
5g	$2a(a - 4) - 8$
	Equation items
9a	$3x - 2 = 10$
9b	$3x - 2 = 4 + x$
9c	$2 - 3x = 7 - x$

Table 4.2: Test Items in focus

As a reminder, the samples used for the analysis presented in Chapter 6 are 1671 Grade 9 learners and 464 Grade 10 learners from 23 schools in Gauteng. In Chapters 7-10 the sample is a subset, consisting of 150 Grade 9 and 150 Grade 10 learners from the 23 schools. All 23 schools are urban schools where learners come from poor socio-economic backgrounds, and while the language of learning and teaching is English, it is not the home language of the majority of the learners.

4.2.5. Justification of the quality of the data

The WMCS data is worthy to be used for this study for two main reasons:

4.2.5.1. It is data we can trust

I am confident that I can trust the data that was collected in terms of the instrument use, the procedures used for collection, the initial coding, and the capturing of the data. The items of the test were developed with the Solo taxonomy (Biggs & Collis, 2014) in mind. The instrument was piloted and a Rasch analysis performed to confirm the items were suitable for the purposes of the test. The collection of data was done under test conditions and so we are confident that copying answers is not something to be concerned about. I can also see from the initial coding that there are not rows of identical responses of correct, incorrect or missing. I am also confident that learners mostly did not respond to the items purely because they felt they needed to write something down, something that is seen as a common practice in primary school (Hoadley, 2008). I can say this because there are many missing responses. Many responses did not make sense and what I have done by introducing a sequential mixed-methods study is develop a way to create some order from the chaos. The coding and capturing of 14% of the data was moderated by myself and two undergraduate students I trained.

4.2.5.2. It is a good representation of schools in Gauteng

In 2016, education MEC Panyaza Lesufi stated that schools would be categorised into four categories (poor, fair, good, and great) depending on their matric pass rates, maths and science pass rates, and bachelor passes. The matric results showed that 25% of Gauteng schools are 'poor', 41% are 'fair', 22% are 'good' and only 12% are 'great' (Macupe, 2016). This means that of the 853 schools in Gauteng (Hamann, 2016), 214 schools are 'poor' achieving schools. WMCS tested about 10% of these schools: the resulting data collected, and that I am using as my sample, is therefore a good representative of the 'poor' schools in Gauteng that are underperforming and not getting above 50% pass rates.

In addition to the above, the WMCS data is a large data set, and my involvement in the project and with the data for the project, provides me with the unique opportunity to use the data for my study. I would never be able to collect such a large amount of trustworthy data on my own for my PhD. Although the WMCS data set consists of over 7000 learners from 40 schools, I draw on a subset of 2135 learners from 23 schools.

4.2.6. Contribution to WMCS

As mentioned earlier, my research focuses on errors made by PD learners. My research will contribute to the WMCS project as it will identify the types of errors these learners make. It also provides opportunities for future research to compare errors made by PD learners and errors made by the comparison group. One of the foci of the PD was making teachers aware of the types of errors found in literature, especially in integers, and so it will be useful for the WMCS project to see whether there is a shift in the errors made in comparison with what is in the literature.

This section has given the background, or rather, the heritage, into which my research and data were born. I have explicitly highlighted that although the focus of the WMCS project is twofold (PD and learner attainment) my study focuses only on learner attainment and errors.

4.3. Description and contents of each phase of the analysis

In order to answer my research questions, I conducted my study in two phases of data analysis that combine qualitative and quantitative methods. In this chapter, I elaborate on each phase, discussing what analysis the phase consists of, how it contributes to answering my research questions, and what sample is used for each phase.

4.3.1. Phase 1: Quantitative: Trends and patterns

This phase acts as a backdrop, provides the context in terms of learner performance, and begins to talk to my first research question about the overview of learners' overall test performance. It is descriptive and informative. It involves a quantitative analysis of categorical data (codes 0, 1 and 2), as well as scale variables (scores for a collection of correct items per learner as well as scores for the number of correct responses per item). The sample used for this phase is 2135 learners comprising the data of 1671 Grade 9 and 464 Grade 10 learners from 23 schools. All 23 schools are urban schools where learners come from poor socio-economic backgrounds, and while the language of learning and teaching is English, it is not the home language of the majority of the learners. The original test had 45 items but I only focus on 18 as they relate to integers, expressions, equality and equations.

This phase discusses analyses from three points of view. I conduct a descriptive analysis, a response pattern analysis, and inferential statistics analysis. These are discussed in more detail below.

4.3.1.1 Descriptive analysis of the data set

The scale variables will describe the general direction or trend of this particular population, while the ordinal data will illuminate patterns in learner responses. Using descriptive statistics such as percentages and frequency counts, I identify the items and topics that were best (and worst) answered in both the pre- and post-tests, as well as in Grades 9 and Grade 10. Measures of central tendency (mean, mode and median) will describe the way the group relates to a central value, while measures of variability (range, quartiles and the interquartile range, variance and standard deviation) will describe how similar (or varied) the data is for a particular test item (or question/topics).

4.3.1.2 Response pattern analysis of learner responses

A response pattern analysis similar to that used by Steinle, Gvozdenko, Price, Stacey, and Pierce (2009) and Steinle and Stacey (2003) will be performed. An example can be seen in Figure 4.1 alongside. The vertical black line separates the pre-test from the post-test. Both sets of questions were ordered from easiest to most difficult based on the number of correct responses from the whole sample. The data was then sorted so that the correct responses (green) were shown at the top. For the three pre-test items, we can see that learners responded as expected: learners who couldn't correctly respond to the more difficult question were also not responding correctly to the easier questions. On the right-hand side, however, we do not see the same pattern, which we would expect to see. A response pattern analysis will reveal patterns in learners' correct and/or incorrect responses and hence determine groups of closely related responses. This form of analysis will also allow me to investigate how learners perform on certain items. Counting the number of observations where patterns occur will inform me of which patterns are prominent for the sample.

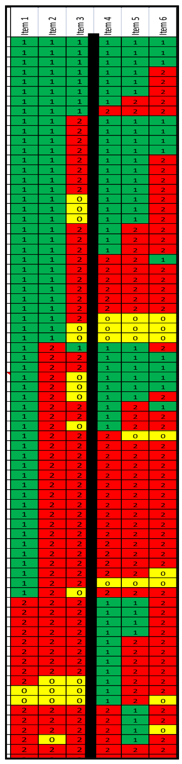


Figure 4.1: Example RPA

4.3.1.4 Inferential statistics analysis of the data

Having both a pre- and post-test enables me to compare results and determine what shifts (if any) have occurred in terms of overall general performance. To be specific, multiple T-tests were used to test for significant differences between groups of learners (pre- and post-Grade 9, pre- and post-Grade 10, as well as between Grades 9 and Grade 10). Analysing the data by using descriptive analysis, response patterns and inferential statistics will provide an overview of the performance of these Grade 9 and Grade 10 learners. I will not only gain insight into the trends but also the patterns of responses. The difference between trends and patterns is that where trends indicate the general direction, patterns identify a series of data points that repeat in a predictable or recognisable way.

4.3.2. Phase 2: Concurrent triangulation design

Phase 2 consists of a quantitative part followed by an independent, qualitative part. The quantitative section consists of descriptive statistics, T-tests, effect size and a response pattern analysis while the qualitative section consists of an error analysis of learner responses.

A T-test is a statistical test that tells you the significance of the difference between the means of two groups. The significance tells you whether the difference is due to chance or real change. If a change is statistically significant, the p-value will be less than 0.05, meaning that there is a 95% confidence interval that the change is statistically significant. Two types of T-tests were conducted, depending on the type of sample used. A paired T-test was used when the samples were directly related, for example, the Grade 9 pre-test learners were the same as the Grade 9 post-test learners. This means that the number of learners in each group was the same but more importantly the actual learners were the same. When I compared two different samples, for example the Grade 9 pre-test group and the Grade 10 pre-test group, these learners were not the same and hence the size of the sample was also not the same, in this situation a standard T-test was conducted.

The effect size is a value that tells you the strength of a relationship between two variables. The value of an effect size varies between 0 and 1. A small effect size is less than 0.2 and suggests a weak relationship. A large effect size is considered to be larger than 0.8 and suggests a strong relationship (Cohen, Manion, & Morrison, 2007).

A response pattern analysis was discussed earlier. In phase 1, the response pattern analysis is based on the relationships between topics as well as within topics for a large sample. In phase 2, the response pattern analysis is based on the sub-sample of 150 learners in Grade 9 and 150 learners in Grade 10 and is based on each topic, looking at the group of learners that get the items correct or incorrect.

The qualitative part is a document analysis that involves an in-depth error analysis on 18 of the 45 items from the WMCS test. Although there is literature on the types of errors made in integers, expressions, equality and solving linear equations, the quantity and prominence of errors made is not known for my sample. A mix between a typological and an inductive data analysis was used to analyse the data for the qualitative part of Phase 2 (Hatch, 2002). Learners' responses to test items were coded systematically using both inductive and deductive codes. I used categories derived from literature as a preliminary way to classify the errors. Additional subcategories were added in order to categorise errors found that are not found in the literature.

Frequency counts of the errors made show what errors change over time. A change in the errors made is seen within questions, across questions, within a grade from pre- to post-test and across grades from Grade 9 to Grade 10. The frequency counts also show whether certain errors are more prominent in certain items (and questions/topics) and whether they are carried through into different items (and different questions/topics). This part of phase 2 is not only necessary, but requires great rigour and a lot of time to complete in order to address the research questions.

I have structured this phase in terms of the topics investigated. This means that for each topic (equations, equality, integers and expressions) I conduct the quantitative part and then the qualitative part rather than conducting the quantitative analysis for all topics and the qualitative analysis for all topics. This decision was made to enhance the coherence of the analysis.

In various sections (for example in Chapter 9) I have used an error map to show the movement of errors in a particular item. An error map is a diagram that provides the tracking of the errors. It contains two columns, one for the pre-test and one for the post-test. In these two columns are the codes for the errors and a line connects the code from an error in the pre-test to an error in the post-test. For example, if a line with the percentage 24%, for example, goes from error 1 in the pre-test to error 3 in the post-test, it means that 24% of the learners who made error 1 moved on to error 3 in the post-test. See Figure 4.3 as an example. For this particular item there were three errors made; these can be seen in each of the pre- and post-test columns. Error 1 has $\frac{54}{150}$ learners (36%) that make that error in the pre-test; 63% of them however continued to make error 1 in the post-test. This is shown by the bold green line with n=34(63%) on it. A total of 58 learners made error 1 in the post-test, which means that 59% ($\frac{34}{58}$) of error 1 in the post-test was made up of learners who made the same error in the pre-test.

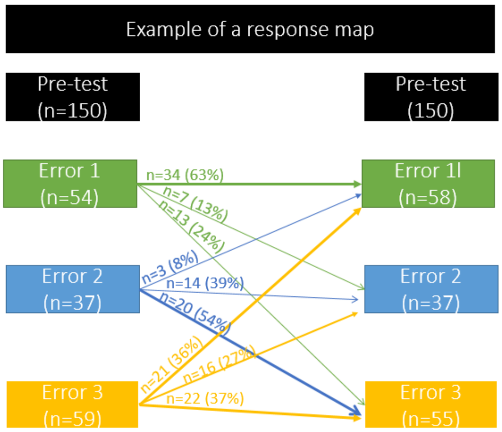


Figure 4.1: Error map example

4.4. Item analysis

As mentioned earlier, I analysed 18 of the 45 items. There are six items relating to integers (Questions 1 and 3); two items on equality (Question 4), three on equations (Question 9) and seven items on algebraic expressions (Question 5). These items are displayed in Figures 4.3 to 4.7. These items were chosen because they relate to my research topic.

4.4.1. Integer items

1) Write these numbers in order from smallest to largest:

30 -35 -2 -500 -10 4

Figure 4.2: Integer items: descending order

a) $5 - 7 =$
b) $5 + (-7) =$
c) $-5 + 7 =$
d) $6 - (-10) =$
e) $-7 - 5 =$

Figure 4.3: Integer items

Item 1 tests whether learners are able to order integers while items 3a-3e test learners' ability to add and subtract integers. There are commonalities between items, for example, 3a, b, c and 3 all use the numerals 5 and 7. Item 3d is the only item that uses different numerals. However, 3d is similar to 3b in that they both contain brackets, which literature says is a difficulty for learners. Items 3c and 3e are similar in that both have a learding negative. Items 3a and 3b are structured differently yet reduce to the same value. These errors within these commonalities are investigated.

4.4.2. Equality items

4) Write down the missing number in the space provided:

a) $7 + 5 = \_\_\_ + 2$

b) $4747 + 3945 = \_\_\_\_ + 3943$

Figure 4.4: Equality items

These two items test learners' view of the equal sign. The difference between the two items is that item 4b uses larger values and hence expects learners to approach the item using structure sense rather than recalling number facts. These items are different to the equation items in that they do not use letters and purely rely on arithmetic. Obviously, learners could approach the items using knowledge of equations but that was not the purpose of the items.

4.4.3. Expression items

5) Simplify:

a) $2a + 5a =$
b) $2a + 5b + a =$
c) $(a + b)b =$
d) $a + 4 + a - 4$
e) $3a - (b + a) =$
f) $a + b + a - b =$
g) $2a(a - 4) - 8 =$

Figure 4.5: Expression items

There are seven items dedicated to the simplification of terms. The majority of the items involve addition. Item 5a, b, d and f all involve the adding and subtracting of like terms while items 5c and g require learners to use the distributive law in order to simplify the expressions. Item 5a and 5b are very similar in that they both use the numerals 2 and 5, but the difference is that in 5b there are unlike terms. Item 5f is a generalised version of item 5d and once again, errors within these commonalities are investigated.

4.4.4) Equation items

9) Solve for the unknown:

a) $3x - 2 = 10$
b) $3x - 2 = 4 + x$
c) $2 - 3x = 7 - x$

Figure 4.6: Equation items

These three items test learners' ability to solve a basic linear equation. The equations involve only addition and subtraction, there are no brackets and no fractions in the structure of the items. Item 9a and 9b are similar in that they have the same left-hand side, but they are different in that 9b has a letter on both sides of the equal sign. Another important difference is that item 9a can be solved using arithmetic instead of inverse operations. Item 9b and 9c are similar in that they both contain a letter on both sides of the equal sign but what is different in their structure is that the letter is subtracted in item 9c, on both sides of the equal sign. In retrospect, there are many other equations that could fit between items 9b and 9c, however I used an already set and tested test. If I were to repeat this study, I would develop my own test and incorporate many different equations, especially ones that fit between 9b and 9c, for example, $3x - 2 = 4 - x$; $4 - x = 3x - 2$; $2 - 3x = 4 - x$.

4.5. Validation criteria and ethical considerations

In both qualitative and quantitative research, there are two types of validation criteria that need to be addressed: validity and reliability (Creswell, 2014). Reliability is about the quality of measure. A study is deemed reliable if it would yield the same results if conducted again in a similar context. Without this measure, quantitative research would be considered invalid and therefore reliability is a pre-condition for validity (Cohen et al., 2007) in a quantitative study. Validity is about the legitimacy of the findings and hence Onwuegbuzie and Johnson (2006) propose the term legitimation instead of validity.

The term validity has been problematised by researchers such as Teddlie and Tashakkori (2009) because of its multiple meanings (35 different meanings are cited by Teddlie and Tashakkori (2009) within both the qualitative and quantitative paradigms). Yet, despite the multiple meanings, the process is similar across the two orientations (Teddlie & Tashakkori, 2009).

Guba and Lincoln (1982) propose the term trustworthiness instead of validity, which includes credibility, transferability and confirmability. Since trustworthiness is similar across both qualitative and quantitative research and incorporates aspects of reliability within its transferability tenet, and since I am doing a mixed-methods study, I will draw on trustworthiness to discuss the legitimacy of my findings from a quantitative as well as a qualitative perspective (Caruth, 2013; Venkatesh, Brown, & Bala, 2013).

4.5.1. Trustworthiness

4.5.1.1. Credibility

Credibility refers to the accuracy of describing the setting (choosing the appropriate subjects), the data and the manner in which the study was conducted. This study can be considered credible. The topic investigates the change in errors across Grade 9 and 10 learners and Grade 9 and 10 learners were used as the sample. As discussed in section 4.2, the data is not only a good representation of the full data set but is also data that can be trusted. The test was written under examination conditions and was piloted and analysed using Rasch analysis. The study itself was conducted ethically, and the methodology has been logically and systematically laid out for the reader to follow.

4.5.1.2. Transferability

Transferability is about the generalisability or external validity of a study (Cohen et al., 2007; Guba & Lincoln, 1982). For the qualitative section of my study, some transferability of the findings is possible through purposive sampling and providing a 'thick description' (Guba & Lincoln, 1982) of the study in order for it to be transferred to a similar context. Phase two of my study uses purposive sampling, and I have provided an accurate and in-depth account of the phase and what it consists of and how the data was not only collected but also analysed so that the study can be transferred to a second and similar setting. My findings should therefore be transferable to other township schools in Johannesburg, using Grade 9 and Grade 10 learners who perform in a similar fashion, with similar mathematical errors and error patterns. The quantitative findings are transferable in that percentages, frequency counts and T-tests can all be replicated with the knowledge of each learner's performance.

Dependability is synonymous with the notion of reliability in a rationalist paradigm (Cohen et al., 2007; Guba & Lincoln, 1982), which is about the extent to which a study can be replicated. Using Guba and Lincoln's (1982) terms, dependability is about the stability of a study and is achieved through the use of 'overlapping methods', which is to use some form of triangulation. In this study I use multiple methods such as frequency counts, response pattern analyses, box plots and T-tests to make conclusions. This study is therefore a triangulation study in that quantitative and qualitative methods are triangulated and multiple methods of analyzing and interpreting the quantitative results are used. This is done in order to increase the reliability and trustworthiness of the results. Increasing dependability means to, for example, having a paper trail that points to each stage or phase of one's study and clearly outlines each methodological step and decision made. Since I have the original tests, I am able to confirm every percentage and performance pattern and provide proof of every error made. My study therefore exhibits aspects of dependability.

4.5.1.3. Confirmability

Confirmability is the last tenet of Guba and Lincoln's (1982) notion of trustworthiness. This is concerned with being objective rather than subjective. In terms of the quantitative findings, it is easy to be objective but for the qualitative findings it means that all my findings and interpretations need to be reasonable and evident to others. For my quantitative data to be confirmable, I must ensure that others would code the errors in the same way that I did. In the chapter that follows, I explain each code but also provide examples of errors that were coded as such.

Details of each code were given in detail and so other researchers should be able to recode or use my codes to replicate the study. Confirmability is also about being clear and upfront about the epistemological and ontological assumptions of a study, as well as the limitations. I have been clear about the epistemological and ontological assumptions of my study, and the limitations were discussed in Chapter 1.

4.5.2. Ethical considerations

Being a part of a larger project has meant that I received group ethical clearance to collect and analyse this data (Protocol number: H17/01/01). I applied for my own protocol number for this study (H19/02/28).

Two key ethical principles for conducting research are 'autonomy' and 'non-maleficence' (Bertram & Christiansen, 2013, p. 66). Autonomy relates to voluntary participation and non-maleficence relates to the researcher making every effort to protect the participants and not bring any harm to them.

WMCS requested and received ethical clearance from the Gauteng Department of Education (GDE) as well as from the Wits Ethics Committee to conduct and analyse the data collected. After receiving permission from the principals of the 25 schools, learners and parents were informed of the nature of the WMCS research, and what would be expected of them if they agreed to participate. Learners and parents who agreed returned a consent letter acknowledging that their involvement was voluntary and confidential. Hence, autonomy was achieved. With the learners, schools and teachers being anonymous, and conducting research that does not involve interacting with the learners, both autonomy and non-maleficence can be achieved.

4.6. Conclusion

This chapter has focused on the design on my study. I began by giving the background of where my data came from and how it is linked to the Learning Gains study. I discussed my sample and how it was chosen and how it differs in the two phases of my study. I provided justifications of why I use a mixed-methods design. I described what each phase of my study entails and also provided an item analysis of the items used. Validation criteria and ethical considerations were also discussed. In Chapter 5 I discuss the codes used and how learners' responses were coded. I provide many examples of learner work to show that I have been systematic and rigorous.

Chapter 5 : Codes and coding of learners' responses

As mentioned in the methodology chapter, I am investigating 18 items and hence am coding 18 learner responses. These 18 responses are separated into four topics: equality, integers, expressions and solving equations. I have a different coding scheme for each topic. When coding the responses to equations I did so from five different angles, three of which include the view of the equal sign, negatives and letters which strongly relate to the equality, integers and expression items and hence there is some overlap between those codes. I begin this chapter by explaining the coding process. Since there are many items to be coded, this chapter is long and can become tedious to read. I, therefore, present a summary of the codes and if more information is needed, the reader can consult sections 5.3-5.6, where I discuss the coding schemes used for the linear equations (Part 1). Part 2 is then the coding scheme for the equality items, Part 3 is the coding scheme for the integer items, and finally Part 4 is the coding scheme for the expression items.

5.1. The coding process

Coding took place on two levels: I coded learner responses for the quantitative analysis and then coded learner responses for the qualitative analysis.

For the quantitative analysis I coded responses as correct (1) incorrect (2) or missing (0). These were then recorded in an Excel spreadsheet and totals for each code were calculated. Using these values, the total number of correct responses for each item was calculated as well as the total number of correct responses per learner. In addition, the total number of correct responses per topic was calculated and all calculations were converted into percentages, bar graphs and line graphs. Box plots were also created. The totals for each item were also used to perform the T-tests. For the response pattern analysis, the spreadsheet with the correct, incorrect and missing codes was colour-coded (correct: green; incorrect: red; and missing: yellow) and then I filtered out the relevant sections.

The qualitative coding of responses to 18 items was not a trivial or short task, it was complex and lengthy, especially since I coded the responses to the equation items in five different ways. There were two different types of responses required from learners: firstly, the responses to the integer, equality and expression items required a single answer from learners and secondly, solving linear equations required multiple lines of working out. For the single answer responses, the coding was more straightforward and required mapping an answer to a code. For the equation responses, coding required more interpretation. How each response was assigned a code is thoroughly and systematically discussed in sections 5.3-5.6. However, I provide a summary of the codes and a short description of each in the section that follows.

5.2. Summary of codes

Part 1: Coding of equation responses			
Code		Description	Derivation of code
Lens 1: Approach	Pure arithmetic approach	Use of numbers in a logical approach	Own
	Pseudo-arithmetic approach	Use of numbers, not logical	Own
	Algebraic approach with equations	Attempt to use inverses and maintain balance	Own
	Pseudo-algebraic with equations approach	Maintained two sides of an equation but steps do not follow each other	Own
	Algebraic with expressions approach	Convert to an expression at some point	Own
Lens 2: View of the equal sign	Sameness-relational view of the equal sign	Maintained a left- and right-hand side, maintained balance, all inverses are correct	Literature
	Substitutive-relational view of the equal sign	Substituted a value, maintained balance	Literature
	Pre-relational view of the equal sign	The left- and right-hand sides are maintained but there are issues with balance and using the incorrect inverse	Literature
	Operational view of the equal sign	Work from left to right and not consider the whole of the right side of the equal sign	Literature
	Pre-operational view of the equal sign	The left- and right-hand sides of the equal sign are simplified separately	Own
	Unknown view of the equal sign	Unable to say what the view is	Own
Lens 3: Inverses	All inverses correct	All inverses are used and correct	Own
	Part of inverses in/correct	One or two inverses are incorrect	Own
	Inverse with no balance	Applied inverse to only one side	Own
	Wrong sign	Used wrong sign when applying inverse	Own
	Wrong inverse	Used the incorrect inverse	Literature
	Inverse not used	No inverse was used	Own
	Inverse used in an expression	Inverse used within an expression	Own

Lens 4: Negatives	No use of subtraction or negatives	No subtraction or use operating with negatives	Own
	In/correct subtraction	Subtraction in/correct	Own
	In/correct negatives	Operating with negatives is in/correct	Own
	Deletion of a letter	Drop the subtracted letter or have both letters 'cancel' and be left with a constant	Own
Lens 5: Letters	Non-conjoining errors	Did not add unlike terms, other errors	Own
	Conjoining	Juxtaposed a constant and x; add unlike terms, add coefficients and attach letter; add the constant and coefficient of x, which is 1	Literature
	Dropping a letter	Added unlike terms but dropped the letter	Own

Table 5.1: Summary of qualitative codes part 1

Part 2: Coding the Equality responses			
	Code	Description	
Items 4a and 4b	Operational view of the equal sign	Work from left to right and not consider the whole of the right side of the equal sign.	Literature
	Relational view of the equal sign	When both the left- and right-hand sides of the equal sign are considered and balance is maintained	Literature
	Unknown view of the equal sign	Neither of the two above	Own
Part 3: Coding the integer responses			
Item 1	Absolute value	Order integers as though they were all positive	Own
	Reversed negatives	Order the negative integers in the reverse order	Own
	Other	Not one of the two above	Own
Items 3a-3e	Right to left reasoning	Operate from right to left	Literature
	Avoidance of the minus symbol	Ignore a minus symbol	Own
	Signs rule as sign and operation	Use signs rule for multiplication as the sign and operation to perform	Literature
	Detachment	Detaching the minus symbol and operating on what is left, then reattaching the minus symbol	Literature
	Wrong operation: multiplication	Multiplying the two numbers rather than adding or subtracting them	Own

	Unknown	None of the above	Own
Part 4: Coding the expression responses			
Items 5a-5g	Non-conjoining errors	Did not add unlike terms, other errors	Own
	Conjoining	Juxtaposed a constant and x; added unlike terms, added coefficients and attached letter; added the constant and coefficient of x, which is 1	Literature
	Dropping a letter	Adds unlike terms but drops the letter	Own

Table 5.2: Summary of the qualitative codes part 2

5.3. Part 1: Codes for responses to Linear equations items

5.3.1. Lens 1: Approaches to solving equations

When looking at approaches the correctness of an item is not of interest. The pure arithmetic approach is where learners only use numeric values to attempt to solve the equation, except possibly in the last statement, where the solution is in the form $x = n.$ Numbers are used in place of, or instead of, letters. Arithmetic responses are evident when a learner operates with numbers only. This manifests, for example, where the left-hand side equates to the first term on the right. For example, if a learner substituted $x = 2$ into $3x - 2 = 4 + x$ to obtain $3(2) - 2 = 4 + (2)$ we can see that $3(2) - 2$ does yield 4 if the x is not operated with (see Figure 5.1). Another example is where a learner uses numeric values to manipulate the equation to balance (see for example Figure 5.2). An arithmetic approach is also identified in a string or 'run-on' equation (Kieran, 1992). For example, in Figure 5.3, the learner has also substituted 2 but did not maintain equality and has produced a string of operations. Other manifestations of an arithmetic approach involve correct substitution and obtain the correct answer.

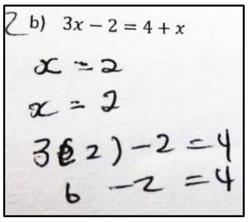


Figure 5.1: Arithmetic approach

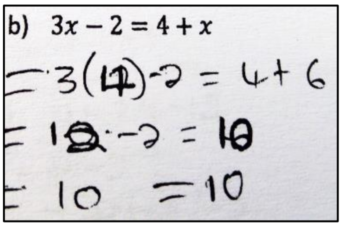


Figure 5.2: Manipulated arithmetic approach

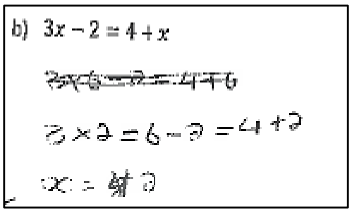


Figure 5.3: Run-on arithmetic approach

Pseudo-arithmetic approaches are arithmetic approaches where I am unable to explain how the learner got to their answer. The responses suggest that an arithmetic approach was used. I identified these types of responses in three ways: firstly, through incorrect inspection (See Figure 5.4), which is where a learner got the correct answer without showing any working out. Secondly, where learners substitute a value but do not maintain balance (see Figure 5.5). Although the learners appear to know that they should have both x values the same, balance was not maintained. Many of these arithmetic responses appeared to be arbitrary placements of numbers and operations on numbers (see Figure 5.6) and hence I cannot explain them beyond the fact that only numbers were used.

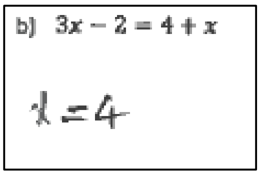


Figure 5.4: Pseudo-arithmetic response: incorrect inspection

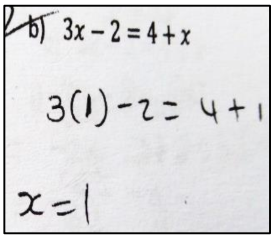


Figure 5.5: Pseudo-arithmetic response: balance not maintained

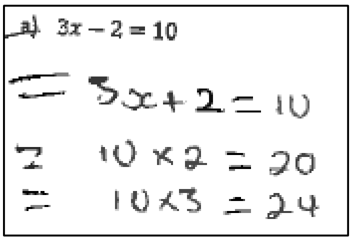


Figure 5.6: Pseudo-arithmetic response: arbitrary placement of values

5.3.1.1. Algebraic approaches

a) Algebraic approach with equations[1]

An algebraic approach is one that uses algebra as a means to solve the equation. I have included the phrase 'with equations' because not all algebra involves equality. This category is based on the balance method of solving equation. This approach is where learners attempt to 'do to the left as they do to the right'. Learners continued to have both a left and a right side of the equation but in attempting to solve the equation they made errors relating to balance, inverse and conjoining, or left their simplifications incomplete. Many learners who used this approach made some error: either on maintain balance, using the incorrect inverse or simplifying either side of the equation. An example is shown in Figure 5.7, where we see a learner who obtained $4x$, presumably from $3x + x$, implying s/he made an inverse error. We see the same error when the learner 'moves' 2 to the other side. The final error the learner makes is simplifying $\frac{2}{4}$ to 2, making a numeric error. Another example is seen in Figure 5.8, where the learner has attempted to maintain equality; s/he approached the item algebraically with equality, the only error the learner made was to add two and not subtract two on the right-hand side.

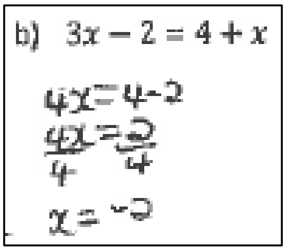


Figure 5.7: Algebraic approach example 1

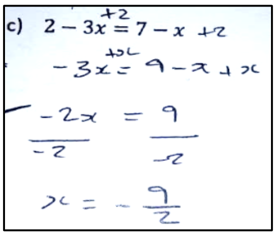


Figure 5.8: Algebraic approach example 2

b) Pseudo-algebraic approach with equality

Some algebraic approaches continued to be in the form of an equation, with a left and right side of the equation rather than, for example, converting it to an expression. Despite it continuing to look like an equation, I was unable

[1] I use the term equality when referring to a statement that has a left and right side of an equal sign where the intention is to solve for x, obtaining a statement of the form $x = k$

to explain the response. Within the responses, although a left and right side were visible, there was no evidence of attempting to balance or use inverses; hence, a pseudo-algebraic with equality approach was inferred. Examples are shown in Figures 5.9 and 5.10:

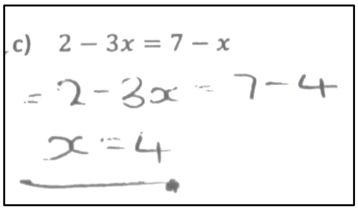


Figure 5.9: Pseudo-algebraic approach example 1

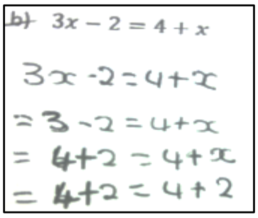


Figure 5.10: Pseudo-algebraic approach example 2

As mentioned earlier, I take the view that an approach is the way in which a learner attempts to solve an equation. Many learners used an algebraic approach, meaning they used letters and the rules of algebra to answer the question, however they converted the equation into an expression instead of using inverses and balance. Whereas an arithmetic approach and an algebraic approach with equality can result in the correct answer, this approach will not work, it is an incorrect approach, not a less sophisticated approach. Within this approach, it appears that learners have no sense of what it means to solve an equation nor what it means to be an equation. The example in Figure 5.11 shows how the learner converted the equation into an expression and incorrectly joined like terms by using laws of exponents.

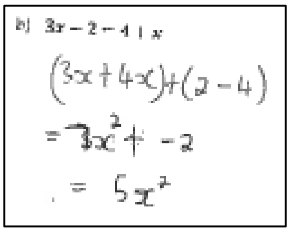


Figure 5.11: Algebraic approach with expressions

5.3.2. Lens 2: View of the equal sign

Jones et al. (2012) discuss three views of the equal sign: *sameness-relational; substitutive relational* and *operational*. These were discussed in length in Chapter 3. This framework was useful as it allowed me to disaggregate the relational code and separate out those who used substitution. I then found categories that fit in-between relational and operational. Besides these three categories, I identified a further three: a *pre-relational*; *pre-operational* and an *unknown* category. Coding learners' view of the equal sign based on their responses meant that I was not concerned with the errors made, the approach used or even the procedure used but rather in how the learner appeared to deal with the equal sign. The six categories are discussed below:

5.3.2.1. Sameness-relational view of the equal sign

A sameness-relational view of the equal sign is one of equivalence where the equal sign is used to represent sameness. The criteria used to operationalise a sameness-relational view is to see whether the learners operated on both sides of the equation. If a learner used the correct additive/multiplicative inverse property as well as the correct balancing inverse (implicitly or explicitly), the response was categorised as a sameness-relational view. This is not to say that only correct answers were classified as such. A response that had a reduction error, a dropped negative, a dropped letter, or an un-simplified equation was still coded as sameness-relational because they were able to use the inverses correctly and maintain two sides of the equation, operating on both. The errors mentioned above are not related to equality or balance and hence are not related to a relational view. The errors are procedural, and related to reducing expressions rather than maintaining balance. See for example, Figures 5.12, 5.13 and 5.14 and the discussion which follows.

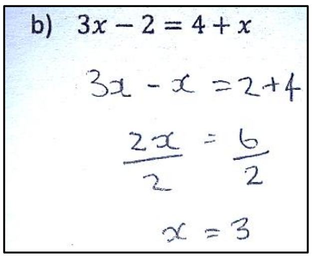


Figure 5.12: Sameness-relational view example 1

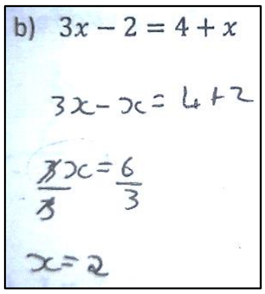


Figure 5.13: Sameness-relational view example 2

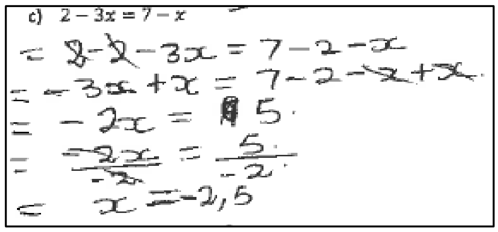


Figure 5.14: Sameness-relational view example 3

In Figure 5.12, the learner has applied the correct additive inverse implicitly and the balancing inverse explicitly $(-x)$. There are no errors in this example. In Figure 5.13, the learner has also applied the additive inverse implicitly and the balancing inverse explicitly. However, the learner has reduced $3x - x$ to $3x$. This error is not related to the view of the equal sign but rather related to reducing an expression and hence it was still categorised as a sameness-relational view. Figure 5.14 shows how a learner has explicitly used additive and balancing inverses. I cannot be certain of a 'true' relational view as I am only analysing what was written. To be certain of a sameness-relational view I would need to conduct interviews, which is beyond the scope of my study. Although learners may have mastered the procedure for solving equations, or learnt how to apply inverse operations, some are still unable to answer arithmetic sentences that require a relational view for example, $7 + 4 = \_ + 5$. This is an indication that some learners do not have a 'real' relational view despite being able to apply both the additive/multiplicative inverse and the balancing inverse.

5.3.2.2. Substitutive-relational view of the equal sign

This category is where a learner has substituted a value for x *and* maintained balance. Although it becomes an arithmetic equation, which is less sophisticated for Grade 9 learners, it is still a valid way of obtaining an answer. Figure 5.15 is a numeric example of a substitutive-relational view. The learner substituted the correct value, which made the equation balance rather than for example substituting 2 into $3x - 2 = 4 + x$ such that $3(2) - 2 = 4$ where only the first term is considered and the x is ignored.

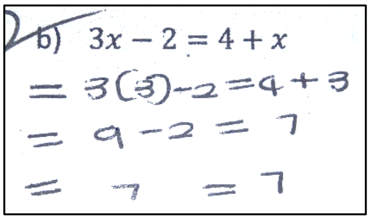


Figure 5.15: Substitutive-relational view

5.3.2.3. Pre-relational view of the equal sign

A pre-relational view is one that is not yet relational but is also not operational. It differs from a sameness-relational view because it allows for incorrect inverses. This category can be identified by i) the equation remaining an equation and not becoming an expression and ii) inverses are partially correct or incorrect. For example, when solving $3x - 2 = 4 + x$ and the wrong inverse property is applied: $3x - 2 - 3 = 4 + x - 3$. The learner here has used the additive rather than the multiplicative inverse – the wrong inverse property. Alternatively, the learner could apply the incorrectly signed inverse, for example $3x - 2 - 2 = 4 + x - 2$ where $-2 - 2$ is expected to become zero. This would be partially correct as the numeral is correct but the operation is wrong. Another example of a partially correct inverse is when the learner has only applied the additive/multiplicative inverse and balancing inverses for one of the terms in the equation. For example, in the above example, applying the inverse of -2 but not of x. In the examples mentioned here, the learners have made errors relating to inverses. Again, this view differs from a sameness-relational view because it involves incorrect or partially correct inverses, whereas a sameness-relational view involves the correct use of all inverses. Examples are shown in Figures 5.16; 5.17 and 5.18. In Figure 5.16, the learner has applied the additive inverse +2 rather than -2, but maintained balance by applying +2 as the balancing inverse. Although the learner made a reduction error ($2 + 2 \to 0$), a conjoining error ($7 - x + 2 \to 9x$) and another reduction error $\frac{9x}{-3} \to -3$, it does not influence the view of the equal sign. The equation remains an equation and is not (incorrectly) transformed into an expression, and the wrong signed inverses were used. In Figure 5.17, the learner applied the wrong inverse property but did maintain balance by doing the same to the right side. Although the learner dropped the x and left the equation incomplete, these errors do not affect the view of the equal sign. Figure 5.18 is an example of undoing, although incorrectly. This particular example was categorised as pre-relational because the learner used the incorrect inverse property as well as the incorrect sign. Undoing is not a trivial form of finding a solution as learners need to know how the letters and numbers are held together- by which operations - and then learners need to know which operation to do first in order to undo the equation because it goes against the order of operations.

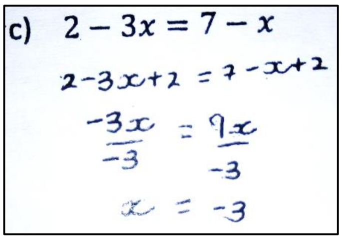


Figure 5.16: Pre-relational example 1

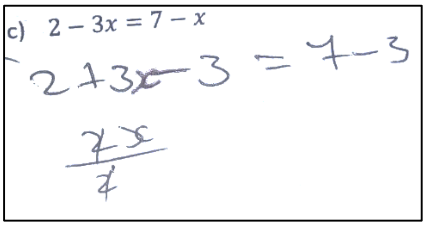


Figure 5.17: Pre-relational example 2

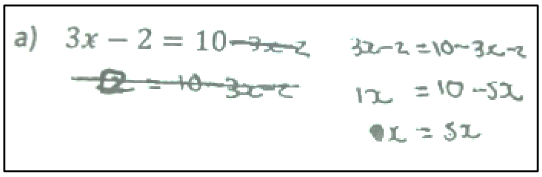


Figure 5.18: Pre-relational example 3

5.3.2.4. Operational view of the equal sign

An operational view of the equal sign is where learners used the symbol as a 'give me the answer' symbol, or an 'and then I do this' symbol. A typical 'gives me' response is where a learner has substituted a value on the left side to obtain the first value on the right hand side, and disregards the second term, for example in response to $3x - 2 = 4 + x$, the learner writes $3(2) - 2 = 4$. A learner does not operate on both sides of the equation but rather only focuses on one side. The above example shows a working from left to right but the x on the right is not operated on. A different form of 'gives me the answer' is a run-on equation, where the learner uses a string of computations, for example in response to $3x - 2 = 10,$ the learner writes $x = 10 - 3 = 7 - 2 = 5$. A third type of 'gives me the answer' is one not found in literature. It is where learners add/subtract the left and right sides together, creating an expression, and then simplify the expression to a single term, giving a single answer. Figures 5.19- 5.24 exemplify an operational view. Figure 5.19 is an example of a vertical run-on, where each computation is done on the next line. The equal sign here acts as an 'and then I do this' symbol. Each computation is an expression and the final answer is a single number, typical of an operational view. Figure 5.20 is similar to that of Figure 5.19 in that it is a horizontal run-on, where each step is written next to the other. The equal sign is being used as a 'gives me' and 'and then I do this' symbol. Starting with the first term, the learner writes that $3x \times 2$ *gives* $6x$, *and then* subtracts 2, which *gives* 4, *and then* brings down the x.

Figure 5.21 is an example of when a learner operates on one side only, the left side, treating the equal sign as a 'gives me the answer' symbol. Figures 5.22, 5.23 and 5.24 all present a form of an operational view that I had previously never encountered, or recognised. The learner from Figure 5.22 reduces both the left and right sides separately, hence never using inverses. S/he then operates on the two sides separately and combines them by adding, subtracting or multiplying. Figure 5.23 is similar but the learner here appears to have 'moved' the four to the left side, creating an expression and a single termed answer. Likewise, in Figure 5.24, the learner has created an expression giving a single termed answer.

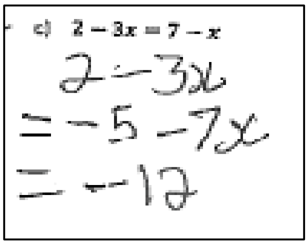


Figure 5.19: Operational example 1

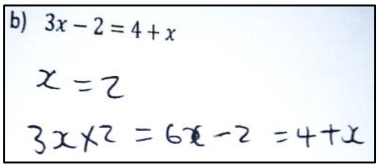


Figure 5.20: Operational example 2

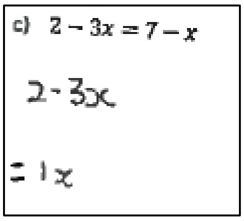


Figure 5.21: Operational example 3

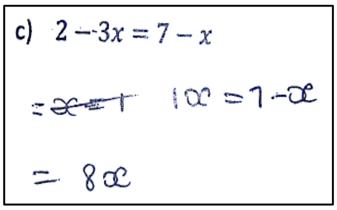


Figure 5.22: Operational example 4

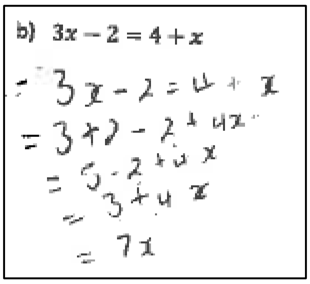


Figure 5.23: Operational example 5

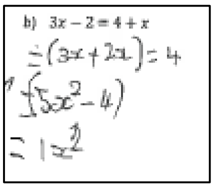


Figure 5.24: Operational example 6

5.3.2.5. Pre-operational view of the equal sign

A pre-operational view is one that is not yet operational but also not suitable for the unknown category. I identified only one type of a pre-operational response: the equal sign as a separator. This view is where learners separate the left and right side of the equation, not working from left to right but rather focusing on each side of the equation as two separate expressions. In both Figures 5.25 and 5.26, the learner has reduced the left side and the right side separately. The only difference between the two examples is that Figure 5.25 is arithmetic and Figure 5.26 remains algebraic. What would be required for this view to become an operational view is that the learner would need to link the left-hand side to the first term of the right-hand side. This means that they would need to move from the left-hand side to the right-hand side of the equation.

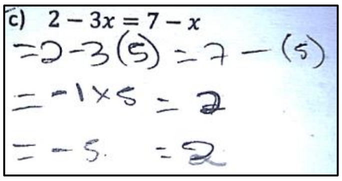


Figure 5.25: Pre-operational example 1

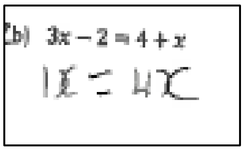


Figure 5.26: Pre-operational example 2

5.3.2.6. Unknown view of the equal sign

An unknown code is one where I am unable to say what the learner's view of the equal sign is. There is no evidence in their response that suggests the learner understands what an equation is, what the equal sign means and what is required to solve an equation. Since this category was the most common, I disaggregated it and identified three different types of unknown responses.

Only the answer was given: When only an answer is given there is no way of telling how the learner arrived there (See Figure 5.27). *Inverse related:* Some responses included an inverse on one side of the equation but I was still unable to say what view the learner adopted (See Figure 5.28). *Other/unknown:* These responses were not relational, pre-relational, operational or pre-operational, and did not fit into one of the other two unknown views (See Figure 5.29).

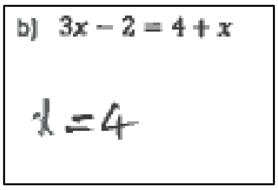


Figure 5.27: Unknown example 1

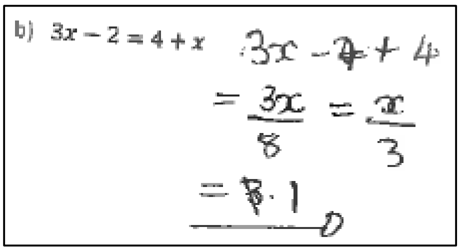


Figure 5.28: Unknown example 2

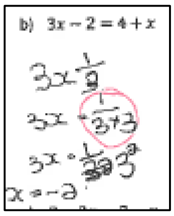
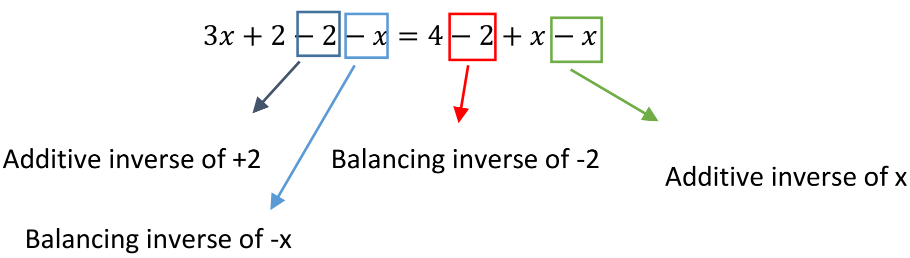


Figure 5.29: Unknown example 3

5.3.3. Lens 3: Learners' use of inverse operations

When solving linear equations, if learners are not using arithmetic approaches then they are expected to use inverses. In all three equation items there were inverses that needed to have been used: the inverse for the constant, for the letter and the multiplicative inverse to isolate x. Some learners used all three inverses correctly and some only applied one or two inverses (correctly or incorrectly). There were no learners who applied all three inverses and applied them incorrectly. Since there is no literature on the types of errors made when applying inverses, I developed my own codes and introduce a new term: balancing inverse.

If one were solving $3x - 2 = 4 + x$, one would apply the additive inverse of -2 on the left side, and what I call the balancing inverse on the right. To apply inverses correctly one needs to apply both, the pair of additive/multiplicative inverse as well as the balancing inverse. For example:



I used this language to help identify what inverse learners used/didn't use. The additive inverse is the inverse that reduces to zero (for example, $-2+2=0$); the multiplicative inverse is the inverse that reduces to 1 (for example $3 \div 3 = 1$). The balancing inverse is the label I gave to the inverse that maintains balance. There would therefore be a balancing additive inverse and a balancing multiplicative inverse. Some learners do not apply the inverse properties explicitly, meaning that they do not necessarily write for example, using the example above, $3x + 2 - 2 = 4 + x - 2$. Instead they would just write $3x = 2 + x$ or $3x = 4 + x - 2$, where the balancing additive inverse is explicit.

I identified eight codes when looking at how learners used inverse operations:

a) All inverses used correctly

This code was given when learners applied all three inverses and applied them correctly. These responses were all correct. There were no instances of all thee inverses being applied correctly and the answer still being wrong. There were some correct responses but inverses were not used, these type of responses were approached arithmetically. Figure 5.30 shows an example of where all inverses were used correctly and the answer was correct. Figure 5.31 shows how all inverses could have been used correctly but the answer was incorrect.

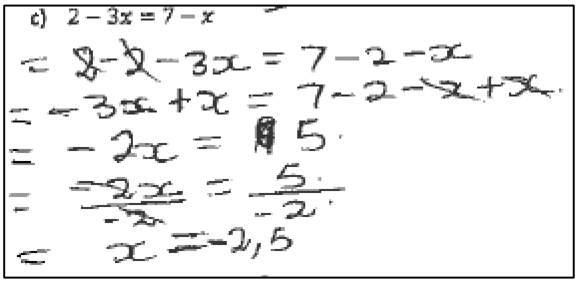


Figure 5.30: Inverses used correctly example 1

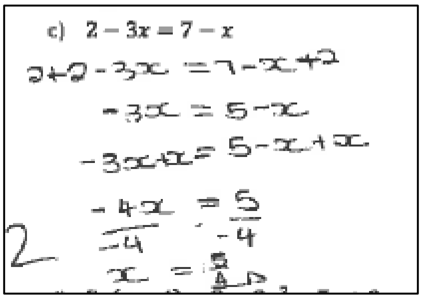


Figure 5.31: Inverses used correctly example 2

b) One or two of the three inverses used and used correctly

Some learners applied only one of the two additive inverses, typically ignoring the x on the right hand side. This meant that all the inverses that they did apply were correct but they didn't apply them all. I do not report on the disaggregation of which inverse was applied and which was not because it was not a worthwhile distinction. See Figure 5.32 for an example. Here the learner has correctly applied the additive and balancing inverse of 2 and the multiplicative inverse of dividing by 3 but because the learner applied conjoining there is no inverse for x.

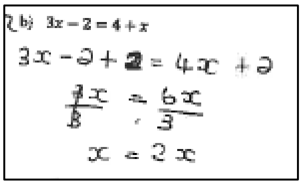


Figure 5.32: One or two of the three inverses used and used correctly

c) Inverse used but no balance

This code was assigned to responses where an inverse was applied on the left or right side of the equal sign and to the other side, meaning that the learner applied the additive inverse and not the balancing inverse. Figure 5.33 is an example of a learner who attempted to apply all three inverses. The learner correctly applied the additive and balancing inverse of 2 as well as the multiplicative inverse of dividing by 3, and hence this part received the code one or two of the three inverses used and used correctly. However, in applying the additive inverse for x, the learner did not apply the balancing inverse and hence this part received the code inverse used but no balance.

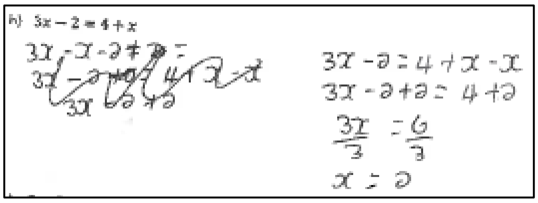


Figure 5.33: Inverse applied with no balance

d) Wrong sign

This category is where a learner applied an inverse but with the wrong sign. The learner still treated the pair of numbers as adding to zero and hence cancelling but mathematically it does not add to zero. See for example in Figure 5.34 where the learner did not change the sign for the inverse of -2.

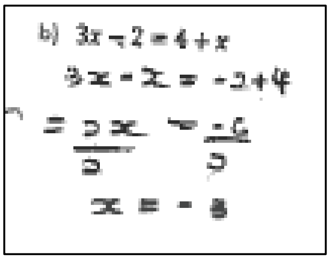


Figure 5.34: Inverses: Wrong sign

e) Wrong inverse operation

This code was assigned to responses where the learners used the additive inverse instead of the multiplicative inverse, or vice versa. In Figure 5.35 we see that the learners have divided by 2 instead of adding 2. The learners used the wrong inverse operation.

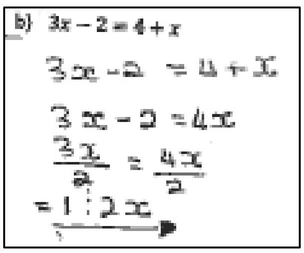


Figure 5.35: Wrong inverse operation

f) Inverse not used

This code was given to all responses that did not have an inverse operation in the response. Figures 5.36 is an example of where a learner did not used inverses in their solutions. Instead the learner has substituted 3 on the left side only.

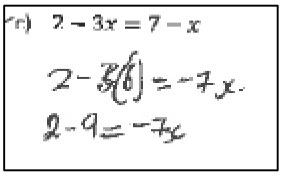


Figure 5.36: Inverse not used

g) Inverse not used-expression

Similar to g) but the response was converted to an expression.

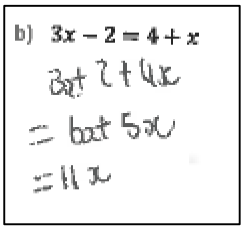


Figure 5.37: Inverse not used-expression

h) Inverse used-expression

This code was given to responses that were converted to an expression by applying an inverse operation at some point during the conversion. In Figure 5.38 the learner has converted the equation into an expression but has also included the additive inverse of 2. Because it is now an expression there is no balancing inverse. In Figure 5.39 the learner has applied the additive inverse of $4x$ and should have obtained $0 = 1x - 4x$ but instead has left it as an expression.

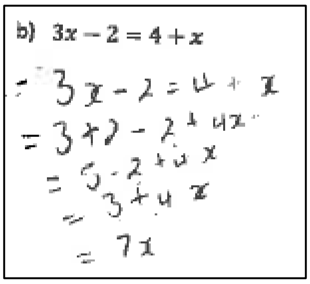


Figure 5.38: Inverse used-expression example 1

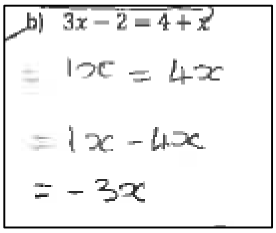


Figure 5.39: Inverse used-expression example 2

5.3.4. Lens 4: Learners’ use of negatives

When coding for learners’ use of negatives I made a note of the number of times learners operated with a minus symbol. This means I have a code for subtraction (correct and incorrect) as well as for dealing with negatives (correct and incorrect).

a) No operations with a minus symbol

This code was assigned to responses where learners did not use subtraction and did not need to deal with negatives. Items 9a and 9b could be solved correctly without dealing with negatives and hence not operating with a minus symbol is not an indication of poor attainment, nor is it an indication of getting the item correct. This code is helpful in that we can know how many learners did operate with a minus symbol and this value is used to determine percentages. For example, knowing that a certain percentage of learners operated with a minus symbol incorrectly would be worthwhile knowing. Figures 5.40 and 5.41 are two examples of no evidence of using subtraction or negativity.

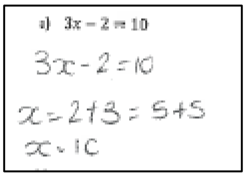


Figure 5.40: No operations with a minus symbol example 1

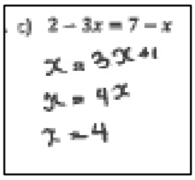


Figure 5.41: No operations with a minus symbol example 2

b) Correct/incorrect subtraction

When learners were confronted with a statement of the form $a - b$ or $ax - bx$, where $a, b > 0$ and $a > b$, their response was coded as either correct or incorrect subtraction. This is separated from dealing with negatives for two reasons: firstly, the minus symbol has a dual function and hence it is apt to have multiple codes. Secondly, dealing with subtraction of whole numbers is a skill I expect all these learners to have mastered whereas dealing with negatives is a skill that I know learners struggle with. Coding these responses was complex in that learners made errors with dealing with the minus symbol as well as conjoining, which is a letter error. Hence, in addition to, for example, the form $a - b$ and $ax - bx$, I included the form $ax - b$ and $a - bx$. The conjoining error would be counted as a letter error (Lens 6) and so I counted only the minus symbol error. See Figures 5.42- 5.44 for examples of learners’ work.

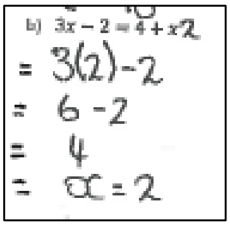


Figure 5.42: Correct subtraction with constants

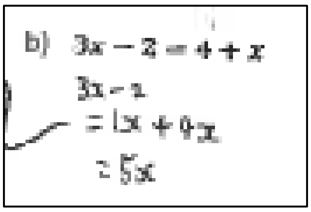


Figure 5.43: Correct subtraction with letters

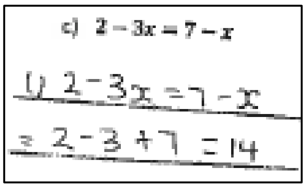


Figure 5.44: Incorrect subtraction

c) Correct/incorrect negativity

This code was assigned to responses where learners were confronted with a statement that required them to deal with negative values and negative answers. If learners correctly dealt with negatives they received the code correct negativity and if incorrect I assigned a further code of either right-to-left reasoning; too many signs, detachment or signs rule as operation and sign (see Part 3 for a discussion on these codes). See Figures 5.45- 5.48 for examples of incorrect negativity.

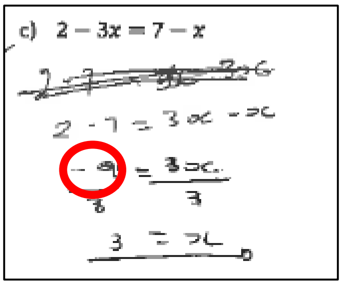


Figure 5.45: Incorrect negativity: detachment

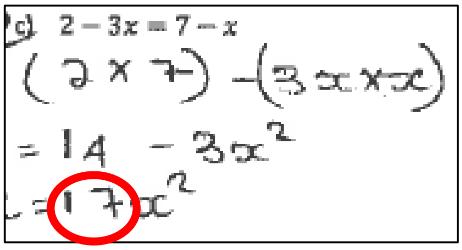


Figure 5.46: In correct negativity: too many signs

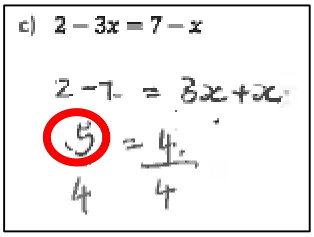


Figure 5.47: Incorrect negativity: right-to-left reasoning

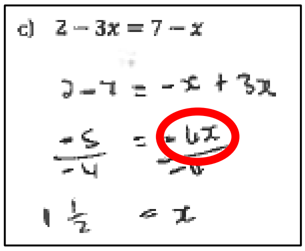


Figure 5.48: Incorrect negativity: bracket reasoning

d) Deletion

This category was assigned to two types of responses: for example $3x - x = 3$ (see Figure 5.50) and $3x - x = 3x$ (see Figure 5.49). In these responses, the learner appears to either delete the second appearance of x or delete the two letters and is left with a constant.

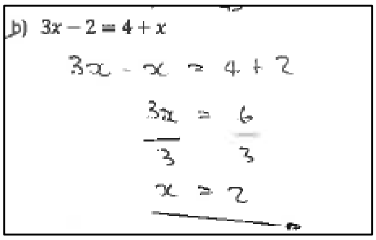


Figure 5.49: Use of negatives: deletion example 1

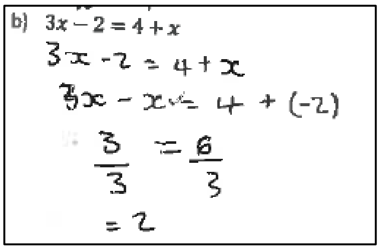


Figure 5.50: Use of negatives: deletion example 2

5.3.5. Lens 5: Learners' use of letters in terms of conjoining

Coding for letter errors involves more than just conjoining; however, for the purpose of the research I chose to code the conjoining errors, and disaggregate them. The purpose of this is to compare these results to those when learners simplified expressions, and investigate whether learners conjoin there too. If so, do they conjoin more or less than when solving equations? In the literature review, I argued for a broader and more explicit definition of conjoining. I therefore have four codes for the different types of conjoining but also have a code for when learners used exponents and exponential laws to add like terms. In addition I have a code for when learners ignored the second appearance of the letter and for when no conjoining took place. As a reminder, some responses had multiple errors, for example $2 - 3x \rightarrow -5x$, where a learner makes a negativity error as well as a conjoining error. For the purpose of this lens's analysis I only noted the conjoining error. The negativity error was noted in lens 4.

a) Conjoining

Conjoining occurred in three ways: the first is where a constant and a letter were joined, for example $4 + x$ or $x - 7$. The responses learners gave when conjoining $4 + x$ were $4x$ and $7x$ or $-7x$ for $x - 7$. These responses suggest that learners either simply concatenated the two terms or that they treated the x as $0x$ because there is 'nothing' in front of the x, and joined to the 4 or 7. For an example, see Figure 5.51 where the learners joined $4 + x \rightarrow 4x$ **.** The second way is where learners operated on the numbers and did not just join them. For example, where $4 + x$ was combined to get $5x$. This, in contrast to conjoining with a $0x$, suggests that learners were aware that x is equivalent to $1x$. This type of response was also visible with subtraction, for example, $x - 7 \rightarrow -6x$. In Figure 5.52 the learners have correctly treated the $-x$ as $-1x$ (otherwise the learner would have obtained a $5x$ or a $9x$ by treating x as $0x$), the learners then also conjoin and obtain $6x$. In addition, the learners make a subtraction error which would have been counted during lens 5 analysis. The third way is when the variable had a coefficient that was not 1 or -1. For example $2 + 3x \rightarrow 5x$. For an example of learners' work, see Figure 5.53 where the learner added (incorrectly) $2 - 3x \rightarrow 1x$. The negativity error that occurs in this response would have been noted when analysing through lens 5, learners' use of negatives. This responses also has a conjoining with $0x$ error, where $7 - x$ was reduced to $7x$**.**

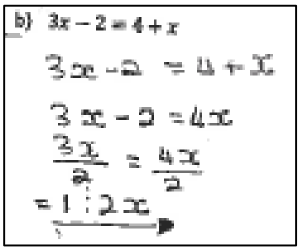


Figure 5.51: Conjoining example 1

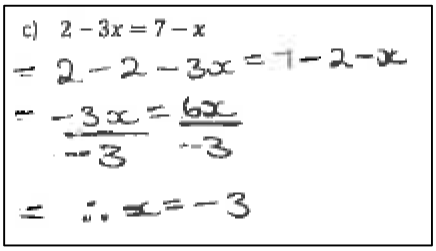


Figure 5.52: Conjoining example 2

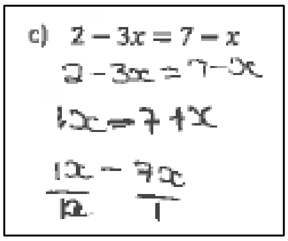


Figure 5.53: Conjoining example 3

b) Dropping a letter

In some responses it appeared that the learner ignored the second appearance of the letter and responded to, for example $4 + x$ with 4 or $4x - x$ with $4x$. These types of responses received the code *dropping a letter*. This code is similar to the deletion error except that it takes into account addition and subtraction. The deletion error is specific to subtraction. See Figure 5.54 as an example of learners' work.

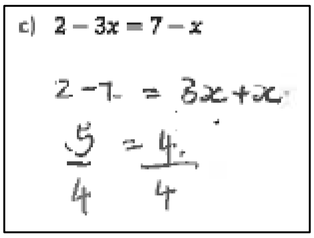


Figure 5.54: Dropping a letter

c) Exponents

Responses where the laws of exponents were overgeneralised were coded as *exponents*. This occurs when like terms are added together incorrectly. For example, see Figure 5.55.

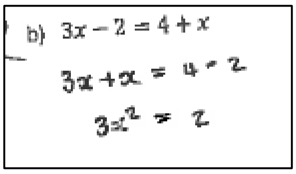


Figure 5.55: Adding like terms with exponents

d) Unknown

When a written response has the letter x and should not have had the letter, then it suggests that some form of conjoining took place. If I was unable to explain how the response came into being then the responses received the code unknown. See for example Figure 5.56.

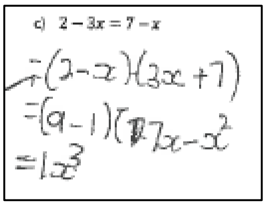


Figure 5.56: Conjoining unknown

e) No conjoining

If no conjoining took place, then the response was coded as 'no conjoining'. This is a helpful code as it allows us to know how many learners did conjoin in some way.

5.4. Part 2: Dealing with equality

The equality items analysed were item 4a: $7 + 5 = \_ + 2$ and item 4b: $4747 + 3945 = \_ + 3943$. These items were developed to test learners' view of the equal sign. Since I am analysing learners' written scripts and not interview data, I infer a view of the equal sign based on a single value as a response. I therefore used only three codes: relational, operational and unknown rather than use, for example the additional codes from Jones, Inglis, Gilmore, and Evans (2013). This is because from a single value I am unable to say whether a relational response was a sameness-relational or substitutive-relational response, hence I have only a relational category. The three codes are discussed below.

5.2.1. Operational response

This label comes from the view of the equal sign. When looking at equality, researchers typically investigated the view of the equal sign rather than the errors made (see for example Carpenter et al. (2003); Kieran (2007)). Because learners' responses are a single value it is difficult to separate the view of the equal sign and the learners' error. An *operational* response is evident when a learner only takes the values on one side of the numeric equation into account. This manifests in three different ways: in responding to $7 + 5 = \_ + 2$ learners who respond with 12 have not taken into account the two on the right. This is typical of an *operational* view of the equal sign where, according to the learners, the answer comes next (Carpenter et al., 2003). Similarly for item 4b, learners who responded with 8692 to $4747 + 3945 = \_ + 3943$ used the blank space as the place where one puts the answer to $4747 + 3945$. The second type of *operational* response is where learners work from right to left and so would respond to, for example, $7 + 5 = \_ + 2$ with $7 + 5 = \mathbf{3} + 2$ because $5 = 3 + 2$, or respond with $7 + 5 = \mathbf{5} + 2$ because $7 = 5 + 2.$ Hence, the learners have not taken the 7 (or 5) on the left into account. Similarly for item 4b, learners responded to $4747 + 3945 = \_ + 3943$ with $4747 + 3945 = 2 + 3943$ because $3945 = 2 + 3943$, or $4747 + 3945 = 804 + 3943$ because $4747 = 804 + 3943$. In both these instances, the learner is ignoring one of the values and working in one direction. Lastly, learners who respond with 14 to $7 + 5 = \_ + 2$

are also working operationally. They have added 7 and 5 and then 2 to get the final answer of 14. This form of an *operational* response is what Kieran (1992) calls a string or run-on equation. It is considered a form of *operational* working because to obtain 14 the equal sign is viewed as a do-something symbol rather than a relational symbol, and according to the interviews conducted by Carpenter et al. (2003), it is done because the learners need to add all the numbers because the question has an addition sign in both places. For item 4b, a response of 12635 would have been obtained by adding all the numbers together. This corroborates Carpenter et al. (2003)'s findings that adding all the numbers together does not dissipate as learners get older. I consider all four examples discussed as variants of an *operational* response. This is consistent with the findings of Essien and Setati (2006), who argued that Grade 8 and 9 learners use the equal sign as a unidirectional symbol and work in only one direction.

5.2.2. Relational response

As with the operational response, this code was also drawn from learners' view of the equal sign. A *relational* response is one where learners consider both sides of the equation. I assume that learners who got the correct answer ($7 + 5 = \mathbf{10} + 2$ and $4747 + 3945 = \mathbf{4749} + 3943$) used a relational approach. I assume this because to obtain 10 (or 4749) the learners were aware that the 2 (or 3943) on the right side needed to be incorporated in their working out. They needed to add 5 and 7 and then subtract 2 (or add the 4747+3945 and subtract 3943), which they have done.

5.2.3. Other/ unknown responses

This category contains responses that were obtained less than 15 times (made by less than 10% of the learners). It also includes all blank responses. There were two exceptions from the pre-test. The operational responses of 3 and 14 were not classified as 'other' in order for me to remain consistent in how I operationalised what an operational response consisted of. In the post-test there were three exceptions: the responses of 2; 804 and 12635 occurred less than 15 times, however in order for me to be consistent with how I coded item 4a I did not classify these responses as other. The responses of 4747 for item 4b was a common response and yet it was not a relational or operational response. It could be that learners did not notice that there was a difference between 3943 and 3945 and so hence would put the same value, 4747, in the blank space. Since I only have three codes, I have categorised it as other. In addition, because of the large numbers, learners may have made numeric errors as well as working in only one direction as described by an operational response. Learners were not allowed calculators during the test and I assume they therefore used the vertical method for addition and subtraction. A learner who responds operationally would respond to $4747 + 3945 = \_ + 3943$ by adding $4747 + 3945$ and not subtracting 3943. When adding $4747 + 3945$ using the vertical method there are many numeric errors that could happen, for example not carrying the 10 from 12:

	4747
+	3945
	8682

Similarly, to answer $4747 + 3945 = \_ + 3943$ correctly and have a relational approach, learners needed to complete two operations. Firstly to add $4747 + 3945$ and secondly to subtract 3943 from the sum of 4747 and 3945. When adding for example using the vertical method, learners could have made procedural errors such as not carrying the 10s when adding, and subtracting the smaller from the larger value. These are the two common errors learners make when doing vertical addition and subtraction (Cox, 1975; Penlington, 2018; Son, 2016). An example of an error that could have been made is:

	4747
+	3945
	8682
-	3943
	5341

The above types of responses, which include procedural errors, were not categorised as being from a relational or an operational approach. This is because, although the responses are operational with a procedural error, there are multiple errors that could have been made, and I am unable to tell what they all are. If they were responses that were given then they occurred less than 4 times each, meaning, for example less than four people gave the response 5341. Hence, these responses were categorised as other approaches. Other examples of *other* responses given to item 4a in both the pre- and post-test are 0; 6; and 9. For item 4b some other examples of responses that were categorised as other are: 2; 3942; 4750; and 9745.

5.5. Part 3: Dealing with negatives

5.3.1. Item 1

Item 1 was an ordering question where learners needed to put a string of numbers in ascending order. There were only two main errors and hence the codes used for item 1 were:

a) absolute value ($-2 \quad 4 \quad -10 \quad 30 \quad -35 \quad -500$), where the integers were placed in ascending order as if they were all positive values.
b) reversed negative ($-2 \quad -10 \quad -35 \quad -500 \quad 4 \quad 30$) is where the learners placed the negative values in the reverse order, so in descending order, but the positive values were correct.
c) other responses: these are responses that did not fit into the other two categories. (They also occurred less than 10% of the time).

5.3.2. Items 3a—e

There are six error codes used to code these items:

a) Right to left reasoning

A response that obtains this code is one where in order to get the response, you need to work from right to left instead of left to right. This type of reasoning only occurs when the subtrahend is bigger than the minuend, meaning that in $a-b$, $a>0$ and $b>0$, $b>a$. This code therefore is limited to only a certain number of items. The item $5+(-7)$ is the same as $5-7$ after deciding that the addition of a negative is the same as subtraction, and hence after this simplification can lead to right to left reasoning. I therefore coded the response of 2 to this item as right to left reasoning. There were responses of 4 to the item 6-(-10) which I assume comes from $10-6$; this too is a form of right to left reasoning and hence I coded it as such. It is possible that the learner applied 'too many signs' reasoning to obtain $6-10$ and then applied right to left reasoning.

b) Too many signs or avoidance of the minus symbol

According to literature, this code is given when for example learners ignore one of the minus symbols because there are too many for them to deal with. Although in all the integer items there are at most only two minus symbols, I have chosen to continue to use this code. The following responses were coded as too many signs:

- $5-7 \rightarrow\ 12$

This item was probably treated as $5+7$, ignoring the subtraction

- $5+(-7) \rightarrow\ 12$

Similar to the one above, this item was most likely treated as $5+(7)$, where the negative sign was ignored.

- $-5+7 \rightarrow\ 12$

The leading negative in this item was most likely ignored to obtain $5+7$.

- $-7-5 \rightarrow\ 2$

Similar to the item above, the learning negative in this item was probably ignored and the item was treated as $7-5$, hence obtaining the answer of 2.

- $6-(-10) \rightarrow\ -4$

If one of the minus symbols is ignored in this item the item would change to be $6-10$, which then correctly is equivalent to -4. This response was therefore also coded as too many signs.

c) Signs rule as sign and operation

The signs rule refers to the multiplication rule, for example, a negative number multiplied by a negative number gives a positive number. This rule is overgeneralised and simplified to a negative and a negative gives a positive and so, when responding to $-7-5$, for example, the learners will give the answer $+12$. This however assumes that the learner uses the rule to obtain the sign of the answer but also to indicate the operation used.

d) Detachment

As discussed in the literature review, I found similarities and differences in the categories called detachment and bracket reasoning. I chose to use detachment as a category instead of bracket reasoning as I view bracket reasoning as a special kind of detachment. Bracket reasoning also only appears with a limited number of items and so would not be present in any item other than 3c and 3e. I therefore decided to use only detachment as the category was more user-friendly across the items. This category was applied to items where it appears that the learner holds the minus symbol in their head, operating on what is left behind, and then reattaches the minus symbol. The responses that received this code are:

- $5 - 7 \rightarrow -12$
- $5 + (-7) \rightarrow -12$

Detaching the minus symbol from both items above, adding and reattaching the negative sign results in

- $-(5 + 7) = -12$
- $-5 + 7 \rightarrow -12$

Holding the leading negative in your head and adding the 5 and 7 and then reattaching the negative sign results in -12

- $-7 - 5 \rightarrow -2$

Holding the leading negative in your head and operating on the 7 and 5 and then reattaching the negative sign results in -2.

Answers		Codes per Item				
3a,b,c,e	3d	Item 3a: 5-7	Item 3b: 5+(-7)	Item 3c: -5+7	Item3d:6-(-10)	Item 3f: -7-5
2	4	Right to left reasoning	Right to left reasoning after simplification	Correct	Right to left reasoning with too many signs	Too many signs
-2	-4	Correct	Correct	Signs rule as sign and operation	Too many signs	Detachment
12	16	Too many signs	Too many signs	Too many signs	Correct	Signs rule as sign and operation
-12	-16	Detachment	Detachment	Detachment	Detachment	Correct
35	60	Wrong operation: multiplication	Wrong operation: multiplication	Wrong operation: multiplication	Wrong operation: multiplication	Wrong operation: multiplication

-35	-60	Wrong operation: multiplication	Wrong operation: multiplication	Wrong operation: multiplication	Wrong operation: multiplication	Wrong operation: multiplication
Any other value	Any other value	Unknown	Unknown	Unknown	Unknown	Unknown

Table 5.3: Negativity codes

e) Wrong operation: multiplication

This category was assigned responses where the learner multiplied the two values instead of adding or subtracting.

f) Unknown

An unknown code was given to all responses where I could not explain how the learner possibly got it as a response. All these responses occurred less than 10% and hence are not worth worrying about.

I present the codes used for items 3a-3e in a table to show what code I gave what response. The first row are all the items and the first two columns are the responses to items 3a, b, c, and e, and then in the second column the responses for item 3d. Item 3d has a different set of responses because it contained the value 6 and -10 rather than $(+-)7$ and $(+-)5$.

5.6. Part 4: Dealing with letters

In coding items relating to simplifying expressions I used the same codes as I did for lens 6: Learners' use of letter. The only code that was present in coding of the expression items that was not in lens 6 was a code termed numeric. This is because noting that learners used arithmetic would have been picked up when investigating the approaches used. The numeric code was assigned to responses where learners substituted values for the unknowns rather than operate on them. See for example Figure 5.57.

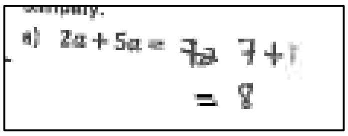


Figure 5.57: Expressions: numeric

5.7. Conclusion

This chapter was purely about the codes used for the qualitative analysis. I have discussed how and why I coded certain responses as well as provided learner work to exemplify the error I would code.

The next six chapters are dedicated to analysis. Chapter 6 is the quantitative analysis of the full sample and Chapters 7-10 are related to the sub sample. It is in these chapters (Chapters 7-10) that I use the codes described in this chapter. Chapter 11 is a synthesis of all six analysis chapters.

Chapter 6 : Trends and patterns, setting the scene

6.1. Introduction

The aim of this study is to investigate Grade 9 and Grade 10 learner performance, and the relationship between the errors learners make when solving linear equations, simplifying expressions, dealing with negatives and dealing with equality. Having both a pre- and post-test for both Grade 9 and 10 learners provides me with the opportunity to investigate the shifts that occur in performance and errors made from Grade 9 into Grade 10. I investigate these shifts in two phases.

This chapter aims to explore my data for the purpose of discovering new insights about learners' performance when adding and subtracting integers, simplifying expressions, dealing with equality and when solving linear equations. It also aims to create sense from the chaos of having much data. Using qualitative and quantitative methods, I analyse Grade 9 and 10 learners' performance from different points of view. This chapter reports on a quantitative analyses on the whole sample, 2135 learners (Phase 1). The learners wrote the same test at two points over one academic school year: one in February 2018 (or 2017) and one in September 2018 (or 2017). It is important to note that the Grade 9 and 10 learners are not the same learners. The learners were chosen based on three criteria: firstly, learners needed to have written both a pre- and a post-test for me to be able to compare results. Secondly, the learners needed to have attempted all the equation items as this is the topic on which my study is based, and thirdly the learners must be taught by teachers who participated in the WMCS professional development. Details of my sample and about the WMCS professional development course were discussed in Chapter 4. In the chapters that follow I provide both a quantitative and qualitative analysis on a subset of the sample (Phase 2).

This data set ($n = 2135$) is a unique opportunity to investigate learners' overall performance on a large scale. Learners' responses to 18 items were recorded as missing, correct or incorrect and a code of 0 (yellow), 1(green), or 2(red) were assigned. Colour coding the categories in this way provided a visual, colour-coded summary of the learners' responses, which has been a simple but powerful tool for an initial analysis. For each test, the total number of correct responses per learner (learner scores) and the total number of correct responses per item (item scores) were calculated. This chapter is dedicated to Phase 1 and involves a quantitative analysis of ordinal data (codes 0, 1 and 2), as well as on scale variables (scores for a collection of items correct per learner as well as scores for the number of correct responses per item). The scale variables describe the general direction or trend of this particular population and the ordinal data illuminate patterns in learner responses. Having both a pre- and post-test for each grade enables me to compare results and determine what shifts (if any) have occurred in terms of overall general performance. In phase 2 (Chapters 7-10) I will be able to comment more on the shifts (if any) in the types of errors learner make. I aim to achieve the following in Phase 1 of the analysis:

- Locate the full sample by mapping out the terrain
- Gain a general understanding and initial overview of the sample's mathematical ability/performance in terms of their procedural fluency.
- Identify general trends and gain some insight into learner difficulties
- Describe how the items performed with this sample.

As mentioned in Chapter 4, the methodology chapter, this thesis is structured in two phases, where the first is a quantitative analysis of the whole sample. The second phase is a sequential triangulation, mixed-methods research design as it contains both statistical analysis and error analysis. Phase 2 involves a subset of the sample, namely, 150 Grade 9 and 150 Grade 10 learners. This chapter is dedicated to Phase 1. I discuss the analysis by using:

a) Descriptive statistics
b) Inferential statistics
c) A response pattern analysis.

In this chapter, I present a discussion on the trends and patterns that were observed in the data. The difference between trends and patterns is that where trends indicate the general direction of the data, patterns identify a series of data points that repeat in a predictable or recognisable way.

6.2. Quantitative analysis of the full sample

6.2.1. Descriptive statistics

Using descriptive statistics such as percentages and frequency counts, I investigate how the learners responded to the 18 test items. By analysing learner scores, I identify the items (and topics) that were best (and worst) answered in both the pre- and post-tests as well as in both Grade 9 and Grade 10. I also investigate how the learners performed as a group. I begin with the finding from the analysis of learners' scores; I then present my findings from the topic analysis and finally discuss the findings from the item analysis.

6.2.1.1. Learner scores

Learner scores are the percentage of items learners got correct. The scores ranged from 0% to 100% across all four groups (Grade 9 pre-test; Grade 9 post-test; Grade 10 pre-test and Grade 10 post-test). The box plots below show the distribution of the learners' scores as a percentage. There is one plot for each group/test. Box plots are an effective way of visualising the differences amongst different groups. They give a visual representation of the five-number summary (minimum, maximum; quartile 1; the median and quartile 3). The lines that come out the box are called whiskers, with each of them representing 25% of the learners. The middle line is the median or halfway mark between the learners. The box below the median and the box above the median also each represent 25% of the learners. The dots that are above the Grade 9 pre- and post-test are the outliers. Typically, one would remove them from one's sample, however I have chosen to keep them in the sample because in the qualitative analysis it will enable us to see the strategies that the learner used when analysing for example the approaches used when solving the equation.

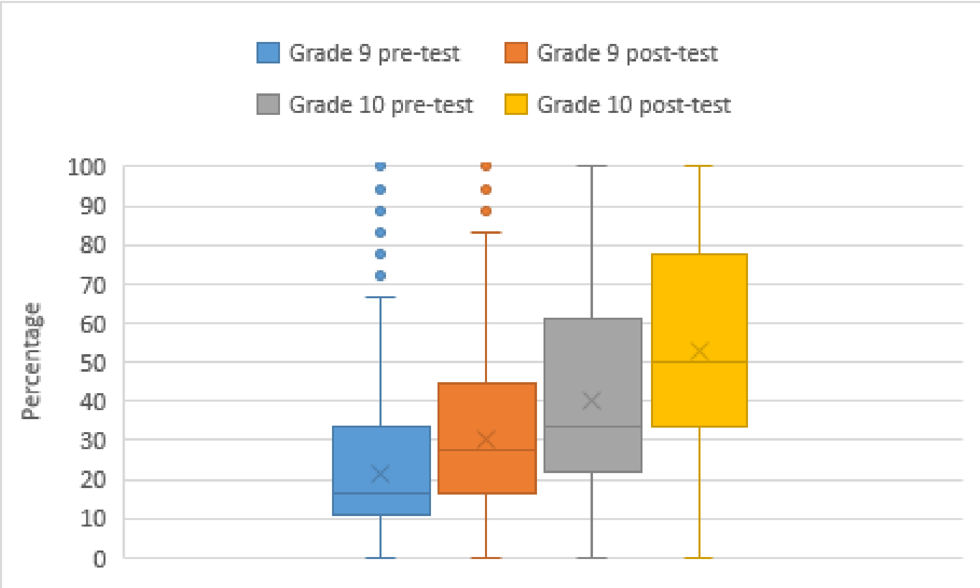


Figure 6.1: Percentage of Grade 9 and 10 learner performance on selected test items

Although, as expected, the mean values increased in every group, the greatest increase occurred from the Grade 10 pre-test to the Grade 10 post-test, with a 13p.p. increase. The increase in the upper quartile from the Grade 10 pre- to Grade 10 post-test also indicates a more significant improvement than within Grade 9. The improvement from the Grade 9 pre- to Grade 10 post-test suggests a significant improvement over the two years (31p.p increase). I say this because more than 75% of the Grade 10 learners performed better in the post-test than the majority of the Grade 9 pre-test scores. In addition, 50% of the Grade 10 learners achieved better scores than 75% of the Grade 9 learners in the post-test. In the Grade 9 pre-test, 75% of learners were achieving less than 33%. However, in the Grade 10 post-test, 75% of learners were achieving more than 28%.

6.2.1.2. More gains by Grade 10 learners on integers and equations

The average score for each set of items pertaining to a topic was calculated. Meaning that, for example, the percentage of correct responses to the six integer items were calculated for each learner and then the average was taken. Figure 6.2 shows the average scores for each test per topic. Learners performed best in dealing with integers and worst in expressions. Learners' performance in equality shows a steady increase from the beginning of Grade 9 to the end of Grade 10. There were more gains made in learners' performance in integers and equations in Grade 10 than in Grade 9. We see this by the steepness of the last third of the graphs. From the Grade 9 pre-test average to the grade 10 post-test average, learners improved the most in equations. Interestingly, in the beginning of Grade 9, equations performance was worse than that of expressions but by the end of Grade 9 and throughout Grade 10, equation performance was better than expressions.

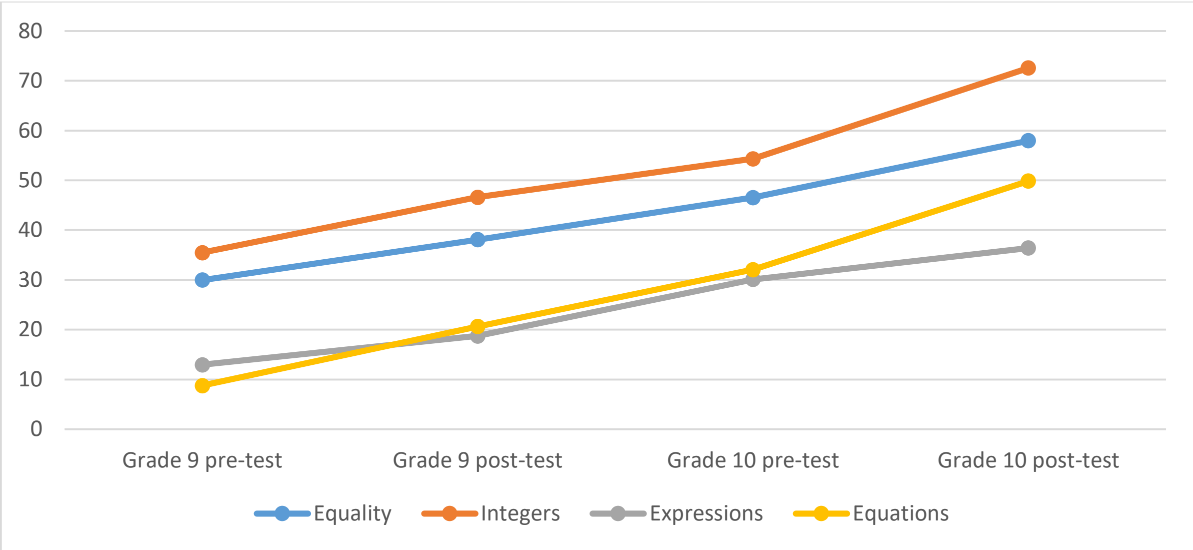


Figure 6.2: Grade 9 and 10 average scores per topic

6.2.1.3. Learners struggle with algebra

The percentage of correct responses per item was calculated and graphed (see Figure 6.3). Each line represents a different group. In all four tests, the items behaved similarly.

The best answered items were item 3a) $5-7$ and item 5a) $2a+5a$ for the Grade 9 pre- and post-test, as well as the Grade 10 pre-test. In the Grade 10 post-test, items 3a and 9a were the best answered items. The worst answered items were 5e and 5g, while in the other three tests, 5e and 9c were the worst answered items. In all four tests the worst answered items were 5e $\left(3a-(b+a)\right)$, with only 1% of learners getting the item correct in the Grade 9 pre-test and 16% in the Grade 10 post-test. Item 9c ($2-3x=7-x$) also had only 1% of learners getting the item correct in the Grade 9 pre-test and 29% in the Grade 10 post-test.

Across the four groups, from the Grade 9 pre-test to the Grade 10 post-test, the items where the percentages improved the most were 9a and 9b as they increased by 44p.p each.

Item 5a was an easy item and learners generally performed well on it (65%; 62%; 56% and 64% for Grade 9 pre-test, Grade 9 post-test; Grade 10 pre-test and Grade 10 post-test respectively). Although these values look like they performed well, the averages for the tests were 22%, 31%, 40% and 53% respectively. This tells us that the

Grade 9 and 10 learners are still struggling with basic algebraic concepts. Item 5a was interesting because although it remained one of the best-answered items, the percentage of correct answers decreased from Grade 9 to Grade 10. It is possible that as learners moved from Grade 9 to Grade 10, new knowledge, for example, exponential rules, interfered with learners' knowledge to add like terms and instead add them such that $2a + 5a = 7a^2$.

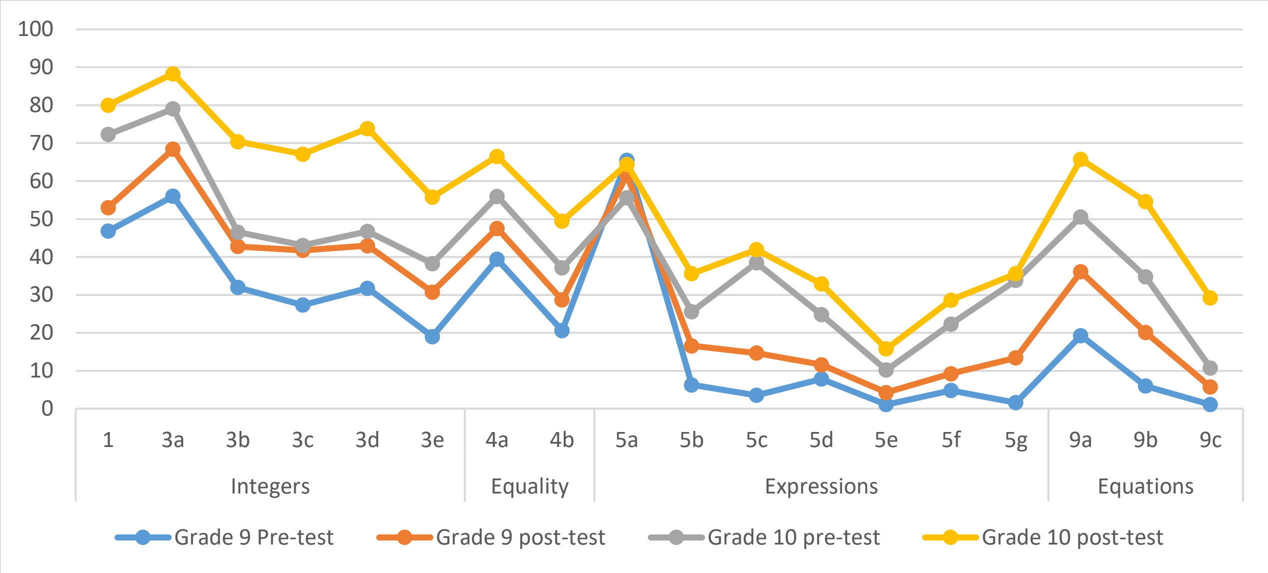


Figure 6.3: Percentage of Grade 9 and 10 learner performance per test item

6.2.2. Statistically significant improvements

To test the hypothesis that the difference in the Grade 9 pre- and post-test and the Grade 10 pre- and post-test scores are statistically significant, a paired sample T-test was performed. A paired T-test looks at the difference between two dependant variables to determine whether the mean scores are statistically significant. This was done between each pre- and post-test for each topic, as well as for the test as a whole. Results show that there was a statistically significant change in the means for all topics and in both grades. The results are summarised in Table 6.1.

	Test as a whole		Equations		Equality		Negatives		Expressions	
	Grade 9	Grade 10	Grade 9	Grade 10	Grade 9	Grade 10	Grade 9	Grade 10	Grade 9	Grade 10
P value	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$	$P < 0.05$
T-value	-22.4	-16.6	-18.3	-12	-8.9	-6.6	-15.1	-14.8	-11.5	-5.9

Table 6.1: Grade 9 and 10 P- and T-values per topic

A correlation statistic was applied to determine the strength of the predictability of a topic against the whole test as well as against equations. None of the topics proved to be predictors of learners' equation performance. See

Tables 6.2 and 6.3 for a summary of results. Results show that in the Grade 10 pre-test, equations, expressions and negatives were relatively stronger predictors of the test results. For Grade 9 learners, negatives were a stronger predictor than for Grade 10 learners. Both the negatives and expression sections contained six and seven items respectively and so it makes sense that these are predictors of test performance of the 18 items. What is surprising is that the Grade 10 equations section, which contained only 3 items, had a correlation coefficient of $r = 0.78$ in both the pre- and post-test.

	Equations		Equality		Negatives		Expressions	
	Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10
Test as a whole	0,62	0,78	0,61	0,58	0,85	0,79	0,61	0,88
Equations			0,29	0,38	0,35	0,51	0,38	0,62

Table 6.2: Correlation statistics for Grade 9 and 10 in the pre-test

	Equations		Equality		Negatives		Expressions	
	Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10
Test as a whole	0,75	0,78	0,60	0,64	0,81	0,76	0,79	0,88
Equations			0,32	0,41	0,44	0,47	0,61	0,63

Table 6.3: Correlation statistics for Grades 9 and 10 in the post-test

This piece of analysis has shown that learners did not perform well in a test that was created using Grade 8 and 9 curriculum items. However, despite the poor performance, there was statistically significant improvement between each test, and learners' equation performance is a relatively strong predictor of learners' test performance.

6.2.3. Correlations between topic scores

Since one of the focuses of this thesis is on relationships between topics, I found the correlations between each topic. These are represented in the correlation matrix in Tables 6.4 and 6.5. In Grade 9, there is a weak correlation between equality items and expression items in the pre-test, and also a weak correlation between the pre-test expression items and the post-test equality and integer items. The strongest correlation was $r = 0.6$, which occurred between the pre- and post-test equality items and between the post-test equation and expression items. In Grade 10 the correlations between topics were stronger than the Grade 9 correlations, with no correlation below $r = 0.3$ and the strongest correlation being between the pre-test expressions topic and the post-test expressions topic ($r = 0.7$). The topic that had the most gains from Grade 9 to Grade 10 was the expressions topic. What this means is that the correlations between the Grade 9 pre-test expressions topics and all the other topics in both the pre- and post-test increased in the correlation value by 0.2 or 0.3 in Grade 10.

		Pre-test full sample				Post-test full sample			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Grade 9	Equation	1,0	0,3	0,4	0,4	0,5	0,3	0,3	0,5
Pre-test full sample	Equality	0,3	1,0	0,4	0,2	0,3	0,6	0,3	0,3
	Integer	0,4	0,4	1,0	0,2	0,4	0,4	0,5	0,4
	Expression	0,4	0,2	0,2	1,0	0,3	0,2	0,2	0,4
Post-test full sample	Equation	0,5	0,3	0,4	0,3	1,0	0,4	0,4	0,6
	Equality	0,3	0,6	0,4	0,2	0,4	1,0	0,4	0,3
	Integer	0,3	0,3	0,5	0,2	0,4	0,4	1,0	0,4
	Expression	0,5	0,3	0,4	0,4	0,6	0,3	0,4	1,0

Table 6.4: Grade 9 correlations between topics

		Pre-test full sample				Post-test full sample			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Grade 10	Equation	1,0	0,4	0,5	0,6	0,6	0,4	0,4	0,6
Pre-test full sample	Equality	0,4	1,0	0,4	0,4	0,3	0,6	0,3	0,4
	Integer	0,5	0,4	1,0	0,5	0,5	0,4	0,5	0,5
	Expression	0,6	0,4	0,5	1,0	0,5	0,4	0,4	0,7
Post-test full sample	Equation	0,6	0,3	0,5	0,5	1,0	0,4	0,5	0,6
	Equality	0,4	0,6	0,4	0,4	0,4	1,0	0,4	0,4
	Integer	0,4	0,3	0,5	0,4	0,5	0,4	1,0	0,5
	Expression	0,6	0,4	0,5	0,7	0,6	0,4	0,5	1,0

Table 6.5: Grade 10 correlations between topics

6.2.4. **Response pattern analysis**

A response pattern analysis shows how every learner responded to (correct, incorrect or missing) the 18 items together. These responses are organised with the pre-test on the left side of the solid black line and the post-test on the right side. Each column represents the answer to one test item. With, for example, 1671 Grade 9 Learners, it is not possible to see all learners' responses at the same time. This is where colouring the correct, incorrect and missing responses is helpful. We can see the extent of the poor results by the large number of red cells and can also see an improvement in the number of correct items by an increase in green cells in the post-test (the right-hand side). It is expected that from pre-to post-test there is an increase in correct responses and so the most interesting results are that in the equations, equality and expressions sections, a large percentage of learners did not change the number of items they did incorrectly. Figure 6.4 shows a snippet of the Grade 9 data for the equation items (hence why there are only three columns on either side of the black line for the three equation items). What is noticeable is that the set of red cells appears to remain in the same position in the post-test, suggesting that the items that were incorrect in the pre-test were the same items that were incorrect in the post-test. This amounted to 62% of the learners. In Grade 10, 48% of learners did not change the number of items they got correct. The other topic where there was a large percentage of learners not changing the number of items correct was in the equality section: in Grade 9, 64% and in Grade 10, 58%. This is evidence of a large proportion of learners appearing not to make progress during the academic year and raises the question as to what errors are being made and whether there are changes in these during the academic year. I elaborate on the errors made in Chapter 10, where the strength of the mixed methods used in this study will become more evident.

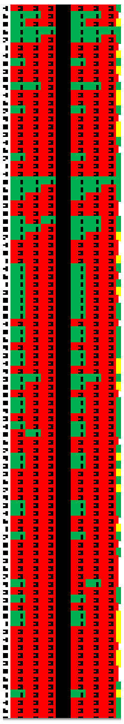


Figure 6.4: Learners who got the same number of items correct in the Grade 9 equations section

Figure 6.5: Learners who got the same number of items correct in the Grade 10 equations section

It is important to note that although the response pattern shows little improvement it relates to the number of correct responses and not the individual items. The line graph shows where each item increased in correct responses. In addition, the response pattern shows only that a large percentage show no/little change but the box plots and T-test results confirm that there was in fact some change. It shows that there was little change which confirms the response pattern analysis and does not contradict it.

6.3. Conclusion

In this chapter I have located the full sample in terms of their mathematical performance. The learner, topic and item scores show how poorly learners responded to the 18 items. However, the scores also show where there was the most improvement. More gains were made in Grade 10 than in Grade 9 on integer and equations items, and the greatest increase in the correlation value occurred between the Grade 9 pre-test expressions topics and all the other topics in both the pre- and post-test. This gives us a general understanding and initial overview of the sample's mathematical ability/performance. I gained some insight into learner difficulties, namely equations where 62% of Grade 9 learners did not improve in the number of equation items they got correct. In Chapter 11, I synthesise the quantitative findings from Phase 2 and compare it to findings from this chapter. This will show that the qualitative results from the sub sample are generalisable to the larger sample, which is a methodological contribution of this study.

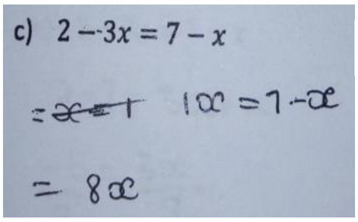
Interlude

One of the focuses of this thesis is the shift in learners' errors when solving three linear equations: $3x - 2 = 10$; $3x - 2 = 4 + x$ and $2 - 3x = 7 - x$. In order to investigate a shift, I analyse responses from pre- and post-tests of Grade 9 and 10 learner test scripts. In addition, I analyse the data using a mixed-methods, sequential triangulation design where I interpret the quantitative and qualitative results at the same time. The purpose of this type of design is to validate the findings from each part with the other part. I therefore split Chapters 7-10 into two: a quantitative part and a qualitative part. As a reminder, the sample used for this analysis is 150 Grade 9 and 150 Grade 10 learners. This sample was randomly selected from the big data set discussed in Chapter 6, with the condition that there were no missing responses in the equation items.

The quantitative analysis of Chapters 7-10 involves descriptive statistics as well as T-tests. Each of the chapters 7-10 are of a different topic, with Chapter 7 being about equations, Chapter 8 about equality, Chapter 9 about integers and chapter 10 about expressions.

A common thread that links Chapters 7-10 is that there are learners who although they show no improvement in the number of items they get correct, do show a change in their written work and the errors made. In addition, throughout Chapters 7-10 I argue that there was a greater improvement in Grade 10, even though the content is explicitly taught in Grade 8. We therefore would expect the greatest improvement in Grades 8 or 9 and a plateauing of results in Grade 10. In terms of teaching, teachers are not expected to revisit basic concepts in Grade 10 but they are revisited in Grade 8 and 9 which is why we would expect more gains in Grade 9 than in Grade 10.

Chapter 7 : Analysis of learners' responses to equation items



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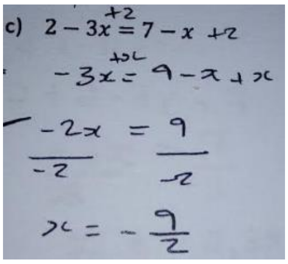

7.1. Introduction

In photography, where the aim of the photographer is to obtain a picture that tells a story, the lens is the light-gathering device of the camera and typically contains a group of compound lenses that allows the picture to be taken. Using this metaphor of a lens, I too want to provide a picture of learners' errors when solving linear equations. I use five different lenses (a group of lenses) that are information-gathering tools that enabled me to comment on what learners do, or don't do, when solving an equation. In addition, a camera captures only a single image and the surrounding context is not seen in that image and hence a variety of pictures gives a more complete story. Similarly, analysing the data in only one way does not give a full picture of learners' attainment. I therefore analyse in multiple ways for the purpose of being able to obtain a more complete understanding of learners' abilities to solve equations. Analysing the data from five different lenses also enables me to say something more about the illogical responses that are not easily amenable to classification and which I am tempted to code as 'other'.

This chapter focuses on the analysis through the five lenses: the approach used to solve equations; learners' views of the equal sign; how learners used inverses; learners' use of negatives; and learners' use of letters. The approach or view of the equal sign and view of a solution are closely related in that the approach taken to solve an equation often leads to a particular view of the equal sign. These two are therefore clustered together. The purpose of the five lenses is to seek order from the chaotic responses that we often get from learners when they solve equations.

The qualitative analysis involves a documents analysis on learners' incorrect responses. When glancing at the response as a whole for each item, there appeared to be numerous responses that were not about solving equations but rather, for example, were converted into expressions or were responses that I could not explain. Many responses were incoherent and coding for errors was complex, which made it difficult to maintain consistency. Coding such responses would have resulted in a large 'other' or 'unknown' category. With this in mind, I decided to analyse the responses according to individual components rather than as a response as a whole and for different types of errors. I zoomed into each response five times, each time investigating something different and call each zoomed-in area a lens. Analysing through the different lenses is my attempt to create order from the chaos.

Findings from this chapter show that:

a) Grade 10 made more gains than Grade 9

b) Although some learners show no improvement in the number of items they get correct, they do show a change in their written work and the errors made

c) At the end of Grade 9, more than half the learners in my sample are unable to solve simple linear equations and at the end of Grade 10, only 37% of learners are able to solve an equation such as $2x - 3 = 7 - x$

d) Only 39% of Grade 9 learners increased the number of equation items they got correct

e) The majority of learners continue to have an operational view of the equal sign.

f) Only 34% of Grade 9 learners solve an equation using inverses

g) Dealing with negative numbers was not a major struggle when solving equations as they were seldom used. When used, however, they were only a stumbling block to item 9c

h) Equations are a possible predictor of learners' scores in algebra tests in Grade 9.

This chapter is structured into a quantitative section followed by a qualitative section. The qualitative section is sectioned into five independent analyses of the linear equations. Figure 7.1 is an overview of the structure of the analysis in this chapter.

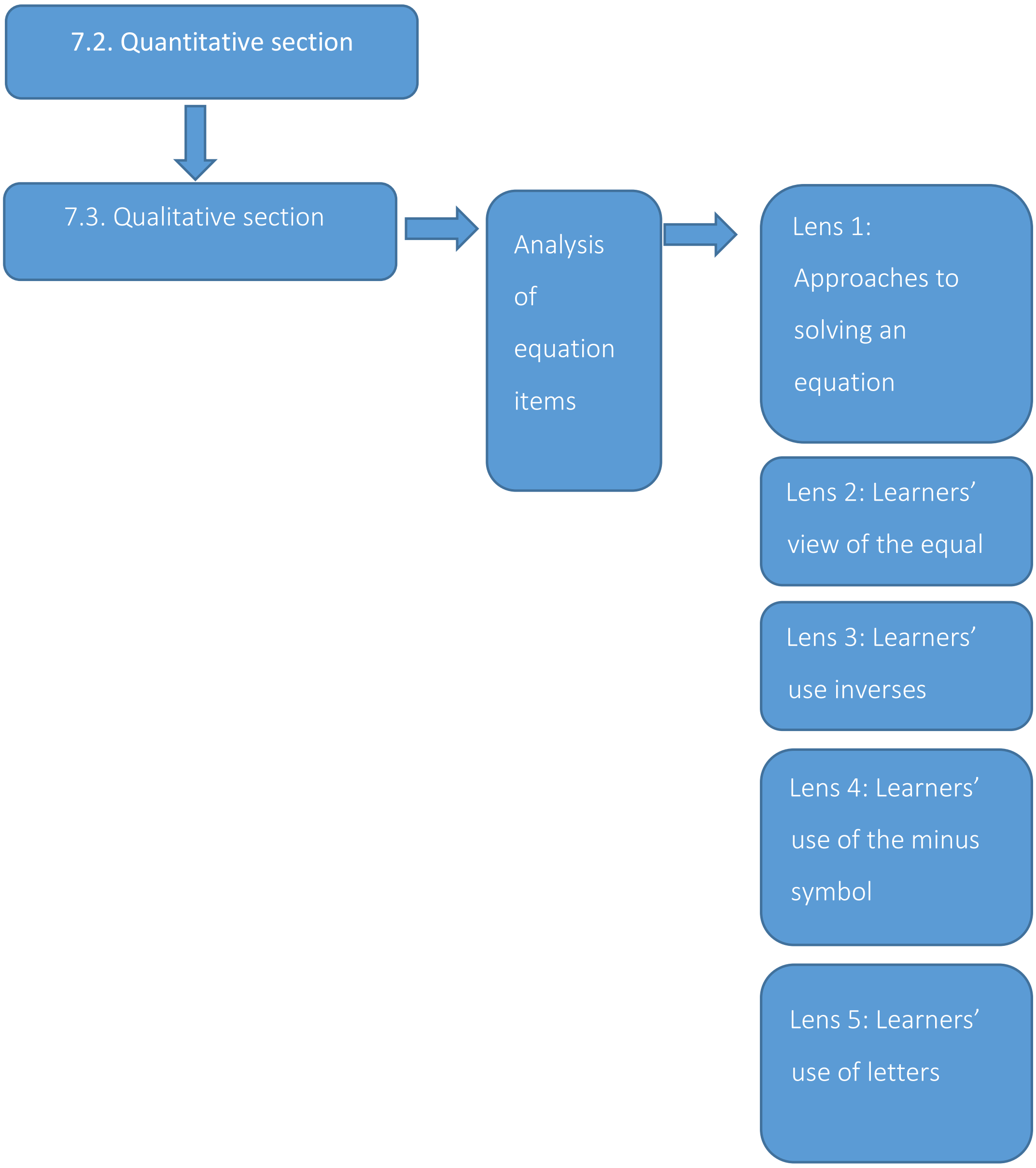

Figure 7.1: Organisation of Chapter 7

7.2. Quantitative analysis

This section is about learners' performance[2] when solving linear equations. I analyse learners' incorrect responses, provide T-test and correlation test results, and provide a response pattern analysis. In this section I provide evidence that despite a statistical significance between mean scores, learners (even at Grade 10 level) are struggling to correctly solve a linear equation. I further show that some learners who get items

[2] Learners' performance is based on the number of correct responses they obtain. Overall test performance is the number of correct responses given in the whole test (18 items)

correct in the pre-test then get the items incorrect in the post-test. In addition, I argue that the large percentage of learners who did not change the number of items they got correct, warrants qualitative analysis to determine whether there are changes in their errors from pre to post. The presence of different errors will show there has been a change in what the learner thinks about the topic. On the other hand, if there is no change in their errors, then it signifies that the learner's way of thinking is entrenched.

7.2.1. Overall performance on the equation items

This section provides evidence of the poor performance of Grade 9 and 10 learners in the topic of solving linear equations. Of the three equation items, item 9a: $3x - 2 = 10$ was answered the best in both Grade 9 and Grade 10; however, despite being the best answered equation item, in Grade 9 only 35% of the population sampled got the item correct in the pre-test and 44% in the post-test (see Figure 7.2).

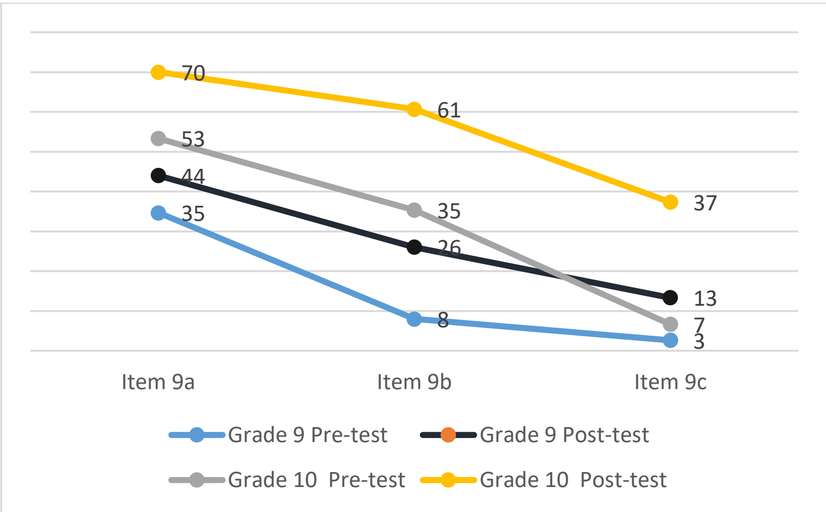

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Figure 7.2: Percentage of correct responses to equation items for Grades 9 and 10

In Grade 10, the percentage of correct responses was higher, with 53% in the pre-test and 70% in the post-test. Compared to item 9a, there was a big difference in the percentage of correct responses to item 9b: $3x - 2 = 4 + x$. Only 8% arrived at the right solution for this equation. There was, however, an 18p.p. increase in the post-test results. In Grade 9 only 3% got item 9c correct in the pre-test and this increased to 13% in the post-test. In the Grade 10 post-test the percentage of correct responses increased to 37%.

The higher percentages of correct responses to item 9a were expected since the item was designed to be less cognitively demanding, with a letter on only one side of the equation. Although the Grade 10 results were better than Grade 9, it is worrying that in Grade 9, less than 13% of the learners are able to solve a basic linear equation, such as $2 - 3x = 7 - x$, correctly. The big improvement in the percentage of correct responses to item 9b and item 9c in both Grade 9 and then again in Grade 10 suggests that learners made progress with dealing with a letter on both sides of the equation in Grade 9 and that Grade 10 learners also continued to make progress. Item 9c: $2 - 3x = 7 + x$ is similar to item 9b: $3x - 2 = 4 + x$ in that, it has a letter on both sides of the equal sign. The difference in item 9c is that the letter is subtracted and hence there is an extra negative to deal with. The large increase in correct responses to item 9c in Grade 10 suggests that learners also made progress in dealing with negatives. The drop in correct responses in the Grade 10 pre-test could suggest that learners get the item correct when they have been practising and using negatives regularly, however the holiday between Grade 9 and 10 and not practising could account for the drop as learners are not exposed to much mathematics.

Table 7.1 shows a summary of the percentages of learners who could answer items correctly. The first two columns show the Grade 9 percentages for the pre- and post-test and the last two columns for Grade 10.

	Grade 9		Grade 10	
Percentage	Pre-test (%)	Post-test (%)	Pre-test (%)	Post-test (%)
None correct	64	53	44	25
One correct	29	23	21	15
Two correct	4	13	31	27
All correct	3	11	4	33

Table 7.1: Number of correct responses to equation items

From Table 7.1 we see that 64% of the Grade 9 sample could not answer a single equation item correctly in the pre-test. By the end of Grade 10, only 25% got all three items incorrect. Only 3% could answer all three items correctly in the pre-test; although there was an improvement in the number of equations learners could answer in the post-test, 53% still could not answer any of the equations correctly by the end of Grade 9. By the end of Grade 10, however, 33% of learners were able to get all three items correct. This is a 29p.p. increase from the pre-test. In addition, 60% of learners were able to get at least two of the three items correct in the post-test.

These percentages illuminate the difficulty that learners face when solving linear equations. The 19 p.p. drop in getting none of the items correct suggests that in Grade 10 learners improve more with solving equations than in Grade 9. It should alarm us that learners could be entering Grade 10 mathematics without knowing how to solve a simple linear equation such as $3x - 2 = 10$ and yet are expected to learn to solve quadratic, exponential, hyperbolic and trigonometric equations. This could further suggest that learners are not getting enough practice and exposure to equations in Grade 9 since they are only becoming comfortable with Grade 8-level equations by the end of Grade 10. This lag time in responses highlights the need for a conceptualisation of learning that takes the extended amount of time for learning into account. This conceptualisation was an outcome of the thesis and is discussed in Chapter 12.

The box plot (Figure 7.3) confirms that there was a greater improvement in Grade 10 with solving equations. The mean increases in each test, starting at 15% in the Grade 9 pre-test and increasing to 28% in the Grade 9 post-test. There was then a small increase to 32% in the Grade 10 pre-test but a much larger increase in the Grade 10 post-test (from 32% to 56%). Comparing the Grade 9 pre-test to the Grade 10 post-test, we see that 50% of the Grade 10 learners performed better than 100% of the Grade 9's in their pre-test (excluding the one outlier in the Grade 9 pre-test). The difference between the Grade 9 pre- and post-test was small, with 75% of the learners in the pre-test obtaining less than 33% and 75% obtaining less than 42% in the post-test. There was a bigger difference between the Grade 10 tests: 50% of the learners performed at the same level as the Grade 9s, obtaining less than 33%, but the improvement was in the 25% of learners who obtained between 33% and 67% in the Grade 10 pre-test. In the Grade 10 post-test there was yet a greater improvement, where 50% of the learners obtained scores greater than 75% of the Grade 10s in the pre-test.

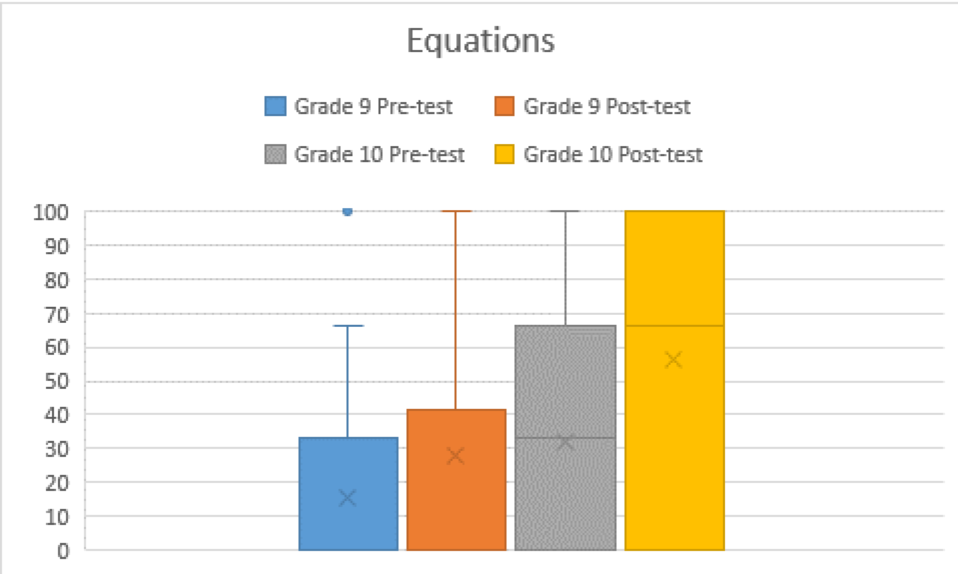

Figure 7.3: Grade 9 and 10 learner performance on equation items

7.2.2. T-test and correlation results

T-tests show that in both grades there was a statistically significant improvement in the learners' performance on the equations items. Grade 9: $t(149) = -4.3\ p < 0.05$, Grade 10: $t(149) = -9.3\ p < 0.05$. However, the effect size of the Grade 10 results suggests that there is a stronger improvement in Grade 10 (Cohens $d = 0.38$ for Grade 9, and $d = 0.68$ for Grade 10). This confirms what we saw in Figures 7.1 and 7.2. The correlation between learners' equation scores (out of 3) and learners' test score (out of 18) was conducted.

The Grade 9 results show a strong positive correlation of $r = 0.8$ between the equation scores and the overall test scores. This means that the Grade 9 equation scores could be a predictor of learners' performance in the test consisting of integers, expressions, equality items and equations. In Grade 10, there was a weak positive correlation ($r = 0.4$ in both the pre and post-test). These results suggest that although learners do better at equations in Grade 10 and make more gains in Grade 10, the lower Grade 9 results are more of an indicator of learners' test results in Grade 9, meaning that Grade 9 equation scores are similar to their test results but in Grade 10 they are not. Perhaps this is because linear equations are not a focus in Grade 10 but other equations are, meaning that the knowledge of the other equations helps learners concretise their knowledge of linear equations.

7.2.3. Response pattern analysis

The figures below are what I have termed *response patterns* as they give the patterns of learners' responses to the items. I am using these response patterns to show how learners responded to a collection of items rather than just one item. As mentioned in the methodology section, the solid black line separates the pre-test from the post-test. The first three columns relate to items 9a; 9b and then 9c, similarly on the post-test side. The green cells represent correct responses and the red cells represent the incorrect responses. As a reminder there are no yellow cells because one of the criteria in randomly selecting these learners was that they had to have attempted the equation items and hence not leave any blank.

What is foregrounded by the response patterns is how solving equations presents a difficulty for learners. There are many more red cells compared to green cells, meaning, there is much more incorrect than there is correct. Even in the post-test there are many incorrect responses. This suggests again that even at the end of Grade 9, learners struggle to solve simple linear equations. What the response patterns show that percentages don't, are how individual learners responded to each item in the pre- and post-test.

We know from Figure 7.1 that 31 Grade 10 learners (21%) got two of the three equation items correct in the pre-test but the response patterns tell us that the majority of these learners got the same items, and others, correct in the post-test (see Figure 7.4). Figure 7.2 also showed that there is a large percentage of learners who got all three items incorrect in the pre-test, however we see from the response pattern that in the post-test there is a large percentage of learners who continue to get incorrect responses for the equation items (see Figure 7.5). Compared to Grade 9, there were more learners in Grade 10 that continued to get more items correct in the post-test (see Figures7.6 and 7.7). Although this is expected, the poor performance is alarming.

The response patterns also highlight learners who improved the number of items they got correct, those that got less items correct in the post-test compared to the pre-test (those that regressed) and those that maintained the same number of items correct/incorrect in both tests. In Grade 9, 47% and in Grade 10, 42% of learners did not change in the number of items they got correct; they remained getting the same number of items correct/incorrect as what they did in the pre-test, ultimately showing no change within the year. This does raise the question as to what errors learners were making and whether these changed during the year. In Grade 9, 15% of learners regressed but in Grade 10 only 7% regressed. Figure 7.8 shows this regression by the number of red cells on the right as compared to the number of red cells on the left of the solid black line. There are more red cells, meaning more incorrect answers, in the post-test that in the pre-test. Figure 7.8 shows the set of Grade 9 learners that improved in the number of items they got correct (38%) by the mass of green cells that are on the right hand side as compared to the left hand side of the solid black line. This percentage increased to 53% in Grade 10.

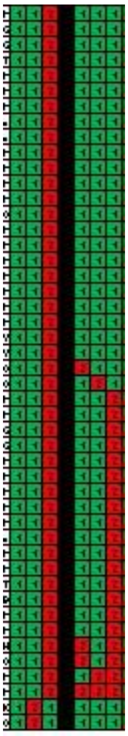

Figure 7.4: Two correct items in Grade 10 pre-test

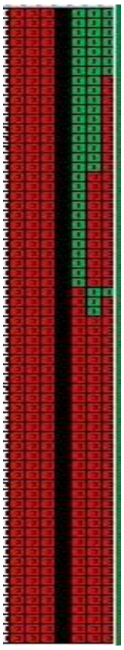

Figure 7.5: No items correct in the Grade 10 pre-test

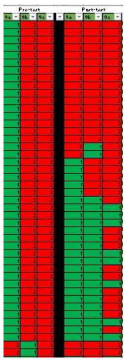

Figure 7.6: One item correct in Grade 9 pre-test

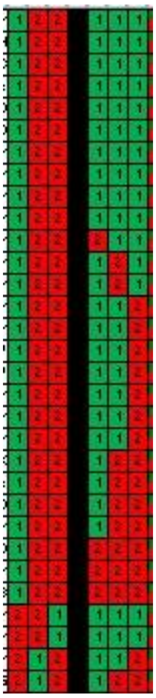

Figure 7.7: One item correct in Grade 10 pre-test

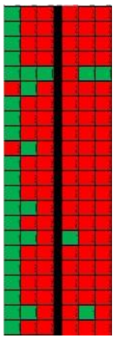

Figure 7.8: Regression from Grade 9 pre to post-test

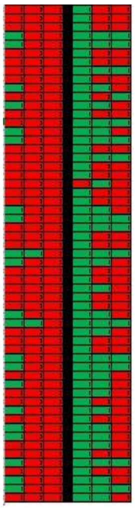

Figure 7.9: Improvement for Grade 9 pre to post-test

The percentages of learners that regressed suggest that their knowledge about solving equations is not stable and could change for the worse. The decrease in this percentage in Grade 10 further suggests that in Grade 10 learners' knowledge about solving equations is much stronger and more stable. It is expected that at Grade 10 level, learners' knowledge is more stable and that their equation-solving skills are stronger, however it is happening later than expected, and later than is needed. This instability is yet another indication of the need for more time and experience to master the content. Again, this is discussed in more detail in Chapter 12.

7.3. Qualitative analysis

In this section I analyse learners' responses to three equation items using six lenses: the approach used to solve the equation; learners' views of the equal sign; learners' views of a solution; how learners used inverses; learners' use of the minus symbol; and lastly learners' use of letters. Learners' approach, view of the equal sign and view of a solution are closely related in that their approach used mostly leads to a particular view of the equal sign. These two are therefore clustered together and are discussed first.

7.3.1. Lens 1: Approaches used to solve equations

I analysed responses to the three algebraic equations by identifying the approaches learners use to solve the equations. I use the word 'approach' to describe the way a learner attempts to solve the equations.

In the Grade 9 pre-test, the most common approach to solving item 9a: $3x - 2 = 10$ was *algebraic-with-equations*, meaning that it was most common for learners to attempt to keep a left and right hand side of an equation. Even though this was the most common it was only done by 34% of the learners (see Chapter 5 for a full discussion on the codes). The second most common approach was to solve the equation using arithmetic (23%). Together these tell us that 57% of the learners attempted a valid and understandable mathematical approach to solve the equation. The remaining 43% of responses were split between being a response that resembled arithmetic (pseudo-arithmetic) being used (14%), having a left and right side but not involving any equality rules (pseudo-algebraic with equations) (12%) and converting the equation into an expression (17%). See Figures 7.10 and 7.11. There was very little difference in the post-test results, with 58% of learners using an algebraic-with-equations approach or an arithmetic approach. Although the percentage of learners who converted an equation to an expression remained the same, there was a 5p.p. decrease in the percentage of learners who used pseudo-arithmetic approaches.

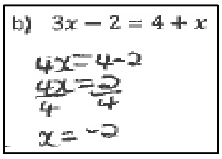

Figure 7.10: Equality with equations

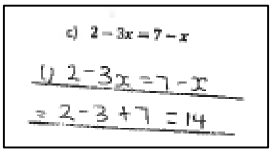

Figure 7.11: Pseudo arithmetic approach

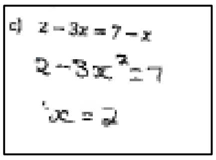

Figure 7.12: Pseudo algebraic with equations approach

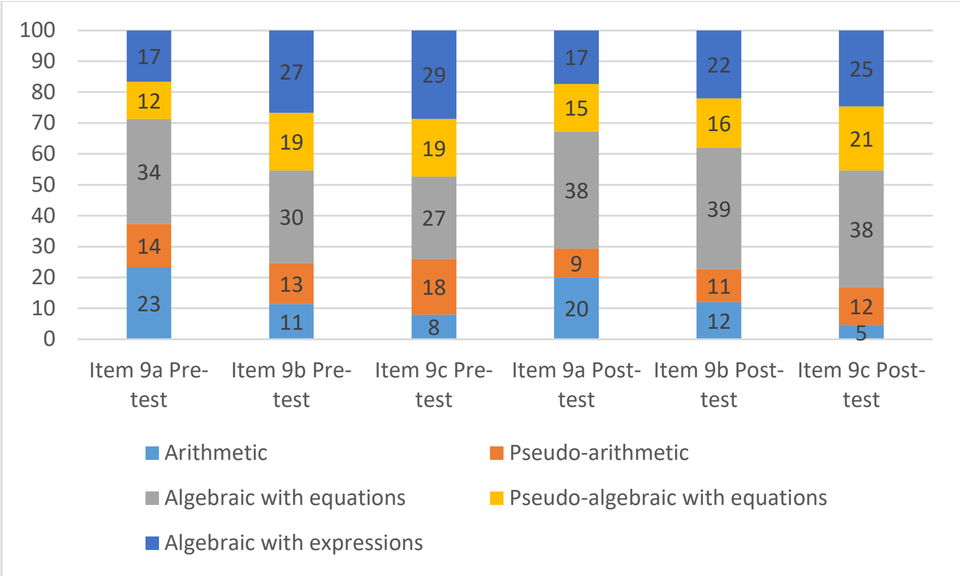

Figure 7.13: Grade 9 approaches to solving equations

In Grade 9 the results from the coding of items 9b and 9c were similar in that the two most common approaches used were algebraic-with-equations and then algebraic-with-expressions (See Figure 7.13).

The percentage of learners who used arithmetic to solve the equations was low at 11% for item 9b and 8% for item 9c. This is not surprising, as research has suggested that when there is a letter on both sides of the equal sign learners' arithmetic approaches fail them (Filloy & Rojano, 1989). As with item 9a, the results for the post-test were very similar to that of the pre-test (see Figure 7.10), suggesting that there was no shift in the approach used to solve equations. Although algebraic-with-equations was the most common approach in all three equations, the percentage of learners who used it never reached above 34% in the pre-test and 39% in the post-test. This means that over 60% of learners are not maintaining a left and right hand side when solving equations. With the small percentage of learners using arithmetic approaches, this leaves about 30% of learners using an approach that is not explainable by the end of Grade 9.

In Grade 10, there was a noticeable difference in the approaches to solving equations. The most common approach was still algebraic-with-equations, with 73% of learners using this approach. There was only 1% of learners that used an arithmetic approach and 9% that used pseudo-arithmetic approaches. The Grade 10 results, in comparison to the Grade 9 results, show that the performance in the pre- and post-test items 9b and 9c are also very similar (see Figure 7.13 and 7.14). In comparison to Grade 9, the algebraic-with-equations approach was used by 72% of the learners in both items in the pre-test and 79% in the post-test. This means that by the end of Grade 10, only about 20% of learners do not have a left and right hand side of an equation, which in contrast to Grade 9, is a big improvement.

These results suggest that when learners reach Grade 10 they are more skilled at solving equations algebraically and have mostly stopped using arithmetic to solve an equation. Although this is a big improvement from Grade 9, it ought to have happened sooner.

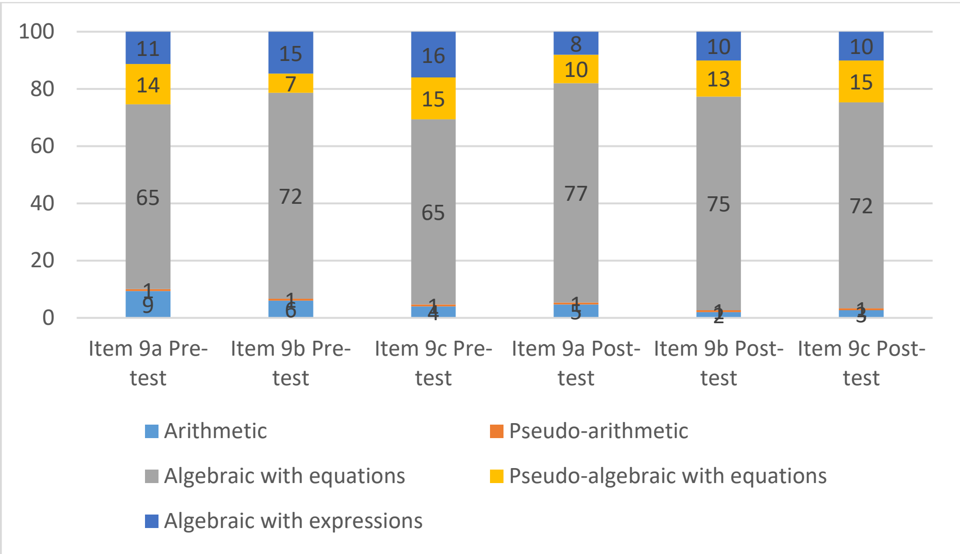

Figure 7.14: Grade 10 approaches to solving equations

7.3.2. Lens 2: View of the equal sign

The analysis of approaches in Lens 1 showed that the majority of the Grade 9 learners did not maintain a left- and right-hand side of an equation. It is therefore not surprising that the analysis of learners' views of the equal sign reveal that the majority of Grade 9 learners do not have a relational view of the equal sign. This concurs with literature that suggests that many learners view the equal sign operationally. In the Grade 9 pre-test, half the sample (51%) attempted to solve $3x - 2 = 10$ operationally, meaning that in trying to isolate x they used the equal sign as an "and then I did this" symbol. This was the most common view in all three items, but also the most common view in the post-test (see Figure 7.15). Figures 7.15 to 7.17 are examples of learners' responses where their view of the equal sign was operational.

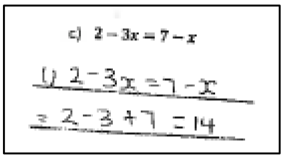

Figure 7.15: Operational view of the equal sign example 1

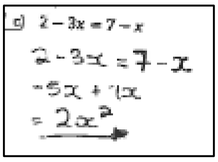

Figure 7.16: Operational view of the equal sign example 2

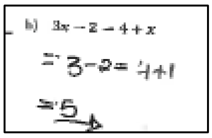

Figure 7.17: Operational view of the equal sign example 3

The three categories that changed the most from the pre- to the post-test were the increase in the relational view of the equal sign, a decrease in the operational view, and an increase in the 'other' view. It is encouraging that learners are making the move from operational to relational, but ending up with more learners in the 'other' category was not expected. This could possibly be due to new knowledge learnt, such as exponents, factorising and quadratic equations. In Grade 10, there was a big increase in the relational view (see Figure 7.18), with 50% of relational responses in item 9a, 50% in item 9b and 40% in item 9c. In the post-test there was an even bigger increase in the relational responses, with 72%, 70% and 60% for items 9a, 9b and 9c respectively.

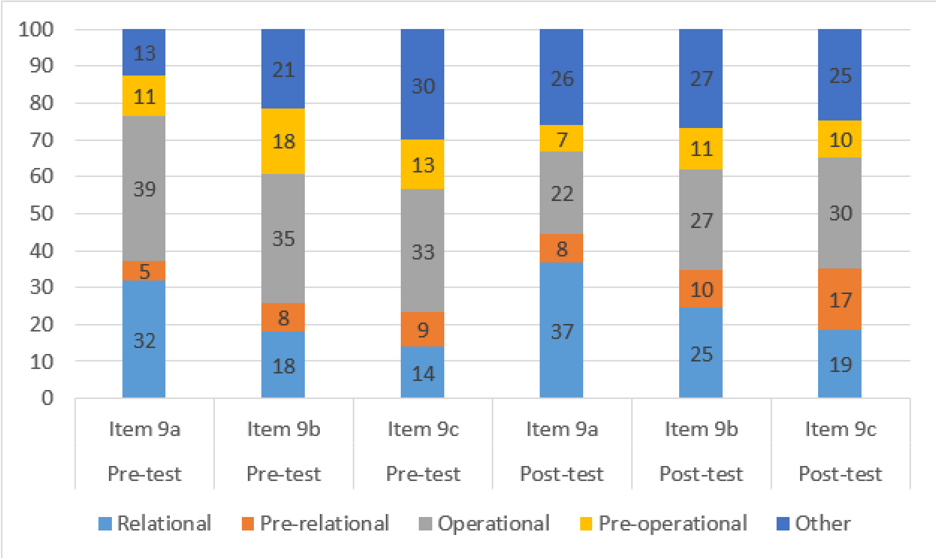

Figure 7.18: Grade 9 View of the equal sign

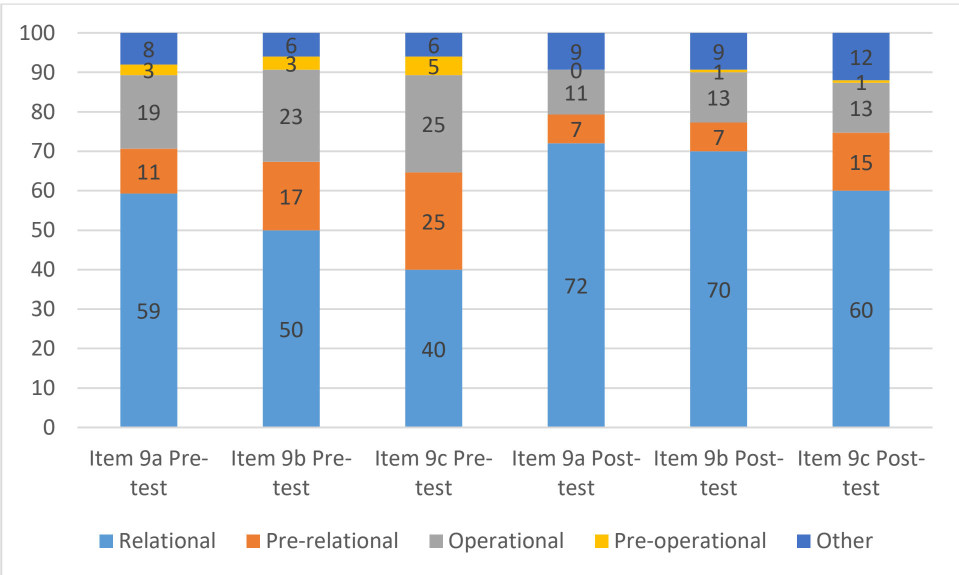

Figure 7.19: Grade 10 view of the equal sign

Figures 7.20 and 7.21 show the Grade 9 and Grade 10 disaggregation of the relational view of the equal sign. In both grades, not only was there an increase but the majority of the relational responses were sameness-relational rather than substitutive-relational in all three items. This is encouraging as more learners are viewing equations in the context of sameness. In Grade 9, in item 9a, the substitutive-relational category showed a very small decrease and there were no learners who had a substitutive-relational view in items 9b and 9c. In Grade 10, in item 9a, there was a big decrease in the substitutive-relational responses. In the Grade 9 pre-test, 38% of the relational responses were substitutive-relational. This decreased to 27% in the post-test but in Grade 10, the percentages were much lower, at 8% and 6% in the pre- and post-test.

These results are further evidence that learners move away from arithmetic methods such as substitution when dealing with equations with a letter on both sides of the equal sign, but also that they move away from substitution as they move into older grades. This too is encouraging because it suggests that learners are moving towards more sophisticated methods of solving equations rather than using substitution.

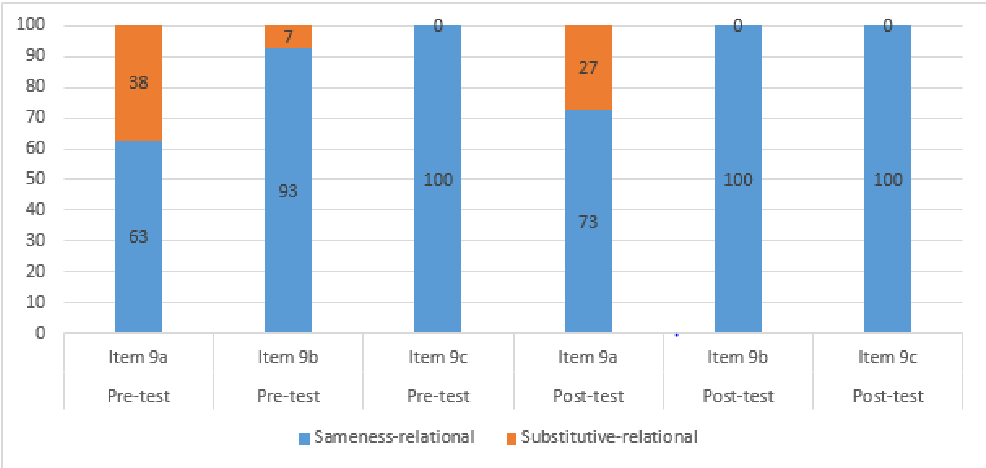

Figure 7.20: Grade 9 disaggregation of relational category

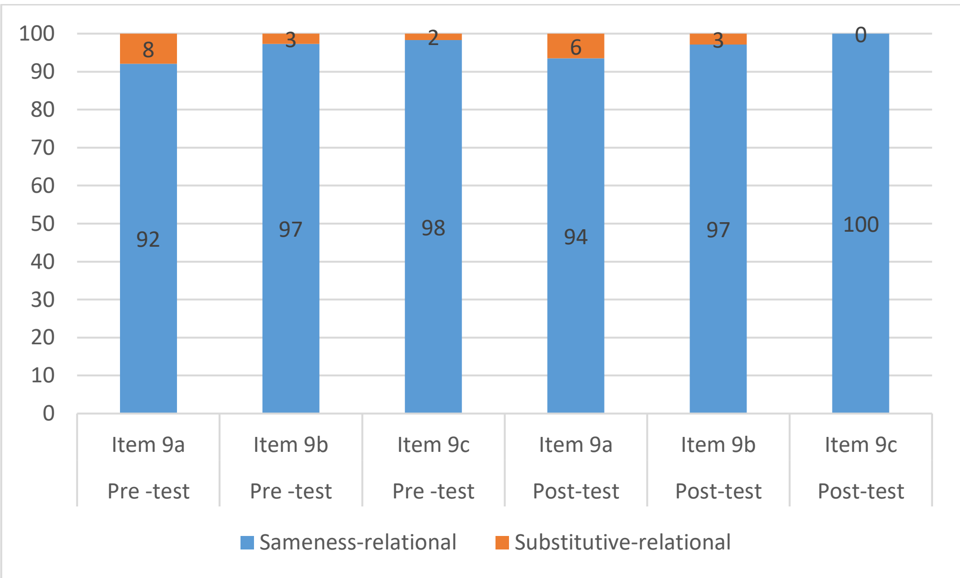

Figure 7.21: Grade 10 relational disaggregation

7.3.3. Lens 3: learners' use of inverses

This lens provides an analysis of the types of inverse errors learners made when they did use inverse operations. When solving linear equations, if learners are not using arithmetic they are expected to use inverses. We know from Lens 1 that the majority of Grade 9 learners did not approach solving an equation from an algebraic-with-equations view, meaning they did not have a left and right side of the equation. This part of the analysis (Lens 3) therefore serves as confirmation of the findings in Lens 1 in that you can't be using inverses if you do not have a left and right side to balance.

Figure 7.22 shows the percentage of learners who either used an inverse in some way (correctly or incorrectly) or didn't use them at all. The first three bars show the pre-test percentages and the last three bars show the post-test percentages. In Grade 9, the majority of learners did not use inverse operations to solve the equations. We know from the analysis of the different approaches used (Lens 1) that many learners used arithmetic when solving item 9a but there were very few who did so in items 9b and 9c, suggesting that the majority of responses to items 9b and 9c were not algebraically sound. I say this because if learners were not using arithmetic but also not using inverses, then what were they using and how were they solving an equation? It is encouraging that from the Grade 9 pre- to the post-test the percentage of inverses not used went down and the percentage of using inverses went up. This suggests that as learners move to the end of Grade 9 they are using inverses more than in the beginning of Grade 9. There was a 10p.p, 12p.p. and 14p.p. increase in the percentage of learners who used all inverses correctly in items 9a, 9b and 9c respectively from pre- to post-test. Although Item 9a was the item that had the most arithmetic approaches used, it is also the item where the most inverses were used correctly. This is possibly due to the fact that item 9a only had two inverses -- one additive and one multiplicative. Items 9b and 9b both had a letter on either side of the equal sign and so have two additive inverses to attend to as well as a multiplicative inverse.

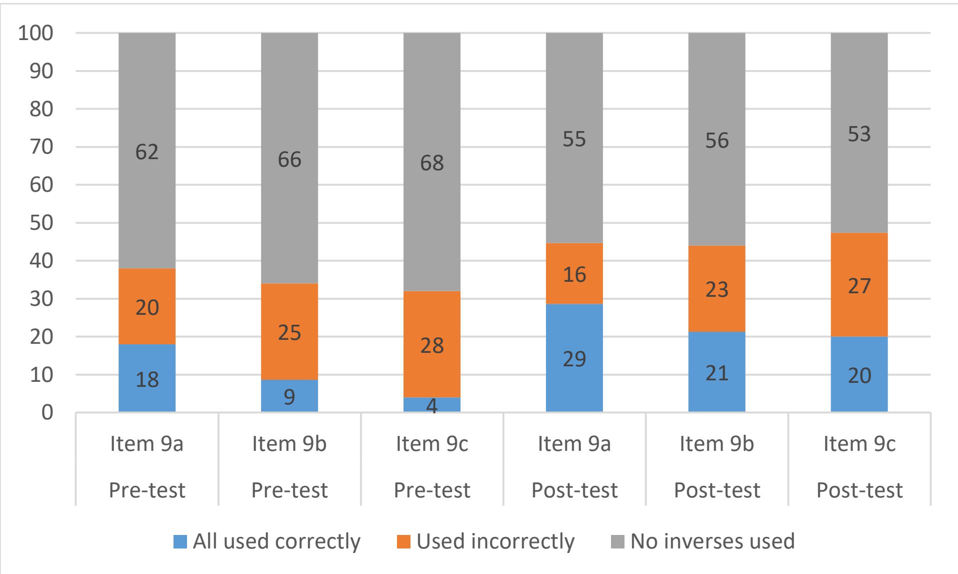

Figure 7.22: Grade 9 learners' use of inverses

In Grade 10, there was a big increase in learners using inverses correctly (see Figure 7.23.). In item 9a, the Grade 9 post-test was 29% but the Grade 10 pre-test was 55% and increased in the post-test to 71%. This means that within one year there was a 42p.p increase in using inverses correctly. There was also a decrease in using inverses incorrectly in all the items in the pre- and post-test, except for item 9c in the pre-test. What is most encouraging is the decrease in the percentage of learners who did not use inverses to solve equations. By the end of Grade 10, 19%; 21% and 25% were still not using inverses to solve equations.

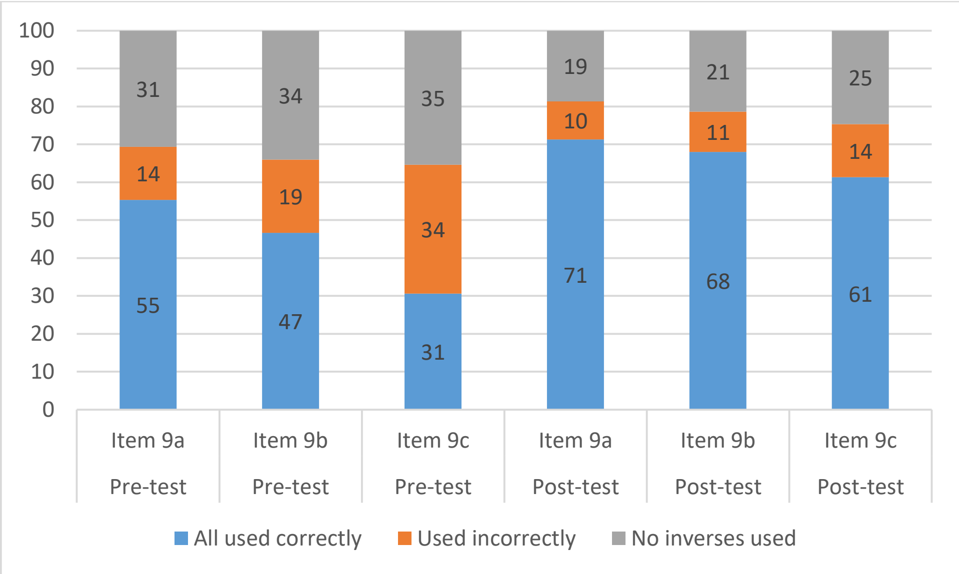

Figure 7.23: Grade 10 learners' use of inverses

Besides coding whether learners used inverses or not, I coded for the different inverse errors. When coding for the learners' use of inverses, they could have multiple error codes because the items require using two inverses in item 9a and three in items 9b and 9c. A learner could have used the wrong sign in one inverse (for example, $+2$ instead of -2) and the wrong inverse for another (for example, multiplying by 3 instead of dividing by 3).

Figure 7.24 shows the errors Grade 9 learners made when not using inverses correctly. In both the pre- and post-test, the most common error when answering item 9a was to use an inverse whilst converting the equation into an expression. This error accounted for 35% in the pre-test and 42% of the errors made in the post-test. In Grade 10, these percentages decreased to 17% in the pre-test and 7% in the post-test (see Figure 7.25). The second most common error in item 9a was for learners using the wrong sign. This error accounted for 27% and 37% of the errors made in the pre- and post-tests respectively. Using the wrong inverse and the balancing error accounted for 19% each in the Grade 9 pre-test. These errors decreased in the post-test to 16% and 5% respectively but increased in the Grade 10 pre-test to 23% and 15% respectively, only to decrease again to 7% for both errors in the post-test. This suggests that at the beginning of a year learners are more susceptible to balance errors and applying the wrong inverse, but as they practice more and progress through the year they make fewer of these errors.

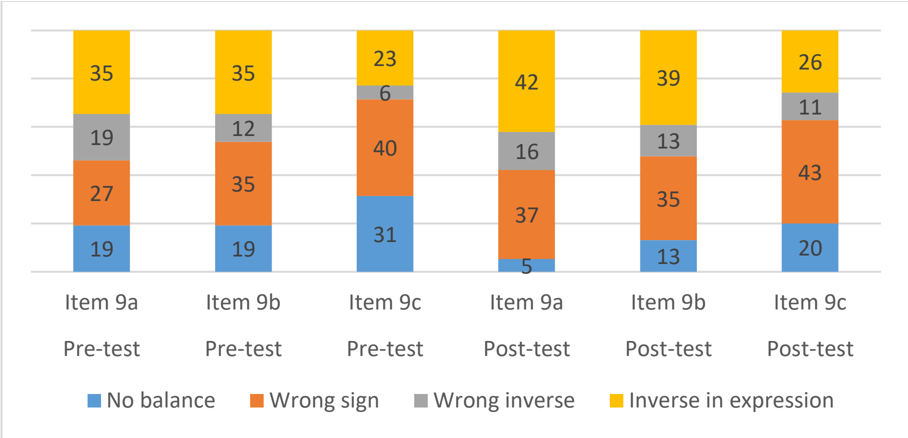

Figure 7.24: Grade 9 inverse errors

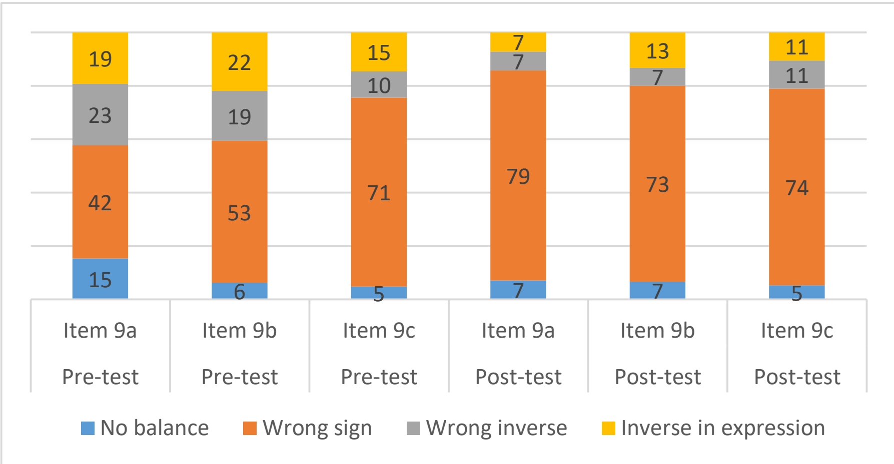

Figure 7.25: Grade 10 inverse errors

In the pre-test of item 9b, both the inverse-in-expression error and the wrong sign error accounted for 35% of the inverse errors, but in the post-test converting to an expression accounted for 39% and the wrong sign error for 35%. An example of the inverse-in-expression error is seen in Figure 7.26, where we see that the learner has correctly applied the additive inverse of $4x$, but instead of being left with an equation $1x - 4x = 0$ the learner has converted it into an expression.

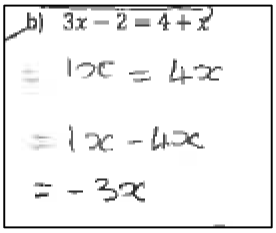

Figure 7.26: Inverse used-expression

In Grade 10 of item 9b the wrong sign continued to be the most common error. However, there was a huge increase in the error, accounting for 53% in the pre-test and 73% in the post-test. For Grade 10, the wrong sign was the most common error in all items for both the pre- and post-test. In addition, it accounted for almost ¾ of the inverse errors. Figure 7.27 gives an example of a learner who applied the wrong sign when applying the inverse of -2. The learner did however correctly apply the additive inverse of x and multiplicative inverse of $2x$.

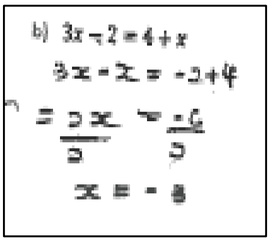

Figure 7.27: Inverse used: wrong sign

The balance error accounted for 19% in the Grade 9 pre-test, decreased to 13% in the post-test and further decreased to 6% and 7% in the Grade 10 pre- and post-test. The wrong inverse error accounted for 12% in the Grade 9 pre-test and 13% in the post-test. This error increased to 18% in the Grade 9 pre-test but then decreased to 7% in the post-test, suggesting that the wrong inverse error is an error that takes longer to dissipate.

In item 9c the wrong sign was the most common error in both tests and in both grades. The wrong inverse error increased from 6% in the Grade 9 pre-test to 11%, 10% and 11% in the Grade 9 post-test, Grade 10 pre-test and Grade 10 post-test respectively. The balance error however decreased from 31% in the Grade 9 pre-test to 20% in the post-test and then further decreased to 5% in the Grade 10 pre- and post-test.

What these results show is that in Grade 9, the majority of learners do not use inverse operations and those who do, do so incorrectly by mainly converting to an expression or by using the wrong sign. In Grade 10 many more learners use inverses, and the wrong sign is the most common error. These results suggest that the more learners are exposed to solving equations the better they get at using inverses.

7.3.4. Lens 4: Learners' use of the minus symbol

Learners' use of the minus symbol includes using it as an operation (subtraction) but also as a sign (negatives). In all three items there were minus symbols, however not all items required the use of subtraction or negativity. That being said, there are multiple ways of solving the equation and some of these require subtraction, use of negatives, both or neither.

The results of the analysis regarding the use of the minus symbol show an overwhelming percentage of learners did not make errors with negativity or subtraction in both Grades 9 and 10. This can be seen in Figure 7.28, which shows the Grade 9 results, and Figure 7.29, the Grade 10 results. The blue bars show the percentage of learners that got the item correct and hence did not make errors using the minus symbol. The orange bars show the percentage of learners where, although they got the item incorrect, the error was not due to subtraction or negativity. In Grade 9, item 9a, only 8% of learners made errors using the minus symbol, with 7% making subtraction errors (operation error) and 1 % negativity errors (signs errors). This decreased to 7% in the post-test and in Grade 10 the percentage was as low as 2% and 3% in the pre- and post-tests respectively. These low percentages were expected in item 9a because the item did not require the use of subtraction or negativity. Item 9b also has low percentages for errors made with negativity and subtraction, but it also did not require any action with negatives, it only required a subtraction. Item 9c has a higher percentage of learners who made errors with the minus symbol. In Grade 9, 36% of the learners made an error involving the minus symbol, but 31% was due to the minus symbol as an operation, a signs error, and 5% due to the minus symbol as a sign. In the Grade 9 post-test, these percentages decreased to 21% making an error with the minus symbol, again with the majority making subtraction errors (24%). In Grade 10, the percentages were similar to that of the Grade 9s, with 28% of learners in the pre-test making an error with the minus symbol and 21% in the post-test. In both these tests, the percentages of errors with subtraction were much higher than that with negativity (24% in the pre-test and 17% in the post-test)

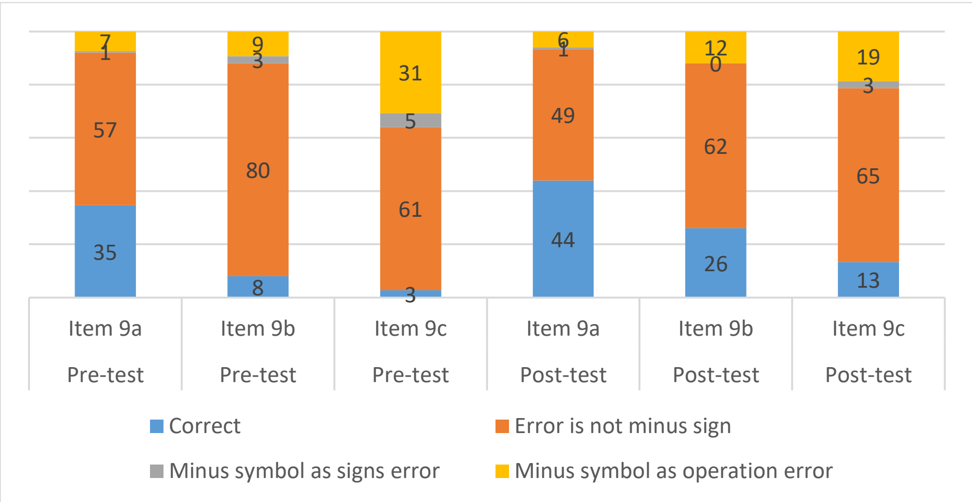

Figure 7.28: Grade 9 use of the minus symbol when solving equations

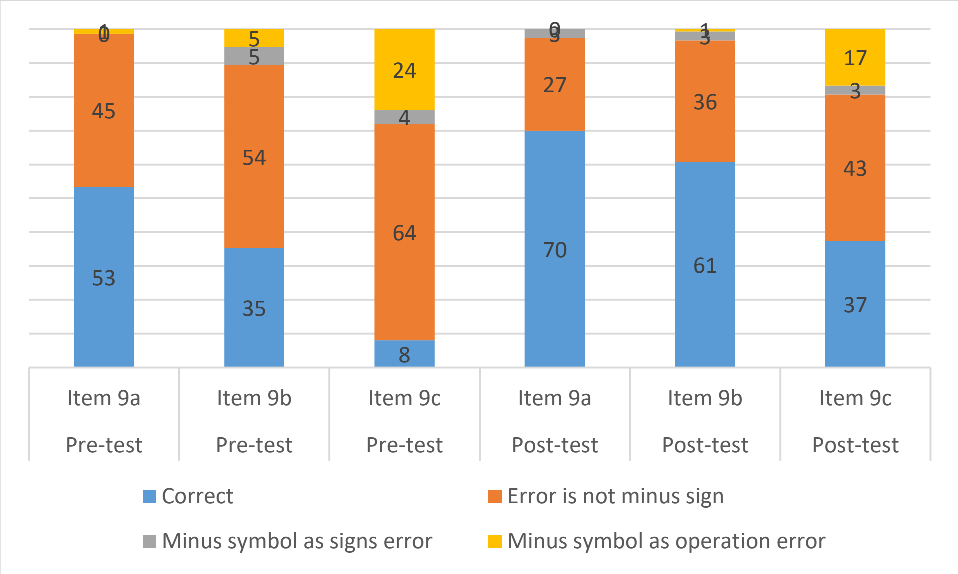

Figure 7.29: Grade 10 use of the minus symbol when solving equations

7.3.5. Lens 5: Learners' use of letters

According to Küchemann (1981), there are multiple ways in which learners can use letters. I am however only concerned with learners' incorrect ways of adding and subtracting terms. Figure 7.30 shows us that in fact the majority of learners in Grade 9 did not conjoin at any point in their response. We see this in the combined large percentage of correct responses and responses that were incorrect but were not wrong because of conjoining. Despite the fact that less than 50% of learners conjoined, there was still a large proportion of learners who did conjoin, especially in items 9b and 9c. There was more conjoining in items 9b and 9c, which is possibly due to the letter on the right-hand side. In items 9b and 9c, there were two places where learners could conjoin whereas in item 9a there was only one place. It therefore makes sense that the percentage of conjoining is more than double in items 9b and 9c. Where there was very little change in the percentage of conjoining in item 9a from pre- to post-test (18% to 15%) and item 9b pre- to post-test (38% to 34%), there was a much bigger drop in items 9c (41% to 26%). This suggests that subtracting x rather than something being subtracted from x is less of an 'encouragement' to conjoin. In Grade 9 there was also a small percentage of learners who misused the exponential law. The results from the Grade 10 analysis were similar to the Grade 9 analysis in that the combined percentages of correct responses with those that did not conjoin outweigh the percentages of learners who conjoined (see Figure 7.31). However, it is clear that there was less conjoining in Grade 10 than in Grade 9, with no more than 11% conjoining in any question, across both tests. In the Grade 10 results we also see that few learners misused the exponential laws. These results suggest that in the context of solving equations, learners are not prone to making as many errors related to simplifying expressions.

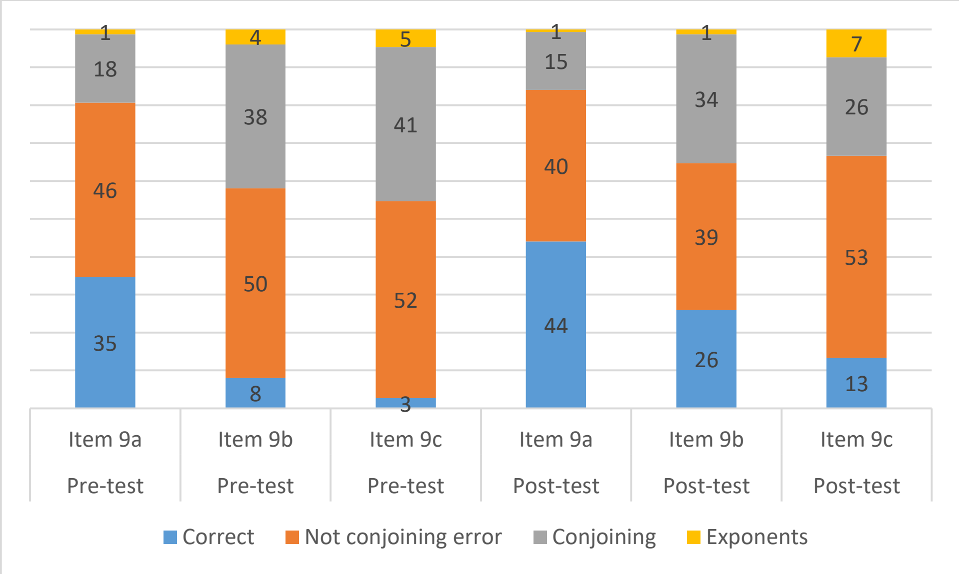

Figure 7.30: Grade 9 use of letters when solving equations

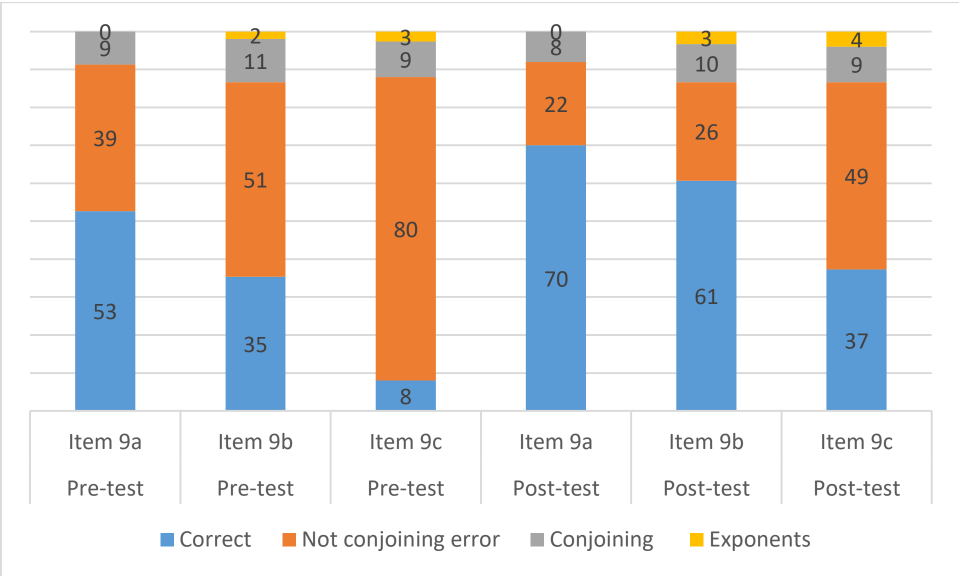

Figure 7.31: Grade 10 use of letters when solving equations

7.4. Correlations between equations and the other topics

In Grade 9, the strongest relationship between equations and the other topic was, in both the pre- and post-test, with expressions ($r = 0.6$) and in the pre-test, with integers $r = 0.6$. This means that there is a relationship between learner scores in the equation items and the integer and expression items in the pre-test. This suggests that as learners improved in simplifying expressions, they also improved with solving equations items. There was a weak correlation between equality scores and equation scores in the pre and post-tests ($r = 0.3$), suggesting that learners' improvements in solving numeric equations did not influence their equation-solving performance.

Grade 9		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equation	1,0	0,4	0,6	0,6	0,5	0,3	0,4	0,6
Post-test	Equation	0,5	0,4	0,4	0,5	1,0	0,3	0,5	0,6

Table 7.1: Grade 9 correlations between equations and other topics

Grade 10		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equation	1,0	0,4	0,5	0,6	0,6	0,4	0,4	0,6
Post-test	Equation	0,6	0,3	0,6	0,5	1,0	0,5	0,5	0,6

Table 7.2: Grade 10 correlations between equations and other topics

7.5. Conclusion

In terms of correctness, learners responded to the three items as expected, with more correct responses in item 9a and fewer in item 9c. However, at the end of Grade 9, more than half the learners in my sample were unable to solve simple linear equations. Learners showed improvement from one test to the other, with the most improvement happening in Grade 10. In Grade 9, only 39% of learners improved the number of equation items they got correct, whereas in Grade 10, 53% increased the number of items they got correct in the post-test. Inferential statistics confirmed that there was a statistically significant increase in learners' mean scores in both grades but also confirmed that the correlation between the pre-and post-test equation scores was stronger in Grade 10.

The error analysis showed that the most common approach learners use to solve an equation was to do so algebraically with equations. The second most common approach was to convert the equation to an expression for items 9b and 9c. In addition, in the pre-test, the most common view of a solution was as a result, meaning that the solution was a consequence of doing various operations with or on letters and numbers. This shifted to viewing a solution as a value that creates a statement that is true. The majority of learners also continued to have an operational view of the equal sign. An important finding was that the majority of learners did not use inverse operations to solve equations, but when they did, it was done incorrectly. Dealing with negative numbers appeared not to be a major struggle when solving equations as they were seldom used. When used, however, they were only a stumbling block in item 9c, which was expected due to the nature of the item. Conjoining dramatically decreased in Grade 10. These results suggest that, in the context of solving equations, learners are less prone to making the type of errors they typically make when simplifying an expression because there are fewer conjoining and negative errors.

In Figure 7.2 we saw that there was a change in the percentage of correct responses within each grade. However, throughout this chapter we saw that there was in fact very little change in the approach used, the view of the equal sign, the view of a solution or even with the errors made with negatives, letters and inverses. There was however a change from Grade 9 to Grade 10 in the approach used, the view of the equal sign, the view of a solution and use of inverses.

I have shown that learners are making multiple errors that involve many things, such as negatives, conjoining, inverses, approaching the equation arithmetically and converting to an expression. There is not a large percentage of learners doing any one of those errors, but rather, all of them. I assert that learners do not actually know what an equation is and hence solving it is just a random set of actions that resemble adding, subtracting and sometimes dividing, but learners seem unsure of when they need to perform various actions. It appears as though they are not sure as to when they need to find a value for x, as in $x = k$, when they need to simplify, or when they need to substitute numeric values. Hence, when solving an equation, they do everything and anything. This points towards learners not having enough time or practice to master the content and highlights the need for a theoretical conceptualisation that accounts for the lag-time in learning.

Chapter 8 : Analysis of responses to numeric equations

8.1. Introduction

In this chapter, I discuss learners' responses to two numeric equations. Analysing them contributed to my understanding of learners' views of the equal sign. This chapter then links with the equations chapter in that in Lens 2, I investigated learners' views of the equal sign in the context of equations. The difference between an algebraic equation and a numeric equation is that a numeric equation is an equation that consists only of numbers and an unknown. Typically, the unknown number is represented by using a box, a question mark or a blank space, for example $9 + 5 = \_ + 3$. I present both quantitative and qualitative results, which together give a better picture of how learners perform when answering these types of questions. The quantitative analysis provides an overview of learners' performance and the qualitative analysis is a deeper analysis as to why learners performed the way they did. The qualitative analysis is concerned with the errors made when filling in the missing value of the two numeric equations. As a reminder, the items are listed below and I refer to them as the *equality* items because they were designed to test learners' view of equality as well as their ability to perceive structure in the relationships between the values.

The items are as follows:

1) Item 4a: $7 + 5 = \_ + 2$
2) Item 4b: $4747 + 3945 = \_ + 3943$

Item 4b is structured the same as item 4a but contains larger values, intending for learners to approach the item from a relational-structural point of view (Jones et al., 2012). This is where, for example with $4747 + 3945 = \_ + 3943$, the number 3945 decreases by two units to get 3943 and therefore, 4747 should increase by two units so that the equality is preserved, hence the 4749 goes in the blank space.

This analysis is organised into two parts. Firstly, I begin by conducting a quantitative analysis and discuss the overall performance on the two equality items. The second part is the qualitative analysis of responses to item 4a and 4b. I discuss the responses and present a response mapping of each item. This is where I track a learner's response to see how their responses changed in the post-test. As a reminder, the sample consists of 150 randomly selected scripts from both Grade 9 and 10 learners.

I argue that despite statistically significant results in Grade 10, there was very little shift in improvement between each test for both grades for item 4a. I also show that in the post-test for item 4b of Grade 10, there is a big jump in correct responses, which suggests that something changed in Grade 10 for more learners to treat the equal sign relationally.

8.2. Quantitative analysis

In this section I analyse the percentage of correct responses and provide a response pattern analysis and a statistical analysis including T-tests and correlations. The results of these analyses highlight that many learners continue to get numeric equations incorrect, and few improve from pre to post-test. This section also highlights that although there is statistical significance in the change in means from pre to post-tests, the relationship of the significance is weak in Grade 9 and stronger in Grade 10.

8.2.1. Correct responses

Figures 8.1 and 8.2 show the percentage of correct responses to items 4a and 4b.

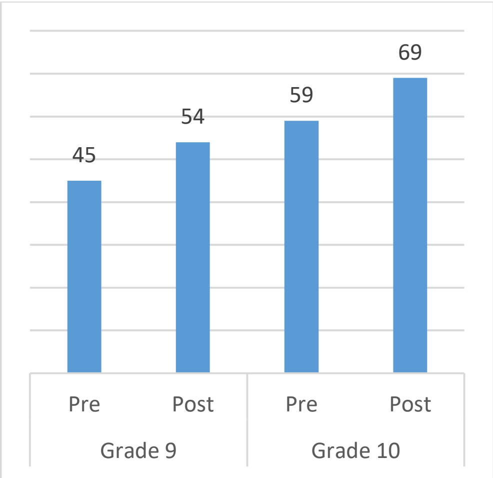

Figure 8.1: Percentage of correct responses to item 4a

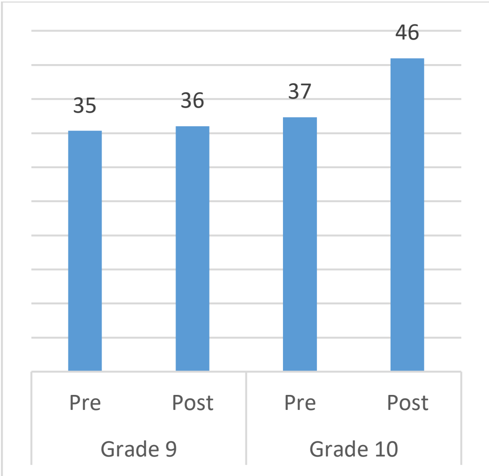

Figure 8.2: Percentage of correct responses to item 4b

Looking across the two years, from the Grade 9 pre-test to the Grade 10 post-test, there was a 53% increase in correct responses to item 4a and a 31% increase in item 4b. In item 4a, the increase was gradual but in item 4b, the increase was from the Grade 10 pre-test to the Grade 10 post-test.

An item of the type $7 + 5 = \_ + 2$ (item 4a) is first encountered in primary school and hence it is expected that Grade 9 learners have mastered such numeric equations. Based on the quantitative results, with only 54% of learners at the end of Grade 9 getting the item correct, learners have clearly not mastered dealing with simple numeric equations. Despite the 53% increase from the Grade 9 pre-test to the Grade 10 post-test, it is worrying that by the end of Grade 10, only 69% of learners can correctly answer $7 + 5 = \_ + 2$. Although the increase is encouraging, it is happening too late and should have been mastered in primary school, at least three years before. Item 4b was structured the same as item 4a but contained larger numbers, which could explain the lower percentage of correct responses. This suggests that at Grade 10 level, learners are still struggling with adding and subtracting whole numbers but also do not have a relational-structural view of the equal sign.

8.2.2. Little change in getting items correct from pre- to post-tests

The aim of the response pattern analysis is to do a quick analysis visually. Seeing colour representing correct and incorrect responses is more salient than ordering values in a table. It provides the reader with a visual representation of how the learners responded to two items at a single time. The response patterns show how learners responded to items 4a and 4b in relation to the pre- and post-test as well as in relation to each other. The two response patterns provided here are a snap shot of the learners in Grade 9 (Figure 8.3) and Grade 10 (Figure 8.4) who did not change in the number of items they got incorrect in the post-test. This means that for those who got the item wrong in the pre-test, they continued to get the item wrong in the post-test, ultimately showing no improvement in the post-test. What we see in both response patterns was that it is generally the same item that learners got incorrect. In Grade 9, 60% of the sample did not show an improvement and in Grade 10, 56% showed no improvement in the number of items they got correct.

What this analysis has highlighted is that there is a large percentage of learners who got both items incorrect but more importantly, that a small percentage of learners improved in their responses from pre- to post test, suggesting that very little learning took place with regards to dealing with numeric equations. Although this is not a topic covered in Grade 9 or Grade 10, it was expected that it would be mastered in primary school.

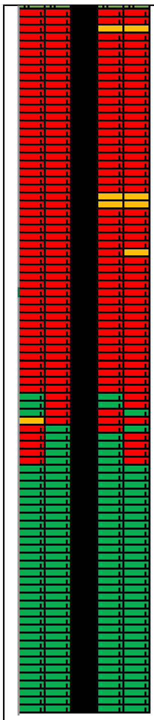

Figure 8.3: Grade 9 learners who did not change the number of items they got correct in the post-test

Figure 8.4: Grade 10 learners who did not change the number of items they got correct in the post-test

8.2.3. Equality items are more of a predictor of overall test performance in Grade 10

A paired T-test on the Grade 9 data revealed that there was no statistically significant improvement from the pre-test scores ($M = 42\%, SD = 43$) to the post-test scores ($M = 46\%, SD = 44$) of items 4a and 4b, $t(149) = -2, p > 0.05.$ A paired T-test on the Grade 10 data, however, revealed a statistically significant improvement from the pre-test scores ($M = 48\%, SD = 43$ to the post-test scores ($M = 57\%, SD = 41$) of items 4a and 4b, $t(149) = -2.8, p < 0.005$. The effect size (Cohen's d) is small, $d = 0.2$, meaning that although statically significant, the difference is trivial.

The correlation between learners' equality scores (out of 2) and learners' test score (out of 18) was conducted. The results show a positive correlation of $r = 0.6$ between the equality scores and the overall test scores in the pre-test and a weaker correlation of $r = 0.5$ for the post-test. Although the correlation for the pre-test was stronger, it is too weak to be a predictor of the test scores. This means that the equality scores are not a predictor of learners' performance in the test consisting of integers, expressions, equality items and equations. In Grade 10, there was a stronger positive correlation in both the pre- and post-test ($r = 0.6$ in the pre-test and $r = 0.7$). This suggests that the equality items are more of a predictor of learner performance when learners are in Grade 10, however it is important to note the limitation of using only two items as predictors.

The quantitative analysis has revealed three important issues: firstly, that by the end of Grade 10, learners still do not have a relational view of the equal sign when answering simple numeric equations. Evidence for this is in the low percentage of learners who can correctly answer the item. Secondly, the response patterns highlight that few learners improved their result from the pre- to post-test. Lastly, the quantitative analysis has revealed that there were no statistically significant results in Grade 9, but there were statistically significant results in Grade 10. The stronger correlation in Grade 10 suggests that the Grade 10 performance is more of a predictor of learner performance than the Grade 9 performance.

8.3. Qualitative analysis of responses to item 4a and 4b

In this section I analyse the responses learners gave to the two items. I first look at the common incorrect responses and then relate that to learners' relational and operational view of the equal sign. I end by providing a response map for each item.

8.3.1. Large variety of incorrect responses

In the pre-test of Grade 9, there were 13 different responses to item 4a $(7 + 5 =? +2)$, which decreased to nine different responses in the post-test. In Grade 10, there were also 13 different responses but only nine of them were the same as those in Grade 9. In the post-test, the number of different responses decreased to eight, with five of them being the same as those in Grade 9. The variety of responses is further evidence of learners not knowing how to approach these types of equations.

In both Grade 9 and Grade 10, the most common incorrect responses to item 4a in the pre- and post-test were 5 and 12, ie $7 + 5 = \mathbf{5} + 2$ and $7 + 5 = \mathbf{12} + 2$. Figure 8.5 shows the percentages of each response for each of the Grade 9 and 10 tests. There was a 28% drop in the response 12 from the Grade 9 pre-test to post-test. This suggests that this error starts to drop in Grade 9. The question is about ensuring that the value in the blank space makes the left and right side equal, but learners appear have to read the question as calculating the sum of 7 and 5. Obtaining the answer 5 or 12 is suggestive of an operational view because learners work from left to right rather than working in both directions. Together, obtaining the answers 5 and 12 amounts to 43%; 37%; 29% and 24% for Grade 9 pre- and post-test and Grade 10 pre- and post-test respectively.

A correct answer to this item is suggestive of a relational view of the equal sign. Interestingly, there was a 9p.p. drop in providing 12 as the answer in Grade 9 but there was also a 9p.p increase in the correct responses (see Figure 8.5). This suggests that the same learners who operated operationally went to working relationally in the Grade 9 post-test.

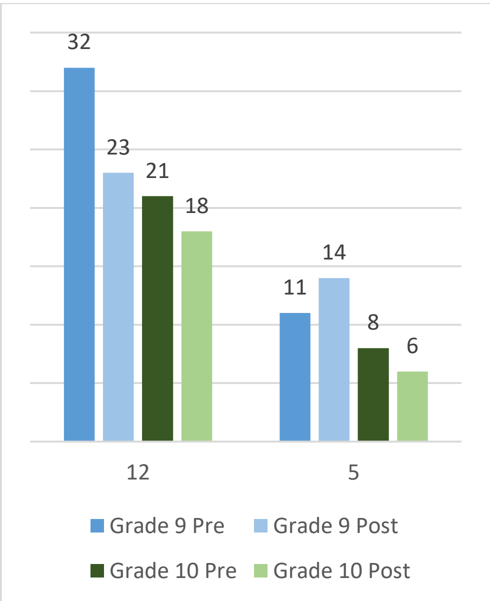

Figure 8.5: Percentage of the most common incorrect responses to item 4a

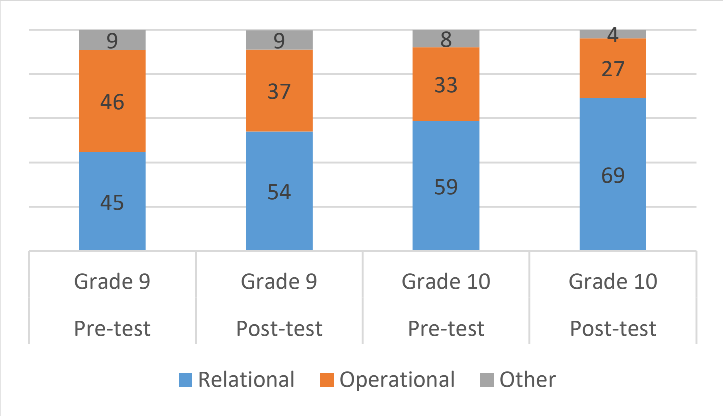

Figure 8.6: Grade 9 and 10 percentage of codes for item 4a

Figure 8.6 shows the percentage of responses per code for item 4a. We see that as many learners responded relationally as did operationally in the pre-test (45% in the pre-test and 46% in the post-test). In Grade 10 the relational category increased by only 5p.p. However, by the end of Grade 10, 67% of learners appeared to be answering relationally while only 29% answered operationally.

Item 4b ($4747 + 3945 = \_ + 3943$) had a larger variety of different responses (58 different responses across the two tests in Grade 9 and 64 in Grade 10) than what was in item 4a. This large variety is possibly due to the use of larger numbers. The most common responses in Grade 9 were 4747; 4749 (correct); and 8692. In Grade 10, however, the most common responses were 4749 (correct); 8692; and surprisingly, a blank response. Since the most common responses were not the same in both grades, I have included the combined most common responses in Figure 8.5.

The answer of 4747 was a surprising one, especially since it was one of the most common responses in Grade 9. It appears that this response is just a repeat of the question as learners possibly just wrote down the same number. This response dissipated in Grade 10 from 12% to 5% but a blank response then become more common in Grade 10, rising from 1% in the Grade 9 pre-test to 9% in the Grade 10 pre-test. In the Grade 9 pre- and post-test as well as the Grade 10 pre-test, the percentages of learners obtaining the correct answer were very similar, with not much change. However, in the Grade 10 post-test, there was a spike in correct responses, from 37% to 46%. This suggests that learners were working more relationally by the end of Grade 10. Learners getting item 4b correct is a stronger indication of working relationally. Although there was a big jump in percentages in correct responses at the end of Grade 10, there was no a big change in the operational responses. From the beginning of Grade 9 to the end of Grade 10, there was only a 3p.p. difference. See Figure 8.7. Figure 8.8 shows the percentage of responses coded for item 4b, which shows the complete category of operational responses rather than just the response 8692. There was little change from pre- to post-test responses that were relational and the percentage of operational responses stayed the same but more than 10p.p. lower than the relational response. The *other* category was a common category and consisted of a variant of responses that could not be explained. However, as mentioned in the methods chapter, with the values being added being large, with four digits each, and with needing to do an addition and a subtraction without a calculator, there are so many places a learner could go wrong.

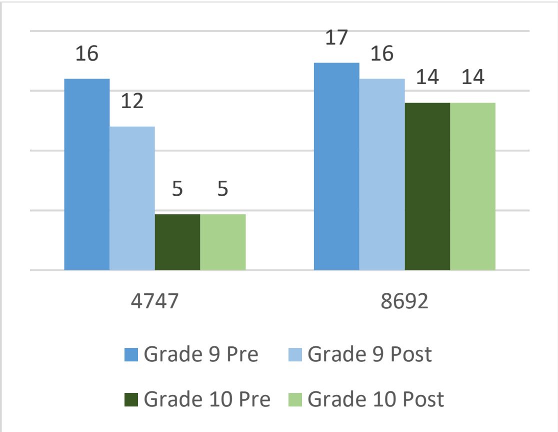

Figure 8.7: Percentage of the most common incorrect responses to item 4b

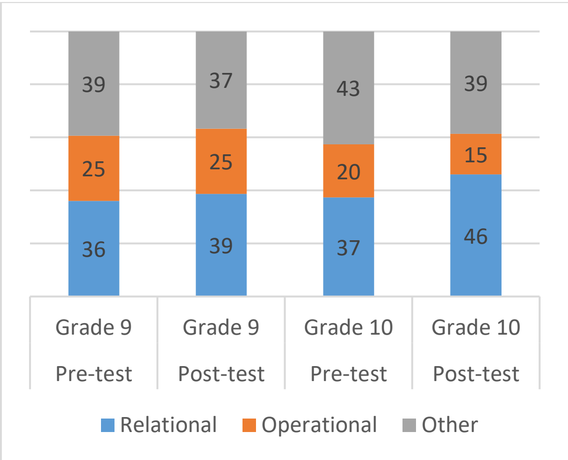

Figure 8.8: Grade 9 and 10 codes for item 4b

The difference in performance of the two items suggests that learners struggled with larger numbers. Learners could have used the vertical column method (instead of number facts) for item 4b seeing the values were larger. The high percentage of incorrect responses therefore would suggest that learners still make errors with adding and subtracting large numbers. It also suggests that the learners are not able to work with the items relationally or structurally.

8.3.2. Learners in a certain category (relations; operational or other) tended to remain in that category in the post-tests.

The figures below are what I have termed 'response maps'. I have tracked learners' responses from the pre- to post-test. There are four response maps, one for item 4a and another for the item 4b for Grades 9 and 10. There are two columns, one for the pre-test and another for the post-test responses. In each column are the three categories of codes that I assigned to each response: relational, operational or other. The lines that go from the pre-test to the post-test show the number (and percentage) of learners' responses that changed in the post-test. The bold lines show where the majority of the responses changed.

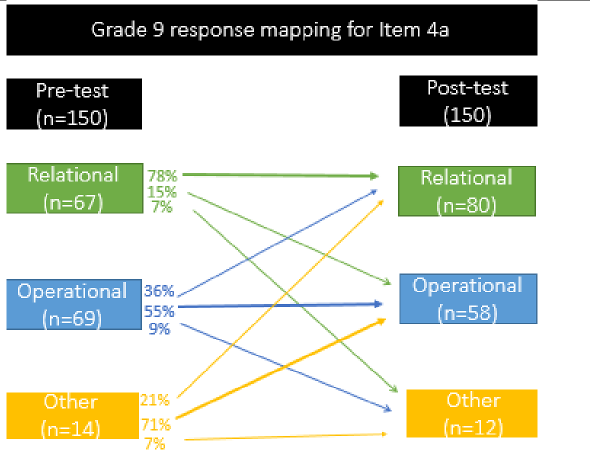

Figure 8.9: Grade 9 response mas for item 4a

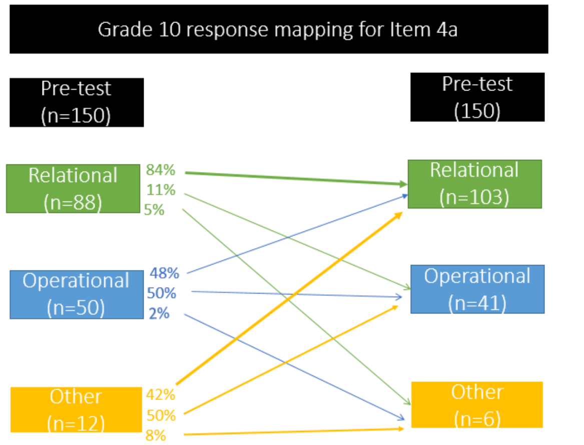

Figure 8.10: Grade 10 response map for item 4a

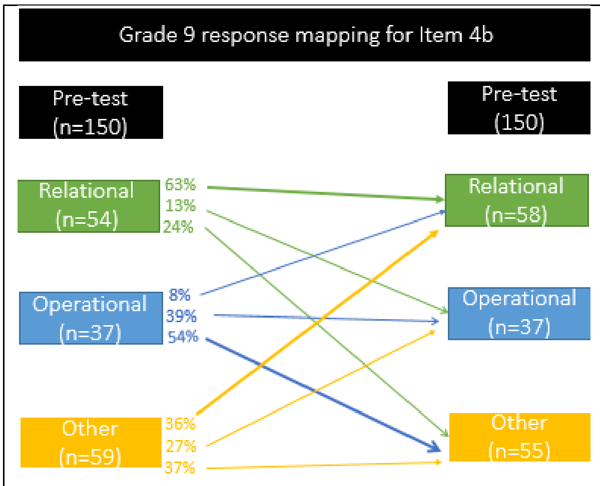

Figure 8.11: Grade 9 response map for item 4b

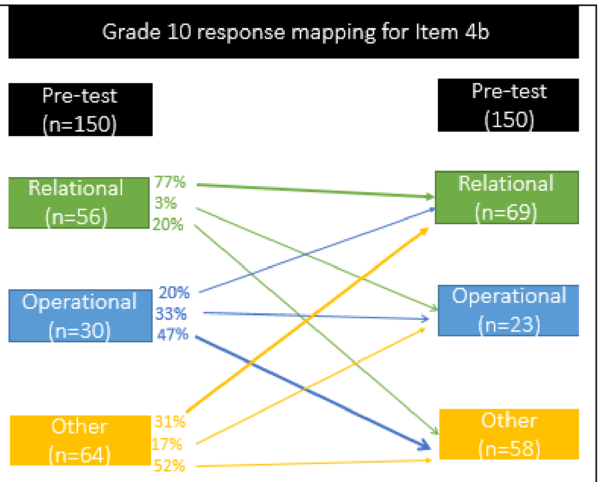

Figure 8.12: Grade 10 response map for item 4b

In Grade 9, for item 4a, 78% of the learners who gave a relational response in item 4a also gave a relational response in the post-test, meaning they stayed in the relational category. This suggests that their knowledge is stable. In Grade 10, the percentage is slightly higher at 84%. However, 22% regressed in Grade 9 and 16% in Grade 10, and either gave an operational response or an 'other' response. Interestingly, 71% of learners in Grade 9 who

gave an 'other' response in the pre-test, responded relationally in the post-test. Only 36% of learners who worked operationally improved to a relational response in the post-test in Grade 9, and 48% did so in Grade 10. The majority of learners in Grade 9 (55%) who gave an operational response, continued to give an operational response in the post-test. In Grade 10 this percentage decreased slightly to 50%. This means that 23% of Grade 9 learners and 19% of Grade 10 learners improved from the pre- to the post-test. In Grade 9, the 40 learners who remained in the same category (excluding those who remained in the relational category) and the 11 learners who regressed are learners that did not improve during the year. In Grade 10, 19% of learners remained in the same category (excluding the relational category) and 10% regressed and hence did not improve.
Overall, the response mappings for Grade 9 and 10 are very similar. The biggest difference between Grade 9 and Grade 10 is that there were more Grade 10 learners who remained in the relational category and more learners who moved from the operational to relational category than in Grade 9.

For item 4b, the response mappings for Grade 9 and Grade 10 are also very similar in terms of patterns of percentages. There is a 14p.p. increase in the percentage of learners who remained in the relational category (from 63% in Grade 9 to 77% in Grade 10). There was also a 16p.p. increase in the percentage of learners who moved to the relational category in the post-test (from 8% in Grade 9 to 20% in Grade 10). Overall, in Grade 10, 25% of learners improved categories and 10% regressed.

Learners who have responded with a relational response for both item 4a and 4b are more likely to have a relational understanding of the equal sign. However, only 20% of Grade 9 and 27% of Grade 10 learners consistently gave a relational response to both item and in both tests. Surprisingly, only 8% of Grade 9 learners and 5% of Grade 10 learners consistently responded with an operational response. I expected more since this category was always relatively large.

More learners in Grade 10 were working relationally echoes McNeil and Alibali (2005), who claim that one's knowledge of the equal sign changes as a function of experience in mathematics and variation in contents. Grade 10s would have been exposed to various equations -- linear, quadratic, exponential and trigonometric -- and this increased exposure and practice could have resulted in an improved way of thinking of equality.

8.4. Equality performance in relation to equation performance

In terms of the quantitative analysis, there were only weak correlations between learners' performance in the equality items and their performance in the equation items. This means that learners' equality performance was not a predictor of equation performance.

In Chapter 7, I did a qualitative analysis of the equation items. I analysed learner solutions from five points of view (five lenses). One of the lenses was the view of the equal sign (lens 2). The difference in coding was that in the equation items, I was able to code for relational, pre-relational, operational, pre-operational and other. However, in the equality items, because the learners provided answers only rather than more detailed responses, I could only code for operational, relational and other responses.

The analysis on the views of the equal sign on three equation items revealed that the majority of Grade 9 learners do not have a relational view of the equal sign. This concurs with literature, which suggests that many learners in Grade 9 view the equal sign operationally. In the Grade 9 pre-test, half the sample (51%) attempted to solve $3x - 2 = 10$ operationally, meaning that in trying to isolate x they used the equal sign as an "and then I did this" symbol. This was the most common view in all three items, but also the most common view in the post-test. In the analysis of the Grade 9 equality items, 43% of learners responded with an answer that suggested an operational view of the equal sign.

By grouping the pre-relational and relational responses as well as the pre-operational and operational responses, I was able to compare responses across items. The percentage of learners who responded relationally to both items 9a and 4a in the Grade 9 pre-test was calculated. Only 33% responded relationally to both items and even fewer for items 9b and 4a; and 9c and 4a. This suggests that a third of the learners responded relationally in the two items. When looking at operational responses, however, 41%; 48% and 39% responded with an answer that was suggestive of an operational view of the equal sign for items 9a and 4a; 9b and 4a; and 9c and 4 respectively. The relationships were even weaker in the post-test. In the Grade 10 pre- and post-tests, 81%; 80% and 77% of learners responded relationally to items 4a and 9a; 4a and 9b; and 4a and 9c respectively. In terms of operational responses, 38%; 44%; and 46% responded operationally to both items 4a and 9a; 4a and 9b; and 4a and 9c respectively. These results suggest that it is only in Grade 10 that learners respond in the same way across topics.

8.5. Correlations between equality items and other topics

In Grade 9, the strongest relationship between topics was between the pre- and post-test equality items ($r = 0.6$) The weakest correlations were between expressions in the pre- and post-test and equality in the pre- and post-test ($r = 0.2$). What this suggests is that improving in expressions did not influence learners' equality responses.

The biggest difference between the Grade 9 and Grade 10 correlations was that the relationship between the equality scores and the expression scores improved by 0.2 in the pre-test comparison and by 0.3 in the post-test comparison. This suggests that in Grade 10, learners' responses to equality items could be influencing their responses to expression items. The fact that most of the correlations were below $r = 0.6$ does suggest that learners' responses to the equality items had no bearing on the other topics. Perhaps this is because there were only two equality items.

Grade 9		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equality	0,4	1,0	0,5	0,2	0,4	0,6	0,2	0,3
Post-test	Equality	0,3	0,6	0,4	0,2	0,3	1,0	0,3	0,2

Table 8.1: Grade 9 correlations between equality and other topics

Grade 10		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equality	0,4	1,0	0,4	0,4	0,3	0,5	0,4	0,5
Post-test	Equality	0,4	0,5	0,4	0,5	0,5	1,0	0,5	0,5

Table 8.2: Grade 10 correlations between equality and other topics

8.6. Conclusion

This piece of analysis was based on the responses to two equality items. I presented two main arguments related to the fact that few learners showed an improvement in terms of the number of items they got correct in the post-test and secondly, that in Grade 10 learners *started* to work relationally.

The quantitative results show that by the end of Grade 9, only 54% of learners were able to correctly answer item 4a: $7 + 5 = \_ + 2$. However, by the end of Grade 10, this increased to 70%. It also showed that less than 50% of Grade 9 and 10 learners could correctly answer item 4b: $4747 + 3945 = \_ + 3943$. T-tests reveal that although statistically significant in Grade 10, the difference in means was trivial. In addition, the correlation between the overall equality performance and the equation performance was weak in both Grades 9 and 10. These results are influenced by the fact that there were only two items.

In the qualitative analysis, responses were coded and assigned to one of three categories: relational, operational or other. Findings reveal that there were many 'other' responses in answering item 4b; this could be as a result of having large numbers to deal with without a calculator. The most common error made in Grades 9 and 10 in both the pre- and post-test for item 4a, was 5 and 12, both responses reveal an operational view of the equal sign. For item 4b, the most common error was 8692 which is also indicative of an operational view of the equal sign. Although it was expected that this would be the case, what is surprising is that on average there are 41% of Grade 9 learners and 30% of Grade 10 learners who are working operationally with a basic numeric equation. Even more surprising is the percentage of learners who remain consistent in answering operationally: 55% of Grade 9 learners and 50% of Grade 10 learners for item 4a, and 39% for Grade 9 and 33% of Grade 10 learners for item 4b.

In relation to the equation items, results showed that there was a strong relationship between relational responses across topics and items. Literature has suggested that one's view of the equal sign when dealing with arithmetic sentences influences one's equation-solving abilities, but I found that the correlation between equality and equations was weak.

Chapter 9 : Analysis of responses to integer items

9.1. Introduction

This section of analysis is about learner responses to the six integer items. This analysis is important for my study because it allows me to compare the results with the results from the analysis of equation items where learners made errors relating to negatives (See Chapter 7). The analysis is presented in two parts: a quantitative analysis of the overall performance of integer items, and a qualitative analysis consisting of an error analysis, with an error map for some items. As mentioned previously, the sample consists of 150 Grade 9 and 10 learners whose responses to items in a pre-test as well as a post-test were analysed. I begin by reminding the reader of the six items and of the codes used for analysis. A detailed discussion of the codes used and justification for why they were used appears in Chapter 5.

The quantitative and qualitative results, together, give the reader a better picture of how learners perform when answering integer questions. The quantitative analysis gives an overview of learners' performance and the qualitative analysis gives a deeper analysis as to why learners performed the way they did. The qualitative analysis focuses on an error analysis and reveals the most prominent errors as well as the shifts in errors made.

In this chapter I suggest that learners are not consistent in how they view the minus symbol. The concurrent triangulation design of the analysis was used to determine whether similar results are found with both methodologies. The qualitative results do support the quantitative results in that we see a general decrease in the errors made across the four tests. Doing both analyses was also beneficial as it enabled me to see that there was improvement despite the different types of errors made. Although there was an encouraging improvement in Grade 10, I argue that the improvement is happening too late. Bishop, Lamb, Philipp, Whitacre & Schappelle (2014, p. 49) interview a learner who says that "negative numbers aren't really numbers.… they're just acting like other numbers except there is a minus in front of them…" This view of negative numbers explains my findings in the qualitative analysis. The analysis showed that learners detach the minus symbol and avoid multiple occurrences of it. It is possible that when learners detach the minus symbol it is because they deal with the numbers without the minus symbol, holding the symbol in their head, only to reattach it once they have operated on the positive numbers.

As a reminder the six items:

1) Item 1: Arrange from smallest to largest: 30; −35; −2; −500; −10; 4
2) Item 3a: $5 - 7$
3) Item 3b: $5 + (-7)$
4) Item 3c: $-7 + 5$
5) Item 3d: $6 - (-10)$
6) Item 3e: $-7 - 5$

9.2. Quantitative analysis of responses to integer items

This section discusses learner performance on the integer items in terms of the items as a group, by comparing groups and by looking at the items individually. I use T-tests and a response pattern analysis to substantiate my claims about the differences in the performance, and show that overall learner performance improves in integer items more in Grade 10 than in Grade 9. In addition, I show that when learners get items incorrect/correct in the pre-test, they are not necessarily the same items they got incorrect/correct in the post-test, suggesting that learners' knowledge is not consistent.

9.2.1. Overall performance on the integer items

The overall performance on the integer items is better in Grade 10 than in Grade 9. From the boxplot in Figure 9.1, we can see this better performance in three ways. Firstly, in both the Grade 9 pre- and post-tests, $\frac{3}{4}$ of the learners obtained less than 67% for the items as a group but in Grade 10, half the learners in the pre-test and $\frac{3}{4}$ of the learners in the post-test got above 50% for the integer items. Secondly, although the mean increases in every test, in Grade 10 it shows a large increase from 52% to 74%. Lastly, the median also increases in each test but shows a much larger increase (from 50% to 83%) in the Grade 10 post-test. The boxplot also shows two interesting features where there is no change in quartiles. In Grade 9, the upper quartile remains at 67% in both the pre and post-test, suggesting that for the top 25% of learners there was no improvement in Grade 9. The lower quartile of the Grade 9 post-test and the Grade 10 pre-test also remains the same at 33%, suggesting that the bottom 25% of learners in each of the grades did not improve.

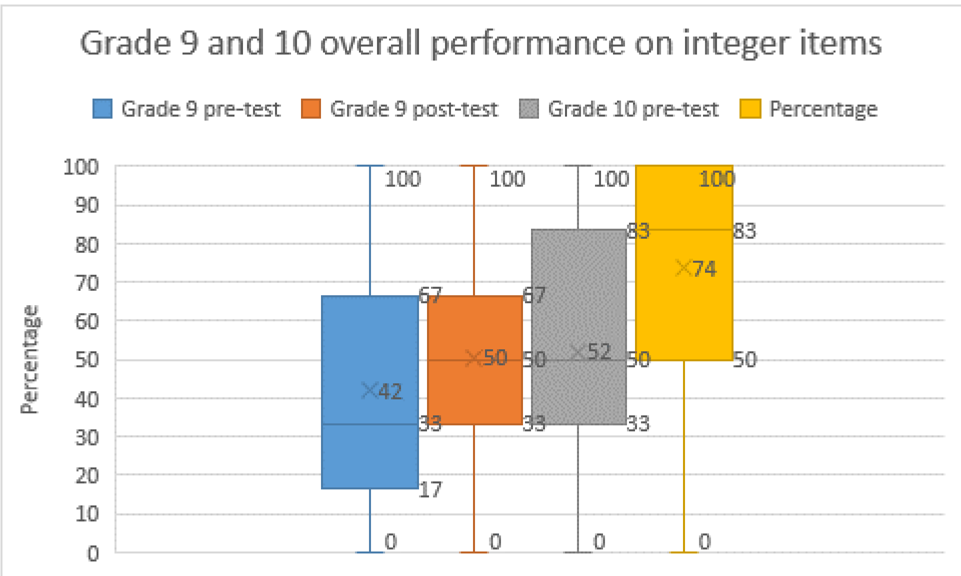

Figure 9.1: Grade 9 and 10 learner performance on integer items

9.2.2. Comparison of learners' performance in the integer items

A paired T-test was conducted to compare Grade 9 learners' performance in integer items in the pre-test in and learners' performance in integer items in the post-test. There was a significant difference in the scores for learners' performance in integer items in the pre-test (M= 42, SD=29) and post-test (M=50, SD=28); $t(149) = -3.4,\ p < 0.0005$. These results suggest that the increase in the post-test scores was not due to chance. Cohen's d was used to calculate the effect size because it provides a measure of effect size that is weighted according to paired data and hence equal sample sizes. The effect size was small ($d = 0.28$), suggesting that even though there was a statistically significant improvement, the strength of the improvement was weak. Another paired T-test was conducted on the Grade 10 data and it was found that there was a significant difference in the scores for learners' performance in integer items in the pre-test (M=52, SD=29) and learners' performance in integer items in the post-test (M=74, SD=28); $t(149) = -9,\ p < 0.0005$. These results suggest that the increase in the post-test scores was not due to chance but due to learning. The effect size was large ($d = 0.77$), suggesting that the strength of the difference is strong. The statistics therefore tell us that the improvement we see in each subsequent test is a result of learning rather than by chance, but also that the difference in Grade 10 is greater than in Grade 9.

9.2.3. Learner performance on individual items

The claim made in this section concurs with that made in the previous section, where the greatest improvements appear to be in Grade 10 rather than Grade 9. This section discusses learner performance in each of the six items. In four of the six items, no more than 38% of learners could correctly answer the questions in the Grade 9 pre-test. Figure 9.2 shows the percentage of correct responses for the six items for Grade 9 and Grade 10's pre- and post-test. It shows an overlap between Grade 9 post-test and Grade 10 pre-test results, with very little difference between them. The figure also shows that there is a bigger increase in correct responses in Grade 10 than what there was in Grade 9. Although this was found in the analysis of the overall performance of integer items, we now see that the big increase in Grade 10 was due to the same four items, where learners obtained at most 38%. This is discussed in more detail in the sections that follow.

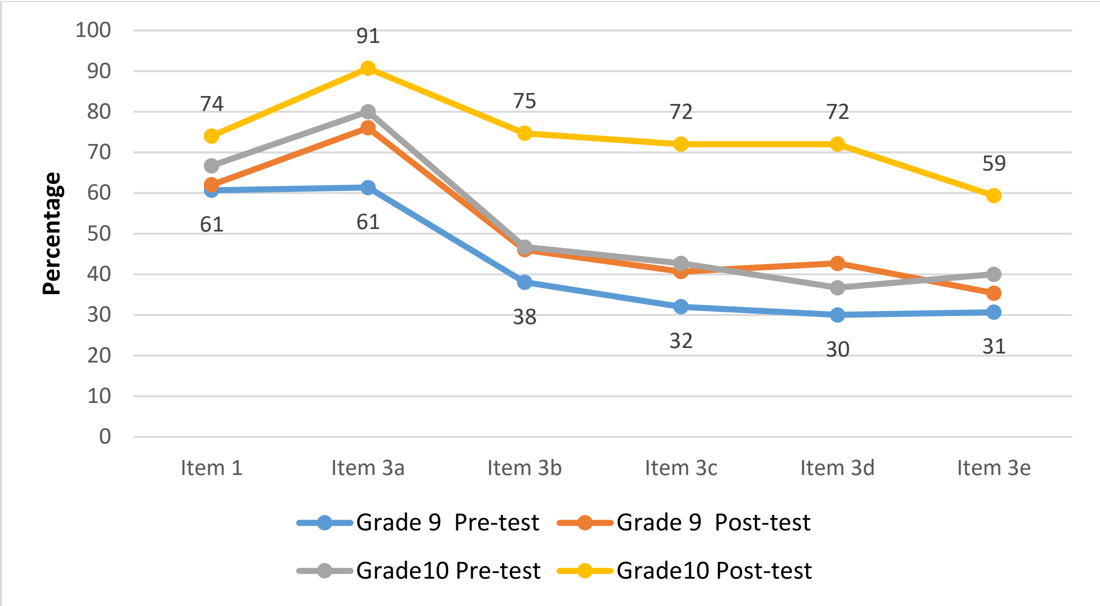

Figure 9.2: Grade 9 and 10 percentage of correct responses to integer items

The analysis of Figure 9.2 reveals the poor integer results and shows that the best- and worst-answered items remained the same in Grade 9 and in Grade 10. Although there are big improvements in Grade 10, we see that learners appear to have particular problems with items that involve brackets (items 3b and 3d) and those that have leading negatives (item 3c and 3e). Item 3b is a different way of writing item 3a and the decline in the percentage of correct responses (in both Grade 9 and Grade 10) further suggests that learners struggle with questions where there are brackets. Item 3d also had brackets and learners' performance was also weak in Grade 9 and in the Grade 10 pre-test. The percentage of correct responses however increased to 72% in the Grade 10 post-test. There was a big difference in the percentage correct between item 3a $(5 - 7)$ and item 3e: $(-7 - 5)$: with 91% in item 3a and 59% in item 3e in Grade 10 and 61% in item 3a and 31% in item 3e in Grade 9.

Where Figure 9.2 shows the average learner performance per item, a response pattern shows how every learner responds to the six integer items together. The response pattern analysis reveals that there are learners who got items incorrect in the post-test but got them correct in the pre-test. It also shows that learners who get the same number of items correct in both the pre- and post-tests, on the whole, got the same items correct in the post-test (see Figure 9.3 and 9.4).

The solid black line separates the pre-test from the post-test. Each column relates to an integer item, in the order 1; 3a; 3b; 3c; 3d and 3e. The green cells represent correct responses; the red cells represent the incorrect responses; and the yellow cells the blank responses.

There is a large percentage (45% in Grade 9 and 65% in Grade 10) of learners who improved, getting more items correct in the post-test. This concurs with the findings from the boxplot in Figure 9.1 as well as the bar graph in Figure 9.2. Although this was expected, it is concerning that 31% of Grade 9 and 20% of Grade 10 learners showed no improvement in the number of items they got correct in the post-test. What is more concerning is that although the number of items correct did not change, in 27% of the cases, the individual item/s that learners got correct did change. Figures 9.3 and 9.4 show the set of learners in Grade 9 and 10 respectively that had the same number of items correct in both the pre- and post-test. However, in Grade 9, 53% and in Grade 10, 37% of the cases show that the patterns are not the same. For example, in Grade 9 (Figure 9.3), if you look down the first column on the left side (pre-test side) of the black line you will see that learners who got that item (item 1) incorrect are at the top of the pile; but if you look down the first column on the right side of the black line, there are four instances where the responses to the items are different in the two tests. The decrease in the percentage of Grade 10 learners that changed their responses in the post-test suggests that Grade 10 learners' knowledge of integers is more stable because learners are more consistent in their response. A smaller percentage of learners, 24% of Grade 9 and 15% of Grade 10 learners, decreased in the number of items they got correct from the pre- to the post-test. Of the learners who regressed in the number of equation items they got correct in the post-test, 69% of learners in Grade 9 and 45% in Grade 10 also dropped in their overall post-test scores. This further suggests that learners' algebra knowledge, but in particular, their integer knowledge, is unstable.

These results highlight instances where learners were inconsistent in their responses and hence demonstrate their unstable integer knowledge. This suggests that the errors they make are also inconsistent. The errors learners make when answering integer items are discussed in the qualitative section.

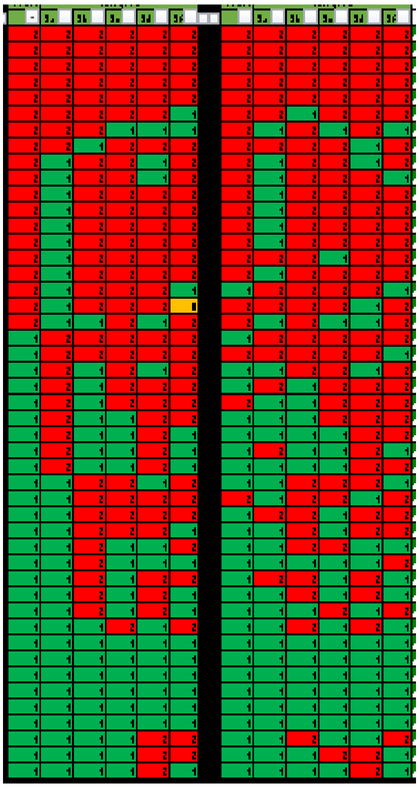

Figure 9.3: Grade 9 learners who did not change the number of items correctly

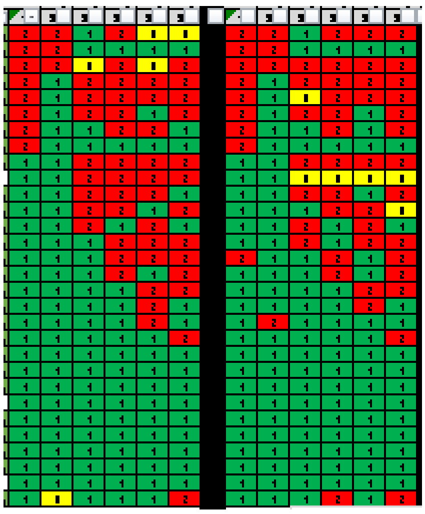

Figure 9.4: Grade 10 learners who did not change the number of items correctly

9.3. Qualitative analysis of responses to integer items

In coding the integer items, all answers were recorded on a spreadsheet because learners only gave single valued answers. Recoding them in this way made it possible to use the filter function and assign the same answers to the same code.

Having done a systematic analysis of all the data, I present the findings of the error analysis by means of themes and errors rather than by each item. I foreground the main findings. In this analysis, I treated item 1 as a theme on its own because it required sorting integers in ascending order rather than adding and subtracting of integers. The items therefore had different error codes and the findings are distinct from the other themes I will discuss. I begin this section by giving the findings of item 1, its own theme, and then continue to discuss the errors made in the other five items by discussing avoiding the minus symbol, detaching the minus symbol, right to left reasoning, the misapplication of the signs rule, and using the wrong operation (multiplication).

It is important to note upfront that some items do not lend themselves to certain errors, whereas some do, for example $-7-5$ does not lend itself to right to left reasoning because of the leading negative but it does lend itself to the detachment error. The item 5-7 lends itself to right to left reasoning but not to the detachment error. Consequently, learners' errors in negativity tend to be closely related to the structure of the item.

9.3.1. Becoming aware of the cardinality of numbers

The findings from item 1 were that there were two main errors made: an absolute-value error and a negatives-reversed error. The absolute-value error is where the learners order the integers as though they are all positive values (for example, 1;-2; 3;-4). They ignore the cardinality of the numbers, hence ignoring the minus symbol in front of the numerals. The negatives-reversed error is where the learners order the values with the negatives on the left hand side of the positives but they place them in the reverse order to what was asked. In other words, this error is a consequence of separating the positives from the negatives and then treating the negatives as whole numbers (for example, when asked to rank the numbers from smallest to largest, learners give -2; -4; 1; 3). In the Grade 9 pre- and post-test, there was not much difference in the percentages between the two errors. However, there was a big drop in percentage in Grade 10 where the absolute value error decreased from 17% in Grade 9 to 5% in Grade 10, but the negatives-reversed error increased from 15% in Grade 9 to 20% in Grade 10. The difference in responses from Grade 9 to Grade 10 suggests that learners are more aware of the cardinality of numbers, but ordering negative numbers is still a difficulty for some.

A reminder of the codes used for items 3a-3e

Code	Brief explanation	Example
Avoidance of the minus symbol	Ignoring the minus symbol, not using it or operating with it.	$-7+5 \rightarrow 12$
Detachment	Detaching the minus symbol and operating on what remains, then reattaching the minus symbol	$-7+5 \rightarrow -12$

Right to left reasoning	Perform operations from right to left rather than left to right	$5-7 \rightarrow 2$
Signs rule as sign and operation	Using the signs rule to provide the sign and operation that needs to be performed	$-7+5 \rightarrow -2$
Wrong operation: multiplication	Multiplying the two numbers rather than adding or subtracting them	$-7+5 \rightarrow -35$
Other	None of the above	$-7+5 \rightarrow -1$

Table 9.1: Codes used for integer items 3a-3e

9.3.2. Avoiding the minus symbol

Avoiding the minus symbol was an error that occurred in all five items, but was least common in item 3a, most likely because there was only one operation sign. The error was most common when there were two minus symbols, for example in items 3d and 3e. An example of this error is $5+(-7) \rightarrow 12$ where the negative in front of the 7 is ignored and -7 is treated as a positive number. In item 3d, an example of this error is $6-(-10) \rightarrow -4$ where one of the minus symbols is ignored, or, in other words there were too many minus symbols and so they treated the two as one minus symbol. Similarly, in item 3e, $-7-5$ was given the answer of 2 because the leading negative was ignored. This error gradually decreased in Grade 9 and in Grade 10. Of the Grade 9 learners who got the item incorrect, 33% shifted their response from avoiding the minus symbol to a correct response; 33% however continued to make the same mistake in the post-test. In item 3b of Grade 10, 53% of the learners who avoided the minus symbol went on to getting the item correct in the post-test and 21% continued to make the same mistake (see Figure 9.5). In item 3d, learners ignore the negative signs altogether or combine the multiple negative signs to be a single subtraction. This error was made by 29% of the Grade 9 learners and this decreased to 21% in the post-test.

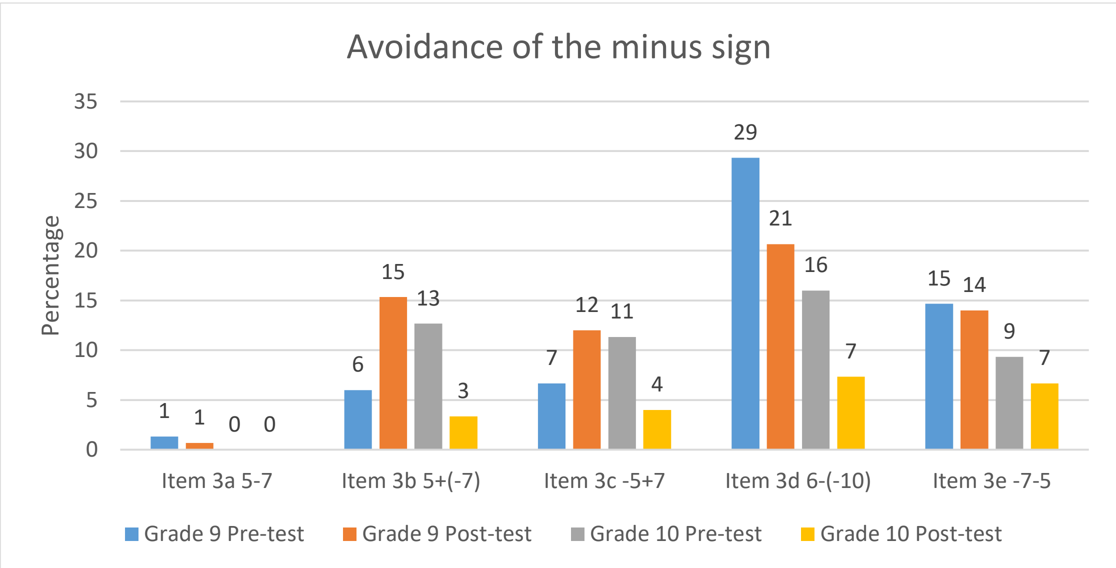

Figure 9.5: Avoidance of the minus symbol per item

9.3.3. Detaching the minus symbol

Detaching the minus symbol is another error that occurred in all five items. It was the most common error in items 3b, 3c and 3e. In item 3b of the Grade 9 pre-test, 51% of the integer errors were from detaching the minus symbol and in item 3b of the Grade 9 pre-test, 71%. These high percentages were double-checked. In both item 3b and 3c, the percentages dropped by 36p.p. and 48p.p. respectively at the end of Grade 9. In Grade 10 the error continued to decrease. It was surprising that item 3d did not have a high Grade 9 pre-test percentage because it, as in item 3b, had two operation signs next to each other. This could therefore suggest that it is when there are two different signs in an expression that learners are more likely to detach the minus symbol but that when the signs are the same, and in particular negative, then learners ignore the one.

In looking at the shift from the Grade 9 pre-test to post-test, 41% of learners who made the detachment error shifted to getting the item correct in the post-test. However, 21% continued to detach the signs from the numerals and 16% shifted their error to avoiding the minus symbol (see Figure 9.7). Figure 9.7 only shows two of the errors made as these were the ones that changed the most. The difference in the Grade 10 responses (see Figure 9.8) is that there is a big decrease in the number of instances where learners made the detachment error. In addition, 67% of learners who detached the minus symbol shifted their response to a correct response and only 11% continued to make the same mistake.

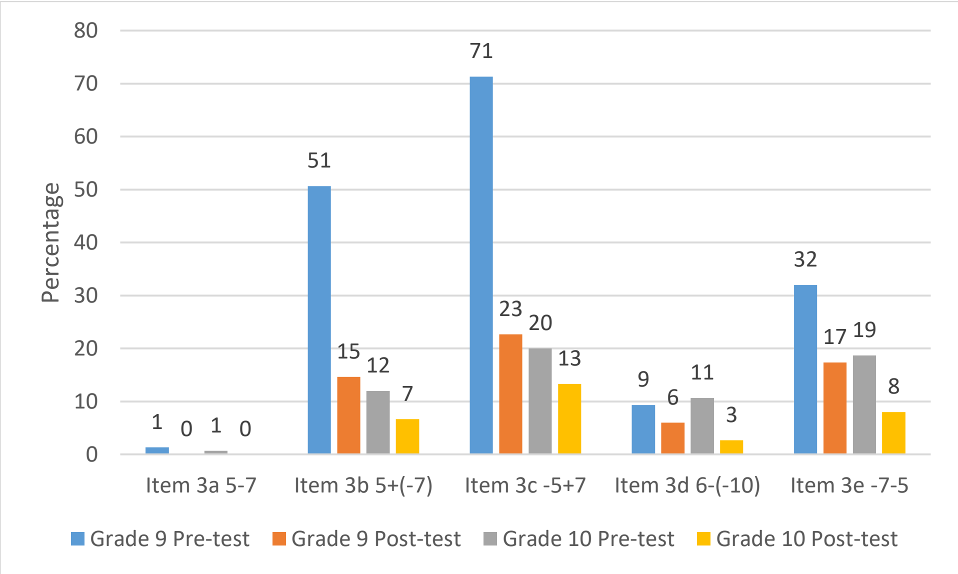

Figure 9.6: Detachment of the minus symbol error in each item

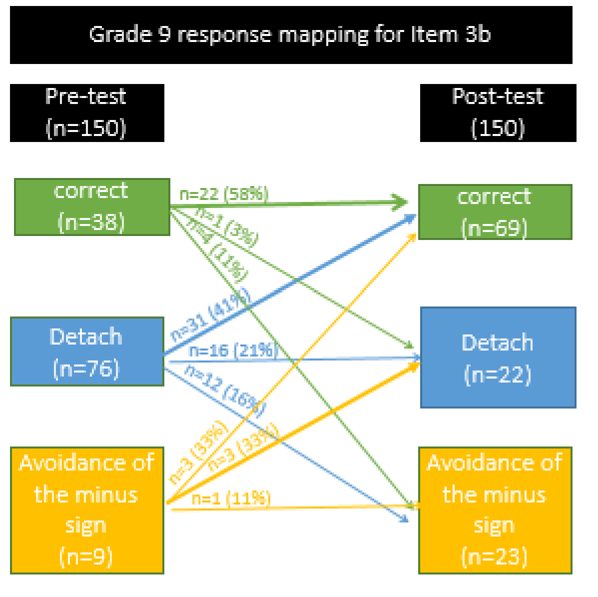

Figure 9.7: Grade 9 response mappings for item 3b

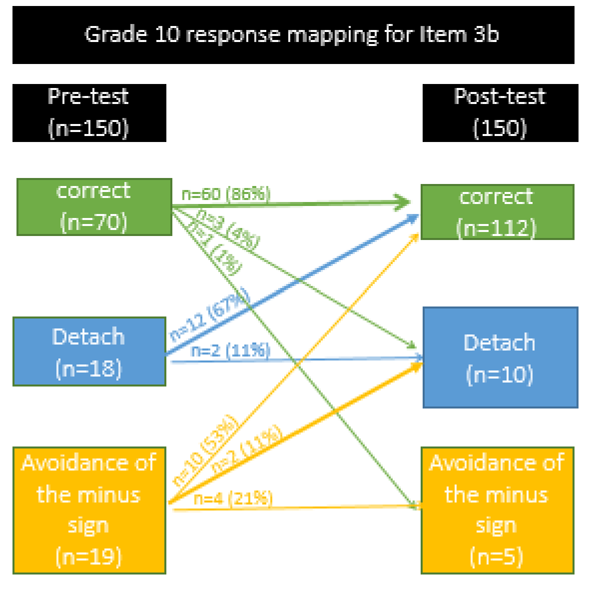

Figure 9.8: Grade 10 response mappings for item 3b

For item 3c, the percentage of learners who shifted their response from detachment to a correct response also increased from Grade 9 to Grade 10, from 40% to 60%. The percentage of learners that continued to make the same detachment error stayed the same at 23%. In item 3c, in the both the pre- and post-test, the most common error was again detachment, where 71% of learners obtained the answer -12. This decreased to 23% by the end of Grade 9 yet it remained the most common error in Grade 10. In Grade 9, 23% of the learners who were assigned the code went on to get the item correct in the post-test and in Grade 10, this increased to 54%. The percentage of learners who continued to make the same error decreased from Grade 9 to Grade 10, from 23% to 14%.

9.3.4. Right-to-left reasoning

When learners apply right-to-left reasoning, they read the question from right to left, ultimately reversing the order of the numbers to obtain a positive answer. Vlassis (2004) argues that learners reverse the order (i.e. they treat $5 - 7$ as $7 - 5$) so that they operate on something more 'comfortable' (p. 477). Learners find $7 - 5$ more "comfortable" than $5 - 7$ because it results in a positive answer. This suggests that learners avoid getting negative numbers as answers and hence have not accepted the unary function of the minus symbol. In item 3a, this was the most common error across Grade 9 and Grade 10. This is possibly due to the structure of the item. There was a consistent decrease in the percentage of learners who applied right-to-left reasoning in subsequent tests. There was a decrease from 23% in the Grade 9 pre-test to 7% in the Grade 10 post-test (see Figure 9.9). The fact that other types of errors were made by so few learners (for example in item 3b) suggests that perhaps the design of the item encourages right-to-left reasoning. In item 3d, a right-to-left reasoning error was when they not only moved from right to left, but also ignored one of the minus symbols and gave $6 - (-10) \rightarrow 4$ as an answer.

What is interesting is that 50% of the Grade 9 learners and 66% of the Grade 10 learners who applied right-to-left reasoning in the pre-test gave the correct answer in the post-test. There were 41% of Grade 9 and 33% of Grade 10 learners who continued to answer with right-to-left reasoning. This error did not appear in items 3c and 3e. In item 3c $(-5+7)$ this is possibly the case because the 7 is added to -5 and not subtracted from -5. In item 3e $(-7-5)$ right-to-left reasoning possibly did not occur because $5<7$; again, if it were $-5-7$ then more right-to-left reasoning would occur.

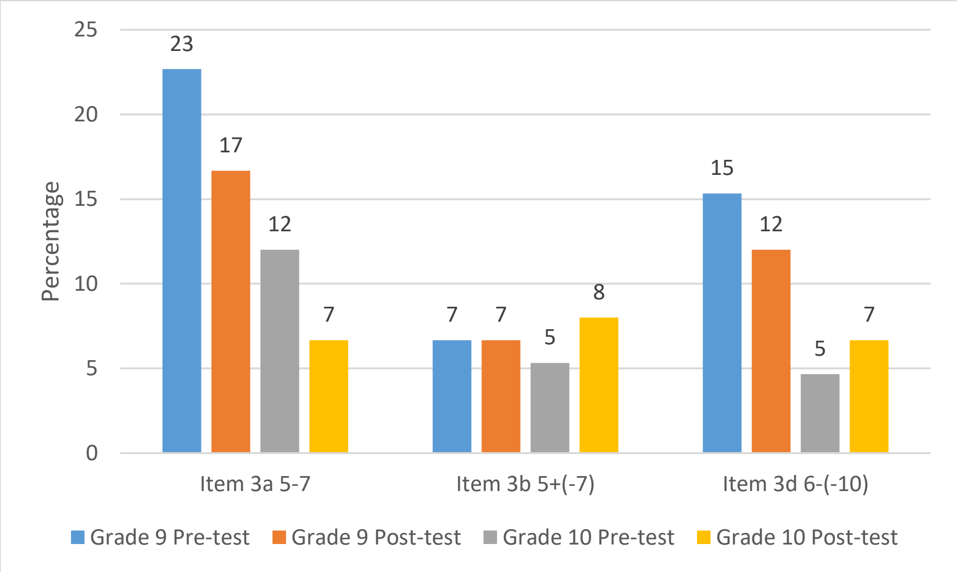

Figure 9.9: Right-to-left reasoning

9.3.5. Signs rule

The signs rule is an error that only occurred in item 3c and 3e, both of which had a leading negative. In item 3c, only 4% of learners made the error in the pre-test but in the post-test, this increased to 15%. In Grade 10 the percentage of learners who made the signs rule error remained at a similar percentage to the Grade 9 percentage at 14% but then decreased to 9% in the post-test. This suggests that as learners gain more knowledge about the multiplication rule for integers they start to use the rule when adding and subtracting. The signs rule was the most common error in item 3e in the post-test, increasing from 15% to 23% (see Figure 9.10). The signs rule error was the most common error in the Grade 10 tests, with 23% in the pre-test and 17% in the post-test.

Of the learners who made the signs rule error in Grade 9, 43% went on to get the correct answer in the post-test but 35% continued to give the same answer. In Grade 10, these percentages did not really change, with 44% shifting to get the correct answer and 35% continuing to get the same incorrect answer. This suggests that the signs rule is a persistent error where there was little change from Grade 9 to Grade 10.

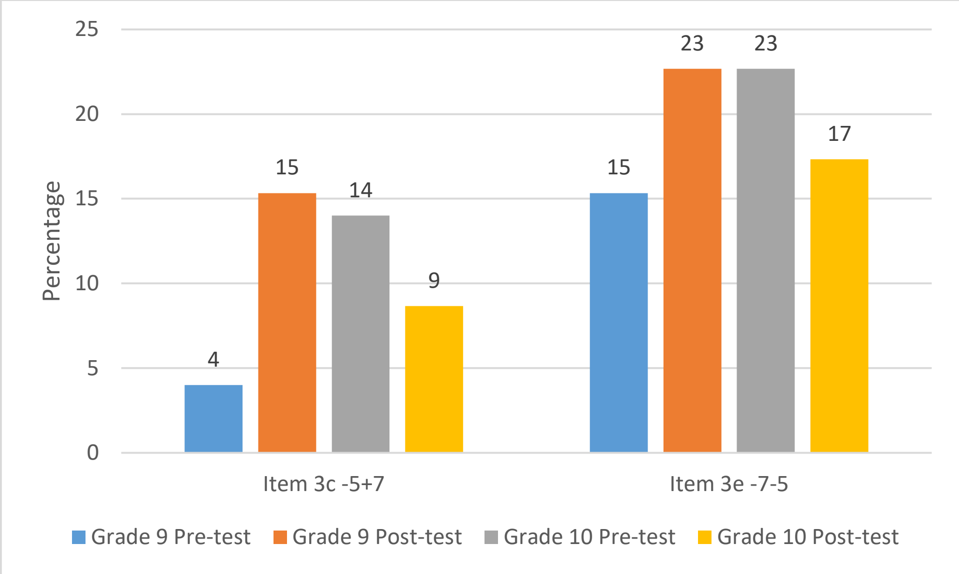

Figure 9.10: Signs rule

9.3.6. Wrong operations: multiplication

Applying the wrong operation (multiplication) is likely to occur when an item contains brackets and the learners multiply the numbers, possibly because they think the brackets mean to multiply. In items 3a and 3c, the error was made by only 1% of the sample and it didn't occur at all in item 3e. It was however prominent in items 3b and 3d. What is common between items 3b and 3e is that they both contain brackets. Item 3b is another way of asking item 3a, and yet there is a big difference in how learners responded to them. In contrast to item 3a, only 25% of learners in Grade 9 got the item correct; however, in Grade 10 post-test, 75% were getting the item correct. The brackets used in item 3d are for subtraction and not multiplication, however, I anticipated that learners would multiply the numbers because of the misconception that brackets mean multiply. Surprisingly, the error 'wrong operation: multiplication' was not a common error but it did increase a little from pre- to post-test in Grade 9 (7% to 9%). It was a common error in Grade 10, however, with 25% of learners multiplying the six and 10 together but this decreased to 8% by the end of Grade 10. In Grade 10, in item 3d, using the wrong operation was the second most common error. Of the learners who were assigned this code, 70% went on to get the item correct in the Grade 10 post-test and 14% continued to give the same response. This error was not made when answering item 3e, possibly because there were no brackets in the item.

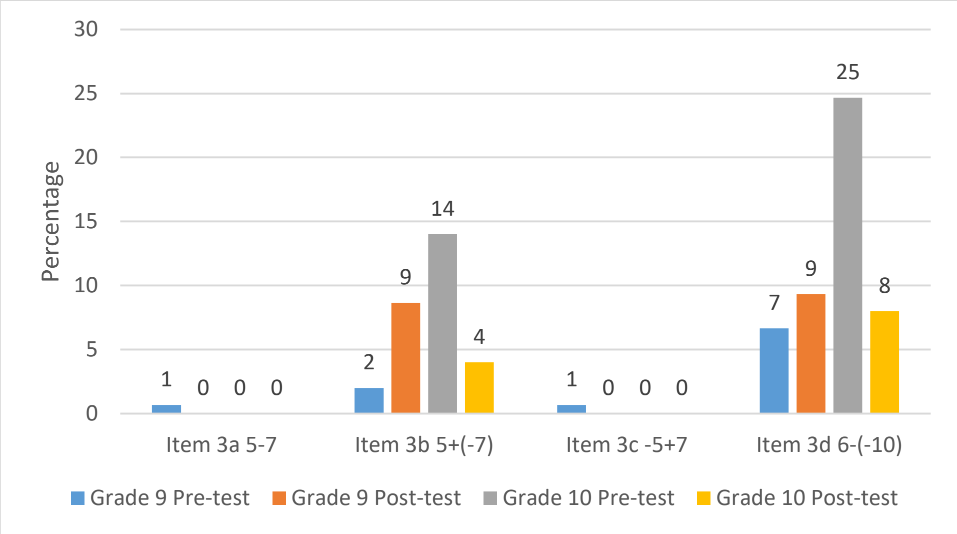

Figure 9.11: Wrong operation: multiplication

9.4. Discussion of errors found

What the above analysis shows is that learners are not consistent in how they operate with the minus symbol. Learners' inconsistencies suggest learners' knowledge is unstable. We see this in the percentage of learners that not only shift from a correct response to an incorrect response but also in the percentage of learners that shift from one error in the pre-test to a different error in the post-test. This happened in, for example, item 3b, where 33% of learners who avoided the minus symbol in the Grade 9 pre-test later detached the minus symbol in the post-test.

Some learners do not use the minus symbol in its correct function, meaning that when it is a subtraction they don't subtract and when it's a sign they don't use it as a negative number. Items 3a, 3d and 3e involved subtraction and yet there wasn't an overriding error that stood out. Item 3a was prone to right-to-left reasoning, having learners use the subtraction but take the smaller numeral from the bigger numeral for the purpose of getting a positive answer, suggesting they are not familiar with the unary function. Item 3d was a subtraction of a negative number and the most common response in Grade 9 was $6-(-10) \rightarrow -4$, where learners avoided the multiple minus symbols and treated them as one operation. These learners were able to deal with the unary function. In Grade 10, however, the most common error was to multiply the values and signs and obtain the value 60. This suggests that learners were aware of the unary function but not the binary function, which is different to item 3a. Item 3e was about subtracting a positive value from a negative value and in the beginning of Grade 9 the most common error was to detach the minus symbol, meaning the learners acknowledged its existence but did not use it in its binary or unary function. However, from the Grade 9 post-test to the Grade 10 post-test, the most common error was to use the signs rule, suggesting the learners did more than just acknowledge the minus symbol. So when there is a subtraction, sometimes learners acknowledge the operation and other times ignore it; sometimes they obey it and sometimes they have their own incorrect pattern or theory to answer the question and do so by subtracting the smaller numeral from the larger. This suggests that learners do different things with items with different structures that involve subtraction.

Items 3b, 3c, 3d and 3e all involve the minus symbol as a structural signifier. In addition, items 3c and 3e both have a leading negative. Although detachment was the most common error in Grade 9 for both items, in Grade 10 the common error differed. In item 3c it remained detachment but for item 3e it became the signs rule, suggesting that learners moved from detaching the minus symbol to applying it in its unary function -- yet, incorrectly. Literature (Vlassis, 2004) has suggested that a leading negative encourages detachment and specifically bracket reasoning (which I have explained in the literature review, is viewed as a special form of detachment). I assert that an item of the form $-a+b$ (where a and $b>0$ and $b>a$) does not encourage the signs rule in the way that $-a-b$ does. Perhaps it is the appearance of two minus symbols that encourages the signs rule. And yet, despite what literature says, the learners in my sample have done different things with a leading negative when in Grade 10. This further supports my stance that much of learners' understandings are unstable.

Item 3d: 6-(-10) involved subtracting a negative and item 3b: $5+(-7)$ involved adding a negative, and both contain brackets around the second number. Again, learners dealt with these differently. For item 3b, Grade 9s detached the minus symbol to get -12 but Grade 10s multiplied the values to get 35. While -4 was the most common error amongst the grade 9s, in Grade 10, the most common error was again to multiply the values to get 60. An error that I was expecting to see more of was the signs rule because there were two symbols next to each other: $+(-)$ and $-(-)$. However, this was not present in item 3e because if the signs rule was applied, the correct answer would have been obtained. This could mean that many of the correct answers that were obtained were done so with incorrect thinking.

9.5. Integer performance in relation to equation performance

From the analysis presented in this chapter, it is clear that negative numbers are difficult for learners. We saw many errors, with some made by 71% of the sample (the detachment error in item 3c of the Grade 9 pre-test) in the equations chapter. However, there were fewer negative errors. This is partially due to the structure of the equation items. For example, in 9a: $3x-2=10$ does not result in negative numbers being used because learners 'take over the 2'. When looking at the relationship between the negative errors in chapter 7 and the negative errors in this chapter, there was little to no relationship between them, with less than 20% of learners making a negative error in both the negative items and equation items. There was, however, a strong relationship between not making negative errors in the two sets of items. In the Grade 9 pre-test, a large percentage of learners made no negative errors in either items. For example, in item 3e, 69%; 56% and 47% of learners also did not make a negative error in the equation items, items 9a, 9b and 9c respectively. Amongst the Grade 10 learners post-test, almost 100% of learners who did not make a negative error in the integer items also did not make an error in the equations.

In terms of the quantitative analysis, there were only strong correlations between the overall integer results and the results of the test as a whole, and weak relationships between the overall results of the integer items and the overall equation items.

9.6. Correlations between the integer items and the other topics

In Grade 9 the strongest correlation was between the pre-test integer scores and the pre-test equation scores ($r = 0.6$). This went down when comparing the post-test scores ($r = 0.4$). This suggests that there was a relationship between the two topics. It suggests that an improvement in integers influenced the equations scores. In Grade 10 the strongest correlation was between the post-test equation scores and the pre-test integer scores, again suggesting that an improvement in integers influenced the equations scores.

We saw in the equation analysis that very few learners were making negativity errors, perhaps because of the improvement we saw in the integer scores in this chapter.

Grade 9		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Integer	0,6	0,5	1,0	0,5	0,4	0,4	0,5	0,4
Post-test	Integer	0,4	0,2	0,5	0,4	0,5	0,3	1,0	0,4

Table 9.2: Grade 9 correlations between integers and other topics

Grade 10		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Integer	0,5	0,4	1,0	0,5	0,6	0,4	0,5	0,5
Post-test	Integer	0,4	0,4	0,5	0,4	0,5	0,5	1,0	0,4

Table 9.3: Grade 10 correlations between integers and other topics

9.7. Conclusion

In analysing the integer items, I drew on Bishop et al. (2014); Gallardo (2002) and Vlassis (2004) as my analytical framework. The SA curriculum states that learners are introduced to integers in Grade 7. It is therefore surprising that so many learners could not answer many of the integer items correctly.

In this chapter I showed that there is a bigger increase in correct responses in Grade 10 than there was in Grade 9. Although this was found in the analysis of the overall performance of integer items, we now see that the big increase in Grade 10 was due to the same four items where learners obtained at most 38%. Although there are big improvements in Grade 10, we see that learners appear to have particular problems with items that involve brackets (items 3b and 3d) and those that have leading negatives (item 3c and 3e). The weak Grade 9 and Grade 10 pre-test results further corroborate that working with integers is difficult for learners and suggests that negative numbers are not as straightforward to learn as what educators and researchers may think.

Chapter 10 : Analysis of algebraic expression items

10.1. Introduction

This chapter presents an analysis of expression items. It focuses on the connections made between the errors in an individual topic of expressions and the errors made when solving simple linear equations. For example, if a learner adds unlike terms when given an expression such as $5x + 2$, do they also add unlike terms when given an equation such as $5x + 1 = 3x - 6$? L. Booth (1988) has suggested that one of the most prominent and persistent errors in solving simple linear equations and algebraic expressions is related to conjoining. This is not an error that disappears as learners get older (Tirosh, Even, & Robinson, 1998). From Chapter 7, we saw that some Grade 10 learners were still conjoining unlike terms. In this chapter, I report on the analyses of learner responses to seven items related to simplifying algebraic expressions. The focus is on the incorrect addition and subtraction of terms because conjoining was one of the more prominent errors when solving equations. Findings suggest that just as conjoining is a prominent and persistent error, the misuse of the exponential laws is just as prominent and persistent.

A comparison of pre- and post-test scores of the expression items shows that although many learners get the same number of items incorrect in the pre- and post-tests, the errors they make differ. The change in errors also occurs when solving equations, but fewer learners made conjoin errors when grappling with equations. This is to be expected since the algebraic manipulation required to solve the linear equations items was generally simpler than for items discussed here. In what follows, I discuss the change in errors; the overall performance on the expression items; the structure of some of the expression items; and the way students respond to the operations of the expression items.

In terms of learners' overall performance in the expression items, I will show that larger gains are made early in Grade 10 and not during Grade 9 as might be expected. In terms of the change in the errors made, I will present evidence that new knowledge (of dealing with exponents) interferes with old knowledge (of adding and subtracting like terms) and I will show that these errors are still present at the end of Grade 10. Thirdly, I will show that when brackets are present, learners operate on what is in the brackets first even if they are unlike terms. The fourth argument I present is that learners do not pay attention to the operations within an expression and instead attach on the letter/s after, by default, adding the numbers present rather than performing the given operations. Before doing the analysis, I remind the reader of the seven items and the codes used for the analysis. In the South African Mathematics curriculum, these seven items are considered Grade 8-level items. They require learners to add and subtract like terms and apply the distributive law. The seven items are:

1) Item 5a: $2a + 5a =$
2) Item 5b: $2a + 5b + a =$
3) Item 5c: $(a + b)b =$
4) Item 5d: $a + 4 + a - 4 =$
5) Item 5e: $3a - (b + a) =$
6) Item 5f: $a + b + a - b =$
7) Item 5g: $2a(a - 4) - 8 =$

The codes I used to analyse student responses to algebraic expressions were presented in the methodology section, Chapter 5. In this chapter, I present the findings of having applied the codes to learners' written work. A reminder of the codes used to analyse the responses is provided in Table 10.1.

Code	Description	Example
Conjoining	Includes different types of conjoining: juxtaposing a constant and a letter; adding unlike terms; adding coefficients and attaching the letter; adding the constant and invisible 1 in front of the letter.	$2a + b \rightarrow 2ab$ $4 + a \rightarrow 4a$
Exponents	Combining like terms and applying laws of exponents	$2a + b + a \rightarrow 2a^2 + b$ $a + 4 + a - 4 \rightarrow 4^2a^2$
Non-conjoining errors	Did not add unlike terms but rather made other errors, for example substituting values or incorrectly rewriting the question as the product of binomials.	$a + 4 + a - 4 \rightarrow (a + 4)(a + 4)$ $2(1) + 5(2) + (1) = 13$

Table 10.1: Codes used for expression items

The analysis is presented in two parts: a quantitative analysis of the overall performance on expression items and a qualitative error analysis. In the quantitative analysis I show that despite the poor performance, learners deal with expressions better in Grade 10, and that the increase in Grade 10 could point towards learners needing time for the content to sink in.

10.2. Quantitative analysis

10.2.1. Comparison of learners' performance on the expression items for the pre- and post-test

Based on the continuous nature of any mathematics curriculum, where every grade builds on the content taught in the previous grade, it seems obvious that learners should be better skilled at dealing with expression items in Grade 10 as compared to Grade 9. Contrary to reasonable expectation, there was a big difference in the percentage of correct responses *between* the Grade 9 and Grade 10 scores. This is particularly surprising when one considers that the items presented to both groups of students are Grade 8-level items. I anticipated seeing greater gains in Grade 8 or 9 when the content was explicitly taught.

What was not expected was that there was a big difference in the percentage of correct responses *between* the Grade 9 and Grade 10 scores. This is surprising because the items are Grade 8 level and hence one might reasonably expect greater gains to be made in Grade 8 or 9 when the content is first explicitly taught. The fact that learners are only starting to master adding like terms, as an example, suggests that they needed time for the content to settle or rather that learners needed *synk-time,* time for the content to *sink in.*

In this section I compare pre- and post-test results in three different ways. Firstly, I looked at a boxplot of the learners' overall expression scores, which shows that there were large gains made in Grade 10. Secondly, I present response maps and highlight that a large percentage (45%) of learners had the same number of incorrect responses in the post-test, suggesting there was little improvement and contradicting the box plot results. This apparent contradiction is resolved in the fact that the remaining 55% of learners are getting items *correct* in the post-test and suggests that the learners getting items correct are improving more than learners who get items incorrect. What I mean is that learners who are stronger improve more than weaker students. This means that the increase in the boxplots highlights the increase of the strong learners but the response pattern highlights the performance of the weak students. Thirdly, using inferential statistics I show that there is a statistically significant change in the means between the tests written but also that the expression results are a relatively strong predictor of the test performance.

10.2.2. Reasoning for increased Grade 10 pre-test performance

When comparing the pre- and post-test results of the expression items, the boxplot (see Figure 10.1) shows a considerable change in the Grade 10 performance on expression items. Although the mean percentage increased in every test, there was a large increase from the Grade 9 post-test mean (21%) to the Grade 10 pre-test mean (33%).

This increase might be attributed to three ideas:

1. The learners being exposed to the content in the first few weeks of Grade 10, before data collection etc. This suggests that at the time of testing, learners were exposed to the content and hence had recent practice in dealing with expressions. According to South Africa's Curriculum and Assessment Policy Statement (CAPS), learners would have been practising working with binomials and so perhaps being constantly left with three unlike terms (for example $x^2 + 5x + 6)$ was reinforcement that unlike terms should not be added together.
2. The gains in Grade 10 are also suggestive of learners needing more time to mature in their knowledge and understanding of mathematics. I say this because in Grade 8, the curriculum included number work such as ratios and proportion as well as algebra, but in Grade 9 learners seem to be overloaded with letters in every topic, including geometry. Not only are learners exposed to letters all the time but they are exposed to them in different mathematical contexts, which is likely to be confusing to them. In Grade 10, perhaps learners are getting used to letters in different contexts and hence we see a bigger improvement in their performance from the beginning. The repeated use of letters in different contexts possibly leads to a better differentiation between a similar but distinguishable use of letters. Grade 10s are exposed to a wider range of usage and develop a better understanding of the way letters impact on mathematics as a result.
3. The increase could be attributed to learners taking school more seriously, having chosen mathematics instead of mathematical literacy, and also could be attributed to better teaching by more knowledgeable teachers

From the boxplot in Figure 10.1 it can be seen that in the Grade 9 post-test there was an improvement whereby only half the learners got below 14%. Despite this improvement, half the Grade 9 learners are achieving less than 29% for the expression items. In the Grade 10 pre-test, a quarter of the learners achieved better than what 75% of the learners achieved in the Grade 9 post-test, with a quarter of the learners achieving between 57% and 100%. The Grade 10 post-test, however, shows an even bigger increase, with only a quarter of the learners achieving less than 14%. Although the median for the Grade 10 post-test remains the same (29%) as that in the pre-test, the third quartile has increased to 71%. These results reiterate how poorly learners initially performed on simplifying algebraic expressions, but also highlight the large gains made at the beginning of Grade 10.

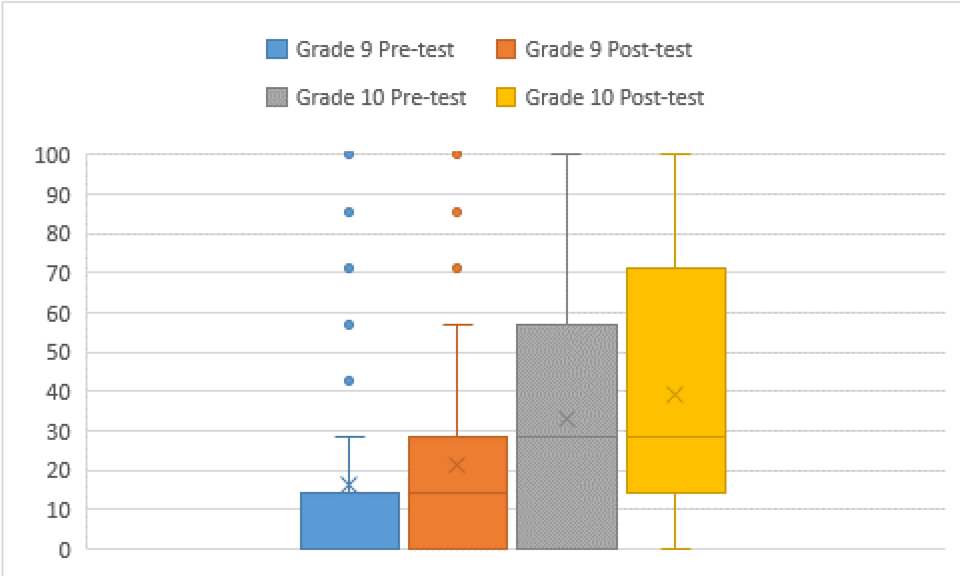

Figure 10.1: Boxplot of learners' performance in expression items

10.2.2.1. Learners with the same number of incorrect responses in the post-test

Where the boxplot shows how the learners responded as a group, a response pattern analysis shows how every learner answered (correct, incorrect or missing) the seven expression items in unison (see Figures 10.2-10.5). The response patterns show how learners dealt with a group of items. These responses are organised with the pre-test on the left side of the solid black line and the post-test on the right side. Each column represents the answer to one test item. We can see the extent of the poor results by the large number of red cells and can also see an improvement in the number of correct items by an increase in green cells in the post-test (the right-hand side). The response patterns reveal that many of the Grade 9 (44%) and 10 learners (45%) are getting the same number of items incorrect in the post-test. In addition, it appears that the majority of the items they got incorrect in the pre-test are the same items they got incorrect in the post-test (see Figure 10.2; 10.3). Figure 10.2 represents the Grade 9 data and Figure 10.3 the Grade 10 data. We see this in the similarity of the number and position of green and red cells in both the pre- and post-test. This is evidence of a large proportion of learners appearing not to make progress during the academic year and raises the question as to what errors are being made and whether there are changes in the errors during the academic year. I elaborate on the errors made in section 10.3., where the strength of the mixed methods used in this study will become more evident.

In Grade 9, a quarter of the learners got fewer items correct in the post-test, and 49% of these learners decreased in their performance on the test as a whole. In Grade 10, 21% of learners got fewer items correct and *all* of these learners performed more poorly overall on the test. These results suggest that the Grade 10 results for expressions influenced the results on the test as a whole and this is confirmed in the section that follows.

The Grade 10 pre-test results are possibly higher than they would have been had we tested the Grade 10s in January, when the school year started. Since the tests were administered in February, learners would have already practised much algebra. This means that the Grade 10 gains are in fact even higher than these results have shown.

Figure 10.2: Response pattern of Grade 9 learners who got the same number of items correct

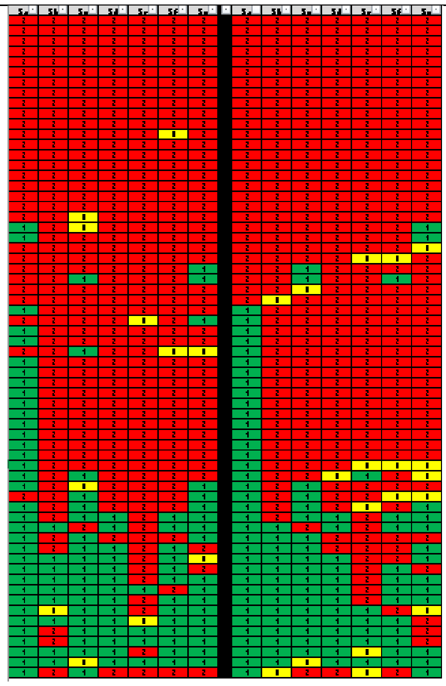

Figure 10.3: Response pattern of Grade 10 learners who got the same number of items correct

10.2.2.2. Expression results being a predictor of the test results

A paired T-test was conducted to compare Grade 9 and 10 learners' performance in simplifying expression items in the pre- and post-test. The results of the paired T-test on both the Grade 9 and Grade 10 data revealed that the change in means was statistically significant but that the strength of the relationship was weak. The Grade 9 results were: $t(150) = -2.9, p < 0.0005$; with $M_1 = 1.1;\ M_2 = 1.5\ ;\ SD_1 = 1.3;\ SD_2 = 1.9$ with Cohen's $d = 0.1$. The Grade 10 results were: $t(150) = -2.9, p < 0.0005$; with $M_1 = 4.7;\ M_2 = 5.3\ ; SD_1 = 2.2;\ SD_2 = 2.3$ with Cohen's $d = 0.2$. In Chapter 7, I explained what the different values mean, but very briefly. Because the p values were less than 0.0005, we can say that there was a statistically significant change between the two mean values.

Although Figure 10.1 suggests there was a greater change between grades, it is important to remind the reader that the Grade 9 and 10 learners are not the same learners and hence a two-sampled T-test (instead of a paired T-test) was conducted for a between-grades comparison. There was indeed a statistically significant change between the means. In addition, the effect size was greater in Grade 10, suggesting that the strength of the relationship was larger but still relatively weak (Cohen's $d = 0.4$). The results for the between-grade comparison were: $t(150) = -3.4, p < 0.0005$; with $M_1 = 1.5;\ M_2 = 5.3\ ; SD_1 = 1.9;\ SD_2 = 2.2$. The Grade 10 expression results did influence the test results as a whole. The correlation between both the pre-test results as a whole and the expression results for Grade 10 and the post-test results as a whole and the expression results was $r = 0.7$, confirming that the expression performance is a relatively strong predictor of performance in the test generally, but not of performance in equations in particular (the correlation was $r \leq 0.5$). In Chapter 6, where I did an analysis on the whole sample and not the subsample of 150 learners on which this chapter is based, the correlation between the test results as a whole and the expression results was $r = 0.8$ in the pre- as well as post-test. So the results hold for the whole sample too.

10.2.2.3. Greater improvements in Grade 10

The previous section focused on the comparison of the pre- and post-test results for the expression items as a whole and compared the gains made in Grade 10 to those in Grade 9. In this section I look at learners' performance on each of the seven items. Figure 10.4 shows the overall performance on the expression items, per item. Overall, we see that the Grade 10 learners, represented by the grey and yellow lines, performed better than the Grade 9s, which we already know from section 10.2.1.

In Grade 9, learners achieved between 10% and 20% for six of the seven items, with item 5a outperforming the others with between 60% and 70% of learners getting the item correct in the pre- and post-test. The item that showed the greatest increase in correct responses was item 5b, with an almost 10p.p. increase. The Grade 10 performance is different, not only in that they performed better, but in that their performance on the items was more varied and that there were greater gains in Grade 10 than in Grade 9. The percentage of correct responses for six of the seven items in the Grade 10 pre- and post-tests were between 10% and 40%, with item 5b having a 15p.p. increase from the pre- to post-test. In both Grades 9 and 10, it was item 5b that showed the most improvement.

Figure 10.4 shows us that the increase in Grade 10 is mainly due to the performance of items 5b and 5d. In Item 5b, learners needed to notice that there were unlike terms in the expression. The improved performance suggests that learners were more attentive to the addition of $5b$ in item 5b $(2a + 5b + a)$ and hence only added $2a$ and a. Similarly, in item 5d it appears they were more aware of unlike terms and fewer learners added the a and 4 together. It was surprising then that 5f did not also have a larger improvement since it had the same structure as item 5d. It is possible though that learners became better at distinguishing unlike terms when numbers were involved rather than only letters, as was the case in item 5f.

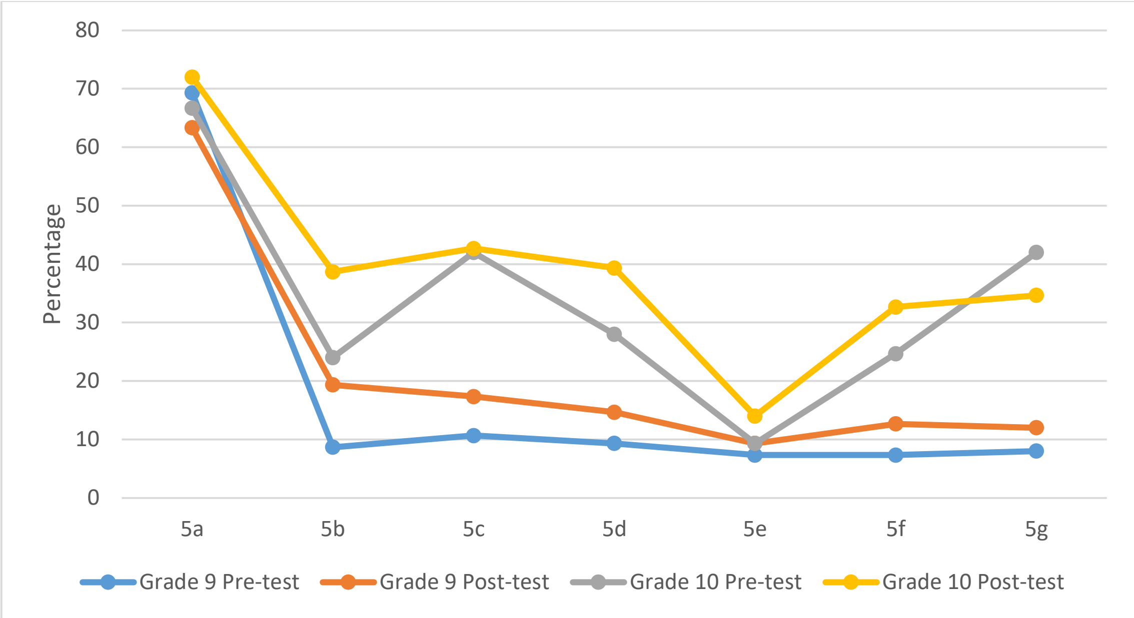

Figure 10.4: Grade 9 and 10 percentages of correct responses to expression items

In items 5b) $2a + 5b + a$; 5d) $a + 4 + a - 4$ and 5f) $a + b + a - b$ we see that Grade 10s improved more than the Grade 9s did, which would coincide with the findings in Chapters 7-9 that more gains are made within Grade 10. What is common in all three items is the increased focus on separating out unlike terms.

Item 5c) $(a + b)b$ and 5g) $2a(a + 4) - 8$ both had more correct responses as learners move to Grade 10. We see this in the big gap between Grades 9 and 10. It is possible that this happened because of the brackets and the increased attention to distributive law in Grade 9, hence more correct responses. Focusing on brackets could also explain the decrease in correct responses in item 5e) $3a - (b + a)$ where it is possible that learners use the distributive law and distribute $3a$ into the brackets. The qualitative error analysis will confirm this.

In items 5a and 5e there was little change in learners' performance, meaning that most incorrect responses in the pre-test were also incorrect in the post-test. The findings from the response patterns highlighted this but now we know it was mainly due to these two items. The error analysis done in this chapter reveals that more learners were using exponents to answer both these items, proving it was worth analysing but also that there was a change in the conjoining errors made despite not changing the overall percentage of correct responses. This shows that despite the small change in correct responses, there are changes in the errors made, which again emphasises the value of the mixed-methods research for deepening our understanding of not only learner performance but of learning and the process of maturation as well.

In this section, I drew on quantitative data to argue learners need time for content to settle in their minds, which is possibly why we are seeing a delayed effect in learning. In addition, I have illustrated the need to consider the nature of errors made, since despite small changes in overall performance, there may be significant changes to the errors being made.

10.3. Qualitative analysis

Conjoining is a prominent and persistent error found in the data. The qualitative analysis focuses on the connections made between the errors in simplifying expressions and the errors made when solving linear equations. In this section I present the results of coding learners' responses to seven expression items. The analysis was a confirmatory one where I set out to confirm what literature has already said about the prominence of conjoining. I therefore coded for conjoining, and classified the other errors as non-conjoining errors. Incorrectly using exponential laws was a common non-conjoining error that led me to focus on two key issues: that of conjoining and that of incorrectly using the exponential laws when simplifying algebraic expressions.

10.3.1. Conjoining of terms

A conjoining error is when the learner adds for example unlike terms such as $2x + y$ and responds with $2xy$ or $3xy$. This error was found in both tests and both grades as well as in all the items, with the exception of Item 5a. It is an error that was made by more than 12% of learners at any stage and by at most 56% of learners for a single item. Although conjoining was most common in Grade 9, it still appeared in Grade 10 (See Figure 10.5). In Grade 9, approximately 80% of learners made at least one conjoining error in the pre-test or post-test. However, in Grade 10, the percentage of learners making at least one conjoining error dropped to 61% and 42% in the pre- and post-tests.

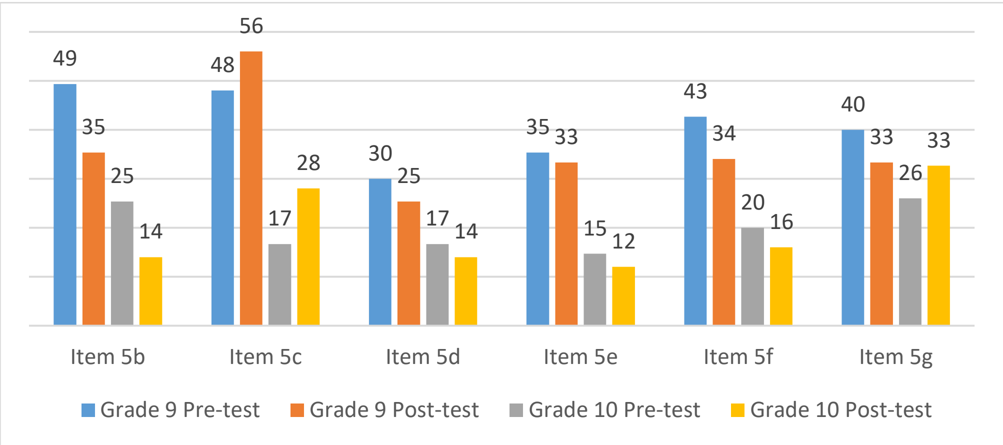

Figure 10.5: Conjoining terms in the expression items in Grade 9 and 10

There were two possible reasons with respect to conjoining errors. Firstly, learners not paying attention to the operation and instead just attaching letters and by default, adding any numbers present. The second reason is that of dealing with brackets first, even if what was in the brackets was unlike terms.

10.3.1.1. Treating addition and subtraction as the same

A finding from the qualitative analysis was that some learners did not appear to take notice of the operator but rather just appended the letters. Although learners did add the coefficients I assert that they did not do so because of the addition operation sign but rather did it by default. In Item 5b) $2a + 5b + a$, it was common for learners to add the 2 and 5 and append the a and b, obtaining $7ab$ (see Figure 10.9), with 49% of Grade 9 learners in the pre-test answering in this way. There were very few learners who replied with $7ba$ (see Figure 10.6), which suggests that learners know that the convention is to use alphabetical order. There were also few learners that replied with $7aba$. Having $7ab$ as more common concurs with what is found in the literature. Pournara et al. (2016) and L. Booth (1982) both used the item "3 add 4n". In Booth's (1982) study, 45% of 13 year olds (Grade 9 learners) responded to the item with $3n4$ or $7n$. And in the study conducted by Pournara et al. (2016), 18% of Grade 10 learners conjoined the terms. In my study, 49% of Grade 9 learners responded with $7ab$; $7aba$ or $8ab$ (see Figure 10.7 and 10.10) and in Grade 10, conjoining dropped to 14% in the post-test for item 5b.

In item 5d) $a + 4 + a - 4$ the correct answer was $2a$, however there were many Grade 9s (30%) that conjoined the a's and the fours to get either $4a$; $4a - 4a$ or $8a$ (see Figure 10.7 and 10.10). This percentage decreased to 25% in the Grade 9 post-test and in Grade 10, the error decreased from 17% to 14% in the post-test. Item 5f) $a + b + a - b$ is the generalised version of item 5d, and similarly there was a decrease in the conjoining errors, with ab being the most common error in both the pre- and post-tests. Figure 10.8 shows a learner that responded with ab, presumably because there was an a and a b and so put them together. This error is slightly different to that of item 5d as few learners responded with $2ab$. This suggests that when a letter is accompanied by a coefficient of $+1$ or -1, learners do not add the coefficients, ultimately not acknowledging that, for example, $x = 1x$.

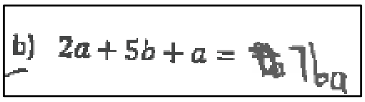

Figure 10.6: Conjoining in item 5b example 1

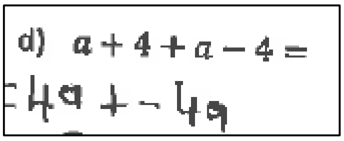

Figure 10.7: Conjoining and bracket reasoning in item 5d example 1

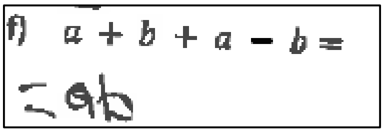

Figure 10.8: Conjoining in item 5f

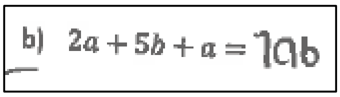

Figure 10.9: Conjoining in item 5b example 2

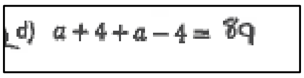

Figure 10.10: Conjoining in item 5d example 2

A possible explanation for this error lies in what Landy et al. (2014) have called visual saliency. The visual alignment of numbers and symbols or of unlike terms next to each other, for e.g. $a + 4 + a - 4$, creates a tension between formal rules and visual patterns from prior learning. With $a + 4 + a - 4$ learners are possibly visually grouping $a + 4$ and $a - 4$ (see Figure 10.7), similarly with item 5f and yielding $2ab$. A second reason for this error is what

Sfard and Linchevski (1994) call the process-object duality. This is where learners do not recognise that, for example $3a + 5b$ is an object and instead want to process it and add them together. I however am arguing that learners are not adding or subtracting because of the operations they see but rather that they are merely putting the numbers and letters together, and where there are multiple numbers they, by default, add them. It is likely that learners concatenate (Matz, 1980) the letters and numbers because they bring in knowledge from previously learnt content. For example when they add fractions such as $2 + \frac{1}{2}$ they concatenate the numbers and get $2\frac{1}{2}$. This example also highlights the object-process duality where the two objects added together result in what looks very much like the original two objects but without the operation sign. This explains why learners would add $a + b$ and get ab.

10.3.1.2. Operating on brackets first

Three items involved brackets and it was common for learners to conjoin what was inside the brackets even though they were not like terms.

In item 5c) a common error was to obtain ab^2 or some form of expression that contains ab, for example, b^2a; abb; $2ab^2$; or ab (see Figures 10.11 and 10.12). These combinations are all achieved by first conjoining $a + b$ to ab. In Figures 10.11 and 10.12 we see two learners responding to the item in a different way: the learner who answered abb was clearly conjoining what is in the brackets first and then added, not multiplied, what was outside the bracket, showing that they do not know which operations to use when. Although I anticipated that learners would conjoin what was in the brackets first, I had not expected 48% of learners in the pre-test and 56% in the Grade 9 post-test to do so. This item had the most conjoining errors compared to other items. Remembering Figure 10.4, we note that in item 5c there was a big increase in correct responses in Grade 10. The big drop in conjoining errors could account for that increase in correct responses since 18% of learners went from conjoining in the pre-test to getting the answer correct in the post-test of Grade 10.

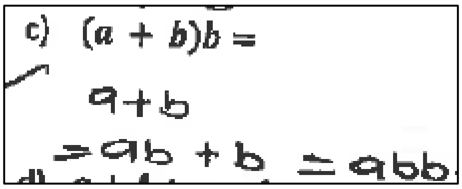

Figure 10.11: Conjoining in Item 5c example 1

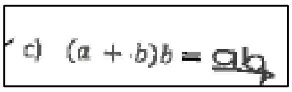

Figure 10.12: Conjoining in Item 5c example 2

In item 5e) $3a - (b + a)$ I correctly anticipated that learners would conjoin the $b + a$ in the brackets to become ab and then conjoin again to obtain $3ab$ (see Figure 10.13). A similar, but not common, conjoining error was $7ba$ (see Figure 10.14) where the learners did not put the letters into alphabetical order. This suggests that they were focusing of what was in the bracket in the order that it appeared in the bracket. We see that in Grade 9 conjoining is very common, with 35% in the pre-test and then 33% in the post-test. The little change between Grade 9 pre- and post-test suggests that this is a persistent error yet again. However, in Grade 10 there was a big drop in the percentage of learners who conjoin. This is further evidence that although we do not see much change in learners' percentages they are making changes, even if it is changing errors. Concerning the change in errors, 50% of learners who conjoined in the pre-test went on to use exponential rules to add terms.

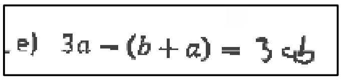

Figure 10.13: Conjoining in item 5e example 1

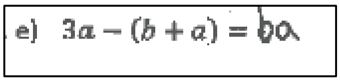

Figure 10.14: Conjoining in Item 5e example 2

In item 5g) $2a(a-4)-8$ conjoining was common throughout Grade 9 and 10, with many responses such as $8a^2-8$; $14a; 14a^2$; $16a$ or $16a^2$(see Figure 10.15) all of which include adding what is in the brackets first. In Figure 10.4 there was a big difference in the Grade 9 and 10 performance in this item, however there was not as much difference in the percentage of conjoining errors. This suggests that conjoining was more persistent in Grade 10 than in Grade 9 for this particular item. As with Item 5c, it is plausible that the increase in conjoining is due to an over-generalisation that you must operate on what is in the brackets first, even if it is unlike terms.

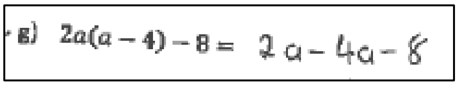

Figure 10.15: Conjoining brackets first in item 5g

For the majority of the items (5 of the 7 items) more than 40% of Grade 9 learners continued to conjoin in the post-test, meaning that at least 40% of learners were not changing their error made. This is further evidence of the persistent nature of the conjoining error. In Grade 10 we see an improvement where in four of the items, at least 36% of learners continued to conjoin. This is not an error that is reported in the Grade 12 Diagnostic Report (DBE, 2020), which identifies common errors in the final-year examination. This suggests that eventually conjoining does disappear as a prominent error.

When learners conjoin terms, they are misreading the intention of the question, symbols and operations. They have not understood the meaning behind what is asked. Where the question requires the use of the distributive law they are attempting to add unlike terms. This means that they have their own theory that, for example, $a+b$ is the same as ab which could come from prior knowledge, for example $4+\frac{1}{2}$ is the same as $4\frac{1}{2}$ where the numbers are appended together and the operation sign appears to just drop off. In addition, what these incorrect answers suggest is that learners have not yet internalised that you cannot add unlike terms. This suggests that learners have not yet understood that when they conjoin and get, for example, ab, it actually refers to multiplication.

10.3.2. Use of exponential laws

Using the exponential laws, and in particular the product rule, was used to add like and unlike terms. This is an error that is found in all items, across both tests as well as both grades. It is an error made by at least 19% of learners and so in that regard, could be considered more common than conjoining. Where in Figure 10.5 we saw a general decrease in conjoining errors from the Grade 9 pre-test to the Grade 10 post-test, in Figure 10.6 we see a general increase in exponential errors from the Grade 9 pre-test to the Grade 10-post-test. This could be evidence of curriculum effects (Pournara et al., 2016) on old knowledge. Learners are exposed to the exponential laws in Grade 9 and hence could be substituting this new knowledge for the previously learnt addition of like terms. Learners are taught to add the exponents but this idea is over-generalised and learners add exponents without considering that the bases need to be the same and multiplied. When adding like terms, learners are required to keep the base and add the coefficients but are instead adding the coefficients as well as adding the exponents. For example, $2x + 3x = 5x^2$. This change in trend in error from conjoining to exponents suggests that learners have changed the theory they had about adding and subtracting algebraic expressions. From adding coefficients and appending the letters they now add coefficients and exponents. It appears that they have found an alternative pattern ($a^m.a^n = a^{m+n}$) that they have now generalised to the addition of unlike terms ($a^m + a^n = a^{m+n}$).

Although there were no conjoining errors in item 5a) $2a + 5a$, there were many exponent errors where learners answered $7a^2$. Figure 10.17 shows the same type of error but for item 5d, where the learner has added $a + a$ by using the exponential laws and giving a^2 as the answer. Another example is given in Figure 10.20 for item 5b. In this example it appears that the learner mistook the 5b for a 5a and then used the exponential product rule to yield $7a^3$. Figure 10.16 gives an example from item 5b of a learner who appears to know that unlike terms cannot be added but then adds the like terms incorrectly by using the product rule.

These types of errors were the most common type of non-conjoining error made and were made by 20% or more of the learners in all four groups (See Figure 10.21).

In the Grade 10 responses, there was a new error involving products of binomials. Although it was an error made by few students, it is important because it highlights that new knowledge could be hindering learners' understandings of current knowledge. An example of this error can be found in Figure 10.18 where the learner has tried to manipulate 2a+5a into the product of two binomials. In addition, we see in Figure 10.19 a learner that has explicitly tried to add the letters as exponents, mimicking the product rule $a^m.a^n = a^{n+m}$. In Figure 10.20 we see an example of a learner who most probably mistook the $5b$ for a $5a$ in item 5b and then used the exponential rule to add them.

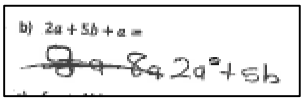

Figure 10.16: Exponents in item 5b example 1

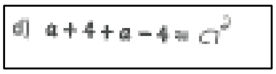

Figure 10.17: Exponents in item 5d example 1

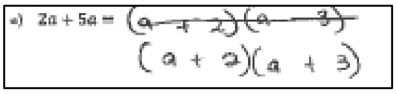

Figure 10.18: Error involving binomials

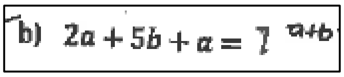

Figure 10.19: Exponents in item 5b example 2

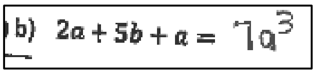

Figure 10.20: Exponents in item 5b example 2

This suggests that the new knowledge being learnt (multiplication of binomials) was interfering with old knowledge. Even though the responses linked to the product of binomials were not common errors, the exponents error and the product of binomials errors suggest that learners use the new rules or procedures learnt to answer items that they previously probably got correct.

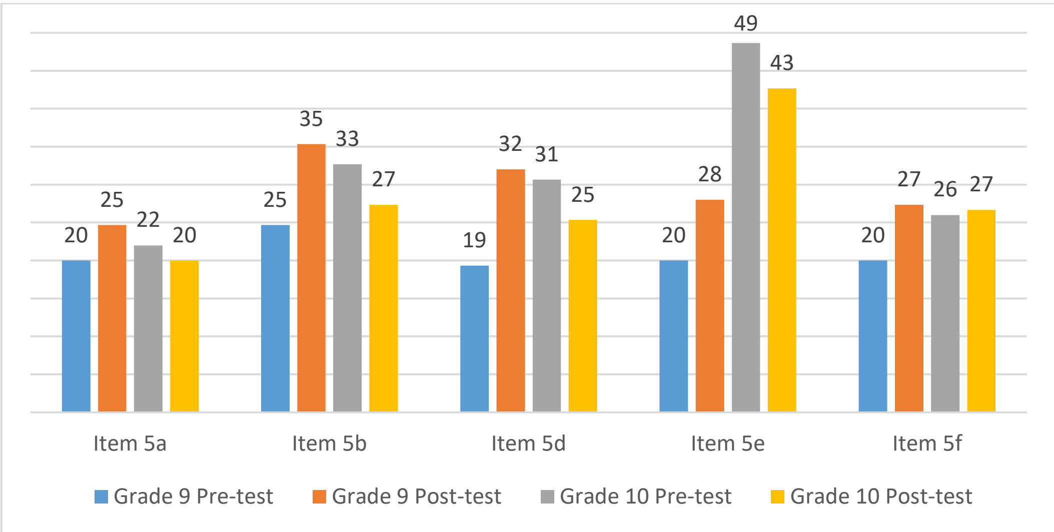

Figure 10.21: Use of exponential laws in Grades 9 and 10 for addition and subtraction of terms

When investigating the difference in errors made in the pre- and post-test, I found that approximately 25% of learners in Grade 9 went from making a conjoining error in the pre-test to misusing the exponential laws in the post-test in items 5b; 5d; 5e and 5f. Item 5c) and 5g) both required the distributive law and hence required the use of the multiplication law of powers. It is therefore understandable that the percentage of learners who misused the exponential laws was much lower. In Grade 10, in all items except item 5b), there was a larger percentage of learners who misused the exponential laws in the post-test after making conjoining errors in the pre-test. In Grade 10, item 5e had almost double the percentage of learners that misused the exponential laws in the post-test after conjoining in the pre-test. Of the learners that conjoined in the pre-test, 50% of them changed to using exponential laws in the post-test. Results show that, overall, a larger percentage of learners in both

Grades 9 and Grade 10 continue to misuse the exponential laws compared to those who continued to conjoin. This suggests that just as conjoining is a prominent and persistent error, the misuse of the exponential laws is just as prominent and persistent. This is a particular problem in Grade 10, where up to 61% of learners continue to misuse the exponential laws.

10.4. Comparing performance on expression items and conjoining within equations

Since an equation can be considered as two expressions set equal to each other, it is logical to assume that the errors that learners make when simplifying expressions are transferred to solving equations. However, conjoining was less prominent when learners were dealing with equations (see Chapter 7). When looking at the percentage of learners that conjoined in both topics it was found that in the Grade 9 pre-test, there was between 30% and 40% of learners that conjoined in for example item 5d and 9c. In the Grade 10 pre-test however, there was a much larger percentage of learners that did not conjoin in both sets of items. For example, 69% of learners did not conjoin in items 5b and 9c; 77% of learners did not conjoin in items 5g and 9c. The percentage was always stronger when comparing with item 9c rather than item 9a or 9b. This suggests that when the letter was subtracted, it caused more learners to conjoin.

The quantitative results showed that there was a weak correlation between the expression performance and the equation performance, and a stronger correlation in Grade 10 between the expression performance and the test as a whole.

Overall, these results suggest that in the context of equations, learners are less likely to conjoin. This is possibly due to the fact that when dealing with equations there is a new set of 'rules' to be obeyed, such as what you do to the left, you do to the right. This also means that they have different theories of dealing with letters in the different topics. I suspect that when learners are solving equations they are preoccupied with 'moving' the constants 'to the other side' and thus they are less likely to conjoin. In terms of the use of exponents as discussed in the equations chapter, there was a very small percentage of learners that used the exponential laws to add unlike terms. This again suggests that new rules learnt to solve an equation are more dominant in a learner's mind when solving the equation, even if those new rules are not applied correctly.

10.5. Correlations between expression items and other topics

In Grade 9 the strongest correlation between expressions and other topics was with equations (in both the pre- and post-test, $r = 0.6$). The weakest correlation was with equality, which we saw in Chapter 8. In Grade 10 the strongest correlation was also between expressions and equations, with the weakest between equality and expressions. However, the correlation between expressions and equality in Grade 10 did improve between 0.1

and 0.2. As mentioned in Chapter 7, what this suggests is that there is a relationship between expression and equation scores, and no relationship between expression and equality scores in Grade 9. The fact that the correlation between the pre- and post-test expression scores is $r = 0.7$ suggests that over the year, learners improved in simplifying expressions. Since simplifying expressions is mainly practiced in Grades 8 and 9 it does point towards a delayed development which in turn points towards practice with other mathematics being helpful in dealing with expressions. This delayed development is discussed in Chapter 12.

Grade 9		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Expression	0,6	0,2	0,5	1,0	0,5	0,2	0,4	0,6
Post-test	Expression	0,6	0,3	0,4	0,6	0,6	0,2	0,4	1,0

Table 10.2: Grade 9 correlations for expressions

Grade 10		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Expression	0,6	0,4	0,5	1,0	0,5	0,5	0,4	0,7
Post-test	Expression	0,6	0,5	0,5	0,7	0,6	0,5	0,4	1,0

Table 10.3: Grade 10 correlations for expressions

10.6. Conclusion

This chapter focused on the responses to expression items. I presented quantitative data that highlighted that more gains were made in the beginning of Grade 10 rather than during Grade 9. It was found that there were statistically significant, but weak, improvements between the four tests, suggesting that there was only a little improvement in performance on the expression items. I also showed that, despite learners continuing to get items incorrect, there was a change in errors made. The errors in focus were related to the addition and subtraction of like and unlike terms. The error analysis showed that conjoining, although still a common error, decreased substantially in Grade 10. Although there were fewer conjoining errors in Grade 10, the lowest percentage in the post-test was 12% in item 5e and the maximum percentage was 33% in item 5g, suggesting that conjoining is a persistent error.

In Grade 10, there was an increase in learners misapplying the exponential laws when adding like and unlike terms. These errors were common in both grades and throughout the test. This chapter also highlighted that learners operate on what is inside of brackets even though they are unlike terms. The interference of new knowledge was also a finding from the data analysis. I highlighted the difference in findings between this chapter and the equations chapter, Chapter 7, and how conjoining may be context-related as it was not as prominent when solving equations.

Chapter 11 : Synthesis of quantitative findings

11.1.Introduction

This short chapter synthesises the quantitative findings from the full and the sub-sample. The aim is to show the similarities of the findings, confirming that the sub-sample was indeed a good representation of the full sample and that the qualitative findings can be generalised to the bigger sample. I discuss the learner scores, topic scores, item scores, correlations, response patterns and inferential statistics.

11.2.Learner scores

Learner scores were calculated by adding the number of correct responses divided by the number of items. Box plots using Excel were then created. Figures 11.1 and 11.2 show the distribution of learner scores. The figures are almost identical, except that the full sample has more outliers. Besides the means and quartiles being similar, what is noticeably the same is that there is a larger jump in the Grade 10 post-test scores.

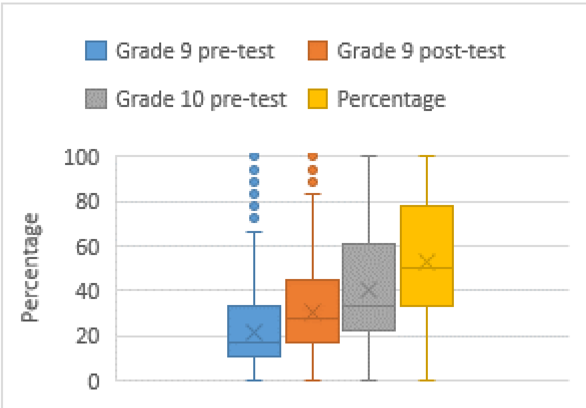

Figure 11.1: Learner scores for the full sample

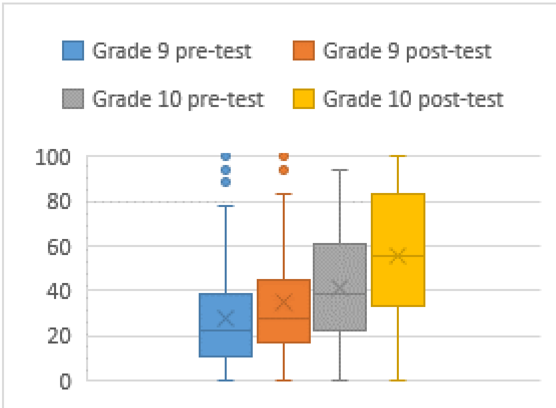

Figure 11.2: Learner scores for the sub-sample

11.3.Topic scores

Topic scores are the average scores per topic. These were calculated by adding the number of correct responses per topic and then taking the average for each test and each topic. A line graph was then created. The two figures, Figure 11.3 and 11.4, are very similar. Both show that integers was the best answered topic, then equality, and then equations and expressions. Both samples show that expressions were better than equations in the Grade 9 pre-test and that in the Grade 10 post-test, there was a big increase in the integer and equation scores.

Figure 11.3: Topic scores for the full sample

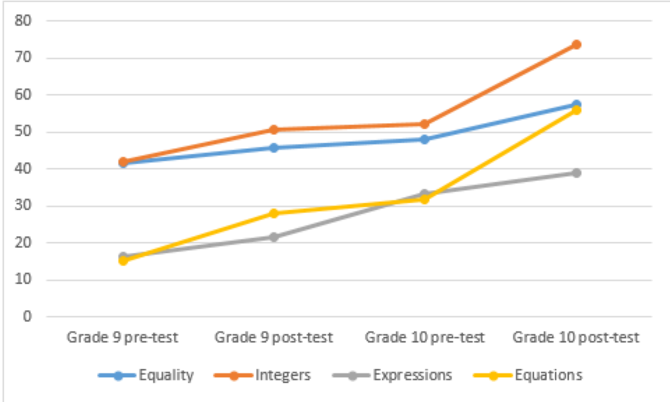

Figure 11.4: Topic scores for the sub-sample

11.4.Item scores

The item scores are the number of correct responses to each item. These were calculated by adding the number of correct responses and dividing by the number of learners in the sample. These values were then graphed. Figure 11.5 and 11.6 show the graphs for the full sample and sub-sample respectively. The overall pattern looks very similar for both sets of samples, with the sub-sample percentages being slightly elevated. With the integer items it is clearer that there was a bigger jump in the Grade 10 post-test, as Figures 11.3 and 11.4 showed earlier. It is also clearer that there was little difference between the Grade 9 post-test and the Grade 10 pre-test. The results for items 4a and 4b, the equality items were slightly different in the sub-sample, with Grade 10 results being more noticeably higher than the other three tests. In the expression items the behaviour of each item was also similar for both sets of samples. In item 5a, in all four tests, both sets of samples' percentage of correct responses congregated around 60%. We also see in item 5e the percentages congregate between 0% and 20%. One noticeable difference between the two samples is the behaviour of item 5g. In the sub-set the pre-test percentage is better than the post-test percentage. The equation results were also similar in both sample sets but there was a larger increase in the Grade 10 post-test scores in the sub-sample.

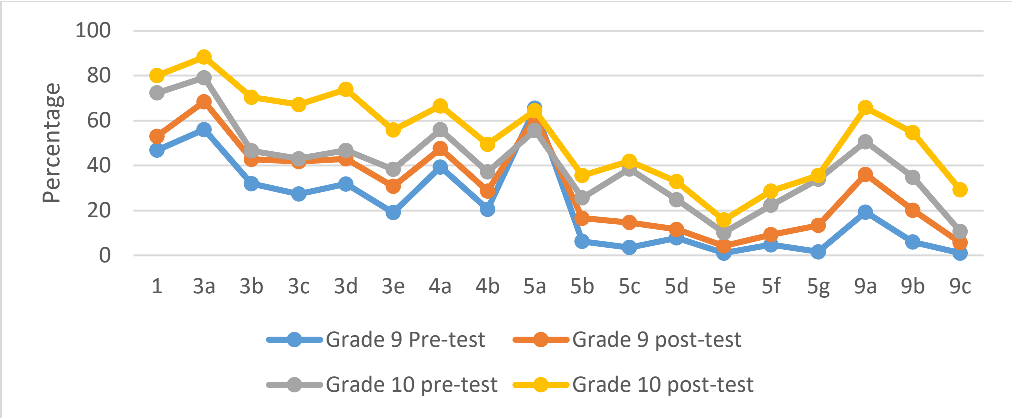

Figure 11.5: Item scores for the full sample

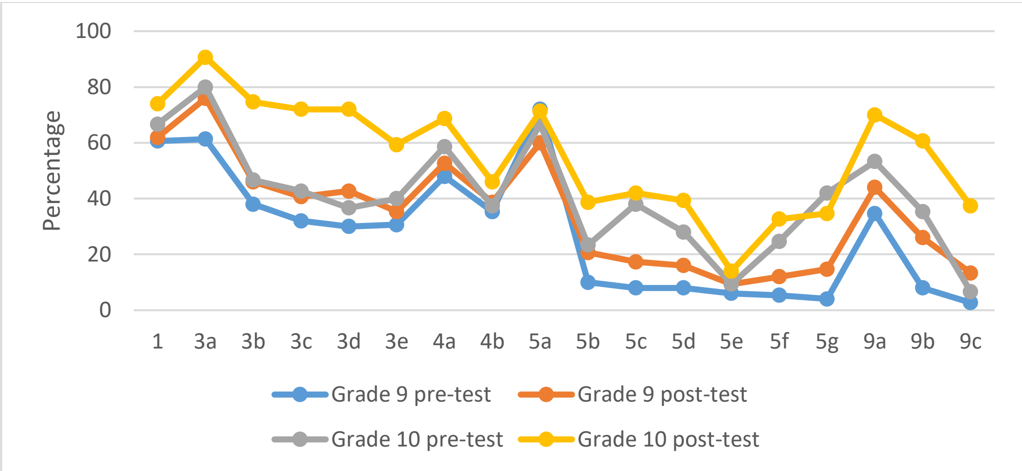

Figure 11.6: Item scores for the sub-sample

11.5.Correlations between topics

The aim of this chapter is to show that there is little difference in the quantitative results between the sub-sample and the full sample. This is important to do because it shows how the sub-sample was indeed random and represents the full sample. It points towards my results being generalisable. It in turn suggests that I can infer what errors the full sample make, having analysed the errors from the sub-sample.

The correlations between each topic were calculated and reported in the relevant chapters (See Chapters 7-10). I now compare those correlations with the correlations from the full sample. I took the absolute value of the difference between the two samples and for consistency, rounded to one decimal place. The differences were then colour-coded to show how different or similar the results are (see Tables 11.1-11.2). The darker red the cell, the more difference there was and the lighter the cell, the less difference there was. White cells show that there was no difference.

In Grade 9 there are noticeably lighter-coloured cells between the post-tests correlations, meaning that there was less difference when the topics were compared against one another in the post-tests. This suggests that the Grade

9 post-test results for the sub-sample are more generalisable than the pre-test results. However, the maximum difference between any two topics did not reach above 0.2. In Grade 10 there was hardly any difference between the correlations of the sub-sample and the full sample. Both the Grade 9 and Grade 10 comparisons showed very little difference, suggesting that the samples are very similar. This is yet another example of how the results of the sub-sample can be generalised to the full sample. The correlations for both grades can be found in the Appendix.

Grade 9 Difference		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equation	0,0	0,1	0,2	0,2	0,0	0,0	0,0	0,1
	Equality	0,1	0,0	0,1	0,1	0,0	0,0	0,1	0,0
	Integer	0,2	0,1	0,0	0,2	0,0	0,0	0,0	0,1
	Expression	0,2	0,1	0,2	0,0	0,1	0,0	0,2	0,2
Post-test	Equation	0,0	0,0	0,0	0,1	0,0	0,1	0,0	0,0
	Equality	0,0	0,0	0,0	0,0	0,1	0,0	0,1	0,1
	Integer	0,0	0,1	0,0	0,2	0,0	0,1	0,0	0,0
	Expression	0,1	0,0	0,1	0,2	0,0	0,1	0,0	0,0

Table 11.1: Grade 9 difference in correlations between topics

Grade 10 Difference		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equation	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
	Equality	0,0	0,0	0,0	0,0	0,0	0,0	0,1	0,0
	Integer	0,0	0,0	0,0	0,0	0,1	0,0	0,0	0,0
	Expression	0,0	0,0	0,0	0,0	0,0	0,1	0,0	0,0
Post-test	Equation	0,0	0,0	0,1	0,0	0,0	0,1	0,0	0,0
	Equality	0,0	0,0	0,0	0,1	0,1	0,0	0,0	0,1
	Integer	0,0	0,1	0,0	0,0	0,0	0,0	0,0	0,0
	Expression	0,0	0,0	0,0	0,0	0,0	0,1	0,0	0,0

Table 11.2: Grade 10 difference in correlations between topics

11.6. Response patterns

Tables 11.1-11.3 compare the percentages of learners who increased, had the same, or decreased the number of correct responses in the post-tests of Grades 9 and 10. There are percentages for each topic area as well as for the whole test. What is noticeable is that the percentages for the full and sub-sample are very similar. For example, in Table 11.1, in Grade 9, the biggest difference is 7p.p. for equations. There are bigger gaps in Grade 10 but they do not exceed 10p.p. We see similar patterns in the other two tables. These tables are presented as evidence of the systematic analysis that was done. The aim of this chapter is to show how the sub-sample performed almost identically to the full sample and hence findings can be generalised.

a) Percentage of increased correct responses in the post-test

	Grade 9		Grade 10	
	Full sample	Sub-sample	Full sample	Sub-sample
Whole test	61	61	71	80
Equations	31	38	43	53
Equality	24	25	30	29
Negatives	49	45	59	65
Expressions	34	31	41	36

Table 11.3: Table of percentages of learners that increased the number of correct responses in the post-tests

b) Same number of in/correct responses in the post-test

	Grade 9		Grade 10	
	Full sample	Sub-sample	Full sample	Sub-sample
Whole test	18	16	13	8
Equations	62	47	48	42
Equality	64	59	58	56
Negatives	29	31	27	20
Expressions	43	44	36	43

Table 11.4: Table of percentages of learners that had the same number of correct responses in the post-tests

c) Decreased correct responses in post-test

	Grade 9		Grade 10	
	Full sample	Sub-sample	Full sample	Sub-sample
Whole test	21	23	16	12
Equations	7	15	9	5
Equality	12	17	12	15
Negatives	22	24	15	15
Expressions	23	25	23	21

Table 11.5: Table of percentages of learners that decreased the numbers of correct responses in the post-tests

11.7.Inferential statistics

The difference in the pre- and post-test means showed statistically significant improvement ($p < 0.05$) in all topics as well as the whole test for both grades as well as for both the full and sub-sample. The only exception was the Grade 9 sub-sample in the topic equality, where there was a p value of 0.12. Again this shows how the full and sub-sample performed in a similar fashion.

Correlations between the topics performance and the test as a whole as well as between the topics and equations were calculated. As can be seen in Tables 11.4 and 11.5, the correlations are also very similar, with the exception of the Grade 9 pre-test. The correlation between equations and the test as a whole showed $r = 0.6$ for the full sample but $r = 0.9$ for the sub-sample.

Pre-test correlations		Equations		Equality		Negatives		Expressions	
		Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10
Test as a whole	Full sample	0,6	0,8	0,6	0,6	0,9	0,8	0,6	0,9
	Sub-sample	0,9	0,7	0,6	0,6	0,9	0,8	0,8	0,9
Equations	Full sample			0,3	0,4	0,4	0,5	0,4	0,6
	Sub-sample			0,4	0,4	0,6	0,5	0,6	0,6

Table 11.6: Correlations from the pre-test for each topic

Post-test correlations		Equations		Equality		Negatives		Expressions	
		Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10	Gr 9	Gr 10
Test as a whole	Full sample	0,8	0,8	0,6	0,6	0,8	0,8	0,8	0,9
	Sub-sample	0,8	0,8	0,5	0,7	0,8	0,8	0,8	0,9
Equations	Full sample			0,3	0,4	0,4	0,5	0,6	0,6
	Sub-sample			0,3	0,5	0,5	0,5	0,6	0,6

Table 11.7: Correlations from the post-test for each topic

11.8.Conclusion

This short chapter was dedicated to showing that the sub-sample and the full sample behaved in a similar fashion and hence the findings of the sub-sample (both quantitative and therefore qualitative) can be generalised to the full sample. I showed this through learner scores, topic scores, item scores, a response pattern, and through inferential statistics.

What this enables, is that all the qualitative findings can be generalised to the full sample. This means that we can deduce how more than 2000 Grade 9 and 10 learners responded when answering questions that relate to equality, integers, expressions and equations. We can know that the majority of the Grade 9 learners, for example do not use inverses; or that in Grade 10, there are big shifts in the way learners approach an equation, view the equal sign, and view a solution. We can also say that learners appear to have different patterns or theories that they use to answer questions in different topics. This means they make different errors depending on the context or area of mathematics.

In the chapter that follows, I provide my concluding remarks and recommendations for future research.

Chapter 12 : Theoretical contribution

12.1. Introduction

Throughout Chapters 7 to 10, a key finding was that Grade 10 learners made more gains than Grade 9 learners. This was not expected since the items tested Grade 8 and 9 content. It was expected that the Grade 10s would perform better but not that they would make more gains. So how do we explain this? As I got deeper into the analysis process, I started talking about *synk-time* to suggest that learners need more time for the mathematics they are earning to 'sink-in' and thus synchronise internally. The term synk a combination of sink and sync. Using this led me to search for theory that may explain this notion. There are three theoretical constructs that have informed the notion of synk-time. These are Vygotsky's (1978) notion of *internalisation*, Tomasello's (2003) notion of *intention reading* and *pattern finding*, and Gopnik and Meltzoff's (1997) notion of the *Theory Theory*. These theories all describe the process of 'coming to know' but do so at different levels. Where Vygotsky's is a broad theory, Tomasello, Gopnik and Meltzoff build on from Vygotsky, focus on practical learning and provide complementary concepts to describe how the learner acquires skills and knowledge. Building a theoretical position of how these three constructs work together, how they apply to the learning of mathematics, and how they account for a delay in learning is therefore an outcome of this study.

I argue that part of what enables internalisation is synk-time, and that it involves theory testing and pattern-finding, and crucially, participation in mediatory activities. Mediatory activities refers to the actions learners take to understand the content. It refers to what the learners are doing and is not limited to what the teacher is teaching at that moment. Whether the learner is working on homework or answering questioning verbally or on paper, the learner is encountering mathematical content that could be aiding understanding of another topic or different content. The activity and learning the learner is encountering may not be the same as the goal of the teacher's lesson but the activity does trigger some understanding of other topics. Later in this chapter, in Figure 12.5, I exemplify mediatory activities for the content of adding and subtracting integers. In addition to being an outcome of this study, integrating the theories and applying it to my data is a contribution to the mathematics education community as I discuss how the three theories work together and complement one another.

In this chapter I start by discussing the three theories in isolation and then present a discussion on their integration. This leads me to define synk-time and then drawing on it to explain the delay in mastering mathematical concepts in Grade 8 and 9

12.2. Vygotsky's notion of internalisation

Central to Vygotsky's (1978) notion of internalisation is the idea that everything (all concepts) exist in two ways: on the external level (i.e. in books, conversations, activities, on the blackboard, etc.) and on the internal level (the individual's own thinking). Learning takes place when what is external is transformed into the internal. This may well be a slow process but depending on the content could also be a quick one. In other words, internalisation is the process of owning content, coming to know something for one's self, and it incorporates the meaning behind the signs into one's mental processes (Berger, 2002). What is internalised is the "specific structure of human interaction, mediated by cultural tools among which language is the most powerful" (Arievitch & Van der Veer, 1995, p. 114). It is not a simple and straightforward process but requires the individual to play an active role in the "internal reconstruction of [the] external operation" (Vygotsky, 1978, p. 56), meaning that internalising content or skills requires deliberate reconstruction of the external. Because Vygotsky argues that learning occurs through social interactions, it is therefore dependent on experience and can take time. Radford (2013) explains that in this process the internalisation of (mathematical) actions involves the reorganisation of mental activity on the basis of *signs*. To solve an equation and execute the procedure correctly, a learner would need to have internalised the correct meaning of the equal sign as well as how to use inverse operations. A learner who has internalised content or a skill incorrectly will produce errors and would need to relook at what they thought was correct. Figure 12.1[3] shows a diagrammatic view of the process of internalisation. Subsequent diagrams build on Figure 12.1 but the final diagram is Figure 12.4. The windy road represents the process, emphasising that it is not a linear process. The flags represent the mediatory activities that are used to enhance learning. The journey starts with the external/social function and reaches a point where the content/skills are internalised and made one's own.

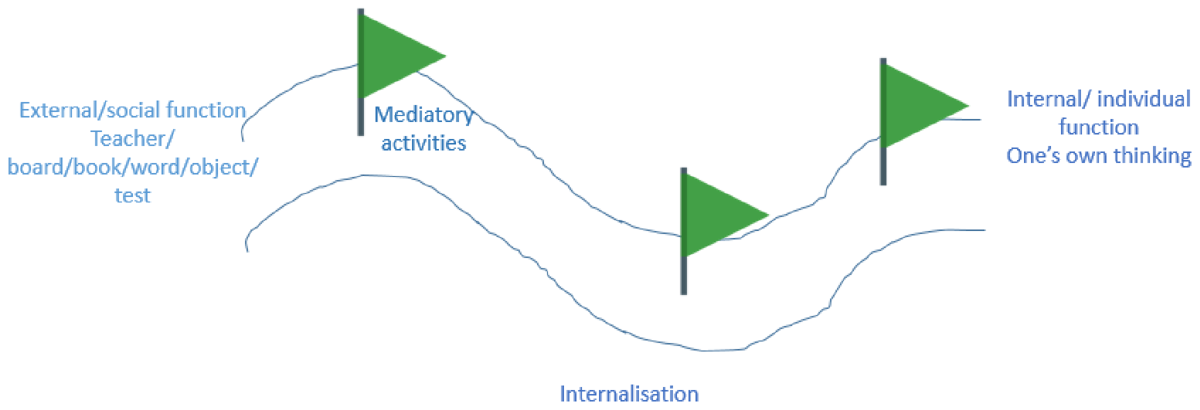

Figure 12.1: Diagram of internalisation

Vygotsky's theory does not account for the process of re-internalising content and hence Gopnik and Meltzoff's Theory Theory is useful. Internalisation offers a way of talking about what takes place but does not offer a

[3] A reader who is familiar with Confrey's (2006) work on the conceptual corridor will see a resemblance in my diagrams. I am however not dealing with learning trajectories and the conceptual corridor.

mechanism (a *how)* to account for what takes place. This is where Tomasello's theory of language acquisition is useful. While Vygotsky's notion of internalisation presents a useful description of how social interaction leads to learning, Tomasello's theory introduces concepts that are useful to any theory of learning that involves the use of symbolic data to mediate learning. Later I will argue that where Tomasello provides two skills needed for learning (intention-reading and pattern-finding), Gopnik and Meltzoff offer a notion that explains how these skills are used to promote learning.

12.3. Tomasello's theory of language acquisition as a theory of learning

Tomasello (2007) is a linguist and primarily concerned with how a child acquires language. He proposes a usage-based approach to language acquisition. This approach assumes that one acquires liguistic expressions from the language used in one's environment. It further assumes that cognitive abilities such as intention-reading and pattern-finding are used to create more complex expressions (Brink, 2020). He argues that single words are initially learnt with no grammatical rules attached to them (intention-reading). These words form the basic building blocks of language and with a collection of words, the child begins to form the ability to construct longer and more complex expressions through the help of pattern-finding. He adds that constructing these more complex expressions happens gradually and in a "piecemeal fashion" (Tomasello, 2007, p. 258). These two skills emphasise the role of social interaction. Tomasello asserts that at the heart of language is its symbolic dimension, which relies on the human ability to understand intention (Ghalebi & Sadighi, 2015). Although he specifically talks about language acquisition, I will show that his theory can be applied to the learning of mathematics. In the sections below, I discuss intention-reading and pattern-finding in relation to language acquisition and afterwards link it to learning mathematics.

Intention-reading refers to a child's ability to understand other people's communicative intentions. It is about reading someone else's intention and sharing the attention with them. For example, an adult points to an aeroplane in the sky and says the words "there is an aeroplane". The child then associates the words with the adult bringing their attention to the object, the aeroplane. With multiple uses of the words "there is" the child begins to realise that "there is" is a term used to bring attention to an object.

Intention-reading is therefore about the *meaning* behind the intention and where "meaning is use" (Heidar, 2012, p. 414). This means that meaning only comes with use of the word/s. Intention-reading skills are necessary for a child to acquire in order to deal with linguistic symbols and complex expressions and constructions. It is, therefore, considered the functional (or semantic) dimension of language acquisition (Tomasello, 2003) and can therefore be considered the functional dimension of learning any new skill. The functional dimension enables abstraction processes by comparing elements that have a similar function, which leads to pattern-finding.

In contrast to intention-reading, pattern-finding is what a child ought to do in order to go beyond the individual utterance (in language acquisition). It is where a child attempts to categorise similar objects or occurrences, for example, where a child will put all types of fruit into one category. While reading the adult and sharing their attention, the child begins to piece together the necessary components needed for language acquisition, for example, recognising speech patterns such as subject-verb-object (Tomasello, 2003), which is the pattern-finding skill. The child then uses these pieces to create new constructions. For example, in hearing "there is a dog", "there is a chair" etc., the child recognises the patterns and is able to create his/her own sentence: "there is a toy". Once patterns are created, children are able to create abstract constructions. Another example is where the child goes from hearing "put it on the table" many times and will read the intention behind "put it on" and then abstract its meaning and create his/her own sentence: "put it on the floor", realising the pattern that the noun comes at the end. This pattern-finding skill is necessary for a child to determine the patterns that adults use so they can construct grammatically correct constructions, and hence this skill is referred to as the grammatical dimension of language acquisition. Where intention-reading could be summarised as 'meaning is use', pattern-finding can be summarised as where structure emerges from use. In addition to the meaning-form pair that these skills highlight, we also notice that it is only through usage that one can acquire language. It is important to note that intention-reading and pattern-finding are intertwined. Where intention-reading allows the child to piece together bits of language, pattern-finding is about the identification of patterns, but intention-reading then enables the child to make use of the patterns they recognise. It is therefore a cyclical process where one feeds into the other.

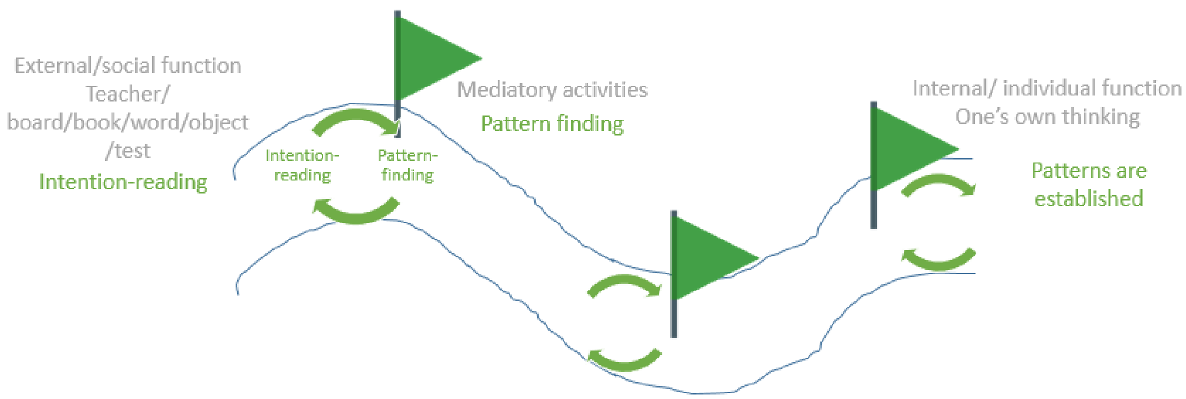

Figure 12.2: Process of intention-reading and pattern-finding

Figure 12.2 shows how I picture Tomasello's skills needed for language acquisition. Although intention-reading and pattern-finding are largely intertwined, the process of learning, a language in his case, starts with reading the intention of others and searching for patterns. The road depicts the journey taken to go from the initial discovery of the meaning behind words to having established patterns. These patterns could be incorrect, in which case the child would need to read the intentions and find alternative patterns. This is where Gopnik and Meltzoff's Theory

Theory is useful: it talks about theory testing and hypothesising that which is acquired. If what is acquired is incorrect then a new theory is developed.

Based on this usage-based theory, an awareness of structure emerges through usage (Ghalebi & Sadighi, 2015). This means that it is only through constant use of language that its structure or grammatical tenets emerge. In terms of language acquisition, awareness of language structure emerges from language use, and children become aware of the structure while trying to identify the intentions and patterns of the other speaker's language (Ghalebi & Sadighi, 2015). After patterns are identified, they are generalised to form abstract linguistic categories. It is possible that patterns could be generalised incorrectly, in which case the child will go back to reading the intentions and attempt to find a better pattern. This cyclical process of moving between intention-reading and pattern-finding is what I assert happens during internalisation and hence during synk-time. It is during this time that the initial emergence of structure develops and learners start becoming aware of patterns that enable them to then generalise and form abstract categories. Tomasello (2007) states that pattern-finding skills are not specific to language learners and hence can be generalised to the learning of mathematics. Therefore, it is only through the consistent exposure to the mathematics that the structural and grammatical rules of mathematics will emerge and potentially become visible. For example, the more a learner is exposed to different contexts that involve like and unlike terms, the more the learner will start to piece together what like terms are and the structure of like terms will emerge. Hence, learning to use mathematics takes time and consistent usage. The heart of acquiring a new skill is in the usage of its symbolic forms and these rely on the ability to understand the objective of what they are being exposed to.

The parallel skill to intention-reading in learning mathematics is where learners are able to have a shared understanding with others as to what are certain mathematical activities[4], it is about finding the meaning behind the intentions, the meaning of words, concepts, symbols and skills. For example, knowing what it means to *simplify* as opposed to what it means to *solve*. Having this functional understanding allows for a meaning-making dimension of the content.

The parallel skill to pattern-finding in mathematics is that learners categorise similar mathematical actions together but understand how they differ. For example, adding like terms in an expression versus adding them in an equation where you often have to first apply the additive inverse before adding like terms. Mathematically speaking, pattern-finding is where a learner is able to acknowledge that a certain skill (such as adding like terms) is applied in other topics, such as solving equations but at the same time recognise how it is different, so in solving

[4] I define *mathematical activities* as any activity that elicits learners' engagement in mathematics. It could constitute working on a worksheet, completing examples, doing homework, class work, talking mathematics, thinking or doing mathematics.

equations learners need to first apply inverse operations before adding like terms. Since pattern-finding is largely about form, in mathematics education one could say that it is about structure and generalisation. Mathematically it could therefore be considered the connections-making dimension. For example, connecting factorising with area, or solving simultaneous equations with the intersection of two functions.

12.4. Gopnik and Meltzoff learning theory: The Theory Theory

Gopnik and Meltzoff (1997) offer a theory called the Theory Theory (TTT), which asserts children's conceptual structures are theories and that "their conceptual development is theory formation and change" (p.11). They argue that both children and adults learn by creating, hypothesising and testing theories. The overarching idea of TTT is that children (and adults) develop "abstract, coherent, systems of entities and rules, particularly causal entities and rules" (Gopnik, 2003, p. 6), which in essence is a theory. These rules and abstract entities enable us to not only make predictions about new information but enable us to interpret and explain the information. While experiencing the world, we test the theories we created and any counter-evidence found makes us reinterpret or seek alternative theories (Gopnik, 2003). For example, sharing the attention with someone that a four-legged creature is a dog, a child may test his/her theory and point to a cat but say dog. Correction from the other person will lead the child to reinterpret his/her theory. Figure 2.3 shows the process of theory testing.

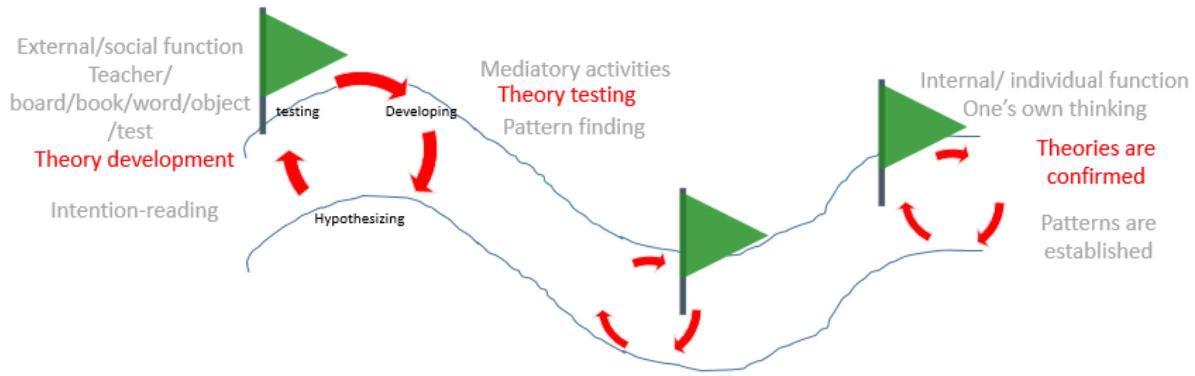

Figure 12.3: Process of theory testing

This theory fits well with intention-reading and pattern-finding in that it explains what happens as a child comes to know something. This means that we test our theories made about the form of the information and this results in creating meaning from the original information, the information received at the social level. It therefore serves a similar purpose to internalisation, where the external is being tested and internalised, where meaning is made from the information given at the social level. Just as multiple mediatory activities may be needed for the external to become internalised, so too constant hypothesising is needed for awareness of structure to emerge and to be able to categorise different pieces of information.

12.5. Towards an intergrated theory of the three theories

The theories discussed above all describe the process of 'coming to know'. They do so at different levels (Vygotsky is a broad theory and Tomasello and Gopnik are more focused on practical learning). They do, however, provide complementary concepts to describe how the learner acquires skills and knowledge. This process, internalisation, goes from the external to the internal; from intention-reading to established patterns and from developing theories to confirming theories. The process involves mediation in terms of mediatory activities, theory testing and hypothesising as well as pattern-finding. It is within this mediatory process that I place synk-time; this is because it describes the process while testing theories and finding patterns in the content.

A major finding from this study is that learners make more gains in Grade 10 than in Grade 9 on Grade 8 work. Higher levels of mathematical proficiency are demonstrated much later than one would expect and, also, much later than the assumptions built into the curriculum.

Using Vygotsky's theory of learning in coming to know mathematics, we can say learners have encountered the content twice: first in a social setting, where a teacher or more knowledgeable other (Vygotsky, 1978) introduces the concept or skill and second on an individual level, where the learner encounters the content or is required to use the skill. Encountering the content on an individual level could happen while in class, when doing homework, when dealing with other content (for example, encountering negative numbers while solving an equation), or when thinking, talking or writing mathematics.

Evidence of internalisation having taken place would be a learner obtaining the correct answers as this shows that a learner has understood certain content and is able to execute a procedure. Mediating the external (the content and skills taught) involves semiotic mediation, which is mediation through the use of signs and symbols In terms of my study. These would include the operation signs, the equal sign and variables. Learners are exposed to these signs and symbols while actively participating in mathematics, whether it is in a classroom while dealing with other topics, at home while doing homework, or talking to a peer about the work done in class. All these involve repeated exposure to the content or skill that was initially introduced in an external, social setting.

In essence, what I am arguing is that part of what enables internalisation is synk-time and that it involves theory-testing and pattern-finding, and very importantly, participation. ME Jones and Bond (2019) state that one way to learn a new language is through cultural immersion. Similarly, one way to learn mathematics is through consistent participation (Boaler (1999); Krauss, Baumert, and Blum (2008)).

Figure 12.4 is a diagrammatic view of how I see the different learning theories working together. Vygotsky's theory essentially tells at a broad level what we are working with, the external or the internal. The theory also provides a language to talk about the process (internalisation). Tomasello's theory offers us a way of talking about what happens during that process, and Gopnik and Meltzoff's theory helps in describing the process of learning.

In Figure 12.4 I display the three theories relevant to my study in a way that enables me to also talk about synk-time. Each theory is in a different-coloured text: Vygotsky: blue; Gopnik and Meltzoff: red; and Tomasello: green. On the left-hand side of the diagram I represent all the notions that deal with the learner encountering information. This information could be from a teacher, a test, a book, it could be hearing a new word for the first time and it is in Gopnik's terms, where theories would start to develop. For example, in mathematics this could be where learners are introduced to negative numbers in the social setting of the classroom. In this example, learners would be trying to read the intention of the example $-2-5$: what is it that they are supposed to do? Receiving information is the start of a journey, it's the start of a process. The process may be long or short, and in mathematics, the length of the journey may depend on prior knowledge, the strength of one's basic mathematics skills or the intensity of the exposure they have with content.

The journey is shown by a windy road, attempting to show that learning is not a linear process. It represents internalisation, which is the process for going from the external to the internal. Since internalisation incorporates mediatory activities, I have placed flags to represent different mediatory activities that aid internalisation. I see these as helpful nudges along the way. It is a stimulus trying to work out that which is being internalised. These stimuli are what helps formulate or categorise the information and hence pattern-finding is one. Pattern-finding is the way in which we test hypotheses and synk-time is the time dedicated to activities during pattern-finding. Whether you are exposed to content, practising, repeating or participating, what you are doing is looking for patterns within the content; patterns in the procedure. Synk time therefore incorporates mathematical activities that one does or is exposed to that help internalise the content.

The right-hand side of the diagram is the destination, it is where one has come-to-know and can provide a correct answer. However, it is important to note that I do not see the end product as being static, I see it as a work in progress, that we never really fully know. Especially in the context of mathematics education, there is always another context in which the skill can be applied that is not known to the learner.

The windy road as well as the circular arrows in the road represent the idea that learning is not a linear process and that it involves cyclical processes between developing theories, hypothesising and testing the theories. There are three sets of this cyclic process to reinforce that during internalisation sometimes we need more than one mediatory instance to internalise the information. This is why the process of internalisation is different for everyone.

It is important to remember that learning is a complex process and no diagram can depict this process in its entirety.

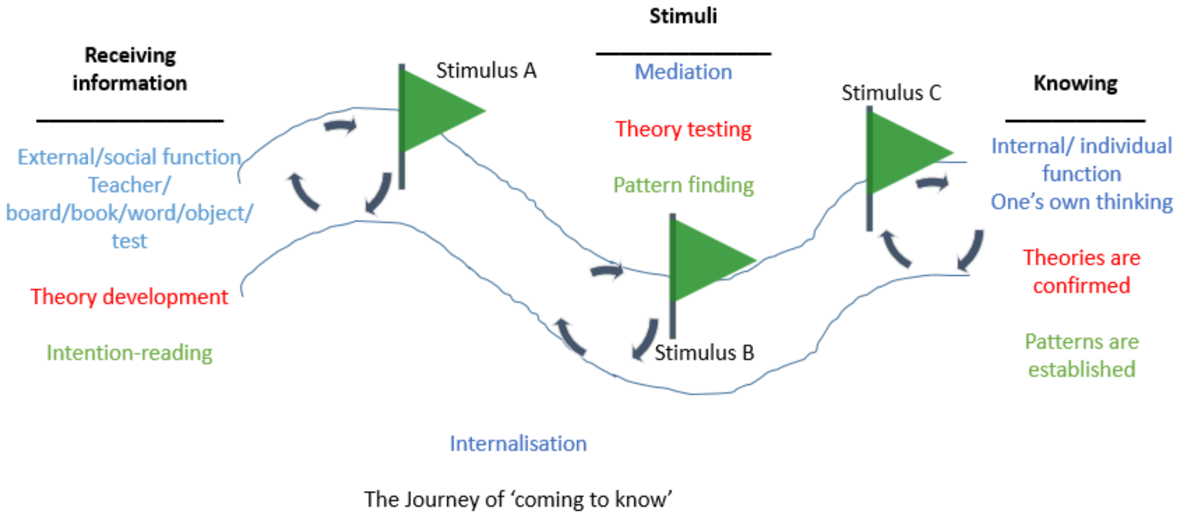

Figure 12.4: Diagrammatic conceptualisation of my theoretical framework

In order to exemplify the three theories and synk-time in respect to a mathematical topic, a diagrammatic version of the addition and subtraction of integers is shown in Figure 12.5. The content that is being learnt is the addition and subtraction of integers which we see at the start of the windy road. The first mediatory activity could the first exposure to the topic, where the teacher teaches and gives examples, exercises or activities to be done. In participating in these activities, learners find patterns, develop theories and attempt to read the intention of each activity. The patterns found, theories developed and intention reading may not be correct and these are revised the more the learner participates and the more exposure given to the learner. After learning about integers learners are exposed to like terms that involve integers, for example $-7x - 5x = -12x$. Although the teacher teaches simplification of like terms, the addition and subtraction of integers content remains in the background, where previous theories and patterns are tested, and revised if found to not work. Later, learners are exposed to equations which again involve integer addition and subtraction and where again patterns and theories are tested and revised or confirmed. In Grade 9 learners then encounter factorising where they are explicitly dealing with integer addition and subtraction as well as integer multiplication. For example when factorising $x^2 - 12x + 35$, learners need to find factors of 35 that when added together give -12. One could continue and show that in Grade 10 when learners are exposed to drawing quadratic functions they are again exposed to integers. All of these exposures are beneficial for the learners' development in integer addition and subtraction. My data shows that it is only in Grade 10 that we see the gains made in integer addition and subtraction. This suggests that the continuous exposure to integers, in different contexts aided the learners' development and points towards teachers being more mindful of the type of activity they could expose their learners to.

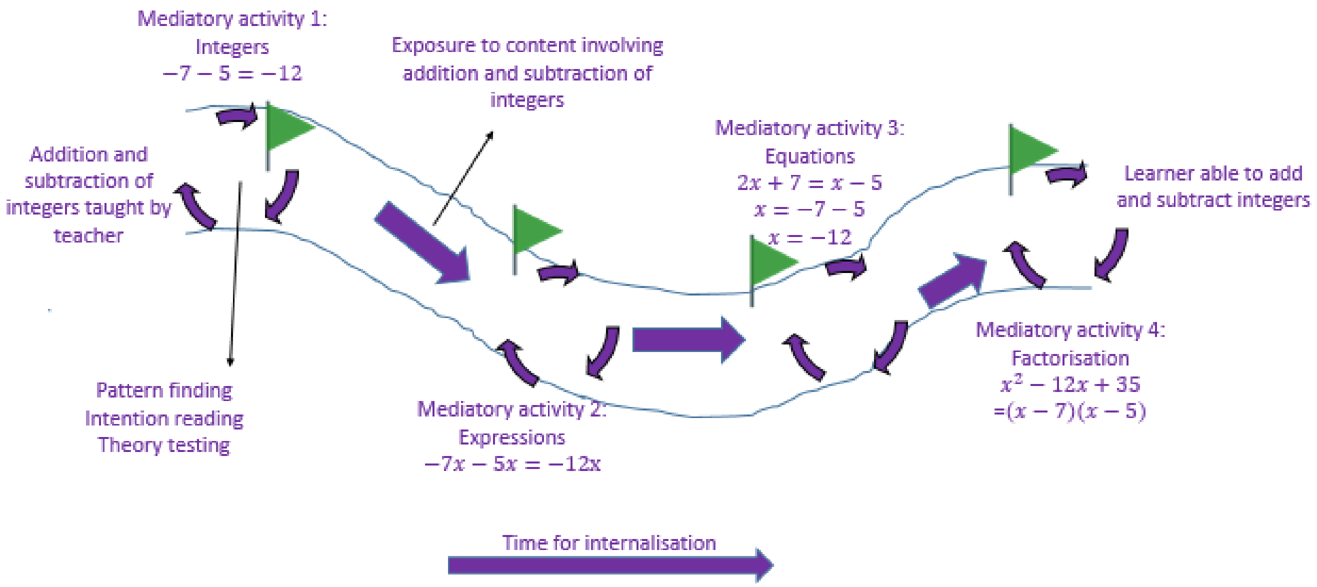

Figure 12.5: Exemplification of the addition and subtraction of integers in terms of the three theories

12.6. Defining synk-time

I define synk-time as the process of internalisation that depends on opportunities (implicit and explicit mediatory activities) to be actively working on mathematical content that will lead to the maturation of concepts. The key elements of synk-time are 1) a process that occurs over time; 2) explicit and implicit mediatory activities and 3) involves the learner taking action.

Every mediatory activity is an opportunity for learners to encounter mathematical content. At first this encounter happens externally, from the teacher in the form of explicit teaching, but after that any mathematics that involves the same concept that is actively worked on by the learner is an implicit mediatory activity. Central to the idea of synk-time is action by the individual. This action could be thinking or talking about the concept, doing homework, or answering questions on a worksheet.

'Synk' is a combination of 'sync' and 'sink'. 'Sync' relates to something being synchronised or matched up with (Stevenson, 2010) and so could involve the synchronisation of the external to the internal. The term 'sink' or 'sink-in' means to realise something or understand it fully (Stevenson, 2010). Synk therefore incorporates both. It refers to a process that occurs over time rather than just a duration of time. Synk-time synchronizes the form and meaning parts of information and enables information to be sufficiently understood to lead to correct answers. Synk-time is, therefore, active time, time used to engage with, participate in and be exposed to different mathematical content that helps understand other mathematical content. ‘Synk-time’ refers to the maturation of concepts, or in other words, the time learners need for content to ‘sink-in’. This time that learners need is not passive time but rather requires active participation, practice, repetition and exposure to the content and skills. It is about the opportunities the learners get to re-conceptualise the content they are working on. I am deliberately using ‘synk-time’ rather than ‘synk-in’ or ‘synking’ because I want to emphasise that in order for synchronisation to happen, the learners need some amount of time to do so. Taking longer for something to sink in suggests that more synk-time was needed and that the internalisation process was longer.

12.7. Evidence in my data of the need for synk-time

Most learners in my study appear to internalise content somewhere between one and two years after their first encounter with this. According to the curriculum, the content they have internalised should have been mastered at least a year earlier. This suggests that the activities done in Grades 8 and 9 were not enough for it to become internal. The fact that Grade 10 learners showed more gains suggests that the mathematical activities and actions performed in Grade 10 were more conducive to learning Grade 8 content. In my opinion, some of the mathematical actions performed in Grade 10 encouraged a better understanding of content. For example, regularly working with different types of equations in Grade 10, such as $y = x^2 - 5x - 6$ or $y = 2^x - 8$, reinforces that unlike terms cannot be added together. When learners are exposed to functions, whether $y = 3x - 2$ or $y = x^2 - 5x + 6$, learners learn to read the intention behind the question and do not add the different terms together. This could be benefiting their performance in simplifying expressions where by Grade 10 learners are not adding, for example, $2a$ and $5b$ together when given $2a + 5b$. Another example is that the constant practice and exposure to factorising where one needs to be careful with the signs used could explain why learners in Grade 10 had more gains in working with negatives than learners in Grade 9. For example factorising $x^2 - x - 6$ and obtaining $(x - 3)(x + 2)$ because $-3x + 2x = -x$ rather than $(x + 3)(x - 2)$ which would produce a middle term of $+x$; or even $(x - 3)(x - 2)$ which would produce a middle term of $-5x$.Having to decide which factors to use is based on integer addition and subtraction and hence the exposure to factorisation could be benefiting the learners’ performance in integers.
In terms of my data, it is only at the end of Grade 10 that we see many more learners getting the item $4747 + 3945 = ? + 3943$ correct, meaning that it is only at the end of Grade 10 that learners are approaching an equation structurally. This is happening much later than expected. The improvement may be due to regular focus on the structures of different functions and knowing what for example the effect of a and c are in $ax^2 + bx + c$.

It is also possible that the synk-time is so long because of the contextual factors of my sample. My sample consists of learners from poor socio-economic backgrounds and hence they are possibly exposed to fewer mediatory instances. For example, they would not be able to afford extra lessons, and their schools are less resourced so they do not experience a variety of contexts that the content or skill could be seen in, meaning that if they had access to, for example, different textbooks, they would be exposed to similar questions being asked in different ways. Besides the contextual factors, it is possible that there are more gains in Grade 10 because the curriculum provides more, and different, repetition of a similar skill. For example, dealing with quadratics still involves adding of like terms. Another reason includes better teaching in Grade 10. Anecdotal evidence suggests that teachers who teach at a higher level are better qualified, have more experience, and hence are able to make more connections when teaching. My experience suggests that teachers who teach Grades 8 and 9 are often out-of-field teachers with little experience and are less likely to be able to make connections. A further reason for more gains in Grade 10 is increased motivation on the part of the learner. These learners chose Mathematics over Mathematical Literacy and hence may be more motivated to do well.

12.8. Interpreting my findings in relation to synk-time

In this section I will discuss four findings from each of the qualitative analysis chapters, Chapters 7-10, in terms of the three theories. I do this to elaborate on the integration of the three theories by linking it to my data.

In Chapter 7 I provided an analysis of the equation items, using five different lenses. One of the main findings was that Grade 9 learners are not using inverse operations to solve an equation, but Grade 10 learners are using inverse operations more. In Grades 8 and 9 learners are to taught to solve an equation using inverses, but when confronted with an equation in the test, they did not use inverses, suggesting that they have not internalised the concept of inverse operations. The fact that learners continued to get the solutions to the equations incorrect, and that the incorrect solution was different in the pre- and post-test, suggests that learners were continuously trying to find a pattern (Tomasello, 2009) in the solutions to the equations and that they were continuously revising the theories (Gopnik & Meltzoff, 1997) they had built up about how to solve an equation. Figures 12.7 and 12.8 show a learner's response to item 9c in the pre- and post-test. In the pre-test, although the learner correctly added the additive inverse of the constant, 2, the learner did not do the same for the letter $-x$. Instead, the learner conjoined $7 - x$ to obtain $8x$. The learner appears to be working with an incorrect theory that one can add unlike terms, and possibly obtained this pattern from prior knowledge such as adding fractions where 2 $+\frac{1}{2} = 2\frac{1}{2}$. In the post-test we see a shift in the learner's response, a change in pattern-finding and theory-making, and observe that the learner addresses the $-x$ on the right hand side, but adds $-x$ instead of x. This shows a positive change in theory development and pattern finding and suggests that during the year, the more exposure to equation solving helped the learner see the pattern in applying inverses to both the numerical constants and the letters.

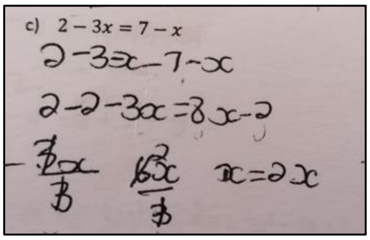

Figure 12.6: Learner A's response to item 9c in the pre-test

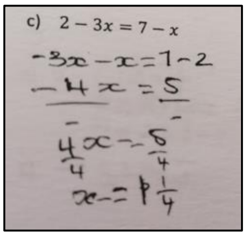

Figure 12.7: Learner A's response to item 9c in the post-test

In Grade 10, there were many more learners that used inverses, suggesting that having a year to internalise the concept, hence, having a year of synk-time, was beneficial. This points towards spending more time on equations in order to expose learners to inverses more often. Figures 12.9 and 12.10[5] show learner B's responses to items 9b and c in the pre- and post-test. In the pre-test, the learner appears to have a theory that involves adding like terms as well as what to divide by to isolate x. We see the learner uses inverses in the post-test, hence changing his/her theory. Although the learner correctly applied the inverse operations in the post-test, s/he has incorrectly added like terms by misapplying the laws for multiplying powers. The learner does this in both items 9b and 9c in the post-test, showing his/her theory for how to add like terms.

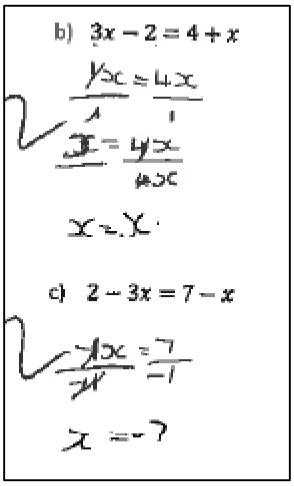

Figure 12.8: Learner B's responses in the pre-test

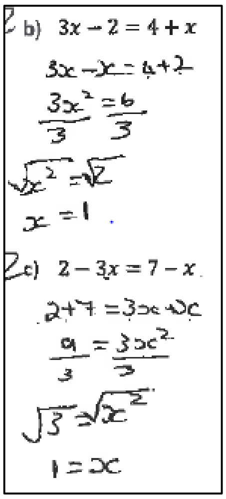

Figure 12.9: Learner B's responses in the post-test

In chapter 8 I discussed the errors made in Grade 9 and Grade 10 when answering numeric equations where the items tested the view of the equal sign, for example $7 + 5 = ? + 2$. It was noted that there were many different responses to the items. The variety of responses is further evidence of learners not knowing how to approach these types of equations. It suggests that learners have multiple theories; that there are many different theories

[5] The 2's on the left hand side of the learners response was an initial code given than meant the response was incorrect. If it was correct a 1 was given and if it was left blank, 0 was given.

across all learners (Gopnik & Meltzoff, 1997); and that learners have established many incorrect patterns (Tomasello, 2009). It was also found that many learners gave the same response in the pre- and post-test, for example for both the pre- and post-test they answered $7 + 5 = ? + 2$ with 12, which is an operational view of the equal sign. What these results tell us is that the majority of the learners did not change the theory (Gopnik & Meltzoff, 1997) they were using to get to the answer. If they had changed their theory, we would have seen a higher percentage of learners going from wrong to right or even right to wrong. The large percentage of learners that made no gains is also suggestive of them not looking for or attempting to find better patterns (Tomasello, 2009) to aid them to get the right answer. The results from item 4b) $4747 + 3945 = \_ + 3943$, with 46% of Grade 10 learners answering relationally and only 15% operationally, suggest that more Grade 10 learners approach equality items relationally, and suggesting that they have internalised the meaning of equality.

In Grade 9, learners were still mainly working operationally but in the Grade 10 post-test there was a change in the responses given, suggesting that the correct patterns were found and correct theories were tested and hence brought about correct answers. The fact that this change happened is encouraging but it should have occurred in Grade 8. This is evidence that learners need more time to be exposed to the content, more time for pattern finding and theory testing. This all points towards a need for synk-time. Literature (e.g. Knuth et al., 2005) has suggested that one's view of the equal sign when dealing with arithmetic sentences influences one's equation-solving abilities, but I found that this is only the case when learners are in Grade 10. This points towards learners having different theories for different topics that involve the same content in Grade 9 but then having similar theories in Grade 10. It also suggests that in Grade 9, learners find patterns for different contexts and don't transfer the knowledge from one topic into another, but in Grade 10 their responses to different topics that involve the same content are more aligned.

In Chapter 9 I focused on learner responses to integer items. The errors learners made: right-to-left reasoning, avoiding the minus symbol etc., all point to learners having their own incorrect theories and established patterns. Learners who maintained their error in subsequent tests are more likely to have the incorrect patterns entrenched in them, having incorrectly internalised knowledge. Learners who change their errors in subsequent tests point towards having revised their theories. In Figures 12.11 and 12.12, learner C has maintained his/her theory of adding integers of the structure $a - b$ where $b > a$ and does so by adding from right to left. Also in item 3c, Learner C maintains his/her pattern and theory of adding integers where there is a leading negative, and applied bracket reasoning. In item 3b we see the learner change his/her pattern for adding integers where in the pre-test s/he avoids the negative sign and merely adds 5 and 7. In the post-test the learner applies bracket reasoning, again confirming that s/he has developed a pattern in dealing with items where there is a leading negative.

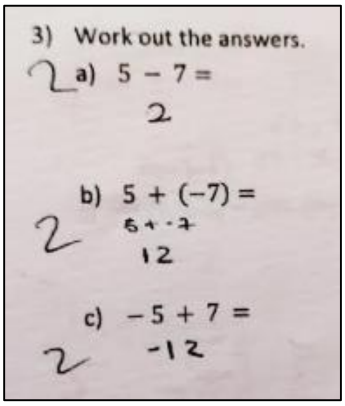

Figure 12.10: Learner C's responses in the pre-test

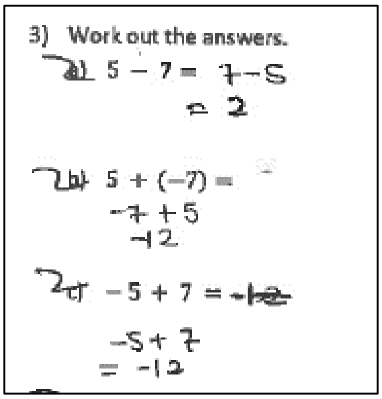

Figure 12.11: Learner C's responses in the post-test

Correct answers are suggestive of knowledge and skills internalised correctly. One of the main findings from the analysis of integer responses is that there was, again, a big improvement in Grade 10. This points, again, to learners needing more synk-time, more time to develop the correct theories and patterns.

In Chapter 10 I focused on learner responses to expression items. In items 5b) $2a + 5b + a$; 5d) $a + 4 + a - 4$ and 5f) $a + b + a - b$ we see that Grade 10s improve more than the Grade 9s did, which would coincide with the findings in Chapters 7-9 that more gains are made within Grade 10. What is common in all three items is the increased focus on separating out unlike terms. This suggests that in Grade 10, learners are more comfortable with unlike terms, meaning they get more items correct, which confirms my argument that learners need synk-time to adjust to what they learn. It is however surprising that they needed two years of synk-time since this is Grade 8 content. The fact that learners are only starting to master the addition of like terms, suggests that they needed time for the content to settle in their minds, that is, learners need *synk-time* time for content to *sink-in.*

All four analysis chapters focused on the errors made. In terms of the three theories an error is evidence of content not being internalised correctly. It is also indicative of learners having established the incorrect patterns from incorrect theories developed. A change in error shows that patterns and theories are changing and a correct answer suggests content has been correctly internalised and that the correct theories and patterns have been developed. I argued that a possible reason for the delay in improved performance is that learners need time to mature, or what I call synk-time.

12.9. Conclusion

Learning is a complex process and no diagram or thesis can capture it in its entirety. In this chapter I have introduced an integration of three theoretical constructs that account for a delay in learning. While Vygotsky's (1978) notion of internalisation was useful it did not provide enough language to talk about what happened during internalisation. Tomasello (2009), as well as Gopnik and Meltzof (1997), provided a language to discuss the process of internalisation. The process of internalisation is what I have termed synk-time.

In this section, I argued that learners need time to practise mathematics and immerse themselves in mathematical activities to internalise mathematical content and skills. The amount of time learners need to internalise mathematical content is individual but also dependant on the type of content to be learnt, for example, learning to factorise takes longer than learning about like and unlike terms. The time needed may be a few minutes, an hour, a day, or a few months. In arguing that learners need more exposure to mathematical content, I drew on three theoretical constructs. I discussed each construct individually with a diagram and then produced a final diagram showing how the three theories fit together.

Chapter 13 : Concluding chapter

13.1. Introduction

As mentioned in the introductory chapter, my motivation for this study (and for originally wanting to be a mathematics teacher) came from the deep personal desire to help learners improve the level of their mathematical achievement. As a teacher I want to understand my learners better, to understand what their problems are. My connection with WMCS meant I had access to a significant amount of data that I could process and investigate what learners' problems were. The large amount of data meant that in terms of my PhD I needed to focus on some aspects to narrow the scope of the thesis. I chose to focus on integers, expressions, equality and equations.

This thesis is located in the context of the persistent poor mathematics attainment in the majority of schools in South Africa, and in particular the poor performance in algebra and solving linear equations. The so-called crisis in secondary school mathematics learning in South Africa is evident in the poor learner attainment in local (DBE, 2016) and international assessments (Reddy et al., 2012) and is evident in my data as well.

Grounded on the use of three symbols: the equality symbol; the minus symbol and letters, this study had two foci: changes in errors and the relationships between the errors made when solving linear equations and errors made in other topics

Learners' poor performance is often characterised by test responses that make little or no sense to the reader. In this sense, responses are chaotic. This study involved seeking order in the chaos of learner errors. I developed a methodology to quieten the chaos while providing insights into learner performance and learner error. A mixed-methods research design was employed in order to achieve this. Quantitative methods were conducted on a large sample of Grade 9 and 10 learners (n =2135) as well as on a randomly selected subset consisting of 150 Grade 9 learners and 150 Grade 10 learners from 23 different schools. Techniques such as T-tests and correlations were used to analyse pre- and post-tests results. Taking a social constructivist approach to learning and learner error, the subset of data was also analysed qualitatively. Both typological and inductive data analysis were employed in the topics of equality, integers, expressions and equations. Having found common errors, a change in errors was explored, and the relatedness between errors was investigated.

I begin this chapter by answering the research questions and providing a summary of the findings. I then highlight the contributions of this thesis and close with a reflection on the research and recommendations for future studies.

13.2. Revisiting the research questions

This study was guided by three research questions. The first question was: What is the general performance of a cohort of Grade 9 and 10 learners in terms of simplifying integers, simplifying expressions, solving equations and understanding the dual role of the equal sign? This question interrogated the general performance of a cohort of Grade 9 and 10 learners. In Chapter 6 I provided evidence of learners' poor performance on a large scale with a large sample. The learner, topic and item scores showed how poorly learners responded to the 18 test items. However, the scores also showed where there was the most improvement. More gains were made in Grade 10 than in Grade 9 on integer and equations items. Chapter 6 provided a general understanding and initial overview of the sample's mathematical ability/performance. I also gained insight into learner difficulties, for example in solving equations, where 62% of Grade 9 learners did not improve in the number of equation items they got correct.

The second research question investigated what errors were made and how these changed over time. Two sub-questions were asked: firstly, what errors are made when dealing with integers, expressions, equality and equations in Grade 9 and Grade 10? And secondly, which errors disappear, are persistent or change over time? In coding the errors, Chapter 7 was the apex of creating order from chaos. Again, Grade 10 learners made more gains than Grade 9 learners with more than a 20% difference in the percentage of correct responses from the Grade 10 pre-test to the Grade 10 post-test. T-tests show that in both grades there was a statistically significant improvement in the learners' performance on the equations items. Although some learners showed no improvement in the number of items they got correct, they did show a change in their response and the errors made. At the end of Grade 9, The majority of my sample were unable to solve a simple linear equation and by the end of Grade 10 only 37% of learners were able to solve an equation with variables/letters on both sides such as $2x - 3 = 7 - x$. The majority of Grade 9 learners continued to have an operational view of the equal sign, but in Grade 10 there were more relational responses. I say this because more learners considered both the left and right sides of the equal sign. Surprisingly, less than 30% of Grade 9 learners used inverses to solve linear equation items but in Grade 10 more than 65% of learners used inverses for all three items. This is evidence that learners have not internalised the need for inverse operations in Grade 9.

In terms of learners' views of the equal sign, I argued two main points. These related to the fact that few learners showed an improvement in terms of the number of items they got correct in the post-test and secondly, that in

Grade 10, learners *started* to work relationally (see Chapter 8). This has important repercussions for solving equations. If learners are working relationally they are more than likely applying inverse operations- which is what we saw in Chapter 7.

When focusing on the errors made in items relating to integers, there was a bigger increase in correct responses in Grade 10 (see Chapter 9). This was also found in the analysis of the overall performance of integer items but in this chapter we saw that the big increase in Grade 10 was due to the same four items where learners obtained at most 38%. Although there were big improvements in Grade 10, we saw that learners appeared to have particular problems with items that involve brackets (items 3b and 3d) and in items that have leading negatives (item 3c and 3e).

Another finding was that conjoining, although still a common error, decreased substantially in Grade 10 (see Chapter 10). There were fewer conjoining errors in Grade 10, where the lowest percentage in the post-test was 12% in item 5e and the maximum percentage was 33% in item 5g, suggesting that conjoining is a persistent error. In Grade 10, there was an increase in learners misapplying the exponential laws when adding like and unlike terms. I also highlighted that learners operate on what is inside of brackets even though they are unlike terms.

The third research question was about the relationship between errors in the different topics. Based on qualitative analyses, no relationship was found between solving equations and the other topics. This means that learners are approaching the mathematics in different topics differently. Errors made when dealing with integers are different to the integer errors made when solving equations. A possible reason for this is that when learners begin to solve equations algebraically they are consumed with new rules such as 'do to the right as you do to the left'. Another possible reason for this is that the items were not specifically designed for this study (this is a limitation and is discussed later in the chapter). In solving equations there was a change from an operational to a relational view in Grade 10 but this was not the same when analysing the equality items as there was very little change from Grade 9 to Grade 10. Conjoining was a prominent error when analysing the expression items, and the error decreased substantially in Grade 10. When investigating the conjoining error when solving equations, however, there were fewer conjoining errors than in the expression items. This means that learners were less likely to conjoin when solving equations as compared to when they are asked to simplify expressions. A possibly explanation for this is that they are too busy trying to 'move terms to the other side' rather than add unlike terms together. The same was found for negativity errors, less errors relating to negatives were found when solving equations as compared to when learners dealt with integer items.

One of the consistent findings in learner performance on the individual topics was that Grade 10 learners made more gains than Grade 9 learners. This was not expected since the items tested Grade 8 and 9 content. It was expected that the Grade 10s would perform better but not that they would make more gains. I introduced the notion of *synk-time* to suggest that learners need more time for the mathematics they are learning to 'sink-in'; and to be synchronised internally. This led me to search for theory that may explain this notion. Three theoretical constructs informed the notion of synk-time: Vygotsky's (1978) notion of internalisation, Tomasello's (2003) notion of intention reading and pattern finding, and Gopnik and Meltzoff's (1997) notion of the Theory Theory. Vygotsky's theory did not account for the process of re-internalising content and hence Gopnik and Meltzoff's Theory Theory was useful. Internalisation offers a way of talking about what takes place but does not offer a mechanism (a *how)* to account for what takes place. This is where Tomasello's theory of language acquisition was useful. Building a theoretical position of how these three constructs work together and account for a delay in learning was therefore an outcome of this study. What I argued is that part of what enables internalisation is synk-time involving theory testing and pattern-finding, and very importantly, participation.

13.3. Contributions of this study

This study makes three contributions to the mathematics education research community. They are methodologically, practically and theoretically. Methodologically I have contributed to the mathematics research community by conducting a mixed-methods study. This was done by looking quantitatively at a large sample (see Chapter 6) as well as with a sub-sample (see Chapters 7-10). I showed that the results from the sub-sample could be generalised to the larger sample. In Chapters 7-10 I also conducted an error analysis, which is strongly qualitative in nature. Practically, in terms of the errors made, I have contributed to the mathematics education research community by, for example, investigating older learners' responses to numeric equations (see Chapter 8), where typically only younger learners' responses are investigated. By investigating older learners' responses I added to the literature, showing that indeed, as learners get older they start to view the equal sign relationally. I also looked at responses to equations in a novel way, through five different lenses (see Chapter 7). This helped me create order from the chaos of responses. Theoretically, I have contributed to the mathematics research community by integrating three theoretical constructs and relating them to my data (see Chapter 12).

13.4. Limitations of the study

No study can capture the full story of learning and hence all studies have their limitations. This study has three main limitations: the first is that data was collected only from a written assessment rather than from interviews as well. Asking follow-up questions or conducting individual interviews with students could have shed further light on students' understandings. The second limitation is that the test instrument was not developed for the purpose of my study and hence a) the items were not designed to investigate links between errors and b) there is a limited number of items per topic. In addition, many of the errors related to negativity are related to the structure of the item, and therefore it is a limitation for my study. The third main limitation relates to decisions I made in the coding of qualitative data. Due to the scope of my study I decided to focus on five ways of looking at the responses to equations. Also, when analysing the letter errors I deliberately only investigated conjoining and exponential errors, this was beneficial for my study but remains a limitation on the basis that I could have looked deeper at some of the responses.

13.5. Point to future research and recommendations for teaching

Having conducted such a large study, I end with multiple recommendations for future research and for teaching In terms of recommendations for research, there are methodological approaches that could be addressed as well as alternative foci to consider and lastly recommendations for probing more deeply into aspects of my research. In terms of recommendations for teaching, I point the reader towards the curriculum and practice. These are elaborated on below.

13.5.1. Recommendations for research

13.5.1.1. Methodological approaches

Interviews

Following from the above-mentioned limitations, a point towards future research would be to conduct a similar study but to include interviews with learners. This would get into their heads, and understand why they made the errors they did. Interviews could be conducted in February and then a follow-up interview in October. This will enable the researcher to see changes in learners talk, and thought processes.

Additional test items

In addition, where one of my limitations was that the test instrument was not designed for the purpose of my study, an idea for future research would be to conduct a similar study but to purposefully design a test that has more items with the same structure that are designed for the purpose of linking errors across different topics. In particular, I would have more equation items, for example place $2-3x=x+4$ and $x+4=3x-2$ between items 9b) $3x-2=x+4$ and 9c) $2-3x=7-x$. These two additions would be beneficial because from $3x-2=x+4$ to $2-3x=x+4$ we are only changing the order of the terms on the left hand side, this could give insight into whether having the algebraic term subtracted is more difficult for the learners. Inserting $x+4=3x-2$ would be key in determining whether learners know the reflective property of solving equations.

13.5.1.2. Alternative foci

Hierarchy of errors

Based on my focus on errors, I suggest that future research includes conducting a similar study with the lens of determining whether there is a hierarchy of errors. This would enable us to see improvement amidst poor performance.

Intervention study

When looking across items and across the pre- and post-test, there is a large percentage of learners who have unstable knowledge. Literature suggests that this can be addressed with pedagogic intervention using technology (Bardini, Oldenburg, Stacey, & Pierce, 2013), much classroom discussion (Saenz-Ludlow & Walgamuth, 1998), and exposure to different types of numeric equations (Behr et al., 1980), for example the non-standard equation $7 = 7$ or $1 + 3 = 2 + 2$. An intervention study is therefore another point toward future research. An intervention study that focuses on language such as the difference between sign and operation when learning about integers could be beneficial to see whether it makes a difference to learners' performance in integer item.

13.5.1.3. Deeper probing into my research

Inverses

One of the most interesting findings from my data analysis was that Grade 9 learners do not use inverses but in Grade 10 they do. It would therefore be beneficial to conduct a study that only investigates the use of inverses and why Grade 9 learners use other methods to solve equations. Developing codes for the different type of inverse errors and investigating whether more errors are made when dealing with letter inverses rather than constant inverses could be beneficial.

Synk-time

Another recommendation I would make for future research is to investigate my idea of synk-time. This could be done by investigating what topics learners study during the two years and how it may aid understanding in solving equations, simplifying expressions or adding and subtracting integers. It may also be beneficial to analyse the curriculum to see which topics and pieces of content aid understanding in solving equations.

13.5.2. Recommendations for teaching

13.5.2.1. Curriculum

Curriculum coverage and content

A further recommendation for future research points towards an investigation into the amount of time given for backlogs in mathematics, as well as what revision is covered in the curriculum.

The fact that I saw gains made on Grade 8 and 9 work in Grade 10 suggests that there are not enough opportunities for Grade 8 and 9 learners to internalise content. This study has shown that even in Grade 10 learners struggle with solving equations with a letter on both sides, how then can we expect learners to solve quadratic equations in Grade 9? A suggestion to curriculum developers is to take out unnecessary equations from the Grade 9 syllabus (such as quadratic and exponential equations) and dedicate more time to solving equations with a letter on both sides.

13.5.2.2. Practice

A recommendation for teaching equality is that more opportunities are given to learners to see non-standard equations, for example 7=3+4 rather than the conventional 3+4=7. In addition emphasising the equal sign as 'the same as' rather than as a 'gives me' symbol. The errors seen when dealing with integer addition and subtraction suggest that learners do not know the difference between a sign and an operation. I therefore suggest that focusing on this language will benefit learners and improve performance. A recommendation for when teaching equations is to provide learners with more opportunities to solve equations using inverses and being explicit that inverses are used rather than 'moving to the other side'.

13.6. Conclusion

This mixed-methods study of learner errors revealed that Grade 10 learners make more gains than Grade 9 learners where a possible reason for learners' delayed achievement can be understood through the notion of synk-time. The four main takeaways of this study relate to: 1) Grade 10s making more gains than Grade 9s. This was a finding that was carried through from Chapters 6-10. 2) Generalisability. It was shown in Chapter 11 that my sub-sample results are very similar to the quantitative results from the full sample, and hence the results are generalisable. 3) The relationship between errors. I wanted to know whether the errors learners made in certain topics were carried through to equations and it was found that they make more errors in the individual topics, for example, there were more negativity errors in the integer items than made in responding to solving an equation. 4) The use of inverses. A surprising and important finding is that Grade 9 learners are less likely to use inverse operations but by the end of Grade 10, inverses are used by almost everyone.

These findings resulted in creating a theoretical framework where I integrated Vygotsky's (1978) notion of internalisation, Tomasello's (2003) notion of intention reading and pattern finding, and Gopnik and Meltzoff's (1997) notion of The Theory Theory.

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Appendices

Grade 9 Full sample		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
	Equation	1,0	0,3	0,4	0,4	0,5	0,3	0,3	0,5
	Equality	0,3	1,0	0,4	0,2	0,3	0,6	0,3	0,3
	Integer	0,4	0,4	1,0	0,2	0,4	0,4	0,5	0,4
Pre-test	Expression	0,4	0,2	0,2	1,0	0,3	0,2	0,2	0,4
	Equation	0,5	0,3	0,4	0,3	1,0	0,4	0,4	0,6
	Equality	0,3	0,6	0,4	0,2	0,4	1,0	0,4	0,3
	Integer	0,3	0,3	0,5	0,2	0,4	0,4	1,0	0,4
Post-test	Expression	0,5	0,3	0,4	0,4	0,6	0,3	0,4	1,0

Grade 9 Sub-sample		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
	Equation	1,0	0,4	0,6	0,6	0,5	0,3	0,4	0,6
	Equality	0,4	1,0	0,5	0,2	0,4	0,6	0,2	0,3
	Integer	0,6	0,5	1,0	0,5	0,4	0,4	0,5	0,4
Pre-test	Expression	0,6	0,2	0,5	1,0	0,5	0,2	0,4	0,6
	Equation	0,5	0,4	0,4	0,5	1,0	0,3	0,5	0,6
	Equality	0,3	0,6	0,4	0,2	0,3	1,0	0,3	0,2
	Integer	0,4	0,2	0,5	0,4	0,5	0,3	1,0	0,4
Post-test	Expression	0,6	0,3	0,4	0,6	0,6	0,2	0,4	1,0

Grade 10 Full sample		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
	Equation	1,0	0,4	0,5	0,6	0,6	0,4	0,4	0,6
	Equality	0,4	1,0	0,4	0,4	0,3	0,6	0,3	0,4
	Integer	0,5	0,4	1,0	0,5	0,5	0,4	0,5	0,5
Pre-test	Expression	0,6	0,4	0,5	1,0	0,5	0,4	0,4	0,7
	Equation	0,6	0,3	0,5	0,5	1,0	0,4	0,5	0,6
	Equality	0,4	0,6	0,4	0,4	0,4	1,0	0,4	0,4
	Integer	0,4	0,3	0,5	0,4	0,5	0,4	1,0	0,5
Post-test	Expression	0,6	0,4	0,5	0,7	0,6	0,4	0,5	1,0

Grade 10 Sub-sample		Pre-test				Post-test			
		Equation	Equality	Integer	Expression	Equation	Equality	Integer	Expression
Pre-test	Equation	1,0	0,4	0,5	0,6	0,6	0,4	0,4	0,6
	Equality	0,4	1,0	0,4	0,4	0,3	0,5	0,4	0,5
	Integer	0,5	0,4	1,0	0,5	0,6	0,4	0,5	0,5
	Expression	0,6	0,4	0,5	1,0	0,5	0,5	0,4	0,7
Post-test	Equation	0,6	0,3	0,6	0,5	1,0	0,5	0,5	0,6
	Equality	0,4	0,5	0,4	0,5	0,5	1,0	0,5	0,5
	Integer	0,4	0,4	0,5	0,4	0,5	0,5	1,0	0,4
	Expression	0,6	0,5	0,5	0,7	0,6	0,5	0,4	1,0