Stability Analysis of Dynamic Nonlinear Systems by means of Lyapunov Matrix-Valued Functions

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Declaration

I declare that this research report is my own, unaided work. It is being submitted for the Degree of Master of Science in Engineering in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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signed this ___________________ day of ______________________2009

Abstract

The method of Lyapunov matrix-valued functions is critically examined for its capability, applicability and overall functionality in the adequate construction and development of an appropriate Lyapunov function for the stability analysis of dynamic nonlinear systems. This method provides an analytical methodology of Lyapunov function construction by effectively exploiting indispensible information relating to the internal dynamics of the nonlinear system, gained by means of hierarchical nonlinear system decomposition. While relatively computationally intensive in its application, when compared to traditional scalar Lyapunov function construction techniques, as well as to vector Lyapunov function approaches, in terms of practical applicability and the successful acquirement of a suitable Lyapunov function, it is found that the method of Lyapunov matrix-valued functions outperforms its predecessors for both linear and nonlinear dynamic systems. Furthermore, in order to present a comprehensive investigation and analysis of the researched methodology, a linear system simplification is proposed, and two variations on the Lyapunov matrix-valued function method are also put forward. A critical analysis of the investigated technique ensues, whereby both its virtues and weaknesses in terms of practical applicability and relative improvement on pre-existing techniques are highlighted. Finally, the stability of a practical, real-world case study is analysed, namely, the Buckling Beam system, where it is found that the combination of the Lyapunov matrix-valued function approach with Aizerman's method proves to be extremely successful in the construction of an appropriate Lyapunov function. The research report is concluded with a series of recommendations for prospective research areas in both Lyapunov matrix-valued function theory development, as well as its extension to practical, real-world applications.

Acknowledgements

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To Moo, Gran and Abu

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List of Symbols

R	the set of all real numbers
$R_+ = [0, +\infty) \subset R$	the set of all non-negative numbers
R^k	the kth dimensional real vector space
$R \times R^n$	the Cartesian product of R and R^n
$\|x\|$	the Euclidian norm of x in R^n
$\chi(t; t_0, x_0)$	a motion of a system at $t \in R$ iff $x(t_0) = x_0$, $\chi(t_0; t_0, x_0) \equiv x_0$
$B_\epsilon = \{x \in R^n : \|x\| < \epsilon\}$	open ball with centre at the origin and radius $\epsilon > 0$
$\mathcal{N}$	time invariant neighbourhood of the origin of R^n
$f : R \times \mathcal{N} \to R^n$	a vector function mapping $R \times \mathcal{N}$ onto R^n
$C(\tau \times \mathcal{N})$	the family of all functions continuous on $\tau \times \mathcal{N}$
$C^{(i,j)}(\tau \times \mathcal{N})$	the family of all functions i-times differentiable on τ and j-times differentiable on $\mathcal{N}$
$D^+v(t,x)$ $\left(D^-v(t,x)\right)$	the upper right (left) Dini derivative of v along $\chi(t; t_0, x_0)$ at (t,x)
$D_+v(t,x)$ $\left(D_-v(t,x)\right)$	the lower right (left) Dini derivative of v along $\chi(t; t_0, x_0)$ at (t,x)
$D^*v(t,x)$	denotes that both $D^+v(t,x)$ and $D_+v(t,x)$ can be used
$Dv(t,x)$	the Eulerian derivative of v along $\chi(t; t_0, x_0)$ at (t,x)
$\lambda_i(\cdot)$	the ith eigenvalue of a matrix $(\cdot)$
$\lambda_M(\cdot)$	the maximal eigenvalue of a matrix $(\cdot)$
$\lambda_m(\cdot)$	the minimal eigenvalue of a matrix $(\cdot)$

Chapter 1

Introduction

"What is it indeed that gives us the feeling of elegance in a solution, in a demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details."

Poincaré, 1902

In 1892, in his doctoral thesis entitled: "*A general task about the stability of motion"*, Russian born academician, Aleksandr Mikhailovich Lyapunov founded modern stability theory and provided a powerful technique with which to obtain general system stability [1]. More than 100 years later, this technique, which enables one to determine the stability of the equilibrium points or states of any dynamic nonlinear system, still proves to be extremely robust in the qualitative analysis of nonlinear system stability. Lyapunov effectively developed two nonlinear system stability analysis techniques. His first method, entitled, *Lyapunov's First Method* [2] required the linearization of the particular nonlinear system around an operating point. While this method does have its benefits, it ultimately serves as a *local* stability analysis technique, providing qualitative information about the stability *only around the operating point in question.* While clearly a desired property of the system, it provides no information on the *global* stability of the nonlinear system as, for the general nonlinear system, instability of one or more of its equilibrium states does not infer global instability, discussed further in Section 2.2.

Lyapunov's second method, entitled, *Lyapunov's Second or Direct Method*, proves to be a more general and powerful approach, enabling the potential global stability of the general nonlinear system to be investigated and therefore does not suffer from the drawbacks incurred by Lyapunov's first method. In essence, this approach calls for the construction of a *Lyapunov function*; a concept inspired by the intuitive knowledge that if the energy near an equilibrium state of a physical system is always decreasing, it implies the equilibrium is stable [3]. A Lyapunov function is simply a manifestation of this energy concept, whereby the stability analysis of a particular nonlinear system's equilibrium state is reduced to the investigation of the properties of its corresponding Lyapunov function. A clear advantage of this method is that it does not demand an analytic or numerical solution, consequently possessing great power in applications [4]. The problem however, is that the required Lyapunov function, as well as its time derivative, must satisfy rigid constraints, and, to date, there exist no formulated methodologies of obtaining such functions. Furthermore, the inability to obtain a particular Lyapunov function for a given nonlinear system, does not infer the equilibrium point or, potentially, the *global* nonlinear system under investigation is unstable, thereby making this technique *sufficient*, but not *necessary*, for stability. To remedy this problem, various different approaches and generalisations have been suggested in the literature. These methodologies include, among many others, the development of appropriate candidate Lyapunov functions by means of *scalar, vector and matrix functions* [4,5,6,7,8,9,10], *evolutionary programming optimisation approaches* [11,12], *sum of squares algorithms* [13,14,15.], *variable gradient method* [16], *Zubov's method*[16,17].

In the late 1970's, renowned Ukrainian physicist and mathematician, Anatoly Andreevich Martynyuk, developed what is known as the *matrix-valued Lyapunov function method,* a technique by which an appropriate *scalar* or *vector* Lyapunov function is developed using the dynamic properties of the system's states. The method of matrix-valued Lyapunov functions, as well as the method of vector Lyapunov functions (discussed in Section 3.6), attempts to relax the otherwise stringent constraints imposed by Lyapunov's direct method, making it a more adaptable and methodical approach for nonlinear system stability analysis. By taking into account the dynamic interconnections of the constituent subsystems of the whole nonlinear system, it serves to provide a more intuitive approach to the Lyapunov function construction. This method reduces the original Lyapunov theorem and constraints imposed on the Lyapunov function to the property of having a fixed sign of special matrices [9,10].

In summary, the development of the Lyapunov matrix function method, presents the discovery of a two-index system of functions of suitable structure for construction of appropriate scalar or vector Lyapunov functions. As a direct result of this technique, Martynyuk and his students developed new efficient stability conditions for the analysis of a number of broad classes of systems of equations, namely [18],

- systems with lumped parameters
- singularly perturbed systems including Lur'e-Postnikov systems

- systems with random parameters including singularly perturbed stochastic systems
- impulsive systems
- large scale discrete systems
- large-scale power systems modelled by ODE's
- hybrid systems
- systems with delay
- systems in Banach and metric spaces
- systems modelling population dynamics

The primary aim of the research report is to introduce and critically analyse the method of Lyapunov matrix-valued functions for the stability analysis of dynamic, nonlinear systems. In doing so, its virtues and flaws are highlighted and contextualised with respect to existing Lyapunov function construction frameworks. In order to achieve this objective, Chapter 2 familiarises the reader with the necessary nonlinear system terminology and the definitions required to fully appreciate the method of Lyapunov matrix-valued functions. From there, Chapter 3 introduces the concept of *Lyapunov Stability* and *Lyapunov Stability Theory* in the context of continuous, dynamic nonlinear systems. Lyapunov's two fundamental stability analysis theories are formally presented and their various advantages and disadvantages discussed.

Chetaev's Theorem for Instability and *LaSalle's Invariance Principle* are also included to supplement the existing Lyapunov stability theory. Vector Lyapunov functions are then introduced as a precursor to the matrix-valued Lyapunov functions, as certain principles implemented in the vector Lyapunov approach are used in the matrix Lyapunov function theory. Chapter 4 initiates the reader with the main body of work, namely the introduction and implementation of Lyapunov matrix-valued functions in the stability analysis of dynamic nonlinear systems. Here, the primary method of the matrix function's construction, namely, the *hierarchical Lyapunov matrix-valued function approach,* is introduced and implemented on various nonlinear systems. A linear system simplification of these methods is then introduced and accompanied with corresponding linear system examples for analysis. Following this simplification, two variations on the hierarchical method are presented and discussed. Finally, in Chapter 5, an experimental case study is presented, namely the Buckling Beam system, where the applicability and practicality of the Lyapunov matrix-valued function method to a real world system is examined.

Chapter 2

Preliminaries

"Today the network of relationships linking the human race to itself and to the rest of the biosphere is so complex that all aspects affect all others to an extraordinary degree. Someone should be studying the whole system, however crudely that has to be done, because no gluing together of partial studies of a complex nonlinear system can give a good idea of the behaviour of the whole."

Gell-Mann, 1998

2.1 Introduction

Nonlinearity, as defined in [19], is the deviation of any functional relationship from direct proportionality, typically characterised by a system which does not subscribe to the principle of superposition. Simply put, the output of a nonlinear system does not linearly depend on its inputs, or multiples thereof, often making it difficult to predict and frequently resulting in a rich variety of complex behaviour where a wide range of output values result from a relatively small input set. There exist two broadly distinct streams in the analysis and solution of nonlinear systems of differential equations. The first being the attempt to obtain a closed-form, definite solution of the nonlinear system. The problem with this technique however, is that an analytic solution is rarely possible whereas numerical methods only provide an approximate solution and could potentially result in computationally expensive and often unrealisable processes depending on the nonlinear system under investigation. These techniques are traditionally classed as *quantitative* methods. The second technique, initiated by Poincaré in around 1880 [3], attempts to obtain global information about the system as a whole, trying to estimate complicated and intricate information concerning the general behaviour of the solution trajectories around the system's operating points.

These techniques, termed *qualitative* analyses, prove to be powerful analysis tools as they extract information directly from the nonlinear system's differential equations, thereby avoiding the need to obtain an analytic solution. Since Poincaré, much effort has been dedicated to the qualitative analysis of nonlinear systems in and around their various regions of operation, owing to its ease of application. These techniques include, but are not limited to, *phase plane analysis*; where the system's states are orthogonally plotted against each other, often in a two or three dimension Euclidian space by means of the *method of isoclines, Poincaré maps;* a dimensional reduction of the nonlinear system's governing differential equations, effectively determining how the system trajectories intersect a cross section of the phase space and *bifurcation diagrams*; a method by which a system's qualitative behaviour varies as a function of a given parameter. Section 2.2 familiarises the reader with some of these rich nonlinear system phenomena as well as the corresponding, and most appropriate, qualitative analysis techniques commonly implemented to study these systems. The concepts of an *autonomous system* and an *equilibrium point*, as well as its various classifications, are introduced in Section 2.3.1. Finally, a number of mathematical preliminaries required in the development of subsequent chapters are presented from Section 2.3.2 to Section 2.3.4.

2.2 Nonlinear System Dynamics

Nonlinear systems, unlike their linear system counterparts, are known to exhibit rich dynamics owing to their unique and complex structures. These phenomena, traditionally considered in a qualitative context, abound in numerous, naturally occurring as well as man-made systems from socio-economic and population-based models, to electrical, mechanical, chemical, aeronautical and biological systems [20]. Clearly, a study of these various behaviours (in either a quantitative *or* qualitative setting) is essential for a comprehensive understanding of the nonlinear system as a whole. All the states and their time derivatives introduced throughout the remainder of this research report and analysed by means of their dynamic differential state equations are all functions of time, even if not explicitly stated. Examples of such nonlinear phenomena include:

Multiple equilibria or multiple operating points, where there exist countless examples of dynamic nonlinear systems with two or more equilibrium or operating points. These systems include, but are certainly not limited to, chemical reaction based systems, power flow differential equations, modelling the flow of real and reactive power, and digital circuits for binary logic. An explanatory example of the lightly damped nonlinear pendulum is included for the purpose of illustration, see Figure 2.1, where mathematically there exist an infinite amount of equilibrium points. However, in reality, there exist only two equilibrium points, corresponding to the stable downward position of the pendulum arm, $\theta = 0$, and the unstable, vertically upward position of the pendulum arm, $\theta = \pi$.

From Newton's laws of motion, the nonlinear differential equations modelling the system are

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -k sin\theta - c\omega \end{aligned} \qquad (2.1)$$

where θ represents the angle made between the pendulum arm and the vertical, increasing in an anti-clockwise direction, ω represents the angular velocity of the pendulum arm, increasing in an anti-clockwise direction, k is a constant given by, $k = \frac{g}{\ell}$, where g is the acceleration due to gravity, and c represents the damping coefficient. ℓ and m represent the length of the rigid rod and the mass of the pendulum ball respectively.

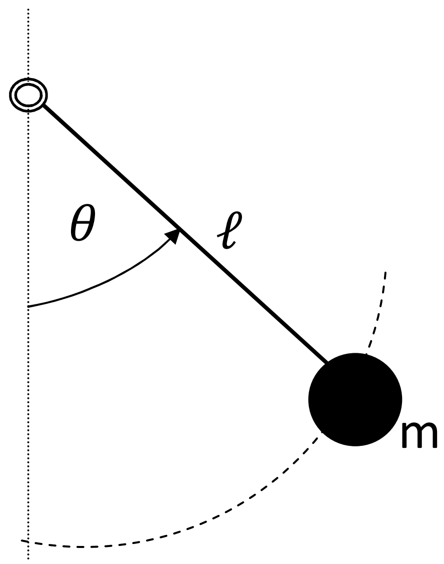


Figure 2.1. Graphical representation of nonlinear pendulum

This model is beautiful in its simplicity while still exhibiting rich dynamic behaviour and will be revisited a number of times throughout the course of this research report. For the lightly damped case, the condition $k > c$ must be satisfied. Therefore, the arbitrary assignment $k = 2$ and $c = 1$ is chosen.

The corresponding parameterised phase plane portrait of the system (2.1) is illustrated in Figure 2.2. Clearly, the system has an infinite amount of alternating *stable* and *unstable* equilibrium points given by $(\theta, \omega) = (n\pi, 0)$, where $n = 1,2, \ldots$ The stable equilibrium points are seen in Figure 2.2 for $n = 2k$ and the unstable equilibrium points for $n = 2k + 1$ where $k \in \mathcal{Z}$.

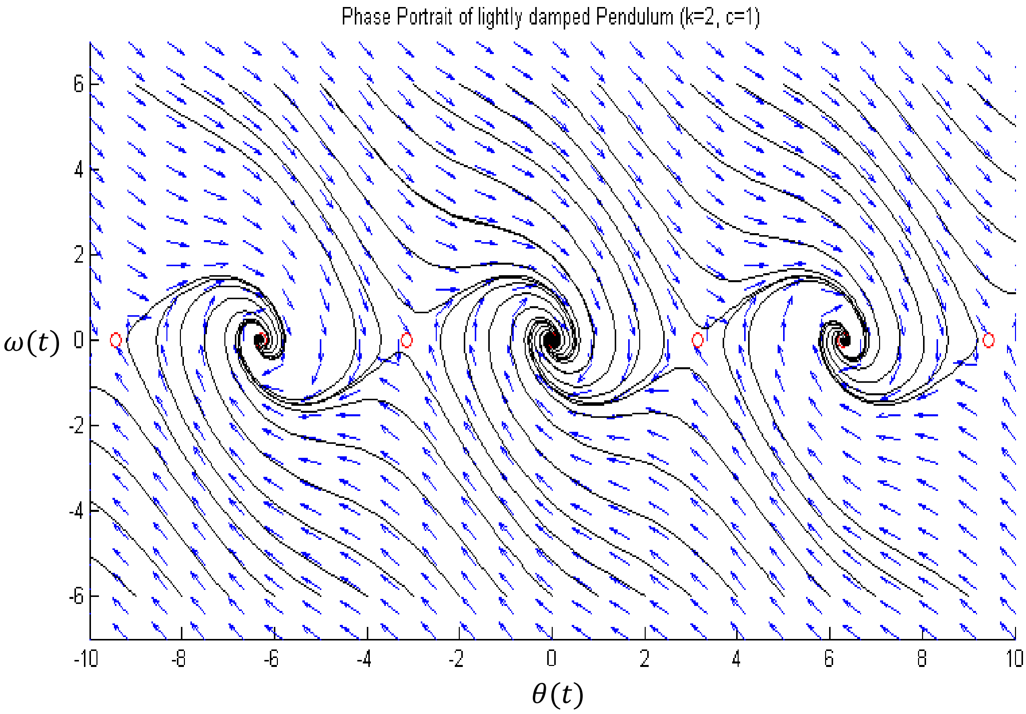


Figure 2.2. Phase portrait of lightly damped pendulum (k=2, c=1). Multiple initial conditions and their corresponding solution trajectories reveal the system's multiple equilibrium points.

In contrast, consider the differential equation model for the *linear* system

$$\frac{dx(t)}{dt} = Ax(t) \tag{2.2}$$

where $x \in R^n$ and $A \in R^{n\times n}$, a constant matrix. The point $x = 0$ is an equilibrium point of the system, that is, if the initial state at time $t = 0$ of the differential equation (2.2) is 0, i.e., $x(0) = 0$, then the state of the equation remains 0 for all t, i.e., $x(t) \equiv 0$. Under the assumption that A is a non-singular matrix, $x = 0$ is the only equilibrium point of the linear system. Clearly, the linear model represented by (2.2), does not permit the existence of multiple equilibrium points, making this a purely nonlinear phenomenon.

Limit cycles, or periodic variations of state variables, are found in a number of nonlinear systems. For such systems, there exists a closed trajectory in phase space to which nearby trajectories approach as either $t \to \infty$ or $t \to -\infty$. The former is considered a *stable* or *attractive* limit cycle, whereas the latter an *unstable* limit cycle. By way of example, consider the *unforced Van der Pol oscillator*, modelled by the nonlinear differential equations

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = -\mu(x_1^2 - 1)x_2 - x_1 \tag{2.3}$$

where μ is a constant and the system exhibits a stable limit cycle for $\mu > 0$. Figure 2.3 shows the phase plane diagram of system (2.3), where the stable limit cycle can clearly be seen.

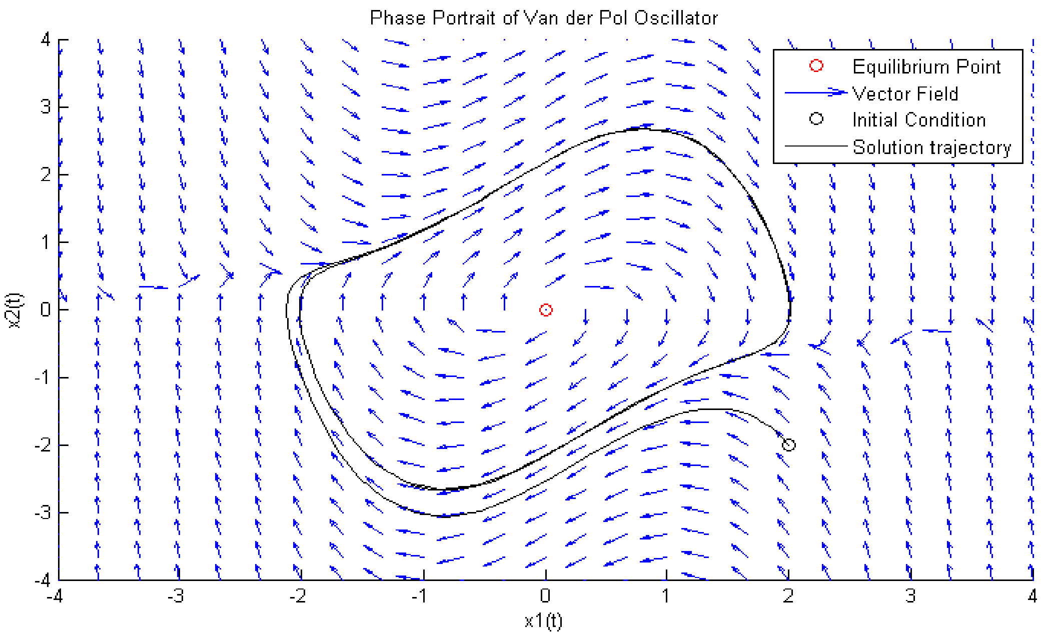


Figure 2.3. Phase portrait of Van der Pol oscillator showing the equilibrium point (0, 0), vector field, initial condition (2, -2) and the resulting solution trajectory

Stable limit cycles imply self sustained oscillations. Any small perturbation from the closed trajectory, as a result of a small parameter change, would ultimately result in the system returning back to the stable limit cycle. In contrast, linear systems, represented by (2.2), could potentially fluctuate between stable and unstable behaviour as a function of a particular parameter. The equilibrium point of such a system is termed a *centre* where the system's eigenvalues are located on the $j\omega$ axis. This system is considered *critically stable* as small perturbations would most likely move the eigenvalues off the $j\omega$ axis, completely changing the qualitative behaviour around the equilibrium point from critically stable, either to asymptotically stable (eigenvalues to the *left* of the $j\omega$ axis) or to unstable (eigenvalues to the *right* of the $j\omega$ axis). Figure 2.4 provides a graphical explanation of this behaviour for a simple 2×2 linear system, where the resulting parabola is the locus of points $\big(trace(A), det(A)\big)$ for which the discriminant of the characteristic equation,

$$\lambda^2 - trace(A)\lambda + determinant(A)$$

is zero, i.e., the equation of the parabola is $det(A) = \frac{[trace(A)]^2}{4}$. Here, the *determinant of A* is plotted against the *trace of A*.

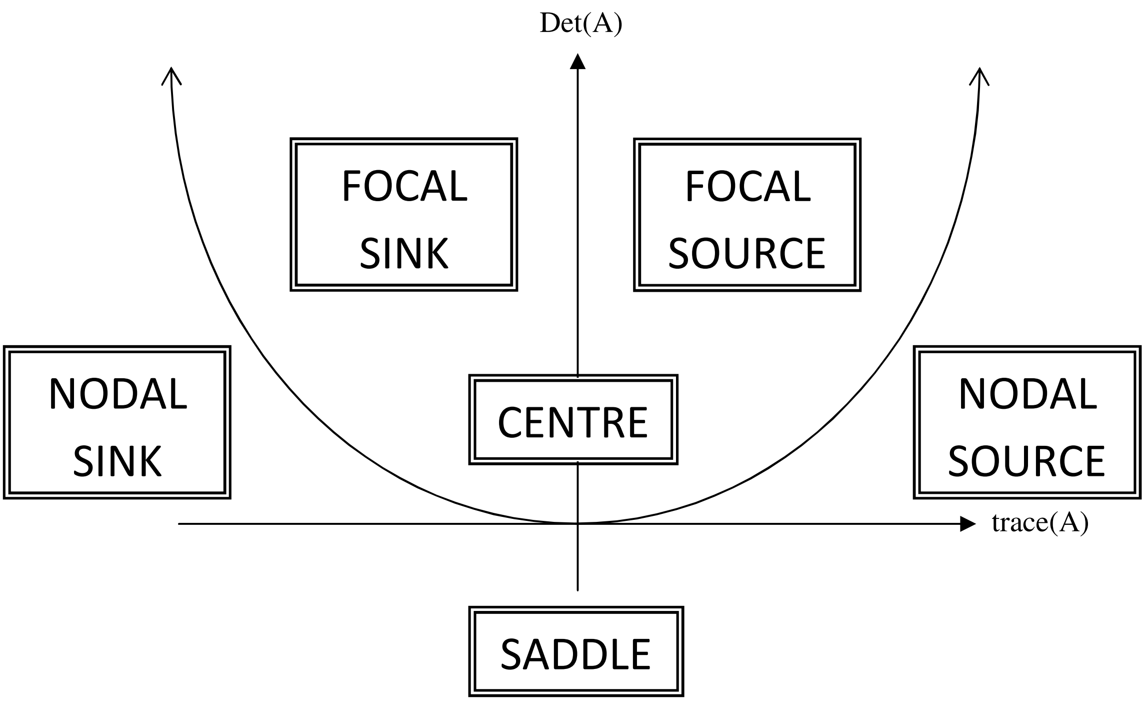


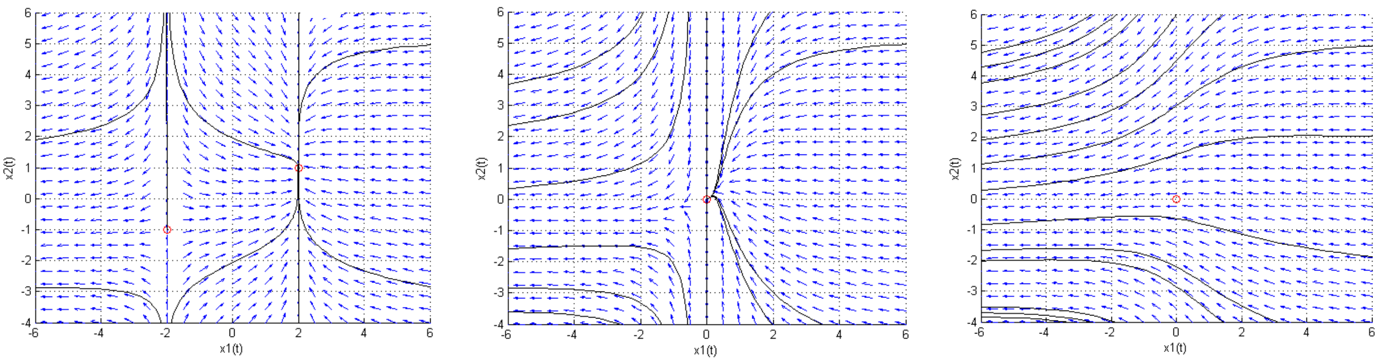
Figure 2.4. Graphical representation of the relationship between the trace and the determinant of a 2×2 linear system.

From Figure 2.4 one can see that equilibrium points which exhibit centre dynamic behaviour are located on the $trace(A) = 0$ axis, $Det(A) > 0$. Clearly, any small perturbation would result in the equilibrium point deviating from the $trace(A) = 0$ line and either exhibiting stable focal, or unstable focal behaviour, thereby emphasizing the *inability* of a linear system to exhibit limit cycle behaviour.

Bifurcations; a dynamic change in the qualitative behaviour of the nonlinear system, under parametric variations of the model. This change could be in the number of equilibrium points, the number of limit cycles, or the stability of one or many of these features. Consider the second-order, continuous nonlinear system

$$\frac{dx_1}{dt} = \beta - x_1^2$$
$$\frac{dx_2}{dt} = -x_2 + 0.5x_1 \tag{2.4}$$

which depends on the parameter β. As seen in Figure 2.5, for varying values of β the system exhibits dramatic qualitative changes in both the number of equilibrium points, as well as their respective stabilities'. When $\beta > 0$, there exist two equilibrium points, $(\sqrt{\beta}, \frac{\sqrt{\beta}}{2})$ and $(-\sqrt{\beta}, -\frac{\sqrt{\beta}}{2})$, the former being a stable node whereas the latter being a saddle, seen in (a) for $\beta = 4$. As β decreases, the two equilibrium points approach each other, collide at $\beta = 0$, seen in (b) and disappear for $\beta < 0$, seen in (c). Such a change in qualitative behaviour is called a *bifurcation*, where the β parameter is called the *bifurcation parameter*.



(a) (b) (c)

Figure 2.5. Phase portrait of Saddle-Node Bifurcation of system (2.4) for (a) $\beta > 0$, (b) $\beta = 0$, and (c) $\beta < 0$

Another graphical analysis tool is the *bifurcation diagram* where the relative norm, or magnitude of the equilibrium point(s) is plotted against the bifurcation parameter which, in this case, is β. Here, a stable node, focus, or limit cycle is represented by a solid line and an unstable node, focus, limit cycle or saddle is represented by a dashed line [17]. The bifurcation diagram of system (2.4) is plotted in Figure 2.6.

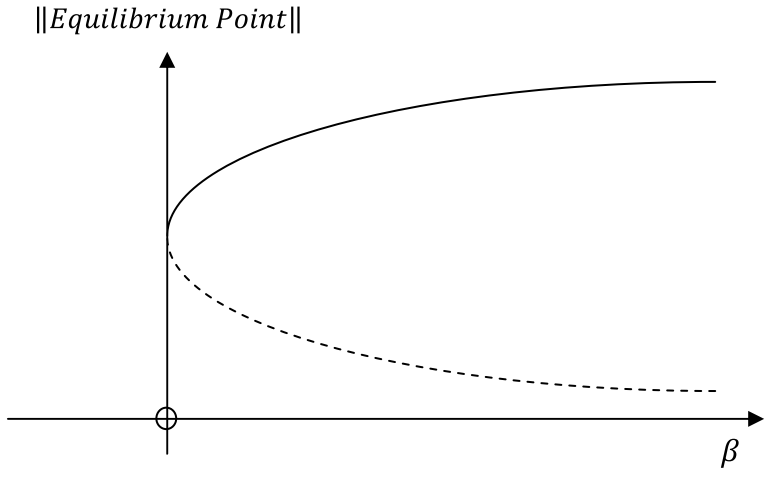


Figure 2.6. Bifurcation Diagram of Saddle-Node Bifurcation

The bifurcation exhibited by system (2.4) is called a *saddle-node* bifurcation as it results from the collision of a saddle and a node [17]. Many other types of bifurcations exist such as *transcritical bifurcations, subcritical pitchfork bifurcations* and *Hopf bifurcations.* While the discussion of these bifurcations is beyond the scope of this research report, the *Hopf bifurcation* is exhibited in the Lorenz system, discussed in the subsequent paragraph. Here, a *stable focus* for a certain range of parameter values, changes into an *unstable focus* once the bifurcation parameter crosses a certain critical point. For the Lorenz system this point occurs when $\rho \cong 24.74$, where at this point the system starts to exhibit *chaotic behaviour.*

Chaos, a phenomenon first documented by renowned meteorologist *Edward Lorenz* in the 1960's, describes the complex nonlinear dynamic behaviour exhibited by a system with extreme sensitivity to initial conditions. Chaotic behaviour is said to exist between the realms of deterministic periodicity and randomness, where characteristic unstable, aperiodic behaviour is exhibited in deterministic, nonlinear, dynamical systems [21]. A typical example of a nonlinear system exhibiting chaotic dynamics is the *Lorenz* system, described by the following differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= \sigma(x_2 - x_1) \\ \frac{dx_2}{dt} &= x_1(\rho - x_3) - x_2 \\ \frac{dx_3}{dt} &= x_1 x_2 - \beta x_3 \end{aligned} \tag{2.5}$$

where σ is known as the *Prandtl number* and ρ the *Rayleigh number*. Here, for values $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$ the system (2.5) exhibits a *strange attractor*, a complex, globally-bounded pattern traced out by the state variable trajectories in the phase space, indicative of chaotic behaviour. This pattern is seen in Figure 2.7. where the state x_3 is plotted against the state x_1.

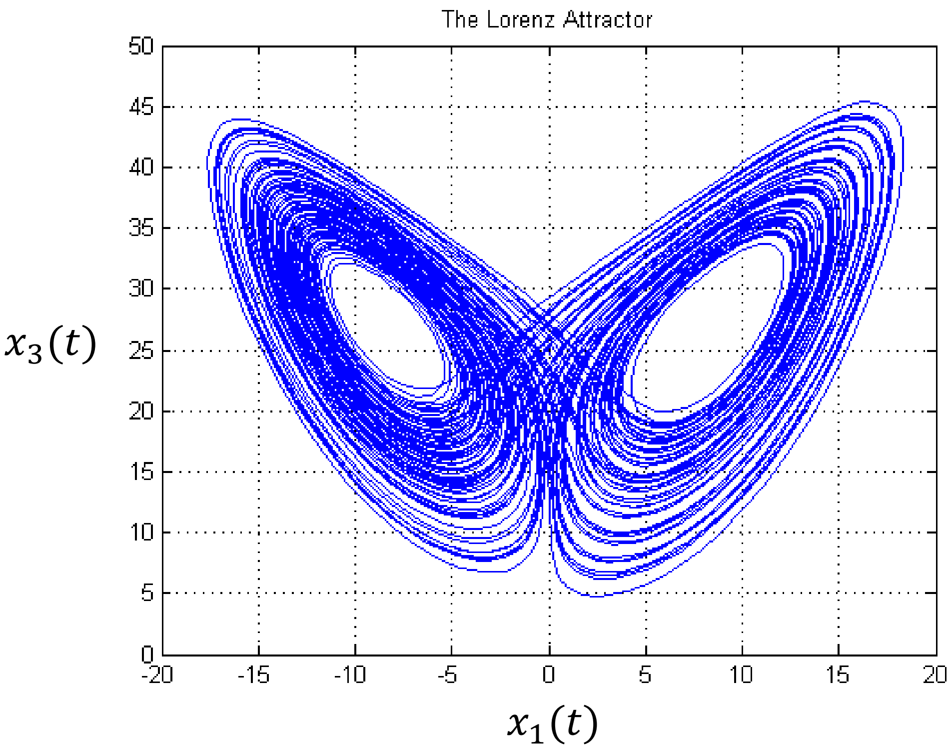


Figure 2.7. Strange attractor of the Lorenz system for $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$

Since the solution to any linear system is simply the sum of exponentials, with exponents determined by the eigenvalues of the A matrix from (2.2), which either decay to the equilibrium point, or diverge from it (assuming the eigenvalues are not on the $j\omega$ axis), a linear system's qualitative behaviour around its equilibrium point remains independent of its initial conditions, thereby enforcing the claim that a linear system's dynamics is simply not rich enough to capture such intricately complex behaviour exhibited by a chaotic system. A common property of a chaotic system, and in fact of nonlinear systems in general, is local instability around its equilibrium point(s) does not infer global instability of the entire system, a property not shared with linear systems in general. This property is clearly exhibited by the Lorenz system as a set of simple calculations would show that the origin is locally unstable, however the system as a whole exhibits global stability.

Finite escape time, the nonlinear phenomenon where the state of a nonlinear system can potentially tend to *infinity* in a *finite* time, as opposed to an unstable linear system, where its trajectories tend to infinity as time approaches infinity. Consider the scalar, nonlinear system

$$\frac{dx}{dt} = -x^2, \quad x(0) = -1 \tag{2.6}$$

having the unique solution

$$x(t) = \frac{1}{t-1}$$

which exists for $t \in [0\,,\infty) - \{1\}$. As $t \to 1,\ x(t) \to \infty$. The system (2.6) is said to have a finite escape time at $t = 1$. There exist various other nonlinear phenomena such as *jump resonance* and *quasiperiodicity* [17], however these phenomena will not be discussed further, as they are deemed beyond the scope of this research report.

2.3 Mathematical Preliminaries

The following mathematical preliminary concepts are presented in order to develop the definitions and theorems introduced in subsequent chapters.

2.3.1 General Nonlinear System Concepts

Consider the following system, where $\dot{x}$ denotes the time derivative of x,

$$\dot{x} = f(t, x) \tag{2.7}$$

Here, $x \in R^n$ and f is some nonlinear function of the states, $f \in C(R_+ \times R^n, R)$.

DEFINITION 2.1 AUTONOMOUS SYSTEM. The system (2.7) is said to be *autonomous* if $f(t, x)$ is not explicitly dependent on time t.

DEFINITION 2.2 EQUILIBRIUM POINT AT $\boldsymbol{t_0}$. $x^* \in R^n$ is said to be an *equilibrium point* of the system (2.7) at time t_0 iff

$$f(t, x^*) = 0 \quad \forall \quad t \geq t_0$$

Note that if x^* is an equilibrium point of system (2.7) at t_0 and $x(t_0) = x^*$ then it implies that $x(t) \equiv x^* \; \forall \; t \geq t_0$. Conversely, if $f(t, x^*) = 0$ for all t, it then implies that x^* is an *equilibrium point* of system (2.7).

If the system is *autonomous,* finding the equilibrium point(s) simply corresponds to solving the nonlinear algebraic equation

$$f(x) = 0 \tag{2.8}$$

Equation (2.8) could have multiple solutions, no solutions or an infinite amount of solutions depending on the nature of the nonlinear system.

A very important concept in nonlinear system investigation is that of *continuity*. Essentially, a function is considered a *continuous function* if small changes in the input result in small changes in the output. In general, some function f is *continuous at some point* α in its domain if and only if the following condition holds.

DEFINITION 2.3 CONTINUOUS FUNCTION. The function $f(t,x) \in C(R_+ \times \mathcal{N}, R)$ $\mathcal{N} \subset R^n$ is considered a *continuous function at point* α, $\alpha \in \mathcal{N}$, iff

$$\lim_{x \to \alpha} f(t,x) = f(t,\alpha) \tag{2.9}$$

holds.

A function is considered *continuous* if it is *continuous* at every point in its domain. In general, a function is considered *continuous* on some given subset of its domain if it is *continuous* at every point of that subset. *Rudolf Lipschitz* a German born mathematician, developed a stronger criteria for continuity whereby his *Lipschitz continuous function* is limited by how fast it can change. Intuitively, the slope joining any two points on the graph of this function will never be steeper than a certain number called the *Lipschitz constant* of the function. This idea is mathematically formulated as follows.

DEFINITION 2.4 LIPSCHITZ CONTINUOUS FUNCTION. A function $f(t,x)$ is considered a *locally Lipschitz continuous function* with respect to x if for each point in $R_+ \times \mathcal{N}$ there exists a neighbourhood $R_+ \times \mathcal{S}$ and a positive constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y| \tag{2.10}$$

for any $(t,x) \in R_+ \times \mathcal{N}$, $(t,y) \in R_+ \times \mathcal{S}$

Here, L is called the *Lipschitz constant* of the function $f(t,x)$. A definition for *globally Lipschitz continuous functions* requires the condition (2.10) to hold for $x, y \in R$.

2.3.2 Dini and Eulerian Derivative

The *Dini derivative,* introduced by Italian mathematician and politician *Ulisse Dini*, presents a technique of calculating the derivative of *non-differentiable*, commonly referred to as *non-smooth* continuous functions. This definition is important as the Lyapunov functions developed using vector and matrix-valued functions for the analysis of the general nonlinear system are not necessarily *smooth* or *infinitely differentiable* and therefore require a more generalised definition of the derivative to apply the developed theory.

DEFINITION 2.5 DINI AND EULERIAN DERIVATIVE. Let v be a locally Lipschitz continuous scalar, vector or matrix function defined on an open set in $R_+ \times \mathcal{N}$, $v(t,x) \in C(R_+ \times \mathcal{N}, R^{s \times s})$ and let solutions χ of the system (2.7) exist and be defined on $R_+ \times \mathcal{N}$. Then, for all $(t,x) \in R_+ \times \mathcal{N}$,

(i) $D^+v(t,x) = \lim_{\theta \to 0^+} \sup \left\{\frac{v[t+\theta,\chi(t+\theta,x)]-v(t,x)}{\theta}\right\}$ is the *upper right Dini derivative* of v along the motion of χ

(ii) $D_+v(t,x) = \lim_{\theta \to 0^+} \inf \left\{\frac{v[t+\theta,\chi(t+\theta,x)]-v(t,x)}{\theta}\right\}$ is the *lower right Dini derivative* of v along the motion of χ

(iii) $D^-v(t,x) = \lim_{\theta \to 0^-} \sup \left\{\frac{v[t+\theta,\chi(t+\theta,x)]-v(t,x)}{\theta}\right\}$ is the *upper left Dini derivative* of v along the motion of χ

(iv) $D_-v(t,x) = \lim_{\theta \to 0^-} \inf \left\{\frac{v[t+\theta,\chi(t+\theta,x)]-v(t,x)}{\theta}\right\}$ is the *lower left Dini derivative* of v along the motion of χ

(v) The function v has *Eulerian derivative,* $\dot{v}$, $\dot{v}(t,x) = \frac{d}{dt}v(t,x)$ at (t,x) along the motion of χ iff

$$D^+v(t,x) = D_+v(t,x) = D^-v(t,x) = D_-v(t,x)$$

denoted by $\dot{v}(t,x) = Dv(t,x)$

The implementation of the Dini derivative in conjunction with comparison functions (introduced in Section 2.3.4) enables one to determine $D^+v(t,x)$ without having to explicitly solve for the system's motions. Definition 2.5 is derived from [9]

Take, for instance, the scalar function,

$$f(x) = |x| \tag{2.11}$$

which has a derivative defined for every point $x \in R$ except $x = 0$. The derivative of function (2.11) with respect to x is given as

$$\frac{d}{dx}f(x) = \begin{cases} -1 & ; \quad x < 0 \\ undefined & ; \quad x = 0 \\ 1 & ; \quad x > 0 \end{cases} \tag{2.12}$$

Calculating the upper right Dini Derivative of function (2.11) at $x = 0$ yields

$$\begin{aligned} D^+f(x)|_{x=0} &= \lim_{\theta \to 0^+} sup \left\{ \frac{|x+\theta| - |x|}{\theta} \right\} \Bigg|_{x=0} \\ &= \lim_{\theta \to 0^+} sup \left\{ \frac{|\theta|}{\theta} \right\} \end{aligned} \tag{2.13}$$

One can clearly deduce that approaching $\theta = 0$ from the right would result in $|\theta| = \theta$, thereby making the upper right Dini derivative of function (2.11) at $x = 0$ equal to 1. This results in the upper Dini derivative of a conventionally *undifferentiable* function (2.11) being defined for all $x \in R$, and given by

$$D^+f(x) = \begin{cases} -1, & x < 0 \\ \;\;1, & x \geq 0 \end{cases} \tag{2.14}$$

This simple, yet descriptive, example illustrates the power and adaptability of the Dini derivative to non-differentiable functions and provides a more generalised approach of finding the derivative of any given function.

2.3.3 Function and Matrix Sign Definiteness

The following definitions present the necessary conditions required to satisfy the property of *sign-definiteness* for both functions and matrices. These properties are of fundamental importance to the adequate development of Lyapunov's matrix-valued function method for the stability analysis of nonlinear systems. While not explicitly stated, these definitions will be consistently used throughout the course of this research report where, for example, should a function or matrix be referred to as *positive definite*, the function or matrix in question would adhere to the strict definitions presented in this section. The conditions required for function sign definiteness (semi-definiteness) are presented first, which are followed by matrix sign definiteness (semi-definiteness) conditions.

Let B_h be an n dimensional hyperball of radius h, centred at the origin, i.e.

$$B_h = \{x \in R^n : \|x\| < h\}$$

DEFINITION 2.6 SIGN SEMI-DEFINITE FUNCTION. A continuous function $v(t,x): R_+ \times B_h \mapsto R$ is called a

(i) *locally positive semi-definite function (l.p.s.d.f)* iff for some time-invariant neighbourhood B_h of $x = 0,\ B_h \subseteq R^n,\ h > 0$

$$v(t,0) = 0 \ \ and \ \ v(t,x) \geq 0 \ \forall \ x \in B_h,\ \ t \geq 0 \tag{2.15}$$

(ii) *locally negative semi-definite function (l.n.s.d.f)* iff for some time invariant neighbourhood B_h of $x = 0,\ B_h \subseteq R^n,\ h > 0$

$$v(t,0) = 0 \ \ and \ \ -v(t,x) \geq 0 \ \forall \ x \in B_h,\ \ t \geq 0 \tag{2.16}$$

(iii) *globally positive semi-definite function (g.p.s.d.f)* iff (i) holds for $B_h = R^n$

(iv) *globally negative semi-definite function (g.n.s.d.f)* iff (ii) holds for $B_h = R^n$

DEFINITION 2.7 SIGN DEFINITE FUNCTION. A continuous function $v(t,x)$: $R_+ \times B_h \mapsto R$ is called a

(i) *locally positive definite function (l.p.d.f)* if there exists some time-invariant neighbourhood B_h of $x = 0$, $B_h \subseteq R^n$, $h > 0$, such that it is both positive semi-definite on B_h and $v(t,x) > 0 \;\forall\; (x \neq 0) \in B_h$

(ii) *locally negative definite function (l.n.d.f)* iff $(-v)$ is positive definite .

(iii) *globally positive definite function (g.p.d.f)* iff (i) holds for $B_h = R^n$

(iv) *globally negative definite function (g.n.d.f)* iff (ii) holds for $B_h = R^n$

Definitions 2.6 and 2.7 are concerned with the sign semi-definiteness and sign definiteness of functions. Definitions 2.9 and 2.10 now state these properties in terms of matrices.

DEFINITION 2.8 SYMMETRIC MATRIX. A $n \times n$ matrix A is a *symmetric matrix* iff the following condition holds

$$A = A^T \tag{2.17}$$

where A^T denotes the transpose of A.

DEFINITION 2.9 SIGN SEMI-DEFINITE MATRIX. A $n \times n$ matrix A is

(i) *positive semi-definite* if the quadratic form, $x^T Ax \geq 0$ for all $x \in R^n$, $x \neq 0$

(ii) *negative semi-definite* if the quadratic form, $x^T Ax \leq 0$ for all $x \in R^n$, $x \neq 0$

where x^T denotes the transpose of vector x.

DEFINITION 2.10 SIGN DEFINITE MATRIX. A $n \times n$ matrix A is

(i) *positive definite* if the quadratic form, $x^T Ax > 0$ for all $x \in R^n$, $x \neq 0$

(ii) *negative definite* if the quadratic form, $x^T Ax < 0$ for all $x \in R^n$, $x \neq 0$

where x^T denotes the transpose of vector x.

Note that if matrix A is symmetric one need only calculate the eigenvalues of A, to determine its sign definiteness. Should *all* the eigenvalues be positive definite, matrix A is *positive definite* whereas should *all* the eigenvalues of matrix A be negative, matrix A is *negative definite.* In the case of matrix A having complex elements, the above property translates into; should matrix A be a *Hermitian matrix*, that is

$$A = A^{\intercal^*} \tag{2.18}$$

where $A^{\intercal^*}$ denotes the *conjugate transpose* of matrix A, matrix A is positive definite if *all* its eigenvalues are positive definite and, conversely, negative definite if all its eigenvalues are negative definite. A practical test for matrix positive definiteness that does not require explicit calculation of the eigenvalues is the *principal minor test*. The kth leading principal minor is the determinant formed by deleting the last $n - k$ rows and columns of the matrix, A. A necessary and sufficient condition that a symmetric $n \times n$ matrix be positive definite is that all n leading principal minors, Δ_k, are positive. An analogous statement for determining matrix sign *semi-definiteness* does not exist, i.e. all n leading principles minors of matrix A being positive semi-definite does not infer matrix A is positive semi-definite. However, as stated by Glandorf [22], the necessary and sufficient condition for a matrix to be positive (negative) semi-definite is that *all possible* principle minors are nonnegative (nonpositive).

Let a kth order principle minor of the matrix $H = (h_{ij}) \in R^{n\times n}$ be denoted by

$$H\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{bmatrix} = \begin{bmatrix} h_{i_1 i_1} & h_{i_1 i_2} & \cdots & h_{i_1 i_k} \\ h_{i_2 i_1} & h_{i_2 i_2} & \cdots & h_{i_2 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_k i_1} & h_{i_k i_2} & \cdots & h_{i_k i_k} \end{bmatrix}$$

where

$$i_j \in \{1,2,\dots,n\}, \qquad i_j < i_{j+1}, \qquad j = 1,2,\dots,k, \qquad k = 1,2,\dots,n$$

The leading principle minor of the kth order of H is

$$H\begin{bmatrix}1 & 2 & \cdots & k\\ 1 & 2 & \cdots & k\end{bmatrix} = \begin{bmatrix}h_{11} & h_{12} & \cdots & h_{1k}\\ h_{21} & h_{22} & \cdots & h_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ h_{k1} & h_{k2} & \cdots & h_{kk}\end{bmatrix}, \qquad k = 1,2,\dots,n$$

The following theorem is well known and provides the necessary and sufficient conditions for both matrix sign semi-definiteness and matrix sign definiteness.

THEOREM 2.1 MATRIX SIGN SEMI-DEFINITENESS [9]. It is necessary and sufficient for the symmetric $n \times n$ matrix H to be:

(i) positive semi-definite if all its principal minors are non-negative, i.e.

$$H\begin{bmatrix}i_1 & i_2 & \cdots & i_k\\ i_1 & i_2 & \cdots & i_k\end{bmatrix} \geq 0, \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \qquad k = 1,2,\dots,n$$

(ii) negative semi-definite if both all its even order principal minors are non-negative and all its odd order principal minors are non-positive, i.e.

$$H\begin{bmatrix}i_1 & i_2 & \cdots & i_k\\ i_1 & i_2 & \cdots & i_k\end{bmatrix} \begin{cases}\geq 0, & k = 2,4,\dots\\ \leq 0, & k = 1,3,\dots\end{cases}$$

(iii) positive definite if all its leading principle minors are positive, i.e.

$$H\begin{bmatrix}1 & 2 & \cdots & k\\ 1 & 2 & \cdots & k\end{bmatrix} > 0, \qquad k = 1,2,\dots,n$$

(iv) negative definite if both its first order leading principal minor is negative and all its leading principal minors are alternatively negative and positive, i.e.

$$(-1)^k H\begin{bmatrix}1 & 2 & \cdots & k\\ 1 & 2 & \cdots & k\end{bmatrix} > 0, \qquad k = 1,2,\dots,n$$

2.3.4 Comparison Functions

The use and development of comparison functions is primarily attributed to Hahn [23]. The implementation however, of these functions in the context of Lyapunov's direct method is attributed to Yoshizawa [24,25].

DEFINITION 2.11 COMPARISON FUNCTIONS [9]. A function φ, $\varphi: R_+ \to R_+$ belongs to

(i) the *class* $\mathcal{K}_{[0,\alpha)}$, $0 < \alpha \leq +\infty$ iff it is defined, continuous and strictly increasing on $[0, \alpha)$ and $\varphi(0) = 0$;

(ii) the *class* $\mathcal{K}$ iff (i) holds for $\alpha = +\infty$, $\mathcal{K} = \mathcal{K}_{[0.+\infty)}$;

(iii) the *class* $\mathcal{KR}$ iff both it belongs to the class $\mathcal{K}$ and $\varphi(\varsigma) \to +\infty$ as $\varsigma \to +\infty$;

(iv) the *class* $\mathcal{L}_{[0,\alpha)}$ iff it is defined, continuous, and strictly decreasing on $[0, \alpha)$ and $\lim_{\varsigma \to +\infty} \varphi(\varsigma) = 0$;

(v) the *class* $\mathcal{L}$ iff (iv) holds for $\alpha = +\infty$, $\mathcal{L} = \mathcal{L}_{[0,+\infty)}$

It will be shown in Section 3.6, as well as Chapter 4, that the use of these functions in the re-expression of the definitions introduced in Section 2.3.3 serves to provide an indispensible framework in which to develop Lyapunov *vector* and *matrix-valued* functions, which ultimately offer more adaptable approaches to the formulation of an appropriate Lyapunov function for the stability analysis of nonlinear systems.

Chapter 3

Lyapunov Stability Theory

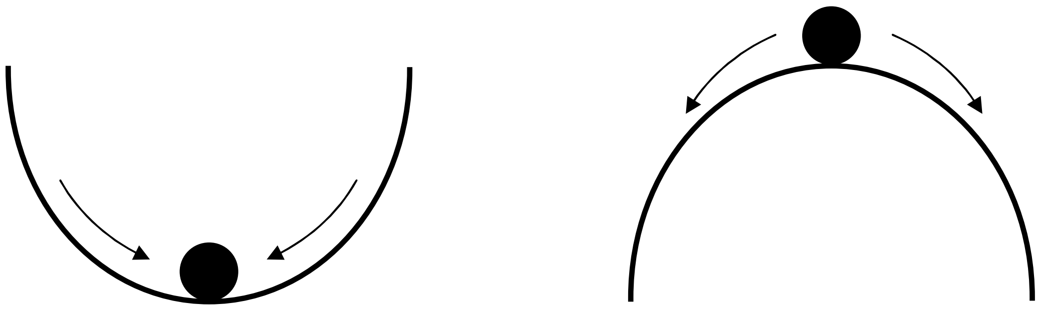
"Stability is an absolutely universal attribute of nature and therefore it has to be reflected in the basic laws of nature. If the knowledge can be constructed on the basis of small perturbations then scientific thinking could be based on some type of Lyapunov function. In any case this function always exists from postulate of stability".

Chetaev, 1936

3.1 Introduction

As mentioned in the introductory quotation, *stability* is an inherent property of nature and a systematic, mathematical approach to determine the stability of a particular process is imperative to the understanding of the natural world. Stability in general is considered in a *qualitative* context, whereby the stability of the equilibrium point(s) of a system is investigated *locally* and potentially extended *globally,* should certain prevailing criteria be met. In terms of qualitative analysis methods, stability analysis of a particular solution trajectory can be rephrased as: given a solution is a curve or trajectory C in some space, if the trajectories D starting near C tend to remain near, or converge to C, then C is *stable*. If these trajectories diverge from C, then C is *unstable*. This rather crude, yet effective, definition succinctly sums up the problem of stability, as well as its unmistakable importance in the comprehensive appreciation of general nonlinear systems. Stability is clearly a central requirement for the useful implementation and manipulation of nonlinear systems as trajectories which either oscillate in a neighbourhood around an operating point, or tend towards the operating point, is a particularly desirable property, and one definitely worth investigating.

A common and effective example to illustrate the concept of stability is shown in Figure 3.1. The system is simply a spherical marble placed within either a *concave up* hemispherical bowl (seen in (a)), or on top of a *concave down* hemispherical bowl (seen in (b)). By simple, logical analysis one can conclude that should the marble be placed anywhere *near* the equilibrium point of system (a), the marble will tend towards this equilibrium point exhibiting *stable* behaviour (more correctly, *asymptotically stable* behaviour, assuming the presence of friction), whereas should the marble be placed anywhere *near* the equilibrium point of the bowl in (b), the marble will diverge from this point, exhibiting what is referred to as *unstable* behaviour. The marbles depicted in figures (a) and (b) are at their respective equilibrium points as should these marbles start at these positions at time $t = 0$, they will remain at these positions for all time $t > 0$.



(a) (b)

Figure 3.1 Simple, illustrative example of a (a) stable and (b) unstable system

There exist a number of definitions of stability, initially formulated and developed by *Lyapunov*, and presented in Section 3.2. From there, two founding methodologies on the investigation of the stability of nonlinear systems are presented, namely, *Lyapunov's first method* and *Lyapunov's second method*, which are introduced in Section 3.3.Section 3.4 deals with general instability theorems and Section 3.5 presents *LaSalle's Invariance Principle*, a method of obtaining the property of *asymptotic stability* by relaxing a certain criterion of Lyapunov's second method. Finally, Section 3.6 introduces the basic idea and methodology behind *vector Lyapunov functions* and its advantages over the standard scalar function approach in the stability analysis of nonlinear systems.

3.2 Definitions of Stability

Consider the continuous, nonlinear, dynamical system.

$$\dot{x}(t) = f\big(t, x(t)\big) \tag{3.1}$$

where $x(t) \in B_h \subseteq \mathrm{R^n}$ denotes the system state vector, B_h an open hyperball of radius $h > 0$ containing the equilibrium point, and $f: B_h \to R^n$ is locally Lipschitz continuous on B_h. For the system in (3.1), the equilibrium state $x = a$ will be treated for stability. For the autonomous case, where f is not an explicit function of time t, it can be shown, without loss of generality, by means of the simple linear transformation of coordinates,

$$y = x - a \quad \Rightarrow \quad x = y + a$$

that (3.1) can be replaced by

$$\dot{y} + \dot{a} = f(y + a)$$

which takes the general form

$$\dot{y}(t) = g\big(y(t)\big), \quad g(0) = 0 \tag{3.2}$$

thereby transforming the equilibrium state $x = a$ to the origin. Therefore, for convenience, all definitions and theorems are stated for the case when the equilibrium point is at the origin of R^n, i.e. $a = 0$. If multiple equilibrium points exist, the stability of each equilibrium point is analysed separately by appropriately shifting each one to the origin.

DEFINITION 3.1 STABILITY IN THE SENSE OF LYAPUNOV. The equilibrium point $x = 0$ is called a *stable* equilibrium point of the system (3.1) if for all $t_0 > 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that

$$|x_0| < \delta(t_0, \epsilon) \Rightarrow |x(t)| < \epsilon \;\; \forall \;\, t \geq t_0 \tag{3.3}$$

where $x(t_0) = x_0$ represents the solution trajectories' initial conditions starting arbitrarily close to the origin. This definition is diagrammatically captured in Figure 3.2 illustrated by the trajectory labelled (a). An equilibrium point exhibiting this type of behaviour is also referred to as *stable in the sense of Lyapunov.*

DEFINITION 3.2 UNIFORM STABILITY. The equilibrium point $x = 0$ is called a *uniformly stable* equilibrium point of system (3.1) if in Definition 3.1, δ can be chosen independent of t_0.

Intuitively, this concept ensures that for every $\epsilon > 0$, the dimensions of the hyperball B_h of radius $h = \delta(t_0, \epsilon)$ containing all the initial conditions x_0 do not tend to zero as $t \to \infty$. Represented mathematically, an equilibrium point is considered *uniformly stable* iff Definition 3.1 holds and for every $\epsilon > 0$ the corresponding maximal δ_M obeying Definition 3.1 satisfies

$$inf[\delta_M(t, \epsilon)] > 0 \tag{3.4}$$

DEFINITION 3.3 ASYMPTOTIC STABILITY. The equilibrium point $x = 0$ is an *asymptotically stable* equilibrium point of system (3.1) iff

(i) Definition 3.1 holds for equilibrium point $x = 0$

(ii) equilibrium point $x = 0$ is *attractive*, i.e. for all $t_0 \geq 0$ there exists $\delta(t_0)$ such that

$$|x_0| < \delta(t_0) \Rightarrow \lim_{t\to\infty} |x(t)| = 0 \tag{3.5}$$

This behaviour is illustrated in Figure 3.2 by the trajectory labelled (b).

DEFINITION 3.4 UNIFORM ASYMPTOTIC STABILITY. The equilibrium point $x = 0$ is a *uniformly asymptotically stable* equilibrium point of the system (3.1) if

(i) Definition 3.2 holds for equilibrium point $x = 0$

(ii) The trajectory $x(t)$ converges uniformly to 0, i.e. there exists $\delta > 0$ and a function $\gamma(t, x_0)$ where $\gamma \in C(R_+ \times R^n, R_+)$ such that $\lim_{t\to\infty} \gamma(t, x_0) = 0$ for all $x_0 \in B_h$ and

$$|x_0| < \delta \;\; \Rightarrow \;\; |x(t)| \leq \gamma(t - t_0, x_0) \;\; \forall \;\; t > t_0 \tag{3.6}$$

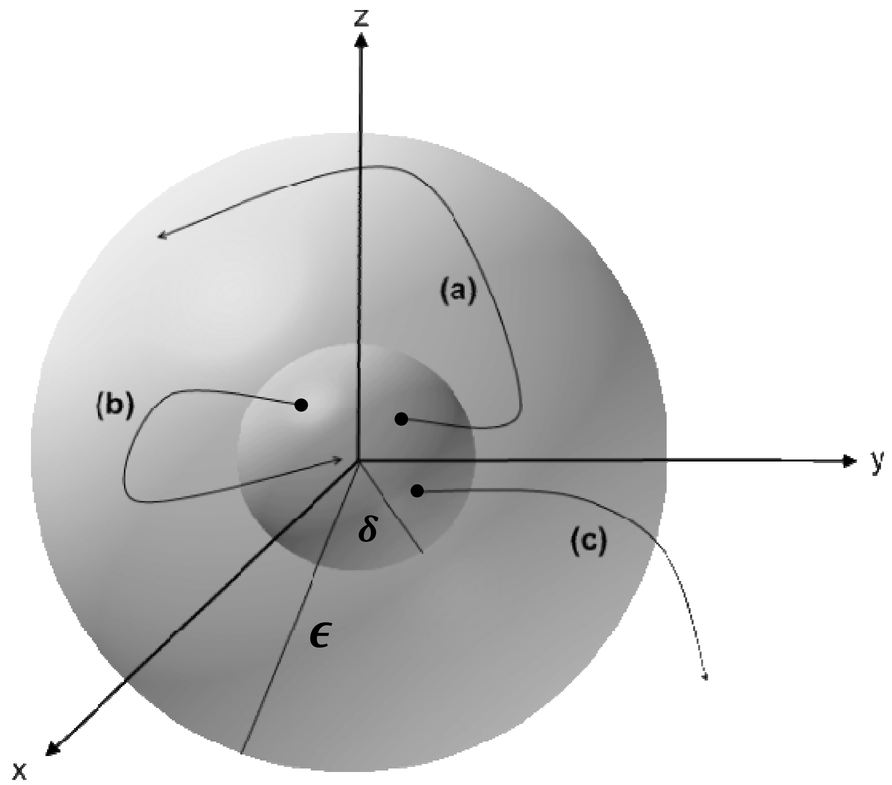


Figure 3.2 Three dimensional phase plane diagram of trajectories which are (a) stable in the sense of Lyapunov, (b) asymptotically stable and (c) unstable

While all three dynamic behaviours illustrated in Figure 3.2 could not possibly occur for the same equilibrium point, the diagram nevertheless serves as a graphical aid in the visualisation of these various stability principles. Definition 3.1 to Definition 3.4 describe the local stability around the equilibrium point $x = 0$. Definition 3.5 and Definition 3.6 will now describe the stability of the equilibrium point $x = 0$ in a global sense, effectively defining the global stability of system (3.1).

DEFINITION 3.5 GLOBAL ASYMPTOTIC STABILITY. The equilibrium point $x = 0$ is a *globally asymptotically stable* equilibrium point of system (3.1) iff

(i) Definition 3.1 holds for all $x_0 \in R^n$

(ii) Definition 3.3 holds for all $x_0 \in R^n$

DEFINITION 3.6 GLOBAL UNIFORM ASYMPTOTIC STABILITY. The equilibrium point $x = 0$ is a *globally uniformly asymptotically stable* equilibrium point of system (3.1) if it is *globally asymptotically stable* according to Definition 3.5 and the convergence to the origin of trajectories $x(t)$ is uniform in time, i.e. there exists a function $\gamma \in C(R_+ \times R^n, R)$ such that

$$|x(t)| \le \gamma(t - t_0, x_0) \quad \forall \quad t \ge t_0 \tag{3.7}$$

Global uniform asymptotic stability is occasionally referred to in the literature as *complete stability.*

DEFINITION 3.7 EXPONENTIAL STABILITY, ESTIMATE OF RATE OF CONVERGENCE. The equilibrium point $x = 0$ is an *exponentially stable* equilibrium point of system (3.1) if there exist $m, \alpha > 0$ such that

$$|x(t)| \le m e^{-\alpha(t-t_0)}|x_0| \quad \forall \quad x_0 \in B_h, \quad t \ge t_0 \ge 0 \tag{3.8}$$

The constant α is considered an estimate for the rate of convergence of the solution trajectories towards the equilibrium point $x = 0$. For the equilibrium point to be *globally exponentially stable* $x_0 \in B_h$ is replaced by $x_0 \in R^n$.

While an intuitive definition of equilibrium point *instability* might be one where an equilibrium point does not subscribe to any of the above mentioned *stability* definitions, i.e. an *unstable* equilibrium point is *not stable,* a strictly formal definition is presented nonetheless, and illustrated graphically as trajectory (c) in Figure 3.2.

DEFINITION 3.8 INSTABILITY. The equilibrium point $x = 0$ is called an *unstable* equilibrium point of system (3.1) if for all $t_0 > 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that

$$|x_0| < \delta(t_0, \epsilon) \Rightarrow |x(t)| \ge \epsilon \quad \forall \; t \ge t_0 \tag{3.9}$$

Instability, by necessity, is a local concept. One point worth mentioning is in order for an equilibrium point to be unstable, it does not require every initial condition starting arbitrarily close to the origin to exit a neighbourhood of the origin, it simply requires *one* trajectory to exhibit this behaviour.

In summary, a number of stability conditions have been introduced to which neighbouring trajectories of a nonlinear system's equilibrium point must subscribe in order to fall under that particular stability category. So far, only *definitions* of stability have been provided whereas no formal methods of obtaining the stability have been given. Various methods are now introduced in Section 3.3 which provide ways by which to determine whether a given equilibrium point does in fact fall into one or, quite possibly, many of these stability classes. The results and definitions introduced in Section 3.2 are attributed primarily to [20]

3.3 Lyapunov Stability Theorems

Lyapunov essentially developed two different methods in solving the problem of stability analysis of general nonlinear systems. In his paper, *General Problem of Stability of Motion*, he states [2]:

> *All ways, which we can present for solving the question we are interested in, we can divide into two categories.*
>
> *With one we associate all those, which lead to a direct investigation of a perturbed motion and in the basis of which there is a determination of general and particular solutions of the differential equation (3.1).*
>
> *In general the solutions should be searched in the form of infinite series. They are series ordered in terms of integer powers of fixed variables. However, we shall meet series of another character in the sequel. The collection of all ways for the stability investigation, which are in this category, we call the first method.*
>
> *With another one we associate all those which are based on principles independent of a determination of any solution of the differential equations of a perturbed motion. All these ways can be reduced to a determination and investigation of integrals of the equation (3.1), and in general, in the basis of all of them, which we shall meet in the sequel, there will be always a determination of functions of variables* $x_1, x_2, \ldots, x_n, t$ *according to given conditions, which should be satisfied by their total derivatives in* t*, taken under an assumption that* $x_1, x_2, \ldots, x_n$ *are functions of* t *satisfying the equation (3.1).*
>
> *The collection of all ways of such a category we shall call the second method.*

Not only did Lyapunov formulate the strict mathematical definitions of the various types of stability, he too developed two nonlinear system stability analysis techniques, the likes of which are still being implemented to this very day. While both methods offer powerful *local* stability analysis frameworks, only Lyapunov's second method possesses the valuable potential to be extended to the *global* system stability analysis, a property sought after by many the engineer, physicist, mathematician and general stability theorist.

3.3.1 Lyapunov's First Method

Lyapunov's first method involves a quantitative investigation of a linearised model of the nonlinear system, in the neighbourhood surrounding its equilibrium point(s). Lyapunov hypothesized that for small deviations from the equilibrium point(s), the performance of the system is approximately governed by the low-order linear terms [8]. These terms dominate and thus determine the system's localised stability. This method however does present two fundamental drawbacks. Firstly, this is principally a *local* stability analysis technique. No generalizations may be drawn regarding the global stability of the equilibrium point under investigation. While not refuting the power and applicability of this method, the global stability of the equilibrium point in question is of extreme importance in both applications and theory as it effectively defines whether the system as a whole is stable or unstable.

The second fundamental weakness of Lyapunov's first method is that should the resulting linearised system exhibit *centre* dynamic behaviour, i.e. the eigenvalues of the resulting linear system (*Jacobian matrix*) are imaginary, the qualitative behaviour of the linearised system around the equilibrium point is not qualitatively identical to that of the original nonlinear system. Therefore, no satisfactory conclusion may be drawn concerning the local stability of the equilibrium point under investigation, where further analysis would be required to determine its stability.

Consider the autonomous nonlinear system

$$\dot{x} = f(x) \tag{3.10}$$

with $f(0) = 0 \;\; \forall \;\; t > 0$, and $\dot{x} = \frac{dx}{dt}$.

Define

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} \tag{3.11}$$

to be the *Jacobian* matrix of $f(x)$ with respect to x evaluated at the origin. The system

$$\frac{dz}{dt} = Az \tag{3.12}$$

is referred to as the *linearization* of system (3.10) about the origin. Assuming the matrix A has no eigenvalues on the imaginary axis, the stability of system (3.12) completely determines the local stability around the origin of system (3.10). The following theorem specifies conditions to which the linearized system (3.12) must subscribe in order to draw a conclusion about the local stability of the origin of system (3.10). The theorem is commonly referred to as *Lyapunov's First or Indirect Method.*

THEOREM 3.1 LYAPUNOV'S FIRST OR INDIRECT METHOD [20]. Let $x = 0$ be the equilibrium point for the nonlinear system (3.10). Let

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$$

and λ_i, $i = 1,2,\ldots,m$ denote the eigenvalues for the m^{th} order system. Then,

(i) The origin is asymptotically stable iff $Re(\lambda_i) < 0$ for eigenvalues λ_i of A

(ii) The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues λ_i of A

For $\lambda_i = 0$ no conclusion can be drawn from Theorem 3.1.

For purposes of illustration, an example is now included.

EXAMPLE 3.1 Consider the nonlinear pendulum equation, discussed in Section 2.2, expressed using the dimensionless variables,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\operatorname{asin} x_1 - bx_2 \end{aligned} \tag{3.13}$$

This system has two physical equilibrium points; $(0,0)$ and $(\pi, 0)$. The stability of both equilibrium points is now analysed by means of Theorem 3.1. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -acosx_1 & -b \end{bmatrix}$$

To determine the stability of the origin, the Jacobian at $(0,0)$ is evaluated.

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The eigenvalues of A are

$$\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 - 4a}$$

For the lightly damped case, $a > b > 0$, the eigenvalues satisfy condition (i) in Theorem 3.1, making the equilibrium point (0,0) locally asymptotically stable. To investigate the stability of equilibrium point $(\pi, 0)$, it needs to be shifted to the origin by the transformation of co-ordinates

$$z_1 = x_1 - \pi, \quad z_2 = x_2$$

The Jacobian, $\partial f/\partial z$ at $z = 0$ is

$$\tilde{A} = \left.\frac{\partial f(x)}{\partial x}\right|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

The eigenvalues of $\check{A}$ are

$$\check{\lambda}_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 + 4a}$$

For all $a > b$, $a > 0$ and $b \geq 0$, there is one eigenvalues in the open right-half plane, i.e. $Re(\lambda) > 0$. Therefore, the equilibrium point $(\pi, 0)$ is locally unstable. Note that in the absence of friction, $(b = 0)$, the eigenvalues lie on the imaginary axis, making $Re(\lambda_{1,2}) = 0$. In this case, the stability of the origin cannot be determined using linearization techniques. Another, more powerful approach needs to be applied, namely, *Lyapunov's Second Method.*

3.3.2 Lyapunov's Second or Direct Method

Lyapunov's second or *direct* method is a qualitative, potentially global, stability analysis technique effectively defining a framework by which the stability of any general nonlinear system can be investigated. In essence, this theorem generalises the basic concept that some measure of *energy dissipation* or, the rate of change of energy in a system, enables one to ascertain system stability. A major advantage of this method is stability can be deduced without having to find an explicit solution to the system's defining nonlinear differential equations. This is a powerful concept since the analytic solution of a nonlinear system is very difficult to find and, in most cases, impossible to obtain. Simply put, the idea is to establish stability properties of the equilibrium point or, in the global case, of the nonlinear system, by studying how certain carefully selected scalar (or vector) functions of the state evolve as the system state evolves. In order to present Lyapunov's direct method for the stability analysis of nonlinear systems, a number of necessary preliminary definitions were introduced in Section 2.3. These definitions lay the appropriate groundwork for the development of the theorem.

Definitions 2.6 and 2.7 introduced in Section 2.2.3 are now expressed in terms of *comparison functions*, which were discussed in Section 2.3.4. These functions are used as upper and lower estimates of the Lyapunov function, v, and its time derivative. The main purpose for the following definitions, will become more apparent in Section 3.6 and in Chapter 4 where the method of Lyapunov *vector* functions and the method of Lyapunov *matrix-valued* functions are explored, and further developed.

DEFINITION 3.9 LOCALLY POSITIVE DEFINITE FUNCTION. A continuous function $v(t,x): R_+ \times R^n \mapsto R$ is called a *locally positive definite function* if for some time-invariant neighbourhood B_h of $x = 0$, $B_h \subseteq R^n$, $h > 0$ and some $\alpha(\cdot)$ of class $\mathcal{K}$,

$$v(t,0) = 0 \;\; and \;\; v(t,x) \geq \alpha(|x|) \;\; \forall \;\; x \in B_h, \;\; t \geq 0 \tag{3.14}$$

DEFINITION 3.10 GLOBALLY POSITIVE DEFINITE FUNCTION. A continuous function $v(t,x): R_+ \times R^n \mapsto R$ is called a *globally positive definite function* if for some $\alpha(\cdot)$ of class $\mathcal{KR}$,

$$v(t,0) = 0 \;\; and \;\; v(t,x) \geq \alpha(|x|) \;\; \forall \;\; x \in R^n, \;\; t \geq 0 \tag{3.15}$$

and, in addition, $\alpha(p) \to \infty$ as $p \to \infty$

Definitions 3.9 and 3.10 do not specify an upper bound as t varies. The following definition takes this into regard.

DEFINITION 3.11 DECRESCENT FUNCTION. A continuous function $v(t,x): R^n \times R_+ \mapsto R_+$ is called a *decrescent function* if there exists some time-invariant neighbourhood B_h of $x = 0$, $B_h \subseteq R^n$, $h > 0$ and some function $\beta(\cdot)$ of class $\mathcal{K}$, such that,

$$v(t,x) \leq \beta(|x|) \;\; \forall \;\; x \in B_h, \;\; t \geq 0 \tag{3.16}$$

From Definitions 3.9 to 3.11 Lyapunov's direct or second method for stability can now be stated. This method calls for a specific *positive semi-definite* scalar function with continuous first partial derivatives, whose time derivative along the trajectories of (3.1) is *negative definite*. A function satisfying these constraints is aptly referred to as a *Lyapunov function.* The time derivative taken along the trajectories of (3.1) is given by

$$\left.\frac{dv(t,x)}{dt}\right|_{(3.1)} = \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t,x) \tag{3.17}$$

THEOREM 3.2 LYAPUNOV'S DIRECT METHOD. Let $x = 0$ be an equilibrium point of system (3.1) and contained within the n dimensional hyperball, $B_h \subseteq R^n$, $h > 0$. Let $v(t, x): R_+ \times R^n \mapsto R$ be a *continuously differentiable* function such that

(i) $v(t, 0) = 0 \quad \forall\ t > 0$

(ii) $v(t, x) > 0 \quad \forall\, x \in B_h,\ t > 0$

(iii) $\left.\frac{dv(t,x)}{dt}\right|_{(3.1)} \leq 0 \quad \forall\ x \in B_h,\ t > 0$

Then, the equilibrium state $x = 0$ is *stable in the sense of Lyapunov.* Moreover, if

$$\left.\frac{dv(t,x)}{dt}\right|_{(3.1)} < 0 \quad \forall\ x \in B_h, \qquad t > 0$$

then the equilibrium state $x = 0$ is *asymptotically stable.* Note that should the Lyapunov function be autonomous, i.e., not an explicit function of time, the restriction $x \neq 0$ must be applied to the above asymptotic stability condition. Furthermore, should $x(t)$ be radially unbounded, i.e., $t \to \infty \Rightarrow x(t) \to \infty$, ($B_h = R^n$) then the equilibrium state $x = 0$ is uniformly asymptotically stable *in the whole*.

Proof.

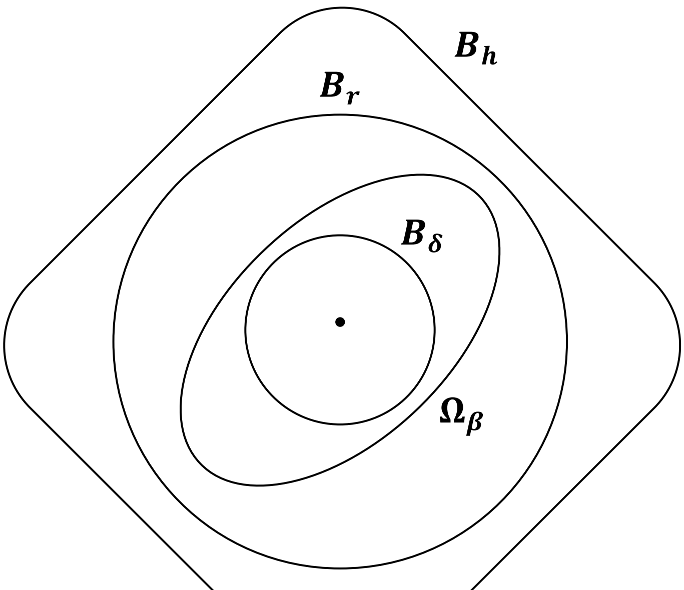


Figure 3.3. Geometric Representation of sets used in proof of Theorem 3.2

Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \leq r\} \subset B_h$$

Let $\alpha = \min_{\|x\|=r} v(x)$. Then, $\alpha > 0$ by condition (i) and (ii) of Theorem 3.2. Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \mid v(x) \leq \beta\}$$

Then, Ω_β is in the interior of B_r, as seen in Figure 3.3. The set Ω_β has the property that any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all $t > 0$. This follows from condition (iii) of Theorem 3.2 since

$$\left.\frac{dv(x(t))}{dt}\right|_{(3.1)} \leq 0 \Rightarrow v\big(x(t)\big) \leq v\big(x(0)\big) \leq \beta \;\; \forall\, t > 0$$

It can be shown that since Ω_β is a *compact set*, system (3.1) has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. As $v(x(t))$ is continuous, and $v(0) = 0$, there exists $\delta > 0$ such that

$$\|x\| \leq \delta \;\; \Rightarrow \;\; v\big(x(t)\big) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \;\; \Rightarrow \;\; x(0) \in \Omega_\beta \;\; \Rightarrow \;\; x(t) \in \Omega_\beta \;\; \Rightarrow \;\; x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \;\; \Rightarrow \;\; \|x(t)\| < r \leq \epsilon, \qquad \forall \;\, t \geq 0$$

which implies that the equilibrium point $x = 0$ is *stable,* according to Definition 3.1. In order to prove the last statement of Theorem 3.2, i.e. *asymptotic stability,* assume that this statement holds, i.e.

$$\left.\frac{dv(t,x)}{dt}\right|_{(3.1)} < 0 \;\; \forall \;\, x \in B_h, \qquad t > 0$$

It is therefore required to show $x(t) \to 0$ as $t \to \infty$; that is, for every $\alpha > 0$, there is a $T > 0$ such that $\|x(t)\| < \alpha \ \forall \ t > T$. According to the previous arguments, for every $\alpha > 0$, there exists $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $v(x(t)) \to 0$ as $t \to \infty$. Since $v(x(t))$ is monotonically decreasing and bounded from below by zero,

$$v(x(t)) \to c, \quad c \geq 0 \ \ as \ t \to \infty$$

To show that $c = 0$, a contradiction argument is used. Suppose $c > 0$. By continuity of $v(x(t))$, there exists a $d > 0$ such that $B_d \subset \Omega_c$. The limit $v(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{v}(x)$. By the last statement of Theorem 3.2, $-\gamma < 0$. It follows that

$$v(x(t)) = v(x(0)) + \int_0^t \dot{v}(x(\tau))d\tau \leq v(x(0) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the original assumption that $c > 0$, thereby showing that $c = 0$ and

$$v(x(t)) \to 0 \ \ as \ \ t \to \infty$$ □

The proof of Theorem 3.2 is derived from [17]. Theorem 3.2 is concisely summarized in Table 3.1, in terms of the definitions introduced in Section 2.3.3.

Table 3.1. Lyapunov's Direct Method summarized in table form

Conditions of $v(t, x)$	**Conditions on $\dot{v}(t, x)$**	**Conclusions**
1. l.p.d.f	l.n.s.d.f	Stable (*i.s.o Lyapunov)*
2. l.p.d.f, decrescent	l.n.s.d.f	Uniformly Stable
3. l.p.d.f, decrescent	l.n.d.f	Uniformly Asymptotically Stable
4. g.p.d.f, decrescent	g.n.d.f	Globally, Uniformly Asymptotically Stable

where $\dot{v}(t,x) \triangleq \left.\frac{dv(t,x)}{dt}\right|_{(3.1)}$

From Table 3.1 one can see that the condition of *uniform stability* is directly derived from the *decrescentness* property of the candidate Lyapunov function, $v(t,x)$.

Consider the linear, autonomous system

$$\dot{x}(t) = Ax(t) \tag{3.18}$$

A common candidate scalar Lyapunov function is the quadratic function

$$v(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j \tag{3.19}$$

where P is a real, symmetric matrix. What makes this function particularly useful is the ease at which its sign definiteness can be checked. $v(x)$ is *positive definite* (*positive semi-definite*) if and only if all the *leading principle minors* of P are *positive* (*non-negative*). For the specific case of a symmetric matrix, as is the case for P, $v(x)$ is *positive definite* (*positive semi-definite*) if and only if all the eigenvalues of P are *positive* (*non-negative*). The time derivative of $v(x)$ along the trajectories of system (3.18) is given by

$$\begin{aligned} \dot{v}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x \end{aligned} \tag{3.20}$$

By letting

$$A^T P + PA = -Q \tag{3.21}$$

equation (3.20) becomes

$$\dot{v}(x) = -x^T Q x \tag{3.22}$$

Therefore, as long as matrix Q is positive definite and symmetric, condition (iii) of Theorem 3.2 is satisfied. Equation (3.21) is referred to as the *Lyapunov matrix equation*, where should its unique solution, P, be found to be positive definite, system (3.18) is globally asymptotically stable. Theorem 3.3 presents this property.

THEOREM 3.3 LYAPUNOV MATRIX THEOREM [9]. For the real parts of all eigenvalues of matrix A, $A \in R^{n \times n}$ to be negative it is necessary and sufficient that for any positive definite, symmetric matrix Q, $Q \in R^{n \times n}$ there exists the unique solution P, $P \in R^{n \times n}$ of the *Lyapunov matrix equation* (3.21), which is also a positive definite, symmetric matrix.

For solving the Lyapunov matrix equation, see Aliev and Larin [27], Barbashin [28], Barnett and Storey [16]. To provide further understanding of these concepts, the following example is presented.

EXAMPLE 3.2. Consider the linear, autonomous system (3.18) where

$$A = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \tag{3.23}$$

By letting Q be the 2×2 identity matrix, the Lyapunov matrix equation is

$$\begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} P + P \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.24}$$

Solving (3.24) for P yields

$$P = \begin{pmatrix} \frac{7}{20} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} \end{pmatrix} \tag{3.25}$$

Since P is positive definite and symmetric, by Theorem 3.3, the real part of all the eigenvalues of A are negative and therefore, by Theorem 3.1, system (3.18) is globally asymptotically stable.

EXAMPLE 3.3. Consider the nonlinear pendulum discussed in Example 3.1. The stability of the origin is now analysed using Lyapunov's direct method.

Consider the candidate Lyapunov function

$$v(x) = a(1 - cosx_1) + \frac{1}{2}x_2^2 \tag{3.26}$$

This function is derived from *energy concepts*, where the total energy of the pendulum system is taken as the sum of the potential and kinetic energies, with the reference of the potential energy chosen such that $E(0) = 0$, i.e.

$$E(x) = v(x) = \int_0^{x_1} asinydy + \frac{1}{2}x_2^2 = a(1 - cosx_1) + \frac{1}{2}x_2^2$$

The derivative of $v(x)$ along the trajectories of (3.13), denoted $\dot{v}(x)$, is given by,

$$\dot{v}(x) = a\dot{x}_1 sinx_1 + x_2\dot{x}_2 = -bx_2^2 \tag{3.27}$$

The derivative $\dot{v}(x)$ is negative semi-definite for all $x_2 \in R^n$. Taking $B_h = \{x \in R^2 \mid |x_1| < \pi\}$, $v(x)$ is locally positive definite, and therefore from Theorem 3.2 the origin is *stable in the sense of Lyapunov.* However, as seen in Figure 2.2, for the lightly damped case, $a > b > 0$, the origin is in fact *asymptotically stable* as all the trajectories tend to the origin as $t \to \infty$. The energy Lyapunov function fails to show this fact. It will be shown in Section 3.5 that *LaSalle's Invariance Principle* enables one to arrive at this, more appropriate, conclusion. Another option is to try and obtain another, more suitable, Lyapunov candidate function.

This example highlights a very important, and in fact, fundamental drawback of Lyapunov's direct method; namely, the theorem's conditions are only *sufficient* and not *necessary.* Failure to obtain a suitable Lyapunov function to ascertain a particular equilibrium point's stability or asymptotic stability *does not* imply instability. It simply means that the equilibrium point's stability property cannot be established using *that* particular Lyapunov candidate function and more investigation, in the form of another candidate Lyapunov function, another method of the Lyapunov function construction, etc, needs to be pursued. A great number of methods have been explored from the very development of Lyapunov's stability theorems to form a methodical and algorithmical approach to the formulation of an appropriate Lyapunov function. However, to date, there exists no such method. A primary reason is attributed to the underlying complex and unique structure of a nonlinear system, whereby global instability cannot be inferred from local instability.

3.3.3 Exponential Stability Theorem

Theorems 3.1, 3.2 and 3.3 provide a structured approach for obtaining the stability as well as the asymptotic stability of a nonlinear system's equilibrium point. However, these theorems stop short of providing explicit rates of convergence of solution trajectories to equilibrium points. Since *exponentially stable* equilibrium points are robust to perturbations, they are a desirable property from the viewpoint of applications. Theorem 3.4 provides the necessary and sufficient conditions for the existence of an exponentially stable equilibrium point.

THEOREM 3.4 EXPONENTIAL STABILITY THEOREM [17]. Let $x = 0$ be an equilibrium point of system (3.1) and contained within the n dimensional hyperball, $B_h \subseteq R^n$, $h > 0$. Let $v(t,x): R_+ \times R^n \mapsto R$ be a *continuously differentiable* function and there exists some constants $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 > 0$ such that

$$\begin{aligned} &\text{(i)} \quad \alpha_1 |x(t)|^2 \le v(t,x) \le \alpha_2 |x(t)|^2 \\ &\text{(ii)} \quad \left.\frac{dv(t,x)}{dt}\right|_{(3.1)} \le -\alpha_3 |x(t)| \\ &\text{(iii)} \quad \left|\frac{dv(t,x)}{dt}\right| \le \alpha_4 |x(t)| \end{aligned} \tag{3.28}$$

Then the equilibrium point $x = 0$ is an *exponentially stable* equilibrium point of system (3.1)

The idea of exponential stability has the potential to be extended to the estimation of an equilibrium point's *region of asymptotic stability*, as the rate at which a particular trajectory approaches an asymptotically stable equilibrium point is indicative of the equilibrium point's *strength of attraction*. While the notion of region of asymptotic stability is not explored in this research report, Chapter 6 suggests the extension of Lyapunov's matrix-valued function method to the estimation of the region of asymptotic stability as a potential area of future research.

3.4 Instability Theorems

Theorems 3.1, 3.2 , 3.3 and 3.4 have been involved with the direct determination of whether the equilibrium point $x = 0$ is *stable, asymptotically stable* or *exponentially stable* in either the local or global sense. There do also exist *instability theorems* for establishing whether an equilibrium point is *unstable.* The most well known of these theorems is *Chetaev's Instability Theorem.* However, just as Lyapunov's direct method provides *sufficient* conditions for the stability of an equilibrium point, so too does Chetaev's theorem provide sufficient criteria for the instability investigation of the equilibrium point. As defined in Definition 3.8 the equilibrium point $x = 0$ is called an *unstable* equilibrium point of system (3.1) if for some $t_0 > 0$ and $\epsilon > 0$ there exists $\delta(t_0, \epsilon)$ such that $|x_0| < \delta(t_0, \epsilon)$ implies $|x(t)| \geq \epsilon \ \ \forall \ t \geq t_0$. Theorems 3.5 and 3.6 formally develop the criteria required to determine the instability of an equilibrium point.

THEOREM 3.5 INSTABILITY THEOREM. The equilibrium point $x = 0$ is an *unstable* equilibrium point of system (3.1), at time t_0, if there exists a *decrescent function,* $v : R_+ \times R^n \ \rightarrow \ R$ such that

(i) $\dot{v}(t, x)$ is a *locally positive definite*

(ii) $v(t, 0) = 0$ and there exist points x arbitrarily close to $x = 0$ such that $v(t_0, x) > 0$

Proof. Given that there exists the function $v(t, x)$ such that

$$v(t, x) \leq \beta(|x|) \quad x \in B_r, \ \ r > 0 \qquad \text{(Decrescence)}$$

$$\dot{v}(t, x) \geq \alpha(|x|) \quad x \in B_s, \ \ s > 0 \qquad \text{(Local Positive Definiteness)}$$

It is required to prove that for some $\epsilon > 0$, there is no δ such that $|x_0| < \delta(t_0, \epsilon)$ implies $|x(t)| \geq \epsilon \ \ \forall \ t \geq t_0$. Choose $\epsilon = \min{(r, s)}$. Given $\delta > 0$ choose x_0 with $|x_0| < \delta$ and $v(t_0 . x_0) > 0$. This choice is possible based on requirements imposed on $v(t_0, x)$. As long as some trajectory, $\phi(t, t_0, x_0)$, lies in B_ϵ, $\dot{v}\big(t, x(t)\big) \geq 0$ which implies that

$$v\big(t, x(t)\big) \geq v(x_0, t_0) > 0$$

This implies that $|x(t)|$ is bounded away from $x = 0$ and therefore $v\big(t, x(t)\big)$ is also bounded away from $x = 0$. Thus $v\big(t, x(t)\big)$ will exceed $\beta(\epsilon)$ in finite time and therefore this guarantees that $|x(t)|$ will exceed ϵ in finite time. □

THEOREM 3.6 CHETAEV'S INSTABILITY THEOREM. The equilibrium point $x = 0$ is called an *unstable* equilibrium point of the system (3.1) at time t_0 if there exists a *decrescent function,* $v : R^n \times R_+ \rightarrow R$ such that

(i) $\dot{v}(t, x) = \lambda v(t, x) + v_1(t, x)$, where $\lambda > 0$ and $v_1(t, x) \geq 0 \;\; \forall \;\; t \geq 0$, $x \in B_r$.

(ii) $v(t, 0) = 0$ and there exists points x arbitrarily close to $x = 0$ such that $v(t_0, x) > 0$

Proof. Choose $\epsilon = r$ and given $\delta > 0$ pick x_0 such that $|x_0| < \delta$ and $v(t_0, x_0) > 0$. When $|x(t)| \leq r$,

$$\dot{v}(t, x) = \lambda v(t, x) + v_1(t, x) \geq \lambda v(t, x)$$

By multiplying the above inequality by an integrating factor $e^{-\lambda t}$, it follows that

$$\frac{dv(t, x)e^{-\lambda t}}{dt} \geq 0$$

Integrating this inequality from t_0 to t yields,

$$v\big(t, x(t)\big) \geq e^{\lambda(t - t_0)} v(t_0, x_0)$$

Thus $v(t, x(t))$ grows without bounds. Since $v(t, x)$ is decrescent,

$$v(t, x) \geq \beta(|x|)$$

for some function $\beta \in \mathcal{K}$, so that for some t_δ, $v\big(t, x(t)\big) > \beta(\epsilon)$, establishing that $|x(t_\delta)| > \epsilon$ □

The following example of the simplified *Lotka-Volterra predator-prey* model is used to illustrate the implementation of the above mentioned instability theorem. This model is frequently used to describe the dynamics of biological systems in which two species, one *predator*, one *prey*, interact, and was independently proposed by *Alfred J. Lotka* in *1925* and *Vito Volterra* in *1926.*

EXAMPLE 3.4. Consider the system representing the simplified *Lotka-Volterra* predator-prey model

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - x_1 x_2 \\ \frac{dx_2}{dt} &= x_1 x_2 - x_2 \end{aligned} \tag{3.29}$$

which has equilibrium points at the origin and at $(x_1, x_2) = (1,1)$. For the equilibrium point at the origin consider the candidate function,

$$v(x) = \frac{1}{2}(x_1^2 - x_2^2) \tag{3.30}$$

The time derivative of (3.30) along the trajectories of (3.29), denoted by $\dot{v}(x)$, is given by

$$\begin{aligned} \dot{v}(x) &= x_1(x_1 - x_1 x_2) - x_2(x_1 x_2 - x_2) \\ &= x_1^2 + x_2^2 - x_1^2 x_2 - x_1 x_2^2 \end{aligned} \tag{3.31}$$

From (3.31) it is clear that $\dot{v}(x)$ is *locally positive definite* for sufficiently small x_1 and x_2, and $v(x)$ can be positive for points arbitrarily near the origin. Therefore by Theorem 3.5, the origin is *unstable.* For the equilibrium point (1,1), the following linear translation of coordinates is applied

$$\gamma_1 = x_1 - 1, \quad \gamma_2 = x_2 - 1$$

Now, (3.29) becomes

$$\begin{aligned} \frac{d\gamma_1}{dt} &= -\gamma_2 - \gamma_1 \gamma_2 \\ \frac{d\gamma_2}{dt} &= \gamma_1 \gamma_2 + \gamma_1 \end{aligned} \tag{3.32}$$

which has an equilibrium point at the origin. Choose the arbitrary candidate function

$$v(\gamma_1.\gamma_2) = \gamma_1 + \gamma_2 - \ln(1+\gamma_1) - \ln\,(1+\gamma_2) \qquad (3.33)$$

with a series expansion of

$$v(\gamma) = \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2 + \cdots \;\; (H.O.T) \qquad (3.34)$$

is *locally positive definite* sufficiently close to the origin. The time derivative of (3.33) along the trajectories of (3.32) is given by

$$\begin{aligned} \dot{v}(\gamma_1,\gamma_2) &= \dot{\gamma}_1 + \dot{\gamma}_2 - \frac{\dot{\gamma}_1}{1+\gamma_1} - \frac{\dot{\gamma}_2}{1+\gamma_2} \\ &= -\gamma_1\gamma_2 - \gamma_2 + \gamma_1\gamma_2 + \gamma_1 + \gamma_2 - \gamma_1 \\ &= 0 \end{aligned} \qquad (3.35)$$

Thus, $v(\gamma_1,\gamma_2)$ is a *Lyapunov function* of system (3.33) and hence the equilibrium point (1,1) is stable *in the sense of Lyapunov*. To verify these results, the phase portrait of the solution trajectories of system (3.29) is plotted in Figure 3.4.

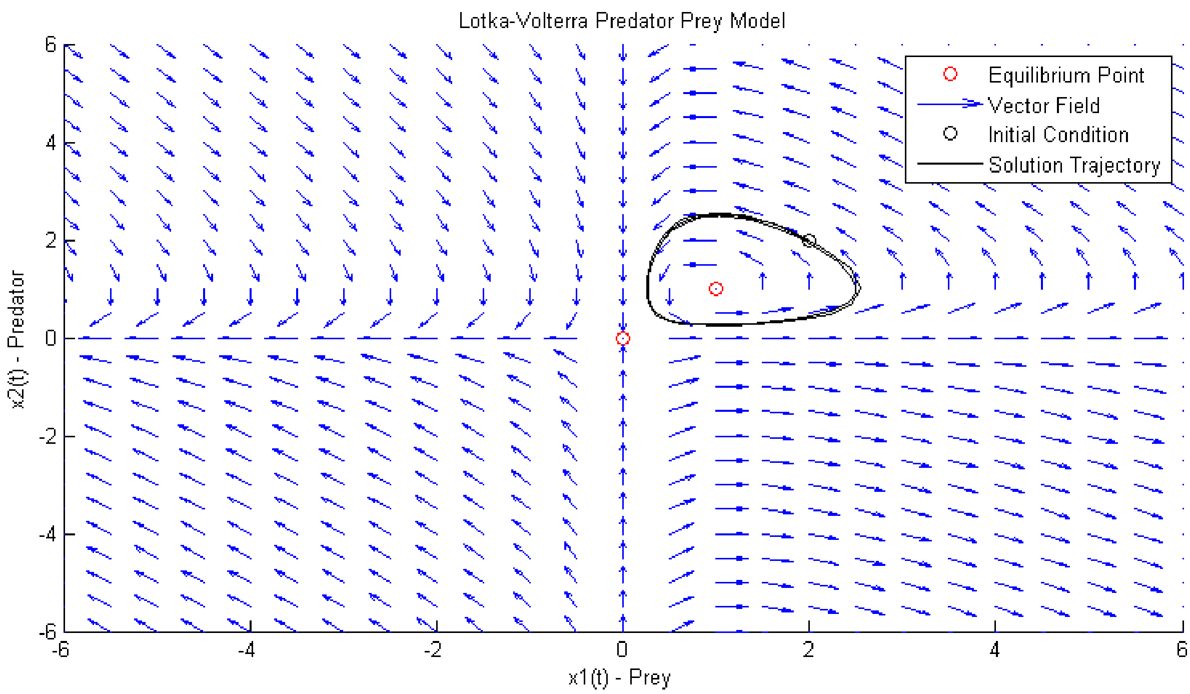


Figure 3.4. Phase Portrait of Lotka-Volterra Predator Prey Model

Clearly, a *saddle* point exists at the equilibrium point $x = 0$. This could also be verified by analysing the eigenvalues of the linearised Jacobian matrix about the origin, discussed in Section 3.3.1, where these two eigenvalues would be found to be $\lambda_{1,2} = \pm 1$. As for applying Lyapunov's first method to the stability analysis of the second equilibrium point $(1,1)$, it is found that the eigenvalues of the Jacobian matrix about the point $(1,1)$ are $\lambda_{1,2} = \pm\, i$. No stability properties can therefore be concluded for the equilibrium point $(1,1)$. As seen in Figure 3.4, a equilibrium point $(1,1)$ exhibits *stable limit cycle* behaviour, thereby reinforcing the result obtained from Example 3.2 that the point $(1,1)$ is *stable in the sense of Lyapunov* and not *asymptotically stable.* Furthermore, as seen in (3.27), the time derivative of the Lyapunov function along the trajectories of system (3.21) is zero, i.e. $\dot{v}(x) = 0$. This could possibly indicate the *limit cycle behaviour* inherent in the system, however there currently exists no theorem exposing this fact.

3.5 LaSalle's Invariance Principle

In Section 3.3.1 and Section 3.3.2 both Lyapunov's first and second methods are introduced as effective methods in determining the stability property of a particular equilibrium point, considered to lie at the origin. The concept of asymptotic stability and the criteria required, from both methods, in determining whether the equilibrium point is asymptotically stable was also introduced, namely, for Lyapunov's first method, the eigenvalues of the linearized A matrix need to lie in the negative real half plane whereas for Lyapunov's second or direct method, the candidate Lyapunov function needs to satisfy the constraints of being a locally positive definite function as well as its time derivative satisfy the constraints of being a locally negative definite function.

LaSalle's invariance principle, developed in 1960 by *J.P LaSalle*, in fact first introduced by *Barbashin* and *Krasovskii* in a special case in 1952, and later by *Krasovskii* in the general case in 1959 [17], attempts to relax the constraints imposed on the candidate Lyapunov function for the condition of asymptotic stability. It effectively develops a method of obtaining asymptotic stability without the need of the time derivative of the Lyapunov function to be a locally negative definite function, but in fact only requires it to be a locally *semi-definite* function. The principle essentially asserts that if a Lyapunov function exists within the neighbourhood of the origin, with a negative semi-definite time derivative along the trajectories of the system and, in addition, it can be established that no trajectory can stay identically at points where $\dot{v}(t,x) = 0$, *except* at the origin, then the origin is *asymptotically stable* . In order to formally state *LaSalle's invariance principle* a few preliminary definitions need to be introduced. It is noted that this principle applies primarily to *autonomous* or *periodic systems* and therefore, system (3.10) will be considered.

DEFINITION 3.12 POSITIVE LIMIT SET. A set $S \subset R^n$ is the *positive limit set* of a trajectory $\phi(\cdot, x_0, t_0)$ if for every $y \in S$, there exists a sequence of times $t_n \to \infty$ such that $\phi(t_n, x_0, t_0) \to y$

DEFINITION 3.13 INVARIANT SET. A set $M \subset R^n$ is said to be an *invariant set* with respect to (3.10) if whenever $y \in M$ there exists

$$\phi(t, y, t_0) \in M \quad \forall \ t \in R_+$$

That is, if a solution trajectory belongs to M at some time instant, then it belongs to M for all $t \in R_+$.

DEFINITION 3.14 POSITIVE INVARIANT SET. A set $M \subset R^n$ is said to be a *positively invariant set* with respect to (3.10) if whenever $y \in M$ there exists

$$\phi(t, y, t_0) \in M \quad \forall \ t > 0$$

PROPOSITION 3.1. If $\phi(\cdot, x_0, t_0)$ is a *bounded trajectory,* its positive limit set is *compact.* Further, $\phi(t, x_0, t_0)$ approaches its positive limit set as $t \to \infty$.

PROPOSITION 3.2. Let S be the positive limit set of any trajectory solution of system (3.10), then S is *invariant.*

Proof. Let $y \in S$ and $t_1 \geq 0$ be arbitrary. Required to prove that $\phi(t, y, t_1) \in S \ \ \forall \ t \geq t_1$. Since $y \in S$ implies that there exists $t_n \to \infty$ such that $\phi(t_n, t_0, x_0) \to y$ as $n \to \infty$. Since trajectories are continuous irrespective of initial conditions, it follows that

$$\phi(t, y, t_1) = \lim_{n \to \infty} \phi(t, \phi(t_n, t_0, x_0), t_1)$$

$$= \lim_{n \to \infty} \phi(t + t_n - t_1, x_0, t_0)$$

since the system is autonomous. Now, $t_n \to \infty$ as $n \to \infty$ so by Proposition 3.1 the right hand side converges to an element of S. □

PROPOSITION 3.3 LASALLE'S PRINCIPLE. Let $v : R^n \to R$ be *continuously differentiable* and consider that

$$\Omega_c = \{x \in R^n \mid v(x) \leq c\}$$

is bounded and that $\dot{v}(x) \leq 0$ for all $x \in \Omega_c$.

Define $S \subset \Omega_c$ by

$$S = \{x \in \Omega_c \mid \dot{v}(x) = 0\}$$

and let M be the largest *invariant set* in S. Then, whenever $x_0 \in \Omega_c$, $\phi(t, x_0, 0)$ approaches M as $t \to \infty$.

Proof. Let $x_0 \in \Omega_c$. Since $v(\phi(t, x_0, 0))$ is a nonincreasing function of time, $\phi(t, x_0, 0) \in \Omega_c \;\forall\, t \in R$. Further, since Ω_c is bounded, $v(\phi(t, x_0, 0))$ is also bounded from below. Let

$$c_0 = \lim_{t \to \infty} v(\phi(t, x_0, 0))$$

and let L be the *positive limit set* of the trajectory. Then $v(y) = c_0$ for $y \in L$. Since L is invariant, $\dot{v}(y) = 0 \;\forall\; y \in L$ so that $L \subset S$. Since M is the largest invariant set inside S, $L \subset M$. Since $s(t, x_0, 0)$ approaches L as $t \to \infty$, $s(t, x_0, 0)$ approaches M as $t \to \infty$. □

THEOREM 3.7 LASALLE'S INVARIANCE PRINCIPLE FOR ASYMPTOTIC STABILITY. Let $v : R^n \to R$ be such that on $\Omega_c = \{x \in R^n \mid v(x) \leq c\}$, be a *compact set* $\dot{v}(x) \leq 0$. Define

$$S = \{x \in \Omega_c \mid \dot{v}(x) = 0\}$$

Then, if S contains no trajectories other than $x = 0$, then the equilibrium point at the origin is *asymptotically stable.*

The proof is derived directly from preceding proof of Proposition 3.3. LaSalle's invariance principle is now extended to the case of *global asymptotic stability.*

THEOREM 3.8 LASALLE'S INVARIANCE PRINCIPLE FOR GLOBAL ASYMPTOTIC STABILITY. Let $v : R^n \to R$ be a *globally positive definite function* and $\dot{v}(x) \leq 0$ for all $x \in R^n$. In addition, let the set

$$S = \{x \in R^n \mid \dot{v}(x) = 0\}$$

contain no *nontrivial* trajectories. Then the equilibrium point $x = 0$ is *globally asymptotically stable.*

To illustrate Theorem 3.7, refer back to Example 3.2, where the *stability* of the origin of the nonlinear pendulum is found to be *stable* using Lyapunov's direct method however, *asymptotic stability* could not be obtained. Recall that the candidate Lyapunov function is defined as

$$v(x) = a(1 - cosx_1) + \frac{1}{2}x_2^2$$

with its corresponding time derivative along the trajectory of system (3.13) given as $\dot{v}(x) = -bx_2^2$. Let $B_h = \{x \in R^2 \mid |x_1| < \pi\}$; $v(x)$ is a *locally positive definite function* in B_h and $\dot{v}(x) \leq 0$, i.e. $\dot{v}(x)$ is *locally negative semi-definite.* To find $S = \{x \in B_h \mid \dot{v}(x) = 0\}$ note that

$$\dot{v}(x) = 0 \Rightarrow x_2 = 0$$

Hence, $S = \{x \in B_h \mid x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow sinx_1 \equiv 0 \Rightarrow x_1(t) \equiv 0$$

since $|x_1(t)| < \pi$. Hence the only solution that can stay identically in S is the trivial solution, $(x_1, x_2) = (0,0)$. Therefore, by Theorem 3.7, the origin is *asymptotically stable.*

The results, theories and definitions expressed in Section 3.5 are attributed primarily to [33, 17, 20, 34]

3.6 Vector Lyapunov Functions

Many attempts have been made to weaken the constraints imposed on a candidate Lyapunov function. These efforts can be broadly classed into three categories,

(i) Attempt to weaken the sign-definiteness property of the auxiliary functions, i.e. replace the positive definite function requirement from condition (ii) of Theorem 3.2 with a positive semi-definite requirement;

(ii) Modify or generalise the assumptions on the total derivative of the scalar auxiliary function along the solution trajectories of the system;

(iii) Implement multicomponent auxiliary functions in the form of either vector or matrix-valued functions.

The third category incorporates the preceding two categories, as the construction and development of a multicomponent auxiliary function stems from the weakening and generalisation of the Lyapunov function criteria.

Vector Lyapunov functions introduce the method of using *multiple* Lyapunov functions to relax the rigid sign-definiteness constraints imposed by Lyapunov's direct method [18]. These multiple Lyapunov functions potentially satisfy weaker requirements individually, however together, present a more structured and adaptable approach to the study of nonlinear system stability. This method calls for the development of a *comparison system*, where the stability of the original nonlinear system can be inferred from the stability of this comparison system. In spite of this method's vast applicability, a major drawback still persists, that is, the resulting comparison system must satisfy the property of being both *quasimonotone* and *nondecreasing*. This problem is dealt with by means of the development of a *cone-valued Lyapunov function*, which acts on the premise that a suitable cone can be found relative to the comparison system, which itself is quasimonotone.

The following theorem provides the sufficient conditions in terms of vector Lyapunov functions for the stability analysis of system (3.1)

THEOREM 3.9 VECTOR LYAPUNOV FUNCTION [29]. Suppose that

(i) $V \in C(R_+ \times S(\rho), R_+^N)$, $V(t,x)$ is locally Lipschitzian in x and $Q(V(t,x))$ is positive definite and decrescent where $Q \in C(R_+, R_+^N)$, $Q(u)$ is nondecreasing in u and $Q(0) = 0$, $S(\rho) = \{x \in R^n \ : \ \|x\| < \rho\}$;

(ii) $g \in C[R_+ \times R_+^N, R^N]$, $g(t,o) \equiv 0$ and $g(t,u)$ is *quasimonotone nondecreasing* in u for each $t \in R_+$,

$$D^+V(t,x) \leq g(t, V(t,x))$$

where D^+ denotes the upper right Dini derivative.

Then the stability properties of the null solution of the comparison system,

$$u' = g(t,u), \qquad u(t_0) = u_0 \geq 0 \tag{3.36}$$

imply the corresponding stability properties of the trivial solution of system (3.1). The advantage of vector Lyapunov functions over the conventional scalar function approach can be seen in the following example [29].

EXAMPLE 3.5. Consider the nonlinear, nonautonomous system,

$$\begin{aligned} \dot{x} &= e^{-t}x + ysint - (x^3 + xy^2)\sin^2 t \\ \dot{y} &= xsint + e^{-t}y - (x^2y + y^3)\sin^2 t \end{aligned} \tag{3.37}$$

By choosing the auxiliary scalar Lyapunov function, $v(x,y) = x^2 + y^2$ the following differential inequality results in,

$$D^+v(x,y) = 2[e^{-t}(x^2 + y^2) + \sin t(x^2 + y^2) - \sin^2 t(x^2 + y^2)] \tag{3.38}$$

Using $2|ab| \leq a^2 + b^2$ and observing that $\sin^2 t(x^2 + y^2) \geq 0$, equation (3.38) simplifies to,

$$D^+v(x,y) \leq 2(e^{-t} + |\sin t|)v(x,y) \tag{3.39}$$

Since the null solution of

$$u' = 2(e^{-t} + |\sin t|)u, \qquad u(t_0) = u_0 \geq 0 \tag{3.40}$$

is *not* stable, nothing can be inferred about the stability of the trivial solution of system (3.37). In contrast, by using the vector function: $(V_1(x,y) \quad V_2(x,y))$,

$$V_1(x,y) = \frac{1}{2}(x+y)^2, \qquad V_2(x,y) = \frac{1}{2}(x-y)^2 \tag{3.41}$$

the conditions of Theorem 3.9 are satisfied with

$$Q(u) = \sum_{i=1}^{N} u_i\,, \qquad N = 2$$

where $u_1 = V_1(x,y)$ and $u_2 = V_2(x,y)$; $Q(u)$ is *nondecreasing* and

$$g = (g_1 \quad g_2) = (2(e^{-t} + \sin t)u_1 \quad 2(e^{-t} - \sin t)u_2)$$

where the vector comparison function $g(u)$ is *quasimonotone nondecreasing in* u. The null solution of (3.36) with this comparison function is stable. Therefore, by Theorem 3.9, the trivial solution of (3.37) is stable.

A natural progression of vector Lyapunov functions to the stability analysis of large scale systems is clear. A large dynamic system, consisting of multiple interconnected subsystems, continually changes its stability properties over long periods of time owing to the constant disconnection and reconnection of its constituent parts. Such volatile behaviour severely impacts the structure and stability of the overall system which may ultimately cause the system to fail. To avoid this, large scale systems need to be built with sufficient stability properties in mind to withstand structural perturbations. Vector Lyapunov functions lend themselves perfectly to this application, where each individual subsystem's stability as well as its interconnections can be independently analysed using a subsystem specific Lyapunov function. The stability of the system as a whole is then analysed using the aggregation of all these independent Lyapunov functions. This implementation is further discussed in the analysis of dynamic large scale systems by means of the matrix-valued Lyapunov function.

Chapter 4

Lyapunov Matrix-valued Functions

“True stability results when presumed order and presumed disorder are balanced. A truly stable system expects the unexpected, is prepared to be disrupted, waits to be transformed.”

Robbins, 1985

4.1 Introduction

As discussed in Section 3.6, matrix Lyapunov functions, as is the case of vector Lyapunov functions, attempt to weaken the otherwise conservatively stringent constraints imposed by Lyapunov's direct method by providing a systematic approach of the Lyapunov function's construction. In general, the Lyapunov matrix function approach leads to either the construction of a vector function or a scalar Lyapunov function. This research report will focus on the implementation of the Lyapunov matrix function approach in the composition of an appropriate scalar function adhering to Lyapunov's direct method. The matrix function is formulated by taking the system's dynamic interactions into account. Both independent state dynamics, as well as interlinking states' dynamics are captured in the matrix function, where their respective stability's are analysed. These stabilities ultimately provide an estimation of the *overall* dynamic system's stability. In general, the Lyapunov matrix function is a two-indices system of functions of the form:

$$U(t,x) = [v_{ij}(t,x)], \qquad i,j = 1,2,\dots,m \tag{4.1}$$

where $U \in C(R_+ \times R^n, R^{m\times m})$, and has elements:

$$v_{ii}(t,x) \in C(R_+ \times R^n, R_+), \qquad i = 1,2,\dots,m \tag{4.2}$$

and $$v_{ij}(t,x) \in C(R_+ \times R^n, R), \qquad i \neq j \tag{4.3}$$

The above elements must also satisfy the following conditions

(i) $v_{ij}(t,x)$ are locally Lipschitzian in x that is, for each point in $R_+ \times \mathcal{N}$ there exists a neighbourhood $R_+ \times S$ and a positive number $L > 0$ such that:

$$|v(t,x) - v(t,y)| \leq L|x - y|$$

for all $(t,x) \in R_+ \times S$, $(t,y) \in R_+ \times S$

(ii) $v_{ij}(t,0) = 0$ for all $t \in R_+$, $i,j = 1,2,\dots,m$

(iii) $v_{ij}(t,x) = v_{ji}(t,x)$ in any open connected neighbourhood $\mathcal{N}$ of point $x = 0$ for all $t \in R_+$

Let $y \in R^m$, $y \neq 0$ be given. By means of the vector y and matrix-valued function (4.1), the following scalar function is introduced:

$$v(t,x,y) = y^T U(t,x) y \qquad (4.4)$$

The sign of $v(t,x,y)$ and its time derivative is dependent on both the vector y as well as the matrix $U(t,x)$. The vector y however is conventionally chosen as $y \in R_+^n$ thereby transferring the sign dependence of $v(t,x,y)$ solely onto $U(t,x)$. Also note that if $v_{ij} \equiv 0$ for all $(i \neq j) \in [1,m]$, then $U(t,x) = diag\big(v_{11}(t,x),\dots,v_{mm}(t,x)\big)$ and

$$V(t,x) = U(t,x)e, \qquad e \in R^m \qquad (4.5)$$

is a vector function, and therefore can be used in conjunction with Theorem 3.9 to establish stability properties of a given nonlinear system. Thus, the two-indices system of functions (4.1) forms a basis for construction of both scalar and vector Lyapunov functions. The definitions and theories presented in this chapter are attributed primarily to Martynyuk [6, 8, 9, 10]

4.2 Mixed hierarchical subsystems

This methodology is based on the *hierarchical decomposition technique*, whereby the non-diagonal elements of the matrix function (4.1) are derived from the generalised *second level decomposition* of the dynamical system, whereas the diagonal elements are derived from the *first level mathematical decomposition*. Unless otherwise stated, the generalised method of Lyapunov matrix-valued functions will refer to the *hierarchical* Lyapunov matrix-valued approach. The $v_{ii}(t, x)$ elements take independent state dynamic interactions into account while the $v_{ij}(t, x), (i \neq j)$ elements are derived from the interdependent state dynamics.

4.2.1 Mixed hierarchical decomposition

Consider the dynamical system described by the equation:

$$\frac{dx}{dt} = f(t, x), \qquad x(t_0) = x_0 \tag{4.6}$$

where $x \in R^n$, $f \in C(R_+ \times R^n, R^n)$ and the solution $x(t, t_0, x_0)$ exists for all initial values of $(t_0, x_0) \in R_+ \times R^n$. The equilibrium state $x = 0$, is the unique equilibrium state of the system (4.6), provided that $f(t, 0) = 0$ for all $t \in R_+$. Should the system have more than one equilibrium state, each equilibrium state is analysed separately within a bounded region around the equilibrium point. As discussed in Section 3.2, should the equilibrium point not lie at the origin, a simple linear transformation is used to translate the equilibrium points to the origin for further analysis. The system (4.6) can be interpreted as a physical composition of some systems or as a large scale system admitting mathematical decomposition into several free subsystems [10].

The independent subsystems are of the form:

$$\frac{dx_i}{dt} = g_i(t, x_i) \qquad i = 1,2, \dots, m \tag{4.7}$$

where $x_i \in R^{n_i}$ and $g_i \in C(R_+ \times R^{n_i}, R^{n_i})$. The following functions combine with (4.7) to form the original system (4.6). These *link functions* are:

$$h_i = (t, x_1, x_2, \dots, x_s) \quad (4.8)$$

where $h_i \in C(R_+ \times R^{n_i} \times \dots \times R^{n_s}, R^{n_i})$. Therefore, system (4.6) can be completely described by both its independent and link functions, i.e.

$$\frac{dx_i}{dt} = g_i(t, x_i) + h_i(t, x_1, x_2, \dots, x_s) \quad i = 1,2, \dots, m \quad (4.9)$$

The transformation of system (4.6) to system (4.9) is referred to as the *mathematical first level decomposition* of system (4.6).

For the second level decomposition, (i, j) couples for all $(i \neq j) \in [1, m]$ are singled out:

$$(i,j) \text{ couple} \quad \begin{cases} \dfrac{dx_i}{dt} = q_i(t, x_i, x_j) \\ \dfrac{dx_j}{dt} = q_j(t, x_i, x_j) \end{cases} \quad (4.10)$$

where, $x_i \in R^{n_i}$, $x_j \in R^{n_j}$, $q_i \in C(R_+ \times R^{n_i} \times R^{n_j}, R^{n_i})$, $q_i \in C(R_+ \times R^{n_i} \times R^{n_j}, R^{n_i})$. Without loss of generality, system (4.6) can be redefined as:

$$\frac{dx_i}{dt} = f_i(t, x), \quad i = 1,2, \dots, m \quad (4.11)$$

The second level link functions therefore have the following designations for all $(i \neq j) \in [1, m]$:

$$\begin{aligned} h_i^*(t, x_1, x_2, \dots, x_s) &= f_i(t, x) - q_i(t, x_i, x_j) \\ h_j^*(t, x_1, x_2, \dots, x_s) &= f_j(t, x) - q_j(t, x_i, x_j) \end{aligned} \quad (4.12)$$

where

$$h_i^* \in C(R \times R^{n_1} \times R^{n_2} \times \dots \times R^{n_s}, R^{n_i})$$
$$h_j^* \in C(R \times R^{n_1} \times R^{n_2} \times \dots \times R^{n_s}, R^{n_j})$$

System (4.6) expressed as interacting (i,j) couples of subsystems is therefore:

$$(i,j) \text{ couple } \begin{cases} \dfrac{dx_i}{dt} = q_i\big(t, x_i, x_j\big) + h_i^*(t, x_1, x_2, \dots, x_s) \\ \dfrac{dx_j}{dt} = q_j\big(t, x_i, x_j\big) + h_j^*(t, x_1, x_2, \dots, x_s) \end{cases} \quad (4.13)$$

where $\frac{dx_i}{dt}$ and $\frac{dx_j}{dt}$ are defined for all $(i \neq j) \in [1, s]$. The transformation of system (4.6) to system (4.13) is called the *mathematical second level decomposition* of system (4.6), where system (4.13) represents a *mixed hierarchical structure of subsystems.* Systems (4.7) and (4.10) provide an indication of an independent state's contribution towards its own time evolution, whereas systems (4.8) and (4.12) provide an indication of the dynamic interactions between states. This information proves to be extremely beneficial in the stability analysis of the system (4.6), and ultimately makes the construction of a hierarchical matrix Lyapunov function for the stability analysis of system (4.6) possible.

4.2.2 Hierarchical Matrix Function Structure

From the *mathematical first level decomposition* (4.9), the following functions are constructed:

$$v_{ii} \in C(R_+ \times R^{n_i}, R_+), \;\; i = 1,2, \dots, m \quad (4.14)$$

where $v_{ii}(t, 0) = 0$ for all $t \in R_+$ and functions v_{ii} are locally Lipschitzian in x_i. From the *mathematical second level decomposition* (4.13), the functions:

$$v_{ij} \in C(R_+ \times R^{n_i} \times R^{n_j}, R), \;\; (i \neq j) = 1,2, \dots, m \quad (4.15)$$

are constructed, where $v_{ij}(t, 0,0) = 0 \;\; \forall \;\; t \in R_+$ and functions v_{ij} are locally Lipschitzian in x_i and x_j. The matrix-valued function, $U(t, x)$ is formulated in terms of (4.14) and (4.15):

$$U(t, x) = \big[v_{ij}(t, \cdot)\big], \;\; i, j = 1,2, \dots m \quad (4.16)$$

where $U \in C(R_+ \times R^{n_i} \times R^{n_j}, R^{m\times m})$. The matrix-valued function in (4.16) is considered *hierarchical* since it takes first and second mathematical decomposition into account. The following conditions serve to constrain the matrix-valued function in order to formulate a suitable candidate Lyapunov function for the stability analysis of system (4.6). Lyapunov's first condition imposed on a candidate Lyapunov function, that of *positive definiteness*, is considered in Assumption 4.1 and expressed in terms of the elements (4.14) and (4.15) of matrix function (4.16)

ASSUMPTION 4.1. If there exists:

(i) an open, time-invariant neighbourhood $\mathcal{N}_i$ of the point $x_i = 0$, $\mathcal{N}_i \subseteq R^{n_i}$, $i = 1,2,\dots,m$

(ii) a vector $\eta \in R_+^m$, $\eta > 0$ and positive, semi-definite functions $u_i(x_i)$, $u_i(0) = 0$, $w_i(x_i)$, $w_i(0) = 0$, $i = 1,2,\dots,m$, positive constants $\underline{c}_{ii} > 0$, $\overline{c}_{ii} > 0$ and arbitrary constants $\underline{c}_{ij}$, $\overline{c}_{ij}$ $(i \neq j)$ such that:

(a) $\underline{c}_{ii} u_i^2(x_i) \le v_{ii}(t, x_i) \le \overline{c}_{ii} w_i^2(x_i)$ for all $(t, x_i) \in R_+ \times \mathcal{N}_i$

(b) $\underline{c}_{ij} u_i(x_i) u_j(x_j) \le v_{ij}(t, x_i, x_j) \le \overline{c}_{ij} w_i(x_i) w_j(x_j)$
for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_i \times \mathcal{N}_j$

then, if matrices

$$A = [\underline{c}_{ij}], \quad B = [\overline{c}_{ij}], \quad i,j = 1,2,\dots,m$$

are positive definite, then hierarchical matrix-valued function (4.16) is positive definite and decreasing.

Proof. The function $v(t,x,\eta) = \eta^T U(t,x)\eta$ expressed in component form is

$$v(t,x,\eta) = \sum_{i=1}^{m} \eta_i^2 v_{ii}(t,x_i) + \sum_{\substack{i,j=1 \\ i\neq j}}^{m} \eta_i \eta_j v_{ij}(t,x_i,x_j)$$

Under the conditions (ii)(a) and (ii)(b) of Assumption 4.1, the following estimation holds:

$$v(t,x,\eta) \geq \sum_{i=1}^{m} \eta_i^2 \underline{c}_{ii} u_i^2(x_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \eta_i \eta_j \underline{c}_{ij} u_i(x_i) u_j(x_j)$$

$$= \sum_{i=1}^{m} \left[\eta_i^2 \underline{c}_{ii} u_i^2(x_i) + \sum_{\substack{j=1 \\ i \neq j}}^{m} \eta_i \eta_j \underline{c}_{ij} u_i(x_i) u_j(x_j) \right]$$

$$= \eta_1^2 \underline{c}_{11} u_1^2(x_1) + \left[\eta_1 \eta_2 \underline{c}_{12} u_1(x_1) u_2(x_2) + \cdots + \eta_1 \eta_m \underline{c}_{1m} u_1(x_1) u_2(x_2) \right]$$
$$+ \eta_2^2 \underline{c}_{22} u_2^2(x_2) + \left[\eta_2 \eta_1 \underline{c}_{21} u_2(x_2) u_1(x_1) + \cdots + \eta_2 \eta_m \underline{c}_{2m} u_2(x_2) u_m(x_m) \right]$$

$$+ \cdots +$$

$$\eta_m^2 \underline{c}_{mm} u_m^2(x_m) + [\eta_m \eta_1 \underline{c}_{m1} u_m(x_m) u_1(x_1) + \cdots + \eta_m \eta_{m-1} \underline{c}_{m(m-1)}$$

$$u_m(x_m) u_{m-1}(x_{m-1})$$

Which can be rewritten in the form:

$$(u_1 \quad u_2 \quad \cdots \quad u_m) \begin{pmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_m \end{pmatrix} \begin{pmatrix} \underline{c}_{11} & \underline{c}_{12} & \cdots & \underline{c}_{1m} \\ \underline{c}_{21} & \underline{c}_{22} & \cdots & \underline{c}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{c}_{m1} & \underline{c}_{m2} & \cdots & \underline{c}_{mm} \end{pmatrix} \begin{pmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

By assigning $u = \left(u_1(x_1), \dots, u_m(x_m)\right)^T$, $H = diag(\eta_1, \dots, \eta_m)$ and $A = [\underline{c}_{ij}]$ $i,j = 1,2, \dots, m$, the above equation can be written as:

$$v(t,x,\eta) \geq u^T H^T A H u \tag{4.17}$$

Similarly, from the upper bound constraints:

$$v(t,x,\eta) \leq w^T H^T B H w \tag{4.18}$$

where $w = \left(w_1(x_1), \dots, w_m(x_m)\right)^T$ and B= $[\overline{c}_{ij}]$, $i,j = 1,2, \dots, m$.

Since u and w are positive semi-definite and H is positive definite according to Assumption 4.1, inequalities (4.17) and (4.18) would result in hierarchical matrix-valued function (4.16) being positive definite if and only if matrix A is positive definite and decreasing and matrix B is positive definite. □

4.2.3 Hierarchical Matrix Function Derivative

The second constraint imposed on a candidate Lyapunov function is the condition that its time derivative along the trajectories of the system is *negative semi-definite*, or in the case of asymptotic stability, *negative definite.* Assumptions 4.2 and 4.3 attempt to adapt this condition to the method of Lyapunov hierarchical matrix-valued functions.

ASSUMPTION 4.2.

The independent subsystems (4.7) from first level decomposition and their corresponding link functions (4.8) adhere to the following conditions:

(i) there exist functions $v_{ii} \in C(R_+ \times R^{n_i}, R_+)$, satisfying conditions of Assumption 4.1;

(ii) there exist functions $\varphi_i \in K$ and constants p_{ii}^0, μ_{ik}, $i, k = 1,2, \dots, m$ for which the following conditions are satisfied:

(a) $D_t^+ v_{ii} + (D_{x_i}^+ v_{ii}) g_i(t, x_i) \leq p_{ii}^0 \varphi_i^2(\|x_i\|)$

(b) $(D_{x_i}^+ v_{ii}) h_i(t, x) \leq \varphi_i(\|x_i\|) \sum_{k=1}^{m} \mu_{ik} \varphi_k(\|x_k\|)$

for all $(t, x) \in R_+ \times \mathcal{N}$, $\mathcal{N} = \mathcal{N}_1 \times \dots \times \mathcal{N}_m$

Note that if conditions (i) and (ii)(a) are satisfied with $p_{ii}^0 < 0$, $i \in [1, m]$ this implies uniform asymptotic stability of the *i*th independent subsystem. $p_{ii}^0 = 0$, implies uniform stability of the state $x_i = 0$, whereas $p_{ii}^0 > 0$ implies instability.

ASSUMPTION 4.3

From the second level decomposition, the independent (i,j) couples of subsystems (4.10) and their corresponding link functions (4.12) adhere to the following conditions:

(i) there exist functions $v_{ij} \in C(R_+ \times R^{n_i} \times R^{n_j}, R)$, which satisfy the conditions of Assumption 4.1

(ii) there exist functions $\varphi_i \in K$ and constants $p_{ij}^1, \ p_{ij}^2, \ p_{ij}^3, \ v_{kp}^{ij}$; $i,j,k,p = 1,2,\dots,m$ for which the following conditions are satisfied:

(a) $D_t^+ v_{ij} + (D_{x_{ij}}^+ v_{ij})^T q_{ij}(t, x_{ij}) \leq p_{ij}^1 \varphi_i^{\mathbf{2}}(\|x_i\|) + 2p_{ij}^2 \varphi_i(\|x_i\|)\varphi_j(\|x_j\|) + p_{ij}^3 \varphi_j^{\mathbf{2}}(\|x_j\|)$

(b) $\left(D_{x_{ij}}^+ v_{ij}\right)^T H_{ij}(t,x) \leq \sum_{k,p=1}^{s} v_{kp}^{ij} \varphi_k(\|x_k\|)\, \varphi_p(\|x_p\|)$

for all $(t, x_{ij}) \in R_+ \times \mathcal{N}_i \times \mathcal{N}_j$ for all $i \neq j$, $(i,j) = 1,2,\dots,m$

where $x_{ij} = \left(x_i, \ \ x_j\right)^T, \ D_{x_{ij}}^+ = \left(D_{x_i}^+, \quad D_{x_j}^+\right)^T$, for all $(i \neq j) \in [1,m]$;

$$q_{ij} = \left(q_i\left(t, x_i, x_j\right), \ \ q_j\left(t, x_i, x_j\right)\right)^T, \ \ H_{ij} = \left(h_i^*(t,x), \ \ h_j^*(t,x)\right)^T,$$

$$\mathcal{N}_i \subseteq R^{n_i}, \quad \mathcal{N}_j \subseteq R^{n_j}$$

For purposes of clarity, the mathematical operator $(\cdot)^{\mathbf{2}}$ is written in bold and not to be confused with the superscript, 2. Note that the dynamic properties of the (i,j) couples of subsystems (4.10) are captured in the following matrix:

$$Q_{ij} = \begin{pmatrix} p_{ij}^1 & p_{ij}^2 \\ p_{ij}^2 & p_{ij}^3 \end{pmatrix} \tag{4.19}$$

Namely, if matrix (4.19) is negative semi-definite (negative definite) for all $(i \neq j) \in [1,m]$, then states $x_i = x_j = 0$ of couples (i,j) of subsystems (4.10) are uniformly stable (uniformly asymptotically stable).

PROPOSITION 4.1 [10]

If all conditions of Assumptions 4.2 and 4.3 are satisfied, then for the function:

$$D^+v(t,x,\eta) = \eta^T D^+ U(t,x)\eta$$

estimate

$$D^+v(t,x,\eta) \leq \varphi^T(\|x\|)S\varphi(\|x\|) \tag{4.20}$$

holds, where $\varphi(\|x\|) = \left(\varphi_1(\|x_1\|), \dots, \varphi_m(\|x_m\|)\right)^T$, $S = \frac{1}{2}(B + B^T)$ and elements b_{ij} of matrix B are defined by:

$$b_{qq} = \eta_q^2\left(p_{qq}^0 + \mu_{qq}\right) + \eta_q\left(\sum_{\substack{i=1\\i\neq q}}^{m}\eta_i p_{qi}^1 + \sum_{\substack{j=1\\j\neq q}}^{m}\eta_j p_{jq}^3\right) + \sum_{i,j=1}^{m}\eta_i\eta_j v_{qq}^{ij}$$

$$b_{ql} = \eta_q^2\mu_{ql} + 2\eta_q\eta_l p_{ql}^2 + \sum_{\substack{i,j=1\\i\neq j}}^{m}\eta_i\eta_j v_{ql}^{ij} \tag{4.21}$$

Proof. The function $D^+v(t,x,\eta)$ in coordinate form reads

$$D^+v(t,x,\eta) = \sum_{i,j=1}^{s} D^+v_{ij}(t,\cdot)\eta_i\eta_j, \qquad i,j = 1,2,\dots,m \tag{4.22}$$

By substituting estimates from Assumptions 4.2 and 4.3 into (4.22), one obtains estimate (4.20) □

PROPOSITION 4.2 [10]

If in Assumptions 4.2 and 4.3, conditions (ii)(a) and (ii)(b) are satisfied with reverse sign and constants $\tilde{p}_{ii}^0,\ \tilde{\mu}_{ik},\ \tilde{p}_{ij}^1,\ \tilde{p}_{ij}^2,\ \tilde{p}_{ij}^3,\ \tilde{v}_{kp}^{ij},\ \ i,j,k,p = 1,2,\dots,m$, then for the function, $D^+v(t,x)$ the following estimate holds:

$$D^+v(t,x,\eta) \geq \varphi^T(\|x\|)\tilde{S}\varphi(\|x\|) \tag{4.23}$$

where $\tilde{S} = \frac{1}{2}\left(\tilde{B} + \tilde{B}^T\right)$ and elements $\tilde{b}_{ij}$ are defined similarly to elements b_{ij} of matrix B.

Proposition 4.2 adapts the instability theorem, Theorem 3.5, to be used in connection with the method of Lyapunov matrix-valued functions.

4.2.4 Stability and Instability Conditions

The stability and instability conditions of the state $x = 0$ of system (4.6) are established in terms of the conditions asserted in Assumptions 4.1, 4.2 and 4.3. Owing to the construction of the hierarchical matrix-valued function, as well as its derivative, the conditions imposed on the Lyapunov function are expressed as conditions on the constituent elements of the Lyapunov matrix-valued function.

THEOREM 4.1

Let vector function f in system (4.6) be continuous on $R_+ \times R^n$. If the following conditions are satisfied:

(i) all conditions of Assumptions 4.1, 4.2 and 4.3 hold
(ii) matrices A and B (see Assumption 4.1) are positive definite
(iii) matrix $S \in R^{m\times m}$ in inequality (4.20) is:
  (a) negative semi-definite;
  (b) negative definite;
(iv) matrix $\tilde{S} \in R^{m\times m}$ in inequality (4.23) is:
  (a) positive definite;

Then

(a) conditions (i), (ii), and (iii)(a) imply uniform stability of the state $x = 0$ of system (4.6)
(b) conditions (i), (ii), and (iii)(b) imply uniform asymptotic stability of the state $x = 0$ of system (4.6)
(c) conditions (i), (ii) and (iv)(a) imply instability of the state $x = 0$ of system (4.6)

If, in addition, in conditions of Assumptions 4.1, 4.2 and 4.3, $\mathcal{N}_i = R^{n_i}$, $i = 1,2, \ldots, m$, and the functions

$$u_i(x_i) = \varphi_i(\|x_i\|) \in KR, \qquad w_i(x_i) = \psi_i(\|x_i\|) \in KR$$

Then

(d) conditions (i), (ii), and (iii)(a) imply uniform stability in the whole (global uniform stability) of the state $x = 0$ of system (4.6)

(e) conditions (i), (ii), and (iii)(b) imply uniform asymptotic stability in the whole (global uniform asymptotic stability) of the state $x = 0$ of system (4.6)

Proof. Assertions (a) and (b) of Theorem 4.1 are derived directly from Theorem 3.2 as should these criteria be met, all the conditions of Theorem 3.2 are satisfied, namely, function $v(t,x)$ is positive definite and decreasing and its derivative $D^+v(t,x)$ is negative semi-definite (negative definite). Assertion (c) of Theorem 4.1 follows directly from Theorem 3.5. Assertions (d) and (e) follow from Theorem 3.2. □

Section 4.2.5 now illustrates the methodology employed to implement the above presented theorem, adapting the stability and instability conditions to nonlinear systems.

4.2.5 Methodology

Consider the 2nd order, nonlinear, autonomous system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_2^3 \end{aligned} \tag{4.24}$$

System (4.24) has one equilibrium point situated at $(0, 0)$. The stability of the origin is analysed by means of the *Lyapunov hierarchical matrix-valued function method.* For the diagonal elements (4.14) of the matrix function (4.16), construct auxiliary functions, $v_{ii}(x_i)$, as the quadratic forms

$$v_{ii}(x_i) = x_i^T P_{ii} x_i \tag{4.25}$$

By taking $P_{11} = P_{22} = diag(1,1)$,

$$v_{11}(x_1) = x_1^2, \quad v_{22}(x_2) = x_2^2 \tag{4.26}$$

For the nondiagonal elements (4.15), consider the bilinear form

$$v_{ij}(x_i, x_j) = v_{ij}(x_j, x_i) = x_i^T P_{ij} x_j$$

and arbitrarily choose,

$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{4.27}$$

Furthermore, for Assumptions 4.2 and 4.3, assign

$$\varphi_1(\|x_1\|) = |x_1| \quad \text{and} \quad \varphi_2(\|x_2\|) = |x_2|$$

The mathematical decomposition of system (4.24) into its independent and inter-dependent subsystems follows. From *first level decomposition*, the independent subsystems (4.7) are:

$$\begin{aligned} g_1(x_1) &= -3x_1 \\ g_2(x_1) &= -x_2 - x_2^3 \end{aligned} \tag{4.28}$$

and the corresponding link functions (4.8) are:

$$h_1(x) = x_2$$
$$h_2(x) = x_1 \tag{4.29}$$

For *second level decomposition* of system (4.24), there is only one (i,j) couple. The independent second level subsystem (4.10) is:

$$q_1(x_1,x_2) = -3x_1 + x_2$$
$$q_2(x_1,x_2) = x_1 - x_2 - x_2^3 \tag{4.30}$$

Notice that since system (4.24) is a second order system, the second level link function (4.12) is equal to zero. From condition (ii)(a) of Assumption 4.2,

$$2x_1(-3x_1) \le p_{11}^0 |x_1|^2$$

$$\Rightarrow\ -6x_1^2 \le p_{11}^0 |x_1|^2 \tag{4.31}$$

and

$$2x_2(-x_2 - x_2^3) \le p_{22}^0 |x_2|^2$$

$$\Rightarrow\ -2x_2^2 - 2x_2^4 \le p_{22}^0 |x_2|^2 \tag{4.32}$$

Choosing $p_{11}^0 = -6$ satisfies inequality (4.31). Since the stability of the origin is being investigated, x_1 and x_2 are arbitrarily close to zero in the neighbourhood surrounding the origin, resulting in the left-hand side of the inequality (4.32) being dominated by the term $-2x_2^2$. Therefore, by ignoring the high order $-4x_2^4$ term, inequality (4.32) becomes

$$-2x_2^2 \le p_{22}^0 |x_2|^2 \tag{4.33}$$

Choosing $p_{22}^0 = -2$ satisfies inequality (4.33). It is clear from the comments on Assumption 4.2 that since both p_{11}^0 and p_{22}^0 are negative definite, both independent subsystems are asymptotically stable. Now, considering condition (ii)(b) of Assumption 4.2

$$2x_1(x_2) \le \mu_{11}|x_1|^2 + \mu_{12}|x_1||x_2| \tag{4.34}$$

Choosing $\mu_{11} = 0$ and $\mu_{12} = 2$, satisfies inequality (4.34). For the second link function,

$$2x_2(x_1) \leq \mu_{21}|x_2||x_1| + \mu_{22}|x_2|^2 \tag{4.35}$$

Choosing $\mu_{21} = 2$ and $\mu_{22} = 0$ satisfies inequality (4.35).

From the mathematical second level decomposition, conditions (ii)(a) of Assumption 4.3 for $i = 1$ and $j = 2$ are,

$$\begin{aligned} &(x_1 \quad x_2)\begin{pmatrix} -3x_1 + x_2 \\ x_1 - x_2 - x_2^3 \end{pmatrix} \\ &\leq p_{12}^1|x_1|^2 + 2p_{12}^2|x_1||x_2| + p_{12}^3|x_2| \\ &\Rightarrow -3x_1^2 + 2x_1x_2 - x_2^2 - x_2^4 \\ &\leq p_{12}^1|x_1|^2 + 2p_{12}^2|x_1||x_2| + p_{12}^3|x_2|^2 \end{aligned} \tag{4.36}$$

Let $p_{12}^1 = -3$ and $p_{12}^2 = 1$. Now, inequality (4.36) simplifies to

$$-x_2^2 - x_2^4 \leq p_{12}^3|x_2|^2 \tag{4.37}$$

Since $|x_1|, |x_2| \ll 1$, the left-hand side of inequality (4.37) is again dominated by the $-x_2^2$ term, and therefore by ignoring the high order term, $-x_2^4$, (4.37) simplifies to

$$-x_2^2 \leq p_{12}^3|x_2|^2 \tag{4.38}$$

By letting $p_{12}^3 = -1$, inequality (4.38) is satisfied. It can be shown that owing to the symmetric nature of the matrix function (4.16), i.e. $v_{12} = v_{21}$,

$$p_{21}^1 = p_{12}^3 = -3$$

$$p_{21}^2 = p_{12}^2 = 1$$

$$p_{21}^3 = p_{12}^1 = -1$$

Note that from the comments on Assumption 4.3,

$$Q_{12} = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix} \tag{4.39}$$

and

$$Q_{21} = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} \tag{4.40}$$

which are both negative definite, thereby implying that the states $x_1 = x_2 = 0$ of the $(1,2)$ couple are uniformly asymptotically stable. Since all the constants satisfying Assumptions 4.2 and 4.3 have been obtained, the S matrix from inequality (4.20) is therefore given by

$$S = \begin{pmatrix} -12 & 4 \\ 4 & -4 \end{pmatrix} \tag{4.41}$$

with $\eta = (1,1)^T$. Matrix S in inequality (4.41) is negative definite with eigenvalues $\lambda_1 = -13.66$ and $\lambda_2 = -2.34$. Furthermore, the conditions in Assumption 4.1 are satisfied with the constants

$$\begin{aligned} \underline{c}_{11} = \overline{c}_{11} = \underline{c}_{22} = \overline{c}_{22} = 1 \\ \underline{c}_{12} = \overline{c}_{12} = \underline{c}_{21} = \overline{c}_{21} = 0 \end{aligned} \tag{4.42}$$

taking $u_i(x_i) = w_i(x_i) = |x_i|$ for $i = 1,2$. The A and B matrices in Assumption 4.1 are both the 2×2 identity matrix. By Theorem 4.1 condition (b) has been satisfied, thereby implying uniform asymptotic stability of the equilibrium point (0,0). Furthermore, since in Assumptions 4.1, 4.2 and 4.3, the time-invariant neighbourhood $\mathcal{N}$ surrounding the origin can be extended to R^n and the following assignments hold,

$$u_i(x_i) = \varphi(\|x_i\|) \in KR \text{ and } w_i(x_i) = \psi_i(\|x_i\|) \in KR$$

condition (d) of Theorem 4.1 is also satisfied, implying *global* uniform asymptotic stability of equilibrium point $(0,0)$. To verify these results, the phase portrait of system (4.23) is illustrated in Figure 4.1

The Lyapunov function for system (4.24), obtained from (4.26) and (4.27) is therefore given by, with $\eta = (1,1)^T$,

$$v(t,x,\eta) = (1 \quad 1)\begin{pmatrix} x_1^2 & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 & x_2^2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.43}$$

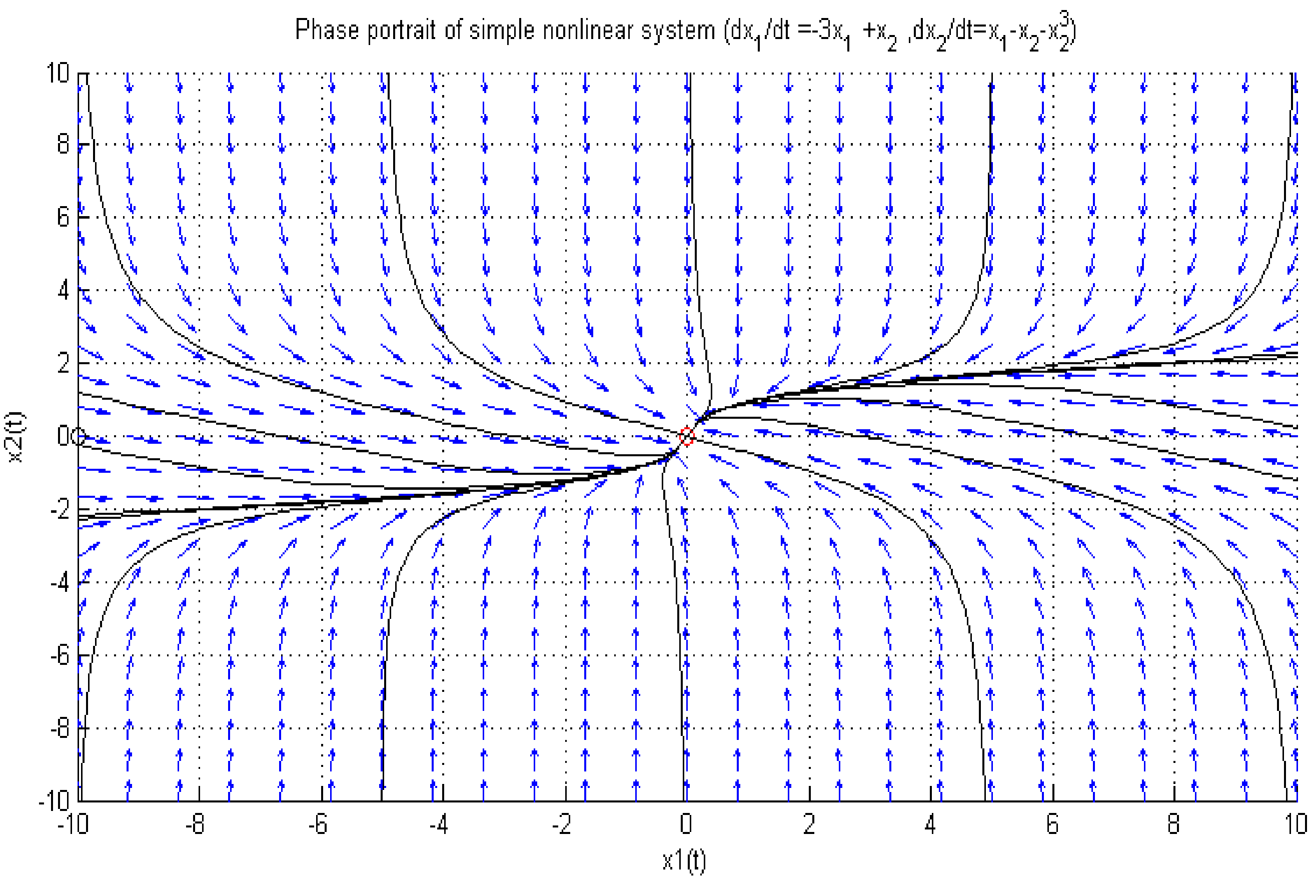


Figure 4.1. Phase portrait of system (4.24) illustrating the global asymptotic stability of the origin.

To draw a comparison between the development of a candidate Lyapunov function for system (4.24) from the hierarchical matrix-valued function approach and the conventional scalar Lyapunov function approach, consider the following conventional candidate Lyapunov function,

$$v(t,x) = ax_1^2 + bx_2^2 \tag{4.44}$$

where $a, b > 0$. The time derivative of (4.44) along the trajectories of system (4.24) is given by

$$\begin{aligned} \dot{v}(t,x) &= 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2 \\ &= -6ax_1^2 + 2(a+b)x_1x_2 - 2bx_2^2 - 2bx_2^4 \end{aligned} \tag{4.45}$$

By performing some rather involved mathematical manipulation, equation (4.45) can be rewritten as

$$\dot{v}(t,x) = -6a\left(x_1 - \frac{a+b}{6a}x_2\right)^2 + \frac{a-5b}{6a}x_2^2 - \frac{4b^2}{a}x_2^2 - 2bx_2^4 \tag{4.46}$$

By letting $a = 5$ and $b = 1$, equation (4.46) simplifies to

$$\begin{aligned} \dot{v}(t,x) &= -30\left(x_1 - \frac{1}{5}x_2\right)^2 - \frac{4}{5}x_2^2 - 2x_2^4 \\ &< 0 \quad \forall \quad x_1, x_2 \neq 0 \end{aligned} \tag{4.47}$$

Therefore, by Theorem 3.2, the equilibrium point (0,0) of system (4.24) is asymptotically stable. Since the Lyapunov function, $v(x) = 5x_1^2 + x_2^2$, is positive definite for all $x_1, x_2 \in R^n$, and its time derivative (4.47) is negative definite for all $x_1, x_2 \in R^n$, by Theorem 3.2, the origin of system (4.23) is globally asymptotically stable. This is the expected result based on the result obtained from the Lyapunov matrix-valued function method as well as from the phase portrait in Figure 4.1.

Here, the conventional Lyapunov function methodology required a fair deal of mathematical manipulation in order for it to be presented in an appropriate form for sign analysis. On the other hand, the hierarchical Lyapunov matrix-valued function method while still requiring a rather cumbersome amount of computation, does not require any degree of mathematical ingenuity, thereby making it straight-forward in its application. Note, by omitting the nondiagonal elements, $v_{ij}(x_i, x_j)$, which corresponds to the vector Lyapunov function approach [30] discussed in Section 3.6, the following S matrix is obtained

$$S = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix} \tag{4.48}$$

The S matrix in (4.48) is negative definite. Therefore, the vector Lyapunov function approach also determines the stability of system (4.24).

As another example, consider the nonlinear, 2nd order, autonomous system

$$\begin{aligned} \dot{x}_1 &= -15x_1 + \frac{15}{2}x_2 + x_1(2x_1 - x_2) \\ \dot{x}_2 &= -\frac{25}{2}x_2 + x_2\left(x_1 + \frac{4}{17}x_2\right) \end{aligned} \tag{4.49}$$

System (4.49) has four equilibrium points: $(0,0), (7.5,0), (8.5,17), (7.5\,,21.25)$. The stability of the origin is analysed by means of the *Lyapunov matrix valued function method.* For the diagonal elements (4.14) of the matrix function (4.16) arbitrarily choose the auxiliary functions

$$v_{11}(x_1) = x_1^2, \qquad v_{22}(x_2) = x_2^2 \tag{4.50}$$

and for the nondiagonal elements (4.15), arbitrarily choose:

$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = x_1^2 + x_2^2 \tag{4.51}$$

Furthermore, for Assumptions 4.2 and 4.3, assign

$$\varphi_1(\|x_1\|) = |x_1| \quad \text{and} \quad \varphi_2(\|x_2\|) = |x_2|$$

The mathematical decomposition of system (4.49) into its independent and inter-dependent subsystems follows. From *first level decomposition*, the independent subsystems (4.7) are,

$$\begin{aligned} g_1(x_1) &= -15x_1 + 2x_1^2 \\ g_2(x_1) &= -\frac{25}{2}x_2 + \frac{4}{17}x_2^2 \end{aligned} \tag{4.52}$$

and the corresponding link functions (4.8) are

$$\begin{aligned} h_1(x) &= -x_1x_2 + \frac{15}{2}x_2 \\ h_2(x) &= x_1x_2 \end{aligned} \tag{4.53}$$

For *second level decomposition* of system (4.49), there is only one (i,j) couple. The independent second level subsystem (4.10) is

$$
\begin{aligned}
q_1(x_1, x_2) &= -15x_1 + \frac{15}{2}x_2 + x_1(2x_1 - x_2) \\
q_2(x_1, x_2) &= -\frac{25}{2}x_2 + x_2\left(x_1 + \frac{4}{17}x_2\right)
\end{aligned} \tag{4.54}
$$

Notice that since system (4.49) is a second order system, the second level link function (4.12) is equal to zero. From condition (ii)(a) of Assumption 4.2,

$$
2x_1(-15x_1 + 2x_1^2) \leq p_{11}^0 |x_1|^2
$$

$$
\Rightarrow \; -30x_1^2 + 4x_1^3 \leq p_{11}^0 |x_1|^2 \tag{4.55}
$$

and

$$
2x_2\left(-\frac{25}{2}x_2 + \frac{4}{17}x_2^2\right) \leq p_{22}^0 |x_2|^2
$$

$$
\Rightarrow \; -25x_2^2 + \frac{8}{17}x_2^3 \leq p_{22}^0 |x_2|^2 \tag{4.56}
$$

Since the stability of the origin is being investigated, x_1 and x_2 are arbitrarily close to zero in the neighbourhood surrounding the origin, resulting in the left-hand side of the inequalities (4.55) and (4.56) being dominated by the terms $-30x_1^2$ and $-25x_2^2$ respectively. Therefore, by ignoring the $4x_1^3$ term in (4.55) and the $\frac{8}{17}x_2^3$ term in (4.56), inequalities (4.55) and (4.56) become

$$
-30x_1^2 \leq p_{11}^0 |x_1|^2 \tag{4.57}
$$

$$
-25x_2^2 \leq p_{22}^0 |x_2|^2 \tag{4.58}
$$

Choosing $p_{11}^0 = -30$ and $p_{22}^0 = -25$ satisfies both inequalities (4.57) and (4.58). It is clear from the comments on Assumption 4.2 that since both p_{11}^0 and p_{22}^0 are negative definite, both independent subsystems are asymptotically stable. Now, considering condition (ii)(b) of Assumption 4.2

$$
\begin{aligned}
&2x_1\left(-x_1x_2 + \frac{15}{2}x_2\right) \leq \mu_{11}|x_1|^2 + \mu_{12}|x_1||x_2| \\
&\Rightarrow -2x_1^2x_2 + 15x_1x_2 \leq \mu_{11}|x_1|^2 + \mu_{12}|x_1||x_2|
\end{aligned} \tag{4.59}
$$

Choosing $\mu_{12} = 15$, (4.59) becomes,

$$-2x_1^2x_2 \le \mu_{11}|x_1|^2 \quad (4.60)$$

Since $x_1^2 \equiv |x_1|^2$, (4.60) further simplifies to,

$$-2x_2 \le \mu_{11} \quad (4.61)$$

Now, since the stability around the origin is being investigated, let $|x_2| = 0.01$. Consider both cases where $x_2 > 0$ and $x_2 < 0$, therefore,

$$\begin{aligned} \mu_{11} &\ge -0.02 \ ; \ x_2 > 0 \\ \mu_{11} &\ge 0.02 \quad ; \ x_2 < 0 \end{aligned} \quad (4.62)$$

Therefore, by letting $\mu_{11} = 0.02$, it satisfies both inequalities (4.62). The value for μ_{11} could potentially provide an indication of the region of asymptotic stability. This notion is beyond the scope of this research report however, worth mentioning, for possible future research areas.

For the second link function,

$$\begin{aligned} 2x_2(x_1x_2) &\le \mu_{21}|x_2||x_1| + \mu_{22}|x_2|^2 \\ \Rightarrow 2x_1x_2^2 &\le \mu_{21}|x_2||x_1| + \mu_{22}|x_2|^2 \end{aligned} \quad (4.63)$$

Let $\mu_{22} = 0$. Inequality (4.63) simplifies to

$$2x_1x_2^2 \le \mu_{21}|x_2||x_1| \quad (4.64)$$

Now, consider the four cases, with $|x_1| = |x_2| = 0.01$.

$\boldsymbol{x_1} > 0\,, \boldsymbol{x_2} > 0$	$\boldsymbol{x_1} > 0\,, \boldsymbol{x_2} < 0$
$2|x_1||x_2|^2 \le \mu_{21}|x_2||x_1|$ $\mu_{21} \ge 2|x_2|$ $\mu_{21} \ge 0.02$	$2|x_1||x_2|^2 \le \mu_{21}|x_2||x_1|$ $\mu_{21} \ge 2|x_2|$ $\mu_{21} \ge 0.02$
$\boldsymbol{x_1} < 0\,, \boldsymbol{x_2} > 0$	$\boldsymbol{x_1} < 0\,, \boldsymbol{x_2} < 0$
$-2|x_1||x_2|^2 \le \mu_{21}|x_2||x_1|$ $\mu_{21} \ge -2|x_2|$ $\mu_{21} \ge -0.02$	$-2|x_1||x_2|^2 \le \mu_{21}|x_2||x_1|$ $\mu_{21} \ge -2|x_2|$ $\mu_{21} \ge -0.02$

From the above cases it is clear that $\mu_{21} = 0.02$ satisfies all four conditions.

From the mathematical second level decomposition, conditions (ii)(a) of Assumption 4.3 for $i = 1$ and $j = 2$ are,

$$(2x_1 \quad 2x_2)\begin{pmatrix} -15x_1 + 2x_1^2 - x_1x_2 + \frac{15}{2}x_2 \\ -\frac{25}{2}x_2 + x_1x_2 + \frac{4}{17}x_2^2 \end{pmatrix}$$
$$\leq p_{12}^1|x_1|^2 + 2p_{12}^2|x_1||x_2| + p_{12}^3|x_2| \tag{4.65}$$
$$-30x_1^2 + 4x_1^3 - 2x_1^2x_2 + 15x_1x_2 - 25x_2^2 + 2x_1x_2^2$$
$$+\frac{8}{17}x_2^3 \leq p_{12}^1|x_1|^2 + 2p_{12}^2|x_1||x_2| + p_{12}^3|x_2|^2$$

Let $p_{12}^1 = -30$ and $p_{12}^3 = -25$. Now,

$$4x_1^3 - 2x_1^2x_2 + 15x_1x_2 + 2x_1x_2^2 + \frac{8}{17}x_2^3 \leq 2p_{12}^2|x_1||x_2| \tag{4.66}$$

Since the stability of the origin is being investigated, $|x_1|, |x_2| \ll 1$, and therefore the left-hand side of inequality (4.66) is dominated by the $15x_1x_2$ term. By ignoring the other terms,

$$15x_1x_2 \leq 2p_{12}^2|x_1||x_2| \tag{4.67}$$

By letting $p_{12}^2 = \frac{15}{2}$, inequality (4.67) is satisfied. It can be shown that owing to the symmetric nature of the matrix function (4.16), i.e. $v_{12} = v_{21}$,

$$p_{21}^1 = p_{12}^3 = -25$$

$$p_{21}^2 = p_{12}^2 = \frac{15}{2}$$

$$p_{21}^3 = p_{12}^1 = -30$$

Therefore, all the constants satisfying Assumptions 4.2 and 4.3 have been obtained. The S matrix from inequality (4.20) is given by

$$S = \begin{pmatrix} -89.98 & 22.501 \\ 22.501 & -75 \end{pmatrix} \tag{4.68}$$

with $\eta = (1,1)^T$, which is negative definite with eigenvalues $\lambda_1 = -106.2$ and $\lambda_2 = -58.8$. Furthermore, the conditions in Assumption 4.1 are satisfied with the constants

$$\underline{c}_{11} = \overline{c}_{11} = \underline{c}_{22} = \overline{c}_{22} = 1$$
$$\underline{c}_{12} = \overline{c}_{12} = \underline{c}_{21} = \overline{c}_{21} = 0 \tag{4.69}$$

taking $u_i(x_i) = w_i(x_i) = |x_i|$ for $i = 1,2$. The A and B matrices in Assumption 4.1 are both the 2×2 identity matrix. By Theorem 4.1 condition (b) has been satisfied, thereby implying uniform asymptotic stability of the equilibrium point (0,0). To verify this result, the phase portrait of the system (4.42) is illustrated in Figure 4.2

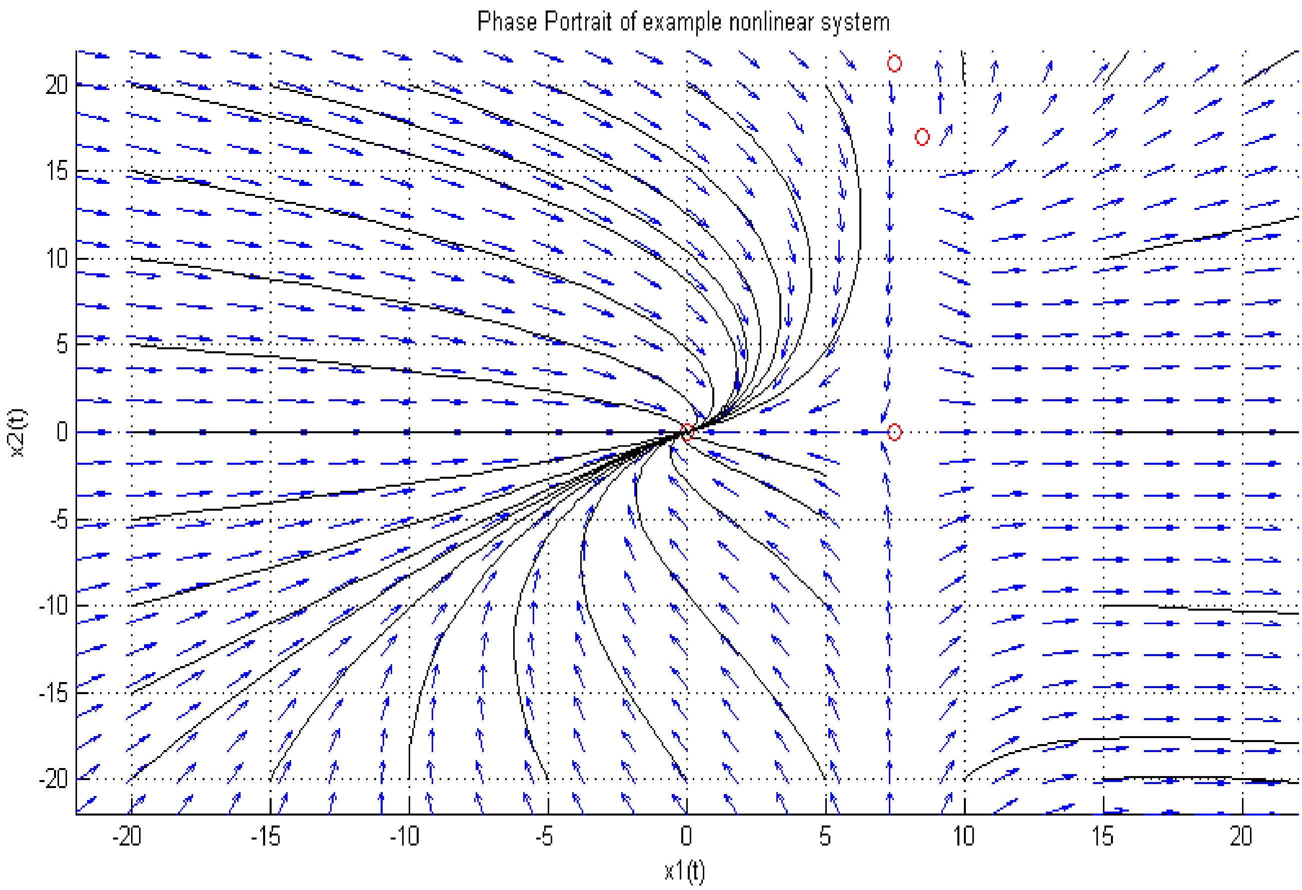


Figure 4.2. Phase portrait of system (4.49) illustrating the four equilibrium points (circles) as well as the asymptotic stability property of the origin.

Now, to draw a comparison between the method of Lyapunov matrix-valued functions and a conventional scalar Lyapunov function approach, as was done in the first example, consider the candidate Lyapunov function

$$v(x) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 \tag{4.70}$$

which corresponds to a candidate Lyapunov function obtained from *Aizerman's Method* [26]. The time derivative of (4.70) along the trajectories of (4.49) is given by

$$\dot{v}(x) = 2a_{11}x_1\dot{x}_1 + a_{12}(x_2\dot{x}_1 + x_1\dot{x}_2) + 2a_{22}x_2\dot{x}_2 \tag{4.71}$$

which simplifies to

$$\begin{aligned}\dot{v}(x) = {} & -30a_{11}x_1^2 + \left(\frac{15}{2}a_{12} - 25a_{22}\right)x_2^2 + 4a_{11}x_1^3 \\ & + \frac{8}{17}a_{22}x_2^3 + \left(15a_{11} - \frac{55}{2}a_{12}\right)x_1x_2 + 3a_{12}x_1^2x_2 \\ & + \left(-\frac{13}{17}a_{12} + 2a_{22}\right)x_1x_2^2\end{aligned} \tag{4.72}$$

Now, there exists the formidable task of choosing the appropriate values for a_{11}, a_{12}, and a_{22} such that not only $\dot{v}(x)$ in equation (4.72) is negative definite, but also the matrix

$$P = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \tag{4.73}$$

is positive definite. Clearly, this is no easy task and a fair amount of mathematical ingenuity and manipulation needs to be employed in order to obtain an appropriate form for sign analysis. Other classical scalar function approaches such as *Szego's method* [26], for example, would yield similar results. Therefore, a clear advantage of the Lyapunov matrix-valued function approach over traditional scalar Lyapunov function techniques can be seen.

In order to investigate the stability of the equilibrium point (7.5,0) using Lyapunov matrix-valued functions, the system (4.49) needs to undergo the linear transformation

$$\bar{x}_1 = x_1 - 7.5, \quad \bar{x}_2 = x_2$$

shifting the equilibrium point under investigation to the origin. Therefore, the linearly translated system is given by

$$\begin{aligned} \dot{x}_1 &= 15x_1 + x_1(2x_1 - x_2) \\ \dot{x}_2 &= -5x_2 + x_2\left(x_1 + \frac{4}{17}x_2\right) \end{aligned} \tag{4.74}$$

where the origin is now investigated for stability. By assigning the diagonal elements as before, i.e.

$$v_{11}(x_1) = x_1^2 \qquad v_{22}(x_2) = x_2^2$$

and the nondiagonal elements

$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = ax_1^2 + bx_2^2$$

From first level decomposition and Assumption 4.2 the following inequalities are obtained

(i) $2x_1(15x_1) \geq \tilde{p}_{11}^0|x_1|^2$

(ii) $2x_2(-5x_2) \geq \tilde{p}_{22}^0|x_2|^2$

(iii) $2x_1(-x_1x_2 + 2x_1^2) \geq \tilde{\mu}_{11}|x_1|^2 + \tilde{\mu}_{12}|x_1||x_2|$

(iv) $2x_2\left(x_1x_2 + \frac{4}{17}x_2^2\right) \geq \tilde{\mu}_{21}|x_2||x_1| + \tilde{\mu}_{22}|x_2|^2$

From (i) and (ii), it is fairly easy to obtain the constants $\tilde{p}_{11}^0 = 30$ and $\tilde{p}_{22}^0 = -10$. Both terms from the left-hand side of inequalities (iii) and (iv) are of order three, and since the stability of the origin is being investigated, $|x_1|, |x_2| \ll 1$, and the constants $\tilde{\mu}_{11}, \tilde{\mu}_{12}, \tilde{\mu}_{21}, \tilde{\mu}_{22} \approx 0$.

From second level decomposition and Assumption 4.3,

$$(2ax_1 \quad 2bx_2)\begin{pmatrix} 15x_1 - x_1x_2 + 2x_1^2 \\ -5x_2 + x_1x_2 + \frac{4}{17}x_2^2 \end{pmatrix}$$

$$= 30ax_1^2 - 2ax_1^2x_2 + 4ax_1^3 - 10bx_2^2 + 2bx_1x_2^2 + \frac{8}{17}bx_2^3$$
$$\geq \tilde{p}_{12}^1|x_1|^2 + 2\tilde{p}_{12}^2|x_1||x_2| + \tilde{p}_{12}^3|x_2|^2 \tag{4.75}$$

Since $|x_1|, |x_2| \ll 1$, the left-hand side of inequality (4.75) is dominated by the $-30ax_1^2$ and $-10bx_2^2$ terms. Therefore, inequality (4.75) simplifies to

$$30ax_1^2 - 10bx_2^2 \geq \tilde{p}_{12}^1|x_1|^2 + 2\tilde{p}_{12}^2|x_1||x_2| + \tilde{p}_{12}^3|x_2|^2 \tag{4.76}$$

Inequality (4.76) is satisfied by the constants,

$$\tilde{p}_{12}^1 = 30a \qquad \tilde{p}_{12}^2 = 0 \qquad \tilde{p}_{12}^3 = -10b$$

Matrix $\tilde{S}$ from inequality (4.23) is therefore given by

$$\tilde{S} = \begin{pmatrix} 30 + 60a & 0 \\ 0 & -10 - 20b \end{pmatrix} \tag{4.77}$$

with $\eta = (1,1)^T$. Clearly, from (4.77), in order for matrix $\tilde{S}$ to be positive definite,

$$30 + 60a > 0 \quad \Rightarrow \quad a > -\frac{1}{2}$$

$$-10 - 20b > 0 \quad \Rightarrow \quad b < -\frac{1}{2}$$

Choosing $a = 1$ and $b = -1$ satisfies the above inequalities. Since equation (4.69) still holds, matrices A and B from Assumption 4.1 are positive definite. Therefore, condition (c) from Theorem 4.1 is satisfied, implying that the equilibrium point (7.5,0) is unstable. This result can be verified from the phase portrait in Figure 4.2. A similar technique, whereby the remaining equilibrium points are translated to the origin and analysed separately, would be employed to investigate their stability.

Also note that had the vector Lyapunov function approach be employed, where the nondiagonal elements, $v_{12} = v_{21} = 0$, this would result in the following $\tilde{S}$ matrix,

$$\tilde{S} = \begin{pmatrix} 30 & 0 \\ 0 & -10 \end{pmatrix} \tag{4.78}$$

which is not positive definite, no discernable conclusion can be drawn about the stability of the equilibrium point (7.5,0) using this method. Since the Lyapunov matrix valued function approach allows for the pseudo-arbitrary development and assignment of both functions and constants, it provides a more adaptable and dynamic approach to nonlinear system stability analysis.

4.2.6 Simplifications

A number of simplifications can be made based on the structure of the Lyapunov matrix function as well as on the system under investigation. These simplifications can be grouped into three broad categories, namely *linear system simplifications*, *symmetry simplifications*, and *second order system simplifications.* The first simplification, derived from the application of Lyapunov matrix function method to linear systems, affects the methodology of the development of the matrix function itself. With nonlinear systems, the assignment of the diagonal, $v_{ii}(x_i)$, and the nondiagonal, $v_{ij}(x_i, x_j)$, elements is essentially an arbitrary assignment, where the necessary constants are found based on whether they satisfy the given constraints expressed in terms of the originally, arbitrarily chosen functions. However, in the case of linear systems, there exists a more structured and methodical approach to this assignment.

Consider the linear, autonomous system (originally presented in [10])

$$\dot{x} = Ax, \quad x(t_0) = x_0 \tag{4.79}$$

where $x \in R^3$ and

$$A = \begin{pmatrix} -3 & -2 & 2 \\ 3 & -4 & 1 \\ 3 & 3 & -4 \end{pmatrix}$$

For the independent first level decomposition,

$$\begin{aligned} g_1(x_1) &= -3x_1 \\ g_2(x_2) &= -4x_2 \\ g_3(x_3) &= -4x_3 \end{aligned} \tag{4.80}$$

and the first level link functions

$$
\begin{aligned}
h_1(x) &= -2x_2 + 2x_3 \\
h_2(x) &= 3x_1 + x_3 \\
h_3(x) &= 3x_1 + 3x_2
\end{aligned}
\tag{4.81}
$$

the following auxiliary functions are chosen

$$
v_{11}(x_1) = x_1^2, \qquad v_{22}(x_2) = x_2^2, \qquad v_{33}(x_3) = x_3^2 \tag{4.82}
$$

Therefore, from condition (ii)(a) of Assumption 4.2, the following constants can be obtained.

$$
\begin{aligned}
p_{11}^0 &= -6, & p_{22}^0 &= -8, & p_{33}^0 &= -8 \\
\mu_{11} &= 0, & \mu_{12} &= -4, & \mu_{13} &= 4 \\
\mu_{21} &= 6, & \mu_{22} &= 0, & \mu_{23} &= 2 \\
\mu_{31} &= 6, & \mu_{32} &= 6, & \mu_{33} &= 0
\end{aligned}
\tag{4.83}
$$

Note $\mu_{ii} = 0\ \forall\, i = 1,2,3$ since system (4.79) is linear. Now, instead of an arbitrary assignment of the nondiagonal elements, elements v_{12}, v_{13} and v_{23}[1] are derived from second level decomposition. The second level decomposition of system (4.79) is

$$
\begin{aligned}
&\textit{(1,2)}\text{ couple} & q_1(x_1,x_2) &= -3x_1 - 2x_2 \\
& & q_2(x_1,x_2) &= 3x_1 - 4x_2 \\
&\textit{(1,3)}\text{ couple} & q_1(x_1,x_3) &= -3x_1 + 2x_3 \\
& & q_3(x_1,x_3) &= 3x_1 - 4x_3 \\
&\textit{(2,3)}\text{ couple} & q_2(x_2,x_3) &= -4x_2 + x_3 \\
& & q_3(x_2,x_3) &= 3x_2 - 4x_3
\end{aligned}
\tag{4.84}
$$

which can be rewritten as

$$
\textit{(1,2)}\text{ couple} \qquad \begin{pmatrix} q_1(x_1,x_2) \\ q_2(x_1,x_2) \end{pmatrix} = B_{12} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

[1] Elements v_{21}, v_{31} and v_{32} do not need to be calculated since, $v_{21} = v_{12}$, $v_{31} = v_{13}$ and $v_{32} = v_{23}$

(1,3) couple $$\begin{pmatrix} q_1(x_1, x_3) \\ q_3(x_1, x_3) \end{pmatrix} = B_{13} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \tag{4.85}$$

(2,3) couple $$\begin{pmatrix} q_2(x_2, x_3) \\ q_3(x_2, x_3) \end{pmatrix} = B_{23} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

where

$$B_{12} = \begin{pmatrix} -3 & -2 \\ 3 & -4 \end{pmatrix}, \qquad B_{13} = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}, \qquad B_{23} = \begin{pmatrix} -4 & 1 \\ 3 & -4 \end{pmatrix}$$

For the nondiagonal elements, consider the auxiliary functions,

$$\begin{aligned} v_{12} = v_{21} &= (x_1 \quad x_2) P_{12} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_{13} = v_{31} &= (x_1 \quad x_3) P_{13} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \\ v_{23} = v_{32} &= (x_2 \quad x_3) P_{23} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{4.86}$$

where $P_{ij},\ i,j = 1,2,3\ \ i \neq j$ are determined from the following Lyapunov matrix equations

$$\begin{aligned} B_{12}^T P_{12} + P_{12} B_{12} &= -G_{12} \\ B_{13}^T P_{13} + P_{13} B_{13} &= -G_{13} \\ B_{23}^T P_{23} + P_{23} B_{23} &= -G_{23} \end{aligned} \tag{4.87}$$

with the arbitrary assignment

$$G_{12} = 252 diag(1,1), \qquad G_{13} = 84 diag(1,1), \qquad G_{23} = 104 diag(1,1)$$

where $diag(i,j)$ denotes the 2×2 diagonal matrix with i and j along its main diagonal. Any positive definite, symmetric matrix could be substituted for G_{ij}. These values are chosen as they yield rational elements for the corresponding B_{ij} matrices.

Therefore,

$$P_{12} = \begin{pmatrix} 43 & 1 \\ 1 & 31 \end{pmatrix}, \qquad P_{13} = \begin{pmatrix} 31 & 17 \\ 17 & 19 \end{pmatrix}, \qquad P_{23} = \begin{pmatrix} 19 & 8 \\ 8 & 15 \end{pmatrix}$$

which results in

$$
\begin{aligned}
v_{12} &= v_{21} = 43x_1^2 + 2x_1x_2 + 31x_2^2 \\
v_{13} &= v_{31} = 31x_1^2 + 34x_1x_3 + 19x_3^2 \\
v_{23} &= v_{32} = 19x_2^2 + 16x_2x_3 + 15x_3^2
\end{aligned}
\tag{4.88}
$$

Conditions (ii)(a) and (ii)(b) of Assumption 4.3 are satisfied by the following constants

$$
\begin{aligned}
&p_{12}^1 = -252, \quad p_{12}^2 = 0, \quad p_{12}^3 = -252 \\
&p_{13}^1 = -84, \quad p_{13}^2 = 0, \quad p_{13}^3 = -84 \\
&p_{23}^1 = -104, \quad p_{23}^2 = 0, \quad p_{23}^3 = -104 \\
&v_{12}^{12} = v_{21}^{12} = 0, \quad v_{13}^{12} = v_{31}^{12} = 87, \quad v_{23}^{12} = v_{32}^{12} = 33 \\
&v_{12}^{13} = v_{21}^{13} = -11, \quad v_{13}^{13} = v_{31}^{13} = 0, \quad v_{23}^{13} = v_{32}^{13} = 23 \\
&v_{12}^{23} = v_{21}^{23} = 81, \quad v_{13}^{23} = v_{31}^{23} = 69, \quad v_{23}^{23} = v_{32}^{23} = 0
\end{aligned}
\tag{4.89}
$$

$$
v_{11}^{12} = v_{22}^{12} = v_{33}^{12} = 0, \quad v_{11}^{13} = v_{22}^{13} = v_{33}^{13} = 0, \quad v_{11}^{23} = v_{22}^{23} = v_{33}^{23} = 0
$$

Note that the last row of constants, as well as the constant values $v_{12}^{12} = v_{21}^{12} = v_{13}^{13} = v_{31}^{13} = v_{23}^{23} = v_{32}^{23} = 0$ are a direct result of the linearity of the system (4.79). The values $p_{12}^2 = p_{13}^2 = p_{23}^2 = 0$ are a result of the calculated nondiagonal elements v_{ij}, derived from the Lyapunov matrix equations (4.87). Furthermore, owing to the symmetrical nature of the Lyapunov matrix function,

$$
\begin{aligned}
&p_{21}^1 = p_{12}^3, \quad p_{21}^2 = p_{12}^2, \quad p_{21}^3 = p_{12}^1 \\
&p_{31}^1 = p_{13}^3, \quad p_{31}^2 = p_{13}^2, \quad p_{31}^3 = p_{13}^1 \\
&p_{32}^1 = p_{23}^3, \quad p_{32}^2 = p_{23}^2, \quad p_{32}^3 = p_{23}^1 \\
&v_{ij}^{ij} = v_{ji}^{ij} = v_{ij}^{ji} = v_{ji}^{ji} \ \forall\, (i \neq j) \in [1,3]
\end{aligned}
\tag{4.90}
$$

In general, the symmetrical property simplification can be summarised as

$$
\begin{aligned}
&p_{ij}^1 = p_{ji}^3, \quad p_{ij}^2 = p_{ji}^2, \quad p_{ij}^3 = p_{ji}^1 \\
&v_{ij}^{ij} = v_{ji}^{ij} = v_{ij}^{ji} = v_{ji}^{ji} \quad \forall\, (i \neq j) \in [1,3]
\end{aligned}
\tag{4.91}
$$

Using the constants (4.90) and the symmetrical simplification (4.91), equation (4.21) simplifies to

$$b_{qq} = \eta_q^2 p_{qq}^0 + \eta_q \left(\sum_{\substack{i=1 \\ i \neq q}}^{m} \eta_i p_{qi}^1 + \sum_{\substack{j=1 \\ j \neq q}}^{m} \eta_j p_{jq}^3 \right)$$

$$b_{ql} = \eta_q^2 \mu_{ql} + 2\eta_q \eta_l p_{ql}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \eta_i \eta_j v_{ql}^{ij} \tag{4.92}$$

Note that should the methodology (4.87) be employed for the construction of the nondiagonal elements, and the system (4.79) be a second order system, the second equation of (4.92) becomes,

$$b_{ql} = \eta_q^2 \mu_{ql} \tag{4.93}$$

This dramatic simplification serves to shorten the computation of the S matrix. Using (4.92) and its simplification (4.93), the corresponding S matrix is

$$S = \begin{pmatrix} -678 & 141 & 317 \\ 141 & -720 & 116 \\ 317 & 116 & -384 \end{pmatrix} \tag{4.94}$$

It can be verified that matrix S is negative definite, and therefore by Theorem 4.1, system (4.79) is globally uniformly asymptotically stable.

By omitting all the nondiagonal elements in matrix function (4.16), i.e. $v_{ij} = v_{ji} = 0 \;\; i,j = 1,2,3$, which corresponds to the vector approach, the corresponding S_1 matrix is

$$S_1 = \begin{pmatrix} -6 & 1 & 5 \\ 1 & -8 & 4 \\ 5 & 4 & -8 \end{pmatrix} \tag{4.95}$$

which is negative definite. Therefore the vector Lyapunov function approach also determines the stability of system (4.79).

In contrast, consider another linear system of the form (4.79), with an A matrix,

$$A = \begin{pmatrix} 0 & -2 & -2 \\ 3 & -4 & 1 \\ 5 & 4 & -8 \end{pmatrix} \tag{4.96}$$

By assigning the diagonal elements in the same way as the previous example, i.e.

$$v_{11}(x_1) = x_1^2 \quad v_{22}(x_2) = x_2^2 \quad v_{33}(x_3) = x_3^2$$

and from first level decomposition, the following constants from Assumption 4.2 are obtained

$$p_{11}^0 = 0 \quad p_{22}^0 = -8 \quad p_{33}^0 = -8$$

$$\mu_{11} = 0 \quad \mu_{12} = -4 \quad \mu_{13} = -4$$

$$\mu_{21} = 6 \quad \mu_{22} = 0 \quad \mu_{23} = 2$$

$$\mu_{31} = 6 \quad \mu_{32} = 6 \quad \mu_{33} = 0$$

Now, by implementing the Lyapunov vector approach and omitting the nondiagonal elements, the following S_2 matrix satisfying the estimate (4.20) is given by

$$S_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -8 & 4 \\ 1 & 4 & -8 \end{pmatrix} \tag{4.97}$$

which has eigenvalues $\lambda_1 = -12$, $\lambda_2 = -4.45$ and $\lambda_3 = 0.45$. Matrix S_2 is *not* negative definite and therefore, the vector function approach *does not* determine the stability of the equilibrium point $x = 0$ of system (4.79). However, by assigning the nondiagonal elements values using the method employed in the first example, i.e. using the results obtained from second level decomposition, conditions (4.86) and (4.87) are satisfied with

$$P_{12} = \begin{pmatrix} 3.875 & -1 \\ -1 & 1.25 \end{pmatrix} \qquad P_{13} = P_{12} \qquad P_{23} = \begin{pmatrix} 2.375 & 1 \\ 1 & 1.875 \end{pmatrix} \tag{4.98}$$

where

$$G_{12} = 6diag(1,1) \qquad G_{13} = G_{12} \qquad G_{23} = 13diag(1,1) \tag{4.99}$$

From (4.92) and (4.93) the following matrix S_3 satisfying the estimate (4.20) is obtained,

$$S_3 = \begin{pmatrix} -24 & -13.75 & 1.75 \\ -13.75 & -46 & 26 \\ 1.75 & 26 & -46 \end{pmatrix} \qquad (4.100)$$

The matrix S_3 in (4.100) is negative definite and therefore, by Theorem 4.1, the equilibrium point $x = 0$ of system (4.79) with an A matrix defined in (4.96) is globally uniformly asymptotically stable.

The above example illustrates the power of the hierarchical Lyapunov matrix-valued function method over the method of Lyapunov vector functions. However, an underlying quandary still exists. Why would one attempt to investigate the stability of a *linear* system using the method of Lyapunov matrix-valued functions if there already exists a vast array of linear system stability analysis techniques, one of which being the simple and easily implementable, *eigenvalue analysis*, discussed in Section 3.3.1 - Lyapunov's Indirect Method? It seems rather cumbersome to perform this lengthy analysis if in just a few simple calculations, one can obtain the eigenvalues of a linear system and consequently, the overall system's global stability.

The key point when attempting to provide insight into a possible answer to this question, is the fact that both these methods are extremely problem specific. If one required a deeper, more insightful understanding of the internal dynamics of the system, then the Lyapunov matrix-valued function method would be implemented as, by means of the system decomposition, it does precisely that. This could be extremely beneficial in a control context, and is discussed further in Section 4.4. However, should one simply require the stability of a particular equilibrium point, indifferent to the system's dynamics, then eigenvalue analysis is ideal. Therefore, in order to determine the most appropriate method of stability analysis, one should perform a small degree of pre-analysis on the system as this will not only save time, but also yield a more appropriate result. This topic, as well as more involved discussions, are presented in Section 4.4.

4.3 Variations

There exist a number of variations on the hierarchical Lyapunov matrix-valued method found in the literature [9]. These variations differ predominantly from the hierarchical method, as well as from each other, in the manner by which the dynamic nonlinear system is decomposed. As a direct consequence, the construction of the diagonal and nondiagonal elements, as well as their respective time derivatives, differ, since these elements are derived directly from the corresponding decomposition technique. It was decided to present two of these variations in order to provide a well-rounded discussion on the method of Lyapunov matrix-valued functions. These variations include, for lack of more appropriate terminology, *large scale systems* and *overlapping decomposition.* Other variations, such as *regular hierarchical decomposition* exist, but will not be discussed further.

4.3.1 Large Scale Systems

A criticism of the method presented in Section 4.2 is that while it does present a more adaptable approach to Lyapunov function construction, it still requires a certain degree of arbitrary choice, particularly concerning the assignment of the nondiagonal elements. The following nonlinear system decomposition and resulting matrix-valued function construction remedies the above arbitrary assignment by providing a structured method of choosing the nondiagonal elements of the matrix-valued function.

Consider the following decomposition of system (4.6)

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(t, x_1, \dots, x_m), \qquad i = 1,2, \dots, m \tag{4.101}$$

where $x_i \in R^{n_i}$, $t \in R_+$, $f_i \in C(R^{n_i}, R^{n_i})$, $g_i \in C(R_+ \times R^{n_i} \times \dots \times R^{n_m}, R^{n_i})$.

Introduce the designation,

$$G_i(t,x) = g_i(t,x_1,\dots,x_m) - \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(t,x_i,x_j) \tag{4.102}$$

where $g_{ij}(t,x_i,x_j) = g_i(t,0,\dots,x_i,\dots,x_j,\dots,0)$ for all $(i\neq j)\in[1,m]$. System (4.101) can now be rewritten as

$$\frac{dx_i}{dt} = f_i(x_i) + \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(t,x_i,x_j) + G_i(t,x) \tag{4.103}$$

As was developed in Section 4.2.2, the decomposition methodology presented above is now adapted to the construction of the *estimates* of the diagonal, $v_{ii}(x_i)$, and nondiagonal, $v_{ij}(t,x_i,x_j)$, elements of both the Lyapunov matrix-valued function, as well as their time-derivatives along the solution trajectories. Assumption 4.4 is concerned with the estimates of the diagonal elements and their time derivatives; Assumption 4.5 presents the construction of the nondiagonal elements themselves and Assumption 4.6 develops estimates on both the derivatives of the diagonal and nondiagonal elements.

ASSUMPTION 4.4 If there exists

(i) an open, connected neighbourhood, $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$

(ii) functions, $v_{ii}\in C(R^{n_i},R_+)$, comparison functions φ_{i1}, φ_{i2} and ψ_i of class $\mathcal{K}$ and real numbers $\underline{c}_{ii}>0$, $\overline{c}_{ii}>0$ and γ_{ii} such that

  (a) $v_{ii}(x_i) = 0$ for all $(x_i = 0)\in\mathcal{N}_i$;

  (b) $\underline{c}_{ii}\varphi_{i1}^2(\|x_i\|) \leq v_{ii}(x_i) \leq \overline{c}_{ii}\varphi_{i2}^2(\|x_i\|)$;

  (c) $\left(D_{x_i}v_{ii}(x_i)\right)^T f_i(x_i) \leq \gamma_{ii}\psi_i^2(\|x_i\|)$ for all $x_i\in\mathcal{N}_i$, $i = 1,2,\dots m$

ASSUMPTION 4.5 If there exists

(i) an open, connected neighbourhood, $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$

(ii) functions, $v_{ij} \in C(R_+ \times R^{n_i} \times R^{n_j}, R)$, comparison functions $\varphi_{i1}, \varphi_{i2}$ of class $\mathcal{K}$, positive constants $(\eta_1, \ldots, \eta_m)^T \in R^m$, $\eta_i > 0$ and arbitrary constants $\underline{c}_{ij}, \overline{c}_{ij}$, $(i \neq j) \in [1, m]$ such that

(a) $v_{ij}(t, x_i, x_j) = 0$ for all $(x_i, x_j) = 0 \in \mathcal{N}_i \times \mathcal{N}_j$, $t \in R_+$, $(i \neq j) \in [1, m]$

(b) $\underline{c}_{ij}\varphi_{i1}^2(\|x_i\|)\varphi_{j1}^2(\|x_j\|) \leq v_{ij}(t, x_i, x_j) \leq \overline{c}_{ij}\varphi_{i2}^2(\|x_i\|)\varphi_{j2}^2(\|x_j\|)$

(c) $D_t v_{ij}(t, x_i, x_j) + \left(D_{x_i} v_{ij}(t, x_i, x_j)\right)^T f_i(x_i) + \left(D_{x_j} v_{ij}(t, x_i, x_j)\right)^T f_j(x_j) + \frac{\eta_i}{2\eta_j}\left(D_{x_i} v_{ii}(x_i)\right)^T g_{ij}(t, x_i, x_j) + \frac{\eta_j}{2\eta_i}\left(D_{x_j} v_{jj}(x_j)\right)^T g_{ji}(t, x_i, x_j) = 0$

ASSUMPTION 4.6 If there exists

(i) an open, connected neighbourhood, $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$

(ii) comparison functions $\psi_i \in \mathcal{K}$, $i = 1,2, \ldots, m$, real numbers α_{ij}^1, α_{ij}^2, α_{ij}^3, ν_{ki}^1, ν_{kij}^1, μ_{kij}^1, μ_{kij}^2, $i, j, k = 1,2, \ldots, m$, such that

(a) $\left(D_{x_i} v_{ii}(x_i)\right)^T G_i(t, x) \leq \psi_i(\|x_i\|)\sum_{k=1}^m \nu_{ki}^1 \psi_k(\|x_k\|) + R_1(\psi)$ for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_i \times \mathcal{N}_j$;

(b) $\left(D_{x_i} v_{ij}(t, \cdot)\right)^T g_{ij}(t, x_i, x_j) \leq \alpha_{ij}^1 \psi_i^2(\|x_i\|) + \alpha_{ij}^2 \psi_i(\|x_i\|)\psi_j(\|x_j\|) + \alpha_{ij}^3 \psi^2(\|x_j\|) + R_2(\psi)$ for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_i \times \mathcal{N}_j$

(c) $\left(D_{x_i} v_{ij}(t, \cdot)\right)^T G_i(t, x) \leq \psi_j(\|x_j\|)\ \sum_{k=1}^m \nu_{ijk}^2 \psi_k(\|x_k\|) + R_3(\psi)$ for all $(t, x_i, x_j) \in R_+ \times \mathcal{N}_i \times \mathcal{N}_j$

(d) $\left(D_{x_i} v_{ij}(t, \cdot)\right)^T g_{ik}(t, x_i, x_k) \leq$

$\psi_j(\|x_j\|)\left(\mu_{ijk}^1\psi_k(\|x_k\|)+\mu_{ijk}^2\psi_i(\|x_i\|)\right)+R_4(\psi)$ for all $(t,x_i,x_j)\in R_+\times\mathcal{N}_i\times\mathcal{N}_j$ and $k\neq j$

Here $R_s(\psi)$ are polynomials in $\psi=\left(\psi_1(\|x_1\|),\dots,\psi_m(\|x_m\|)\right)$ in a power higher than three, $R_s(0)=0$, $s=1,\dots,4$

For function

$$v(t,x,\eta)=\eta^T U(t,x)\eta=\sum_{i,j=1}^{m} v_{ij}(t,\cdot)\eta_i\eta_j \tag{4.104}$$

conditions (ii)(b) from Assumptions 4.4 and 4.5 can be reformulated by means of the bilateral estimate

$$u_1^T H^T \underline{C} H u_1 \leq v(t,x,\eta) \leq u_2^T H^T \overline{C} H u_2 \tag{4.105}$$

where

$$u_1=\left(\varphi_{11}(\|x_1\|),\dots,\varphi_{m1}(\|x_m\|)\right)^T,$$

$$u_2=\left(\varphi_{12}(\|x_1\|),\dots,\varphi_{m2}(\|x_m\|)\right)^T,$$

$$H=diag(\eta_1,\dots,\eta_m),$$

$$\underline{C}=\left[\underline{c}_{ij}\right],\qquad \overline{C}=\left[\overline{c}_{ij}\right],\qquad i=1,2,\dots,m$$

which holds true for all $(t,x)\in R_+\times\mathcal{N}$, $\mathcal{N}=\mathcal{N}_i\times\dots\times\mathcal{N}_m$. From conditions (ii)(c) from Assumptions 4.4 and 4.5 and conditions (ii)(a)-(d) from Assumption 4.6, the following estimate on the time derivative of (4.88) along the solution trajectories of (4.6) is established

$$Dv(t,x,\eta)|_{(4.5)} \leq u_3^T M u_3 \tag{4.106}$$

where $u_3=\left(\psi_1(\|x_1\|),\dots,\psi_m(\|x_m\|)\right)^T$ holds for all $(t,x)\in R_+\times\mathcal{N}$.

Elements σ_{ij}, $i,j=1,\dots,m$ of matrix M in inequality (4.106) are given by

$$\sigma_{ii} = \eta_i^2 \gamma_{ii} + \eta_i^2 v_{ii} + \sum_{\substack{k=1 \\ k \neq i}}^{m} \left(\eta_k \eta_i v_{kii}^2 + \eta_i^2 v_{kii}^2 \right) + 2 \sum_{\substack{j=1 \\ j \neq i}}^{m} \eta_i \eta_j \left(\alpha_{ij}^1 + \alpha_{ji}^3 \right);$$

$$\sigma_{ij} = \frac{1}{2} \left(\eta_i^2 v_{ji}^1 + \eta_j^2 v_{ij}^1 \right) + \sum_{\substack{k=1 \\ k \neq j}}^{m} \eta_k \eta_j v_{kij}^2 + \sum_{\substack{k=1 \\ k \neq i}}^{m} \eta_i \eta_j v_{kij}^2 + \eta_i \eta_j \left(\alpha_{ij}^2 + \alpha_{ji}^2 \right)$$

$$+ \sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{m} \left(\eta_k \eta_j \, \mu_{kji}^1 + \eta_i \eta_j \mu_{ijk}^2 + \eta_i \eta_k \mu_{kij}^1 + \eta_i \eta_j \mu_{jik}^2 \right),$$

$$i, j = 1, 2, \ldots, m, \qquad i \neq j$$

The stability of the equilibrium state $x = 0$ of system (4.6) is now established in terms of the sign definiteness property of the matrices $\underline{C}$, $\overline{C}$, and M.

THEOREM 4.2

Assume that the perturbed motion equation (4.6) adhere to all the conditions of Assumptions 4.4, 4.5 and 4.6. Furthermore, if

(i) matrices $\underline{C}$ and $\overline{C}$ in Assumption 4.4 are positive definite and

(ii) matrix M in inequality (4.90) is negative semi-definite (negative definite)

Then the equilibrium state $x = 0$ of system (4.6) is *uniformly stable* (*uniformly asymptotically stable*). If, in addition, for the conditions of Assumptions 4.4 - 4.6 all estimates are satisfied for $\mathcal{N}_i = R^{n_i}$ and comparison functions $(\varphi_{i1}, \varphi_{i2}, \psi_i) \in$ $\mathcal{KR}$-class, then the equilibrium state $x = 0$ of system (4.6) is *globally uniformly stable* (*globally uniformly asymptotically stable*).

Proof. If all the conditions of Assumptions 4.4 and 4.5 are satisfied, then it is possible to construct the function $v(t, x, \eta)$ for system (4.101) which together with its total derivative $Dv(t, x, \eta)$ satisfies the inequalities (4.105) and (4.106). Assertion (i) of Theorem 4.2 implies that function $v(t, x, \eta)$ is positive definite and decreasing for all $t \in R_+$. Assertion (ii) of Theorem 4.2 implies function

$Dv(t, x, \eta)$ is negative semi-definite (definite). Therefore, the conditions of Theorem 3.2 are satisfied. The proof of the second part of Theorem 4.2 is also based on Theorem 3.2. □

This variation of the hierarchical Lyapunov matrix-valued function approach presents a sufficient extension to the stability analysis of *linear* systems however, does not present a significant improvement to the stability analysis of *nonlinear* systems as condition (ii)(c) in Assumption 4.5, where a systematic procedure is presented to obtain the nondiagonal elements, is very complicated to solve. Here, a nonlinear algebraic equation is formulated which may potentially have *no* solution. Therefore, no significant improvement is established from this particular decomposition methodology for the stability analysis of nonlinear systems.

4.3.2 Overlapping Decomposition

This particular methodology can be thought of as an extension to the decomposition technique introduced in Section 4.3.2. To illustrate this decomposition methodology, consider the dynamical linear system

$$\frac{dx}{dt} = Ax, \qquad x(t_0) = x_0 \tag{4.107}$$

where $x(t) \in R^n$, $t \in R_+$, and A is a $n \times n$ constant matrix. Vector x is divided into three subvectors x_i, $i = 1,2,3$ such that $x = (x_1^T, x_2^T, x_3^T)^T \in R^n$ and $x_i \in R^{n_i}$. Matrix A of system (4.107) is represented in the form

$$A = [A_{ij}], \qquad i, j = 1,2,3 \tag{4.108}$$

where submatrices $A_{ij} \in R^{n_i} \times R^{n_j}$. Now, the three components of vector x are transformed into two components of vector y by

$$y_1 = (x_1^T, x_2^T)^T, \qquad y_2 = (x_2^T, x_3^T)^T \tag{4.109}$$

By means of the linear nondegenerate transform,

$$y = Tx \tag{4.110}$$

where T is a $\tilde{n} \times n$ matrix of the form

$$T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

I_1, I_2 and I_3 are identity matrices whose dimensions correspond to the dimensions of subvectors x_1, x_2 and x_3 respectively. System (4.107) therefore reduces to

$$\begin{aligned} \frac{dy_1}{dt} &= \tilde{A}_{11}y_1 + \tilde{A}_{12}y_2 \\ \frac{dy_2}{dt} &= \tilde{A}_{21}y_1 + \tilde{A}_{22}y_2 \end{aligned} \qquad (4.111)$$

where $\tilde{A} = [\tilde{A}_{ij}],\ \ i, j = 1,2,3$ with

$$\tilde{A}_{11} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad \tilde{A}_{12} = \begin{pmatrix} 0 & A_{13} \\ 0 & A_{23} \end{pmatrix}$$

$$\tilde{A}_{22} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \qquad \tilde{A}_{21} = \begin{pmatrix} A_{21} & 0 \\ A_{31} & 0 \end{pmatrix}$$

Note that

$$\tilde{A} = TAT^I + M \qquad (4.112)$$

where

$$T^I = \begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_2 & \frac{1}{2}I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & \frac{1}{2}A_{12} & -\frac{1}{2}A_{12} & 0 \\ 0 & \frac{1}{2}A_{22} & -\frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{22} & \frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{32} & \frac{1}{2}A_{32} & 0 \end{pmatrix}$$

This method of decomposition, in conjunction with the Lyapunov vector function, was first introduced by Ikeda and Šiljak [31].

DEFINITION 4.1. OVERLAPPING DECOMPOSITION SYSTEM EXTENSION. System (4.111) is an *extension of system* (4.107) if there exists a linear transformation of maximal rank (4.110) such that for $y_0 = Tx_0$

$$x(t, x_0) = T^I y(t, y_0), \qquad t \geq t_0 \tag{4.113}$$

where $y(t, t_0)$ is a solution of system (4.111)

It is proved (Ikeda and Šiljak [31]) that system (4.111) is an extension of system (4.107) in the sense of Definition 4.1 iff one of the conditions, $MT = 0$ or $T^I M = 0$ is satisfied.

For the matrix function construction, consider system (4.111). Since this is a linear system, the linear system simplification, developed in Section 4.3 can be employed. For the diagonal elements, $v_i(y_i) = y_i^T P_{ii} y_i$, matrices P_{ii} are found by solving the Lyapunov matrix equations

$$\tilde{A}_{ii}^T P_{ii} + P_{ii} \tilde{A}_{ii} = G_{ii}, \qquad i = 1,2 \tag{4.114}$$

The nondiagonal elements, $v_{ij}(y_i, y_j) = y_i^T P_{ij} y_j$, are found by solving the linear system simplification of condition (ii)(c) of Assumption 4.5 for P_{12}. This condition is given as,

$$\tilde{A}_{11}^T P_{12} + P_{12} \tilde{A}_{22} = -\frac{\eta_1}{\eta_2} P_{11} \tilde{A}_{12} - \frac{\eta_2}{\eta_1} \tilde{A}_{21}^T P_{22} \tag{4.115}$$

where $\eta_1, \eta_2 > 0$. For both diagonal and nondiagonal elements, the following estimates hold

$$\begin{aligned} v_{11}(y_1) &\geq \lambda_m(P_{11}) \|y_1\|^2 \\ v_{22}(y_2) &\geq \lambda_m(P_{22}) \|y_2\|^2 \\ v_{12}(y_1, y_2) &\geq -\lambda_M^{\frac{1}{2}}(P_{12} P_{12}^T) \|y_1\| \|y_2\| \end{aligned} \tag{4.116}$$

where $\lambda_m(P_{11})$, and $\lambda_m(P_{22})$ are the minimum eigenvalues of matrices P_{11} and P_{22} respectively, and $\lambda_M^{\frac{1}{2}}(\cdot)$ is the norm of matrix $P_{12} P_{12}^T$, coordinated with the vector norm. Based on the above estimates, function

$$v(y,\eta) = \eta^T U(y)\eta, \qquad \eta \in R_+^2, \qquad \eta > 0 \tag{4.117}$$

satisfies the estimate

$$v(y,\eta) \geq u^T H^T C H u \tag{4.118}$$

where $u = (\|y_1\|, \|y_2\|)^T$, $H = diag(\eta_1, \eta_2)$ and

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{\frac{1}{2}}(P_{12}P_{12}^T) \\ \lambda_M^{\frac{1}{2}}(P_{12}P_{12}^T) & \lambda_m(P_{22}) \end{pmatrix} \tag{4.119}$$

It follows that for all $(y_1, y_2) \in R^{n_1} \times R^{n_2}$ the inequality

$$Dv(y,\eta) \leq u^T S u \tag{4.120}$$

holds, where the matrix $S = [\sigma_{ij}]$, $i,j = 1,2$ has the elements

$$\sigma_{11} = \lambda\eta_1^2 + \varkappa\eta_1\eta_2, \qquad \sigma_{22} = \beta\eta_2^2 + \chi\eta_1\eta_2, \qquad \sigma_{12} = \sigma_{21} = 0$$

where

$$\lambda = \lambda_M(\tilde{A}_{11}^T P_{11} + P_{11}\tilde{A}_{11}), \qquad \beta = \lambda_M(\tilde{A}_{22}^T P_{22} + P_{22}\tilde{A})$$

$$\varkappa = \lambda_M(P_{12}\tilde{A}_{21} + \tilde{A}_{21}^T P_{12}^T), \qquad \chi = \lambda_M(\tilde{A}_{12}^T P_{12} + P_{12}^T\tilde{A}_{12})$$

Estimates (4.118) and (4.120) now enables one to establish the necessary conditions for the stability analysis of the equilibrium point $x = 0$ of system (4.107).

THEOREM 4.3 Assume

(i) system (4.111) is the extension of system (4.107) in the sense of Definition 4.1

(ii) there exist solutions to the algebraic equations (4.114) and (4.115);

(iii) in estimate (4.118), matrix C is positive definite;

(iv) in estimate (4.120), matrix S is negative semi-definite (negative definite)

Then the equilibrium state $x = 0$ is stable (asymptotically stable).

Proof. According to condition (i) of Theorem 4.3, system (4.111) is an extension of system (4.107) according to Definition 4.1. Therefore, to investigate the stability of system (4.107) it is sufficient to investigate the stability of system (4.111). Under Assertion (ii), one can construct the diagonal and nondiagonal elements of the matrix-valued function which, under condition (iii), make (4.117) positive definite. From condition (iv) of Theorem 4.3, the total derivative of function (4.117) is negative semi-definite (negative definite). Thus, all the conditions of Theorem 3.2 are satisfied for the stability of the equilibrium state $y = 0$ of system (4.95) and therefore this is sufficient for the stability of the equilibrium state $x = 0$ of system (4.107) □

While this method claims to be a powerful extension to the method of Lyapunov matrix-valued functions, the overlapping decomposition of nonlinear systems proves to be messy and computationally cumbersome in its practical implementation. Therefore, while this variation is extremely useful in the stability analysis of linear systems, effectively decrementing the system order by one, it provides no significant benefit to the stability analysis of nonlinear systems, and is merely shown to provide an overall view of the differing methods of decomposition and corresponding matrix function construction.

4.4 Discussion and Critical Analysis

Section 4.2 introduces the method of hierarchical Lyapunov matrix-valued functions for the stability analysis of nonlinear systems. This method is then supplemented by its linear system simplification in Section 4.2.6 and by two variations on the method, in Section 4.3. A number of observations can be made with regards to this method's applicability to nonlinear systems, as well as its performance in terms of pre-existing Lyapunov function construction methodologies. Two fundamental advantages over classical scalar Lyapunov function construction are evident, namely, *arbitrary assignment reduction* and the use of *decomposition* in the matrix-valued function construction, discussed in Sections 4.4.1 and 4.4.2 respectively. While this method does present a number of improvements over classical techniques, it still presents some disadvantages, which will be discussed in Section 4.4.3

4.4.1 Arbitrary Assignment Reduction

One of the major advantages of the Lyapunov matrix-valued function method is that while one needs to be moderately mathematically inept for its adequate implementation, it serves to reduce the monumental task of arbitrary scalar function selection to one of pseudo-arbitrary choice of constants, required to satisfy a set of less stringent constraints to that of the classical scalar Lyapunov function. Prior to the development of this method, should one be required to analyse the stability of a general nonlinear system, which needn't be particularly complex, according to Lyapunov's direct method, one needs to obtain a positive definite function which has a negative semi-definite time derivative along the trajectories of the original system. Without expert knowledge of the particular process, or possibly years of experience, this function is extremely difficult, and in most cases, impossible to obtain. The more complicated the nonlinear process, the more difficult the task is of finding this elusive energy function.

It is a considerably easier task to obtain arbitrary constants which satisfy a set of more relaxed constraints, than obtaining the above mentioned Lyapunov function. Coupled with system decomposition, obtaining an appropriate Lyapunov function for the stability analysis of the general nonlinear system is no longer reserved for experts in the field, but rather anyone who has the need. Furthermore, as seen in the second example in Section 4.2.5, by applying *Aizerman's method* in the stability analysis of the origin, a candidate scalar Lyapunov function could simply not be obtained whereas the Lyapunov matrix-valued function method managed to obtain such a function, proving to be a more powerful technique in the stability analysis of nonlinear systems. Furthermore, while the method of vector Lyapunov functions also provides a method by which to relax the constraints imposed on a classical Lyapunov function, it is not as universally applicable as that of the Lyapunov matrix-valued function, as seen in the second example in Section 4.2.5, as well as the linear system (4.96).

4.4.2 Decomposition

The method of decomposing a nonlinear system into its constituent independent and inter-dependent parts not only provides a structured framework upon which the diagonal and nondiagonal elements are constructed, it too serves to divulge certain dynamic characteristics about the nonlinear system under investigation, which would otherwise remain dormant if implementing the classical scalar Lyapunov function approach. It is for this reason that the nonlinear system examples were supplemented with a number of linear systems examples as, unlike the conventional linear system stability approach, that of *eigenvalues analysis,* this method provides additional insight into the internal structure of these systems. By exploiting this information, one may not only obtain an equilibrium state's stability, but also potentially utilise this information in a control context.

By means of decomposition, one may be able to isolate a specific problematic nonlinear, or quite possibly, linear term in an open-loop system which either renders the system unstable, or introduces undesirable characteristics, and consequently be more likely to develop an optimised controller to either cancel out this term, or possibly, control it. In this way, the potential for the development of a *control Lyapunov matrix-valued function* is evident, where its implementation in overall system stabilisation and possible control of both nonlinear and linear systems would be extremely powerful as it exploits knowledge of the internal dynamic structure of the system being controlled.

Another advantage of system decomposition is that of its applicability to large scale systems. This method proves to be particularly powerful in its application to large-scale *linear* systems, as the investigation of the stability of such systems by means of Lyapunov matrix-valued functions proves to be less computationally expensive, the larger the system, than that of conventional linear system stability, eigenvalues analysis. Consider the large-scale linear system,

$$\begin{aligned}\frac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ \frac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ \frac{dx_3}{dt} &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3\end{aligned} \tag{4.121}$$

where $x_i \in R^{n_i}$, $i = 1,2,3$ and the constant matrix $A_{ij} \in R^{n_i \times n_j}$, $i, j = 1,2,3$. The stability of system (4.121) is obtained from the estimates on the diagonal and nondiagonal elements, which are derived directly from the first and second level decomposition of system (4.121). Depending on whether one decides to implement the hierarchical matrix valued function method, or its large-scale system variant, this system's stability is obtained directly from its dynamic equations (4.121), irrespective of the size x_i and A_{ij}. In contrast, if investigating the stability of system (4.121) by means of conventional eigenvalue analysis, one needs to calculate all nine eigenvalues, or in the case of the general $n \times n$ linear system, all n eigenvalues, in order to obtain the stability of the equilibrium point $x = 0$. Clearly, a reduction in computational complexity is evident between the Lyapunov matrix-valued function approach and conventional linear system stability analysis, where the greater the system order, the greater the reduction.

4.4.3 Disadvantages

Throughout the course of the research assignment, two major disadvantages of the method of Lyapunov matrix-valued functions are frequently visited in the literature, as well as the practical implementation. The first is one which, quite naively, was presented as an advantage in Section 4.4.1, that of *arbitrary assignment reduction*. The problem however, is that this method provides a reduction in the complexity of the arbitrary function assignment, but does not eliminate it. This method still leaves an element of arbitrary choice. For non-experts in the required field, this may prove detrimental to the adequate stability investigation of the nonlinear system in question as, owing to the sufficiency, but not necessity, condition of Lyapunov's direct method, the inability to obtain an appropriate Lyapunov function does not infer instability of the system's equilibrium state, and therefore, no conclusion can be drawn concerning the stability of the equilibrium state.

The first variation of the Lyapunov matrix-valued function method, introduced in Section 4.3.1, that of large scale systems, attempts to solve this problem by providing a methodical manner in which to obtain the nondiagonal elements of the matrix function. The method of obtaining these nondiagonal elements is by solving an algebraic equation, given in condition (ii)(c) of Assumption 4.5. This leads into the second fundamental drawback of this method, that of, mathematical elegance does not translate into practical realisation. The method of hierarchical Lyapunov matrix-valued functions while mathematically neat in theory, proves to be practically challenging in its application. In general, this conclusion was drawn for the application of the Lyapunov matrix-valued function method, or any of its variants, to the stability analysis of *nonlinear* systems. It was found that examples in the literature tend to avoid the analysis of nonlinear systems for, one suspects, this very reason. Therefore, while the large scale system variant attempts to remedy the arbitrary function assignment problem, it tends to augment the practical unrealisability of the method, thereby causing more of a hindrance than a solution.

Chapter 5

Case Study –
The Buckling Beam

5.1 The Buckling Beam – Due to Euler

Consider a thin metal bar, fixed at one end, under the effect of centric axial loading.

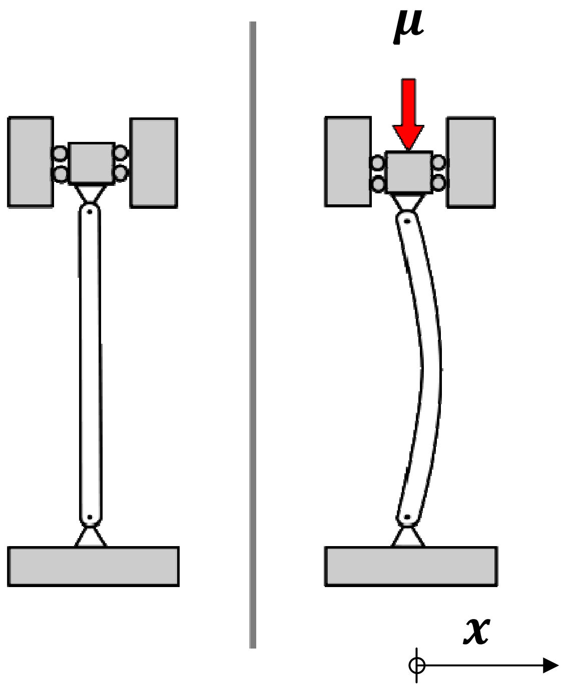


Figure 5.1. The Buckling Beam system

When the applied axial force, μ, is small, corresponding to the left diagram in Figure 5.1, the bar is slightly compressed, but unbuckled. In this configuration, should the bar be pushed to one side and released, it will oscillate back and forth about the axis of loading, assuming little to no damping. Under a heavier load however, it will buckle, as seen in the right diagram in Figure 5.1. Assuming the beam is symmetric about the axis of loading, there are two symmetric buckled states, around which the beam oscillates, should it be perturbed. To mathematically model this qualitative behaviour, *Euler* developed the dynamic equation,

$$m\ddot{x} + d\dot{x} - \mu x + \lambda x + x^3 = 0 \tag{5.1}$$

where m represents the mass of the beam, x represents the one-dimensional deflection of the beam, normal to the axis of loading, μ is the applied axial force, $\lambda x + x^3$ models the restoring spring force in the beam, and d is the damping force. Equation (5.1), represented in state-space form is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{d}{m}x_2 + \frac{\mu}{m}x_1 - \frac{\lambda}{m}x_1 - \frac{1}{m}x_1^3 \end{aligned} \tag{5.2}$$

where the states x_1 and x_2 represent the beam deflectional displacement and deflectional velocity respectively. For the unbuckled state, $\mu \leq \lambda$, the system has one equilibrium point, $x_1 = x_2 = 0$. For the buckled state, $\mu > \lambda$, the system has three equilibrium points, namely, $(0,0)$ and $\left(\pm\sqrt{\mu - \lambda}, 0\right)$. For the stability analysis of system (5.2), the buckled state is considered, i.e. $\mu > \lambda$, where the stability of each equilibrium point is analysed separately. To begin with, the stability of the equilibrium point $x_1 = x_2 = 0$ is investigated.

From first level decomposition of system (5.2), the independent subsystems are

$$\begin{aligned} g_1(x_1) &= 0 \\ g_2(x_2) &= -\frac{d}{m}x_2 \end{aligned} \tag{5.3}$$

and their corresponding link functions are given as

$$\begin{aligned} h_1(x) &= x_2 \\ h_2(x) &= \frac{\mu}{m}x_1 - \frac{\lambda}{m}x_1 - \frac{1}{m}x_1^3 \end{aligned} \tag{5.4}$$

From second level decomposition, the following independent (1,2) couple of system (5.2) is given as

$$\begin{aligned} q_1(x_1, x_2) &= x_2 \\ q_2(x_1, x_2) &= -\frac{d}{m}x_2 + \frac{\mu}{m}x_1 - \frac{\lambda}{m}x_1 - \frac{1}{m}x_1^3 \end{aligned} \tag{5.5}$$

Since (5.2) is a 2nd order system, the second level link functions, $h_1^*(x) = h_2^*(x) = 0$. By assigning the diagonal elements,

$$v_{11}(x_1) = x_1^2, \qquad v_{22}(x_2) = x_2^2 \tag{5.6}$$

and the nondiagonal elements,

$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{5.7}$$

the following estimates are obtained from (5.3) and (5.5) and Assumption 4.2

$$\begin{aligned}
&\text{(i)} \quad 2x_1(0) \geq \tilde{p}_{11}^0 |x_1|^2 \\
&\text{(ii)} \quad 2x_2\left(-\tfrac{d}{m}x_2\right) \geq \tilde{p}_{22}^0 |x_2|^2 \\
&\text{(iii)} \quad 2x_1(x_2) \geq \tilde{\mu}_{11}|x_1|^2 + \tilde{\mu}_{12}|x_1||x_2| \\
&\text{(iv)} \quad 2x_2\left(\tfrac{\mu}{m}x_1 - \tfrac{\lambda}{m}x_1 - \tfrac{1}{m}x_1^3\right) \geq \tilde{\mu}_{21}|x_2||x_1| + \tilde{\mu}_{22}|x_2|^2
\end{aligned} \tag{5.8}$$

where $\varphi_1(\|x_1\|) = |x_1|$ and $\varphi_2(\|x_2\|) = |x_2|$. Inequalities (5.8) are satisfied with the following constants,

$$\begin{aligned}
&\text{(i)} \quad \tilde{p}_{11}^0 = 0 \\
&\text{(ii)} \quad \tilde{p}_{22}^0 = -\tfrac{2d}{m} \\
&\text{(iii)} \quad \tilde{\mu}_{11} = 0, \quad \tilde{\mu}_{12} = 2 \\
&\text{(iv)} \quad \tilde{\mu}_{21} = \tfrac{2(\mu-\lambda)}{m}
\end{aligned} \tag{5.9}$$

With constant (iv) of (5.9), inequality (iv) of (5.8) reduces to

$$-\frac{2}{m}x_2 x_1^3 \geq \tilde{\mu}_{22}|x_2|^2 \tag{5.10}$$

Since the stability of the origin is being investigated, the left-hand side of inequality (5.10) can be approximated to zero. Therefore, $\tilde{\mu}_{22} \approx 0$.

From second level decomposition (5.5) and Assumption 4.3, the following estimates are obtained,

$$
\begin{aligned}
&(x_1 \quad x_2)\begin{pmatrix} x_2 \\ -\frac{d}{m}x_2 + \frac{\mu}{m}x_1 - \frac{\lambda}{m}x_1 - \frac{1}{m}x_1^3 \end{pmatrix} \\
&= \left(1 + \frac{\mu}{m} - \frac{\lambda}{m}\right)x_1x_2 - \frac{d}{m}x_2^2 - \frac{1}{m}x_1^3x_2 \\
&\geq \tilde{p}_{12}^1|x_1|^2 + 2\tilde{p}_{12}^2|x_1||x_2| + \tilde{p}_{12}^3|x_2|^2
\end{aligned} \tag{5.11}
$$

By assigning,

$$
\tilde{p}_{12}^2 = \frac{1}{2}\left(1 + \frac{\mu}{m} - \frac{\lambda}{m}\right), \qquad \tilde{p}_{12}^3 = -\frac{d}{m} \tag{5.12}
$$

inequality (5.11) simplifies to

$$
-\frac{1}{m}x_1^3x_2 \geq \tilde{p}_{12}^1|x_1|^2 \tag{5.13}
$$

Since $|x_1|^2 = x_1^2$, inequality (5.13) can be further simplified as

$$
-\frac{1}{m}x_1x_2 \geq \tilde{p}_{12}^1 \tag{5.14}
$$

The left-hand side of inequality (5.14) can be approximated to zero by the same reasoning employed in (5.10). Therefore, $\tilde{p}_{12}^1 \approx 0$. Matrix $\tilde{S}$ in estimate (4.22) is calculated using the constants (5.9), (5.12) and (5.14), and is given by

$$
\tilde{S} = \begin{pmatrix} 0 & \frac{2[m + (\mu - \lambda)]}{m} \\ \frac{2[m + (\mu - \lambda)]}{m} & -\frac{4d}{m} \end{pmatrix} \tag{5.15}
$$

The problem however, is that the sign definiteness property of matrix $\tilde{S}$ cannot be established since its first leading principal minor is 0 and therefore, at best, can only be determined to be positive semi-definite, which is an insufficient conclusion for Theorem 4.1 condition (c). A more appropriate matrix function needs to be employed. To do this, a combination of *Aizerman's method* and Lyapunov matrix-valued functions is implemented.

Consider the diagonal elements of the matrix-valued function, as before, i.e.

$$v_{11}(x_1) = x_1^2, \qquad v_{22}(x_2) = x_2^2$$

However, now consider the nondiagonal elements to be of the form

$$v_{12}(x_1, x_2) = v_{21}(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 \tag{5.16}$$

which is of the form presented by Aizerman [26].

The constants obtained from first level decomposition remain unchanged, however the constants derived from the second level decomposition are calculated from the following estimate

$$\begin{aligned}
&(2ax_1 + bx_2 \quad 2cx_2 + bx_1)\begin{pmatrix} x_2 \\ -\frac{d}{m}x_2 + \frac{\mu - \lambda}{m}x_1 - \frac{1}{m}x_1^3 \end{pmatrix} \\
&= \frac{b(\mu - \lambda)}{m}x_1^2 + \left(2a + \frac{2c(\mu - \lambda)}{m} - bd\right)x_1x_2 \\
&+ \left(b - \frac{2cd}{m}\right)x_2^2 - \frac{b}{m}x_1^4 - \frac{2c}{m}x_1^3x_2 \\
&\leq \tilde{p}_{12}^1|x_1|^2 + 2\tilde{p}_{12}^2|x_1||x_2| + \tilde{p}_{12}^3|x_2|^2
\end{aligned} \tag{5.17}$$

By applying similar justification as that applied earlier, the terms $-\frac{b}{m}x_1^4$ and $-\frac{2c}{m}x_1^3x_2$ can be ignored. Therefore, the modified constants derived from (5.17) are

$$\begin{aligned}
\tilde{p}_{12}^1 &= \frac{b(\mu - \lambda)}{m}; \\
\tilde{p}_{12}^2 &= \frac{1}{2}\left(2a + \frac{2c(\mu - \lambda)}{m} - bd\right); \\
\tilde{p}_{12}^3 &= b - \frac{2cd}{m}
\end{aligned} \tag{5.18}$$

From the above constants, and those obtained in (5.9), the following modified $\tilde{S}$ matrix is derived,

$$\tilde{S} = \begin{pmatrix} \dfrac{2b(\mu - \lambda)}{m} & \dfrac{m(1+2a) - bd + (1+2c)(\mu - \lambda)}{m} \\ \dfrac{m(1+2a) - bd + (1+2c)(\mu - \lambda)}{m} & \dfrac{2bm - 2d(1+2c)}{m} \end{pmatrix} \tag{5.19}$$

where $\eta = (1,1)^T$. In order for matrix $\tilde{S}$ in (5.19) to be positive definite, its leading principal minors need also be positive definite (Theorem 2.1). Therefore,

$$\frac{2b(\mu - \lambda)}{m} > 0 \tag{5.20}$$

$$\frac{2b(\mu - \lambda)\big(2bm - 2d(1+2c)\big) - [m(1+2a) - bd + (1+2c)(\mu - \lambda)]^2}{m^2} > 0$$

From the first inequality, since the buckled state is being considered, $(\mu - \lambda) > 0$, and $m > 0$. This implies that $b > 0$ in order to satisfy the inequality. Looking back at the original choice for the nondiagonal elements, $b = 0$, which violates this inequality and therefore, understandably, could not provide satisfactory results. Now consider the second inequality. Without loss of generality, assign $m = 1$. Therefore, (5.20) simplifies to

$$2b(\mu - \lambda)\big(2b - 2d(1+2c)\big) - [(1+2a) - bd + (1+2c)(\mu - \lambda)]^2 > 0 \tag{5.21}$$

By arbitrarily assigning $a = \frac{1}{2}$ and $c = \frac{1}{2}$, (5.21) becomes

$$4b(\mu - \lambda)(b - 2d) - [2 - bd + 2(\mu - \lambda)]^2 > 0 \tag{5.22}$$

For purposes of simplification, introduce the designation $\Delta = \mu - \lambda$. Therefore,

$$\begin{gathered} 4b\Delta(b - 2d) - [2 - bd + 2\Delta]^2 > 0 \\ \Rightarrow (4\Delta - d^2)b^2 + 4d(1 - \Delta)b - 4(\Delta + 1)^2 > 0 \end{gathered} \tag{5.23}$$

The quadratic inequality (5.23) has two critical values, found by replacing the inequality sign with an equality sign and solving the resulting equation. The values which satisfy the intersection of this inequality (5.23) and the inequality derived from the condition on the first leading principal minor, $b > 0$, are the values which are greater than the larger of the two critical values, i.e.

$$b > \frac{-2d(1-\Delta) + 2\sqrt{d^2(1-\Delta)^2 + (4\Delta - d^2)(\Delta+1)^2}}{4\Delta - d^2} \quad (5.24)$$

Table 5.1 provides the values above which b should be in order to satisfy condition (c) of Theorem 4.1, i.e. if b is greater than the value provided in the table for the corresponding values of Δ and d, then the matrix $\tilde{S}$ in (5.19) is positive definite, and therefore, by Theorem 4.1, the equilibrium point (0,0) is unstable.

Table 5.1. Lower bounds on b which yield equilibrium point (0,0) of system (5.2) to be unstable

Δ	**d**										
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	2.00	2.31	I.V	N.R	N.R	N.R	N.R	N.R	N.R	N.R	N.R
2	2.12	2.57	4.16	-6.00	N.R	N.R	N.R	N.R	N.R	N.R	N.R
3	2.31	2.80	4.00	10.11	-4.00	N.R	N.R	N.R	N.R	N.R	N.R
4	2.50	3.01	4.06	7.14	I.V	-3.33	N.R	N.R	N.R	N.R	N.R
5	2.68	3.21	4.16	6.41	18.00	-13.93	-3.00	N.R	N.R	N.R	N.R
6	2.86	3.39	4.29	6.13	12.04	-98.00	-7.94	-2.80	N.R	N.R	N.R
7	3.02	3.56	4.42	6.03	10.11	42.03	-16.00	-5.95	-2.67	N.R	N.R
8	3.18	3.72	4.55	6.00	9.20	22.09	-39.97	-9.53	-4.96	-2.57	N.R
9	3.33	3.87	4.67	6.02	8.70	16.72	I.V	-15.21	-7.14	-4.36	-2.50
10	3.48	4.01	4.80	6.06	8.40	14.26	56.15	-25.93	-9.98	-5.90	-3.97

where N.R = Nonreal and I.V = Invalid.

For example, for the undamped case, $d = 0$, choosing $b = 4$ satisfies Δ values from 1 to 10, and therefore provides the satisfactory conditions required to prove the equilibrium point (0,0) is unstable. Note, the results in Table 5.1 correspond to the values $a = c = \frac{1}{2}$ and $m = 1$.

By comparison, consider Aizerman's original conjecture, where an attempt is made to obtain an appropriate scalar Lyapunov function of the form

$$v(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + cx_2^2 \quad (5.25)$$

From (5.25)

$$\dot{v}(x_1,x_2) = 2a_{11}x_1\dot{x}_1 + a_{12}(\dot{x}_1x_2 + x_1\dot{x}_2) + 2a_{22}x_2\dot{x}_2$$

$$= 2a_{11}x_1x_2 + a_{12}\left(x_2(x_2) + x_1\left(-\frac{d}{m}x_2 + \frac{\mu-\lambda}{m}x_1 - \frac{1}{m}x_1^3\right)\right)$$
$$+ 2a_{22}x_2\left(-\frac{d}{m}x_2 + \frac{\mu-\lambda}{m}x_1 - \frac{1}{m}x_1^3\right) \tag{5.26}$$
$$= \left(2a_{11} - \frac{a_{12}d}{m} + \frac{2a_{22}(\mu-\lambda)}{m}\right)x_1x_2 + \frac{a_{12}(\mu-\lambda)}{m}x_1^2$$
$$+ \left(a_{12} - \frac{2a_{22}d}{m}\right)x_2^2 - \frac{a_{12}}{m}x_1^4 - \frac{2a_{22}}{m}x_1^3x_2$$

Now, in order to prove the instability of the equilibrium point (0,0), constants a_{11}, a_{12} and a_{22} need to be chosen in order to ensure $\dot{v}(x_1,x_2)$ in (5.26) is positive definite, thereby satisfying Theorem 3.5. Furthermore, the matrix

$$P = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

must also be positive definite, according to Aizerman's conjecture. This is no menial task, and a fair degree of mathematical knowledge, experience and ingenuity would be required to manipulate (5.26) into an appropriate form for sign analysis. As a further note, had the nondiagonal elements v_{12} and v_{21} been omitted, corresponding to the vector Lyapunov function approach (according to Šiljak [30]), the corresponding $\tilde{S}$ matrix would be

$$\tilde{S} = \begin{pmatrix} 0 & \frac{m+(\mu-\lambda)}{m} \\ \frac{m+(\mu-\lambda)}{m} & -\frac{2d}{m} \end{pmatrix} \tag{5.27}$$

which is not positive definite, making this particular method of vector Lyapunov functions ineffectual in establishing the stability property of the origin. This is not to say that another, more appropriate vector Lyapunov function method wouldn't work for this system, or any other innovative Lyapunov function construction approach for that matter, it simply emphasizes the ability of the Lyapunov matrix-valued function approach to obtain the appropriate stability properties where the two, above mentioned techniques could not. To investigate the stability of the remaining two equilibrium points, system (5.2) is required to be linearly translated, positioning the equilibrium point of interest at the origin. Consider the equilibrium point, $(\sqrt{\mu-\lambda}, 0)$. The linearly translated system according to the transformation

$$(\bar{x}_1, \bar{x}_2) \rightarrow (x_1 - \sqrt{\mu-\lambda}, 0) \tag{5.28}$$

is given by

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 \\ \dot{\bar{x}}_2 &= -\frac{d}{m}\bar{x}_2 - \frac{2(\mu-\lambda)}{m}\bar{x}_1 - \frac{3\sqrt{\mu-\lambda}}{m}\bar{x}_1^2 - \frac{1}{m}\bar{x}_1^3 \end{aligned} \tag{5.29}$$

The origin of system (5.29) is now investigated for stability. By performing first level decomposition on system (5.29), and considering the diagonal elements, $v_{11}(x_1) = x_1^2$ and $v_{22}(x_2) = x_2^2$, from Assumption 4.2, the following constants are obtained,

$$\begin{aligned} p_{11}^0 = 0, \qquad p_{22}^0 = -\frac{2d}{m}, \\ \mu_{11} = 0, \qquad \mu_{12} = 2, \qquad \mu_{21} = -\frac{4(\mu-\lambda)}{m}, \qquad \mu_{22} = 0 \end{aligned} \tag{5.30}$$

Where the higher order terms $-\frac{6\sqrt{\mu-\lambda}}{m}\bar{x}_1^2\bar{x}_2$ and $-\frac{2}{m}\bar{x}_1^3\bar{x}_2$ are ignored. Since the constant $p_{22}^0 = -\frac{2d}{m}$, from the comments on Assumption 4.2 one can see that in order for the second independent subsystem to be stable, d must equal zero, whereas for asymptotic stability, $d > 0$. This can be intuitively verified as an unforced dissipative system, i.e. a system with a positive damping force, would exhibit asymptotic stability, whereas a system in which energy is conserved, i.e. a conservative system, assuming no input, would exhibit critical stability, corresponding to the $d = 0$ case.

For the nondiagonal elements, consider the same function employed in (5.16). From second level decomposition and Assumption 4.3,

$$
\begin{aligned}
&(2a\bar{x}_1 + b\bar{x}_2 \quad 2c\bar{x}_2 + b\bar{x}_1)\begin{pmatrix} \bar{x}_2 \\ -\frac{d}{m}\bar{x}_2 - \frac{2(\mu-\lambda)}{m}\bar{x}_1 - \frac{3\sqrt{\mu-\lambda}}{m}\bar{x}_1^2 - \frac{1}{m}\bar{x}_1^3 \end{pmatrix} \\
&= -\frac{2b(\mu-\lambda)}{m}\bar{x}_1^2 + \left(2a + \frac{4c(\mu-\lambda)}{m} - \frac{bd}{m}\right)\bar{x}_1\bar{x}_2 + \left(b - \frac{2cd}{m}\right)\bar{x}_2^2 \\
&\quad - \frac{6c\sqrt{\mu-\lambda}}{m}\bar{x}_1^2\bar{x}_2 - \frac{2c}{m}\bar{x}_1^3\bar{x}_2 - \frac{3b\sqrt{\mu-\lambda}}{m}\bar{x}_1^3 - \frac{b}{m}\bar{x}_1^4
\end{aligned}
\tag{5.31}
$$

By ignoring the last four higher order terms of (5.31), the following constant designations are made

$$
\begin{aligned}
p_{12}^1 &= -\frac{2b(\mu-\lambda)}{m} \\
p_{12}^2 &= \frac{1}{2}\left(2a - \frac{4c(\mu-\lambda)}{m} - \frac{bd}{m}\right) \\
p_{12}^3 &= b - \frac{2cd}{m}
\end{aligned}
\tag{5.32}
$$

With these constants, and the constants obtained in (5.30), the following S matrix is obtained

$$
S = \begin{pmatrix} -\frac{4b(\mu-\lambda)}{m} & \frac{m(1+2a) - 2(1+2c)(\mu-\lambda) - bd}{m} \\ \frac{m(1+2a) - 2(1+2c)(\mu-\lambda) - bd}{m} & \frac{-2d(1+2c) + 2bm}{m} \end{pmatrix}
\tag{5.33}
$$

For simplification, and without loss of generality, let

$$
a = c = \frac{1}{2}, \qquad \Delta = \mu - \lambda, \qquad m = 1
$$

Therefore matrix S simplifies to,

$$
S = \begin{pmatrix} -4b\Delta & 2 - 4\Delta - bd \\ 2 - 4\Delta - bd & -4d + 2b \end{pmatrix}
\tag{5.34}
$$

In order to analyse the stability of the origin, two cases are considered, namely, the cases $d = 0$ and $d > 0$. Considering the first case, when $d = 0$, equation (5.34) simplifies to

$$S = \begin{pmatrix} -4b\Delta & 2-4\Delta \\ 2-4\Delta & 2b \end{pmatrix} \tag{5.35}$$

Now, for the matrix S in (5.35) to be negative semi-definite, the conditions from Theorem 2.1 state that both all its odd order principal minors are non-positive and all its even order principal minors are non-negative. Therefore,

(i) $\quad -4b\Delta \leq 0 \;\Rightarrow\; b \geq 0$

(ii) $\quad 2b \leq 0 \;\Rightarrow\; b \leq 0$

(iii) $\quad -8b^2 - 4(2\Delta - 1)^2 \geq 0$

The only value for b which satisfies both (i) and (ii) is $b = 0$. Substituting this into (iii) yields $\Delta = \frac{1}{2}$. From this result, the following conclusion can be drawn. Should one be required to analyse the stability of the equilibrium point $(\sqrt{\mu - \lambda}, 0)$ of the undamped system (5.2) ($d = 0$), given the difference between the applied axial force, μ, and the linear component of the restoring spring force, λ, is a half, i.e., $\Delta = \frac{1}{2}$, then the Lyapunov matrix valued function,

$$v(t, x, \eta) = \eta^T U(t, x)\eta \tag{5.36}$$

with

$$U(t, x) = \begin{pmatrix} v_{11}(x_1) & v_{12}(x_1, x_2) \\ v_{21}(x_1 x_2) & v_{22}(x_2) \end{pmatrix}, \qquad \eta = (1\,,1)^T \tag{5.37}$$

where

$$\begin{gathered} v_{11}(x_1) = x_1^2, \qquad v_{22}(x_2) = x_2^2, \\ v_{12}(x_1, x_2) = v_{21}(x_2, x_1) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \end{gathered} \tag{5.38}$$

satisfies condition (a) of Theorem 4.1 and therefore, the equilibrium point at the origin of system (5.29), corresponding to the equilibrium point, $(\sqrt{\mu - \lambda}, 0)$, of system (5.2), is uniformly stable for $d = 0$, $m = 1$, and $\Delta = \frac{1}{2}$.

This is not a particularly practical result as it is too parameter specific and not robust enough to parameter variation. It is suspected however, that by varying the a and c values in the nondiagonal elements of the matrix function, different values for Δ would be obtained. These suspicions are confirmed, where a simple calculation reveals that, for example, the values $a = 2$ and $c = 1$ yield $\Delta = \frac{5}{6}$ whereas the values $a = 3$ and $c = 1$ yield $\Delta = \frac{7}{6}$. In general, to determine the stability of the origin of system (5.29) for *any* value of Δ, keeping $b = 0$, the following relationship is derived,

$$\Delta = \frac{1 + 2a}{2(1 + 2c)} \tag{5.39}$$

This result cannot be extended to asymptotic stability as the conditions (i) and (ii) above would have to be replaced with negative definite inequality signs, thereby ridding any possible value for b. Furthermore, no conclusion can be drawn concerning the global stability of this equilibrium point as the assumptions asserted concerning the higher order terms close to the origin are no longer valid as the further one travels away from the origin, the more influence these terms have on the assigned constants, thereby having a more influential impact on the S matrix.

To determine the stability of the origin of system (5.29) for the case when $d > 0$, consider the S matrix in (5.34). For this matrix to be negative semi-definite, the following conditions must hold

$$\begin{aligned}
&\text{(i)} \quad -4b\Delta \leq 0 \;\Rightarrow\; b \geq 0 \\
&\text{(ii)} \quad -4d + 2b \leq 0 \;\Rightarrow\; b \leq 2d \\
&\text{(iii)} \quad |S| = 8b\Delta(2d - b) - (2 - 4\Delta - bd)^2 \geq 0 \\
&\qquad \Rightarrow (-8\Delta - d^2)b^2 + 4db(2\Delta + 1) - 4(2\Delta - 1)^2 \geq 0
\end{aligned} \tag{5.40}$$

To solve condition (iii) of inequality (5.40) for b, the left-hand side is plotted against b for an arbitrary value of $d > 0$, see Figure 5.2. A family of curves corresponding to constant values of Δ, is shown. Here, one can see that a value of $b = 4$ satisfies condition (iii) of inequality (5.40). Also, owing to the downward trend of the $\Delta = 1$ and $\Delta = 2$ curves, no values for b can be chosen in this current

Lyapunov matrix-valued function configuration, which would determine equilibrium point stability. Further analysis needs to be conducted in order to cater for these circumstances, where a possible suggestion may be to alter the values of a and c. Since the chosen value for b also satisfies the condition of matrix sign negative definiteness, according to Theorem 2.1, one can conclude that condition (d) of Theorem 4.1 has been satisfied, thereby proving the equilibrium point (0,0) of system (5.29), corresponding to the equilibrium point $(\sqrt{\mu - \lambda}, 0)$ of system (5.2) is universally asymptotically stable for $d > 0$.

Figure 5.3 shows the resulting family of Δ curves when $d < 0$. Clearly, no value for $b > 0$ can be chosen to satisfy conditions (i) and (iii) of inequality (5.40) and therefore, another Lyapunov matrix-valued function configuration needs to be considered in order to adequately analyse the equilibrium point's stability. Note, however, that this result does not infer equilibrium point *instability* but merely an inability to obtain an appropriate Lyapunov function.

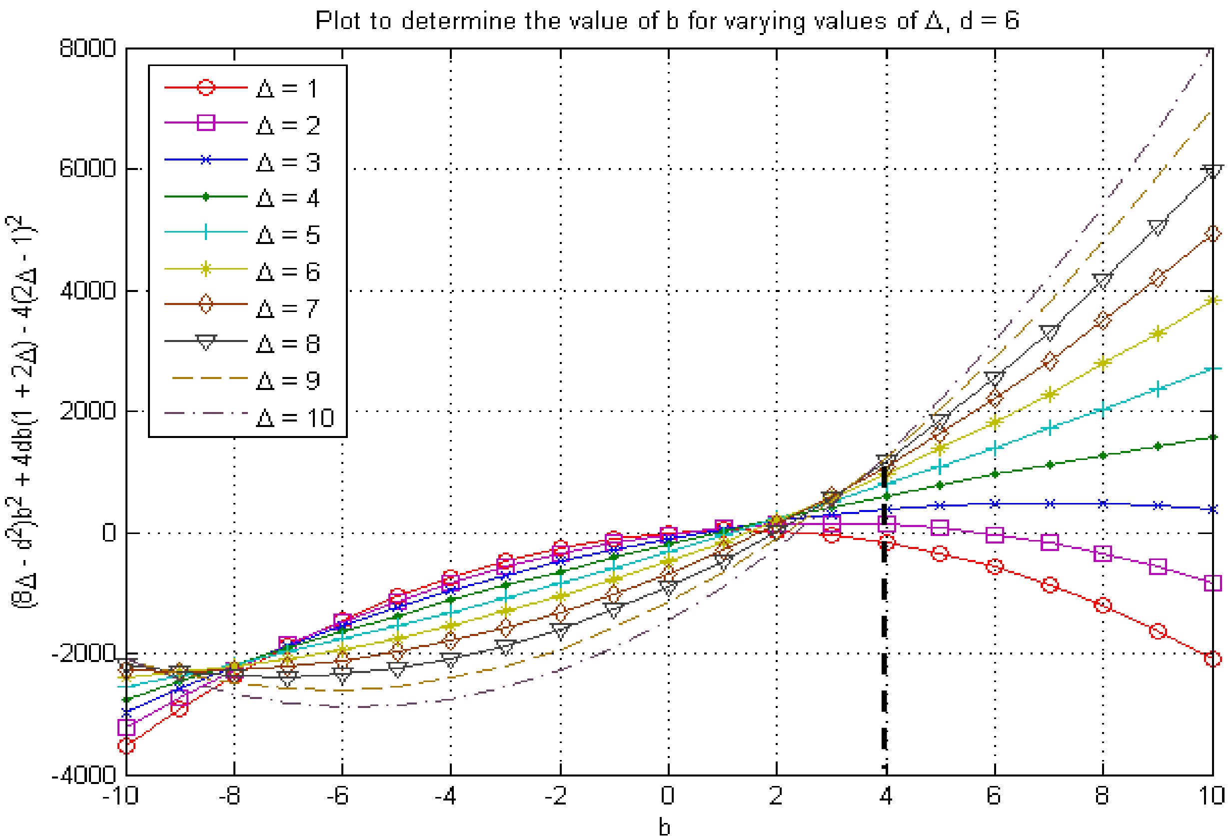


Figure 5.2 Plot of inequality (5.36) for $d = 6$

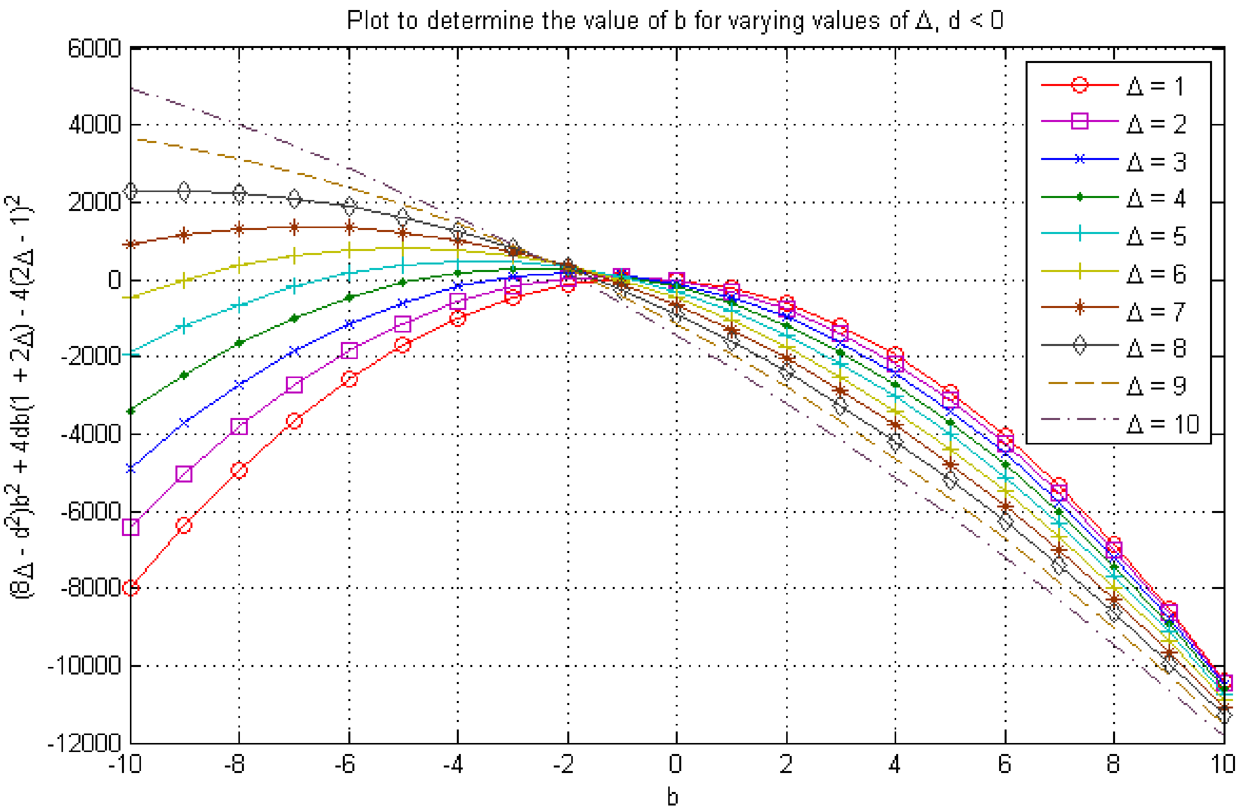


Figure 5.3 Plot of inequality (5.36) for $d < 0$

In order to investigate the stability of the third and final equilibrium point, $\left(-\sqrt{\mu-\lambda}, 0\right)$, the system (5.2) must undergo the linear translation,

$$\hat{x}_1 = x_1 + \sqrt{\mu-\lambda}, \qquad \hat{x}_2 = x_2 \tag{5.41}$$

where the resulting system is given as

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= -\frac{d}{m}\hat{x}_2 - \frac{2(\mu-\lambda)}{m}\hat{x}_1 + \frac{3\sqrt{\mu-\lambda}}{m}\hat{x}_1^2 - \frac{1}{m}\hat{x}_1 \end{aligned} \tag{5.42}$$

The origin of system (5.42) is now investigated for stability. System (5.42) is identical to system (5.29) except for the $+\frac{3\sqrt{\mu-\lambda}}{m}\hat{x}_1^2$ where this term has the opposite sign in system (5.29), i.e. $-\frac{3\sqrt{\mu-\lambda}}{m}\hat{x}_1^2$. Since the terms derived from the terms $+\frac{3\sqrt{\mu-\lambda}}{m}\hat{x}_1^2$ and $-\frac{3\sqrt{\mu-\lambda}}{m}\hat{x}_1^2$ are higher order terms, they are ignored when obtaining the constants from both first and second level decomposition. Therefore, the analyses conducted and conclusions obtained in the investigation of the stability of the equilibrium point, $\left(\sqrt{\mu-\lambda},0\right)$, are identical to that of the equilibrium point $(-\sqrt{\mu-\lambda},0)$. Figures 5.4 and 5.5 illustrate the phase portrait of system (5.2), where the results obtained from the analysis of all three equilibrium points can be verified. Figure 5.5 demonstrates the undamped buckling beam i.e.,$d=0$, and Figure 5.6 illustrates the dissipative system, with the damping coefficient, $d>0$. The values $\mu=2$, $\lambda=1$ and $m=1$ are used in both diagrams.

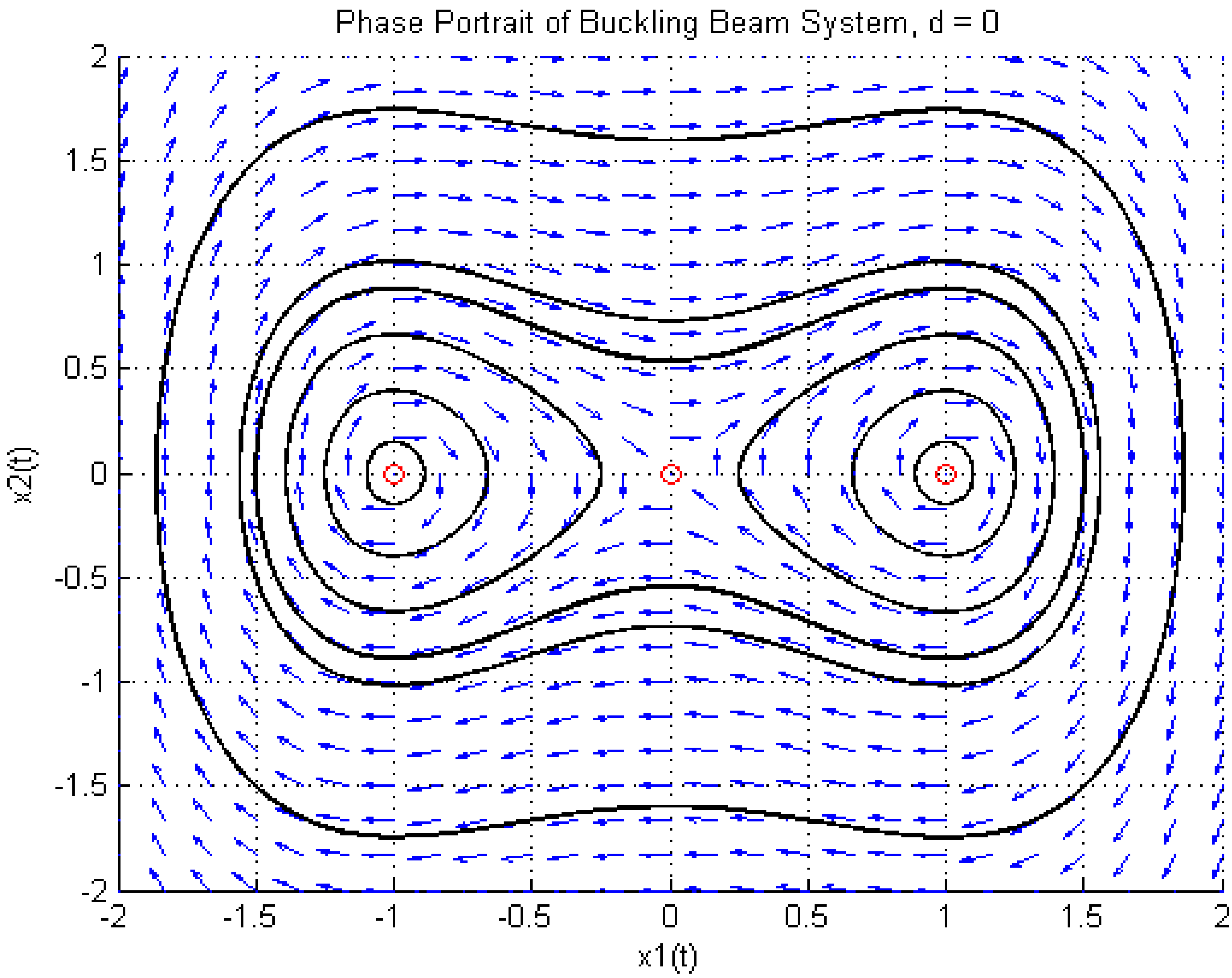


Figure 5.4 Phase Portrait of Buckling Beam System (5.2) for $d=0$

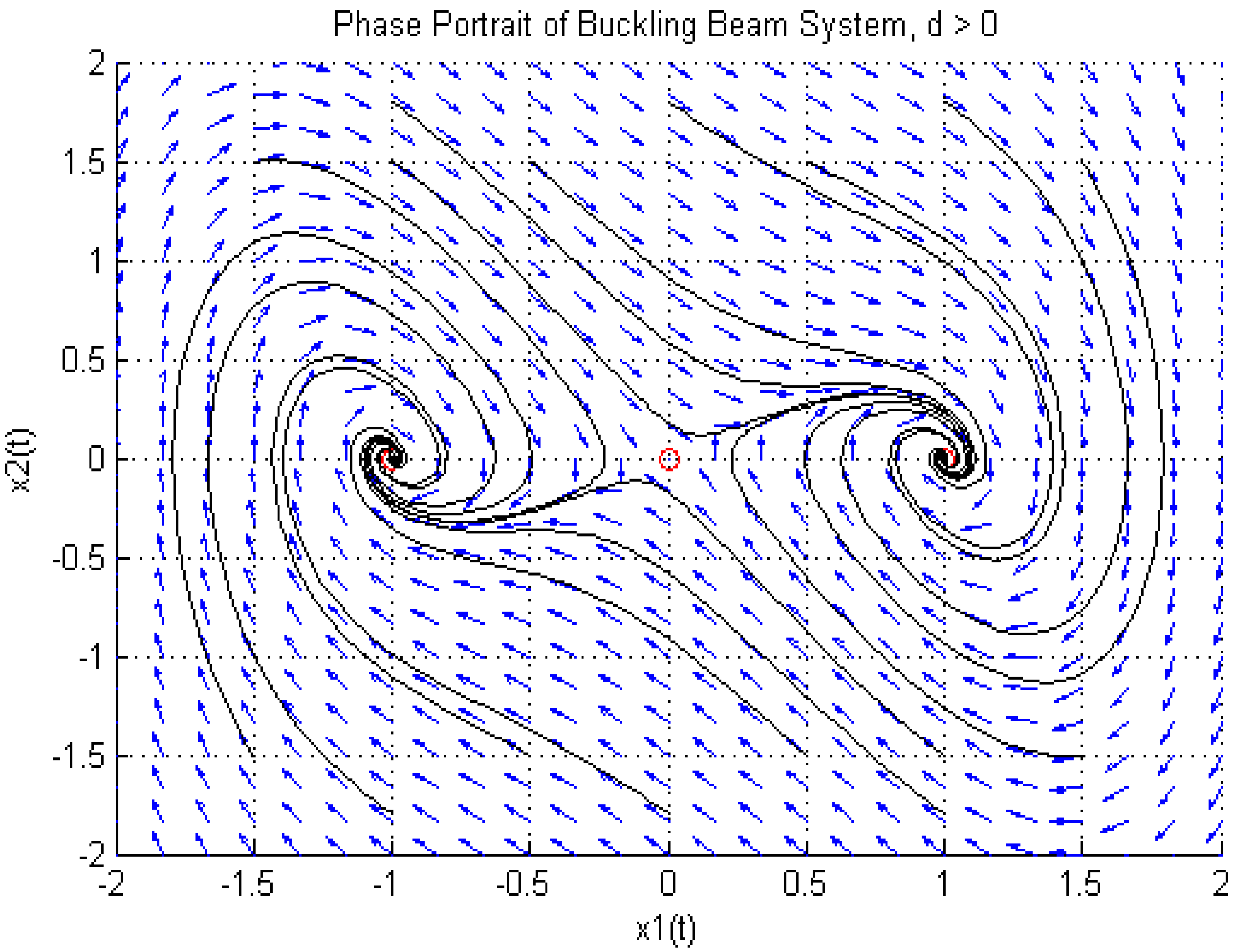


Figure 5.5 Phase Portrait of Buckling Beam System (5.2) for $d > 0$

From Figures 5.4 and 5.5, one can clearly see that for both the case when $d = 0$ and the $d > 0$, the origin is unstable. This result correlates with the results obtained when using the Lyapunov matrix-valued function approach, since an appropriate value for b can be found from Table 5.1, for both $d = 0$ and $d > 0$, thereby satisfying the required conditions for equilibrium point instability. Furthermore, for the undamped case, $d = 0$ the equilibrium points $(1, 0)$ and $(-1, 0)$ in Figure 5.4 are critically stable, as predicted by the Lyapunov matrix-valued function method. Also, for the dissipative system, $d > 0$, the equilibrium points $(1, 0)$ and $(-1, 0)$ are asymptotically stable, also as predicted by the Lyapunov matrix-valued function method. Therefore, the Lyapunov matrix-valued function method correctly established the stability properties of all the equilibrium points of system (5.2).

Chapter 6

Conclusions and Recommendations

The method of hierarchical Lyapunov matrix-valued functions in the stability analysis of dynamic nonlinear systems has proved to be both extremely powerful and insightful in application, applicable where other methods fail, but also computationally intensive and cumbersome in its implementation. This awkwardness is attributed primarily to the extremely complex and unique nature of each dynamic nonlinear system, and, in general, not to the employed structure or methodology of the Lyapunov matrix-valued function approach. This methodology was successfully applied to a number of nonlinear systems, where both the stability and instability of the equilibrium points are found, and verified by means of phase portrait illustrations. Furthermore, the researched methodology is compared to both classical scalar Lyapunov function construction techniques, as well as the vector Lyapunov function approach. It was found that, for a *less complex* nonlinear system, the traditional scalar approach succeeded, after a fair amount of mathematical manipulation, while for a *more complex* nonlinear system, one with multiple equilibrium points, the traditional scalar approach is unable to obtain an appropriate Lyapunov function for the stable equilibrium point, while the vector Lyapunov function approach is unable to obtain an appropriate function to validate the instability of an unstable equilibrium point. In contrast, the Lyapunov matrix-valued function approach successfully obtained the stability and instability of the investigated equilibrium points, which is then verified diagrammatically in the system's phase portrait. From these examples and many others, it is clear that the method of Lyapunov matrix-valued functions proves to be a powerful tool in the stability analysis of nonlinear systems, often outperforming older, more established methodologies.

A number of simplifications of the hierarchical Lyapunov matrix-valued function methodology for linear systems are then presented, where a method is shown of methodically obtaining both the diagonal and nondiagonal elements of the matrix function, thereby eliminating any element of arbitrary choice, a fundamental setback of the conventional scalar Lyapunov function approach. These simplifications are then applied to a set of linear systems in order to verify their validity, which was found to be both applicable and insightful. Once again, the investigated technique is compared to other, more recognised methodologies, only to find it surpasses these techniques in its stability analysis capabilities.

Owing to the required formulation of the matrix-valued function, a hierarchical decomposition methodology is employed. This decomposition serves to divulge certain dynamic characteristics of the nonlinear system under investigation, characteristics which would otherwise remain dormant, thereby presenting a formidable advantage over classical linear system stability analysis techniques. Two variations on the hierarchical Lyapunov matrix-valued function approach are also presented, which are particularly constructive in the stability analysis of linear large-scale systems. However, while the first technique attempts to present a formalised methodology of constructing the nondiagonal elements of the matrix function, for nonlinear systems, it reduces the problem of arbitrary function assignment to solving a nonlinear algebraic equation which was found to be both problematic and, in some cases, impossible. The second variation presented an alternative to the hierarchical method of decomposition, namely, *overlapping decomposition*. The problem, however, is that while this method provides a proficient technique of linear system decomposition, it proves to be both cumbersome and often inapplicable to nonlinear systems. Therefore, while these variations present possible improvements to the stability analysis of linear systems, they tend to have the opposite effect on the stability analysis of nonlinear systems, making the original hierarchical method the most appropriate and widely applicable technique for nonlinear system stability analysis. Finally, the Buckling Beam system is presented, where it was found that the application of the Lyapunov matrix-valued function method in conjunction with Aizerman's method proves to be highly effective in obtaining the equilibrium points' stability.

Throughout the course of this research report, attention has been drawn to the comprehensive investigation, appropriate application and overall development of the method of the Lyapunov matrix-valued functions. In terms of recommendations for future research, three potential research areas are presented, each offering both wide-spread practical applicability as well as the potential for the extension of the theory, thereby improving on the current Lyapunov matrix-valued function philosophy. These three distinct, yet closely linked, directions are; the development of a *control Lyapunov matrix-valued function,* a *modification for Lyapunov's indirect method*, enabling one to determine the stability of non-hyperbolic fixed points, and finally, the determination of an estimate for the *region of asymptotic stability*.

As mentioned in Section 4.4.2, an area worthy of exploration is that of applying the investigated Lyapunov matrix-valued function technique to the development of both linear and nonlinear controllers. This method would be extremely beneficial in this context as it both exploits the underlying internal dynamics of the system requiring control, while consistently ensuring the stability of the closed-loop system is maintained. By dynamically manipulating one or many of the arbitrary constants presented in this technique, one could obtain a robust control paradigm able to analyse even the most highly nonlinear processes. *Control Lyapunov functions*, used in the control of dynamic systems, are not a new concept, theorised and developed by *Z. Artstein* and *E.D Sontag* in the 1980's and 1990's [32]. However, to date, the adaptation of the method of Lyapunov matrix-valued functions for the control of dynamic systems has not been seen in the literature, and therefore presents an exciting area of future research.

Another potential area of future development is that of the improvement of Lyapunov's indirect method. A fundamental drawback of this method is that should the equilibrium point in question be *non-hyperbolic*, i.e. the resulting linearised system or *Jacobian* around the required equilibrium point has purely imaginary eigenvalues, no conclusion can be drawn regarding the stability of the equilibrium point of the original nonlinear system, as the qualitative behaviour of the linearised system is non-identical to that of the nonlinear system. Section 4.2.6 draws attention to the important question of the reasoning behind applying Lyapunov matrix-valued functions to the stability analysis of linear systems, where the method of eigenvalue analysis would simply suffice. Using the Lyapunov matrix-valued function approach in conjunction with Lyapunov's indirect method, the stability of non-hyperbolic equilibrium points can now be analysed, as the Lyapunov matrix valued function approach does not share the inability to obtain the stability of non-hyperbolic equilibrium points with the conventional eigenvalue analysis method. Therefore, in order to investigate the stability of each equilibrium point in a nonlinear system, one would first apply

Lyapunov's indirect method, linearising the nonlinear system around the equilibrium point under investigation. If the equilibrium point is found to be hyperbolic, then conventional eigenvalue analysis of the resulting linearised system can be pursued. However, should the equilibrium point under investigation be non-hyperbolic, then the Lyapunov matrix-valued function approach would be implemented. This technique could be used as a potential substitute to the Centre Manifold Theorem however, considerable more research and investigation is required.

Both Lyapunov's indirect and direct methods, including the Lyapunov matrix valued function approach, provide satisfactory conditions or methodologies for investigating the local, and potentially global, stability of a nonlinear system's equilibrium point or points. However, none of the above mentioned methodologies provide a technique of, should the equilibrium point under investigation be found to be stable, obtaining the equilibrium point's *region of stability* or *region of asymptotic stability.* An area of paramount importance in the stability investigation of dynamic nonlinear systems is that of estimating the region of asymptotic stability. This notion has significant relevance in practical, real-world applications, where one could determine how far a particular process can deviate from its operating point before becoming unstable. This idea is intrinsically linked to the concept of exponential stability, whereby the rate of attraction towards an equilibrium point can be analysed, from which an estimate of the region of asymptotic stability can be established. The extension and adaptation of the pre-existing theory of exponential stability to the method of Lyapunov matrix-valued functions, can provide a systematic approach of estimating this region of asymptotic stability. In this vain, another potential area of exciting new research would be to aggregate the rich existing theory of LaSalle's invariance principle with the methodology of the Lyapunov matrix-valued function approach. In this way, the stringent constraints imposed by Lyapunov's direct method for equilibrium point asymptotic stability could be relaxed by both the application of the Lyapunov matrix-valued function method and LaSalle's invariance principle, making this hybrid technique a powerful tool in the stability investigation of nonlinear systems.

References

[1] K. Hedrih. "Nonlinear Dynamics and Alexandr Mikhailovich Lyapunov", Scientific Technical Review; Mechanics, Automatic Control and Robotics, Vol. 6, No. 1, 2007, pp. 211-218

[2] A. M. Lyapunov. *General Problem of Stability of Motion*, Markov: Math. Soc., 1892. (Published in Collected Papers, Moscow – Leningrad, Academy of Science, USSR, 1956, pp. 5 – 263) [Russian]

[3] J. La Salle, S. Lefschetz. *Stability by Liapunov's Direct Method with Applications*, New York, Academic Press, 1961

[4] V. Lakshmikantham, V. M. Matrosov, S. Sivasundaram. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Netherlands, 1991

[5] L. Vu, D. Liberzon. "Common Lyapunov functions for families of commuting nonlinear systems", Systems and Control Letters, Vol. 54, 2005, pp. 405 – 416

[6] A. A. Martynyuk. "Analysis of Stability Problems via Matrix Lyapunov Functions", Journal of Applied Math. and Stochastic Analysis, Vol. 3, No. 4, 1990, pp. 209 – 226

[7] I. K. Russinov. "A Direction in the Method of Matrix Lyapunov Functions and Stability in terms of Two Measures", Bulgaria Scientific Works, Vol. 34, Book 3, 2004, pp. 69 – 76.

[8] A. A. Martynyuk, V. I. Slyn'ko. "Solution of the Problem of Constructing Liapunov Matrix Functions for a class of Large Scale Systems", Nonlinear Dynamics and Systems Theory, Vol. 1, No. 2, 2001, pp. 193 – 203

[9] A. A. Martynyuk. *Stability by Liapunov's Matrix Function Method with Applications*, Marcel Dekker, Inc, 1998

[10] A. A. Martynyuk. *Qualitative Methods in Nonlinear Dynamics – Novel Approaches to Liapunov's Matrix Functions.* Marcel Dekker, Inc, 2002

[11] C. Banks. "Searching for Lyapunov Functions using Genetic Programming", Virginia Polytech Institute, *unpublished*

[12] Y. A Pykh. "Lyapunov Functions for Lotka-Volterra Systems: An Overview and Problems", Proc. of 5th IFAC Symposium, Nonlinear Control Systems, 2001, pp. 1655 – 1660

[13] A. Papachristodoulou, S. Prajna. "On the Construction of Lyapunov Functions using the Sum of Squares Decomposition", Proc. of IEEE CDC, 2002,

[14] D. P. Spanos, J. M. Goncalves. "Finding Surface Lyapunov Functions through Sum-of-Square Programming", Control and Dynamic Systems, California Inst. Of Tech, *unpublished*

[15] A. Papachristodoulou, S. Prajna. "The Construction of Lyapunov Functions using the Sum of Squares Decomposition", In Proceedings of the 41st IEEE Conference on Decision and Control, 2002, pp. 3482–3487

[16] S. Barnett, C. Storey. *Matrix Methods in Stability Theory,* Nelson, 1970

[17] H. K. Khalil. *Nonlinear Systems*, Prentice Hall, Third Edition, 2002

[18] V. Lakshmikantham, Y. A. Mitropolskii. "Personage in Science – Professor A. A. Martynyuk", Nonlinear Dynamics and Systems Theory, Vol. 6, No. 1, 2006, pp. 1 – 15

[19] L. O. Chua, C. A. Desoer, E. S. Kuh. *Linear and Nonlinear Circuits*, McGraw-Hill Series, 1987

[20] S. Sastry. *Nonlinear Systems – Analysis, Stability, and Control*, Springer-Verlag New York, Inc, 1999

[21] C. Frazier, K. M. Kockelman. "Chaos Theory and Transportation", Presented at the 83rd Annual Meeting of the Transportation Research Board, January 2004

[22] J. E. Prussing. "The Principal Minor Test for Semidefinite Matrices", American Institute of Aeronautics and Astronautics, Inc, Jan/Feb 1986, pp. 121 – 122

[23] W. Hahn. *Stability of Motion*, Springer-Verlag, Berlin, 1967

[24] T. Yoshizawa. "Stability Theory of Liapunov's Second Method", The Math. Soc. of Japan, 1966

[25] T. Yoshizawa. *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Springer-Verlag, New York – Heidelberg – Berlin, 1975

[26] B. Wigdorowitz. *Nonlinear Modelling, Estimation and Control: Course notes. Chapter 2: Classical Nonlinear Analysis and Control Methods,* University of the Witwatersrand, February 2007

[27] F. A. Aliev, V. B Larin. *Optimisation of Linear Control Systems: Analytical Methods and Computational Algorithms*, Amsterdam, Gordon and Breach Publishers, 1998

[28] Ye. A. Barbashin. *The Liapunov Functions.* Moscow: Nauka, 1967 [Russian]

[29] S. Leela. "Current State of the Method of Vector Lyapunov Functions", World Congress of Nonlinear Analysts '92, Proc. of 1st World Congress of Nonlinear Analysts, Tampa, Florida, Vol. 4, August, 1992

[30] D. D. Šiljak. *Decentralised Control of Complex Systems*, Boston, Academic Press, 1991

[31] M. Ikeda, D. D. Šiljak. "Generalised Decomposition of Dynamic Systems and Vector Lyapunov Functions", IEEE Trans. on Automatic Control, Vol. AC-26, No. 5, October, 1981

[32] R. A. Freeman, P. V. Kokotović. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*, Springer, 1996

[33] J. P LaSalle. "An Invariance Principle in the Theory of Stability", Proc. of the 24th Seminar on Space Flight and Guidance Theory, Alabama, 1966, pp. 111 – 130

[34] X. Liao, L. Wang, P. Yu. *Stability of Dynamical Systems*, Elsevier, First Edition, 2007