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# Gauged permutation invariant matrix quantum mechanics: partition functions

**Denjoe O'Connor**[a] **and Sanjaye Ramgoolam**[b,c]

[a]*School of Theoretical Physics, Dublin Institute of Theoretical Physics, 10 Burlington Road, Dublin 4, Ireland*

[b]*School of Physics and Astronomy, Centre for Theoretical Physics, Queen Mary University of London, London E1 4NS, U.K.*

[c]*School of Physics and Mandelstam Institute for Theoretical Physics, University of Witwatersrand, Wits, 2050, South Africa*

*E-mail:* denjoe@stp.dias.ie, s.ramgoolam@qmul.ac.uk

Abstract: The Hilbert spaces of matrix quantum mechanical systems with $N \times N$ matrix degrees of freedom $X$ have been analysed recently in terms of $S_N$ symmetric group elements $U$ acting as $X \to UXU^T$. Solvable models have been constructed uncovering partition algebras as hidden symmetries of these systems. The solvable models include an 11-dimensional space of matrix harmonic oscillators, the simplest of which is the standard matrix harmonic oscillator with $\mathrm{U}(N)$ symmetry. The permutation symmetry is realised as gauge symmetry in a path integral formulation in a companion paper. With the simplest matrix oscillator Hamiltonian subject to gauge permutation symmetry, we use the known result for the micro-canonical partition function to derive the canonical partition function. It is expressed as a sum over partitions of $N$ of products of factors which depend on elementary number-theoretic properties of the partitions, notably the least common multiples and greatest common divisors of pairs of parts appearing in the partition. This formula is recovered using the Molien-Weyl formula, which we review for convenience. The Molien-Weyl formula is then used to generalise the formula for the canonical partition function to the 11-parameter permutation invariant matrix harmonic oscillator.

Keywords: Discrete Symmetries, Gauge Symmetry, Lattice Quantum Field Theory, Matrix Models

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## Contents



## 1 Introduction

Following the long tradition of diverse applications of traditional matrix models with continuous symmetries to real world data ranging across nuclear physics, condensed matter physics and statistical finance (see e.g. [1–3]), permutation invariant matrix models with Gaussian action have been classified and applied to the statistical analysis of ensembles of matrices representing real-world data [4–7, 10]. Permutation invariant sectors of quantum mechanical systems with matrix degrees of freedom have been characterised using partition algebras and solvable models have been described [11] by adapting techniques based on Schur-Weyl duality for matrix systems with continuous symmetries [12–15]. The development of these quantum systems is motivated by the search for similarities in the physics between models with permutation symmetry and those models which have known holographic duals such as [16, 17]. This can inform the search for holographic duals of matrix quantum mechanics models with finite group symmetries. In [18] permutation invariance in matrix quantum mechanics has been realised as a gauge symmetry. The path integral has been described as a limit of a lattice construction and, for the case of the matrix harmonic oscillator, the Molien-Weyl formula for invariants has been obtained from the path integral.

In this paper the first system we consider is the standard matrix harmonic oscillator with Hamiltonian

$$H = \sum_{i,j=1}^{N} A^{\dagger}_{ij} A_{ij} \tag{1.1}$$

There is an action on the matrix harmonic oscillator by permutations $\sigma$ in the symmetric group of all permutations $\sigma : \{1, 2, \cdots , N\} \to \{1, \cdots , N\}$:

$$A^{\dagger}_{ij} \to A^{\dagger}_{\sigma(i)\sigma(j)} \tag{1.2}$$

Defining the permutation matrix

$$(U_\sigma)_{ij} = \delta_{i,\sigma^{-1}(j)} \tag{1.3}$$

we have

$$((U_\sigma)^T)_{ij} = \delta_{j,\sigma^{-1}(i)} = \delta_{i,\sigma(j)} \tag{1.4}$$

The action (1.2) can be expressed in terms of matrix multiplication as

$$A^{\dagger} \to U_\sigma A^{\dagger} U^T_\sigma \tag{1.5}$$

The oscillators transform as $V_N \otimes V_N$ where $V_N$ is the natural representation of $S_N$. $U_\sigma$ is real and $A_{ij}$ transform in the same way. In the subspace of the Fock space with $k$ oscillators, we have polynomials in $A^{\dagger}$ of degree $k$. The problem of projecting degree $k$ polynomials in $N \times N$ matrices to the $S_N$ invariant subspace was considered in [4] and a formula $\mathcal{Z}(N, k)$ (reproduced here as (2.5)) was derived by using projectors in the group algebras of $S_N$ and $S_k$ along with characters in $V_N^{\otimes k}$. In section 2 we use this micro-canonical formula to derive the canonical partition function for the gauge-invariant subspace of the Hilbert space

$$\mathcal{Z}(N, \beta) = \mathrm{tr}(e^{-\beta H} P_0) = \sum_k e^{-\beta k} \mathcal{Z}(N, k) \equiv \sum_k x^k \mathcal{Z}(N, k) = \mathcal{Z}(N, x) \tag{1.6}$$

We have defined $x = e^{-\beta}$. The projector $P_0$ projects to the $S_N$ invariant subspace. Our first main result is to obtain a formula for $\mathcal{Z}(N, x)$ as a sum over partitions of $N$ (2.11) consisting of an inverse symmetry factor Sym $p$ associated with the partition, multiplied by a factor $\mathcal{Z}(N, p, x)$ (2.14). This latter is a product of factors whose structure depends on elementary number theoretic properties of the parts appearing in the partition. These parts of the partition are cycle lengths of the permutations belonging to the conjugacy class in $S_N$ associated with the partition. Specifically if the partition $p$ consists of parts $a_i$ appearing with multiplicities $p_i$, then $\mathcal{Z}(N, p, x)$ is a product labelled by the parts $a_i$ and another product labelled by pairs $(a_i, a_j)$ of parts in the partition. This second product depends on the least common multiples $\mathrm{LCM}(a_i, a_j) \equiv L(a_i, a_j)$ of the pairs and the greatest common divisors $\mathrm{GCD}(a_i, a_j) \equiv G(a_i, a_j)$ of the pairs.

There is an interesting formula, called the Molien-Weyl formula (see for example [19] for a textbook discussion), for the generating function of bosonic Fock space states invariant under a symmetry group $G$, which is expressed as an inverse determinant involving the matrix representing the action of $G$ on the bosonic oscillators. In section 3, we recall this formula and, for completeness, give a derivation in section 3.1. The derivation is generalised in section 3.2 to the case where the bosonic oscillators transform according to a direct sum of representations, and the Fock space state counting is done with different parameters weighting the different representations. This amounts to having different frequencies for the harmonic oscillators transforming in the different irreducible representations. We also generalise the derivation of section 3.1 to the case of Fermionic Fock spaces in section 3.3.

In section 4, we recover the formula (2.14) using the Molien-Weyl formula discussed in section 3. In section 5, we use the refined Molien-Weyl formula from section 3.2 to calculate the partition function for the $S_N$ Hamiltonian of the general 11-parameter permutation invariant matrix harmonic oscillator system solved in [11]. The partition function depends on 7 of the 11 parameters and has a structure of a sum over partitions of $N$ weighted by an inverse symmetry factor of the partition along with elementary number theoretic properties of the parts appearing in the partition. This is the second main result of this paper.

## 2 Bosonic GPIMQM: partition functions

We find an interesting formula for $S_N$ invariant partition functions with a number theoretic characteristics. It is a sum over partitions of $N$, which correspond to cycle structures of $S_N$ permutations, where the LCM and GCD of pairs of cycle lengths play a role.

For permutations $\sigma \in S_N$, let $U_\sigma$ be the linear operator acting in the natural representation $V_N$ of $S_N$. Matrix bosonic oscillators $A^\dagger_{ij}$ with $i, j \in \{1, 2, \cdots, N\}$ admit an action of $U_\sigma$:

$$A^\dagger_{ij} \to (U_\sigma)_{ik} A^\dagger_{kl} (U^T_\sigma)_{lj} \tag{2.1}$$

or in matrix notation

$$A^\dagger \to U A^\dagger U^T \,. \tag{2.2}$$

The action can also be written as:

$$A^\dagger_{ij} \to A^\dagger_{\sigma(i)\sigma(j)} \tag{2.3}$$

The dimension of the subspace of the Fock space of these oscillators, at degree $k$, which is invariant under the $S_N$ action has been computed in eqn (B.9) of [4]. The discussion in the paper [4] is in the context of polynomial functions of a classical matrix variable $M_{ij}$ invariant under the action

$$M_{ij} \to M_{\sigma(i)\sigma(j)} \tag{2.4}$$



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and the mathematics of this invariant theory question evidently applies equally well to the same action on bosonic oscillators. The dimension of the space of $S_N$ invariants at degree $k$ is given as a sum of partitions of $N$ and $k$

$$\begin{aligned}\mathcal{Z}(N,k) &= \sum_{p\vdash N}\sum_{q\vdash k}\frac{1}{\text{Sym }p}\frac{1}{\text{Sym }q}\prod_{i=1}^{k}\left(\sum_{l|i} lp_l\right)^{2q_i}\\ &= \sum_{p\vdash N}\frac{1}{\text{Sym }p}\mathcal{Z}(N,p,k)\end{aligned} \tag{2.5}$$

where

$$\mathcal{Z}(N,p,k) = \sum_{q\vdash k}\frac{1}{\text{Sym }q}\prod_{i=1}^{k}\left(\sum_{l|i} lp_l\right)^{2q_i} \tag{2.6}$$

The partition $p$ of $N$ is described as a list of cycle lengths $l$ of permutations in $S_N$ (parts of the partition) with multiplicities $p_l$ where $l \in \{1,\cdots,N\}$ and $p_l$ are non-negative integers obeying $\sum_l lp_l = N$. The partition $q$ of $k$ is described as a list of cycle lengths $i$ for permutations in $S_k$ with $i \in \{1,\cdots,k\}$ and $q_i$ are non-negative integers obeying $\sum_i iq_i = k$. The symmetry factor Sym $q$ is given by

$$\text{Sym }q = \prod_{i=1}^{k} i^{q_i} q_i! \tag{2.7}$$

It is the number of permutations $\gamma \in S_k$ which obey $\gamma\sigma\gamma^{-1} = \sigma$ for any $\sigma$ having cycle structure $q$. A useful fact we often use to convert sums over permutations into sums over conjugacy classes is that the number of permutations with cycle structure $q$ is

$$\frac{k!}{\text{Sym }q} \tag{2.8}$$

We define the generating function

$$\begin{aligned}\mathcal{Z}(N,x) &= \sum_{k=0}^{\infty} x^k \mathcal{Z}(N,k)\\ &= \sum_{k=0}^{\infty}\sum_{p\vdash N}\frac{x^k}{\text{Sym }p}\mathcal{Z}(N,p,k)\\ &= \sum_{p\vdash N}\frac{1}{\text{Sym }p}\sum_{k} x^k \mathcal{Z}(N,p,k)\end{aligned} \tag{2.9}$$

It is also useful to define a generating function for fixed $N$ and fixed partition $p$ of $N$ by summing over $k$

$$\mathcal{Z}(N,p,x) = \sum_{k} x^k \mathcal{Z}(N,p,k) \tag{2.10}$$

$\mathcal{Z}(N,x)$ can therefore be written as a sum

$$\mathcal{Z}(N,x) = \sum_{p\vdash N}\frac{1}{\text{Sym }p}\mathcal{Z}(N,p,x) \tag{2.11}$$



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Partitions of $N$ can be described in the form $p = [a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$, where $a_j$ are distinct non-zero parts with $1 \leq a_j \leq N$ and $p_j$ are positive integers, obeying

$$N = \sum_{j=1}^{K} a_j p_j \tag{2.12}$$

It is often useful to think of the parts as being ordered as $a_1 < a_2 < \cdots < a_K$. For example for partitions of 4, we have in this notation

$$\begin{aligned} 4 = 4 &\longrightarrow p = [4] \\ 4 = 3+1 &\longrightarrow p = [3,1] \\ 4 = 2+2 &\longrightarrow p = [2^2] \\ 4 = 2+1+1 &\longrightarrow p = [2,1^2] \\ 4 = 1+1+1+1 &\longrightarrow p = [1^4] \end{aligned} \tag{2.13}$$

We will derive the following formula for $\mathcal{Z}(N, p, x)$ which we refer to as the LCM formula:

**Proposition 1: the LCM formula.**

$$\mathcal{Z}(N,p,x) = \prod_{i=1}^{K} \frac{1}{(1-x^{a_i})^{a_i p_i^2}} \prod_{i<j\in\{1,\cdots,K\}} \frac{1}{(1-x^{L(a_i,a_j)})^{2G(a_i,a_j)p_i p_j}} \tag{2.14}$$

$L(a_i, a_j)$ is the least common multiple of $a_i$ and $a_j$ and $G(a_i, a_j)$ is the greatest common denominator of $a_i, a_j$.

A fact relating the LCM and GCD of a pair of numbers which we will find useful is that

$$a_i a_j = L(a_i, a_j) G(a_i, a_j) \tag{2.15}$$

This is proved in the equation (2.31). The expression $2G(a_i, a_j)p_i p_j$ in (2.14) can also be written as:

$$2G(a_i,a_j)p_i p_j = \frac{2a_i a_j p_i p_j}{L(a_i,a_j)} \tag{2.16}$$

We can also present $p$ as $[i^{p_i}]$ where $i$ are all integers in the set $\{1, \cdots, N\}$ and $p_i$ are non-negative integers (possibly zero). In this case we can write

$$\mathcal{Z}(N,p,x) = \prod_{i=1}^{N} \frac{1}{(1-x^{i})^{i p_i^2}} \prod_{i<j\in\{1,\cdots,N\}} \frac{1}{(1-x^{L(i,j)})^{2G(i,j)p_i p_j}} \tag{2.17}$$

The terms with $p_i = 0$ all give factors of 1 so this immediately reduces to the previous formula (2.14).

Examples of (2.14) are as follows. For $N = 2$, we have two partitions $p = [2]$ and $p = [1^2]$:

$$\begin{aligned} \mathcal{Z}(2,[2],x) &= \frac{1}{(1-x^2)^2} \\ \mathcal{Z}(2,[1^2],x) &= \frac{1}{(1-x)^4} \end{aligned} \tag{2.18}$$

Using (2.11)

$$\mathcal{Z}(N=2,x) = \frac{1}{2(1-x^2)^2} + \frac{1}{2(1-x)^4} \tag{2.19}$$

For $N = 3$, there are three partitions $p = [3], p = [2,1], p = [1,1,1] = [1^3]$ and using (2.14) we have

$$\begin{aligned}
\mathcal{Z}(3,[3],x) &= \frac{1}{(1-x^3)^3} \\
\mathcal{Z}(3,[2,1],x) &= \frac{1}{(1-x^2)^4(1-x)} \\
\mathcal{Z}(3,[1^3],x) &= \frac{1}{(1-x)^9}
\end{aligned} \tag{2.20}$$

Using (2.11) then gives

$$\mathcal{Z}(N=3,x) = \frac{1}{3(1-x^3)^3} + \frac{1}{2(1-x)(1-x^2)^4} + \frac{1}{6(1-x)^9} \tag{2.21}$$

### 2.1 Proof of LCM formula

We will start with equation (2.6). The partition $p$ of $N$ is described as $p = [l^{p_l}]$ with $\sum_l lp_l = N$. The partition $q$ of $k$ is described as $q = [i^{q_i}]$ where $\sum_i iq_i = k$.

$$\begin{aligned}
\mathcal{Z}(N,p;x) &= \sum_{k=0}^{\infty} x^k \sum_{q \vdash k} \frac{1}{\text{Sym } q} \prod_{i=1}^{k} \left( \sum_{l|i} lp_l \right)^{2q_i} \\
&= \sum_{k=0}^{\infty} x^k \sum_{\substack{q_1,q_2\cdots \\ \sum_i iq_i = k}} \frac{1}{\prod_i i^{q_i} q_i!} \prod_{i=1}^{k} \left( \sum_{l|i} lp_l \right)^{2q_i} \\
&= \sum_{q_1,q_2,\cdots} \frac{x^{\sum_i iq_i}}{\prod_i i^{q_i} q_i!} \prod_{i=1}^{\infty} \left( \sum_{l|i} lp_l \right)^{2q_i} \\
&= \prod_i \sum_{q_i} \frac{1}{q_i!} (\frac{x^i}{i})^{q_i} \left( \sum_{l|i} lp_l \right)^{2q_i} \\
&= \prod_i \text{Exp}\left[ \frac{x^i}{i} \left( \sum_{l|i} lp_l \right)^2 \right] \\
&= \text{Exp}\left[ \sum_{i=1}^{\infty} \frac{x^i}{i} \left( \sum_{l|i} lp_l \right)^2 \right]
\end{aligned} \tag{2.22}$$

A minor re-writing gives

$$\mathcal{Z}(N,p;x) = \text{Exp}\left[ \sum_{i\in\mathbb{N}} \frac{x^i}{i} \left( \sum_{l|i} lp_l \right)^2 \right] \tag{2.23}$$

where $\mathbb{N}$ is the set of natural numbers.



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For simplicity we first consider $p = [a_1^{p_1}, a_2^{p_2}]$, i.e. partitions with just two types of parts: $a_1$ with multiplicity $p_1$ and $a_2$ with multiplicity $p_2$. Define the sets

$$\begin{aligned} S_1 &= \text{set of positive integer multiples of } a_1 = \{ma_1 : m \in \mathbb{N}\} \\ S_2 &= \text{set of positive integer multiples of } a_2 = \{ma_2 : m \in \mathbb{N}\} \\ S_{12} &= \text{set of positive integers divisible by both } a_1 \text{ and } a_2 \\ &= \text{set of positive integer multiples of } L(a_1, a_2) = \{mL(a_1, a_2) : m \in \mathbb{N}\} \end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned} S_{12} &\subset S_1 \\ S_{12} &\subset S_2 \end{aligned} \tag{2.25}$$

Consider the expression

$$\sum_{i=1}^{\infty} \frac{x^i}{i} \left( \sum_{l|i} l p_l \right)^2 \tag{2.26}$$

appearing in the exponent of (2.23). Given that $l$ is being summed over divisors of $i$ and the only non-zero $l$'s for $p = [a_1^{p_1}, a_2^{p_2}]$ are $a_1$ and $a_2$, it follows that the powers $i$ appearing in this sum belong to a subset $\mathcal{S}$ of $\mathbb{N}$ which is

$$\mathcal{S} = S_1 \cup S_2 \subset \mathbb{N} \tag{2.27}$$

It is also useful to note that

$$S_1 \cap S_2 = S_{12} \tag{2.28}$$

For all elements $i = ma_1 \in S_1$, the expression $(\sum_{l|i} lp_l)^2$ contains $a_1^2 p_1^2$ plus possibly additional terms. For all elements $i = ma_2 \in S_2$, the same expression contains $a_2^2 p_2^2$ plus possibly additional terms. For the elements in $i = mL(a_1, a_2) \in S_{12}$, the expression evaluates to $(a_1 p_1 + a_2 p_2)^2 = (a_1^2 p_1^2 + a_2^2 p_2^2 + 2a_1 a_2 p_1 p_2)$. We can therefore write the sum over $\mathbb{N}$ in (2.23) as a sum over $S_1$ with weight $a_1^2 p_1^2$ and a sum over $S_2$ with weight $a_2^2 p_2^2$, along with a sum over $S_{12}$ with weight $2a_1 a_2 p_1 p_2$ (since $S_1 \cap S_2 = S_{12}$, the contributions $a_1^2 p_1^2 + a_2^2 p_2^2$ to the coefficient of $x^{mL(a_1, a_2)}$ from the expression $(\sum_{l|i} lp_l)^2$ are already included in the sums over $S_1, S_2$). We can therefore write

$$\begin{aligned} \sum_{i \in \mathbb{N}} \frac{x^i}{i} \left( \sum_{l|i} l p_l \right)^2 &= \sum_{m \in \mathbb{N}} \frac{x^{ma_1}}{ma_1} (a_1 p_1)^2 + \sum_{m \in \mathbb{N}} \frac{x^{ma_2}}{ma_2} (a_2 p_2)^2 + \sum_{m \in \mathbb{N}} \frac{x^{mL(a_1, a_2)}}{mL(a_1, a_2)} (2a_1 a_2 p_1 p_2) \\ &= \sum_{m \in \mathbb{N}} \frac{x^{ma_1}}{ma_1} (a_1 p_1)^2 + \sum_{m \in \mathbb{N}} \frac{x^{ma_2}}{ma_2} (a_2 p_2)^2 + \sum_{m \in \mathbb{N}} \frac{x^{mL(a_1, a_2)}}{m} (2G(a_1, a_2) p_1 p_2) \end{aligned} \tag{2.29}$$

In the last line we used

$$\frac{a_1 a_2}{L(a_1, a_2)} = G(a_1, a_2) \tag{2.30}$$

To see this let us denote $h = G(a_1, a_2)$ and $a_1 = h\hat{a}_1, a_2 = h\hat{a}_2$ where $G(\hat{a}_1, \hat{a}_2) = 1$. Then

$$L(a_1, a_2) = L(h\hat{a}_1, h\hat{a}_2) = h\hat{a}_1\hat{a}_2 = \frac{a_1a_2}{h} = \frac{a_1a_2}{G(a_1, a_2)} \tag{2.31}$$

which proves (2.30).

Using these facts we can write

$$\begin{aligned}
&\mathcal{Z}(N, p; x) \\
&= \text{Exp}\left[\sum_{m=1}^{\infty} \frac{x^{ma_1}}{ma_1}(a_1p_1)^2\right] \text{Exp}\left[\sum_{m=1}^{\infty} \frac{x^{ma_2}}{ma_2}(a_2p_2)^2\right] \text{Exp}\left[\sum_{m=1}^{\infty} \frac{x^{mL(a_1,a_2)}}{mL(a_1, a_2)}(2a_1a_2p_1p_2)\right] \\
&= \text{Exp}\left[\sum_{m=1}^{\infty} \frac{x^{ma_1}}{m}(a_1p_1^2)\right] \text{Exp}\left[\sum_{m=1}^{\infty} \frac{x^{ma_2}}{m}(a_2p_2^2)\right] \text{Exp}\left[\sum_{m=1}^{\infty} \frac{x^{mL(a_1,a_2)}}{m}(2G(a_1, a_2)p_1p_2)\right] \\
&= \text{Exp}[-a_1p_1^2 \log(1 - x^{a_1}) - a_2p_2^2 \log(1 - x^{a_2}) - 2p_1p_2G(a_1, a_2) \log(1 - x^{L(a_1,a_2)})] \\
&= \frac{1}{(1 - x^{a_1})^{a_1p_1^2}} \frac{1}{(1 - x^{a_2})^{a_2p_2^2}} \frac{1}{(1 - x^{L(a_1,a_2)})^{2G(a_1,a_2)p_1p_2}}
\end{aligned} \tag{2.32}$$

**Remarks.**

1. We observe that the above result and derivation also apply when one of $a_1$ or $a_2$ is a multiple of the other. Without loss of generality, take $a_2$ to be $na_1$ for some positive integer $n$. Then $S_2 \subset S_1$, $L(a_1, a_2) = a_2$, $G(a_1, a_2) = a_1$ and $S_{12} = S_2$. The formula in (2.32) then simplifies to

$$\begin{aligned}
\mathcal{Z}(N, p; x) &= \frac{1}{(1 - x^{a_1})^{a_1p_1^2}} \frac{1}{(1 - x^{a_2})^{a_2p_2^2}} \frac{1}{(1 - x^{a_2})^{2a_1p_1p_2}} \\
&= \frac{1}{(1 - x^{a_1})^{a_1p_1^2}} \frac{1}{(1 - x^{a_2})^{a_2p_2^2+2a_1p_1p_2}}
\end{aligned} \tag{2.33}$$

2. Another simplification is the case where $G(a_1, a_2) = 1$. In that case, $L(a_1, a_2) = a_1a_2$ and the formula (2.32) simplifies to

$$\mathcal{Z}(N, p; x) = \frac{1}{(1 - x^{a_1})^{a_1p_1^2}} \frac{1}{(1 - x^{a_2})^{a_2p_2^2}} \frac{1}{(1 - x^{a_1a_2})^{2p_1p_2}} \tag{2.34}$$

The derivation above extends to the case of general partitions of the form $p = [a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$. In this case the sum $(\sum_{l|i} lp_l)^2$ appearing in (2.23) will contain, depending on the choice of $i$, terms of type $a_\alpha^2 p_\alpha^2$ for $\{\alpha \in \{1, 2, \cdots, K\}\}$, as well as terms of the type $2a_\alpha a_\beta p_\alpha p_\beta$ for pairs $\{\alpha < \beta; \alpha, \beta \in \{1, \cdots, K\}\}$. Generalising the observation in the case $p = [a_1^{p_1}, a_2^{p_2}]$ the only $x^i$ appearing in the sum $(\sum_{l|i} lp_l)^2$ have powers $i$ belonging to the subset $\mathcal{S} \in \mathbb{N}$ where

$$\mathcal{S} = S_1 \cup S_2 \cdots \cup S_K \subset \mathbb{N} \tag{2.35}$$

where

$$S_\alpha = \text{ set of positive integer multiples of } a_\alpha = \{ma_\alpha : m \in \mathbb{N}\} \tag{2.36}$$

The term $a_\alpha^2 p_\alpha^2$ will appear as a coefficient for $x^i$ for $i \in S_\alpha \subset \mathbb{N}$. The cross-terms $2a_\alpha a_\beta p_\alpha p_\beta$ will appear in the subset $S_{\alpha\beta} \subset \mathbb{N}$

$$\begin{aligned} S_{\alpha\beta} &= \text{ set of positive integers divisible by both } a_\alpha \text{ and } a_\beta \\ &= \text{ set of positive integer multiples of } L(a_\alpha, a_\beta) = \{mL(a_\alpha, a_\beta) : m \in \mathbb{N}\} \end{aligned} \tag{2.37}$$

Taking advantage of the fact that $S_{\alpha\beta} \subset S_\alpha$ and $S_{\alpha\beta} \subset S_\beta$ we therefore conclude that

$$\sum_{i\in\mathbb{N}} \frac{x^i}{i} \left( \sum_{l|i} l p_l \right)^2 = \sum_{m\in\mathbb{N}} \sum_{\alpha=1}^{K} \frac{x^{ma_\alpha}}{ma_\alpha} (a_\alpha p_\alpha)^2 + \sum_{m\in\mathbb{N}} \sum_{\alpha<\beta\in\{1,\cdots,K\}} \frac{x^{mL(a_\alpha,a_\beta)}}{mL(a_\alpha,a_\beta)} (2a_\alpha a_\beta p_\alpha p_\beta) \tag{2.38}$$

This allows us to obtain

$$\begin{aligned} \mathcal{Z}(N,p,x) &= \mathrm{Exp}\left[ \sum_{i\in\mathbb{N}} \frac{x^i}{i} \left( \sum_{l|i} l p_l \right)^2 \right] \\ &= \mathrm{Exp}\left[ \sum_{\alpha=1}^{K} \sum_{m\in\mathbb{N}} \frac{x^{ma_\alpha}}{m} (a_\alpha p_\alpha^2) + \sum_{\alpha<\beta\in\{1,\cdots,K\}} \sum_{m\in\mathbb{N}} \frac{x^{mL(a_\alpha,a_\beta)}}{m} \left( \frac{2a_\alpha a_\beta p_\alpha p_\beta}{L(a_\alpha,a_\beta)} \right) \right] \\ &= \mathrm{Exp}\left[ \sum_{\alpha=1}^{K} \sum_{m\in\mathbb{N}} \frac{x^{ma_\alpha}}{m} (a_\alpha p_\alpha^2) + \sum_{\alpha<\beta\in\{1,\cdots,K\}} \sum_{m\in\mathbb{N}} \frac{x^{mL(a_\alpha,a_\beta)}}{m} (2G(a_\alpha,a_\beta) p_\alpha p_\beta) \right] \\ &= \prod_{\alpha=1}^{K} \mathrm{Exp}[-(a_\alpha p_\alpha^2) \log(1-x^\alpha)] \prod_{\alpha<\beta\in\{1,\cdots,K\}} \mathrm{Exp}[-(2G(a_\alpha,a_\beta) p_\alpha p_\beta) \log(1-x^{L(a_\alpha,a_\beta)})] \\ &= \prod_{\alpha=1}^{K} \frac{1}{(1-x^{a_\alpha})^{a_\alpha p_\alpha^2}} \prod_{\alpha<\beta\in\{1,\cdots,K\}} \frac{1}{(1-x^{L(a_\alpha,a_\beta)})^{2G(a_\alpha,a_\beta) p_\alpha p_\beta}} \end{aligned} \tag{2.39}$$

This completes the proof of the main proposition (2.14).

### 2.2 Effective-graphs for graph counting

It is known that the counting of permutation invariants at degree $k$ for matrices of size $N$ is equivalent to the counting of directed graphs with $k$ edges and $N$ vertices (see section 2.2 of [6]) and in the range $N \geq 2k$ it is simply the counting of all directed graphs with $k$ edges [5]. We have found a generating function for the graph counting at arbitrary $k$ for fixed $N$ (2.14). For partitions of $N$ of the form $[a_1^{p_1}, a_2^{p_2} \cdots a_K^{p_K}]$, with $p_i$ parts of length $a_i$ which corresponds to cycle lengths of permutations in $S_N$, the formula contains a product with factors associated with the cycle lengths $a_i$ and a product with factors associated with pairs $(a_i, a_j)$. This formula has an interpretation in terms of a weighted counting of graphs, which may be considered as "effective graphs" which produce the generating function for numbers of directed graphs. For a partition $p = [a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$, we consider a labelled graph with $p_1 + p_2 + \cdots + p_K$ nodes. The first $p_1$ nodes are assigned a weight $a_1$ each, the next $p_2$ are assigned a weight $a_2$ and so forth. Consider a set of graphs, each consisting of a single arc joining either a node to itself or a node to another. Consider the product, one for each arc, of the form

$$\prod_{\text{arcs}} (1 - x^{L(a_{\text{start}}, a_{\text{end}})})^{-G(a_{\text{start}}, a_{\text{end}})} \tag{2.40}$$

and define a partition function of weighted graphs as a product over the weighted graphs (defined by the arcs):

$$\mathcal{Z}_{\text{ weighted graphs}}(N,p,x) = \prod_{\text{arcs}} (1 - x^{L(a_{\text{start}},a_{\text{end}})})^{-G(a_{\text{start}},a_{\text{end}})} \tag{2.41}$$

The loops joining nodes labelled $a_i$ to nodes labelled $a_i$ (including self-loops) produce the product

$$\prod_{i=1}^{K} (1 - x^{a_i})^{-a_i p_i^2} \tag{2.42}$$

since $L(a_i, a_i) = a_i$ there are $p_{a_i}^2$ such arcs. The arcs joining nodes of different weights produce

$$\prod_{i<j} (1 - x^{L(a_i,a_j)})^{-2p_i p_j G(a_i,a_j)} \tag{2.43}$$

since there are $2p_i p_j$ such arcs. Taking the product over all the weighted graphs we have

$$\prod_{i=1}^{K} (1 - x^{a_i})^{-a_i p_{a_i}^2} \prod_{i<j} (1 - x^{L(a_i,a_j)})^{-2p_i p_j G(a_i,a_j)} \tag{2.44}$$

which is the formula (2.14). This proves

$$\mathcal{Z}_{\text{ weighted graphs}}(N,p,x) = \mathcal{Z}(N,p,x) \tag{2.45}$$

Figure 1 illustrates the effective graphs used to produce the generating function. The summand $Z(N,p,x)$ of $Z(N,x)$ for the case $N = 3, p = [2,1]$ is constructed from the effective graph with a node labelled 2 and a node labelled 1. The different possible 1-edge directed graphs are shown in figure 1. For the first two, we apply the formula (2.42), while for the last two we apply (2.43) to arrive at the function on the right. Taking the product over the four graphs leads to

$$Z(N = 3, p = [2,1], x) = \frac{1}{(1 - x^2)^4 (1 - x)} \tag{2.46}$$

As a corollary of this discussion, note that

$$\mathcal{Z}(N,p,x) = \prod_{i,j \in \{1,\cdots,K\}} \frac{1}{(1 - x^{L(a_i,a_j)})^{G(a_i,a_j)}} \tag{2.47}$$

The case where $i = j$ leads to the first product in (2.14) while the terms $i \neq j$ lead to the second product in (2.14).

## 3 Review: derivation of Molien-Weyl determinant formula

A standard result which is very useful in the computation of dimensions of subspaces of Fock spaces, or of spaces of polynomials, invariant under a group action is the Molien-Weyl formula. A textbook presentation and derivation is in [19]. In a physics context, this formula has been used for example in [20, 21]. In this section, in the interest of a self-contained presentation,

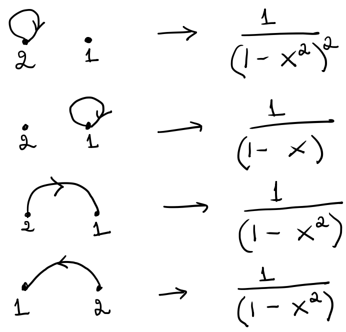


**Figure 1.** Diagrams of effective graphs for the calculation of $Z(N = 3, p = [2, 1], x)$.

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we will give a proof of the Molien-Weyl determinant formula for the generating function of dimensions of state spaces for bosonic Fock spaces using permutations and projector techniques which are used elsewhere in the paper. An alternative derivation proceeds by diagonalising the matrix for $g$ [19]. The derivation of the Molien-Weyl determinant formula from the gauged bosonic matrix quantum mechanics path integral is given in [18]. The permutation and projector techniques admit an immediate generalization to the case of Fock spaces defined using oscillators transforming according to a direct sum of irreducible representations, with the different representations weighted by different parameters. This is described in section 3.2. The generalisation to fermionic Fock spaces is described in section 3.3.

### 3.1 Molien-Weyl determinant formula derivation: bosonic case

Consider a finite group $G$ and a set of bosonic oscillators $A_m^\dagger$ transforming in the representation $V$. The state space of $k$ bosonic oscillators is the symmetrized $k$-fold tensor power $\text{Sym}^k(V)$. The generating function of invariants in the $\text{Sym}^k(V)$ is defined as

$$\mathcal{Z}_{\text{sym}}(G, V; x) = \sum_{k=0}^{\infty} x^k \text{tr}_{\text{Sym}^k(V)}(P_0) \tag{3.1}$$

where the projector to $G$-invariants is

$$P_0 = \frac{1}{|G|} \sum_{g \in G} g \tag{3.2}$$

We therefore have

$$\begin{aligned}\mathcal{Z}_{\text{sym}}(G, V; x) &= \frac{1}{|G|} \sum_{g \in G} \sum_{k=0}^{\infty} x^k \text{tr}_{\text{Sym}^k(V)}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{k=0} \frac{x^k}{k!} \sum_{\tau \in S_k} \text{tr}_{V^{\otimes k}}(g^{\otimes k} \tau)\end{aligned} \tag{3.3}$$

To calculate the trace in the last line, introduce a basis $e_i$ for $V$, with $1 \le i \le \text{Dim}V$. The permutation $\tau$ viewed as a map $\tau : \{1, 2, \cdots, k\} \to \{1, 2, \cdots k\}$ which takes $a \in \{1. \cdots, k\}$ to $\tau(a) \in \{1, \cdots, k\}$ acts as follows on the tensor product states

$$\tau\left(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}\right) = \left(e_{i_{\tau(1)}} \otimes e_{i_{\tau(2)}} \otimes \cdots e_{i_{\tau(k)}}\right) \tag{3.4}$$

On the tensor product vector space $V^{\otimes k}$, the group element acts as $g^{\otimes k}$

$$g^{\otimes k}\left(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}\right) = g_{i_1}^{j_1} \cdots g_{i_k}^{j_k}\left(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}\right) \tag{3.5}$$

with the summation convention of repeated indices $(j_a)$ being understood. The trace is thus

$$\begin{aligned}
\mathrm{tr}_{V^{\otimes k}}(g^{\otimes k}\tau) &= \mathrm{Coeff}\left(e^{i_1} \otimes e^{i_2} \cdots \otimes e^{i_k}, g^{\otimes k}\tau\left(e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_k}\right)\right) \\
&= \mathrm{Coeff}\left(e^{i_1} \otimes e^{i_2} \cdots \otimes e^{i_k}, g^{\otimes k}\left(e_{i_{\tau(1)}} \otimes e_{i_{\tau(2)}} \otimes \cdots e_{i_{\tau(k)}}\right)\right) \\
&= \mathrm{Coeff}\left(e^{i_1} \otimes e^{i_2} \cdots \otimes e^{i_k}, g_{i_{\tau(1)}}^{j_1} g_{i_{\tau(2)}}^{j_2} \cdots g_{i_{\tau(k)}}^{j_k}\left(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}\right)\right) \\
&= g_{i_{\tau(1)}}^{i_1} g_{i_{\tau(2)}}^{i_2} \cdots g_{i_{\tau(k)}}^{i_k}
\end{aligned} \tag{3.6}$$

For a permutation with cycle lengths $a \in \{1, \cdots, k\}$ with multiplicities $q_a$, the expression in the last line can be simplified as

$$g_{i_{\tau(1)}}^{i_1} g_{i_{\tau(2)}}^{i_2} \cdots g_{i_{\tau(k)}}^{i_k} = \prod_{a=1}^{k} (\mathrm{tr}(g^a))^{q_a} \tag{3.7}$$

This expresses the fact that each cycle of length $a$ contributes a factor of $\mathrm{tr}(g^a)$ to the trace in $V^{\otimes k}$. We thus conclude that

$$\mathrm{tr}_{V^{\otimes k}}(g^{\otimes k}\tau) = \prod_{a=1}^{k} (\mathrm{tr}(g^a))^{q_a} \tag{3.8}$$

Using this equation, the bosonic partition function in (3.3) can be expressed as

$$\mathcal{Z}_{\mathrm{sym}}(G, V; x) = \frac{1}{|G|} \sum_{g \in G} \sum_{k=0} \frac{x^k}{k!} \sum_{q \vdash k} \left(\frac{k!}{\mathrm{Sym}\ q}\right) \mathrm{tr}_{V^{\otimes k}}(g^{\otimes k}\tau) \tag{3.9}$$

where we have replaced the sum over all permutations in $S_k$ in equation (3.3) with a sum over partitions $q$ of $\mathrm{tr}_{V^{\otimes k}}(g^{\otimes k}\tau)$ for a fixed $\tau$ with the cycle structure given by $q$. The factor $\left(\frac{k!}{\mathrm{Sym}\ q}\right)$ is the number of permutations with the cycle structure $q$. The equation (3.9) can be manipulated further as follows

$$
\begin{aligned}
Z_{\mathrm{sym}}(G,V;x) &= \frac{1}{|G|}\sum_{g\in G}\sum_{k=0}^{\infty}x^k\sum_{q\vdash k}\frac{1}{\prod_{a=1}^{k}a^{q_a}q_a!}\prod_{a=1}^{k}(\mathrm{tr}(g^a))^{q_a}\\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{k=0}^{\infty}\sum_{q\vdash k}\frac{x^{\sum_a aq_a}}{\prod_{a=1}^{k}a^{q_a}q_a!}\prod_{a=1}^{k}(\mathrm{tr}(g^a))^{q_a}\\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{q_1,q_2,\cdots}\prod_{a=1}^{\infty}\frac{1}{q_a!}\left(\frac{x^a}{a}\right)^{q_a}(\mathrm{tr}(g^a))^{q_a}\\
&= \frac{1}{|G|}\sum_{g\in G}\prod_{a=1}^{\infty}\sum_{q_a=0}^{\infty}\frac{1}{q_a!}\left(\frac{x^a}{a}\right)^{q_a}(\mathrm{tr}(g^a))^{q_a}\\
&= \frac{1}{|G|}\sum_{g\in G}\prod_{a=1}^{\infty}\mathrm{Exp}\left[\frac{x^a}{a}\mathrm{tr}_V(g^a)\right]\\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{Exp}\left[\mathrm{tr}_V\left(\sum_a\frac{x^a}{a}g^a\right)\right]\\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{Exp}\left[-\mathrm{tr}_V\left(\log(1-xg)\right)\right]\\
&= \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-xD^V(g))}
\end{aligned}
\tag{3.10}
$$

We have thus derived the Molien-Weyl formula:

$$
\mathcal{Z}_{\mathrm{sym}}(G,V;x)=\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-xD^V(g))}
\tag{3.11}
$$

### 3.1.1 Projecting to a general representation $R$ of $G$ and fourier transform to constrained holonomy

In the above, we have derived a formula, as a sum over group elements, for the generating function of multiplicities of the trivial representation of $G$ in the bosonic Fock space of states generated by oscillators in the representation $V$ of $G$. The same steps allow the derivation of a formula for the generating function of the multiplicities of a general irreducible representation $R$ of $G$ in the Fock space. This generating function, denoted $\mathcal{Z}_{\mathrm{sym;R}}(G,V;x)$, can be written as a sum over the degree $k$ of the oscillators as

$$
\mathcal{Z}_{\mathrm{sym};R}(G,V;x)=\sum_{k=0}^{\infty}x^k\mathrm{tr}_{\mathrm{Sym}^k(V)}(P_R)
\tag{3.12}
$$

The trace is over the $k$-fold symmetrised tensor power of $V$ of the projector $P_R$ for the irreducible representation $R$. The projector is a sum over group elements:

$$
P_R=\frac{d_R}{|G|}\sum_{g\in G}\chi^R(g^{-1})\ g
\tag{3.13}
$$

i.e. an element of the group algebra, where $d_R$ is the dimension of the irreducible representation and $\chi^R(g)$ is the character of the group element $g$ in the representation $R$. Inserting this



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projector in place of $P_0$ at the start of (3.1) and following the steps of the derivation leads to

$$\mathcal{Z}_{\text{sym};R}(G,V;x) = \frac{d_R}{|G|}\sum_{g\in G}\frac{\chi^R(g^{-1})}{\det(1-xD^V(g))} \tag{3.14}$$

The determinant is a function of the conjugacy class of $g$, so the right hand side can be written as a sum over conjugacy classes. We can also perform a Fourier transform of $\mathcal{Z}_{\text{sym;R}}(G,V;x)$ to obtain a class function $\mathcal{Z}_{\text{sym;C}}(G,V;x)$

$$\begin{aligned}\mathcal{Z}_{\text{sym};C}(G,V;x) &= \sum_R \frac{\chi^R(g_C)}{d_R}\mathcal{Z}_{\text{sym};R}(G,V;x)\\ &= \frac{1}{\det(1-xD^V(g_C))}\end{aligned} \tag{3.15}$$

where $g_C$ is a group element in the specified conjugacy class $C$. We have used the character orthogonality relation

$$\sum_R \chi^R(g_C)\chi^R(g^{-1}) = \frac{|G|}{|C|}\delta(\text{Class(g)},\text{C}) \tag{3.16}$$

where $|C|$ is the size of that conjugacy class $C$, to do the sum over $R$ and to constrain the sum over $g$ in (3.11) to a fixed conjugacy class.

We can apply these observations to the case of $G = S_N$. The generating function for multiplicities of the irrep $R$ specialised to this case is renamed for simplicity as

$$\mathcal{Z}_{\text{sym};R}(G=S_N, V=V_N;x) \to \mathcal{Z}(N,R,x) \tag{3.17}$$

We have, from the above observations,

$$\mathcal{Z}(N,R,x) = \sum_p \frac{\chi^R(g_p^{-1})}{\text{Sym } p}\mathcal{Z}(N,p,x) \tag{3.18}$$

where $\mathcal{Z}(n,p,x)$ is given by the product formula (2.14). The Fourier-transformed, conjugacy class-labelled partition functions are recognised as none other than the $\mathcal{Z}(n,p,x)$ itself:

$$\mathcal{Z}_{\text{sym};C}(G,V=V_N;x) \to \mathcal{Z}(N,p,x) \tag{3.19}$$

In the path-integral formulation developed in [18] the constraint of the sum over $g$ in (3.11) to a fixed conjugacy class can be implemented as a constraint on the product of group elements associated with the links in the discretized thermal circle. This shows that $\mathcal{Z}(n,p,x)$ as given by the product formula (2.14) has an interpretation as a path integral with holonomy around the thermal circle constrained to a fixed conjugacy class.

### 3.2 Molien-Weyl determinant weighted counting of bosons in multiple representations

There is a natural generalisation of (3.11) when we consider a Fock space generated by bosonic oscillators $A^\dagger_{1;m_1}, A^\dagger_{2;m_2}, \cdots, A^\dagger_{s;m_s}$ transforming in a representation $V_1, V_2\cdots, V_s$ with $m_a$

ranging over a basis for irrep $V_l$, the index $m$ ranges over $\dim V_l$ values. We allow distinct parameters $x_1, x_2, \cdots, x_s$ weighting the different representations:

$$\mathcal{Z}_{\text{sym}}(G; V_1, V_2 \cdots, V_s; x_1, x_2, \cdots, x_s) = \frac{1}{|G|} \sum_{g \in G} \prod_{l=1}^{s} \frac{1}{\det(1 - x_l D^{V_l}(g))} \tag{3.20}$$

We prove this for the case of two representations. The extension to the general $s$ case proceeds through an evident generalisation.

$$\begin{aligned}
\mathcal{Z}_{\text{sym}}(G; V_1, V_2; x_1, x_2) &= \frac{1}{|G|} \sum_{g \in G} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} x_1^{k_1} x_2^{k_2} tr_{\text{Sym}^{k_1}(V_1) \otimes \text{Sym}^{k_2}(V_2)}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{x_1^{k_1} x_2^{k_2}}{k_1! k_2!} \sum_{\tau_1 \in S_{k_1}} \sum_{\tau_2 \in S_{k_2}} \text{tr}_{V_1^{\otimes k_1}}(g^{\otimes k_1} \tau_1) \text{tr}_{V_2^{\otimes k_2}}(g^{\otimes k_2} \tau_2) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{k_1, k_2} \sum_{p \vdash k_1} \sum_{q \vdash k_2} \frac{x_1^{k_1} x_2^{k_2}}{\text{Sym } p \ \ \text{Sym } q} \prod_a \text{tr}_{V_1}(g^a)^{p_a} \prod_b \text{tr}_{V_2}(g^b)^{q_b} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{p_1, p_2 \cdots} \sum_{q_1, q_2 \cdots} \frac{x_1^{\sum_a a p_a} x_2^{\sum_b b q_b}}{\prod_a a^{p_a} p_a! \prod_b b^{q_b} q_b!} \prod_a \text{tr}_{V_1}(g^a)^{p_a} \prod_b \text{tr}_{V_2}(g^b)^{q_b} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{p_1, p_2 \cdots} \sum_{q_1, q_2 \cdots} \prod_a \frac{1}{p_a!} \left( \frac{x_1^a \text{tr}_{V_1}(g^a)}{a} \right)^{p_a} \prod_b \frac{1}{q_b!} \left( \frac{x_2^b \text{tr}_{V_2}(g^b)}{b} \right)^{q_b} \\
&= \frac{1}{|G|} \sum_{g \in G} \prod_a \sum_{p_a} \frac{1}{p_a!} \left( \frac{x_1^a \text{tr}_{V_1}(g^a)}{a} \right)^{p_a} \prod_b \sum_{q_b} \frac{1}{q_b!} \left( \frac{x_2^b \text{tr}_{V_2}(g^b)}{b} \right)^{q_b} \\
&= \frac{1}{|G|} \sum_{g \in G} \prod_a \exp\left( \frac{x_1^a}{a} \text{tr}_{V_1}(g^a) \right) \prod_b \exp\left( \frac{x_1^b}{b} \text{tr}_{V_2}(g^b) \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \exp\left( \text{tr}_{V_1} \sum_a \frac{x_1^a D^{V_1}(g^a)}{a} \right) \exp\left( \text{tr}_{V_2} \sum_b \frac{x_2^b D^{V_2}(g^b)}{b} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \exp\left( -\text{tr}_{V_1} \log(1 - x_1 D^{V_1}(g)) \right) \exp\left( -\text{tr}_{V_2} \log(1 - x_2 D^{V_2}(g)) \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - x_1 D^{V_1}(g))} \frac{1}{\det(1 - x_2 D^{V_2}(g))}
\end{aligned} \tag{3.21}$$

It is useful to define

$$Z_V^{(g)}(x) = \frac{1}{\det(1 - x D^V(g))} \tag{3.22}$$

which allows us to write

$$\mathcal{Z}_{\text{sym}}(G; V_1, V_2; x_1, x_2) = \frac{1}{|G|} \sum_{g \in G} Z_{V_1}^{(g)}(x_1) Z_{V_2}^{(g)}(x_2) \tag{3.23}$$

and more generally

$$\mathcal{Z}_{\text{sym}}(G; V_1, V_2 \cdots, V_s; x_1, x_2, \cdots, x_s) = \frac{1}{|G|} \sum_{g \in G} \prod_{a=1}^{s} Z_{V_a}^{(g)}(x_a) \tag{3.24}$$

### 3.3 Molien-Weyl determinant formula derivation: fermionic case

In the fermionic case, the $k$-oscillator state space is the antisymmetrized tensor product $\Lambda^k(V)$. The generating function of fermionic states which are $G$-invariant is given by

$$\mathcal{Z}_{\text{ferm}}(G,V;x) = \sum_{k=0}^{\infty} x^k \text{tr}_{\Lambda^k(V)} P_0 \tag{3.25}$$

where the projector to $G$-invariants is

$$P_0 = \frac{1}{|G|}\sum_{g\in G} g \tag{3.26}$$

The trace over the $k$-fold anti-symmetrized tensor space $\Lambda^k(V)$ can be written as trace over the $k$-fold tensor power $V^{\otimes k}$ along with an anti-symmetrization projector constructed as a sum of permutations $\tau$ in $S_k$, weighted by the sign of the permutation which is denoted $(-1)^\tau$

$$Z_{\text{ferm}}(G,V;x) = \sum_{k=0}^{\infty}\sum_{\tau\in S_k} x^k \frac{(-1)^\tau}{k!}\text{tr}_{V^{\otimes n}}(P_0\tau) \tag{3.27}$$

We convert the sum over permutations in $S_k$ to a sum over conjugacy classes $q$ in $S_k$, specified as $q = [i^{q_i}]$ as in the bosonic case. The sign of a permutation $\tau$ in class $q$ is a class function which can be expressed as

$$(-1)^\tau = (-1)^{\tau^{(q)}} = (-1)^{\sum_i (i-1)q_i} \tag{3.28}$$

This expresses the fact that odd-length cycles are even sign permutations while even-length cycles are odd-sign permutations. The fermionic partition function is thus

$$\begin{aligned}
Z_{\text{ferm}}(G,V;x) &= \sum_{k=0}^{\infty}\sum_{q\vdash k}\frac{(-1)^{\tau^{(q)}}}{\text{Sym } q}\text{tr}_{V^{\otimes n}}(\tau^{(q)}P_0))\\
&= \sum_{k=0}^{\infty}\sum_{q\vdash k}\sum_{g}\frac{1}{|G|}\frac{(-1)^{\tau(q)}}{\text{Sym } q}\text{tr}_{V^{\otimes k}}(g^{\otimes k}\tau^{(q)})\\
&= \sum_{g}\frac{1}{|G|}\sum_{k=0}^{\infty}x^k\sum_{q\vdash k}\prod_{a=1}^{k}\frac{(-1)^{(a-1)q_a}}{a^{q_a}q_a!}(\text{tr}g^a)^{q_a}\\
&= \sum_{g}\frac{1}{|G|}\sum_{k=0}^{\infty}\sum_{q\vdash k}\prod_{a=1}^{k}\frac{(-1)^{(a-1)q_a}x^{aq_a}}{a^{q_a}q_a!}(\text{tr}g^a)^{q_a}\\
&= \sum_{g}\frac{1}{|G|}\sum_{q_1,q_2\cdots}\prod_{a}\frac{(-1)^{q_a}}{q_a!}\left(\frac{(-x)^a\text{tr}g^a}{a}\right)^{q_a}\\
&= \sum_{g}\frac{1}{|G|}\prod_{a}\sum_{q_a=0}^{\infty}\frac{(-1)^{q_a}}{q_a!}\left(\frac{(-x)^a\text{tr}g^a}{a}\right)^{q_a}\\
&= \sum_{g}\frac{1}{|G|}\prod_{a=1}^{\infty}e^{-\frac{(-x)^a}{a}\text{tr}g^a}\\
&= \sum_{g}\frac{1}{|G|}e^{-\sum_{a=1}^{\infty}\frac{(-x)^a}{a}\text{tr}g^a}
\end{aligned}$$



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$$
\begin{aligned}
&= \sum_g \frac{1}{|G|} e^{\sum_{a=1}^{\infty} \frac{(-1)^{a+1} x^a \mathrm{tr} g^a}{a}} \\
&= \sum_g \frac{1}{|G|} e^{\mathrm{tr} \sum_{a=1}^{\infty} \frac{(-1)^{a+1} x^a D^V(g^a)}{a}} \\
&= \sum_g \frac{1}{|G|} e^{\mathrm{tr} D^V \left( xg - \frac{x^2 g^2}{2} + \frac{x^3 g^3}{3} + \cdots \right)} \\
&= \sum_g \frac{1}{|G|} e^{\mathrm{tr} \log(1 + x D^V(g))} \\
&= \sum_g \frac{1}{|G|} \det(1 + x D^V(g))
\end{aligned} \tag{3.29}
$$

We have thus proved the Molien-Weyl formula for $G$-invariant fermionic state counting

$$
Z_{\text{ferm}}(G, V; x) = \sum_g \frac{1}{|G|} \det(1 + x D^V(g)) \tag{3.30}
$$

## 4 Gauged permutation invariant matrix partition function using Molien-Weyl formula

We return to the specific case of interest at hand, the matrix oscillators $A^\dagger_{ij}$ transforming as $(V_N \otimes V_N)$ where $V_N$ is the natural representation of $S_N$ of dimension $N$, with the decomposition into irreducible representations

$$
V_N = (V_0 \oplus V_H) \tag{4.1}
$$

where $V_0$ is the trivial one-dimensional representation and $V_H$ is the $(N-1)$ dimensional representation. In this section, we will reproduce the formula (2.14) for the generating function of bosonic states using the Molien-Weyl formula (3.11). It is useful to recall that the natural representation of dimension $N$ can be described as

$$
V_N = \text{ Span } \{ e_i \ \mid \ i \in \{1, 2, \cdots, n\} \} \tag{4.2}
$$

where permutations $\sigma$ act as linear operators $D^{V_N}(\sigma)$ defined as

$$
D^{V_N}(\sigma) : e_i \to e_{\sigma^{-1}(i)} \tag{4.3}
$$

With the left-to-right multiplication convention

$$
\sigma_1 \sigma_2(i) = \sigma_2(\sigma_1(i)) \tag{4.4}
$$

the linear operators obey the homomorphism condition

$$
D^{V_N}(\sigma_1) D^{V_N}(\sigma_2) = D^{V_N}(\sigma_1 \sigma_2) \tag{4.5}
$$

For a cyclic permutation $\sigma_a = (1, 2, \cdots, a)$ acting on the vector space $V_a$ of dimension $a$, using the action in (4.3). The eigenvalues of this action are multiples of $\omega_a = e^{\frac{2\pi i}{a}}$:

$$
\omega_a^t \qquad\qquad t \in \{0, 2, \cdots, a-1\} \tag{4.6}
$$

$\sigma_a$ generates the cyclic group $\mathbb{Z}_a$ and $V_a$ is isomorphic to the regular representation of $\mathbb{Z}_a$. The corresponding eigenvectors are

$$|\omega_a^t\rangle = e_1 + \omega_a^{-t} e_2 + \cdots + \omega_a^{-(a-1)t} e_a \tag{4.7}$$

As in the derivation of (2.14) in section 2 it is convenient to consider, in the first instance, permutations $\sigma \in S_N$ in a conjugacy class $p = [a_1^{p_1}, a_2^{p_2}]$, i.e. $\sigma$ has $p_1$ cycles of length $a_1$ and $p_2$ cycles of length $a_2$. The permutation $\sigma$ belongs to a subgroup of the form of a product of cyclic groups: $\mathbb{Z}_{a_1}^{\times p_1} \times \mathbb{Z}_{a_2}^{\times p_2} \subset S_N$. We can therefore use the spectrum (4.6) to deduce that the eigenvalues of the matrix $D^{V_N}(\sigma)$ are:

$$\begin{aligned} &\omega_{a_1}^{t_1}, t_1 \in \{0, \cdots, a_1 - 1\} : &&\text{Multiplicity} &= p_1 \\ &\omega_{a_2}^{t_2}, t_2 \in \{0, \cdots, a_2 - 1\} : &&\text{Multiplicity} &= p_2 \end{aligned} \tag{4.8}$$

In the tensor product $V_N \otimes V_N$, the eigenvectors are of two types, as follows:

$$\begin{array}{lll} \multicolumn{3}{c}{\text{Same eigenvalue in both tensor factors}} \\ |\omega_{a_1}^{t_1}; \alpha_1\rangle \otimes |\omega_{a_1}^{t_2}, \alpha_2\rangle & \alpha_1 \in \{1 \cdots p_1\} & \alpha_2 \in \{1 \cdots p_1\} \\ |\omega_{a_2}^{t_1}; \alpha_1\rangle \otimes |\omega_{a_2}^{t_2}, \alpha_2\rangle & \alpha_1 \in \{1 \cdots p_2\} & \alpha_2 \in \{1 \cdots p_2\} \\ \multicolumn{3}{c}{\text{Different eigenvalues in the two tensor factors}} \\ |\omega_{a_1}^{t_1}; \alpha_1\rangle \otimes |\omega_{a_2}^{t_2}, \alpha_2\rangle & \alpha_1 \in \{1 \cdots p_1\} & \alpha_2 \in \{1 \cdots p_2\} \\ |\omega_{a_2}^{t_2}; \alpha_2\rangle \otimes |\omega_{a_1}^{t_1}, \alpha_1\rangle & \alpha_1 \in \{1 \cdots p_2\} & \alpha_2 \in \{1 \cdots p_1\} \end{array} \tag{4.9}$$

The index $\alpha_1$ identifies one of the $Z_{a_1}$ in the product $\mathbb{Z}_{a_1}^{\times p_1}$ while the index $\alpha_2$ identifies one of the $\mathbb{Z}_{a_2}$ factors in the product $\mathbb{Z}_{a_2}^{\times p_2}$. This leads to

$$\det(1 - xD^{V_N \otimes V_N}(g)) = \prod_{t_1,t_2=0}^{a_1-1} (1 - x\omega_{a_1}^{t_1+t_2})^{p_1^2} \prod_{t_1,t_2=0}^{a_2-1} (1 - x\omega_{a_2}^{t_1+t_2})^{p_2^2} \prod_{t_1=0}^{a_1-1} \prod_{t_2=0}^{a_2-1} (1 - x\omega_{a_1}^{t_1}\omega_{a_2}^{t_2})^{2p_1p_2} \tag{4.10}$$

A useful lemma is:

**Lemma 1.**

$$\prod_{t=0}^{a-1} (1 - X\omega_a^t) = (1 - X^a) \tag{4.11}$$

The proof follows by observing that the coefficient of $X^c$ when the product is expanded out is:

$$\begin{aligned} (-1)^c \sum_{t=0}^{a-1} \omega_a^{ct} &= \frac{(1 - \omega_a^{ca})}{(1 - \omega_a^c)} \text{ for } c \in \{1, 2, \cdots, a - 2\} \\ &= 1 \text{ for } c \in \{0, a - 1\} \end{aligned} \tag{4.12}$$

This implies

$$\prod_{t_1,t_2=0}^{a_1-1} (1 - x\omega_{a_1}^{t_1+t_2}) = \prod_{t_1=0}^{a_1-1} (1 - x^{a_1}\omega_{a_1}^{a_1 t_1}) = \prod_{t_1=0}^{a_1-1} (1 - x^{a_1}) = (1 - x^{a_1})^{a_1} \tag{4.13}$$

Similarly

$$\prod_{t_1,t_2=0}^{a_2-1}(1-x\omega_{a_2}^{t_1+t_2}) = \prod_{t_1=0}^{a_2-1}(1-x^{a_2}\omega_{a_2}^{a_2 t_1}) = \prod_{t_1=0}^{a_2-1}(1-x^{a_2}) = (1-x^{a_2})^{a_2} \tag{4.14}$$

To calculate the mixed terms, we will need a second lemma, which we will call the roots and LCM Lemma since it connects the roots of unity eigenvalues and the Least common multiples:

**Lemma 2: roots-and-LCM-lemma.**

$$\prod_{t_1=0}^{a_1-1}(1-x^{a_2}\omega_{a_1}^{a_2 t_1}) = (1-x^{L(a_1,a_2)})^{\frac{a_1 a_2}{L(a_1,a_2)}} = (1-x^{L(a_1,a_2)})^{G(a_1,a_2)} \tag{4.15}$$

We will prove the Lemma shortly, but using the Lemma we can now calculate the mixed terms in (4.10). We note that

$$\prod_{t_1=0}^{a_1-1}\prod_{t_2=0}^{a_2-1}(1-x\omega_{a_1}^{t_1}\omega_{a_2}^{t_2}) = \prod_{t_1=0}^{a_1-1}(1-x^{a_2}\omega_{a_1}^{a_2 t_1}) = (1-x^{L(a_1,a_2)})^{\frac{a_1 a_2}{L(a_1,a_2)}} = (1-x^{L(a_1,a_2)})^{G(a_1,a_2)} \tag{4.16}$$

which gives

$$\prod_{t_1=0}^{a_1-1}\prod_{t_2=0}^{a_2-1}(1-x\omega_{a_1}^{t_1}\omega_{a_2}^{t_2})^{2p_1p_2} = (1-x^{L(a_1,a_2)})^{2p_1p_2G(a_1,a_2)} \tag{4.17}$$

Putting these products together with their $p$-dependent multiplicities as given in (4.9), we have

$$\det(1-xD^{V_N\otimes V_N}(g)) = (1-x^{a_1})^{a_1p_1^2}(1-x^{a_2})^{a_2p_2^2}(1-x^{L(a_1,a_2)})^{2p_1p_2G(a_1,a_2)} \tag{4.18}$$

In other words, for $[g] = [a_1^{p_1}, a_2^{p_2}]$.

$$\frac{1}{\det(1-xD^{V_N\otimes V_N}(g))} = \frac{1}{(1-x^{a_1})^{a_1p_1^2}(1-x^{a_2})^{a_2p_2^2}(1-x^{L(a_1,a_2)})^{2p_1p_2G(a_1,a_2)}} \tag{4.19}$$

**Proof of the roots-and-LCM-lemma.**

For the greatest common divisor of $a_1, a_2$, previously denoted $G(a_1,a_2)$ we will write for simplicity $G(a_1,a_2) = h$. We have

$$\begin{aligned} a_1 &= h\ \hat{a}_1 \\ a_2 &= h\ \hat{a}_2 \\ G(\hat{a}_1,\hat{a}_2) &= 1 \end{aligned} \tag{4.20}$$

for positive integers $h, \hat{a}_1, \hat{a}_2$. In the product over $t_1 \in \{0, 2, \cdots, a_1-1\}$ in (4.15) it is useful to write $t_1 = q\hat{a}_1 + s$, with $s \in \{0, \cdots, \hat{a}_1 - 1\}, q \in \{0, \cdots, h-1\}$. So we have to calculate

$$\prod_{t_1=0}^{a_1-1}(1-x^{a_2}\omega_{a_1}^{a_2 t_1}) = \prod_{q=0}^{h-1}\prod_{s=0}^{\hat{a}_1-1}(1-x^{a_2}\omega_{a_1}^{a_2 t_1}) \tag{4.21}$$

Note that

$$\omega_{a_1}^{a_2 t_1} = e^{\frac{2\pi i a_2 t_1}{a_1}} = e^{\frac{2\pi i \hat{a}_2 t_1}{\hat{a}_1}} = e^{\frac{2\pi i \hat{a}_2 (q\hat{a}_1 + s)}{\hat{a}_1}} = e^{\frac{2\pi i \hat{a}_2 s}{\hat{a}_1}} \tag{4.22}$$

Since $G(\hat{a}_1, \hat{a}_2) = 1$, $e^{\frac{2\pi i \hat{a}_2}{\hat{a}_1}}$ is a primitive root of unity of order $\hat{a}_1$. This means, using Lemma (4.11)

$$\prod_{s=0}^{\hat{a}_1 - 1} (1 - X e^{\frac{2\pi i \hat{a}_2 s}{\hat{a}_1}}) = (1 - X^{\hat{a}_1}) \tag{4.23}$$

We can write

$$\prod_{t_1=0}^{a_1-1} (1 - x^{a_2} \omega_{a_1}^{a_2 t_1}) = \prod_{q=0}^{h-1} \prod_{s=0}^{\hat{a}_1 - 1} (1 - x^{a_2} e^{\frac{2\pi i s \hat{a}_2}{\hat{a}_1}}) = \prod_{q=0}^{h-1} (1 - x^{a_2 \hat{a}_1}) = (1 - x^{a_2 \hat{a}_1})^h \tag{4.24}$$

Now observe that

$$\begin{aligned} L(a_1, a_2) \equiv LCM(a_1, a_2) &= LCM(h\hat{a}_1, h\hat{a}_2) = h\hat{a}_1 \hat{a}_2 \\ a_1 a_2 &= h^2 \hat{a}_1 \hat{a}_2 \\ h &= \frac{a_1 a_2}{LCM(a_1, a_2)} \end{aligned} \tag{4.25}$$

We conclude that

$$\begin{aligned} \prod_{t_1=0}^{a_1-1} (1 - x^{a_2} \omega_{a_1}^{a_2 r_1}) &= (1 - x^{a_2 \hat{a}_1})^h = (1 - x^{h \hat{a}_2 \hat{a}_1})^h \\ &= (1 - x^{L(a_1, a_2)})^{\frac{a_1 a_2}{L(a_1, a_2)}} = (1 - x^{L(a_1, a_2)})^{G(a_1, a_2)} \end{aligned} \tag{4.26}$$

For a general partitition $[a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$, the tensor product states can be organised as in (4.9) according to whether the eigenvalues in the two factors of the tensor product $V_N \otimes V_N$ are the same or different. When the eigenvalues are the same we get, collecting factors of the form in (4.13) and (4.14), a factor in the determinant $\det(1 - x D^{V_N \otimes V_N}(g))$

$$\prod_{i=0}^{a_i - 1} (1 - x^{a_i})^{a_i} \tag{4.27}$$

Collecting contributions to the determinant from cases where the eigenvalues are distinct, noting that the distinct pairs can be any $a_i, a_j$ with $i, j \in \{1, \cdots, K\}$ and using the (4.26) formula, we have

$$\prod_{i<j \in \{1, \cdots, K\}} (1 - x^{L(a_i, a_j)})^{G(a_i, a_j)} \tag{4.28}$$

Collecting both cases and using the multiplicities of eigenvalues for the two cases from (4.9) we obtain

$$\prod_{i=1}^{K} (1 - x^{a_i})^{a_i p_i^2} \prod_{i<j \in \{1, \cdots, K\}} (1 - x^{L(a_i, a_j)})^{2 p_i p_j G(a_i, a_j)} \tag{4.29}$$

which agrees with $\mathcal{Z}(N, p, x)$.

## 5 The refined partition function for general gauged 11-parameter matrix harmonic oscillator system

In this section we derive the partition function for the general gauged permutation invariant matrix harmonic oscillator. The general permutation invariant quadratic function of matrix variables was described in [4] and the representation theoretic diagonalisation was given in [5]. The canonical quantization of the matrix quantum system was described and the canonical partition function of the full model, including permutation invariant and non-invariant sectors was given in [11]. The Molien-Weyl formula, which was recovered physically as a continuum limit of the lattice formulation of the gauge invariant theory in [18], turns out, as we show here, to be a powerful tool for calculating the partition function of the general model.

The partition function of the 11-parameter harmonic oscillator system, with $S_N$ invariance imposed as a gauge symmetry, takes the form

$$\begin{aligned}&\mathcal{Z}(N,\beta_0^{ab},\beta_H^{ab},\beta_2,\beta_3)\\&=\mathrm{tr}_{\mathcal{H}}\mathcal{P}_0\exp\left[-\sum_{a,b=1}^{2}\beta_0^{ab}A^\dagger_{0;a}A_{0;b}-\sum_{a,b=1}^{3}\beta_H^{ab}\sum_{m=1}^{\dim V_H}A^\dagger_{H,a;m}A_{H,b;m}-\beta_2\sum_{m=1}^{\dim V_2}A^\dagger_{2;m}A_{2;m}-\beta_3\sum_{m=1}^{\dim V_3}A^\dagger_{3;m}A_{3;m}\right]\end{aligned}\tag{5.1}$$

The path integral formulation for the quantum system, as a limit of a lattice formulation where the covariant derivative is described in terms of parallel transport using group elements, is given in [18]. The Hilbert space $\mathcal{H}$ is the Fock space generated by the matrix oscillators $A^\dagger_{ij}$. The exponent uses linear combinations of the $A^\dagger_{ij}$ oscillators organised according to the decomposition of the matrix oscillators into irreducible representations of $S_N$:

$$V_N\otimes V_N=2V_0\oplus 3V_H\oplus V_2\oplus V_3\tag{5.2}$$

where $V_0$ is the one-dimensional representation of $S_N$, $V_H$ is the $(N-1)$ dimensional hook representation corresponding to the Young diagram with row lengths $[N-1,1]$, $V_2$ is the representation of dimension $N(N-3)/2$ associated with Young diagram $[N-2,2]$, while $V_3$ is the irrep of dimension $(N-1)(N-2)/2$ associated with the Young diagram $[N-2,1,1]$. The creation operators

$$A^\dagger_{0;a}\,,\qquad A^\dagger_{H,a;m}\,,\qquad A^\dagger_{2;m}\,,\qquad A^\dagger_{3;m}\tag{5.3}$$

transform in the four irreps, respectively, $V_0,V_H,V_2,V_3$ as in (5.2). The $a$ indices run over the multiplicity spaces and these creation operators can be expressed as linear combinations of $A^\dagger_{ij}$ using the Clebsch-Gordan coefficients associated with (5.2), as explained in more detail in [11]. Similar remarks apply to the annihilation operators. The parameters $\beta_0^{ab}$ form a symmetric $2\times 2$ matrix (with 3 parameters), the $\beta_H^{ab}$ form a symmetric $3\times 3$ matrix, so that alongside $\beta_2,\beta_3$ there is a the total number of parameters is $3+6+1+1=11$. By diagonalising the matrices $\beta_0^{Hab}$ and $\beta_H^{ab}$, we find that the thermal partition function of the invariant sector depends on $2+3+1+1=7$ parameters (as does the partition function of the full Hilbert space computed in [11]). We thus find that the most general partition function takes the form

$$\begin{aligned}&\mathcal{Z}(N,\beta_0^{a},\beta_H^{a},\beta_2,\beta_3)\\&=\mathrm{tr}_{\widetilde{\mathcal{H}}}\mathcal{P}_0\exp\left[-\sum_{a=1}^{2}\beta_0^{a}\tilde{A}^\dagger_{0;a}\tilde{A}_{0;a}-\sum_{a=1}^{3}\beta_H^{a}\sum_{m=1}^{\dim V_H}\tilde{A}^\dagger_{H,a;m}\tilde{A}_{H,b;m}-\beta_2\sum_{m=1}^{\dim V_2}\tilde{A}^\dagger_{2;m}\tilde{A}_{2;m}-\beta_3\sum_{m=1}^{\dim V_3}\tilde{A}^\dagger_{3;m}\tilde{A}_{3;m}\right]\end{aligned}\tag{5.4}$$



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where $\beta_0^a, \beta_H^a$ are the eigenvalues of the matrices $\beta_0^{ab}, \beta_H^{ab}$ respectively. The Hilbert space $\widetilde{\mathcal{H}}$ is the Fock space generated by the 7 independent creation operators $\tilde{A}^\dagger_{0;a}, \tilde{A}^\dagger_{H;a}, \tilde{A}^\dagger_2, \tilde{A}^\dagger_3$.

The $S_N$ invariant partition function is a weighted counting of Fock space states generated by the seven types of oscillators

$$\begin{aligned}
&\tilde{A}^\dagger_{0,a}, a \in \{1,2\} \quad \text{weighted by} \quad e^{-\beta_0^a} \\
&\tilde{A}^\dagger_{H,a;m}, a \in \{1,2,3\} \quad \text{weighted by} \quad e^{-\beta_0^a} \\
&\tilde{A}^\dagger_{2;m} \quad \text{weighted by} \quad e^{-\beta_2} \\
&\tilde{A}^\dagger_{3;m} \quad \text{weighted by} \quad e^{-\beta_3}
\end{aligned} \tag{5.5}$$

with the insertion of the $S_N$ projector. As explained in section (3.2), the weighted counting of invariants takes the form

$$\mathcal{Z}(N, \beta_0^a, \beta_H^a, \beta_2, \beta_3) == \frac{1}{N!} \sum_\sigma \left( \prod_{a=1}^{2} Z^{(\sigma)}_{V_0}(e^{-\beta_0^a}) \right) \left( \prod_{a=1}^{3} Z^{(\sigma)}_{V_H}(e^{-\beta_H^a}) \right) Z^{(\sigma)}_{V_2}(e^{-\beta_2}) Z^{(\sigma)}_{V_3}(e^{-\beta_3}) \tag{5.6}$$

where each factor is an inverse determinant of Molien-Weyl form ( see in particular equations (3.24) and (3.22) ).

### 5.1 Molien-Weyl determinants for $V_N, V_0, V_H$

The natural representation $V_N$ decomposes as

$$V_N = V_0 \oplus V_H \tag{5.7}$$

For the Fock space generated by an oscillator in the trivial one-dimensional representation, the counting function is

$$\mathcal{Z}_{V_0}(x) = \sum_p \frac{1}{\text{Sym } p} \frac{1}{(1 - xD^{V_0}(\sigma^{(p)}))} = \sum_p \frac{1}{\text{Sym } p} \frac{1}{(1-x)} \tag{5.8}$$

Next we consider oscillators in the representation $V_N$ and a permutation $\sigma$ with cycle structure of the form $[a_1^{p_1}, a_2^{p_2} \cdots a_K^{p_K}]$. This permutation belongs to a subgroup of the form $\mathbb{Z}_{a_1}^{p_1} \times Z_{a_2}^{p_2} \cdots \times Z_{a_K}^{p_K}$ in $S_N$. The eigenvectors are of the form

$$|\omega_{a_i}^{t_i}; \alpha_i\rangle \quad \text{with} \quad i \in \{1, 2, \cdots, K\} \ , \ \ t_i \in \{0, 1, \cdots, a_i - 1\} \ , \ \ \alpha_i \in \{1, 2, \cdots, p_i\} \tag{5.9}$$

with eigenvalues $\omega_{a_i}^{r_i}$ each with multiplicity $p_i$. The total number of eigenvectors is $p_1 a_1 + p_2 a_2 + \cdots + p_K a_K = N$ and these eigenvectors span $V_N$. Therefore, the determinant is (using (4.11))

$$\det(1 - xD^{V_N}(\sigma)) = \prod_{i=1}^{K} \prod_{t_i=0}^{a_i-1} (1 - x\omega_{a_i}^{t_i})^{p_i} = \prod_{i=1}^{K} (1 - x^{a_i})^{p_i} \tag{5.10}$$

We also have

$$\det(1 - xD^{V_0}(\sigma)) = (1 - x) \tag{5.11}$$

and

$$\det(1 - xD^{V_N}(\sigma)) = \det(1 - xD^{V_0 \oplus V_H}(\sigma)) = \det(1 - xD^{V_0}(\sigma)) \det(1 - xD^{V_H}(\sigma)) \tag{5.12}$$

We conclude that

$$\det(1 - xD^{V_H}(\sigma)) = \frac{1}{(1-x)} \prod_{i=1}^{K} (1 - x^{a_i})^{p_i} \tag{5.13}$$

The Molien-Weyl formula then becomes:

$$\mathcal{Z}_{V_H}(x) = \sum_{p \vdash N} \frac{(1-x)}{\text{Sym } p} \prod_i \frac{1}{(1 - x^{a_i})^{p_i}} \tag{5.14}$$

### 5.2 Using $S^2 V_N$ and $\Lambda^2 V_N$ to get the determinants for $V_2, V_3$

The eigenvectors of $\sigma$, with cycle structure $[a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$ in $V_N \otimes V_N$ are

$$|\omega_{a_i}^{t_i}; \alpha_i\rangle \otimes |\omega_{a_j}^{t_j}; \alpha_j\rangle \tag{5.15}$$

where

$$\begin{aligned} i, j &\in \{1, 2, \cdots, K\} \\ t_i &\in \{0, 1, \cdots, a_i - 1\} \\ t_j &\in \{0, 1, \cdots, a_j - 1\} \\ \alpha_i &\in \{0, 1, \cdots, p_i\} \ , \quad \alpha_j \in \{1, \cdots, p_j\} \end{aligned} \tag{5.16}$$

It is known that the symmetric tensor power decomposes as

$$S^2(V_H) = V_0 \oplus V_H \oplus V_2 \tag{5.17}$$

and

$$\Lambda^2 V_H = V_3 \tag{5.18}$$

These decompositions are described in detail in [10]. A dimension count to check the first equation is

$$\frac{N(N-1)}{2} = 1 + (N-1) + \frac{N(N-3)}{2} \tag{5.19}$$

One also checks that the hook formula for $[N-2, 1, 1]$ agrees with $(N-1)(N-2)/2$.

Since $V_N = V_0 \oplus V_H$, we can use the above to find

$$\begin{aligned} S^2(V_N) &= S^2(V_0) \oplus S^2(V_H) \oplus \text{Sym}\left((V_H \otimes V_0) \oplus (V_0 \otimes V_H)\right) \\ &= V_0 \oplus (V_0 \oplus V_H \oplus V_2) \oplus V_H \\ &= 2V_0 \oplus 2V_H \oplus V_2 \end{aligned} \tag{5.20}$$

and

$$\Lambda^2(V_N) = \Lambda^2(V_H) \oplus \Lambda\left((V_0 \otimes V_H) \oplus (V_0 \otimes V_H)\right) = V_3 \oplus V_H \tag{5.21}$$

These equations imply that

$$\det(1 - xD^{S^2(V_N)}(\sigma)) = \det(1 - xD^{V_0}(\sigma))^2 \det(1 - xD^{V_H}(\sigma))^2 \det(1 - xD^{V_2}(\sigma)) \tag{5.22}$$

and

$$\det(1 - xD^{\Lambda^2(V_N)}(\sigma)) = \det(1 - xD^{V_H}(\sigma)) \det(1 - xD^{V_3}(\sigma)) \tag{5.23}$$

which can be used to calculate $\det(1 - xD^{V_2}(\sigma))$ and $\det(1 - xD^{V_3}(\sigma))$ from the determinants of the symmetrised and anti-symmetrised powers along with those for $V_0, V_H$. The determinants for $V_0$ and $V_H$ are already given in section 5.1. We will now calculate the determinants for the symmetrised and anti-symmetrised square of $V_N$.



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### 5.3 Molien-Weyl determinant for $S^2(V_N)$

Consider $S^2(V_N)$. We list a complete set of eigenvectors, corresponding eigenvalues with multiplicities, and the factor $\det(1 - xD^{S^2(V_N)}(g)$:

**Case 1.** Different powers of the same root of unity $\omega_{a_i}$ on the two tensor factors:

$$\begin{aligned} &\text{Eigenvectors} \quad (|\omega_{a_i}^{t_1};\alpha_1\rangle \otimes |\omega_{a_i}^{t_2};\alpha_2\rangle + |\omega_{a_i}^{t_2};\alpha_2\rangle \otimes |\omega_{a_i}^{t_1};\alpha_1\rangle) \\ &\alpha_1,\alpha_2 \in \{1,\cdots,p_i\}, \quad t_1 < t_2 \in \{0,\cdots,a_i-1\} \\ &\text{Eigenvalues} \quad \omega_{a_i}^{t_1+t_2} \quad \text{Multiplicity} = p_i^2 \end{aligned} \tag{5.24}$$

For each $a_i$ this contributes to the determinant $\det(1 - xD^{S^2(V_N)}(\sigma))$ a factor

$$F_1(a_i) \quad = \prod_{t_1<t_2\in\{0,1,\cdots,a_i-1\}} (1 - x\omega_{a_i}^{t_1+t_2})^{p_i^2} \tag{5.25}$$

**Case 2.** Same power of the same root of unity on the two tensor factors with different $a_i$-subsets chosen from the $p_i$ possibilities:

$$\begin{aligned} &\text{Eigenvectors} \quad (|\omega_{a_i}^{t};\alpha_1\rangle \otimes |\omega_{a_i}^{t};\alpha_2\rangle + |\omega_{a_i}^{t};\alpha_2\rangle \otimes |\omega_{a_i}^{t};\alpha_1\rangle) \\ &\alpha_1 < \alpha_2 \in \{1,\cdots,p_i\} \\ &\text{Eigenvalues} \quad \omega_{a_i}^{2t} \quad \text{Multiplicity} = p_i(p_i-1)/2 \end{aligned} \tag{5.26}$$

This contributes to the MW-determinant a factor

$$F_2(a_i) = \prod_{t=0}^{a_i-1} (1 - x\omega_{a_i}^{2t})^{p_i(p_i-1)/2} \tag{5.27}$$

**Case 3.** Same power of the same root of unity on the two tensor factors with same $a_i$-subsets:

$$\begin{aligned} &\text{Eigenvectors} \quad |\omega_{a_i}^{t};\alpha\rangle \otimes |\omega_{a_i}^{t};\alpha\rangle\,; \qquad t \in \{0,\cdots,a_i-1\}\,; \qquad \alpha \in \{1,\cdots,p_i\} \\ &\text{Eigenvalues} \quad \omega_{a_i}^{2t} \quad \text{Multiplicity} = p_i \end{aligned} \tag{5.28}$$

This contributes a factor

$$F_3(a_i) = \prod_{t=0}^{a_i-1} (1 - x\omega_{a_i}^{2t})^{p_i} \tag{5.29}$$

**Case 4.** Finally we have the case where the two tensor factors contain eigenvectors involving different roots of unity:

$$\begin{aligned} &\text{Eigenvectors} \quad (|\omega_{a_i}^{t_1};\alpha_1\rangle \otimes |\omega_{a_j}^{t_2};\alpha_2\rangle + |\omega_{a_j}^{t_2};\alpha_2\rangle \otimes |\omega_{a_i}^{t_1};\alpha_1\rangle) \quad i \neq j \in \{1,\cdots,K\} \\ &\quad t_1 \in \{0,1,\cdots,a_i-1\}, t_2 \in \{0,1,\cdots,a_j-1\} \\ &\alpha_1 \in \{1,\cdots,p_i\}, \alpha_2 \in \{1,\cdots,p_j\} \\ &\text{Eigenvalues} \quad \omega_{a_i}^{t_1}\omega_{a_j}^{t_2} \quad \text{Multiplicity} \quad p_ip_j \end{aligned} \tag{5.30}$$

This leads to the factor

$$F_4(a_i,a_j) \quad = \quad \prod_{t_1=1}^{a_i-1}\prod_{t_2=1}^{a_j-1}(1-x\omega_{a_i}^{t_1}\omega_{a_j}^{t_2})^{p_1p_2} \tag{5.31}$$

We show in appendix A that

$$F_1(a_i) = \prod_{t_1<t_2\in\{0,\cdots,a_i-1\}}(1-x\omega_{a_i}^{t_1+t_2})^{p_i^2} \quad = \quad \frac{(1-x^{a_i})^{\frac{a_ip_i^2}{2}}}{(1-x^{\frac{a_i}{G(2,a_i)}})^{\frac{G(2,a_i)\ p_i^2}{2}}} \tag{5.32}$$

$$F_2(a_i) = \prod_{t=1}^{a_i-1}(1-x\omega_{a_i}^{2t})^{\frac{p_i(p_i-1)}{2}} = (1-x^{\frac{a_i}{G(2,a_i)}})^{\frac{G(2,a_i)\ p_i(p_i-1)}{2}} \tag{5.33}$$

$$F_3(a_i) = \prod_{t=1}^{a_i-1}(1-x\omega_{a_i}^{2t})^{p_i} = (1-x^{\frac{a_i}{G(2,a_i)}})^{G(2,a_i)p_i} \tag{5.34}$$

$$\begin{aligned} F_4(a_i,a_j) &= \prod_{t_1=1}^{a_i-1}\prod_{t_2=1}^{a_j-1}(1-x\omega_{a_i}^{t_1}\omega_{a_j}^{t_2})^{p_1p_2} = \prod_{t_1=0}^{a_i-1}(1-x^{a_j}\omega_{a_i}^{a_jt_1})^{p_ip_j} \\ &= (1-x^{L(a_i,a_j)})^{G(a_i,a_j)p_ip_j} \end{aligned} \tag{5.35}$$

Collecting terms

$$\frac{1}{\det(1-xD^{S^2(V_N)}(\sigma))} = \prod_i (F_1(a_i)F_2(a_i)F_3(a_i))^{-1}\prod_{i<j}(F_4(a_i,a_j))^{-1} \tag{5.36}$$

Using the equations (5.32), (5.33), (5.34) and (5.35), we have

$$\frac{1}{\det(1-D^{S^2(V_N)}(\sigma))} = \prod_i \frac{1}{(1-x^{\frac{a_i}{G(2,a_i)}})^{G(2,a_i)\frac{p_i}{2}}(1-x^{a_i})^{\frac{a_ip_i^2}{2}}}\prod_{i<j}\frac{1}{(1-x^{L(a_i,a_j)})^{G(a_i,a_j)p_ip_j}} \tag{5.37}$$

### 5.4 Molien-Weyl determinant for $\Lambda^2(V_N)$

We now consider the antisymmetric squared tensor power $\Lambda^2(V_N)$ and list a complete set of eigenvectors, corresponding eigenvalues with multiplicities, and the factor $\det(1-xD^{\Lambda^2(V_N)}(g)$.

**Case 1.** Different powers of the same root of unity $\omega_{a_i}$ in the two tensor factors:

$$\begin{aligned} &\text{Eigenvectors} \quad (|\omega_{a_i}^{t_1};\alpha_1\rangle\otimes|\omega_{a_i}^{t_2};\alpha_2\rangle - |\omega_{a_i}^{t_2};\alpha_2\rangle\otimes|\omega_{a_i}^{t_1};\alpha_1\rangle) \\ &\qquad\qquad \alpha_1,\alpha_2\in\{1,\cdots,p_i\},\quad t_1<t_2\in\{0,\cdots,a_i-1\} \\ &\text{Eigenvalues} \quad \omega_{a_i}^{t_1+t_2} \quad \text{Multiplicity} = p_i^2 \end{aligned} \tag{5.38}$$

For each $a_i$ this contributes to the determinant $\det(1-xD^{S^2(V_N)}(\sigma))$ a factor

$$F_1(a_i) \quad = \quad \prod_{t_1<t_2\in\{0,1,\cdots,a_i-1\}}(1-x\omega_{a_i}^{t_1+t_2})^{p_i^2} \tag{5.39}$$



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**Case 2.** Same power of the same root of unity on the two tensor factors with different $a_i$-subsets chosen from the $p_i$ possibilities:

$$\begin{aligned}
&\text{Eigenvectors} \quad (|\omega_{a_i}^t;\alpha_1\rangle \otimes |\omega_{a_i}^t;\alpha_2\rangle - |\omega_{a_i}^t;\alpha_2\rangle \otimes |\omega_{a_i}^t;\alpha_1\rangle)\\
&\alpha_1 < \alpha_2 \in \{1,2,\cdots,p_i\}\\
&\text{Eigenvalues} \quad \omega_{a_i}^{2t} \quad \text{Multiplicity} = p_i(p_i-1)/2
\end{aligned} \tag{5.40}$$

This contributes to the MW-determinant a factor

$$F_2(a_i) = \prod_{t=1}^{a_i-1} (1 - x\omega_{a_i}^{2t})^{p_i(p_i-1)/2} \tag{5.41}$$

There is no **Case 3** or corresponding $F_3$ factor, unlike the $S^2(V_N)$ case.

**Case 4.** Finally we have the case where the two tensor factors contain eigenvectors involving different roots of unity:

$$\begin{aligned}
&\text{Eigenvectors} \quad (|\omega_{a_i}^{t_1};\alpha_1\rangle \otimes |\omega_{a_j}^{t_2};\alpha_2\rangle - |\omega_{a_j}^{t_2};\alpha_2\rangle \otimes |\omega_{a_i}^{t_1};\alpha_1\rangle) \quad i \neq j \in \{1,\cdots,K\}\\
&\quad t_1 \in \{0,1,\cdots,a_i-1\}, t_2 \in \{0,1,\cdots,a_j-1\}\\
&\alpha_1 \in \{1,\cdots,p_i\}, \alpha_2 \in \{1,\cdots,p_j\}\\
&\text{Eigenvalues} \quad \omega_{a_i}^{t_1}\omega_{a_j}^{t_2} \quad \text{Multiplicity} \quad p_i p_j
\end{aligned} \tag{5.42}$$

This leads to the factor

$$F_4(a_i,a_j) \quad = \quad \prod_{t_1=1}^{a_i-1}\prod_{t_2=1}^{a_j-1} (1 - x\omega_{a_i}^{t_1}\omega_{a_j}^{t_2})^{p_1 p_2} \tag{5.43}$$

Collecting the different cases, for a permutation $\sigma$ with cycle structure $[a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$:

$$\det(1 - D^{\Lambda^2(V_N)}(\sigma) = \prod_i F_1(a_i)F_2(a_i) \prod_{i<j} F_4(a_i,a_j) \tag{5.44}$$

Using the equations (5.32), (5.33), (5.35), the Molien-Weyl generating function for invariants is:

$$\frac{1}{\det(1 - xD^{\Lambda^2(V_N)}(\sigma))} = \prod_i \frac{(1 - x^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i}{2}G(2,a_i)}}{(1 - x^{a_i})^{\frac{a_i p_i^2}{2}}} \prod_{i<j} \frac{1}{(1 - x^{L(a_i,a_j)})^{G(a_i,a_j)p_i p_j}} \tag{5.45}$$

The equations (5.45), (5.37) and (2.14) show that

$$\frac{1}{\det(1 - xD^{\Lambda^2(V_N)}(\sigma))}\frac{1}{\det(1 - xD^{S^2(V_N)}(\sigma))} = \frac{1}{\det(1 - xD^{(V_N \otimes V_N)}(\sigma))} \tag{5.46}$$

which is as expected since

$$\det(1 - xD^{(V \oplus W)}(\sigma)) = \det(1 - xD^{(V)}(\sigma)) \det(1 - xD^{(W)}(\sigma)) \tag{5.47}$$

and $V_N \otimes V_N = S^2(V_N) \oplus \Lambda^2(V_N)$.



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### 5.5 Molien-Weyl determinants for the irreps $V_2, V_3$

Using the decomposition into irreducibles obtained as equation (5.21) from section 5.2

$$\Lambda^2(V_N) = V_H \oplus V_3 \tag{5.48}$$

we have

$$\det(1 - xD^{\Lambda^2(V_N)}(\sigma)) = \det(1 - xD^{V_H}(\sigma)) \det(1 - xD^{V_3}(\sigma)) \tag{5.49}$$

Hence

$$\frac{1}{\det(1 - xD^{V_3}(\sigma))} = \frac{\det(1 - xD^{V_H}(\sigma))}{\det(1 - xD^{\Lambda^2(V_N)}(\sigma))} \tag{5.50}$$

Therefore, for $\sigma$ having a general cycle structure $[a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$, using (5.13) and (5.45), we have:

$$\begin{aligned}\frac{1}{\det(1 - xD^{V_3}(\sigma))} &= \frac{1}{(1-x)} \prod_i (1 - x^{a_i})^{p_i} \times \prod_i \frac{(1 - x^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i}{2} G(2,a_i)}}{(1 - x^{a_i})^{\frac{a_i p_i^2}{2}}} \\ &\quad \times \prod_{i<j} \frac{1}{(1 - x^{L(a_i,a_j)})^{\frac{a_i a_j p_i p_j}{L(a_i,a_j)}}}\end{aligned} \tag{5.51}$$

To obtain the Molien-Weyl determinant for $V_2$ we use the decomposition (5.19) from section 5.2

$$S^2(V_N) = 2V_0 \oplus 2V_H \oplus V_2 \tag{5.52}$$

We have

$$\det(1 - xD^{S^2(V_N)}(\sigma)) = (\det(1 - xD^{V_0}(\sigma)))^2 (\det(1 - xD^{V_H}(\sigma)))^2 (\det(1 - xD^{V_2}(\sigma))) \tag{5.53}$$

This allows us to write $(\det(1 - xD^{V_2}(\sigma)))^{-1}$ in terms of determinants, notably (5.10) and (5.37), which we have already calculated:

$$\begin{aligned}&(\det(1 - xD^{V_2}(\sigma)))^{-1} \\ &\quad = (\det(1 - xD^{V_0}(\sigma)))^2 (\det(1 - xD^{V_H}(\sigma)))^2 \det(1 - xD^{S^2(V_N)}(\sigma))^{-1} \\ &\quad = (\det(1 - xD^{V_N}(\sigma)))^2 \det(1 - xD^{S^2(V_N)}(\sigma))^{-1} \\ &\quad = \left(\prod_i (1 - x^{a_i})^{2p_i}\right) \det(1 - xD^{S^2(V_N)}(\sigma))^{-1} \\ &\quad = \prod_i (1 - x^{a_i})^{2p_i} \prod_i \frac{1}{(1 - x^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i}{2} G(2,a_i)} (1 - x^{a_i})^{\frac{a_i p_i^2}{2}}} \prod_{i<j} \frac{1}{(1 - x^{L(a_i,a_j)})^{G(a_i,a_j) p_i p_j}}\end{aligned} \tag{5.54}$$

### 5.6 Physical partition function

We can now use the results for the Molien-Weyl determinants obtained above to make the formula (5.6) explicit. It is convenient to define $x_1 = e^{-\beta_0^1}, x_2 = e^{-\beta_0^2}$ for the irrep $V_0$,

$x_3 = e^{-\beta_H^1}, x_4 = e^{-\beta_H^2}, x_5 = e^{-\beta_H^3}$ for the irrep $V_H$, $x_6 = e^{-\beta_2}$ For $V_2$ and $x_7 = e^{-\beta_3}$ for $V_3$. The product of Molien-Weyl determinants with the respective $x_i$' s is

$$\frac{1}{\det(1-x_1 D^{V_0}(\sigma))}\frac{1}{\det(1-x_2 D^{V_0}(\sigma))}\frac{1}{\det(1-x_3 D^{V_H}(\sigma))}\frac{1}{\det(1-x_4 D^{V_H}(\sigma))}\frac{1}{\det(1-x_5 D^{V_H}(\sigma))}$$
$$\times\frac{1}{\det(1-x_6 D^{V_2}(\sigma))}\frac{1}{\det(1-x_7 D^{V_3}(\sigma))} \tag{5.55}$$

For $[\sigma] = [a_1^{p_1}, a_2^{p_2}, \cdots, a_K^{p_K}]$

$$\begin{aligned}
&\mathcal{Z}(N,p;x_1,\cdots,x_7)\\
&=\frac{1}{\det(1-x_1 D^{V_0}(\sigma))}\frac{1}{\det(1-x_2 D^{V_0}(\sigma))}\frac{1}{\det(1-x_3 D^{V_H}(\sigma))}\frac{1}{\det(1-x_4 D^{V_H}(\sigma))}\frac{1}{\det(1-x_5 D^{V_H}(\sigma))}\\
&\quad\times\frac{1}{\det(1-x_6 D^{V_2}(\sigma))}\frac{1}{\det(1-x_7 D^{V_3}(\sigma))}\\
&=\frac{(1-x_3)(1-x_4)(1-x_5)}{(1-x_1)(1-x_2)}\prod_i\frac{1}{(1-x_3^{a_i})^{p_i}}\frac{1}{(1-x_4^{a_i})^{p_i}}\frac{1}{(1-x_5^{a_i})^{p_i}}\\
&\quad\times\prod_i(1-x_6^{a_i})^{2p_i}\prod_i\frac{1}{(1-x_6^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i}{2}G(2,a_i)}(1-x_6^{a_i})^{\frac{a_i p_i^2}{2}}}\prod_{i<j}\frac{1}{(1-x_6^{L(a_i,a_j)})^{G(a_i,a_j)p_i p_j}}\\
&\quad\times\frac{1}{(1-x_7)}\prod_i(1-x_7^{a_i})^{p_i}\prod_i\frac{(1-x_7^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i}{2}G(2,a_i)}}{(1-x_7^{a_i})^{\frac{a_i p_i^2}{2}}}\prod_{i<j}\frac{1}{(1-x_7^{L(a_i,a_j)})^{G(a_i,a_j)p_i p_j}}
\end{aligned} \tag{5.56}$$

The general invariant partition function is

$$\mathcal{Z}(N;x_1,\cdots,x_7) = \sum_{p\vdash N}\frac{1}{\text{Sym }p}\mathcal{Z}(N,p;x_1,\cdots,x_7) \tag{5.57}$$

Note that when we set $x_i \to x$, the expression simplifies to (2.14) and we recover, as expected, the partition function of the unrefined theory .

## 6 Summary and outlook

In this paper we have calculated the partition function for matrix quantum mechanics with gauged permutation symmetry. We found explicit formulae for the case of harmonic oscillator systems as a sum over partitions of $N$, where the summands depend on elementary number theoretic functions of the partitions. The phase structure of these partition functions will be studied in detail in upcoming work [22]. The rapid growth of the number of invariant states as a function of the number of oscillators (as evident from the sequences given in [4] and [23]) leads us to expect that the partition function given in (2.11) and (2.14) will have the characteristics of a system with vanishing Hagedorn temperature at large $N$ similar to what is encountered in tensor quantum mechanics with continuous symmetries [24, 25] related to a similar growth of the number of states [23, 26–28]. This is to be contrasted with the finite Hagedorn temperature observed in multi-matrix systems [29], which has been revisited with an interpretation in terms of small black holes in AdS [30–32]. The finite $N$ formulae for both canonical and micro-canonical ensembles available in the systems studied here make these a very interesting set-up to study the approach to the transition at large $N$ and the relation between the behaviours of the canonical and micro-canonical ensembles in this approach.



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Finite group gauge symmetries have been considered in lattice gauge theory [33–36] and comparisons between the phase structure of the quantum mechanical models at hand with phases encountered in lattice gauge theory form additional motivation for the thermodynamic investigations of the partition functions derived here.

It is also worth noting that the canonical partition functions studied here (equations (2.11), (2.17), (5.57) and (5.56)) can be summed up into rational functions of the form $P(x)/Q(x)$ and as such define Hilbert series for commutative rings. The denominators encode information about the generators of the ring. They have a graph-theoretic interpretation, as observed and illustrated in [4], and as studied in more detail in [6]. The numerators encode relations between the generators, as well as higher order relations. Applications of Hilbert series in connection with group invariants with physical applications are an active area of research in particle physics and string theory (see e.g. [37–40]). The structure of these rational forms for the partition functions given here, and the elucidation of the generators and relations encoded, is an interesting subject of further study.

The group-theoretic solutions to the 13-parameter Gaussian matrix model [5] (and related models for symmetric matrices having vanishing diagonals [10], multi-matrix models [6] and tensor models [41]) along with the calculations of partition functions using techniques developed here and the path integral formulation in [18] set up the framework for the study of interesting perturbations of these models by higher order permutation invariant interactions. Any permutation invariant polynomial function of degree greater than two can be viewed as an interaction term. In the matrix data science applications of zero-dimensional matrix models such polynomials of degree 3 and 4 have been used to detect and rank the strengths of non-Gaussianities real-world matrix ensembles [8–10]. Adding the higher ranked polynomial functions to matrix models and performing perturbative calculations of correlators will be useful in these applications. Analogous calculations in the matrix quantum systems will allow us to investigate the effect of interactions on thermodynamic features. It will be interesting to investigate the existence of double scaling limits analogous to the melonic limits of tensor models [42, 43]. The phase structures of the perturbed models should be accessible to hybrid Monte Carlo methods which have been applied to multi-matrix systems in the context of BFSS [16] and BMN [17] models [44–47].

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## A Computation of products over roots of unity

Here we calculate

$$F_1(a_i) \quad = \prod_{t_1<t_2\in\{0,\cdots,a_i-1\}} (1-x\omega_{a_i}^{t_1+t_2})^{p_i^2} \tag{A.1}$$

We first calculate

$$\hat{F}_1(a_i) \equiv \prod_{t_1<t_2\in\{0,\cdots,a_i-1\}} (1-x\omega_{a_i}^{t_1+t_2}) \tag{A.2}$$

Note that $(1-x\omega_{a_i}^{t_1+t_2})$ is unchanged when we swop $t_1, t_2$. This means

$$(\hat{F}_1(a_i))^2 = \prod_{t_1\neq t_2\in\{0,\cdots,a_i-1\}} (1-x\omega_{a_i}^{t_1+t_2}) \tag{A.3}$$

This can be calculated as follows:

$$\begin{aligned}(\hat{F}_1(a_i))^2 &= \prod_{t_1=1}^{a_i-1}\prod_{t_2\neq t_1}(1-x\omega_{a_i}^{t_1+t_2}) \\ &= \prod_{t_1=1}^{a_i-1}\frac{1}{(1-x\omega_{a_i}^{2t_1})}\prod_{t_2=0}^{a_i-1}(1-x\omega_{a_i}^{t_1+t_2}) \\ &= \prod_{t_1}\frac{(1-x^{a_i}\omega_{a_i}^{t_1a_i})}{(1-x\omega_{a_i}^{2t_1})}\end{aligned} \tag{A.4}$$

where we used Lemma 1 (equation (4.11)) for the product over $t_2$. Simplifying further

$$(\hat{F}_1(a_i))^2 = \prod_{t_1=0}^{a_i-1}\frac{(1-x^{a_i}\omega_{a_i}^{t_1a_i})}{(1-x\omega_{a_i}^{2t_1})} = \prod_{t_1}\frac{(1-x^{a_i})}{(1-x\omega_{a_i}^{2t_1})} = (1-x^{a_i})^{a_i}\prod_{t_1}\frac{1}{(1-x\omega_{a_i}^{2t_1})} \tag{A.5}$$

If $a_i$ is odd, $\omega_{a_i}^2$ is a primitive $a_i$'th root of unity and we have (using Lemma 1 ):

$$\prod_{t_1=0}^{a_i-1}\frac{1}{(1-x\omega_{a_i}^{2t_1})} = \frac{1}{(1-x^{a_i})} \tag{A.6}$$

If $a_i$ is even we can write $t_1 = \frac{a_i}{2}r + s$ with $r \in \{0,1\}, s \in \{0,\cdots,\frac{a_i}{2}-1\}$:

$$\begin{aligned}\prod_{t_1=1}^{a_i-1}\frac{1}{(1-x\omega_{a_i}^{2t_1})} &= \prod_{r\in\{0,1\}}\prod_{s=0}^{\frac{a_i}{2}-1}\frac{1}{(1-x\omega_{a_i}^{a_ir+2s})} \\ &= \prod_{r\in\{0,1\}}\prod_{s=0}^{\frac{a_i}{2}-1}\frac{1}{(1-x\omega_{\frac{a_i}{2}}^{s})} \\ &= \prod_{r\in\{0,1\}}\frac{1}{(1-x^{\frac{a_i}{2}})} = \frac{1}{(1-x^{\frac{a_i}{2}})^2}\end{aligned} \tag{A.7}$$

The odd and even cases can be collected by expressing the result in terms of $\mathrm{GCD}(2, a_i) \equiv G(2, a_i)$ which is 2 if $a_i$ is even, and 1 if $a_i$ is odd. Hence

$$\prod_{t_1=0}^{a_i-1} \frac{1}{(1 - x\omega_{a_i}^{2t_1})} = \frac{1}{(1 - x^{\frac{a_i}{G(2,a_i)}})^{G(2,a_i)}} \tag{A.8}$$

We conclude that

$$\prod_{t_1 \neq t_2 \in \{0,\cdots,a_i-1\}} (1 - x\omega_{a_i}^{t_1+t_2}) = \frac{(1 - x^{a_i})^{a_i}}{(1 - x^{\frac{a_i}{G(2,a_i)}})^{G(2,a_i)}} \tag{A.9}$$

and

$$\hat{F}_1(a_i) = \frac{(1 - x^{a_i})^{\frac{a_i}{2}}}{(1 - x^{\frac{a_i}{G(2,a_i)}})^{\frac{G(2,a_i)}{2}}} \tag{A.10}$$

$$F_1(a_i) = \prod_{t_1 < t_2 \in \{0,\cdots,a_i-1\}} (1 - x\omega_{a_i}^{t_1+t_2})^{p_i^2} = \frac{(1 - x^{a_i})^{\frac{a_i p_i^2}{2}}}{(1 - x^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i^2 G(2,a_i)}{2}}} \tag{A.11}$$

The factors $F_2(a_i), F_3(a_i)$ can be computed using (A.8). They are

$$F_2(a_i) = \prod_{t=1}^{a_i-1} (1 - x\omega_{a_i}^{2t})^{p_i(p_i-1)/2} = (1 - x^{\frac{a_i}{G(2,a_i)}})^{G(2,a_i)\ p_i(p_i-1)/2} \tag{A.12}$$

and

$$F_3(a_i) = \prod_{t=1}^{a_i-1} (1 - x\omega_{a_i}^{2t})^{p_i} = (1 - x^{\frac{a_i}{G(2,a_i)}})^{G(2,a_i)p_i} \tag{A.13}$$

Collecting the factors

$$F_1(a_i)F_2(a_i)F_3(a_i) = (1 - x^{a_i})^{\frac{a_i p_i^2}{2}} (1 - x^{\frac{a_i}{G(2,a_i)}})^{\frac{p_i}{2} G(2,a_i)} \tag{A.14}$$

Now consider the factor that depends on $a_i, a_j$, which is simplified using (4.11) and (4.15)

$$\begin{aligned} F_4(a_i, a_j) &= \prod_{t_1,t_2} (1 - x\omega_{a_i}^{t_1}\omega_{a_j}^{t_2})^{p_i p_j} \\ &= \prod_{t_1} (1 - x^{a_2}\omega_{a_1}^{a_2 t_1})^{p_i p_j} \\ &= (1 - x^{L(a_i,a_j)})^{\frac{a_i a_j p_i p_j}{L(a_i,a_j)}} \\ &= (1 - x^{L(a_i,a_j)})^{G(a_i,a_j)p_i p_j} \end{aligned} \tag{A.15}$$

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