PROPERTIES AND ZEROS OF ${}_3F_2$ HYPERGEOMETRIC FUNCTIONS

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Abstract

In this thesis, our primary interest lies in the investigation of the location of the zeros and the asymptotic zero distribution of hypergeometric polynomials.

The location of the zeros and the asymptotic zero distribution of general hypergeometric polynomials are linked with those of the classical orthogonal polynomials in some cases, notably ${}_2F_1$ and ${}_1F_1$ hypergeometric polynomials which have been extensively studied. In the case of ${}_3F_2$ polynomials, less is known about their properties, including the location of their zeros, because there is, in general, no direct link with orthogonal polynomials. Our introduction in Chapter 1 outlines known results in this area and we also review recent papers dealing with the location of the zeros of ${}_2F_1$ and ${}_1F_1$ hypergeometric polynomials.

In Chapter 2, we consider two classes of ${}_3F_2$ hypergeometric polynomials, each of which has a representation in terms of ${}_2F_1$ polynomials. Our first result proves that the class of polynomials ${}_3F_2(-n, a, b; a-1, d; x)$, $a, b, d \in \mathbb{R},\ n \in \mathbb{N}$ is quasi-orthogonal of order 1 on an interval that varies with the values of the real parameters b and d. We deduce the location of $(n-1)$ of its zeros and discuss the apparent role played by the parameter a with regard to the location of the one remaining zero of this class of polynomials. We also prove results on the location of the zeros of the classes ${}_3F_2(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x)$, $b \in \mathbb{R},\ n \in \mathbb{N}$ and ${}_3F_2\left(-n, b, \frac{b-n}{2}+1; b-n, \frac{b-n+1}{2}; x\right)$, $n \in \mathbb{N}$, $b \in \mathbb{R}$ by using the orthogonality and quasi-orthogonality of factors involved in its represen-

tation. We use Mathematica to plot the zeros of these ${}_3F_2$ hypergeometric polynomials for different values of n as well as for different ranges of the parameters. The numerical data is consistent with the results we have proved.

The Euler integral representation of the ${}_2F_1$ Gauss hypergeometric function is well known and plays a prominent role in the derivation of transformation identities and in the evaluation of ${}_2F_1(a, b; c; 1)$, among other applications (cf. [1], p.65). The general ${}_{p+k}F_{q+k}$ hypergeometric function has an integral representation (cf. [37], Theorem 38) where the integrand involves ${}_pF_q$. In Chapter 3, we give a simple and direct proof of an Euler integral representation for a special class of ${}_{q+1}F_q$ functions for $q \geq 2$. The values of certain ${}_3F_2$ and ${}_4F_3$ functions at $x = 1$, some of which can be derived using other methods, are deduced from our integral formula.

In Chapter 4, we prove that the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ asymptotically approach the section of the lemniscate $\left\{z : |z(1-z)^2| = \frac{4}{27}; \mathrm{Re}(z) > \frac{1}{3}\right\}$ as $n \to \infty$. In recent papers (cf. [31], [32], [34], [35]), Martínez-Finkelshtein and Kuijlaars and their co-authors have used Riemann-Hilbert methods to derive the asymptotic distribution of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}$ when the limits $A = \lim_{n\to\infty} \dfrac{\alpha_n}{n}$ and $B = \lim_{n\to\infty} \dfrac{\beta_n}{n}$ exist and lie in the interior of certain specified regions in the AB-plane. Our result corresponds to one of the transitional or boundary cases for Jacobi polynomials in the Kuijlaars Martínez-Finkelshtein classification.

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Chapter 1

Introduction

Consider the second order linear differential equation with three regular singular points at $z = 0$, 1 and ∞

$$z(1-z)\frac{d^2y}{dz^2} + \big[c - (a+b+1)z\big]\frac{dy}{dz} - aby = 0, \tag{1.0.1}$$

called the hypergeometric differential equation. If c is non-integral, this differential equation has two linearly independent solutions in $|z| < 1$, namely

$$y_1(z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{1.0.2}$$

and

$$y_2(z) = z^{1-c}\left(1 + \sum_{k=1}^{\infty} \frac{(a+1-c)_k (b+1-c)_k}{(2-c)_k} \frac{z^k}{k!}\right).$$

Definition 1.0.1 *(cf. [37], Chapter 4) The Gauss hypergeometric function, or* ${}_2F_1$ *hypergeometric function, is defined by*

$$ {}_2F_1\left(a, b; c; z\right) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1, \tag{1.0.3}$$

where a*,* b *and* c *are complex parameters, where* c *is neither zero nor a negative integer and where* $(\alpha)_k$*, is Pochhammer's symbol or the "ascending" factorial* *defined by*

$$(\alpha)_k = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+k-1) & , \quad k \geq 1, \\ 1 & , \quad k = 0,\ \alpha \neq 0 \end{cases}$$

This series converges when $|z| < 1$ and also when $z = 1$ provided that $\mathrm{Re}(c - a - b) > 0$ and when $z = -1$ provided that $\mathrm{Re}(c - a - b + 1) > 0$.

The general hypergeometric function is similarly defined as follows.

Definition 1.0.2 *(cf. [37], Chapter 5) The formal generalised hypergeometric series ${}_pF_q$ with p numerator and q denominator parameters is defined by*

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}, \tag{1.0.4}$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, z$ may be real or complex and $b_1, b_2, \ldots, b_q$ are neither zero nor negative integers.

When $p \leq q$, the series converges for all values of z. When $p > q + 1$, the series converges only for $z = 0$, and is therefore significant only when the series terminates. When $p = q + 1$, the series converges when $|z| < 1$, and also when $z = 1$ provided that $\mathrm{Re}\left(\sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j\right) > 0$, and when $z = -1$ provided that $\mathrm{Re}\left(\sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j + 1\right) > 0$.

It is clear that the Gauss hypergeometric function corresponds the generalised hypergeometric function when $p = 2$ and $q = 1$.

Almost all of the elementary functions of mathematics are either hypergeometric functions or ratios of hypergeometric functions and many of the non-elementary functions that arise in mathematics and physics also have representations as hypergeometric series.

By the definition of Pochhammer's symbol, if k and n are positive integers, then

$$(-n)_k = \begin{cases} (-1)^n \, \frac{n!}{(n-k)!} & , \quad 1 \leq k \leq n, \\ 0 & , \quad k = 0, \ \alpha \neq k \geq n+1. \end{cases}$$

Therefore, if one of the numerator parameters in the hypergeometric function is equal to a negative integer, say $a = -n$, $n \in \mathbb{N}$ in the Gauss hypergeometric function (1.0.3) or $a_1 = -n$, $n \in \mathbb{N}$ in the general hypergeometric function (1.0.4), then the series terminates and the function is a polynomial of degree n in z, provided that no denominator parameter is a negative integer or zero. We note that convergence of the series is no longer an issue since the series terminates.

A natural question to ask is whether anything can be said in general about the location of the zeros of hypergeometric polynomials. One may then extend this question to the location of the zeros of hypergeometric functions that are not necessarily polynomials.

Many results have been proved for the zero location and the asymptotic zero distribution of ${}_2F_1$ hypergeometric functions and polynomials and ${}_1F_1$ hypergeometric polynomials. The reason for extensive knowledge about the zeros of these two classes of hypergeometric functions is their link with classical orthogonal polynomials.

We recall the definition of orthogonality for a sequence of polynomials.

Definition 1.0.3 *A sequence* $p_0,\ p_1,\ p_2, \ldots$ *of real polynomials with deg*(p_k)$=$ k *is orthogonal with respect to a positive weight function* $w(x)$ *on an interval* I *if*

$$\int_I x^l\ p_n(x)w(x)dx = 0 \qquad \textit{for}\, l = 0, 1, \ldots, n-1$$

where $n \in \mathbb{N}$.

It is a well-known classical result (cf. [37]) that all the zeros of $p_n(x)$ are real and simple and lie in the open interval I.

Different classes of ${}_2F_1$ hypergeometric polynomials have well established connections with different types of classical orthogonal polynomials, notably Jacobi polynomials and Gegenbauer or ultraspherical polynomials. Indeed, we have

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right)$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n, and

$$C_n^{\alpha}(x) = \frac{(2\alpha)_n}{n!}\ {}_2F_1\left(-n, 2\alpha+n; \alpha+\frac{1}{2}; \frac{1-x}{2}\right)$$

where $C_n^{\alpha}(x)$ is the Gegenbauer polynomial of degree n. For the ranges of the parameters α and β where the Jacobi polynomials and Gegenbauer polynomials are orthogonal, information about the zeros of the ${}_2F_1$ follows immediately from classical results (cf. [1], [45]). The Legendre polynomial, given by $P_n(x)$, is a special case of the Gegenbauer polynomial with $\alpha = \frac{1}{2}$ and can be written as a ${}_2F_1$ hypergeometric function using Murphy's formula,

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right).$$

The problem of describing the zero behaviour of Gauss hypergeometric polynomials ${}_2F_1(-n, b; c; z)$ where b and c are arbitrary parameters, has in general not been solved. Even when b and c are both real, the only cases that have been fully analysed impose some additional restrictions on b and c. We note here that if b and c are real parameters, then all zeros of ${}_2F_1(-n,b;c;z)$ must occur in complex conjugate pairs and, in particular, if n is an odd integer, ${}_2F_1(-n,b;c;z)$ must always have at least one real zero.

There are twelve classes of ${}_2F_1$ hypergeometric polynomials that admit a quadratic transformation (cf. [46], p.124). These polynomials are given in Table 1.1.

${}_2F_1\left(-n,b;2b;z\right)$	${}_2F_1\left(-n,b;-n-b+1;z\right)$	${}_2F_1\left(-n,b;\frac{-n+b+1}{2};z\right)$
${}_2F_1\left(-n,b;\frac{1}{2};z\right)$	${}_2F_1\left(-n,-n+\frac{1}{2};c;z\right)$	${}_2F_1\left(-n,b;-n+b+\frac{1}{2};z\right)$
${}_2F_1\left(-n,b;\frac{3}{2};z\right)$	${}_2F_1\left(-n,-n-\frac{1}{2};c;z\right)$	${}_2F_1\left(-n,b;-n+b-\frac{1}{2};z\right)$
${}_2F_1\left(-n,b;-2n;z\right)$	${}_2F_1\left(-n,b;b+n+1;z\right)$	${}_2F_1\left(-n,n+1;c;z\right)$

Table 1.1: Hypergeometric polynomials in the quadratic class

The identities

$$ {}_2F_1(-n,b;c;1-z) = \frac{(c-b)_n}{(c)_n}\,{}_2F_1(-n,b;1-n+b-c;z) $$

and

$$ {}_2F_1(-n,b;c;z) = \frac{(b)_n}{(c)_n}\,(-z)^n\,{}_2F_1\left(-n,1-c-n;1-b-n;\frac{1}{z}\right) $$

$b,c \in \mathbb{R}$, $c \notin \{-n+1,-n+2,\ldots,0\}$, link the polynomials listed in Table 1.1 across each row. The hypergeometric polynomials that admit a quadratic transformation are important because the analysis of the zeros of the four classes in the first column essentially deals with the full table. This also leads to corresponding results for the zeros of other related polynomials.

The class ${}_2F_1(-n,b;2b;z)$ is an interesting case and has been discussed in detail in [12] and [13] by Driver and Duren. In summary, they show in [12] that for $b > -\frac{1}{2}$, all of the zeros of ${}_2F_1(-n, b; 2b; z)$ are simple and lie on the circle $|z - 1| = 1$ while for $b < 1 - n$, all the zeros are real and greater than 1. As $b \to -\infty$, the zeros of ${}_2F_1(-n, b; 2b; z)$ converge to the point $z = 2$. For values of b with $1 - n < b < -\frac{1}{2}$, the details of the trajectories of the zeros of ${}_2F_1(-n, b; 2b; z)$ can be found in [13].

As a consequence of the analysis in [12] and [13] and the link between the hypergeometric polynomial ${}_2F_1(-n, b; 2b; z)$ and the Gegenbauer (ultraspherical) polynomial $C_n^\alpha(z)$, namely

$$ {}_2F_1(-n, b; 2b; z) = \frac{n!\; 2^{-2n} z^n}{(b + \frac{1}{2})_n}\, C_n^\alpha\left(1 - \frac{2}{z}\right), \qquad \text{where} \quad \alpha = \frac{1}{2} - b - n, $$

the behaviour of the zeros of $C_n^\alpha(z)$ can be deduced (see [14]).

Two of the other classes of polynomials in the first column of Table 1.1, namely ${}_2F_1\left(-n, b; \frac{1}{2}; z\right)$ and ${}_2F_1\left(-n, b; \frac{3}{2}; z\right)$ can also be expressed in terms of Gegenbauer polynomials and the behaviour of the zeros of each of these polynomials is given in [23] Theorem 2.3 and Theorem 2.4 respectively.

A more challenging question is the location of the zeros of the class of hypergeometric polynomials ${}_2F_1(-n, b; -2n; z)$, $b \in \mathbb{R}$, $n \in \mathbb{N}$. Here, the classical orthogonal polynomials that are linked to this class of hypergeometric polynomials, namely the Jacobi polynomials, give no information regarding the location of the zeros of ${}_2F_1(-n, b; -2n; z)$ as the restrictions on the parameters of the Jacobi polynomials that ensure orthogonality are not satisfied. Different methods are therefore needed.

In [22], Driver and Möller obtain the following result (cf. [22], Theorem 3.1 and Corollary 3.2).

Theorem 1.0.4 *Let* $F = {}_2F_1(-n, b; -2n; z)$ *with* b *real and* $n \in \mathbb{N}$.

- *(i) For* $b > 0$, F *has* n *non-real zeros if* n *is even whereas if* n *is odd,* F *has exactly one real negative zero and the remaining* $(n-1)$ *zeros of* F *are all non-real.*
- *(ii) For* $-n < b < 0$, *if* $-k < b < -k+1$, $k = 1, \ldots, n$, F *has* k *real zeros in the interval* $(1, \infty)$. *In addition, if* $(n-k)$ *is even,* F *has* $(n-k)$ *non-real zeros whereas if* $(n-k)$ *is odd,* F *has one real negative zero and* $(n-k-1)$ *non-real zeros.*
- *(iii) For* $-n > b > -2n$, *if* $-n-k > b > -n-k-1$, $k = 0, 1, \ldots, n-1$, F *has* $(n-k)$ *real zeros in the interval* $(1, \infty)$. *In addition, if* k *is even,* F *has* k *non-real zeros whereas if* k *is odd,* F *has one real zero in* $(0, 1)$ *and* $(k-1)$ *non-real zeros.*
- *(iv) For* $b < -2n$, *all* n *zeros of* F *are non-real for* n *even whereas for* n *odd,* F *has exactly one real zero in the interval* $(0, 1)$.

Their proof involves an asymptotic analysis of the zeros of an appropriate ${}_3F_2$ hypergeometric polynomial using complex analysis methods. Similar methods are used by Driver and Jordaan in [19] to find the asymptotic zero distribution of the class of ${}_3F_2$ hypergeometric polynomials

$$ {}_3F_2\left(-n, n+1, \frac{1}{2}; b+n+1, 1-b-n; z\right). $$

In both these situations, the zeros of the classes of ${}_3F_2$ polynomials involved cluster asymptotically on the unit circle.

There are sixteen classes of hypergeometric polynomials ${}_2F_1(-n, b; c; z)$ that admit a cubic transformation. They are given in Table 1.2.

${}_2F_1\left(-n,-n-\frac{1}{3};-2n;z\right)$	${}_2F_1\left(-n,-n-\frac{1}{3};\frac{2}{3};z\right)$	${}_2F_1\left(-n,n+1;\frac{4}{3};z\right)$
${}_2F_1\left(-n,-n+\frac{1}{3};-2n;z\right)$	${}_2F_1\left(-n,-n+\frac{1}{3};\frac{4}{3};z\right)$	${}_2F_1\left(-n,n+1;\frac{2}{3};z\right)$
${}_2F_1\left(-n,-n-\frac{1}{3};-2n-\frac{2}{3};z\right)$	${}_2F_1\left(-n,-n-\frac{1}{3};\frac{4}{3};z\right)$	${}_2F_1\left(-n,n+\frac{5}{3};\frac{4}{3};z\right)$
${}_2F_1\left(-n,-n+\frac{1}{3};-2n+\frac{2}{3};z\right)$	${}_2F_1\left(-n,-n+\frac{1}{3};\frac{2}{3};z\right)$	${}_2F_1\left(-n,n+\frac{1}{3};\frac{2}{3};z\right)$
${}_2F_1\left(-n,n+1;2n+2;z\right)$	${}_2F_1\left(-n,n+1;-2n;z\right)$	
${}_2F_1\left(-n,-3n-1;-2n;z\right)$	${}_2F_1\left(-n,\frac{-n+1}{3};\frac{-2n+2}{3};z\right)$	

Table 1.2: Hypergeometric polynomials in the cubic class

Once again, identities link the polynomials across each of the first four rows in Table 1.2 so that the analysis of the zeros of the first four polynomials in the first column deals with the first twelve polynomials. The first twelve polynomials in Table 1.2 all have n real zeros lying in specified intervals. The zeros of the remaining three classes of polynomials can be described as follows (cf. [23], Theorem 3.4, Corollary 3.5 and Theorem 3.6).

Theorem 1.0.5 *Let $n \in \mathbb{N}$.*

(a) The zeros of ${}_2F_1\left(-n,n+1;2n+2;z\right)$ all lie on the circle $|z-1|=1$.

(b) The zeros of ${}_2F_1\left(-n,n+1;-2n;z\right)$ all lie on the unit circle $|z|=1$.

(c) The zeros of ${}_2F_1\left(-n,-3n-1;-2n;z\right)$ all lie on the straight line $Re(z)=\frac{1}{2}$.

(d) The zeros of ${}_2F_1\left(-n,\frac{-n+1}{3};\frac{-2n+2}{3};z\right)$ can be described as follows:

  (i) If $n=3k+1$, then ${}_2F_1\left(-n,\frac{-n+1}{3};\frac{-2n+2}{3};z\right)$ reduces to a polynomial of degree k and all its zeros lie on the straight line $Re(z)=\frac{1}{2}$.

  (ii) If $n=3k$, then ${}_2F_1\left(-n,\frac{-n+1}{3};\frac{-2n+2}{3};z\right)$ has k zeros lying on the circle $|z-1|=1$ and the remaining $2k$ zeros are all non-real if k is even and if k is odd, there 2 real zeros and $(2k-2)$ non-real zeros.

(iii) If $n = 3k+2$, then ${}_2F_1\left(-n, \frac{-n+1}{3}; \frac{-2n+2}{3}; z\right)$ has $(k+2)$ zeros lying on the circle $|z-1| = 1$ and the remaining $2k$ zeros are all non-real if k is even and if k is odd, there 2 real zeros and $(2k-2)$ non-real zeros.

In [29], Klein obtained results on the precise number of zeros of hypergeometric functions ${}_2F_1(a, b; c; x)$ that lie in each of the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ by generalising earlier results of Hilbert (1888). These Hilbert-Klein formulas are for hypergeometric functions and not only for polynomials. Szegö recaptured these results for the special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which have a representation as ${}_2F_1$ hypergeometric polynomials (see p.14), in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ (cf. [45], p.145, Theorem 6.72). The number and location of the real zeros of the hypergeometric polynomial ${}_2F_1(-n, b; c; z)$ for b and c real can be deduced (see [20]).

${}_1F_1$ hypergeometric polynomials, sometimes called Whittaker functions, also have connections with the classical orthogonal polynomials of Laguerre and Hermite, where the Laguerre polynomial is given by $L_n^{(\alpha)}(x)$ and can be written as

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_1F_1\left(-n; 1+\alpha; x\right)$$

and the Hermite polynomial $H_n(x)$ can be expressed in terms of Laguerre polynomials by means of the two formulas

$$H_{2n}(x) = (-1)^n 2^{2n} n!\ L_n^{(-1/2)}(x^2)$$

and

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n!\ L_n^{(1/2)}(x^2).$$

Again, knowledge of the nature and location of the zeros of ${}_1F_1$ hypergeometric polynomials can be deduced from the corresponding results for Laguerre and Hermite polynomials.

In the case of ${}_3F_2$ hypergeometric polynomials, much less is known about their properties and the location of their zeros, because there is in general no direct link with orthogonal polynomials. In addition, there are no integral representations, connection formulas, and so on, as in the Gauss case. In rather special cases where a ${}_3F_2$ polynomial can be expressed as a product of two ${}_2F_1$ polynomials, the orthogonality of the relevant ${}_2F_1$ polynomials leads to information about the zeros of the ${}_3F_2$. In [21], Driver and Love consider three classes of ${}_3F_2$ hypergeometric polynomials of degree $2n$ containing one free parameter b that can be written as the product of two ${}_2F_1$ hypergeometric polynomials, each of degree n. They describe the location of the zeros as b varies continuously through real values. These results are extended in [18] to include classes of ${}_3F_2$ hypergeometric polynomials which have simple representations in terms of a ${}_2F_1$ hypergeometric polynomial.

The Askey-scheme (see figure 1.1) is a table for hypergeometric and basic hypergeometric orthogonal polynomials proposed by Richard Askey. It was compiled by Koekoek and Swarttouw in 1998 (see [30]). They consider orthogonal polynomials appearing in the Askey-scheme of hypergeometric orthogonal polynomials and give a q-analogue of this scheme containing basic hypergeometric orthogonal polynomials. For each family of orthogonal polynomials listed in the Askey-scheme, conditions are given on the parameters for which the corresponding weight function is positive. These conditions are mentioned in the orthogonality relations. Many of these orthogonal polynomials are still polynomials (but no longer orthogonal) for other values of the parameters and some can be meaningfully defined for other values as well. Other formulas, such as the generating functions, are only valid for some special values of parameters and arguments. The report [30] is not a full account of all that is known about hypergeometric orthogonal polynomials, however references are given to complete the outline.

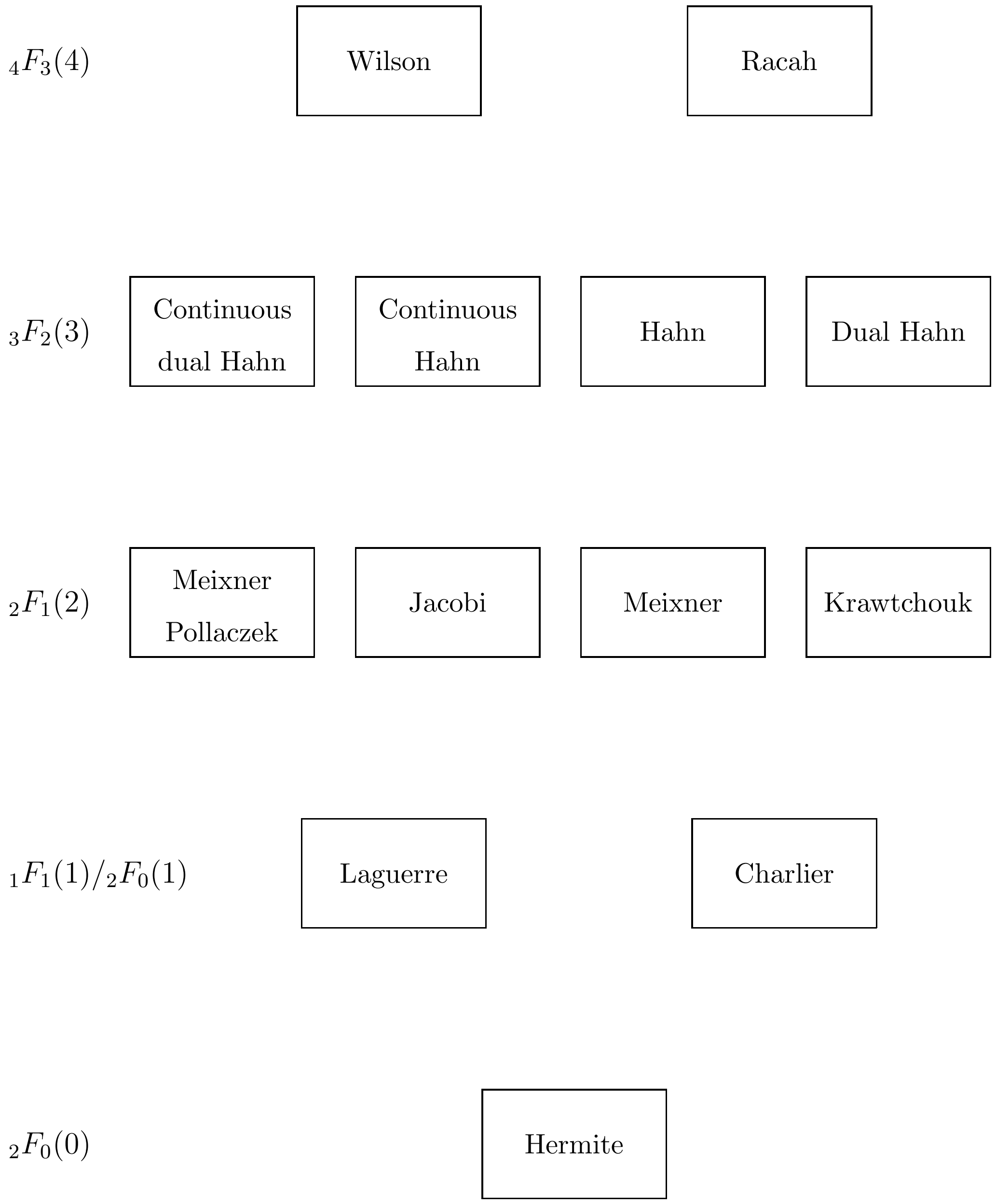

Figure 1.1: The Askey-scheme for hypergeometric orthogonal polynomials.

It is well known that the Jacobi polynomials $P_n^{(\alpha,\beta)}$ are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$ for $\alpha,\beta > -1$ and consequently that their zeros lie in $(-1,1)$ (cf. [1]). As α and/or β decrease below -1, the zeros of $P_n^{(\alpha,\beta)}$ depart from the interval $(-1,1)$. The disappearance of the zeros from the interval of orthogonality occurs in a specific way which can be explained in terms of the concept of quasi-orthogonality given in the following definition.

Definition 1.0.6 *A sequence $r_0,\ r_1,\ r_2,\ldots$ of real polynomials with deg(r_k)$=k$ is quasi-orthogonal of order j, where j is a fixed non-negative integer, with respect to a positive weight function $w(x)$ on an interval I if*

$$\int_I x^l\, r_n(x)w(x)dx = 0 \qquad \textit{for} \qquad l = 0,1,\ldots,n-j-1$$

for $n = j, j+1, j+2, \ldots$.

In addition, at least $(n-j)$ zeros of $r_n(x)$ lie in the interval I (cf. [5]). Note that quasi-orthogonal polynomials $r_n(x)$ are only defined for $n \geq j$. If $j = 0$, then $r_n(x)$ is orthogonal on I with respect to the weight function $w(x)$.

Quasi-orthogonal polynomials have been discussed by Chihara [7], Dickinson [9] and Draux [10] and more recently by Brezinski, Driver and Redivo-Zaglia [5] and Joulak [28].

In [7], Chihara discusses quasi-orthogonality in the context of three-term recurrence relations. He proves that a quasi-orthogonal polynomial of any order j satisfies a three-term recurrence relation whose coefficients are polynomials of appropriate degrees. Draux [10] proved the converse of one of Chihara's results and Dickinson [9] improved Chihara's result by deriving a system of recurrence relations, both necessary and sufficient for quasi-orthogonality. He applies these results to some special cases of Sister Celine's polynomials which are given by

$$f_n(a,x) = {}_3F_2\left(-n, n+1, a; \frac{1}{2}, 1; x\right). \tag{1.0.5}$$

He proves that $f_n\left(\frac{3}{2}, x\right)$ and $f_n\left(2, x\right)$ are quasi-orthogonal of order 1 on the interval $(0,1)$ with respect to the weight functions $(1-x)$ and $x^{-1/2}(1-x)^{3/2}$ respectively. We confirm these results in Chapter 2.

Brezinski, Driver and Redivo-Zaglia [5] consider the location of the zeros of quasi-orthogonal polynomials of order 1 and of order 2. They use classical results such as the Christoffel-Darboux identity and apply these quasi-orthogonality results to Jacobi, Gegenbauer and Laguerre polynomials when the restrictions on the parameters that ensure their orthogonality are not satisfied.

In [28], Joulak studies quasi-orthogonal polynomials and their associated polynomials using tools from linear algebra. He generalises the results in [5] to give more precise results on the location of the zeros, as well as results on the interlacing properties of the zeros of quasi-orthogonal polynomials of order 1, 2 and 3.

Another interesting situation in which quasi-orthogonality arises in a natural way concerns linear combinations of orthogonal polynomials. In his fundamental paper, Shohat [44] proved that if $\{p_n\}$ is a family of orthogonal polynomials on $[a,b]$ with respect to a positive weight function $w(x)$, then a necessary and sufficient condition for a polynomial $r_n(x)$ of degree n to be quasi-orthogonal of order r on $[a,b]$ with respect to w is that

$$r_n(x) = c_0 p_n(x) + c_1 p_{n-1}(x) + \ldots + c_r p_{n-r}(x),$$

where the c_i's are numbers which can depend on n and $c_0 c_r \neq 0$. The zeros of linear combinations of orthogonal polynomials is further explored in [2].

Moving to the question of the asymptotic zero distribution of hypergeometric polynomials, a series of papers by Saff and Varga appeared in the 1970's (cf. [38], [39], [40], [41], [42], [43]). Their primary focus was the asymptotic zero distribution of the polynomials arising in the Padé approximant to the exponential function. It turns out that the numerator and denominator polynomials are both ${}_1F_1$ polynomials (or Whittaker functions) and Saff and Varga obtained the asymptotic zero distribution of the relevant ${}_1F_1$'s as well as proving results about zero-free parabolic regions. Driver and Temme (cf. [24], [25], [26]) use a different approach to describe the asymptotic zero distribution of the polynomials occurring in the Padé approximant to the exponential function. They use uniform expansions of integrals, in which Airy functions arise, to obtain a refinement of the main Saff-Varga result.

There have been several recent developments ([31], [32], [34], [35]) regarding the asymptotic zero distribution of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(z)$ with parameters α_n and β_n that depend on n. Since the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be defined in terms of a hypergeometric polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right),$$

the study of the zeros of Jacobi polynomials therefore gives direct information about the zeros of the corresponding ${}_2F_1$ hypergeometric functions. In [34], Martínez-Finkelshtein, Martínez-González and Orive consider the asymptotic zero distribution of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$ where the limits

$$A = \lim_{n\to\infty} \frac{\alpha_n}{n} \quad \text{and} \quad B = \lim_{n\to\infty} \frac{\beta_n}{n} \tag{1.0.6}$$

exist. They distinguish five cases depending on the values of A and B. The six straight lines $A = 0$, $A = -1$, $B = 0$, $B = -1$, $A + B = -1$ and

$A + B = -2$, divide the AB-plane into different regions and the authors in [34] prove results for the "general" cases where A and B lie in the interior of each of the regions bounded by the lines shown in Figure 1.2.

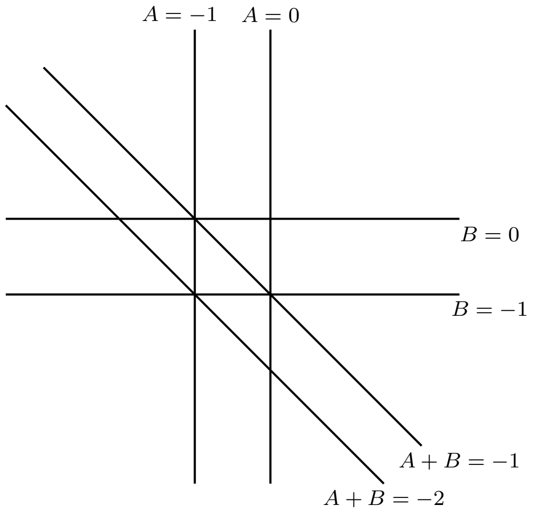

Figure 1.2: The asymptotic zero distribution of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(z)$

The first of the five cases is the case where $A, B > 0$. This corresponds to classical Jacobi polynomials with varying parameters. These polynomials are orthogonal on the interval $[-1, 1]$ and their zeros are real and simple and lie in the interval $(-1, 1)$.

The formulas

$$P_n^{(a,b)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-2n-a-b-1,b)}\left(\frac{x+3}{x-1}\right)$$

and

$$P_n^{(a,b)}(x) = \left(\frac{1+x}{2}\right)^n P_n^{(a,-2n-a-b-1)}\left(\frac{3-x}{x+1}\right)$$

reduce the case $B > 0$ and $A + B < -2$ as well as the case $A > 0$ and $A + B < -2$ respectively to the classical case.

The second case corresponds to where the limits A and B in (1.0.6) satisfy one of the three combinations

$$A < -1,\ A + B > -1$$

or

$$B < -1,\ A + B > -1$$

or

$$A < -1,\ B < -1.$$

In this case, the zeros of the Jacobi polynomials accumulate along an open arc in the complex plane and their asymptotic distribution is discussed in [34].

The third, fourth and fifth cases correspond to combinations of A and B values such that exactly one, exactly two or exactly three, respectively, of the inequalities

$$-1 < A < 0,\ -1 < B < 0,\ \text{and}\ -2 < A + B < -1$$

are satisfied. These cases are treated in [31].

The transitional lines or boundaries namely $A = 0$, $A = -1$, $B = 0$, $B = -1$, $A + B = -1$ and $A + B = -2$ are "non-general" cases in contrast to the aforementioned five general cases. Some of these have been studied in the context of the corresponding hypergeometric function (see [3], [11], [22], [27]).

In [22], Driver and Möller obtain the asymptotic zero distribution of the hypergeometric polynomial

$${}_2F_1(-n, b; -2n; z), \quad n \in \mathbb{N},\ b \in \mathbb{R},$$

which solves a conjecture of Martínez-Finkelshtein, Martínez-González and Orive (cf. [34]) regarding the asymptotic zero distribution of the Jacobi polynomial given by

$$P_n^{(n+b,-n-b)}.$$

This corresponds to the values $A = 1$ and $B = -1$ in (1.0.6). Driver and Möller proved that for $b > 0$, the zeros of

$$z^n \, {}_2F_1\left(-n, b; -2n; \frac{1}{z}\right)$$

asymptotically approach the Cassini curve

$$|(2z-1)^2 - 1| = 1.$$

They raise the possibility that in fact for each n, the zeros of ${}_2F_1\left(-n, b; -2n; \frac{1}{z}\right)$ lie on some Cassini curve – as yet there has been no proof.

In [11], Driver and Duren invoke a theorem of Borwein and Chen [4] to show that the zeros of the hypergeometric polynomial

$${}_2F_1(-n, kn+1; kn+2; z), \qquad k \in \mathbb{N}$$

cluster asymptotically on the loop of the lemniscate

$$\left|z^k(1-z)\right| = \frac{k^k}{(k+1)^{k+1}} \quad \text{with} \quad \operatorname{Re}(z) > \frac{k}{k+1}$$

as $n \to \infty$. This hypergeometric polynomial corresponds to the Jacobi polynomial $P_n^{(kn+1,-n+1)}$ with limit values $A = k \in \mathbb{N}$ and $B = -1$.

In [27], Duren and Guillou prove a conjecture in [11] that the requirement that k be an integer is not necessary. They give an independent direct proof since conditions needed for the Borwein-Chen approach do not apply for arbitrary $k > 0$. They prove that for each $k > 0$, the zeros of the hypergeometric polynomial

$${}_2F_1(-n, kn+1; kn+2; z)$$

cluster on the loop of the lemniscate

$$\left|z^k(1-z)\right| = \frac{k^k}{(k+1)^{k+1}} \quad \text{with} \quad \mathrm{Re}(z) > \frac{k}{k+1}$$

as $n \to \infty$. This hypergeometric polynomial corresponds to the Jacobi polynomial with limits $A = k \in \mathbb{R}^+$ and $B = -1$. Duren and Guillou also show that every point of the curve is a cluster point of zeros.

In [3], Boggs and Duren extend the discussion further. They consider a more general case of a ${}_2F_1$ hypergeometric polynomial and prove that for each $k > 0$ and $l \geq 0$, the zeros of

$${}_2F_1(-n, kn + l + 1; kn + l + 2; z)$$

cluster on the loop of the lemniscate

$$\left|z^k(1-z)\right| = \frac{k^k}{(k+1)^{k+1}} \quad \text{with} \quad \mathrm{Re}(z) > \frac{k}{k+1}$$

as $n \to \infty$. This corresponds to the Jacobi polynomial with limits $A = k \in \mathbb{R}^+$ and $B = -1$. These results are proved by adapting the analysis of the proofs in [27]. They also establish more general statements about zero-free regions of hypergeometric functions that are not necessarily hypergeometric polynomials.

Our primary interest in this thesis lies in two main areas: the location of zeros and the asymptotic zero distribution of ${}_3F_2$ hypergeometric polynomials. In chapter 2, we discuss the location of the zeros of three classes of ${}_3F_2$ hypergeometric polynomials of degree n using the orthogonality and quasi-orthogonality of factors involved in the polynomials' representation. We use Mathematica to plot the zeros of theses ${}_3F_2$ hypergeometric polynomials for different values of n as well as for different ranges of the parameters. The numerical data is consistent with the results we have proved. The results in this chapter are new and have been published (cf. [15]).

In chapter 3, we consider integral representation of various hypergeometric functions and, in particular, the ${}_3F_2$ hypergeometric function with various parameters. We give a simple and direct proof of an integral representation for a special class of ${}_{q+1}F_q$ functions for $q \geq 2$. The values of certain ${}_3F_2$ and ${}_4F_3$ functions at $x = 1$, some of which can be derived using other methods, are deduced from our integral formula. The results in Chapter 3 have been published (cf. [17]).

We use the integral representation formula for ${}_3F_2$ functions proved in Chapter 3 as a tool to find the asymptotic zero distribution of a class of ${}_2F_1$ hypergeometric polynomials in chapter 4 that have not been previously dealt with. In order to prove our result, we follow the method used by Duren and Guillou in [27] with some modifications. This method involves the asymptotic analysis of an integral of the form

$$A_n \int_0^1 [f_z(t)]^n \; dt$$

where A_n is a constant involving n and $f_z(t)$ is a polynomial in the complex variable t and analytic in z. The results in this chapter are new and have been submitted for publication (cf. [16]).

We note briefly that an often more convenient notation for the ${}_pF_q$ hypergeometric polynomial is frequently used. This variation is given by

$${}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}; z\right)$$

and both notations will be used in this thesis purely to facilitate type-setting issues that arise in different contexts.

Chapter 2

Quasi-orthogonality, orthogonality and zeros of some ${}_3F_2$ hypergeometric polynomials

2.1 Introduction

We begin with some definitions and preliminary results which are well known in the literature. Our new results are then divided into three sections.

In Section 2.3, we prove that the hypergeometric polynomial ${}_3F_2(-n, a, b; a-1, d; x)$ is quasi-orthogonal of order 1 on different intervals that depend on the parameters b and d and deduce the location of $(n-1)$ zeros of ${}_3F_2(-n, a, b; a-1, d; x)$. We discuss numerical results obtained by using Mathematica that indicate the role played by the parameter a with regard to the one remaining zero of ${}_3F_2(-n, a, b; a-1, d; x)$.

In Section 2.4, we prove orthogonality and quasi-orthogonality results for each factor involved in a product representation of ${}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$ for different ranges of the real parameter b and deduce information regarding the location of all its zeros.

The class of polynomials ${}_3F_2\left(-n, b, \frac{b-n}{2}+1; b-n, \frac{b-n+1}{2}; x\right)$ is similar in structure to the class discussed in Section 2.4 and we devote Section 2.5 to a discussion of the orthogonality and quasi-orthogonality properties of this class. It should be noted that in all our discussions, $n \in \mathbb{N}$ is restricted by the values of the fixed parameters b and d which means that our sequences of orthogonal and quasi-orthogonal polynomials are finite.

2.2 Preliminary results

We recall the definitions of orthogonality and quasi-orthogonality from Chapter 1. For completeness, we give them again.

Definition 2.2.1 *A sequence $p_0,\ p_1,\ p_2, \ldots$ of real polynomials with deg(p_k)$=$ k is orthogonal with respect to a positive weight function $w(x)$ on an interval I if*

$$\int_I x^l\ p_n(x)w(x)dx = 0 \qquad \textit{for}\, l = 0, 1, \ldots, n-1$$

for $n = 1, 2, \ldots$.

Definition 2.2.2 *A sequence $r_0,\ r_1,\ r_2, \ldots$ of real polynomials with deg(r_k)$=$ k is quasi-orthogonal of order j, where j is a fixed non-negative integer, with respect to a positive weight function $w(x)$ on an interval I if*

$$\int_I x^l\ r_n(x)w(x)dx = 0 \qquad \textit{for} \qquad l = 0, 1, \ldots, n-j-1$$

for $n = j, j+1, j+2, \ldots$.

A well-known consequence of orthogonality is the following result (see for example [37]).

Lemma 2.2.3 *If a sequence* $p_0,\ p_1,\ p_2, \ldots$ *of real polynomials with deg(*p_k*)=* k *is orthogonal with respect to a weight function* $w(x)$ *on an interval* I*, then the zeros of* $p_n(x)$ *are real and simple and lie in the open interval* I*.*

The corresponding consequence of quasi-orthogonality is given, for example, in [5], Theorem 2.

Lemma 2.2.4 *If* $p_n(x)$ *is a real polynomial of exact degree* n *that is quasi-orthogonal of order* j *with respect to a weight function* $w(x)$ *on an interval* I*, then at least* $(n-j)$ *zeros of* $p_n(x)$ *lie in the interval* I*.*

We shall use Rodrigues' formula for the ${}_2F_1$ hypergeometric polynomial (cf. [1], p.99) several times in our proofs.

Theorem 2.2.5 *(cf.[37] p.257-261) Let* $n \in \mathbb{N}$*,* $b, d \in \mathbb{R}$*. The polynomial* ${}_2F_1(-n, b; d; x)$ *is orthogonal to all polynomials with degree less than* n *with respect to the weight function* $w(x) = |x|^{d-1}|1-x|^{b-d-n}$ *on the intervals*

(i) $(0,1)$ *if* $0 < d < b+1-n$*;*

(ii) $(-\infty, 0)$ *if* $d > 0$ *and* $b < 1-n$*; and*

(iii) $(1, \infty)$ *if* $d+n-1 < b < 1-n$*.*

Proof:

(i) Note first that since $x \in (0,1)$, $w(x) = x^{d-1}(1-x)^{b-d-n}$. Rodrigues' formula for the hypergeometric function is given by (cf.[1], p.99, eqn.(2.5.12))

$$\frac{d^n}{dx^n}[x^{n+d-1}(1-x)^{b-d}] = (d)_n\, x^{d-1}(1-x)^{b-d-n}\, {}_2F_1(-n, b; d; x).$$

Denoting $\frac{d}{dx}$ by D, we have for $0 \leq l \leq n-1$,

$$\int_0^1 x^l \, {}_2F_1(-n,b;d;x)\, x^{d-1}(1-x)^{b-d-n}dx$$
$$= \frac{1}{(d)_n} \int_0^1 x^l \, \left\{D^n \left[x^{n+d-1}(1-x)^{b-d}\right]\right\} dx. \tag{2.2.1}$$

Integrating (2.2.1) by parts n times, each time differentiating x^l and integrating $D^n\left[x^{n+d-1}(1-x)^{b-d}\right]$, we obtain the stated result from an elementary analysis of the lowest and highest powers of x and $(1-x)$ respectively. Parts (ii) and (iii) follow by a similar analysis. □

2.3 The polynomial ${}_3F_2(-n,a,b;a-1,d;x)$, $a,b,d \in \mathbb{R}, n \in \mathbb{N}$

The main result of this section is the following theorem.

Theorem 2.3.1 *Let* $F(x) = {}_3F_2(-n,a,b;a-1,d;x)$ *where* $a, d \notin \mathbb{Z}^-$ *and let* $w(x) = |x|^{d-1}|1-x|^{b-d-n+1}$. *Then* F *is orthogonal to all polynomials with degree less than* $n-1$ *with respect to* $w(x)$ *on the intervals*

(i) $(0,1)$ *if* $0 < d < b-n+2$;

(ii) $(-\infty,0)$ *if* $d > 0$ *and* $b < -n$; *and*

(iii) $(1,\infty)$ *if* $d+n-2 < b < -n$.

Proof:
First, from [36], p.497, equation(3), we know that

$$\begin{aligned} &{}_3F_2(-n,a,b;a-1,d;x) \\ &\quad = {}_2F_1(-n,b;d;x) - \frac{nbx}{(a-1)d}\, {}_2F_1(-n+1,b+1;d+1;x). \end{aligned} \tag{2.3.1}$$

In order to prove that

$$\begin{aligned} &I(x) \\ &= \int_s^t x^l \left[{}_2F_1(-n,b;d;x) - \frac{nbx}{(a-1)d}\, {}_2F_1(-n+1,b+1;d+1;x) \right] w(x)\, dx \\ &= 0 \end{aligned} \tag{2.3.2}$$

for $l = 0, 1, \ldots, n-2$, we first rewrite ${}_2F_1(-n,b;d;x)$ using the contiguous relation (cf.[37], p.71, eqn(1))

$$(a-b)\, {}_2F_1(a,b;d;x) = a\, {}_2F_1(a+1,b;d;x) - b\, {}_2F_1(a,b+1;d;x).$$

Putting $a = -n$, we have

$${}_2F_1(-n,b;d;x) = \frac{n}{b+n}\, {}_2F_1(-n+1,b;d;x) + \frac{b}{b+n}\, {}_2F_1(-n,b+1;d;x). \tag{2.3.3}$$

It follows from (2.3.2) and (2.3.3) that we can write $I(x) = I_1(x)+I_2(x)+I_3(x)$ where

$$I_1(x) = \frac{n}{b+n} \int_s^t x^l\, {}_2F_1(-n+1,b;d;x)|x|^{d-1}|1-x|^{b-d-(n-1)}dx, \tag{2.3.4}$$

$$I_2(x) = \frac{b}{b+n} \int_s^t x^l\, {}_2F_1(-n,b+1;d;x)|x|^{d-1}|1-x|^{(b+1)-d-n}dx, \tag{2.3.5}$$

$$\begin{aligned} &I_3(x) = \\ &\frac{-nb}{(a-1)d} \int_s^t x^l\, {}_2F_1(-n+1,b+1;d+1;x)|x|^{(d+1)-1}|1-x|^{(b+1)-(d+1)-(n-1)}dx. \end{aligned} \tag{2.3.6}$$

(i) If $s = 0$ and $t = 1$, then for $d > 0$ and $b > d+n-2$, it follows from Theorem 2.2.5 (i) that $I_1(x) = 0$ for $l = 0, 1, \ldots, n-2$; $I_2(x) = 0$ for $l = 0, 1, \ldots, n-1$; and $I_3(x) = 0$ for $l = 0, 1, \ldots, n-2$. Thus $I(x) = 0$ for $l = 0, 1, \ldots, n-2$ and this proves that for $0 < d < b-n+2$, $F(x)$ is quasi-orthogonal of order 1 with respect to the weight function $w(x)$ on the interval $(0, 1)$.

(ii) If $s \to -\infty$ and $t = 0$, then for $d > 0$ and $b < -n$, we have from Theorem 2.2.5 (ii) that $I_1(x) = 0$ for $l = 0, 1, \ldots, n-2$; $I_2(x) = 0$ for $l = 0, 1, \ldots, n-1$; and $I_3(x) = 0$ for $l = 0, 1, \ldots, n-2$. Thus $I(x) = 0$ for $l = 0, 1, \ldots, n-2$ and this proves that for $d > 0$ and $b < -n$, $F(x)$ is quasi-orthogonal of order 1 with respect to the weight function $w(x)$ on the interval $(-\infty, 0)$.

(iii) If $s = 1$ and $t \to \infty$, then for $d+n-2 < b < -n$, Theorem 2.2.5 (iii) gives that $I_1(x) = 0$ for $l = 0, 1, \ldots, n-2$; $I_2(x) = 0$ for $l = 0, 1, \ldots, n-1$; and $I_3(x) = 0$ for $l = 0, 1, \ldots, n-2$. Thus $I(x) = 0$ for $l = 0, 1, \ldots, n-2$ and this proves that for $d+n-2 < b < -n$, $F(x)$ is quasi-orthogonal of order 1 with respect to the weight function $w(x)$ on the interval $(1, \infty)$. □

The following corollary is a direct consequence of Corollary 2.2.4 along with Theorem 2.3.1.

Corollary 2.3.2 *With $F(x)$ as in Theorem 2.3.1, at least $(n-1)$ zeros of $F(x)$ lie in*

(i) $(0, 1)$ *if* $0 < d < b - n + 2$*;*

(ii) $(-\infty, 0)$ *if* $d > 0$ *and* $b < -n$*;*

(iii) $(1, \infty)$ *if* $d + n - 2 < b < -n$*.*

Proof:

This is an immediate consequence of Theorem 2.3.1 together with Corollary 2.2.4. □

We give some examples generated by Mathematica to illustrate Theorem 2.3.1 and Corollary 2.3.2.

Example 2.3.3 $\quad {}_3F_2(-14, 9.5, 13.53; 8.5, 1.28; x)$

When letting $n = 14$, $a = 9.5$, $b = 13.53$, $d = 1.28$ in ${}_3F_2(-n, a, b; a-1, d; x)$, Corollary 2.3.2 (i) ensures that at least 13 zeros of polynomial

$${}_3F_2(-14, 9.5, 13.53; 8.5, 1.28; x)$$

lie in the interval $(0, 1)$. Indeed, the zeros are

```
{{x → 0.00935228}, {x → 0.0411514}, {x → 0.0944696}, {x → 0.167114},
 {x → 0.255922}, {x → 0.356837}, {x → 0.465082}, {x → 0.575402}, {x → 0.682348},
 {x → 0.780573}, {x → 0.865129}, {x → 0.931735}, {x → 0.977003}, {x → 0.998586}}
```

which is represented by the plot in figure 2.1.

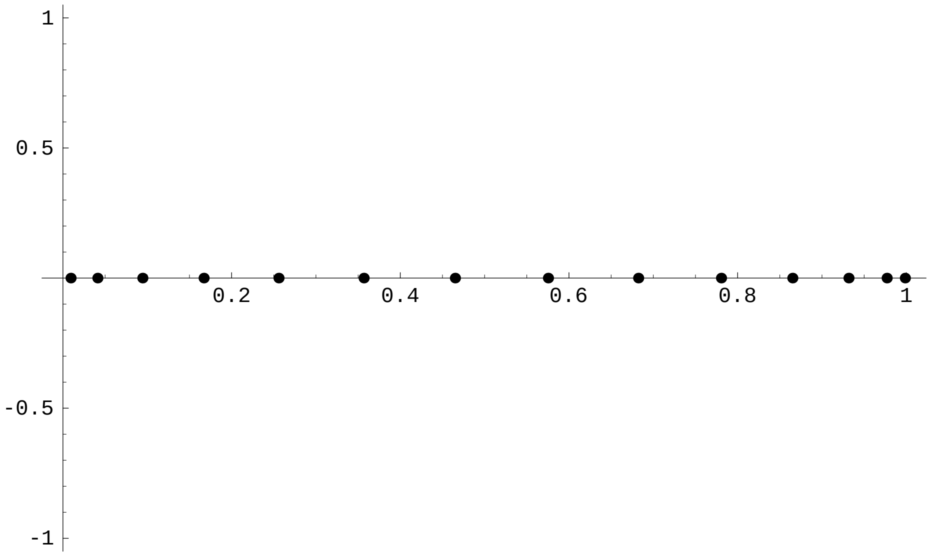

Figure 2.1: The zeros of ${}_3F_2(-14, 9.5, 13.53; 8.5, 1.28; x)$

Example 2.3.4 $\quad {}_3F_2(-10, -5.75, -11.5; -6.75, 9.5; x)$

When letting $n = 10$, $a = -5.75$, $b = -11.5$, $d = 9.5$ in the polynomial ${}_3F_2(-n, a, b; a-1, d; x)$, Corollary 2.3.2 (ii) ensures that at least 9 zeros of polynomial

$${}_3F_2(-10, -5.75, -11.5; -6.75, 9.5; x)$$

lie in the interval $(-\infty, 0)$. Indeed, the zeros are

```
{{x → -33.7586}, {x → -11.1162}, {x → -5.32214}, {x → -2.9769}, {x → -1.79487},
 {x → -1.11695}, {x → -0.694137}, {x → -0.414733}, {x → -0.22095}, {x → 6.18469}}
```

which can be represented by the plot in figure 2.2.

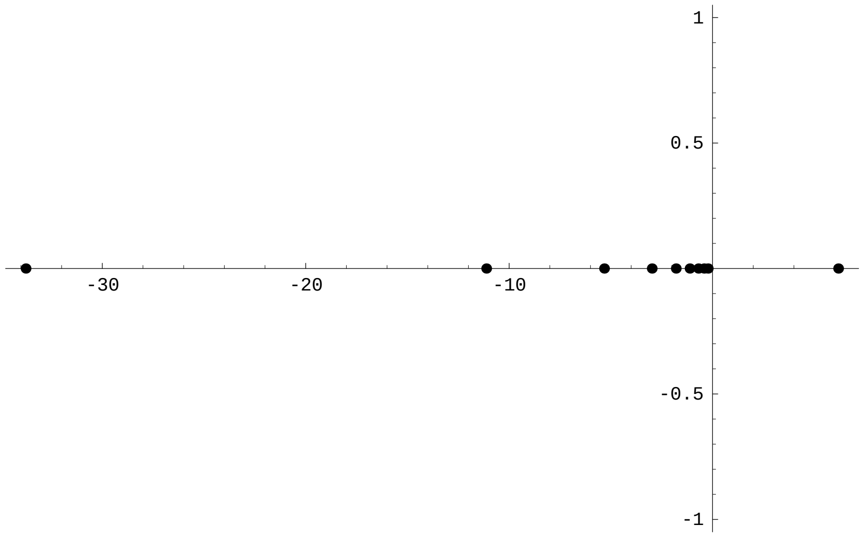

Figure 2.2: The zeros of ${}_3F_2(-10, -5.75, -11.5; -6.75, 9.5; x)$

Example 2.3.5 ${}_3F_2(-10, 0.75, -11.5; -0.25, -19.9; x)$

When letting $n = 10$, $a = 0.75$, $b = -11.5$, $d = -19.9$ in the polynomial ${}_3F_2(-n, a, b; a-1, d; x)$, Corollary 2.3.2 (iii) ensures that at least 9 zeros of polynomial

$$ {}_3F_2(-10, 0.75, -11.5; -0.25, -19.9; x) $$

lie in the interval $(1, \infty)$. Indeed, the zeros are

```
{{x → -0.044764}, {x → 1.00483}, {x → 1.05417}, {x → 1.16618}, {x → 1.36831},
 {x → 1.72093}, {x → 2.36388}, {x → 3.67233}, {x → 6.9528}, {x → 19.8695}}
```

which can be represented by the plot in figure 2.3.

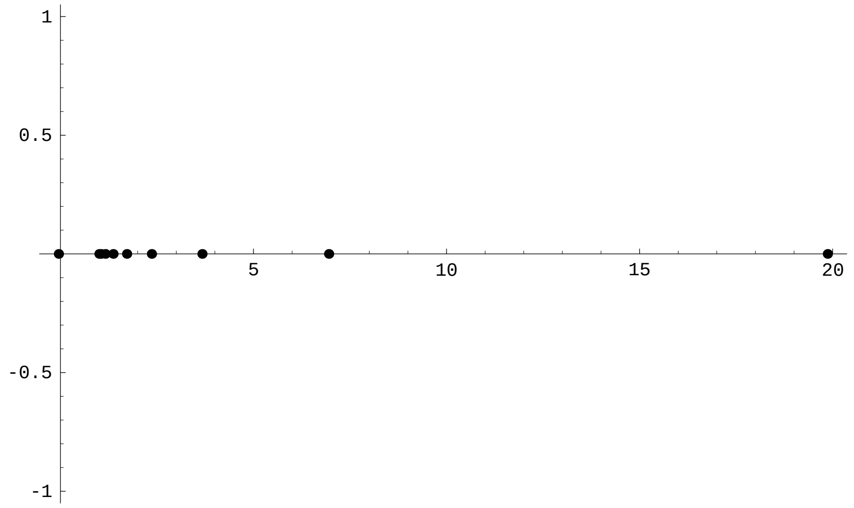

Figure 2.3: The zeros of ${}_3F_2(-10, 0.75, -11.5; -0.25, -19.9; x)$

Note that in examples 2.3.4 and 2.3.5, the "missing zero" is real (which it must be) but lies outside the interval on quasi-orthogonality whereas in example 2.3.3, the missing zero is also, in this case, inside the interval.

Remark: For the ranges of the parameters b and d satisfying the conditions in Corollary 2.3.2 (i), (ii) or (iii), we know that $(n-1)$ zeros of ${}_3F_2(-n, a, b; a-1, d; x)$ lie in a specified interval while the one remaining zero must be real. The weight function $w(x)$ in Theorem 2.3.1 depends on n, b and d but not on the parameter a. One would expect that the value of a plays a role in determining the location of the "missing" zero. Indeed, numerical evidence generated by Mathematica suggests that for $0 < d < b - n + 2$, that is when $(n-1)$ zeros of $F(x)$ lie in $(0, 1)$, the remaining zero lies in $(0, 1)$ when $a > 1$, whereas it lies in $(-\infty, 0)$ when $-n + 1 < a < 1$ and in $(1, \infty)$ when $a < -n + 1$.

This is reminiscent of the results proved in [5] Theorem 3 and [28] Theorem 4 where the location of all n of the zeros of the linear combination $p_n(x) + a_n\, p_{n-1}(x)$, which is quasi-orthogonal of order 1, is proved for different values of a_n. The proofs given in [5] and [28] do not apply here directly since ${}_3F_2(-n, a, b; a-1, d; x)$ does not seem to be expressible as a linear combination of ${}_2F_1$ polynomials (see 2.3.1). In addition, our weight function varies with n.

We note briefly that if we let $a = \frac{3}{2}$, $b = n + 1$ and $d = 1$ in the polynomial ${}_3F_2(-n, a, b; a-1, d; x)$, we obtain a special case of Sister Celine's polynomial defined in equation (1.0.5), specifically $f_n\left(\frac{3}{2}, x\right)$. In [9], Dickinson proved that $f_n\left(\frac{3}{2}, x\right)$ is quasi-orthogonal of order 1 on the interval $(0, 1)$ with respect to the weight function $(1-x)$. Theorem 2.3.1 confirms this result and, similarly with $a = 2$, $b = n + 1$ and $d = \frac{1}{2}$, we can conclude that $f_n(2, x)$ is quasi-orthogonal of order 1 on the interval $(0, 1)$ with respect to the weight function $x^{-1/2}(1-x)^{3/2}$.

2.4 The polynomial ${}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$, $n \in \mathbb{N}$, $b \in \mathbb{R}$.

We now turn to the class of ${}_3F_2$ polynomials that can be expressed as the product of a ${}_2F_1$ and a second factor that is a linear combination of ${}_2F_1$'s. The source of many of our formulas is Prudnikov et al [36].

From [36] p.498, equation (21), we have

$$ {}_3F_2\left(a, b, \frac{a+b}{2}; a+b, \frac{a+b-1}{2}; x\right) = {}_2F_1\left(\frac{a}{2}, \frac{b}{2}; \frac{a+b+1}{2}; x\right) G(x) \tag{2.4.1}$$

where

$$G(x) = 2\,{}_2F_1\left(\frac{a}{2}, \frac{b}{2}; \frac{a+b-1}{2}; x\right) - {}_2F_1\left(\frac{a}{2}, \frac{b}{2}; \frac{a+b+1}{2}; x\right). \tag{2.4.2}$$

Putting $a = -2n$, $n \in \mathbb{N}$ and replacing b by $2b$ for notational convenience, equations (2.4.1) and (2.4.2) become

$$
\begin{aligned}
{}_3F_2&\left(-2n, 2b, b-n; 2b-2n, b-n-\frac{1}{2}; x\right) \\
&= {}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right) G(x) \qquad (2.4.3)
\end{aligned}
$$

where

$$
G(x) = 2\,{}_2F_1\left(-n, b; b-n-\frac{1}{2}; x\right) - {}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right). \qquad (2.4.4)
$$

Thus the left hand side of (2.4.3) is a polynomial of degree $2n$ in x while the right hand side is the product of two polynomials each of degree n in x. Applying Theorem 2.2.5 to ${}_2F_1(-n, b; b-n+\frac{1}{2}; x)$, the first factor on the right hand side of (2.4.3), we immediately obtain the following result.

Lemma 2.4.1 *Let $n \in \mathbb{N}$ and $b \in \mathbb{R}$. The polynomial ${}_2F_1(-n, b; b-n+\frac{1}{2}; x)$ is orthogonal to all polynomials with degree less than n with respect to the weight function $|x|^{b-n-1/2}|1-x|^{-1/2}$ on*

- *(i) $(0, 1)$ if $b > n - \frac{1}{2}$;*
- *(ii) $(1, \infty)$ if $b < 1 - n$.*

The following corollary follows as a direct consequence of Lemma 2.4.1 using Lemma 2.2.3.

Corollary 2.4.2 *The polynomial ${}_2F_1(-n, b; b-n+\frac{1}{2}; x)$, $n \in \mathbb{N}$, $b \in \mathbb{R}$, has n real and simple zeros that lie in*

- *(i) $(0, 1)$ if $b > n - \frac{1}{2}$;*
- *(ii) $(1, \infty)$ if $b < 1 - n$.*

We note that the conditions for b and n that need to be satisfied for

$$ {}_3F_2\left(-2n, 2b, b-n; 2b-2n, b-n-\frac{1}{2}; x\right) $$

to have negative zeros, namely $b < 1-n$ and $b > n-\frac{1}{2}$, are not consistent.

For the polynomial $G(x)$ given in (2.4.4), we have the following lemma.

Lemma 2.4.3 *Let $G(x)$ be as in (2.4.4) where $b \notin \mathbb{Z}^-$, and let $w(x) = |x|^{b-n-1/2}|1-x|^{1/2}$. Then $G(x)$ is orthogonal to all polynomials with degree less than $n-1$ with respect to $w(x)$ on*

(i) $(0,1)$ *if* $b > n+\frac{1}{2}$*;*

(ii) $(1,\infty)$ *if* $b < -n$.

Proof:

(i) Suppose $b > n+\frac{1}{2}$. We first rewrite ${}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right)$ using the contiguous relation (cf. [37], p.71, eqn(13))

$$ (b-c+1){}_2F_1(a,b;c;x) = b{}_2F_1(a,b+1;c;x) - (c-1){}_2F_1(a,b;c-1;x) $$

to obtain

$$ \begin{aligned} {}_2F_1&\left(-n, b; b-n+\frac{1}{2}; x\right) \\ &= \frac{2b}{2n+1}\,{}_2F_1\left(-n, b+1; b-n+\frac{1}{2}; x\right) \\ &\qquad - \frac{2b-2n-1}{2n+1}\,{}_2F_1\left(-n, b; b-n-\frac{1}{2}; x\right) \end{aligned} \tag{2.4.5} $$

Then from (2.4.4) we have

$$ J(x) = \int_0^1 x^l G(x) w(x)dx = J_1(x) + J_2(x) \tag{2.4.6} $$

where

$$
\begin{aligned}
&J_1(x)\\
&=\frac{2b+2n+1}{2n+1}\int_0^1 x^l\,{}_2F_1\left(-n,b;b-n-\frac{1}{2};x\right)|x|^{b-n-1/2}|1-x|^{1/2}dx\\
&=\frac{2b+2n+1}{2n+1}\int_0^1 x^{l+1}\,{}_2F_1\left(-n,b;b-n-\frac{1}{2};x\right)|x|^{b-n-3/2}|1-x|^{1/2}dx,
\end{aligned}
$$

and

$$J_2(x)=\frac{-2b}{2n+1}\int_0^1 x^l\,{}_2F_1\left(-n,b+1;b-n+\frac{1}{2};x\right)|x|^{b-n-1/2}|1-x|^{1/2}dx.$$

Then, from Theorem 2.2.5, we see that $J_1(x)=0$ for $l=0,1,\dots,n-2$ while $J_2(x)=0$ for $l=0,1,\dots,n-1$ and it follows from (2.4.6) that $J(x)=0$ for $l=0,1,\dots,n-2$. This completes the proof of (i) and the proof of (ii) follows the same reasoning. □

Corollary 2.4.4 *Let $G(x)$ be as in (2.4.4) where $b\notin\mathbb{Z}^-$. Then $G(x)$ has $(n-1)$ zeros on*

- *(i)* $(0,1)$ *if* $b>n+\frac{1}{2}$*;*
- *(ii)* $(1,\infty)$ *if* $b<-n$*.*

Proof:

This corollary is a direct consequence of Lemma 2.2.3 and Lemma 2.4.3. □

We are now in a position to state our result.

Theorem 2.4.5 *Let* $H(x)={}_3F_2(-2n,2b,b-n;2b-2n,b-n-\frac{1}{2};x)$ *with* $n\in\mathbb{N}$, $b\in\mathbb{R}$*. Then*

- *(i) for* $b>n+\frac{1}{2}$*, all* $2n$ *zeros of* $H(x)$ *are real and* $(2n-1)$ *of them lie in* $(0,1)$*;*
- *(ii) for* $b<-n$*, all* $2n$ *zeros of* $H(x)$ *are real and* $(2n-1)$ *of them lie in* $(1,\infty)$*.*

Proof:

The result follows immediately from equation (2.4.3) and Corollaries 2.4.4 and 2.4.2. $\square$

Theorem 2.4.5 deals with the case where the polynomial ${}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$ has even degree. In order to analyse the zeros of ${}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$ when n is odd, we put $a = -(2n+1)$ in (2.4.1) and (2.4.2), and replace b by $2b$ to obtain

$$\begin{aligned}&{}_3F_2\left(-2n-1, 2b, b-n-\frac{1}{2}; 2b-2n-1, b-n-1; x\right)\\ &\qquad = {}_2F_1\left(-n-\frac{1}{2}, b; b-n; x\right)\ K(x) \qquad (2.4.7)\end{aligned}$$

where

$$K(x) = 2\ {}_2F_1\left(-n-\frac{1}{2}, b; b-n-1; x\right) - {}_2F_1\left(-n-\frac{1}{2}, b; b-n; x\right). \quad (2.4.8)$$

We note that both factors on the right hand side of (2.4.7) are not polynomials. Applying the identity (cf. [1], p.95)

$${}_2F_1(a, b; c; x) = (1-x)^{c-a-b}\ {}_2F_1(c-a, c-b; c; x). \qquad (2.4.9)$$

to the functions in (2.4.7) and (2.4.8) yields

$$\begin{aligned}&{}_3F_2\left(-2n-1, 2b, b-n-\frac{1}{2}; 2b-2n-1, b-n-1; x\right)\\ &\qquad = \ {}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)\ F(x) \qquad (2.4.10)\end{aligned}$$

where

$$\begin{aligned}&F(x)\\ &= 2\ {}_2F_1\left(-n-1, b-\frac{1}{2}; b-n-1; x\right) - (1-x)\ {}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right).\end{aligned}$$

(2.4.11)

The factors in (2.4.10) and (2.4.11) are now polynomials to which we may apply orthogonality and quasi-orthogonality results. Applying Theorem 2.2.5 to ${}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)$ in (2.4.10), we obtain the following result.

Lemma 2.4.6 *Let* $n \in \mathbb{N}$ *and* $b \in \mathbb{R}$. *The polynomial* ${}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)$ *is orthogonal to all polynomials with degree less than* n *with respect to the weight function* $|x|^{b-n-1}|1-x|^{1/2}$ *on*

(i) $(0,1)$ *for* $b > n$;

(ii) $(1,\infty)$ *for* $b < -n+\frac{1}{2}$.

Corollary 2.4.7 *The polynomial* ${}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)$, $n \in \mathbb{N}$, $b \in \mathbb{R}$, *has* n *zeros on*

(i) $(0,1)$ *for* $b > n$;

(ii) $(1,\infty)$ *for* $b < -n+\frac{1}{2}$.

Next, we have the following result for the polynomial $F(x)$ given in (2.4.11).

Lemma 2.4.8 *Let* $F(x)$ *be as in (2.4.11). Then* $F(x)$ *is orthogonal to all polynomials with degree less than* $n-1$ *with respect to the weight function* $v(x) = |x|^{b-n-1}|1-x|^{-1/2}$ *on*

(i) $(0,1)$ *for* $b > n+1$;

(ii) $(1,\infty)$ *for* $b < -n-\frac{1}{2}$.

Proof:

(i) Suppose $b > n+1$. From (2.4.11), we have

$$T(x) = \int_0^1 x^l F(x) v(x) dx = T_1(x) + T_2(x) \tag{2.4.12}$$

where

$$\begin{aligned} T_1(x) &= 2\int_0^1 x^l\ {}_2F_1\left(-n-1, b-\frac{1}{2}; b-n-1; x\right)|x|^{b-n-1}|1-x|^{-1/2}dx \\ &= 2\int_0^1 x^{l+1}\ {}_2F_1\left(-n-1, b-\frac{1}{2}; b-n-1; x\right)|x|^{b-n-2}|1-x|^{-1/2}dx, \end{aligned}$$

and

$$\begin{aligned} T_2(x) &= -\int_0^1 x^l\ (1-x){}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)|x|^{b-n-1}|1-x|^{-1/2}dx \\ &= -\int_0^1 x^l\ {}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)|x|^{b-n-1}|1-x|^{1/2}dx. \end{aligned}$$

Then, from Theorem 2.2.5, we see that $T_1(x) = 0$ for $l = 0, 1, \ldots, n-1$ while $T_2(x) = 0$ for $l = 0, 1, \ldots, n-1$. So it follows from (2.4.12) that $T(x) = 0$ for $l = 0, 1, \ldots, n-1$. This completes the proof of (i) and the proof of (ii) follows the same reasoning. ☐

Corollary 2.4.9 *The polynomial $F(x)$ has $(n-1)$ zeros on*

(i) $(0, 1)$ *for* $b > n+1$*;*

(ii) $(1, \infty)$ *for* $b < -n - \frac{1}{2}$.

Theorem 2.4.10 *Let* $F = {}_3F_2\left(-2n-1, 2b, b-n-\frac{1}{2}; 2b-2n-1, b-n-1; x\right)$ *where* $n \in \mathbb{N}$, $b \in \mathbb{R}$. *Then*

(i) for $b > n+1$*, all* $(2n+1)$ *zeros of* F *are real and* $2n$ *of them lie in* $(0, 1)$*;*

(ii) for $b < -n - \frac{1}{2}$*, all* $(2n+1)$ *zeros of* F *are real and* $2n$ *of them lie in* $(1, \infty)$.

Proof:

The statements follow immediately from equation (2.4.10) and Corollaries 2.4.7 and 2.4.9. ☐

Combining the results of the even and odd cases of n given in Theorems 2.4.5 and 2.4.10 we obtain the following result for general n.

Corollary 2.4.11 *Let* $E(x) = {}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$ *where* $n \in \mathbb{N}$, $b \in \mathbb{R}$. *Then*

(i) for $b > n+1$, *all* n *zeros of* $E(x)$ *are real and* $(n-1)$ *of them lie in* $(0,1)$*;*

(ii) for $b < -n$, *all* n *zeros of* $E(x)$ *are real and* $(n-1)$ *of them lie in* $(1, \infty)$*.*

We give some examples generated by Mathematica to illustrate the results given in Corollary 2.4.11.

Example 2.4.12 ${}_3F_2(-6, 7.78, 0.89; 1.78, 0.39; x)$

When letting $n = 6$ and $b = 7.78$ in ${}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$, Corollary 2.4.11 (i) ensures that at least 5 zeros of polynomial

$$ {}_3F_2(-6, 7.78, 0.89; 1.78, 0.39; x) $$

lie in the interval $(0,1)$. Indeed, the zeros are

```
{{x → 0.0210759}, {x → 0.176068}, {x → 0.362774},
 {x → 0.601565}, {x → 0.802381}, {x → 0.949022}}
```

which is represented by the plot in figure 2.4.

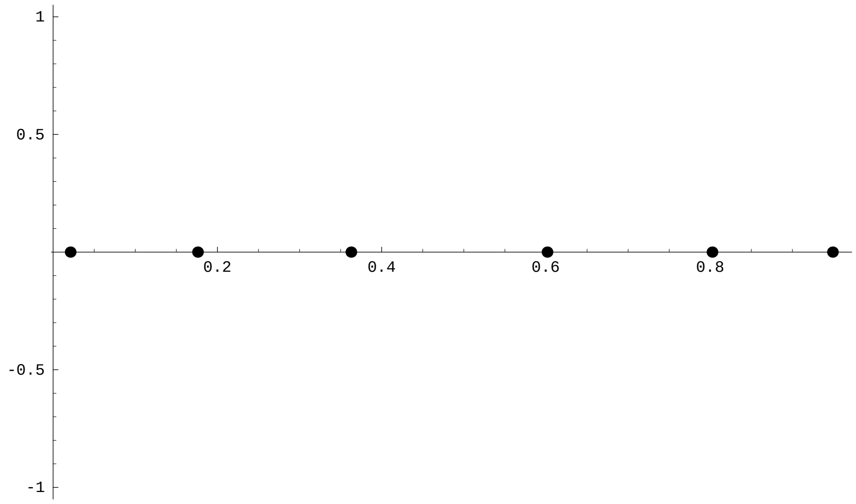

Figure 2.4: The zeros of ${}_3F_2(-6, 7.78, 0.89; 1.78, 0.39; x)$

Example 2.4.13 ${}_3F_2(-10, -14.28, -12.14; -24.28, -12.64; x)$

When letting $n = 10$ and $b = -14.28$ in the hypergeometric polynomial ${}_3F_2\left(-n, b, \frac{b-n}{2}; b-n, \frac{b-n-1}{2}; x\right)$, Corollary 2.4.11 (ii) ensures that at least 9 zeros of polynomial

$$ {}_3F_2(-10, -14.28, -12.14; -24.28, -12.64; x) $$

lie in the interval $(1, \infty)$. Indeed, the zeros are

```
{{x → 1.01604}, {x → 1.06633}, {x → 1.15789}, {x → 1.30511}, {x → 1.53451},
 {x → 1.89856}, {x → 2.50289}, {x → 3.61107}, {x → 5.95427}, {x → 13.5009}}
```

which is represented by the plot in figure 2.5.

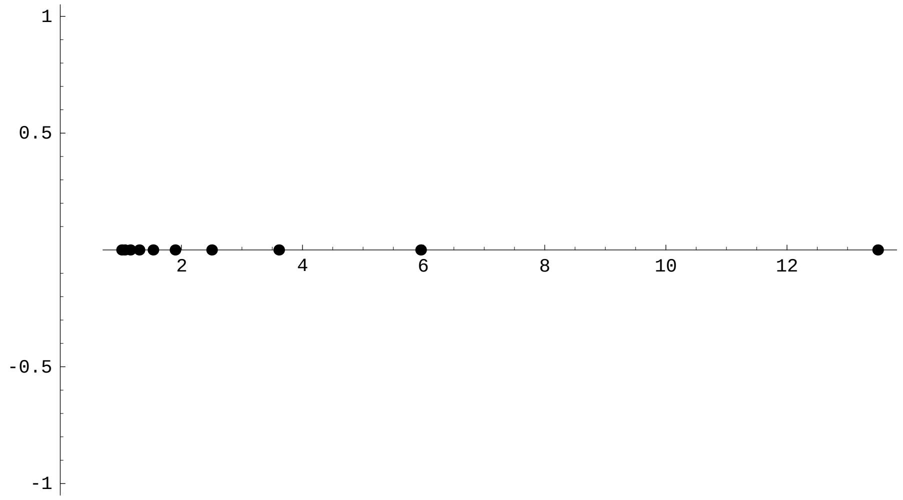

Figure 2.5: The zeros of ${}_3F_2(-10,-14.28,-12.14;-24.28,-12.64;x)$

2.5 The polynomial ${}_3F_2\left(-n,b,\frac{b-n}{2}+1;b-n,\frac{b-n+1}{2};x\right)$, $n\in\mathbb{N}$, $b\in\mathbb{R}$.

A corresponding set of results analogous to Theorems 2.4.5 and 2.4.10 can be proved for the class of ${}_3F_2$ polynomials

$$ {}_3F_2\left(-n,b,\frac{b-n}{2}+1;b-n,\frac{b-n+1}{2};x\right),\ n\in\mathbb{N},b\in\mathbb{R}. $$

From [36] p.499, equation (27), we have

$$ {}_3F_2\left(a,b,\frac{a+b}{2}+1;a+b,\frac{a+b+1}{2};x\right)={}_2F_1\left(\frac{a}{2},\frac{b}{2};\frac{a+b+1}{2};x\right)\ B(x) \tag{2.5.1} $$

where

$$ B(x)=\frac{2(a+b-1)}{a+b}\ {}_2F_1\left(\frac{a}{2},\frac{b}{2};\frac{a+b-1}{2};x\right) $$

$$- \frac{a+b-2}{a+b} \, {}_2F_1\left(\frac{a}{2}, \frac{b}{2}; \frac{a+b+1}{2}; x\right) \tag{2.5.2}$$

We consider each of the cases where a is a negative odd integer or a negative even integer. We first consider the case where a is a negative even integer. Let $a = -2n$, $n \in \mathbb{N}$ and replace b with $2b$ for notational convenience in (2.5.1) and (2.5.2), which become

$$\begin{aligned} {}_3F_2&\left(-2n, 2b, b-n+1; 2b-2n, b-n+\frac{1}{2}; x\right) \\ &= {}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right) B(x) \end{aligned} \tag{2.5.3}$$

and

$$\begin{aligned} B(x) = \frac{2b-2n-1}{b-n} \, &{}_2F_1\left(-n, b; b-n-\frac{1}{2}; x\right) \\ &- \frac{b-n-1}{b-n} \, {}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right) \end{aligned} \tag{2.5.4}$$

Orthogonality results and the location of the zeros of the factor ${}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right)$ on the left of (2.5.3) are given in Lemma 2.4.1 and Corollary 2.4.2 respectively. For the factor $B(x)$ on the right-hand side of (2.5.3), given explicitly in (2.5.4), we utilise similar techniques to those used in Lemma 2.4.3 to obtain the following result.

Lemma 2.5.1 *Let $B(x)$ be as in (2.5.4), $b \notin \mathbb{Z}^-$, and let $w(x) = |x|^{b-n-3/2}|1-x|^{1/2}$. Then $B(x)$ is orthogonal to all polynomials with degree less than n with respect to $w(x)$ on*

- *(i)* $(0,1)$ *if* $b > n + \frac{1}{2}$,
- *(ii)* $(1,\infty)$ *if* $b < -n$.

Proof:

(i) Suppose $b > n + \frac{1}{2}$. We first rewrite ${}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right)$ using the contiguous relation (cf. [37], p.71, eqn(13))

$$(b-c+1){}_2F_1(a,b;c;x) = b{}_2F_1(a,b+1;c;x) - (c-1){}_2F_1(a,b;c-1;x)$$

to obtain the equation as given in (2.4.5), namely

$$\begin{aligned}
&{}_2F_1\left(-n, b; b-n+\frac{1}{2}; x\right) \\
&\qquad = \frac{2b}{2n+1}\,{}_2F_1\left(-n, b+1; b-n+\frac{1}{2}; x\right) \\
&\qquad\qquad - \frac{2b-2n-1}{2n+1}\,{}_2F_1\left(-n, b; b-n-\frac{1}{2}; x\right)
\end{aligned}$$

Then from (2.5.4) we have

$$L(x) = \int_0^1 x^l B(x) w(x) dx = L_1(x) + L_2(x) \tag{2.5.5}$$

where

$$\begin{aligned}
&L_1(x) \\
&= \frac{(2b-2n-1)(n+b)}{(2n+1)(b-n)} \int_0^1 x^l\,{}_2F_1\left(-n, b; b-n-\frac{1}{2}; x\right) |x|^{b-n-1/2}|1-x|^{1/2}dx \\
&= \frac{(2b-2n-1)(n+b)}{(2n+1)(b-n)} \int_0^1 x^{l+1}\,{}_2F_1\left(-n, b; b-n-\frac{1}{2}; x\right) \cdot \\
&\qquad\qquad |x|^{b-n-3/2}|1-x|^{1/2}dx,
\end{aligned}$$

and

$$\begin{aligned}
&L_2(x) \\
&= \frac{-2b(b-n-1)}{(2n+1)(b-n)} \int_0^1 x^l\,{}_2F_1\left(-n, b+1; b-n+\frac{1}{2}; x\right) |x|^{b-n-1/2}|1-x|^{1/2}dx.
\end{aligned}$$

Then, from Theorem 2.2.5, we see that $L_1(x) = 0$ for $l = 0, 1, \ldots, n-2$ while $L_2(x) = 0$ for $l = 0, 1, \ldots, n-1$ and it follows from (2.5.5) that $L(x) = 0$ for $l = 0, 1, \ldots, n-2$. This completes the proof of (i) and the proof of (ii) follows the same reasoning. $\square$

Corollary 2.5.2 *The polynomial $B(x)$ has n zeros on*

(i) $(0,1)$ *if* $b > n + \frac{1}{2}$,

(ii) $(1,\infty)$ *if* $b < -n$.

The results obtained in Corollaries 2.4.2 and 2.5.2 immediately yield the following theorem.

Theorem 2.5.3 *Let* $C(x) = {}_3F_2\left(-2n, 2b, b-n+1; 2b-2n, b-n+\frac{1}{2}; x\right)$ *with* $n \in \mathbb{N}$, $b \in \mathbb{R}$, $b \notin \mathbb{Z}^-$. *Then*

(i) for $b > n + \frac{1}{2}$, *the* $2n$ *zeros of* $C(x)$ *lie in* $(0,1)$,

(ii) for $b < -n$, *the* $2n$ *zeros of* $C(x)$ *lie in* $(1,\infty)$,

If we now consider the case where a is a negative odd integer in (2.5.1) and (2.5.2), the ${}_2F_1$ functions need to be rewritten using the identity (2.4.9) to obtain ${}_2F_1$ polynomials. Putting $a = -2n-1$, $n \in \mathbb{N}$ and replacing b with $2b$ in (2.5.1) and (2.5.2) and applying (2.4.9), yields

$$\begin{aligned} &{}_3F_2\left(-2n-1, 2b, b-n-\frac{1}{2}; 2b-2n-1, b-n; x\right) \\ &\qquad = {}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right)\ D(x) \qquad (2.5.6) \end{aligned}$$

and

$$\begin{aligned} D(x) =& \frac{2(2b-2n-2)}{2b-2n-1}\,{}_2F_1\left(-n-1, b-\frac{1}{2}; b-n-1; x\right) \\ &- (1-x)\frac{2b-2n-3}{2b-2n-1}\,{}_2F_1\left(-n, b+\frac{1}{2}; b-n; x\right) \end{aligned} \tag{2.5.7}$$

Straightforward calculations similar to those used to prove Theorem 2.4.10 give the following result.

Theorem 2.5.4 *Let* $A(x) = {}_3F_2\left(-2n-1, 2b, b-n-\frac{1}{2}; 2b-2n-1, b-n; x\right)$ *where* $n \in \mathbb{N}$ *and* $b \in \mathbb{R}$. *Then,*

(i) for $b > n+1$, *all* $(2n+1)$ *zeros of* $A(x)$ *are real and* $2n$ *of them lie in* $(0,1)$,

(ii) for $b < -n-\frac{1}{2}$, *all* $(2n+1)$ *zeros of* $A(x)$ *are real and* $2n$ *of them lie in* $(1,\infty)$.

Combining the results in Theorems 2.5.3 and 2.5.4 yields the following corollary.

Corollary 2.5.5 *Let* $T(x) = {}_3F_2\left(-n, b, \frac{b-n}{2}+1; b-n, \frac{b-n+1}{2}; x\right)$, $n \in \mathbb{N}$, $b \in \mathbb{R}$. *Then*

(i) for $b > n+1$, *all* n *zeros of* $T(x)$ *are real and* $(n-1)$ *of them lie in* $(0,1)$,

(ii) for $b < -n$, *all* n *zeros of* $T(x)$ *are real and* $(n-1)$ *of them lie in* $(1,\infty)$.

We give some examples generated by Mathematica to illustrate the results given in Corollary 2.5.5.

Example 2.5.6 $\quad {}_3F_2(-6, 7.78, 0.89; 1.78, 0.39; x)$

When letting $n = 8$ and $b = 11.63$ in ${}_3F_2\left(-n, b, \frac{b-n}{2}+1; b-n, \frac{b-n+1}{2}; x\right)$, Corollary 2.5.5 (i) ensures that at least 7 zeros of polynomial

$${}_3F_2(-8, 11.63, 2.815; 3.63, 2.315; x)$$

lie in the interval $(0, 1)$. Indeed, the zeros are

```
{{x → 0.0653866}, {x → 0.177023}, {x → 0.317938}, {x → 0.477212},
 {x → 0.637576}, {x → 0.783351}, {x → 0.899446}, {x → 0.974211}}
```

which is represented by the plot in figure 2.6.

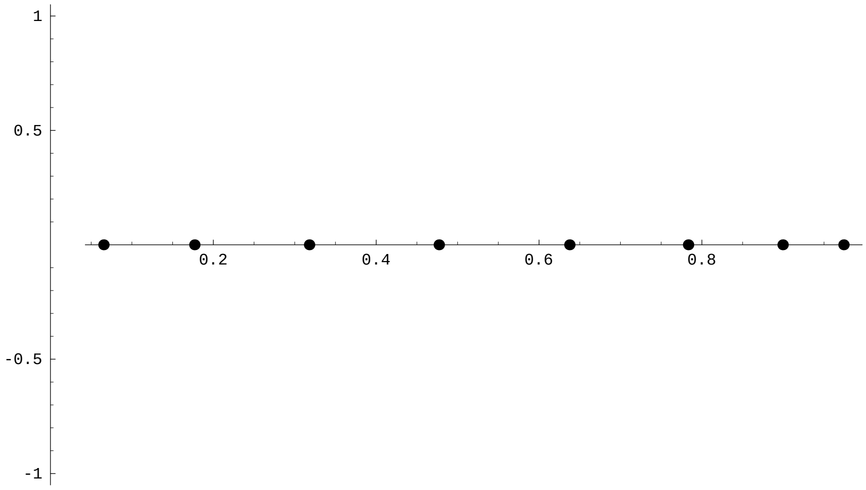

Figure 2.6: The zeros of ${}_3F_2(-6, 7.78, 0.89; 1.78, 0.39; x)$

Example 2.5.7 $\quad {}_3F_2(-12, -17.65, -13.825; -29.65, -14.325; x)$

When letting $n = 12$ and $b = -17.65$ in ${}_3F_2\left(-n, b, \frac{b-n}{2}+1; b-n, \frac{b-n+1}{2}; x\right)$, Corollary 2.5.5 (ii) ensures that at least 11 zeros of polynomial

$${}_3F_2(-12, -17.65, -13.825; -29.65, -14.325; x)$$

lie in the interval $(1, \infty)$. Indeed, the zeros are

```
{{x → 1.01094}, {x → 1.04498}, {x → 1.10464}, {x → 1.19768}, {x → 1.33091}, {x → 1.52873},
 {x → 1.80861}, {x → 2.25059}, {x → 2.91825}, {x → 4.17216}, {x → 6.4541}, {x → 14.789}}
```

which is represented by the plot in figure 2.7.

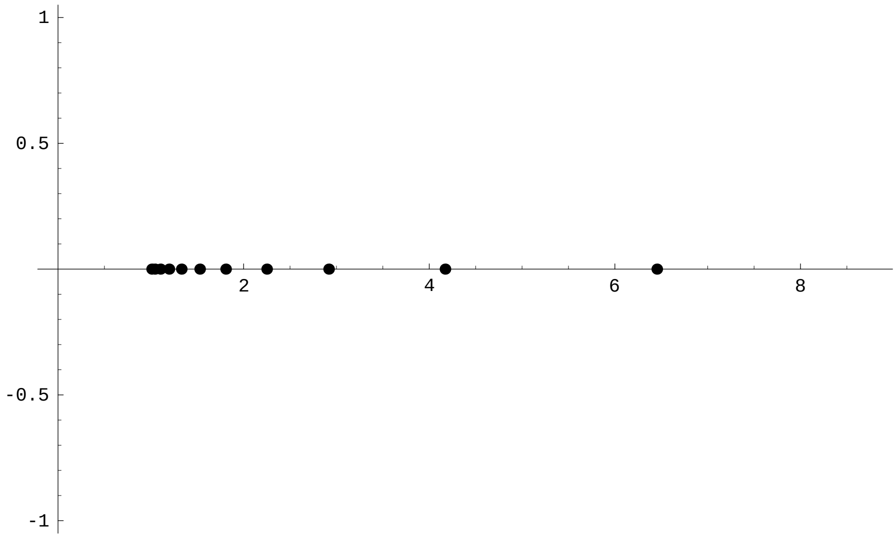

Figure 2.7: The zeros of ${}_3F_2(-12, -17.65, -13.825; -29.65, -14.325; x)$

Chapter 3

Integral representations of some ${}_pF_q$ functions

3.1 Introduction

In this chapter, we begin by discussing the Euler integral representation of the Gauss hypergeometric function, or ${}_2F_1$ hypergeometric function, and its applications. We also mention other integral representations involving the ${}_2F_1$ hypergeometric function such as Barnes's contour integral representation and others.

Some integrals that arise here are really extensions of beta integrals and also appear in the orthogonality relations for some special orthogonal polynomials.

3.2 Preliminary results

The Euler integral representation ${}_2F_1$ function, is well known in the literature (cf. [1], [37]) and is formulated as follows (cf. [1], p.65, Theorem 2.2.1):

If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt \tag{3.2.1} $$

in the x plane cut along the real axis from 1 to ∞. Here, it is understood that $\arg t = \arg\,(1-t) = 0$ and $(1-xt)^{-a}$ has its principal value.

This integral may be viewed as the analytic continuation of the ${}_2F_1$ series for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and it yields the famous Gauss summation formula of 1812 (cf. [1], p.66, Theorem 2.2.2), namely:

For $\mathrm{Re}(c-a-b) > 0$, we have

$$ {}_2F_1(a,b;c;1) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

The case where one of the numerator parameters is a negative integer, thereby making the ${}_2F_1$ a finite sum, is known as the Chu-Vandermonde identity (cf. [1], p.67, Corollary 2.2.3)

$$ {}_2F_1(-n,a;c;1) = \frac{(c-a)_n}{(c)_n}. $$

We have the following more general integral representation for the ${}_{p+1}F_{q+1}$ hypergeometric function (cf. [1] p.67, equation (2.2.2))

$$ {}_{p+1}F_{q+1}\left(\begin{matrix} a_1, & \ldots, & a_p, a_{p+1} \\ b_1, & \ldots, & b_q, b_{q+1} \end{matrix}; x\right) = \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})\Gamma(b_{q+1}-a_{p+1})} \int_0^1 t^{a_{p+1}-1} $$

$$.(1-t)^{b_{q+1}-a_{p+1}-1} {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}; xt\right) dt \tag{3.2.2} $$

when $\mathrm{Re}(b_{q+1}) > \mathrm{Re}(a_{p+1}) > 0$.

By a change of variables, the expression on the right of (3.2.2) also equals

$$ \frac{\Gamma(b_{q+1})x^{1-b_{q+1}}}{\Gamma(a_{p+1})\Gamma(b_{q+1}-a_{p+1})} \int_0^x t^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1} {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}; t\right) dt. $$

We note that (3.2.2) can be used to change the value of a denominator or numerator parameter in ${}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q\end{matrix}; t\right)$. For example, if we let $a_{p+1} = b_q$ in (3.2.2), we obtain

$$\begin{aligned}{}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_{q-1}, b_{q+1}\end{matrix}; x\right) &= \frac{\Gamma(b_{q+1})}{\Gamma(b_q)\Gamma(b_{q+1}-b_q)} \\ &\int_0^1 t^{b_q-1}(1-t)^{b_{q+1}-b_q-1}{}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q\end{matrix}; xt\right) dt.\end{aligned}$$

A special case of this is given in [1] p.68 Theorem 2.2.4 which states that for $\mathrm{Re}(c) > \mathrm{Re}(d) > 0$, $x \neq 1$ and $|\arg(1-x)| < \pi$,

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)}\int_0^1 t^{d-1}(1-t)^{c-d-1}{}_2F_1(a, b; d; xt)dt.$$

This can also be proved using fractional integration by parts (cf. [1] p.113).

The values of ${}_{q+1}F_q$ functions at 1 with $q \geq 2$ encompass the beautiful identities due to Pfaff-Saalschütz, Dougall, Dixon, Rogers, Ramanujan, Whipple and other authors. For further discussion, see [1], Chapters 2 and 3.

A useful property of ${}_2F_1(a, b; c; x)$ is that it is symmetric in the numerator parameters a and b. However, it is not evident that Euler's integral for the ${}_2F_1$ hypergeometric function remains the same when a and b are interchanged. Erdélyi (cf. [1] p.68) presents a double integral from which both representations can be obtained. This double integral is given by

$$\frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\int_0^1\int_0^1 t^{b-1}s^{a-1}(1-t)^{c-b-1}(1-s)^{c-a-1}(1-tsx)^{-c}dt\, ds.$$

Each of the representations ${}_2F_1(a, b; c; x)$ and ${}_2F_1(b, a; c; x)$ can be derived simply from this double integral. To obtain the latter, we first integrate with respect to t.

$$\frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\int_0^1\int_0^1 t^{b-1}s^{a-1}(1-t)^{c-b-1}$$

$$.(1-s)^{c-a-1}(1-tsx)^{-c}dt\,ds$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 s^{a-1}(1-s)^{c-a-1}$$

$$\left[\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tsx)^{-c}dt\right]\,ds$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 s^{a-1}(1-s)^{c-a-1}\left[{}_2F_1(c,b;c;sx)\right]\,ds \qquad \text{by (3.2.1).}$$

This integral is now equal to

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 s^{a-1}(1-s)^{c-a-1}\left[{}_1F_0(b;sx)\right]\,ds$$

since one of the numerator and denominator parameters are the same and can thus be canceled. This is now equal to

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 s^{a-1}(1-s)^{c-a-1}(1-sx)^{-b}\,ds={}_2F_1(b,a;c;x)$$

by (3.2.1).

By changing the order of integration and integrating with respect to s first, we obtain the representation ${}_2F_1(a,b;c;x)$ using similar methods.

$$\frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\int_0^1\int_0^1 t^{b-1}s^{a-1}(1-t)^{c-b-1}$$

$$.(1-s)^{c-a-1}(1-tsx)^{-c}dt\,ds$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}$$

$$\left[\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 s^{a-1}(1-s)^{c-a-1}(1-tsx)^{-c}ds\right]\,dt$$

$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left[{}_2F_1(c,a;c;tx)\right]\,dt \qquad \text{by (3.2.1)}$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left[{}_1F_0(a;tx)\right]\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt$$
$$= {}_2F_1(a,b;c;x) \qquad \text{by (3.2.1).}$$

Erdélyi gives another integral representation for the ${}_2F_1$ hypergeometric function in which the integrand involves the product of two ${}_2F_1$ hypergeometric functions (cf. [1] Theorem 2.9.1). For $\mathrm{Re}(c) > \mathrm{Re}(\mu) > 0$, $x \neq 1$, $|\arg(1-x)| < \pi$, we have

$$\begin{aligned} {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(\mu)\Gamma(c-\mu)} \int_0^1 & t^{\mu-1}(1-t)^{c-\mu-1}(1-xt)^{\mu-a-b} \\ & {}_2F_1(\lambda-a,\lambda-b;\mu;xt)\, {}_2F_1\left(a+b-\lambda,\lambda-\mu;c-\mu;\frac{(1-t)x}{1-xt}\right)\, dt. \end{aligned}$$

An important application of Euler's integral is in the derivation of two transformation formulas of hypergeometric functions due to Pfaff and Euler. These are given by

$${}_2F_1(a,b;c;x) = (1-x)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right) \tag{3.2.3}$$

and

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\, {}_2F_1\left(c-a, c-b; c; x\right) \tag{3.2.4}$$

respectively (cf. [1] p.68 equations (2.2.6) and (2.2.7)). Pfaff's transformation (3.2.3) is proved by replacing t with $(1-s)$ in Euler's integral (3.2.1) and, since the hypergeometric function is symmetric in the numerator parameters, Pfaff's transformation can be applied to itself to obtain Euler's transformation in (3.2.4).

Barnes developed an alternative method of treating the ${}_2F_1$ hypergeometric function. He gives a contour integral representation of ${}_2F_1(a,b;c;x)$, which can be seen as a Mellin inversion formula. Barnes's theorem is proved in [1]

Theorem 2.4.1 and is stated as follows:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}\,(-x)^s ds,$$

$|\arg(-x)|<\pi$. The path of integration is curved, if necessary, to separate the poles $s=-a-n$, $s=-b-n$, from the poles $s=n$, where n is an integer ≥ 0. (Such a contour can always be drawn if a and b are not negative integers).

Note that a Barnes integral for ${}_1F_1$ hypergeometric functions is given in [1] Theorem 4.2.1.

Our new result in this chapter is the proof of a rather simple integral representation for a class of ${}_{q+1}F_q$ functions. It is useful, as we shall see in chapter 4, for expressing a class of ${}_3F_2$ hypergeometric polynomials in terms of an integral which can be analysed asymptotically.

3.3 Results

We prove the following Euler-type integral representation for the special class of hypergeometric functions ${}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};x\right)$.

Theorem 3.3.1 *For Re*$(c)>$ *Re*$(b)>0$*,*

$${}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};x\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^{-a}dt. \tag{3.3.5}$$

Proof:

Suppose that $|x|<1$. Then the left hand side of (3.3.5) becomes

$$\sum_{k=0}^{\infty}\frac{(a)_k\left(\frac{b}{2}\right)_k\left(\frac{b+1}{2}\right)_k}{\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\,k!}\,x^k=\sum_{k=0}^{\infty}\frac{(a)_k(b)_{2k}}{(c)_{2k}\,k!}\,x^k \tag{3.3.6}$$

since (cf. [37], p. 22)

$$(\alpha)_{2k} = 2^{2k} \left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+1}{2}\right)_k .$$

Now

$$\frac{(b)_{2k}}{(c)_{2k}} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b+2k)\Gamma(c-b)}{\Gamma(c+2k)} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b+2k, c-b).$$

Therefore, for Re(c) > Re(b) > 0, $|x| < 1$, the right hand side of (3.3.6) becomes

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k \, x^k}{k!} \int_0^1 t^{b+2k-1}(1-t)^{c-b-1} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \frac{(a)_k \, (xt^2)^k}{k!} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^{-a} dt. \end{aligned}$$

This proves the result for $|x| < 1$. Since the integral is analytic in the cut plane, the result holds for x in this region as well. ☐

Corollary 3.3.2 *For Re*$(c) >$ *Re*$(b) > 0$,

$${}_3F_2\left(-n, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^n dt.$$

Proof:

Substituting $a = -n$ in Theorem 3.3.1 yields this result. ☐

Theorem 3.3.3 *If Re*$(c) >$ *Re*$(b) > 0$ *and Re*$(c-a-b) > 0$*, then*

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} \, {}_2F_1(a, b; c-a; -1). \quad (3.3.7)$$

Proof:

Putting $x = 1$ in equation (3.3.5), the left hand side of (3.3.7) becomes

$$\begin{aligned}
&\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}(1+t)^{-a}dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\sum_{k=0}^{\infty}\binom{-a}{k}t^k dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{-a}{k}\int_0^1 t^{b+k-1}(1-t)^{c-b-a-1}dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{-a}{k}\frac{\Gamma(b+k)\Gamma(c-b-a)}{\Gamma(c-a+k)}
\end{aligned}$$

Now $\binom{-a}{k} = \frac{(-1)^k(a)_k}{k!}$ and $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$. Therefore we have

$$\begin{aligned}
&\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{-a}{k}\frac{\Gamma(b+k)\Gamma(c-b-a)}{\Gamma(c-a+k)} \\
&= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty}\frac{(-1)^k(a)_k}{k!}\frac{(b)_k}{(c-a)_k} \\
&= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a,b;c-a;-1).
\end{aligned}$$

□

Remark: An alternative proof of Theorem 3.3.3 can be found in [47], formula (3.7), which reads

$$ {}_3F_2\left(b,\frac{a}{2},\frac{a+1}{2};\frac{b+c}{2},\frac{b+c+1}{2};1\right) = \frac{\Gamma(c-a)\Gamma(b+c)}{\Gamma(b+c-a)\Gamma(c)}\ {}_2F_1\left(a,b;c;-1\right). \tag{3.3.8}$$

Theorem 3.3.3 emerges once c is replaced with $c-b$ in equation (3.3.8).

Corollary 3.3.4 *If Re*$(c) >$ *Re*$(b) > 0$, *then*

$$ {}_3F_2\left(-n,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};1\right) = \frac{(c-b)_n}{(c)_n}\ {}_2F_1(-n,b;c+n;-1).$$

Proof:

Substituting $a = -n$ in Theorem 3.3.3 yields this result. $\square$

We can generalise Theorem 3.3.1 in the following way.

Theorem 3.3.5 *If* $Re(c) > Re(b) > 0$*, then*

$$
\begin{aligned}
&{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a}dt.
\end{aligned}
$$

Proof:

Suppose that $|x| < 1$. Then the we have

$$
\begin{aligned}
&{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right) \\
&= \sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{b}{q}\right)_k \left(\frac{b+1}{q}\right)_k \cdots \left(\frac{b+q-1}{q}\right)_k}{\left(\frac{c}{q}\right)_k \left(\frac{c+1}{q}\right)_k \cdots \left(\frac{c+q-1}{q}\right)_k} \frac{x^k}{k!} \qquad (3.3.9)
\end{aligned}
$$

Using the property

$$(\alpha)_{qk} = q^{qk} \left(\frac{\alpha}{q}\right)_k \left(\frac{\alpha+1}{q}\right)_k \cdots \left(\frac{\alpha+q-1}{q}\right)_k$$

(cf. [37], p.22, lemma 6) on the right-hand side of (3.3.9) gives

$$
\begin{aligned}
&{}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; x\right) \\
&= \sum_{k=0}^{\infty} \frac{(a)_k (b)_{qk}}{(c)_{qk}\, k!} x^k. \qquad (3.3.10)
\end{aligned}
$$

Now

$$\frac{(b)_{qk}}{(c)_{qk}} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b+qk)\Gamma(c-b)}{\Gamma(c+qk)} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b+qk, c-b).$$

Therefore, for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $|x| < 1$, the right hand side of (3.3.10) becomes

$$\begin{aligned}
&\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k\ x^k}{k!} \int_0^1 t^{b+qk-1}(1-t)^{c-b-1}dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \frac{(a)_k\ (xt^q)^k}{k!} dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a}dt.
\end{aligned}$$

This proves the result for $|x| < 1$. Since the integral is analytic in the cut plane, the result holds for x in this region as well. □

Remark: In [37], Theorem 38, Rainville proves a general integral representation for ${}_{p+k}F_{q+k}$ where k is a positive integer and $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, namely

$$\begin{aligned}
&{}_{p+k}F_{q+k}\left(\begin{matrix} a_1, & \dots & , a_p, & \frac{\alpha}{k}, & \frac{\alpha+1}{k}, & \dots, & \frac{\alpha+k-1}{k} \\ b_1, & \dots & , b_q, & \frac{\alpha+\beta}{k}, & \frac{\alpha+\beta+1}{k}, & \dots, & \frac{\alpha+\beta+k-1}{k} \end{matrix}; ct^k\right) \\
&= \frac{t^{1-\alpha-\beta}}{B(\alpha,\beta)} \int_0^t x^{\alpha-1}\ (t-x)^{\beta-1}{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}; cx^k\right) dx.
\end{aligned} \tag{3.3.11}$$

We can obtain the result in Theorem 3.3.5 from Rainville's theorem as follows:

Letting $t = 1$ in (3.3.11), we obtain

$$\begin{aligned}
&\int_0^1 x^{\alpha-1}\ (1-x)^{\beta-1}{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p \\ b_1, & \dots & , b_q \end{matrix}; dx^k\right) dx \\
&= B(\alpha,\beta)\ {}_{p+k}F_{q+k}\left(\begin{matrix} a_1, & \dots & , a_p, & \frac{\alpha}{k}, & \frac{\alpha+1}{k}, & \dots, & \frac{\alpha+k-1}{k} \\ b_1, & \dots & , b_q, & \frac{\alpha+\beta}{k}, & \frac{\alpha+\beta+1}{k}, & \dots, & \frac{\alpha+\beta+k-1}{k} \end{matrix}; d\right).
\end{aligned}$$

Setting $q = 0$, $p = 1$, $\alpha = b$ and $\alpha + \beta = c$, we remain with

$$\int_0^1 x^{b-1}\,(1-x)^{c-b-1}{}_1F_0\left(a; dx^k\right)dx$$
$$= B(b, c-b)\ {}_{k+1}F_k\left(\begin{matrix} a, \frac{b}{k}, \frac{b+1}{k}, & \ldots, & \frac{b+k-1}{k} \\ \frac{c}{k}, \frac{c+1}{k}, & \ldots, & \frac{c+k-1}{k} \end{matrix}; d\right).$$

Since ${}_1F_0\left(a, -; z\right) = (1-z)^{-a}$, we have that

$${}_{k+1}F_k\left(\begin{matrix} a, \frac{b}{k}, \frac{b+1}{k}, & \ldots, & \frac{b+k-1}{k} \\ \frac{c}{k}, \frac{c+1}{k}, & \ldots, & \frac{c+k-1}{k} \end{matrix}; d\right)$$
$$= \frac{1}{B(b, c-b)} \int_0^1 x^{b-1}\,(1-x)^{c-b-1}(1-dx^k)^{-a}dx.$$

Replacing k with q, d with x and x with t yields Theorem 3.3.5,

$${}_{q+1}F_q\left(\begin{matrix} a, \frac{b}{q}, \frac{b+1}{q}, & \ldots, & \frac{b+q-1}{q} \\ \frac{c}{q}, \frac{c+1}{q}, & \ldots, & \frac{c+q-1}{q} \end{matrix}; x\right)$$
$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}(1-xt^q)^{-a}dt.$$

Letting $q = 2$ in Theorem 3.3.5 yields Theorem 3.3.1.

It also follows from Theorem 3.3.5 with $q = 3$ that for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$${}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; x\right)$$
$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^3)^{-a}dt.$$

This leads to the following result.

Theorem 3.3.6 *For Re$(c-a-b) > 0$,*

$${}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; 1\right)$$
$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k(b)_k}{k!\ (c-a)_k}\ {}_2F_1(-k, b+k; c-a+k; -1). \tag{3.3.12}$$

Proof:

Letting $q = 3$ and $x = 1$ in Theorem 3.3.5, we have

$${}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; 1\right)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t^3)^{-a} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}(1+t+t^2)^{-a} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k}{k!}\, t^k(1+t)^k dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k}{k!} \int_0^1 t^{k+b-1}(1-t)^{c-b-a-1}(1+t)^k dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k}{k!} \int_0^1 t^{k+b-1}(1-t)^{c-b-a-1} \sum_{r=0}^{k} \binom{k}{r} t^r dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(a)_k(-1)^k}{k!} \binom{k}{r} \int_0^1 t^{k+r+b-1}(1-t)^{c-b-a-1} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(a)_k(-1)^k}{k!} \binom{k}{r} \frac{\Gamma(b+r+k)\,\Gamma(c-b-a)}{\Gamma(c-a+k+r)}$$

$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(a)_k(-1)^k}{k!} \binom{k}{r} \frac{(b)_{r+k}}{(c-a)_{r+k}}$$

$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(a)_k(-1)^k}{k!} \binom{k}{r} \frac{(b)_k(b+k)_r}{(c-a)_k(c-a+k)_r} \qquad \text{from [37], p.65}$$

$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k(b)_k}{k!\,(c-a)_k} \sum_{r=0}^{k} \binom{k}{r} \frac{(b+k)_r}{(c-a+k)_r}.$$

Now,

$$\sum_{r=0}^{k} \binom{k}{r} \frac{(b+k)_r}{(c-a+k)_r}$$

$$= \sum_{r=0}^{k} \frac{(-1)^r(-k)_r}{r!}\, \frac{(b+k)_r}{(c-a+k)_r} \qquad \text{since} \quad \frac{k!}{(k-r)!} = (-1)^r(-k)_r$$

$$= \sum_{r=0}^{k} \frac{(-k)_r(b+k)_r}{(c-a+k)_r}\, \frac{(-1)^r}{r!}$$

$$= {}_2F_1(-k, b+k; c-a+k; -1).$$

Hence,

$$\begin{aligned}
&{}_4F_3\left(a, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3}; \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}; 1\right) \\
&= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k(b)_k}{k!\,(c-a)_k}\ {}_2F_1(-k, b+k; c-a+k; -1)
\end{aligned}$$

as stated. □

Another application of Theorem 3.3.1 is the following result.

Theorem 3.3.7 *If Re(c) > Re(b) > 0, then*

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k}\ {}_2F_1(-k, b; c+k; -1).$$

Proof:

From Theorem 3.3.1 with $x = \frac{1}{2}$,

$$\begin{aligned}
&{}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1 - \frac{1}{2}\,t^2\right)^{-a} dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(2-t^2)^{-a}\,dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1+(1-t^2)\right)^{-a} dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty}\binom{-a}{k}(1-t^2)^k\,dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{-a}{k} \int_0^1 t^{b-1}(1-t)^{c-b-1+k}(1+t)^k\,dt \\
&= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\binom{-a}{k} \int_0^1 t^{b-1}(1-t)^{c-b-1+k} \sum_{r=0}^{k}\binom{k}{r} t^r\,dt
\end{aligned}$$

$$= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\sum_{r=0}^{k} \binom{-a}{k}\binom{k}{r} \int_0^1 t^{b+r-1}(1-t)^{c-b-1+k}\,dt$$

$$= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty}\sum_{r=0}^{k} \binom{-a}{k}\binom{k}{r} \frac{\Gamma(b+r)\Gamma(c-b+k)}{\Gamma(c+r+k)}$$

$$= 2^a \sum_{k=0}^{\infty}\sum_{r=0}^{k} \binom{-a}{k}\binom{k}{r} \frac{(b)_r\,(c-b)_k}{(c)_{r+k}}.$$

But since $(c)_{r+k} = (c)_k\,(c+k)_r$ (cf. [37], p.65), we have that

$$\begin{aligned} {}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) &= 2^a \sum_{k=0}^{\infty}\sum_{r=0}^{k} \binom{-a}{k}\binom{k}{r} \frac{(b)_r\,(c-b)_k}{(c)_k\,(c+k)_r} \\ &= 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} \sum_{r=0}^{k} \binom{k}{r} \frac{(b)_r}{(c+k)_r}. \end{aligned}$$

We know that

$$\sum_{r=0}^{k} \binom{k}{r} \frac{(b)_r}{(c+k)_r} = {}_2F_1(-k, b; c+k; -1)$$

and hence

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} {}_2F_1(-k, b; c+k; -1).$$

□

Corollary 3.3.8 *If* $Re(c) > Re(b) > 0$, *then*

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k}\, {}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right).$$

Proof:

From Corollary 3.3.4, we can see that

$${}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) = \frac{(c-b)_k}{(c)_k}\, {}_2F_1(-k, b; c+k; -1)$$

which along with Theorem 3.3.7 gives the result. □

Remark: This corollary is a special case of the transformation

$$\begin{aligned}(1-z)^{-\alpha}\ {}_{q+1}F_q &\left(\begin{matrix}\alpha, a_1, & \ldots, & a_q \\ c_1, & \ldots & , c_q\end{matrix}; \frac{xz}{z-1}\right) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\ z^k\ {}_{q+1}F_q\left(\begin{matrix}-k, a_1, & \ldots, & a_q \\ c_1, & \ldots & , c_q\end{matrix}; x\right)\end{aligned} \tag{3.3.13}$$

given by Chaundy in [6]. Letting $q = 2$ and $x = 1$ in Chaundy's transformation (3.3.13) yields

$$\begin{aligned}(1-z)^{-\alpha}\ {}_3F_2 &\left(\begin{matrix}\alpha, a_1, & \ldots, & a_q \\ c_1, & \ldots & , c_q\end{matrix}; \frac{z}{z-1}\right) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k\ z^k}{k!}\ {}_3F_2\left(\begin{matrix}-k, a_1, & \ldots, & a_q \\ c_1, & \ldots & , c_q\end{matrix}; 1\right).\end{aligned}$$

Then with $\alpha = a$, $a_1 = \frac{b}{2}$, $a_2 = \frac{b+1}{2}$, $c_1 = \frac{c}{2}$ and $c_2 = \frac{c+1}{2}$, we have

$$\begin{aligned}(1-z)^{-a}\ {}_3F_2 &\left(\begin{matrix}a, \frac{b}{2}, & \ldots, & \frac{b+1}{2} \\ \frac{c}{2}, & \ldots & , \frac{c+1}{2}\end{matrix}; \frac{z}{z-1}\right) \\ &= \sum_{k=0}^{\infty} \frac{(a)_k\ z^k}{k!}\ {}_3F_2\left(\begin{matrix}-k, \frac{b}{2}, & \ldots, & \frac{b+1}{2} \\ \frac{c}{2}, & \ldots & , \frac{c+1}{2}\end{matrix}; 1\right).\end{aligned}$$

Finally setting $z = -1$, obtain the result in Corollary 3.3.8, namely

$$\begin{aligned}{}_3F_2 &\left(\begin{matrix}a, \frac{b}{2}, & \ldots, & \frac{b+1}{2} \\ \frac{c}{2}, & \ldots & , \frac{c+1}{2}\end{matrix}; \frac{1}{2}\right) \\ &= 2^a \sum_{k=0}^{\infty} \frac{(a)_k\ (-1)^k}{k!}\ {}_3F_2\left(\begin{matrix}-k, \frac{b}{2}, & \ldots, & \frac{b+1}{2} \\ \frac{c}{2}, & \ldots & , \frac{c+1}{2}\end{matrix}; 1\right) \\ &= 2^a \sum_{k=0}^{\infty} \binom{-a}{k}\ {}_3F_2\left(\begin{matrix}-k, \frac{b}{2}, & \ldots, & \frac{b+1}{2} \\ \frac{c}{2}, & \ldots & , \frac{c+1}{2}\end{matrix}; 1\right).\end{aligned}$$

Chapter 4

Asymptotic zero distribution of some hypergeometric polynomials

4.1 Introduction

In this chapter, we find the asymptotic zero distribution as $n \to \infty$ of

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right). $$

This polynomial cannot be expressed in terms of those considered in [3] or [27]. The limits A and B mentioned in Chapter 1 on page 14 for the corresponding Jacobi polynomials of each of the three cases ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ and the hypergeometric polynomials in [3] and [27], are the same. They are

$$ A = k \in \mathbb{R} \qquad \text{and} \qquad B = -1. $$

In addition, the zeros asymptotically approach the same lemniscate. This suggests that a general result for the asymptotic zero distribution of the class of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(z)$ where

$$ \lim_{n\to\infty} \frac{\alpha_n}{n} = k \text{ and } \lim_{n\to\infty} \frac{\beta_n}{n} = -1 $$

may be possible, but it seems likely that some restrictions may be needed on the parameters α_n and β_n.

We shall prove the following theorem.

Theorem 4.1.1 *The zeros of the hypergeometric polynomial*

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) $$

approach the section of the lemniscate

$$ \left\{ z : \left|z(1-z)^2\right| = \frac{4}{27}; Re(z) > \frac{1}{3} \right\}, $$

as $n \to \infty$.

Figure 4.1 shows numerical plotting of the zeros of the hypergeometric polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ for $n = 70$ and figure 4.2 plots the zeros of the same hypergeometric polynomial for n ranging from 5 to 60. All figures have been generated by Mathematica.

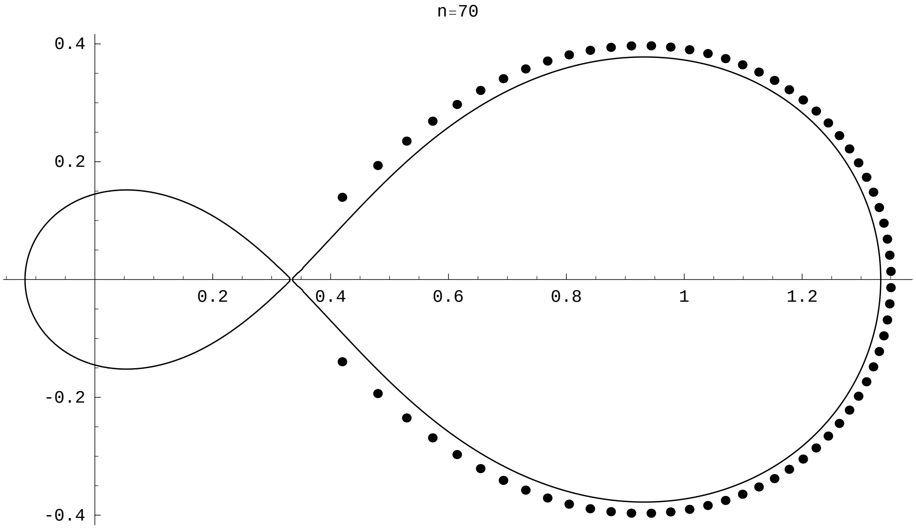

Figure 4.1: The curve $|z(1-z)^2| = \frac{4}{27}$ and the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ for $n = 70$.

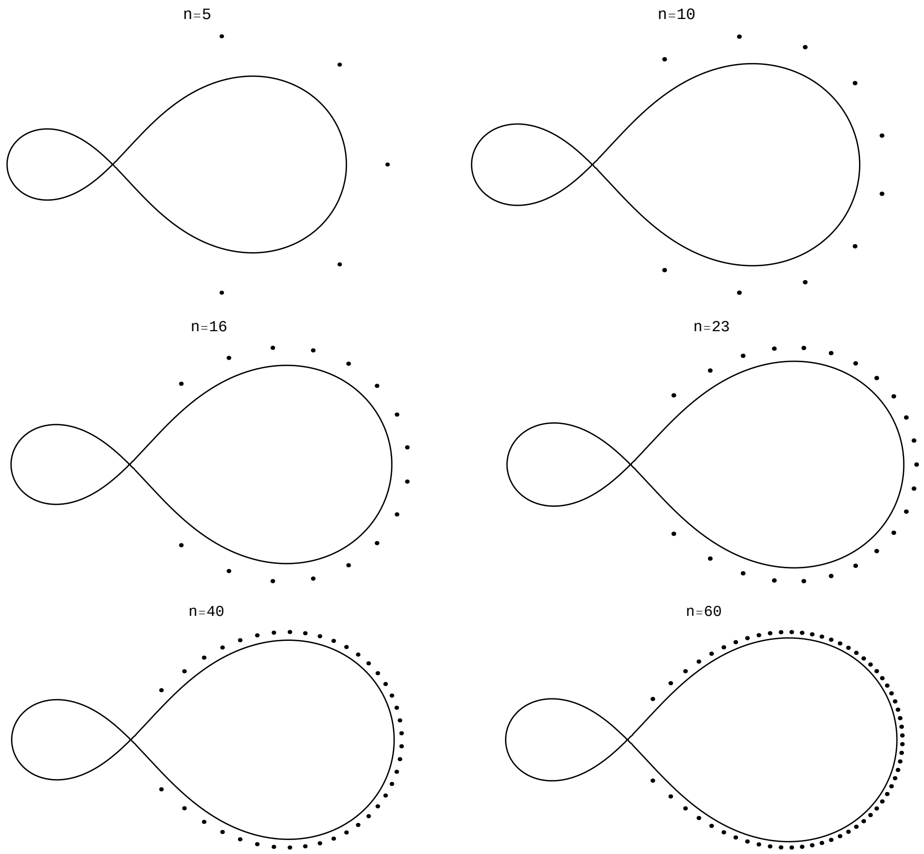

Figure 4.2: The curve $|z(1-z)^2| = \frac{4}{27}$ and the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ for $n = 5, 10, 16, 23, 40, 60$.

In order to prove our result, we follow the method used by Duren and Guillou in [27] with some modifications. This method involves the asymptotic analysis of an integral of the form

$$A_n \int_0^1 [f_z(t)]^n \; dt \tag{4.1.1}$$

where A_n is a constant involving n and $f_z(t)$ is a polynomial in the complex variable t and analytic in z.

The Euler integral formula for ${}_2F_1$ functions (cf. [37], p. 47) is given by

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \; dt.$$

With $a = -n$, $b = \frac{n+1}{2}$ and $c = \frac{n+3}{2}$, we have

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = \frac{n+1}{2} \int_0^1 t^{\frac{n-1}{2}} (1 - zt)^n \, dt $$

which cannot be written in the form given in (4.1.1). Making the transformation from t to t^2 in this integral yields

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = (n+1) \int_0^1 \left[t(1 - zt^2)\right]^n dt. \tag{4.1.2} $$

We could have alternatively used the integral representation formula for ${}_3F_2$ hypergeometric functions from Corollary 3.3.2 on page 51 with $b = n+1$ and $c = n+2$ in order to represent our ${}_2F_1$ in the form of (4.1.1) as given in (4.1.2).

We follow the method outlined in [27] to obtain an asymptotic expansion for the integral

$$ \int_0^1 \left[f_z(t)\right]^n dt $$

where

$$ f_z(t) = t(1 - zt^2) $$

is a polynomial in the complex variable t and analytic in z.

Throughout this paper, we shall denote the two branches of the square root of z by $\pm\sqrt{z}$ where $\sqrt{z}$ is the branch with $\sqrt{1} = 1$.

The function $f_z(t)$ has zeros at $t = 0$ and $t = \pm\frac{1}{\sqrt{z}}$ while the critical points $f_z'(t) = 0$ occur at $t = \pm\frac{1}{\sqrt{3z}}$.

4.2 Preliminary results

In order to prove our main result, we will need the following lemmas.

We recall the classical Eneström-Kakeya theorem (cf. [33] p.136): Given the real polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$, if $a_0 \geq a_1 \geq \ldots \geq a_n > 0$, then $p(z) \neq 0$ for $|z| < 1$.

For our purposes, it is more convenient to use the following version: If $0 < a_0 < a_1 < \ldots < a_n$, then all zeros of the polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ lie in the unit disk $|z| < 1$.

Lemma 4.2.1 *The zeros of* ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ *are contained in the disk* $|z| < n+1$.

Proof:

Let

$$F_n(z) = {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = c_0 + c_1 z + \ldots + c_n z^n,$$

where

$$\begin{aligned} c_m &= \frac{(-n)_m \left(\frac{n+1}{2}\right)_m}{\left(\frac{n+3}{2}\right)_m m!} \\ &= (-1)^m \binom{n}{m} \frac{n+1}{n+2m+1} \end{aligned}$$

It can be shown (by computing a derivative) that the absolute values of the ratios

$$\frac{c_m}{c_{m-1}} = \frac{-(n-m+1)(n+2m-1)}{m(n+2m+1}$$

decrease as m increases. Now

$$\left|\frac{c_n}{c_{n-1}}\right| = \left|\frac{(n-n+1)(n+2n-1)}{n(n+2n+1)}\right|$$

$$= \frac{3n-1}{n(3n+1)} > \frac{1}{n+1}, \quad \text{for } n > 1.$$

Therefore

$$\frac{-(n+1)c_m}{c_{m-1}} > 1, \qquad m = 1, 2, \ldots, n,$$

and it follows immediately that the coefficients of the polynomial

$$\begin{aligned} p(z) = F_n\left(-(n+1)z\right) &= c_0 - c_1(n+1)z + \ldots + (-1)^n(n+1)^n z^n \\ &= a_0 + a_1 z + \ldots + a_n z^n \end{aligned}$$

where

$$a_m = (-1)^m (n+1)^m c_m$$

are positive and increasing: $0 < a_0 < a_1 < \ldots < a_n$. By the Eneström-Kakeya theorem, the zeros of $F_n\left(-(n+1)z\right)$ lie in the unit disk $|z| < 1$ and therefore the zeros of $F_n\left(z\right)$ lie in the disk

$$|z| < n+1.$$

□

Lemma 4.2.2 *The polynomial* ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ *has at least one zero outside the unit circle* $|z| = 1$.

Proof:

We have

$$\begin{aligned} {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) &= c_0 + c_1 z + \ldots + c_n z^n \\ &= c_n(z-k_1)(z-k_2)\ldots(z-k_n) \end{aligned}$$

where k_j is the j^{th} zero of the polynomial, $j = 1, 2, \ldots, n$.

Equating constants on both sides yields

$$c_0 = c_n k_1 k_2 \dots k_n (-1)^n$$

so that

$$|k_1 k_2 \dots k_n| = \left|\frac{c_0}{c_n}\right|.$$

Now $c_0 = 1$ and $c_n = \frac{(-1)^n(n+1)}{3n+1}$, therefore

$$\left|\frac{c_0}{c_n}\right| = \frac{3n+1}{n+1} > 1.$$

It follows that the product of the zeros has modulus greater than 1 and therefore at least one zero of the polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ must be outside the unit circle $|z| = 1$. $\square$

Lemma 4.2.3 *The inequality* $|f_z(1)| > \left|f_z\left(\frac{1}{\sqrt{3z}}\right)\right|$ *holds if and only if*

$$\left|z(1-z)^2\right| > \frac{4}{27}.$$

Similarly, $|f_z(1)| < \left|f_z\left(\frac{1}{\sqrt{3z}}\right)\right|$ *if and only if*

$$\left|z(1-z)^2\right| < \frac{4}{27}.$$

Proof:

The condition $|f_z(1)| > \left|f_z\left(\frac{1}{\sqrt{3z}}\right)\right|$ is equivalent to

$$\begin{aligned} |1-z| &> \left|\frac{1}{\sqrt{3z}}\left(1 - z\left(\frac{1}{\sqrt{3z}}\right)^2\right)\right| \\ \Leftrightarrow |\sqrt{z}(1-z)| &> \left|\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)\right| \\ \Leftrightarrow |z(1-z)^2| &> \frac{4}{27}. \end{aligned}$$

$\square$

We note that Lemma 4.2.3 shows that $|f_z(1)| > \left| f_z\left(\frac{1}{\sqrt{3z}}\right)\right|$ for z outside the lemniscate $|z(1-z)^2| > \frac{4}{27}$, while the inequality is reversed when z is inside either loop of the lemniscate.

Lemma 4.2.4 *If* $Re(\sqrt{z}) > \frac{1}{\sqrt{3}}$*, the function* $|f_z(t)|$ *has a unique path of steepest ascent from* $\frac{1}{\sqrt{z}}$ *to 1. If* $0 < Re(\sqrt{z}) < \frac{1}{\sqrt{3}}$*, there is a unique path of steepest ascent from 0 to 1.*

Proof: The lines through the saddle-points $t = \pm\frac{1}{\sqrt{3z}}$ perpendicular to the linear segment from $-\frac{1}{\sqrt{z}}$ through 0 to $\frac{1}{\sqrt{z}}$ are "continental divides" that separate the t-plane into three basins containing $-\frac{1}{\sqrt{z}}$, 0 and $\frac{1}{\sqrt{z}}$ respectively.

Any point in the 0-basin is joined to 0 by a unique path of steepest decent, orthogonal to the level curves of $f_z(t)$. Points in the $\frac{1}{\sqrt{z}}$-basin and $-\frac{1}{\sqrt{z}}$-basin can be similarly joined to $\frac{1}{\sqrt{z}}$ and $-\frac{1}{\sqrt{z}}$ respectively.

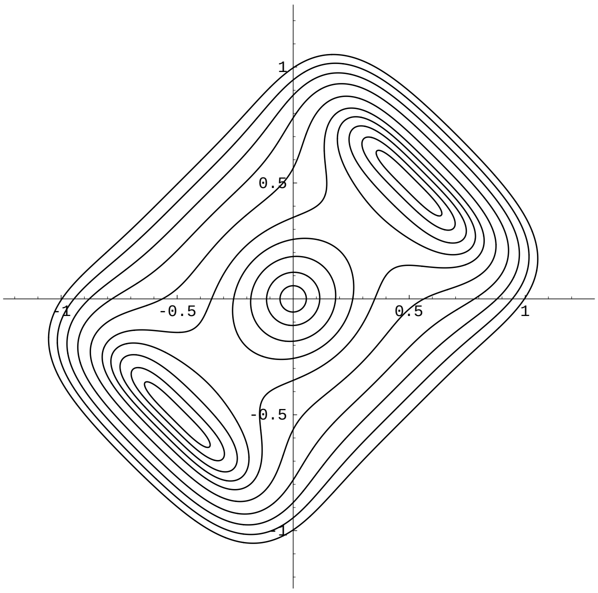

Figure 4.3: The level curves of $f_z(t)$

The point 1 will be located in either the 0-basin or the $\frac{1}{\sqrt{z}}$-basin, but not in the $-\frac{1}{\sqrt{z}}$-basin. First we show that the point 1 lies in the $\frac{1}{\sqrt{z}}$-basin if and only if $\mathrm{Re}(\sqrt{z}) > \frac{1}{\sqrt{3}}$. In order to do so, we multiply each point in the t-plane by $\frac{\sqrt{z}}{|z|}$, which rotates the figure so that all three zeros of $f_z(t)$ move to the real axis and the lines through the saddle-points $-\frac{1}{\sqrt{3z}}$ and $\frac{1}{\sqrt{3z}}$ are carried to the vertical lines through $-\frac{1}{\sqrt{3}|z|}$ and $\frac{1}{\sqrt{3}|z|}$ respectively. The point 1 will then be in the $\frac{1}{\sqrt{z}}$-basin if and only if

$$\mathrm{Re}\left(\frac{\sqrt{z}}{|z|}\right) > \frac{1}{\sqrt{3}|z|}$$

which is equivalent to the condition

$$\mathrm{Re}\left(\sqrt{z}\right) > \frac{1}{\sqrt{3}}.$$

Similarly, the point 1 will then be in the 0-basin if and only if

$$\frac{-1}{\sqrt{3}|z|} < \mathrm{Re}\left(\frac{\sqrt{z}}{|z|}\right) < \frac{1}{\sqrt{3}|z|}$$

which is equivalent to the condition

$$\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}.$$

More precisely, the point 1 will be in the 0-basin if and only if
$0 < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$. ☐

4.3 The asymptotic zero distribution in the region Re$(z) < \frac{1}{3}$

We consider the two possibilities separately. The first is the case where $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$. This section of the z-plane is the area to the left of the parabola with vertex $\frac{1}{3}$ and intercepts $\frac{2}{3}i$ and $-\frac{2}{3}i$. Thus all points z satisfying $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$ will lie to the left of the vertical line $\mathrm{Re}(z) = \frac{1}{3}$ and we shall prove that no zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ are possible for $\mathrm{Re}(z) < \frac{1}{3}$ and, therefore, no zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ can occur for $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$.

Theorem 4.3.1 *For sufficiently large n, the polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ has no zeros in the region $Re(z) < \frac{1}{3}$.*

Proof:

From (4.1.2), we know that

$$\begin{aligned} {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) &= (n+1)\int_0^1 [f_z(t)]^n\, dt \\ &= (n+1)\int_0^1 \left[t(1-zt^2)\right]^n dt. \end{aligned}$$

Lemma 4.2.4 ensures that for $\mathrm{Re}(z) < \frac{1}{3}$, there is a unique path of steepest ascent from 0 to 1. We may thus evaluate the integral involved in (4.1.2) over this path. In order to find the path of steepest ascent, we use that fact that $f_z(t) = t(1-zt^2)$ will have constant argument along this path (cf. [8]) so that we can parametrise the path by letting

$$f_z(t) = f_z(1)r \qquad 0 \le r \le 1,$$

or equivalently

$$t(1-zt^2) = r(1-z). \tag{4.3.1}$$

Then

$$(1-3zt^2)dt = (1-z)dr$$

and our hypergeometric polynomial can be rewritten as

$${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = (n+1)(1-z)^{n+1}\int_0^1 \frac{r^n}{1-3zt^2}dr \tag{4.3.2}$$

where $t = t(r)$ is defined implicitly by (4.3.1), with $t(0) = 0$ and $t(1) = 1$.

Any zeros $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) < \frac{1}{3}$ must satisfy

$$(n+1)(1-z)^{n+1}\int_0^1 \frac{r^n}{1-3zt^2}dr = 0,$$

or equivalently, using (4.3.1)

$$n\int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr = 0. \tag{4.3.3}$$

We will prove that the integral on the left-hand side of (4.3.3) is bounded away from 0 and hence deduce that no zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ can lie in the half-plane $\mathrm{Re}(z) < \frac{1}{3}$.

If the zeros are restricted by the inequality $\left|z - \frac{1}{3}\right| \geq \epsilon$ for some $\epsilon > 0$, then the denominator of the integrand in (4.3.3) satisfies $|1-3zt^2| \geq \delta > 0$, where δ is independent of z. Thus for any fixed ρ with $0 < \rho < 1$, we have

$$\begin{aligned} n\left|\int_0^\rho \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr\right| &\leq n\int_0^\rho \frac{|(1-zt^2)t|}{|1-3zt^2|}\, r^{n-1}dr \\ &\leq Cn\int_0^\rho |(1-zt^2)t|\, r^{n-1}dr \end{aligned}$$

since $\frac{1}{|1-3zt^2|} \leq C$ for some constant C.

Lemma 4.2.1 states that the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ are contained in the disk $|z| < n+1$. Thus

$$\begin{aligned} n\left|\int_0^\rho \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr\right| &\leq Cn\int_0^\rho |(1-zt^2)t|\, r^{n-1}dr \\ &\leq Cn^2 \int_0^\rho r^{n-1}dr \\ &\leq Cn\, \rho^n \to 0 \end{aligned} \tag{4.3.4}$$

as $n \to \infty$ since $0 < \rho < 1$. On the other hand, for ρ sufficiently close to 1, the integral

$$n\int_\rho^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr \tag{4.3.5}$$

is bounded away from zero. To prove this, we first note that since the path $t = t(r)$ must lie on the same side of the "continental divide" as the point 1 (cf. proof of Lemma 4.2.4), we know that $\mathrm{Re}(zt^2) < \frac{1}{3}$. Our restriction on z further ensures that $\left|zt^2 - \frac{1}{3}\right| > \frac{\epsilon}{2}$ for t sufficiently near 1.

Now the linear fractional mapping

$$\omega = \phi(\zeta) = \frac{1-\zeta}{1-3\zeta}$$

sends the region

$$\left\{\zeta : \mathrm{Re}(\zeta) < \frac{1}{3}, \left|\zeta - \frac{1}{3}\right| > \frac{\epsilon}{2}\right\}$$

onto a semidisk to the right of the vertical line $\mathrm{Re}(\omega) = \frac{1}{3}$. It follows that

$$\mathrm{Re}\left\{\frac{(1-zt^2)t}{1-3zt^2}\right\} > \frac{1}{6}$$

when t is close enough to 1. This shows that

$$\mathrm{Re}\left\{n\int_\rho^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}\,dr\right\} > \frac{n}{6}\int_\rho^1 r^{n-1}\,dr > \frac{1}{12}$$

for ρ near 1 and all z satisfying $\mathrm{Re}(z) < \frac{1}{3}$. Combining this with (4.3.4), we see that for sufficiently large n, the polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ can have no zeros in the region $\mathrm{Re}(z) < \frac{1}{3}$. ☐

Thus any zeros in the region $\mathrm{Re}(z) < \frac{1}{3}$ (if they exist) must converge uniformly to the point $\frac{1}{3}$ as $n \to \infty$ (since we had the additional restriction of $\left|z - \frac{1}{3}\right| > \epsilon$ for some $\epsilon > 0$).

4.4 The asymptotic zero distribution in the region Re$(z) > \frac{1}{3}$

We know from section 4.3 that for n sufficiently large, the zeros of the hypergeometric polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ lie in the region $\mathrm{Re}(z) > \frac{1}{3}$.

From Lemma 4.2.2, we know that at least one of the zeros of the hypergeometric polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ lies outside the unit circle $|z| = 1$. Since there are no zeros to the left of $\mathrm{Re}(z) = \frac{1}{3}$, we know that this polynomial has at least one zero in the region $\mathrm{Re}(z) > \frac{1}{3}$.

We are now in a position to prove our theorem.

Theorem 4.4.1 *The zeros of the hypergeometric polynomial*

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) $$

approach the section of the lemniscate

$$ \left\{z : \left|z(1-z)^2\right| = \frac{4}{27}; Re(z) > \frac{1}{3}\right\}, $$

as $n \to \infty$.

Proof:

From (4.1.2), we know that

$$ \begin{aligned} {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) &= (n+1)\int_0^1 [f_z(t)]^n \, dt \\ &= (n+1)\int_0^1 \left[t(1-zt^2)\right]^n \, dt. \end{aligned} $$

Lemma 4.2.4 guarantees that there is a unique path of steepest ascent from $\frac{1}{\sqrt{z}}$ to 1.

We may thus deform the path of integration in (4.1.2) to write

$$ \int_0^1 [f_z(t)]^n \, dt = \int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt + \int_{\frac{1}{\sqrt{z}}}^1 [f_z(t)]^n \, dt $$

following the linear path from 0 to $\frac{1}{\sqrt{z}}$ and then the unique path of steepest ascent from $\frac{1}{\sqrt{z}}$ to 1. The linear path from 0 to $\frac{1}{\sqrt{z}}$ is orthogonal to the level curves of $f_z(t)$ and is therefore the path of steepest ascent from 0 to the saddle-point $\frac{1}{\sqrt{3z}}$, followed by the path of steepest descent to $\frac{1}{\sqrt{z}}$.

Any zero $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) > \frac{1}{3}$ must satisfy

$$\int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt + \int_{\frac{1}{\sqrt{z}}}^1 [f_z(t)]^n \, dt = 0 \tag{4.4.6}$$

where the integrals are taken over paths of steepest ascent or descent.

Making the substitution $s = zt^2$ for $0 \le s \le 1$, we obtain

$$\begin{aligned}\int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt &= \int_0^{\frac{1}{\sqrt{z}}} \left[t(1-zt^2)\right]^n dt \\ &= \frac{1}{2(\sqrt{z})^{n+1}} \int_0^1 s^{(n+1)/2}(1-s)^n ds \\ &= \frac{1}{2(\sqrt{z})^{n+1}} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma(n+1)}{\Gamma\left(\frac{3n+3}{2}\right)} \end{aligned} \tag{4.4.7}$$

Using Stirling's approximation

$$\Gamma(n+1) = e^{-n}n^n\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O\left(\frac{1}{n^3}\right)\right), \qquad n \to \infty,$$

we then have

$$\int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt = \frac{\sqrt{2\pi}}{3\sqrt{n}(\sqrt{z})^{n+1}} \left(\frac{2}{\sqrt{27}}\right)^n \left[1 + O\left(\frac{1}{n}\right)\right], \tag{4.4.8}$$

as $n \to \infty$. The second integral on the right-hand side of (4.4.6) requires that we find the path of steepest ascent from $\frac{1}{\sqrt{z}}$ to 1. We again use the fact that $f_z(t)$ will have constant argument along this path (cf. [8]) so that we can parametrise this path by letting

$$f_z(t) = f_z(1)r \qquad 0 \le r \le 1,$$

yielding

$$\int_{\frac{1}{\sqrt{z}}}^{1} [f_z(t)]^n \, dt = (1-z)^n \int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2} dr \tag{4.4.9}$$

where $t = t(r)$ is defined implicitly by (4.3.1), with $t(0) = \frac{1}{\sqrt{z}}$ and $t(1) = 1$.

Therefore, any zero $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) > \frac{1}{3}$ must asymptotically satisfy

$$\left(\frac{2}{\sqrt{27}}\right)^n \frac{\sqrt{2\pi}}{3\sqrt{n}(\sqrt{z})^{n+1}} \left\{1 + O\left(\frac{1}{n}\right)\right\} + (1-z)^n \int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2} dr = 0,$$

as $n \to \infty$, or equivalently

$$(\sqrt{z})^{n+1}(1-z)^n \int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2} dr = -\left(\frac{2}{\sqrt{27}}\right)^n \frac{\sqrt{2\pi}}{3\sqrt{n}} \left\{1 + O\left(\frac{1}{n}\right)\right\}. \tag{4.4.10}$$

as $n \to \infty$. Taking moduli and n^{th} roots on both sides of (4.4.10), we obtain

$$|\sqrt{z}|^{\frac{1}{n}} \left|\sqrt{z}(1-z)\right| \left|\int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2} dr\right|^{\frac{1}{n}} = \left|\frac{2}{\sqrt{27}}\right| \left(\frac{\sqrt{2\pi}}{3\sqrt{n}}\right)^{\frac{1}{n}} \left\{1 + O\left(\frac{1}{n}\right)\right\}^{\frac{1}{n}}. \tag{4.4.11}$$

It is straightforward to check that, as $n \to \infty$,

$$\left|\int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2} dr\right|^{1/n}$$

converges to 1 uniformly in z and the zeros $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) > \frac{1}{3}$ approach the lemniscate

$$\left|\sqrt{z}(1-z)\right| = \frac{2}{\sqrt{27}}$$

or equivalently

$$\left|z(1-z)^2\right| = \frac{4}{27}.$$

□

In addition, we note that by taking n^{th} roots on both sides of (4.4.10), for large n, there are n points satisfying (4.4.10), distinguished by the n choices of $\sqrt[n]{-1}$. All of these points are zeros of the polynomial, spread out near the right-hand branch of the lemniscate.

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