Exploring Grade 4 learners' use of models and strategies for solving addition and subtraction problems

by

Herman M. Tshesane

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Abstract

The Mathematics Curriculum and Assessment Policy (CAPS) document defines 'mathematics [as] . . . *a human activity*' (DBE, 2011a, p.8). This adoption of a realistic approach to the learning and teaching of mathematics appears to be partial, however, in that at the entry point of the Intermediate Phase, the recommendations of the policy makers are read as prescriptions by practitioners. In particular, the recommendation that 'as the number range for doing calculations increases up to Grade 6, learners should develop more efficient techniques for calculations, including using columns' (DBE, 2011b, p.13) is taken as a prescription to push the standard methods as the way to solving (often de-contextualized) problems from the very start of Grade 4, in disregard to the admonition that 'these techniques should only be introduced and encouraged once learners have an adequate sense of place value and understanding of the properties of numbers and operations' (DBE, 2011b, p.13).

In the background of reports that place South African schools well below international standards with regard to mathematics, with only a third of the learners in grade 3 having attained the minimum standard required of learners at their level in 2011, this report focuses on an exploration into the purported catalytic role that the emergent model of an empty number line can play in shifting learners' attention from counting (*calculation by counting*) towards a focus on the structural properties of number (*calculation by structuring).* The use of emergent models is meant to support and improve upon learners' informal solution strategies whilst seeking to reverse what Freudenthal referred to as the "anti-didactical" use of models in a 'top-down instructional design strategy in which static models are derived from crystallized expert mathematical knowledge' (Gravemeijer and Stephan, 2002, p.146).

With a particular focus on poor performance in numeracy, the Wits Maths Connect-Primary (WMC-P) project was established with the overarching aim of improving the learning and teaching of primary school mathematics. My investigation is located within one Grade 4 class in one of the WMC-P project schools, and in this project, I act as both the teacher of six intervention lessons focused on additive relation problems, as well as researcher of the models and strategies that learners use prior to the intervention lessons, within these lessons, and subsequently. This report presents evidence to illustrate, firstly, that at the entry point of grade 4 level, learners are highly dependent on concrete strategies for solving addition and subtraction problems, and secondly that with proper intervention, learners can make significant shifts towards more abstract calculation.

On the one hand, the key finding that the majority of the problems were tackled using tallies in the pre-test confirms what research has observed regarding the tendency for learners to remain highly dependent on concrete strategies at grade 2 (Venkat, 2011) and grade 3 (Ensor et al., 2009). Also, the results indicate a high proportion of incorrect answers resulting from the use of the column model across all questions in the pre-test and the post-test. On the other hand, the imposition of the use of the empty number line in the delayed-post-test points to the fact that improvements can be achieved in relatively short time frames, and importantly, that these improvements can be retained beyond their immediate coverage in class.

Key words: Number sense, RME, mathematization, emergent models, progression in number, counting and calculating strategies, empty number line, South Africa.

I, HERMAN M. TSHESANE, declare that this assignment is my own work and no part of it has been copied from another source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list.

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Signature

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I wish to acknowledge the invaluable guidance and support provided by the Wits Maths Connect Primary Project team, in particular, by my supervisor and mentor Professor Hamsa Venkat.

Dedication

This piece of work is dedicated to my mother (Gemma Tshesane), my brother (George), my sister (Elizabeth) and the entire extended family at large (Ba-Tubatse ba Mohlogopela) who have had to do without my presence at several important family gatherings while I was completing this project. May the returns on this investment be worth your while.

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Chapter 1: Introduction

1.1 Background to the study

The Wits Maths Connect-Primary (WMC-P) project is a five year project launched in 2011 working with ten government primary schools in one district with the following main objectives:

- ✓ to increase learners' number sense from grade R to grade 6
- ✓ to improve learners' results in the Annual National Assessments (ANA)
- ✓ to improve the quality of Mathematics teaching by supporting teachers with a view to achieving the abovementioned objectives.

In order to establish grade 2 learners' proficiency on number problems, a baseline assessment was conducted through administration of interview-based oral number sense tests drawn from the work of Wright, Martland and Stafford (2006) structured on the basis of their Learning Framework in Number (LFIN). Six learners in each grade 2 class drawn from across the attainment range in all the 10 schools took part in the assessment administered to gather in-depth information regarding learners' early number sense. An overview analysis indicated that three quarters of the learners tested used count-all based strategies in early addition problems (Venkat, 2011). This strategy is often referred to as 'triple concrete counting' (Anghileri, 2006) to highlight the fact that one first counts each one of the addends and then counts the total – an inefficient strategy to use. Only 24% of the learners tested demonstrated an ability to work fluently with *count-on* strategies in the range 1 – 20 (involving counting on from the first number in an addition problem), and fewer could work with 5s and 10s as benchmarks, 'or use 5s and 10s to support their calculation strategies as the number range increased' (Venkat, 2011, p6). The results of the tests provided insights into gaps in fundamental understandings related to additive relations that corroborated Schollar's (2008) findings that learners in the Intermediate Phase reverted to the use of unit tally counting as a strategy to solving addition and subtraction problems.

In light of the widespread prevalence of concrete methods such as tally counting in solving problems in mathematics in the Foundation Phase – methods that Ensor, Hoadley, Jacklin, Kuhne, Schmitt, Lombard, Van Den Heuvel-Panhuizen (2009) have described as keeping learners 'highly dependent on concrete strategies at grade 3 level' – counting strategies such as *counting-on*, *counting-down-from* and *counting-down-to* (described in more detail in

Chapter 3) have been suggested as a means to efficiency in calculation (Wright et al., 2006; Carpenter, Fennema, Franke, Levi & Empson, 1999). Carpenter et al (1999) have noted that whilst counting strategies can be used to produce correct answers in lower number ranges on addition/subtraction problems, they become 'inefficient and distracting' when larger numbers are introduced. This motivates the need to support learners towards appropriate models and strategies that can serve as a means to move them to solving such problems more efficiently.

As learners shift from a reliance on *counting strategies* towards deriving *number facts* from *known facts*, they need to be supported with models that can facilitate this progression: models with more longevity than the ones that currently hold default position in learners' current repertoire of models when faced with solving an addition or subtraction problem. For their inclination to facilitate 'reasoning about relations between number relations' (Gravemeijer & Stephan, 2002, p151), I refer to these superseding models as *relational models.* In this study, my focus is on investigating the models for addition and subtraction that the learners in one Grade 4 class use before, during and after an intervention focused on supporting learners' use of one particular relational model – the empty number line.

1.2 The rationale as informed by literature

The empty number line model has been described as an example of an *emergent model,* whose aim is not at 'modeling as translating contextual problems into a (more) mathematical language' but rather as an 'organizing activity from which the model emerges' (Gravemeijer & Stephan, 2002, p148). In contrast to classical models that support a top-down instructional design approach, such as the use of base-ten blocks, *emergent models* are not designed to fit conventional base-ten position systems. Instead, *emergent models* are meant to ratchet the shift from a model *of* informal solution strategies to a model *for* more formal mathematical reasoning in a bottom-up approach:

> The label emergent refers both to the character of the process by which the models emerge . . . and to the process by which these models support the emergence of formal mathematical ways of knowing (Gravemeijer & Stephan, 2002, p145).

At the heart of my exploration is an investigation of the purported catalytic role that the empty number line can play in shifting learners' attention from the specific numbers in the problem to the mathematical relations between the numbers in the problem. An intervention based on the word problem tasks focused on different types of additive relation situations devised by Askew (2004) within his ‘Big Books’ series was implemented. Part of the research aim was to investigate whether, and if so, how, this intervention impacted on Grade 4 learners’ use of models and strategies for addition and subtraction.

In Askew, Bibby and Brown’s (2001) observations, there were many children – even at the end of primary school in Year 6 in England – who relied more on procedures such as counting to find answers to calculation problems. Similarly, Ensor et al. (2009) concur with Gray and Tall (1994) based on their South African study, regarding the existence of a:

> *proceptual divide* between those who cling to the comfort of counting procedures that, at best, enable them to solve simple problems by counting and those who develop a more flexible form of arithmetic in which the symbols can be used dually as processes or as concepts to manipulate mentally (p.11)

With the discourse of mathematics described as continuously growing through the reification of new mathematical processes into objects (Sfard, 2007), the models that we advance must anticipate what is forthcoming in the curriculum whilst building on what has gone before. I believe that this is what Streefland (1991) meant with the utterance ‘to foresee where and how one can anticipate that which is just coming into view in the distance’ (p.285). Within a framework based on the Dutch Realistic Mathematics Education (RME) theory, we cannot abandon the counting strategies that have seen the incumbents graduate into the intermediate phase. Instead, we need to extend the application-life-span of these counting strategies by scaffolding them with relational models. With the addition of integers looming on the intermediate phase horizon, learners at the exit point of the foundation phase must be introduced to models that can handle operating on integers, so as to make the transition as seamless as possible. The number line is one such model in that it provides access to a continuous model of number within which fractional, decimal and integral values can be accommodated. It can thus operate, as a key linking model in the transition from foundation to intermediate phase: from counting to calculating; from a conception of number as reflecting numerosity to the appreciation of number as an object that can be manipulated in accordance with certain laws (Ensor et al., 2009). Our failure to scaffold this transition will amount to setting our learners up for failure as they will find it difficult to successfully ‘collect like terms’ when they reach the senior phase.

1.3 Problem statement

The column addition/subtraction model is the traditional format used in school to operate on whole numbers. This model is traditionally restricted to whole numbers, leading to the need for different models to be introduced for handling addition/subtraction of fractions, decimals, integers or a mix of these. The point about the column addition/subtraction model is that children only ever need to operate on single digit numbers at a time – i.e. tens and units independently. As a result, learners do not need to have a sense of the relative size or position of numbers. The column addition/subtraction model therefore underplays the need to understand the 'quantity values' (Thompson, 1997) of the numbers involved, and instead, presents addition and subtraction as a series of rules for manipulating individual digits (Wright et al., 2006). It is a formula-based method of operating on whole numbers, a residual of the mechanistic era:

> The struggle against the mechanistic approach to mathematics education has not been conquered completely – especially in classroom practice much work still has to be done in this regard (Van den Heuvel-Panhuizen, 2001, p.2)

1.4 A possible solution

The use of emergent models is meant to support and improve upon learners' informal solution strategies whilst seeking to reverse what Freudenthal referred to as the "anti-didactical" use of models in a 'top-down instructional design strategy in which static models are derived from crystallized expert mathematical knowledge' (Gravemeijer and Stephan, 2002, p.146) which is the lot of the mechanistic school. Gravemeijer and Doorman (1999) concur with Resnick and Omanson (1987) that the use of base-ten blocks is an example of a classical model that represents a top-down instructional design approach in that it necessitates that 'expert knowledge of the decimal system and the conventional column algorithm are taken as points of reference' (p.149).

My hypothesis as I started this study was that learners who have mastered the use of the empty number line as a model for addition and subtraction are better positioned for more seamless transitions between phases given the longevity of this model in application. Fewer difficulties are likely, for example, with larger numbers and with addition of integers because the relational nature of the number line will allow them to start anywhere on the number line and stop anywhere, including the negative integers once the number line is extended to include the negative integers.

1.5 The Purpose statement

I seek to investigate the possibilities of shifting learners to the use of the empty number line model as an alternative to *column addition/subtraction* and *decomposition* models. Further, where this occurs, I would like to find out the extent to which learners armed with the *empty number-line* model, coupled with the ability to discern different addition and subtraction problem types, are better placed, if at all, to deal with addition and subtraction problems.

My investigation is located within one Grade 4 class in one of the WMC-P project schools, and in this project, I act as both the teacher of six intervention lessons focused on additive relation problems and researcher of the models and strategies that learners use prior to the intervention lessons, within these lessons, and subsequently. My conception of what constitutes a strategy will be unpacked in detail in chapters 2 and 3.

1.6 The Research Questions

I. What models and strategies did these learners initially use to solve different types of addition and subtraction problems and how did they perform?

II. In an intervention focused on improving the learners' performance on addition and subtraction problems, what sorts of models and strategies are advanced? What sorts of models and strategies are learners using during the process of the intervention?

III. What effect, if any, has this intervention had on learners' use of models and strategies for solving different types of addition and subtraction problems?

1.7 Methodology

To answer my research questions I requested one of the teachers from one of the ten schools participating in the WMC-P Project to avail his Grade 4 class for purposes of my study. Subsequently, I approached the Principal and each of the learners in his class for consent to participate in the study.

In order to avoid having to translate the word problems and perhaps inadvertently introduce another variable into the intervention advanced by the Big Book Project, an important consideration was that the group of learners had to be sufficiently immersed in the English language, and so the school of choice had to be one whose language of learning and teaching

(LoLT) was English. Consequently, the school that participated was one of the five suburban schools in the project. The teacher whose class was eventually used for this study was one who was open to trying out innovative ways of teaching addition and subtraction to learners on the entry point of their intermediate phase, and concurrently participating in the broader teacher development activity in the WMC-P project.

The data that I worked with came from the pre-test, post-test and delayed-post-test that the learners wrote, the work learners produced during the intervention, the field notes gathered during the intervention. In analyzing the data I appealed to aspects of Realistic Mathematics Education (RME) theory promoted in the idea of guided reinvention through progressive mathematization and the emergent models heuristics.

1.8 Structure of research report

This chapter serves to introduce the area of interest in my research, providing the background against which the investigation is set, the rationale as informed by literature, and the research questions that I sought to answer when I set out on the investigation.

While it is more usual to go into literature and follow it up with theory, I detail the theoretical framework as it outlines the broad assumptions about models and their pedagogical purpose which then establishes a hierarchy of models and strategies as informed by the literature review that then follows. So I first unpack the ways in which I am thinking about models and strategies, and the ways in which I conceptualize teaching and learning with them as part of my theoretical framework in chapter 2. This is followed by chapter 3 wherein I elaborate upon the literature that speaks to RME with a keen interest in how they define and promote the use of models and strategies for addition and subtraction tasks. This graduated process culminates in my analytical framework for this study.

Chapter 4 delineates the methodology adopted in my study, describing the research design and procedure used in sampling, as well as my reasons for the choices made. Along with this is an explanation of the techniques used for data collection and data analysis. Also, ethical issues are addressed over and above matters of validity and reliability. I then present the findings of the study in chapter 5 where I analyze and discuss these in view of the research questions and relevant literature. I conclude in chapter 6 with what I have observed to be the implications and limitations of the study.

Chapter 2: Theoretical framework

2.1 Introduction and rationale

In this chapter I look to illuminate my conception of models and strategies and the use thereof where addition and subtraction problems are concerned, especially in the Foundation and Intermediate Phases of schooling. In order to place on record a priori my broad assumptions regarding models and strategies and their pedagogical purpose, I begin with a chapter outlining my theoretical position in relation to both of these as a precursor to what literature says regarding models and strategies. How these can be used to work with additive relations is addressed in chapter 3 along with evidence of how learners and teachers work with models and strategies as well as progression in their use as informed by literature.

What motivates this look into learners' use of models and strategies as they related to addition and subtraction problems is the contention that the underperformance in the Annual National Assessment (ANA) in 2012 – where the national mean score in Grade 9 was a meagre 13% - is partly due to inadequacies that can be traced back to the formative years of learners' education. In particular, evidence indicates that appropriate intervention at Foundation Phase can go a long way towards constraining the emergent deficits in the early years of learning before they become crystallized in high school (Ensor et al., 2009).

According to Carpenter et al (1999), learners' intuitive knowledge of mathematics can serve as the basis for developing an appreciation for formal school mathematics. As a result, they are in agreement with RME that, when introducing learners to the operations of addition and subtraction, it is best to start with contextualized problems. They propose that, in order to understand how children think about addition and subtraction, it is important to consider differences among problems. Consequently they outline a classification of problems that frames an understanding of the evolution of the strategies children use for solving such problems. It is upon this framework (which is elaborated upon in the chapter that follows) that the intervention used in this study, as authored by Askew (2004) in the Big Book is based.

In this chapter, special attention will be given to the approach of the Dutch Realistic Mathematics Education (RME), with a focus on the character of models and strategies they advance for tackling addition and subtraction problems. Part of this will be to present theory relating to pedagogy for emergent models as a way of setting them apart from mathematical models in general. An important connection made in this chapter is what I perceive to be a one-to-one correspondence between, on the one hand, the levels of thinking/knowledge/understanding that a learner goes through while working with a model, and on the other hand, what I have rephrased as the stages of development in the use of strategies over time. In order to elucidate the nature and relevance of RME to our times, I begin with a historical background.

2.2 A historical background

The belief that the whole world (as well as the parts that constitute it) can be likened to a machine is what is meant by a mechanistic view of the world. In such a world view everything is basically mechanical. This has been the predominant world view of the sciences since the seventeenth century (Sheldrake, 2012); so much so that education researchers today speak of mechanistic approaches to education that characterize what they have coined the 'mechanistic school' (Nellisen & Tomic, 1993). These mechanistic approaches distinguish themselves for their

> procedure-focused way of teaching in which the learning content is split up in meaningless small parts and where the students are offered fixed solving procedures to be trained by exercises, often to be done individually (van den Heuvel-Panhuizen, 2001, p4)

This citation echoes very closely the utterances of Skinner (1954) – the father of behaviorist learning theories – who conceived of learning as the accumulation of stimulus-response associations. He was convinced that

> By making each successive step as small as possible, the frequency of reinforcement can be raised to a maximum, while the possibly aversive consequences of being wrong are reduced to a minimum (Shepard, 2000, p.5)

Concomitant with these learning theories were curriculum and measurement theories that advocated for assessments to be "objective" whilst preferring "formula-based methods" to solving mathematical problems (Shepard, 2000). Aligned with the way of teaching in the mechanistic school, these formula-based methods were procedure-focused. In the spirit of the mechanistic era these methods do not concern themselves with different solution paths, nor do they attend to the different strategies that learners use in search of a solution. On the contrary, the focus of these methods is on getting learners to learn to use conclusive standard procedures that can produce the correct answer (output) from bits of information (input). Consequently, it was sufficient simply to teach the standard algorithms for each of the operations of addition, subtraction, multiplication and division with little or no attention given to learning to solve problems independently (Nelissen and Tomic, 1993).

Nelissen and Tomic (1993) further observed that in any dispensation where the essence of instruction lies in teaching irrefutable procedures to the exclusion of independent thinking, there is a risk that, because ‘children do not consider such formal knowledge “real” or meaningful, they will not be able to apply it or [they will] only do so blindly’ (p.31). In short then, the mechanistic school has been observed to unduly expose learners to the danger of “inert” knowledge when it forces them to learn a formula such as “if the units of the subtrahend are smaller than those of the minuend, borrow from the tens of the minuend to proceed”.

In the wake of the shift in paradigms from a fundamentally mechanistic one to an organic view of the world, the metaphor of a machine has been replaced with that of the crackling of an egg (Sheldrake, 2012). This new metaphor neatly captures the growth that our expanding Universe has been experiencing ever since the Big Bang, and mathematicians and mathematics educators soon felt the need to ‘abandon the static and absolutist theory of mathematics’ (Nelissen and Tomic, 1993, p.20) in favor of a more realistic view of mathematics and approach to mathematics instruction; a type of view and approach to mathematics instruction that will have mathematics as a work continuously under human construction and ‘not a type of finished structure’ (p.20) to be appropriated.

The new understanding is that learning is an active process of mental construction and sense making (Shepard, 2000), effectively reintroducing the concept of mind and making the notion of meaning central to learning. In particular, cognitive theory reveals to us that

> intelligent thought involves self-monitoring and awareness about when and how to use skills, and that *“expertise” develops in a field of study as a principled and coherent way of thinking and representing problems* [italics my emphasis], not just an accumulation of information (Shepard, 2000, p.6)

This understanding forms the basis of what has morphed (from constructivism and socio-culturalism) into what is currently designated social-constructivism, interlacing the individual and social dimensions of learning – individual construction and social interaction –and effectively setting the scene for a realistic approach to mathematics instruction (Nelissen and Tomic, 1993). According to this perspective, the very act of teaching and learning is an exercise in negotiating meaning as opposed to the transmission/acquisition of knowledge

(Bruner, 1986) as the mechanistic school would have it. My emphasis on the words in the citation above is meant to assert my conviction that in the final analysis mathematics inculcates a disciplined way of thinking and that knowledge is created by means of images or representations (Gardner, 1987; Sfard, 1991). As a result, in instruction, didactical use of models can facilitate meaningful mathematization, which according to Freudenthal – the father of Realistic Mathematics Education (RME) – is the core goal of mathematics education:

> What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics (Freudenthal, 1968, p.7)

What makes RME the most promising amongst its contemporaries, in my view, is firstly that its proponents never lose sight of the organic nature of the activity, which keeps it true to the metaphor of mathematics as a field with its own self-organizing abilities, purposes and goals. Not only is the work by learners acknowledged to be always evolving, but so is the RME theory itself. This can be gleaned from statements like 'the development of RME is thirty years old now, and we still consider it as a "work under construction"' (Van den Heuvel-Panhuizen, 2001, p.1); 'the RME theory is not a well-defined, fixed theory. It is an initially global, and to some extent vague theory that is being improved in the process of adaptation and elaboration' (Gravemeijer and Stephan, 2002, p.147).

The second reason that makes RME promising is that it has as its premise that formal mathematics is not something outlandish that the learner somehow has to connect with, but rather something that grows out of the learner's activity: 'common sense develops as the mathematics – which is part of this common sense – develops' (Gravemeijer and Stephan, 2002, p.148). To enable this, it is advised that the instructional design be one that aims at creating the most suitable conditions for the emergence of formal mathematical knowledge. The idea, as Gravemeijer and Doorman (1999) pointed out, is not to try and bridge the gap between learners' informal knowledge and the formal mathematics to be learned but rather to help them 'transcend this dichotomy by aiming at a process in which the formal mathematics emerges from the mathematical activity of the students' (p.116). It is not surprising, therefore, that the models that the theory advances are said to be *emergent*, a coining that serves to capture a perception of modeling as an organizing activity from which the model emerges, and thus accentuates the models' dynamic evolutionary character. This is important for the three reasons outlined below.

Firstly, it is in keeping with the metaphor of an evolving organism as it implies growth in mental faculties in that ‘the idea of mathematizing implies that students develop a high level of intellectual autonomy’ (Gravemeijer and Stephan, 2002, p.147). By extension, learners grow into independent thinkers. Secondly, by insisting that the ‘starting point [be] the exploration of a context problem which can be solved on several levels of understanding’ (Van den Heuvel-Panhuizen, 2001, p13), it guarantees to learners that the activity will always begin on familiar turf. I can see how this can build confidence in learners and the kind of ‘productive disposition’ sought by Kilpatrick, Swafford and Findell (2001). When elaborating the emergence of the model and how that connects with the growth in understanding of learners, Van den Heuvel-Panhuizen (2001) gives credit to Leen Streefland whom she says contributed much for illuminating the mapping between the levels of activity, and the stages in the process of evolution of the model. It is this mapping that forms the basis of my analytical framework.

The last and certainly not the least of reasons that makes RME hold promise stems from the acknowledgement in the second reason above. If *mathematics can and should be learned on one’s own authority and through one’s own mental activities* (Gravemeijer and Stephan, 2002, p.147), then Freudenthal’s conviction that mathematics can be made accessible to every individual is a sound and welcome one. For developing countries like South Africa trying to find their place in a global village (with a rich social and cultural heritage), RME offers a viable option going forward:

> Freudenthal’s most convincing argument for RME is that not all students are future mathematicians but, rather that, for the majority, the mathematics that they will use will be to solve problems in the everyday-life situations (Gravemeijer and Terwel, 2000, p.792)

Freudenthal’s conviction shines a glimmer of hope for young democracies like South Africa that are faced with large classes, as it establishes ‘a strong preference for keeping the class together as a unit of organization [whilst] adapting the education to the different ability levels of the students instead’ (Van den Heuvel-Panhuizen, 2001, p.13). It creates room for individual construction in an interactive environment. In this way the individual does not dissolve into the group but becomes a member of the group whilst remaining an individual. Viewed through a wider lens, it is a theory that can serve as a catalyst to the emergence of classrooms that are interactive, with multiple informal model creation activities in progress and under discussion.

The claims made on its behalf regarding its potential to serve each learner according to their educational needs makes it a viable option for a country with the kind of historical and demographical challenges that South Africa is facing.

2.3 A disjuncture between policy and implementation

Aspects of the RME approach are in evidence in the definition of mathematics provided in the Mathematics Curriculum and Assessment Policy (CAPS) document to the effect that 'mathematics is . . . *a human activity* [italic emphasis mine] that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves' (DBE, 2011a, p.8). This adoption appears to be partial, however, in that at the entry point of the Intermediate Phase, the recommendations of the policy makers are read as prescriptions by those who are tasked with implementing them. For instance, although in the Foundation Phase great emphasis is placed on learners' use of strategies when solving problems in context, the recommendation that 'as the number range for doing calculations increases up to Grade 6, learners should develop more efficient techniques for calculations, including using columns' (DBE, 2011b, p.13) is taken as a prescription to push the standard methods as the way to solving (often de-contextualized) problems from the very start of Grade 4, in disregard of the admonition that 'these techniques should only be introduced and encouraged once learners have an adequate sense of place value and understanding of the properties of numbers and operations' (DBE, 2011b, p.13).

Also, although there is a note that the sequencing and pacing of content are suggestions, I find that the prescriptive sequencing appears to lack the anticipation and responsiveness necessary to keep learners' experience sufficiently organic. In particular, the sequencing in the introduction of models appears not to anticipate what is forthcoming in the curriculum when building on work that is current. For instance, what we see is an absence of models that can support the efficient counting strategies that have been built up in the foundation phase when learners graduate to Grade 4. Rather, we see a premature annexing of the standard algorithms for operating on whole numbers, a move that has been said to sideline the sense that learners should make of the concept of number, especially as it relates to flexibility in computation (Graven, Venkat, Westaway and Tshesane, 2013). I venture to argue, therefore, that at the level of instruction, despite the shift in philosophies and theories of learning and teaching outlined above, algorithmic and formula-based methods still dominate ways of working with problems in the mathematics class. Patterns of performance across the Intermediate Phase in South Africa, as seen in the declining mean ANA scores across Grades 4-6 (ref), suggest that these algorithmic approaches are not producing broadly successful performances.

2.4 Describing models and strategies

Modeling

In the previous section I have delineated the shift in paradigms as it pertains to what mathematicians perceive mathematics to be, as well as what implications this shift has for mathematics education. It is important, therefore, to clarify the differing takes on modelling before proceeding into a description of different models and strategies.

Identified according to the purpose behind their employment, research points out that there are two forms of modelling, namely: *mathematical modelling* and *emergent modelling*. The former is typically employed for purposes of "translating" the problem situation into mathematical expressions that can serve as a model whilst in the latter the model emerges in the process of structuring the problem situation as part of an "organizing" activity (Gravemeijer, 2002). In the case of mathematical modelling, the model and the situation are distinct, and in the emergent modelling case 'the model and the situation modelled co-evolve and are mutually constituted in the course of the modelling activity' (p.2). By virtue of the fact that the model and the situation are seen as distinct in mathematical modelling, it was possible to make use of this type of modelling in the mechanistic era, where problems did not have to be presented in context. Typically, 'if context problems are used in the mechanistic approach, they are mostly used to conclude the learning process [in that they] function only as a field of application' (Van den Heuvel-Panhuizen, 1998, p.3).

Dynamic models

According to the RME approach to learning and teaching, models are interpreted broadly as vehicles to elicit and support the progression from an informal understanding connected to the 'real' or imagined reality to the understanding of formal systems (Van den Heuvel-Panhuizen, 2003). In this way a model can assume the form of materials, visual sketches, paradigmatic situations (like repeated subtraction), schemes, diagrams and even symbols. To be deemed emergent, it is required, therefore, that a model should support such a progression from thinking about acting in the modelled situation to thinking about mathematical relations. Consequently, in order to provide the intended support to learning processes, models must satisfy the following preconditions:

> On the one hand they have to be rooted in realistic, imaginable contexts, and on the other hand they have to be sufficiently flexible to be applied also on a more advanced, or general level (Van den Heuvel-Panhuizen, 2003, p.13)

In this way, the way in which the emergent models used can be connected to (or used to infer) the process of mathematical growth (Gravemeijer, 1997) where the raising of the learners' thinking is regarded as the overarching goal of learning (Van Den Heuvel-Panhuizen, 2001).

Connecting mental strategies to levels of activity

Perhaps due to the reluctance to have RME 'considered a fixed and finished theory of mathematics education' (Van den Heuvel-Panhuizen, 2001, p.2), the notion of a 'strategy' is one that up to now is not well-defined in the literature. Following empirical observations by Carpenter et al. (1999), however, researchers are in agreement that the structure of a problem influences the solution process that unfolds when children attack that particular problem.

Flowing from this observation, Beishuizen (1997) defined strategy as 'the choice out of options related to problem structure', a working definition that incorporates the idea that 'given a collection of numbers to work with, children will select the strategy that is the most appropriate for the specific numbers involved' (Thompson, 1999, p.2). Based on a hierarchy established by Carpenter and colleagues following empirical observations of learners' workings, I hereby propose a framework that can track progress in the level of activity of a learner in the context of addition and subtraction problems. This framing is made possible by what I observe to be a direct relationship between what have been described as the levels of thinking promoted in RME and the progression from the use of *Direct Modeling Strategies* to the use of *Counting Strategies* to the use of *Derived Number Facts* in solving addition and subtraction problems.

For purposes of my study then, I perceive of a broad connection between the levels of activity advanced by RME and what I see as being the stages of development in the use of strategies. So instead of working with the levels of thinking proposed by RME, I choose to work with the more pragmatic stages of development in the use of strategies as a way of tracking the level of activity in the work of the learners.

Strategy	*Stage of development associated with the strategy*
Direct modeling strategy	Stage 1: focus is on directly modeling the action in the (word) problem
Counting strategy	Stage 2: focus is on the number-word sequence in the extracted forward or backward number count.
Derived Number Facts (calculating strategies)	Stage 3: focus is on the mathematical relations between the numbers involved

Table 1: **Connecting mental strategies to levels of activity**

Contexts

As already mentioned, in the realistic approach the use of realistic contexts is an important determining characteristic as they serve the role, not only of application, but they also function as a source for the learning process (Van Den Heuvel-Panhuizen, 2003). The reasoning behind this is that when learners learn mathematics severed from the real-world, learners make little sense of it because the instruction disintegrates into teaching ready-made axiomatics (Freudenthal, 1973), making it difficult for them to apply it in the real world. By beginning in context and having the situation co-evolve with the model, learners are afforded the opportunity to participate in 'the activity of organizing matter from reality or mathematical matter' (Van Den Heuvel-Panhuizen, 2003, p.11) called *mathematization* by the founding fathers of RME. According to Nelissen and Tomic (1993), mathematization is a constructive, interactive and reflective activity, arguing that the point of departure of education is not learning rules and formulas, but rather working with contexts. In other words, problems in context are the basis for mathematization. They define a context as

> a situation which appeals to children and which they can recognize in theory. This situation might be either fictional or real, and forces children to call upon the knowledge they have gained by experience – for example in the form of their own informal working methods – thereby making learning a meaningful activity for them (Nelissen and Tomic, 1993, p.23)

They further posit that working with contexts can set the premise for subsequent abstraction and for conceptualization because thinking must achieve a higher, abstract level that transcends specific contexts. There appears to be consensus amongst researchers on this point, albeit viewed from different vantage points:

> Although both teaching and creating mathematics takes place in social contexts (in different ways), the most essential characteristic of what mathematicians are creating (and what we are demanding our students learn) is its universality and independence of context (Sfard et al., 1993, p43).

It has been found that a well-chosen context can set in motion an active thought process in children, a process by which thinking becomes increasingly formal, known in RME as the process of *progressive mathematization* (Nelissen and Tomic, 1993). The extent to which children's thinking becomes formal is a function of the context chosen and the extent to which the chosen context can be re-contextualized so that it becomes formal in nature. In this regard, because the modeling is emergent – that is, the context (or situation) co-evolves with the model – the use of models is indispensable to a realistic approach to learning.

It makes sense, therefore, that RME theory should characterize itself along three heuristics; namely: *guided reinvention through progressive mathematization, emergent models* and *didactical phenomenology*. To start with, the very choice of the word *heuristic* speaks volumes. The World English (Online) Dictionary has this to say about the adjective *heuristic:*

- ✓ helping to learn; guiding in discovery or investigation
- ✓ (*of a method of teaching*) allowing pupils to learn things for themselves
- ✓ (*maths, science, philosophy)* using or obtained by exploration of possibilities rather than by following set rules
- ✓ (*computing)* denoting a rule of thumb for solving a problem without the exhaustive application of an algorithm: *a heuristic solution*

The idea behind this 'conglomerate of a domain-specific instruction theory' (Gravemeijer and Stephan, 2002, p.148) known as RME seems to be captured in the word heuristic. This is not a coincidence but the result of a concerted effort at staying with the metaphor of an evolving organism: learners do not download pre-structured mathematical knowledge but instead reinvent the mathematics by progressively mathematizing (reality and mathematics) with the use of contextually evolving models and under the guidance of the teacher.

Guided Reinvention through progressive mathematization

In an attempt to make learners' experience of mathematics as organic as possible, Freudenthal insisted that the understanding of formal mathematics that learners should eventually have must stay 'rooted in' their understanding of experientially real everyday-life phenomena (Gravemeijer and Doorman, 1999). Consequently, he worked tirelessly against what he called an '"anti-didactical inversion" where the end results of the work of mathematicians is taken as the starting point for mathematics education' (Gravemeijer and Doorman, 1999, p.116). By upholding learners' own constructions as central to the human activity of mathematics, he advocated for an approach that will see learners 'come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible' (p.116), as opposed to seeing themselves as receivers of ready-made mathematics. The notion of guided reinvention, therefore, advances the belief that the character of the process of learning mathematics should be that of cognitive growth, and not of a stacking of pieces of atomised knowledge.

The activity of mathematization is said to be the main process by which learners reinvent the mathematics for themselves (with the guidance of the teacher); it is in the process of progressive mathematization that learners construct mathematics (Gravemeijer and Doorman, 1999). The adjective 'progressive' is meant to connect with the graduated process by which formal mathematics emerges from the mathematical activity of the learners. In relation to this, Treffers (1987) identifies two ways of mathematizing; namely: horizontal – which speaks to organizing matter from reality – and vertical mathematization – which is a process of organizing mathematical matter (Gravemeijer and Terwel, 2000). In more specific terms,

> Horizontal mathematisation is when learners use their informal strategies to describe and solve a contextual problem and vertical mathematisation occurs when the learners' informal strategies lead them to solve the problem using mathematical language or to find a suitable algorithm (Barnes, 2004, p.54)

With the essence of the learning process being to raise the level of activity of the learner, this progression in mathematization from the horizontal to the vertical can be understood as a shift from an empirical to a structural engagement with the problem. Since the mechanistic school was concerned with having learners appropriate the end result of the work of mathematicians, it can be argued that it was fundamentally (if not wholly) structural. Further evidence of this orientation can be seen in the lack of contextual problems, which meant that learners' activity was confined to vertical mathematization. The danger with this approach, as Barnes (2004) correctly pointed out, is that when learners enter the process at the point of vertical mathematization – that is, without having first gone through a process of horizontal mathematization – in the event that they have forgotten the relevant algorithm, they run the risk of having no recourse to solving the problem. For learners with such an experience, mathematics will forever be ‘a set of indisputable rules and knowledge [with] a fixed structure [which] can be acquired by frequent repetition and memorization’ (Nelissen and Tomic, 1993, p.19). The stark reality is that this is the experience of the majority of learners in South Africa by virtue of the fact that the mechanistic approach in which they have been immersed for several decades did not afford them opportunities to engage in horizontal mathematization (Barnes, 2004). Similarly, the only kind of models that they have been exposed to were fixed, static and pre-structured, leaving little or no room for children’s informal strategies (Klein, Beishuizen and Treffers, 1998).

Emergent models

In the same manner that progressive mathematization facilitates the shift from an empirical to a structural engagement with the contextual problem, emergent models mediate a shift from informal and situated solution procedures to more formal mathematical reasoning (Gravemeijer and Stephan, 2002). In general, the expected shift involves a transition from perceiving numbers as tied to identifiable objects (like ‘10 sweets’) to seeing them as entities in their own right (‘10’). In the process of transforming from a *model-of* to a *model-for* more abstract reasoning on numbers as mathematical entities, a ‘new mathematical reality’ is said to emerge. In this way, the number range in the Foundation Phase can be thought of as ‘constituted by numbers up to 100 as entities in a framework of number relations’ (Gravemeijer and Stephan, 2002, p.158). This resonates with what has been referred to by Sfard (1991) as the ongoing process of reification wherein mathematical processes are continuously re-viewed as mathematical objects. Unsurprisingly, the realistic approach is in sync with the discursive approach for learners’ experience of mathematics to be as close to the historical development of mathematics as possible, so as to afford them as organic an experience as possible.

The relation between emergent modeling and mathematizing

There is no escaping the interface between the two heuristics of guided reinvention through progressive mathematization and that of emergent models. In fact, I see a one-to-one mapping between engaging with a model-of and horizontal mathematization on the one hand, and engaging with a model-for and vertical mathematization on the other. How these mappings connect with learners’ use of models and strategies is motivated by the statement by van den Heuvel-Panhuizen (2003) to the effect that, whilst horizontal mathematization means going from the world of life to the world of symbols, vertical mathematization means moving within the world of symbols:

> The latter implies, for instance, making shortcuts and discovering connections between concepts and *strategies* and making use of these findings . . . In other words, even on the level of counting activities, for example, both forms may occur (p.12)

Consequently, my framing is that the setting up of the model amounts to horizontal mathematization and the use of what is perceived by the learner as the most appropriate strategy is mathematizing vertically. This is summarized in the table below:

Mathematization	Horizontal	Vertical
Implication in the context of solving addition and subtraction problems	Setting up of the model	Use of strategy
Stage in emergence of model	Model-of	Model-for
Nature of engagement	Empirical	Structural

Table 2: Connecting model and strategy to progressive mathematization

This framing has allowed me to analyse the kinds of errors that will arise within learners' use of the empty number line as a model, as well as the strategies accompanying the model used. The approach I have adopted is that problems with the setting up the empty number line point to insufficient exposure to the empirical dimension of moving from the world of life to the world of symbols. On the other hand, where evidence points to problems at the level of strategies this tells us that some of the underlying fluencies required for working with the number line model – for instance, counting forwards or backwards in tens – are not in place.

It is worth mentioning here that some researchers in RME acknowledge that because of the tension between a bottom-up approach that promotes reinvention and the need to 'reach certain given educational goals' and to 'plan instructional activities in advance', often a top-down element is inevitable in instruction (Gravemeijer and Doorman, 1999, p.124). Where this intervention was concerned, although the ideal was to make use of the two heuristics of guided reinvention through progressive mathematization and emergent models, the reality, was that elements of top-down instruction were inevitable.

Chapter 3

Literature review of models and strategies for addition and subtraction culminating in an analytical framework.

In this chapter I review the literature that is relevant to my focus on models and strategies. In particular, I elucidate the entanglement between models and strategies with the aim of highlighting the intrinsic interconnection between them, particularly in the initial stages of learners' encounters with mathematics.

I begin by detailing the lack of progression observed in South African schools and how it can be seen in the lack of shift towards more sophisticated strategies in the move from counting to calculating. With this lack of progression identified, I then discuss the different strategies and the hierarchy that they have been seen to fall into within the research literature. This hierarchy is meant to serve as the foundation stone for an analytical framework designed to grade the level of activity of each learner. The ultimate goal is to arrive at a framing that will provide me with the analytical lens that will enable me to track the nuances within the qualitatively different ways of working on the part of the learners.

Once the progression in the use of strategies as guided by the levels of sophistication of the strategies is established, I then proceed to engage with the models that have been used since the 1960s, which mark the formative years of RME. The motivation behind discussing strategies before models is that early number learning literature has tended to focus more on strategies than models as the key progression trajectory. As a result I begin here with a discussion of the strategies and then trace back to the models that these strategies are associated with in the literature. Each model will then be associated with the strategies it supports in a framing that ensures that their sophistication follows the hierarchy of the strategies already established.

Once I have introduced each of the models and dealt with the limitations of the procedural types I then introduce the empty number line as a relational model, along with the strategies that have been identified as connected to the use of the empty number line in the literature on RME that can be used as an alternative to the traditional standard algorithms that I found to hold sway in learners' ways of working before the intervention. Part of the discussion here is to justify the prizing of the empty number line above all other models by explaining the ways in which a relational model like the empty number line can be useful in overcoming such limitations.

As suggested in the title for this chapter, I conclude this chapter with a table that summarizes the analytical framework that I have used for this study.

The lack of progression

Based on the work by Gelman and Gallistel (1986), Ensor and colleagues delineate a trajectory wherein the role of concrete counting diminishes with time. They identify a progression that begins with counting, is followed by calculation-by-counting, and culminates in calculation-without-counting where children can operate on numbers as mental objects. By counting they refer to (and include) processes such as oral counting which entails counting forwards and backwards in 1s, 2s, 3s, 5s etc. Such processes, they argue, encourage learners to memorise number sequences, as a result, the most that such processes can facilitate is the recalling of facts over the figuring out of new facts.

In his review of research on literacy and mathematics achievement in primary schools in South Africa, Fleisch (2008) makes the startling revelation that at the end of 2001 South African children performed poorly – an average score of 30 per cent – in elementary mathematics. Moreover, when placed against other nations, South African children ranked lower in numeracy achievement than many of their counterparts in SADEC countries, including Botswana, Malawi, Madagascar and Zambia. Ten years later, only a third of the learners in grade 3 attained the minimum standard required of learners at their level in the Annual National Assessments (ANA) of 2011. The inadequacies identified by the ANA report of 2011 related to ***foundational competencies and basic concepts.*** In particular, where numeracy is concerned, learners were found to be unable to ***do simple calculations.*** Ensor et al.'s (2009) key finding from disadvantaged South African schools is that an overemphasis on counting as a means to calculating is part of the reason that students often "remain highly dependent on concrete strategies for solving problems at grade 3 level" (p.8).

This is in line with the observation that whilst counting strategies can be used efficiently to produce correct answers in lower number ranges, they become cumbersome when larger numbers are introduced (Askew et al., 2001; Carpenter et al., 1999; Wright et al., 2006). It is for this reason that researchers argue for a distinction to be made between *calculation by counting* and *calculation by structuring* (Sari, de Haan and Zulkardi, 2009). As the range of numbers increases, learners' attention needs to be gradually shifted from a focus on counting (*calculation by counting*) towards a focus on the structural properties of number (*calculation by structuring).*

Mental strategies and their levels of development

The notion of a 'strategy' is one that is currently not well-defined in the literature. It has been found by Carpenter et al. (1999) that the structure of a problem influences the solution process that unfolds when children attack that particular problem:

> There are important distinctions between different types of addition problems and between different types of subtraction problems, which are reflected in the way that children think about and solve them (p.2)

This observation is probably the reason that has Beishuizen (1997) defining strategy as 'the choice out of options related to problem structure'. Incorporated herein is the idea that 'given a collection of numbers to work with, children will select the strategy that is the most appropriate for the specific numbers involved' (Thompson, 1999, p.2). Based on a hierarchy established by Carpenter and colleagues following empirical observation of learners' workings, the learner will employ a Direct Modeling Strategy, a Counting Strategy or Derived Number Facts to solve the problem. To illustrate the options, I have tabulated below what Carpenter et al. (1999) found to be three different solutions to a problem which they presented to Tanya, Jose and Zena (pseudonyms for the low, middle and high-attainers respectively) in their study:

Problem	
Eliz has 3 dollars to buy cookies. How many more dollars does she need to earn to have 8 dollars?	**Tanya's Direct Modeling strategy** Started with a set of three counters and added more until there was a total of eight counters. She then counted the five that she had added to the initial set to find the answer.
	Jose's Counting strategy Started counting at three and counted "3 [pause], 4, 5, 6, 7, 8", extending a finger with each count. Counted the five extended fingers when he reached eight.
	Zena's possible Derived Fact strategy Learner knows that 3 and 5 make 8. Can see that the missing part is 5 and is able to say so without much delay.

Table 3: The influence of problem-structure on the solution process

The way that Tanya worked with the problem above informed a decision to classify this way of working under *Direct Modeling Strategies* as she is seen to be directly modeling the action or relationship in the (word) problem. Jose, on the other hand, realizes that he does not have to construct and count the initial set of three objects; that he could simply read the answer from the number of fingers extended at the end of the counting sequence. The distinction here is not in the fact that Tanya used counters and Jose used fingers but in the fact that Jose 'could represent the extra dollars needed by the numbers in the counting sequence from four to eight' (Carpenter et al., 1999, p.3). Jose's solution process is more abstract in that all he had to do was to figure out the number of number-words uttered in that sequence, which number he tracked using his fingers.

What makes the use of *derived facts* strategy even more abstract is its focus, not on the counting sequence, but on the relations between the numbers involved. In the possible solution process for the derived facts strategy given above, knowledge of the combinations for or partitions of eight was used to derive the solution. In a nutshell then:

> Most children pass through three levels in acquiring addition and subtraction problem-solving skills. Initially, they solve problems exclusively by Direct Modeling. Over time, Direct Modeling strategies are replaced by the use of Counting strategies, and finally most children come to rely on number facts. . . There is [however] a great variability in the ages at which children use different strategies (Carpenter et al., 1999, p.26)

Brown, Askew & Millet (2003) agree, and they further posit that when learners continuously recycle derived facts into known number facts, they are constantly increasing the range of strategies at their disposal for deriving even more new facts. What Carpenter et al., (1999) have called the *Derived fact* strategy, Askew, Bibby & Brown, (2001) have called the *known facts* and *derived number facts* strategy, whilst Thompson (1999) binds these together into *calculating strategies (using or deriving facts).*

Thompson (1999) finds it necessary to distinguish *counting strategies* from *calculating strategies* for the reason that such a distinction can prove useful in elucidating what is meant by the adjective 'mental' in such phrases as *mental arithmetic* and *mental calculation*, phrases that are increasingly being used in curriculum documents and, consequently, by researchers in their developmental work with teachers. He asserts that the use of the word 'mental' is problematic from the view point that, although it is self-explanatory in the phrase ‘mental recall’ it is rather opaque where the phrase ‘mental strategy’ is concerned. This is no trivial point especially when one takes into account that:

> there is no word for ‘mental’ in The Netherlands and that this leads to their using terms which translate into ‘working *in* your head’ (recalling facts) and ‘working *with* your head’ (figuring out) (Thompson, 1999, p.2)

In elaborating on the processes of *recalling facts* and *figuring out* as the key components of mental calculation – as contrasted from mental arithmetic which is not concerned with the latter of the two aspects – Thompson (1999) rephrases that

> Mental strategies are more about the application of known or quickly calculated number facts in combination with specific properties of the number system to find the solution of a calculation whose answer is not known (p.2).

Part of my argument is that for a move from the use of counting strategies to the more efficient calculating strategies to be effective it needs to be supported by models with a relational character that can bring out the structural properties of number. As Thompson (1999) observed, if we are going to break away from treating single-digit and two-digit addition and subtraction as if they were the same by having learners invariably working in columns and manipulating digits, we need to recognise that mental calculation necessitates a different approach where each area of activity is concerned. More explicitly, working on numbers from 20 to 100 requires an extra set of mental tools (in calculating strategies) over and above the counting strategies that prove sufficient when working on numbers to 20. Tabulated below is a breakdown of mental strategies into the two categories of counting and calculating strategies, listed according to their levels of sophistication as proposed by Thompson (1999):

Counting strategies		***Calculating strategies (using or deriving facts)***	
Name	*Description*	*Name*	*Examples as provided by Thompson (1999)*
Counting all	Entails allocating a number word to each object in turn and beginning the count at one for each of the parts and whole.	*Doubles fact*	(18 – 9) ‘Nine . . . ‘cos I know that nine and nine is 18’
		Near-doubles (addition)	(8+5) ’13 . . . because 8 and 8 is 16 . . . take away 3’
Counting on from the first number	The first number is kept in mind and counting begins where the first number left off	*Near doubles (subtraction)*	(9 – 5) ‘Four . . . because 10 take away 5 is 5 . . . and 9 is one down from 10’

Counting on from the larger	A more cognitively economical counting on strategy for addition where the numbers are compared and the commutative property of addition is invoked so that the smaller number is counted on.	*Subtraction as the inverse of addition*	(7 – 3) 'Four . . . I knew 4 and 3 is 7 . . . and I just took away 3'
		Using fives	(6+7) '13 . . . I took 5 out of the 6 and out of the 7 and I was left with 3'
Counting back from	The number words are recited backwards while keeping track of the number of number words uttered which should be the same as the number being taken away (the subtrahend)	*Bridging through ten (addition)*	(8+6) 'If 8 is 2 less than 10 . . . add two off the 6 . . . then . . . all the leftovers from before . . . so you just put them to 14'
		Bridging through ten (subtraction)	(12 – 4) 'Eight . . . I knew that if you take away 2 . . . that's 10 . . . and you've got another two left . . . take away that and it's 8'
Counting back to	The number words are recited backwards to the subtrahend while keeping track of the difference which should be the same as the number of fingers raised.	*Compensation*	(9+5) 'Fourteen . . . ten and five is 15 . . . and so 9 and 5 would be 14'
Counting up from (complementary addition)	A more cognitively economical counting on strategy for subtraction where the numbers are compared and the inverse relationship between subtraction and addition is invoked so that the difference is counted on.	*Balancing*	'For example, to solve 7+9, Cathy thought of it as 6+10'

Table 4: Hierarchy of counting and calculating strategies

Carpenter et al. (1999) found that in order to understand how children think about addition and subtraction, it is important to consider differences among problems. As a result they devised a classification of problems that frames an understanding of the evolution of the strategies children use for solving such problems. It is upon Carpenter et al.'s (1999) classification that Askew (2004) derived his framework for the Big Book project which formed the basis for the intervention in this study. According to this framing there is 'a relationship between strategies and problem types and the levels at which strategies may be used' (Carpenter et al., 1999, p.30). This they say is a useful device as it 'provides a structure for selecting [addition and subtraction] problems for instruction [and assessment] and [for] interpreting how children solve them' (p.7). This is precisely the purpose for which it was used in this study.

Askew (2004) compresses Carpenter et al.'s (1999) four classes of *Join, Separate, Part-Part-Whole* and *Compare* into the three classes of *Change, Combine and Separate,* as well as *Compare.* What this means is that Carpenter et al.'s (1999) *Part-Part-Whole* is Askew's (2004) *Combine* and *Separate.* Likewise, Carpenter et al.'s (1999) classes of *Join* and *Separate* are collapsed into the *Change* (increase and decrease) class of problems in Askew's (2004) framing.

The table below explicates upon this classification of word problems that places addition and subtraction problems into three categories: *change*; *combining* and *separating*; and *comparison*, as summarized from Askew (2004, years 3 and 4):

Root situation	***Change***		***Combining*** and ***Separating***		***Comparison***
Specification	*Increase*	*Decrease*	*Combining*	*Separating*	*Comparing*
A description of the situation.	Situations where there is an initial quantity and this is increased or decreased in some way.		Situation where two sets are put together to create a new set that did not previously exist	The reverse of combining, where a single set is split into two.	Situations where nothing actually changes or is combined but where two sets are compared.
Example	*I have 10c in my purse and put another 5c. How much is in my purse now?*	*I have 10 jelly beans and eat 6. How many jelly beans do I have left?*	*Gran gave me 10c and granddad gave me 5c. How much do I have altogether?*	*I have 10 books in a pile and put 6 on the shelf. How many books are left in the pile?*	*I have 10c, Penny has 5c. How much more do I have?* *I have 5c, Mark has 10c. How much more do I need to have the same as Mark?*
Calculation implication	Change 10 by ***adding*** 5.	Change 10 by ***taking away*** 6.	***Combine*** 10 and 5.	***Separate*** the 10 into 6 and . . .	What is the ***difference*** between 10 and 5?

Table 5: Classification of the different addition and subtraction problems

Although the study reported in this paper was not designed to test them, it is worth stating that each of the three classes of problems above can be further broken down into three types: *start unknown*, where the initial value is what is sought ([] + 3 = 8); *change unknown*, where the addend or subtrahend is what is sought (5 + [] = 8), and; *result unknown*, where the whole is what not known (5 + 3 = []). Within this study knowledge of the different problem types was used only in as far as the setting of the tests was concerned, as the focus was centrally on models and strategies.

Models

Klein, Beishuizen and Treffers (1998) delineate a trajectory of models used in the Dutch system of education: from the promotion of manipulative models in the 1960s to the proposal for the didactical use of the empty number line by Treffers and De Moor (1990). The general trend has been from the use of rigid manipulatives to the use of graphical representations that could display the flexible thought processes that are implicated when learners are calculating.

Manipulatives

Klein et al., (1998) provide Cramer and Wynberg's (2009) definition of manipulatives as concrete models which, in the process of being used, may create mental representations for these ideas that support [learners'] later work at the symbolic level. In the 1960s in the Dutch system these took the form of multibase arithmetic blocks and Unifix material, which were replaced with the hundred square in the 1980s, with subsequent critiques of these:

> Approaching computation through these materials, however, was criticized because the materials provided a strong conceptual but weak procedural representation of operations on numbers . . . (Klein, Beishuizen and Treffers, 1998, p.444)

Although the hundred square made possible the visualization of the operations of addition and subtraction, its pre-structured character militated against the increasing influence of RME which insisted on the starting point being children's informal strategies.

Part-part-whole

This model is often advanced as a way of solving problems whose language does not suggest an action but rather a static relationship between the component parts of a whole as in Carpenter et al.'s (1999) combine and compare situations. Consequently, like the structured number line that was trialed in the Dutch system in the 1970s, its use by learners has been found to be localized rather than global. A rectangular bar is used to represent the whole and parts pictorially, where the combined length of the parts is equal to the length of the bar representing the whole. Although it can prove handy in making learners visualize the part-whole relationship, it, however, does not build on the counting strategies that have seen learners graduate into the intermediate phase, and hence, its adoption is not organic.

Column (or vertical) models

The column model was institutionalized to support the traditional *column addition* algorithm which is 'designed to mirror the characteristics of the conventional base ten position system' and hence can only be mastered after a learner has acquired 'expert knowledge of the decimal system' (Gravemeijer & Stephan, 2002, p.149). And so the RME recommendation on the use of the column algorithm falls within a recommended sequence of prior skills that include an appreciation of place value. The reality on the ground as confirmed by research is, however, that children do not learn place-value concepts or multi-digit addition and subtraction adequately, mainly because these are presented as algorithms (Dominick & Kamii, 1989; Fuson, 1992):

> Since algorithms are explicit "recipes" for carrying out certain procedures, students are prone to make mistakes in executing the algorithms as they strive to remember details of their implementation, especially in the absence of context (NCTM, 2011).

Kamii & Dominick (1998) provided two reasons for saying that algorithms are 'harmful': they encourage children to give up their own thinking, and they *unteach* place value, thereby preventing children from developing number sense. Moreover, even those that use them to arrive at the correct answer often show little understanding of the procedures they are using. Where the necessary understandings of the decimal system are not in place, working with column models comes to be reduced to working with single digit numbers, providing fertile ground for errors resulting from a tendency to treat tens and units as being independent from each other. A typical error is to subtract the smaller from the larger digit whenever the unit of the subtrahend is larger than that of the minuend. For example, Kilpatrick et al. (2001) found that the US grade 2 national-norms for this kind of problem:

$$\begin{array}{r} 92 \\ \underline{-\,87} \\ 15 \end{array}$$

- are that only 38% of the answers are correct, whilst (Beishuizen, 1997) found that only 55% of Dutch 3rd-graders (Year 4) had sufficient command of such subtraction-with-carrying problems in the National Evaluation Tests conducted in 1987.

Fosnot & Dolk (2001) concur with Kamii & Dominick (1998) that a blanket application of an algorithm to all problems is counterproductive to calculating with number sense. They contend that 'calculating with number sense means that one must look at the numbers first and then decide on a strategy that is fitting – efficient' (p124). By definition, then, number sense is an 'understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and develop useful strategies for handling numbers and operations' (McIntosh, Reys & Reys, 1992, p. 3). From the aforementioned one can appreciate that, for its emphasis on the execution of computational steps related to the numbers in the problem, the *column addition* algorithm advances a procedure which sidelines number and operational relations. It can be argued, therefore, that, instead of playing a supportive role in elaboration of learners' informal strategies and the development of more sophisticated strategies in the manner that the *empty number-line* can support strategies like compensation[1], *column addition* supports a top-down instructional procedure that can work against such development (Gravemeijer and Doorman, 1999; Van den Heuvel-Panhuizen, 2000).

The empty number line

My conception of a relational model is that, first of all, it is graphic. In particular, it must be vectorial by design. It must not only give us the value of the numeral and its position relative to other numbers, it must also relate the operations of addition and subtraction to opposing directions. The power of such a relational model resides in its ability to provide immediate access to the numbers without a reliance on counting by ones. It is the counter's prerogative as to which unit of count (1, 2, 5,10 and so forth) to use in their solution, thus supporting the compressions of counting that have been described as important within progression in early number learning (Gray and Tall, 1994).

One such model is the *empty number-line* developed and successfully used for close on three decades in the emergent model instructional design of Realistic Mathematics Education (RME) based at the Freudenthal Institute. Its graphic nature makes it an image that represents the logical structure of numbers which relates closely to the counting sequence (Anghileri, 2006). As a result, it is the natural choice for modeling the addition and subtraction of larger

[1] Towards the end of this chapter we will see the importance of 'adding more than is required and then compensating' (Thompson, 1999, p.4) as an underlying fluency for the successful use of the empty number line.

quantities. When learners enter the intermediate phase they are ready to have their attention directed to the structural properties of numbers, including the inverse relationship that exists between addition and subtraction (Wright et al., 2006). It is my view that the empty number line can facilitate this redirection. By establishing opposing directions for the operations of addition and subtraction, the *empty number-line* model represents this inverse relationship graphically and so makes it visually available for appropriation into learners' knowledge structures. The *empty number-line* model, therefore, lends itself to a constructivist approach to learning as it looks to build on learners' informal solution methods:

> The model is not employed to steer the students' thinking [but] instead to be adapted by the students to fit *their* thinking (Gravemeijer & Stephan, 2002, p151)

The literature on RME makes a distinction between two broad categories of procedures for solving addition and subtraction problems; namely: *columnwise* and *non-columnwise* processing of numbers (Beishuizen, 1993). The former type is what has been referred to above as the *column addition/subtraction* method, and the latter type of processing can be further elaborated into the two sub-categories of *sequential* and *decomposition* procedures known as the *N10* and the *1010* strategies respectively. The 1010 procedure is also referred to in the literature as the *split method* because when operating on the numbers both operands are partitioned into tens and units to be processed separately, and then recomposed to arrive at the sum or difference. Tabor (2008, p33) provides the following schematization to illustrate solving 32 + 24 using the splitting strategy:

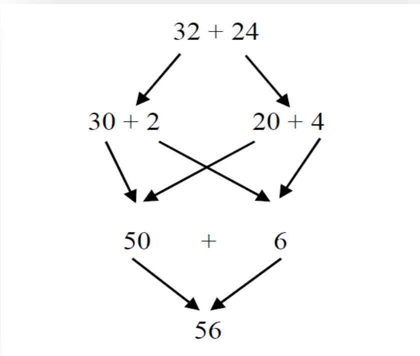


Figure1: the splitting strategy

On the other hand, the N10 procedure is also known as the *jump method* because the tens and units of the second operand are added to or subtracted from the first operand which is kept whole (Beishuizen, 1993; Gravemeijer, 1994; Tabor, 2008). And so for the same question above, a learner could begin with 32, add 20 to it to get 52, and then add 4 to arrive at a solution of 56.

For Dutch children, the N10 strategy has been found to be the more effective of the two, with the 1010 strategy being more prone to errors (Klein, Beishuizen and Treffers, 1998), especially in subtraction problem situations where the units of the subtrahend are smaller than those of the minuend. In my view 1010 is also the less efficient of the two as it is remains a triple-count type of strategy, albeit not necessarily by ones. This stems largely from the fact that the 1010 procedure relies on decomposing the operands into tens and units, resulting in the type of passive “reading off” (Beishuizen, 1993) behavior that is evident in learners’ tackling of the same kind of problems using the column addition/subtraction method. Here are the strategies identified in the literature as put together by Klein, Beishuizen and Treffers, (1998) in the table below:

Mental Computation Procedures for Addition and Subtraction up to 100

Addition (with regrouping): 45 + 39		Subtraction (with regrouping): 65 - 49, 51 - 49	
Sequential procedures:		Sequential procedures:	
N10:	45 + 30 = 75; 75 + 5 = 80; 80 + 4 = 84	N10:	65 – 40 = 25; 25 – 5 = 20; 20 – 4 = 16
N10C:	45 + 40 = 85; 85 – 1 = 84	N10C:	65 – 50 = 15; 15 + 1 = 16
A10:	45 + 5 = 50; 50 + 34 = 84	A10:	65 – 5 = 60; 60 – 40 = 20; 20 – 4 = 16
		A10:	49 + 1 = 50; 50 + 10 = 60; 60 + 5 = 65 answer: 1 + 10 + 5 = 16 (adding-on)
		∩[a]:	51 – 49 = 2 (because 49 + 2 = 51)
Decomposition procedures:		Decomposition procedures:	
1010:	40 + 30 = 70; 5 + 9 = 14; 70 +14 = 84	1010:	60 – 40 = 20; 5 – 9 = 4 (false reversal) 20 + 4 = 24 (false answer)
10s:	40 + 30 = 70; 70 + 5 = 75; 75 + 9 = 84	10s:	60 – 40 = 20; 20 + 5 = 25; 25 – 9 = 16

[a]The Connecting Arc (∩) can be used only for subtraction problems.

Table 6: Mental Computation Procedures for Addition and Subtraction up to 100

For its ability to foster relational thinking and number sense, *non-columnwise* processing of numbers is seen in Dutch primary education as "real" mental arithmetic (Beishuizen, 1993;

Treffers, 1991; Treffers & Goffree, 1985). In particular, for longevity in application and following the observations made by international research regarding the effectiveness of the N10 over the 1010, it was the N10 procedure that was promoted during the intervention because it is a sequential procedure, and so it tends to lend itself to the linear character of the number line. Solving the task by concurrently making jumps on the number line facilitates cognitive involvement with learners' own actions; a forward jump to add and a backward jump to subtract:

> In this way they also keep track of what they are doing, leading to a reduction of the memory load while solving the problem (Klein, Beishuizen and Treffers, 1998, p447)

With facility in the use of strategies receiving priority attention in and across all realistic approaches to the learning and teaching of mathematics, the use of the empty number line is, therefore, motivated by its ability to support these strategies. In view of learners' predominant use of count-all based strategies in early addition problems (Venkat, 2011), there is a need to 'raise the sophistication level of their strategies from counting by ones to counting by tens to counting by multiples of ten' (Klein et al., 1998, p446). It is my conviction that the use of the empty number line can facilitate this shift, as illustrated in the figure below by Beishuizen (1993):

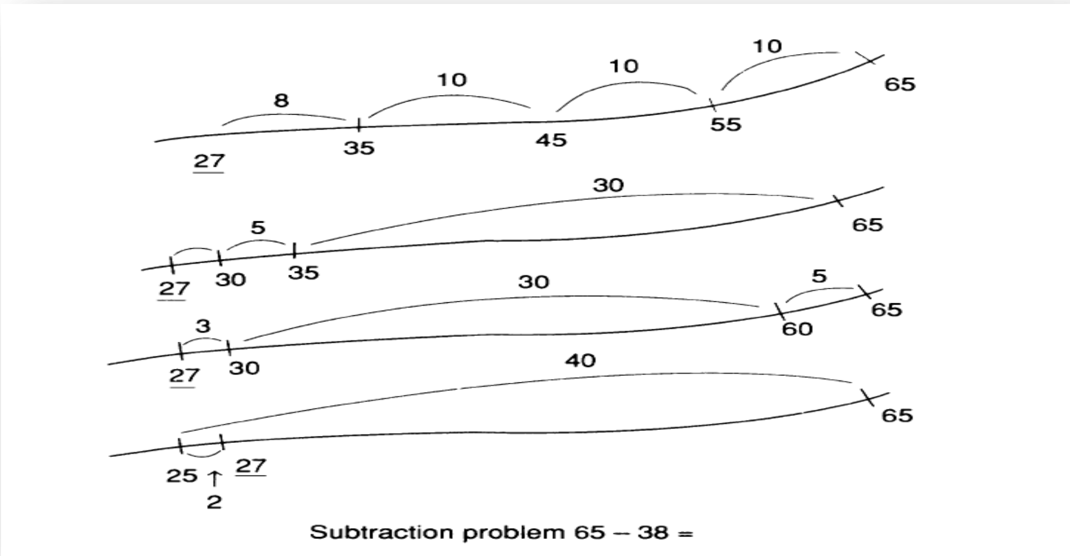


Figure 2: Various N10 strategies demonstrated on the empty number line

It is important to note that the strategy used in the first and second number lines (in figure 2 above) is the N10 strategy for subtraction in its most basic form, and it corresponds to what has been referred to in table 6 as *N10* under the subtraction (with regrouping) column. In the second number line, however, as in the third one, the procedure is a bit more compact in that the three jumps of 10 have been compressed into a single jump of 30. Also, use has been made of the strategy of 'bridging through ten' so that one lands on the 'friendly numbers' that are multiples of ten or so-called 'decuples' (Wright et al., 2006). The only difference between the strategies used in the second as compared to the third number line is that in the third number line the aim of the first jump is to ensure landing on a decuple, and so it corresponds to the *A10* strategy identified in table 6, where *A10* stands for adding through 10 (Tabor, 2008). The fourth number line corresponds to the *N10C* strategy which also harnesses the friendliness of 10 and its multiples. It is even more compact than the three above it as it has reduced the number of jumps to only two.

What is not illustrated in the figure above is what corresponds to what has been referred to in table 6 as the *connecting arc* strategy which, like *N10C*, makes use of the inverse relationship between addition and subtraction. Instead of taking away 38, one could add-onto 38 until one reaches 65, in which case the answer (of 27) will be in the jump as opposed to being in the landing position. Alternatively, one could add-on 30 and then take away 3 to land on 65. This strategy is just as compact as the *N10C* in that one will only need two jumps of 2 and 25 to solve, yet it can be argued that it gets more efficient as the difference between the operands gets smaller, as in when solving 51 – 49 in table 6 above. It is precisely this type of flexibility that the focus on different strategies in the WMC-P project seeks to inculcate on the part of the learners through the teachers on the project. It is the kind of flexibility advocated by Kamii and Dominick (1998) in their call for learners to calculate with number sense by first attending to the numbers in the problem before deciding on a model and strategy that are effective and efficient.

This, then, establishes a hierarchy of strategies for the framework that I have used in this study with respect to solving addition and subtraction problems, with *N10* being the most basic, followed by *N10C* and *A10* which in my view are equally efficient, and are surpassed by the *connecting arc* strategy. For addition problems, the *connecting arc* strategy is not considered as it is not relevant. For the purposes of my study I have considered the 1010 (decomposition) procedure to be relevant only where learners modelled the problem as a horizontal number sentence. Last, but not least, I have regarded the column addition/subtraction algorithm to be the strategy of choice whenever a learner has opted to model the problem vertically.

Following the above summary of strategies and their progression in the context of the empty number line, I offer the table below as an analytical framework that takes the levels of sophistication of the different strategies into account, whilst linking each strategy to the model that literature has described as supporting its employment:

	Model	***Strategy***	***Description of the strategy***	***Underlying fluencies***
Level of sophistication increasing as one moves up the table. Hierarchy within and across models.	Empty Number Line	*Connecting arc*	Adds-on the difference instead of taking away the second operand. For example 51 – 49 will be treated as 49 + 2 = 51 (and the answer is 2)	Inverse relation between addition and subtraction; Compensation; Bridging through ten (up and down); Partitioning single digit numbers; Counting-on AND Counting-back in tens and units
		N10C	Adds-on (or takes away) a value that is bigger than the second operand and then compensates for the amount by which the jump was exceeded	Inverse relation between addition and subtraction; Bridging through ten (up and down); Partitioning single digit numbers; compensation. Counting-on AND Counting-back in tens and units
		A10	Stands for *adding through 10* and is a variation of N10 in which the second operand is decomposed in such a way as to enable the use of the decuples	Bridging through ten (up and down); Partitioning single digit numbers; Counting-on OR Counting-back in tens and units
		N10	'N' represents *the initial number that is kept whole* while '10' represents the series of tens that are then added or subtracted. For example, 38 + 24 might be solved: 38 + 20 = 58, 58 + 4 = 62 as was seen in figure 1 above.	Forwards (OR Backwards) Number-Word Sequencing; Counting-on (OR Counting-back) in tens and ones
	Vertical (Column)	*Column addition (or subtraction) algorithm*	Add (or subtract) the units followed by adding (or subtracting) the tens.	Place value; Regrouping
	Horizontal Number Sentence	*1010 or 10s*	Add (or subtract) the tens separately from the units, and then add the two results.	Place value; Regrouping
	Tallies	*Count-all*	Count each and every tally mark	Forwards (OR Backwards) Number-Word Sequencing; Count by ones
	Pictorial	*Count-all*	Count each and every pictorial representation	Forwards (OR Backwards) Number-Word Sequencing; Count by ones

Table 7: Summary of analytical framework

My justification for placing *connecting arc* and *N10C* at the top of the p is that they require the sophisticated coordination of two processes; faced with a subtraction question, the learner applies its inverse relation by, not only adding-on, but sometimes adding more than is required and then compensating[2]. It is for this reason that the uses of these two strategies are most prized in the analytical framework, as will be seen in the next chapter. This reasoning stems from the understanding that learners need not learn all the strategies but 'because some of them are more important than others, what *is* essential is that [they] become familiar with the key methods' (Thompson, 1999, p.4). The three that he sets apart as needing to be explicitly taught in the foundation phase for later work in the intermediate phase, and hence their inclusion amongst the prerequisites for the *connecting arc* and *N10C* strategies in my theoretical framework, are:

- ✓ Bridging through ten (up and down);
- ✓ Partitioning single digit numbers ; and
- ✓ Compensation (for adding or subtracting nine)

It was with the above framing in mind that I set out to explore learners' use of models and strategies with a keen interest in promoting the *empty number-line* as an alternative to *column addition/subtraction.* By extension, I sought to find out the extent to which learners armed with the *empty number-line* model, are placed at an advantage, if at all, in the face of a range of addition or subtraction problems.

In the next chapter, I detail the method of enquiry used to gather data in search of answers to my research questions.

[2] Hence the C in N10C

CHAPTER 4: RESEARCH METHODOLOGY

4.1. Objectives of the research study.

The main objective of my study was to pilot (in the form of a case study) the introduction and to guide the emergence of a specific relational model – the empty number line – to a class of learners that had not used it previously. This model has been successfully used in the Dutch system for over 30 years, and it is said to hold the promise of applicability beyond the foundation and intermediate phase of the South African school system. Consequently, my hypothesis as I started this study was that learners who have mastered the use of the empty number line as a model for addition and subtraction are better positioned for more seamless transitions between phases given its longevity in application. Fewer difficulties are likely with, for example, addition of integers because the relational nature of the number line will allow them to start anywhere on the number line and stop anywhere, including the negative integers once the number line is extended to include them. The idea, then, was neither for the model to immediately supersede nor replace current ones, but to work as an alternative that has some longevity in terms of later number work.

4.2. Data sources

With a view to answering my research questions, my focus is on one WMC-P Grade 4 class in one of the ten schools in the broader project. This school is a suburban school serving a historically disadvantaged population. The language of learning and teaching is English across all grades. The focal class had 42 learners, with matched initial and post test data for 40 of these learners.

In an exploration of the kinds of models and/or strategies used by learners to solve addition and subtraction problems, the answer to my first research question, namely:

> *What models and strategies did these learners initially use to solve different types of addition and subtraction problems and how did they perform?*

comes from the pre-test. I designed this pre-test with a focus on unearthing the sorts of models and strategies used by learners prior to the intervention. The post and the delayed-post-tests were repeat sittings of the pre-test, with adaptations included based on the intervention pointing to avenues related to learners' use of models that needed following up.

Although at the time not anticipated, the overwhelming majority of the learners defaulted to the vertical model in the post-test. In fact, only one learner willingly used the empty number line after the instruction was given to the effect that learners can use the model if their choice. This had its benefits: firstly, it revealed the extent to which learners were acquainted with the vertical model and its accompanying strategy in the column addition/subtraction algorithm after being exposed to them for ten weeks in their use; secondly, it revealed the extent of the shift toward the use of the more formal horizontal model from tally counting, and the subsequent shift from the use of horizontal models to vertical ones. However, because learners did not willingly gravitate towards the empty number line off their own accord, the post-test could not tell us much about the learners' use of the number line. It was necessary, therefore, to have another sitting where its use would be prescribed. And hence, the learners had to reconvene on the 15th of August 2013 – which was four weeks after the post-test was written – for a delayed post-test that could help provide a sense of retention of learning where the use of the empty number line was concerned.

Given that there were no further lessons between the two post-tests, it made sense to compare performance between the post and the delayed-post-tests for efficiency and accuracy in the use of the column model (which dominated learners' methods in the post test) and that of the empty number line (which was prescribed for the delayed-post-test). In order for the intervention to be deemed successful, learners had to demonstrate facility in setting up the model – which according to my framing corresponds to horizontal mathematization – and in using the relevant strategies – vertical mathematization – to arrive at the correct solution to the problem.

An analysis of the post and the delayed-post-tests at the end of the intervention yielded the opportunity to answer my third research question; namely:

> *What effect, if any, has this intervention had on learners' use of models and strategies for solving different types of addition and subtraction problems?*

The intervention was guided by what literature says regarding the models and strategies that are predominant where learners' current ways of solving addition and subtraction problems are concerned. Data on learners' work with models and strategies during the intervention were drawn from my analysis of learner work on intervention class tasks seen within their exercise books, as well as from field notes written by WMC-P colleagues during intervention lessons. The enabled an answer to the second research question; namely:

In an intervention focused on improving the learners' performance on addition and subtraction problems, what sorts of models and strategies are advanced? What sorts of models and strategies are learners using during the process of the intervention?

4.3. The intervention

Learner responses on the pre and post-tests and on the intervention lesson tasks therefore comprised the key data sources. In between the two tests an intervention program suggested by Askew's (2004) Big Book of Word Problems was rolled out. This was made up of six contact sessions with the Grade 4 class wherein the resources[3] provided by the program were utilised to help move the children from treating 'each problem in isolation from other problems that they have worked on' to treating 'each problem as being one of a generic class of problems' (Askew, 2004, p5). This mirrors the emergent model's transition from being a model *of* informal solution strategies to a model *for* more formal mathematical reasoning, and concurs with a shift in the students' thinking, from thinking about the modeled situation, to a focus on mathematical relations (Gravemeijer, 2004). This attests to the aptness of the choice of RME as the key conceptual framework.

4.3.1: The intervention process, resources and rationales

In line with the RME approach, all the problems that learners had to solve were set in contexts that were sufficiently "real" to South African learners in Grade 4. The tasks were presented as word problems and each lesson had a different theme while preserving the same basic form so that learners could quickly get used to 'what is expected of them in the lesson and they can then concentrate on thinking about the mathematics' (Askew, 2004, p.6). The intervention prescribes four stages to each lesson, namely:

- Solving the Big Book problems
- Linking up the problems
- Follow-up problems
- Wrap-up

Solving the Big Book Problems

The first 15 minutes of the lesson were used to introduce to learners the theme for the lesson and to have learners working individually (or in pairs) to solve each of the three problems

[3] Amongst these resources is the use of the empty line as a model to support the calculation of addition and subtraction problems.

accompanying the introduction. Here is as an example from the first lesson of the Big Book of the kind of contexts (and themes) that formed the basis of the problems encountered by the learners during the intervention:

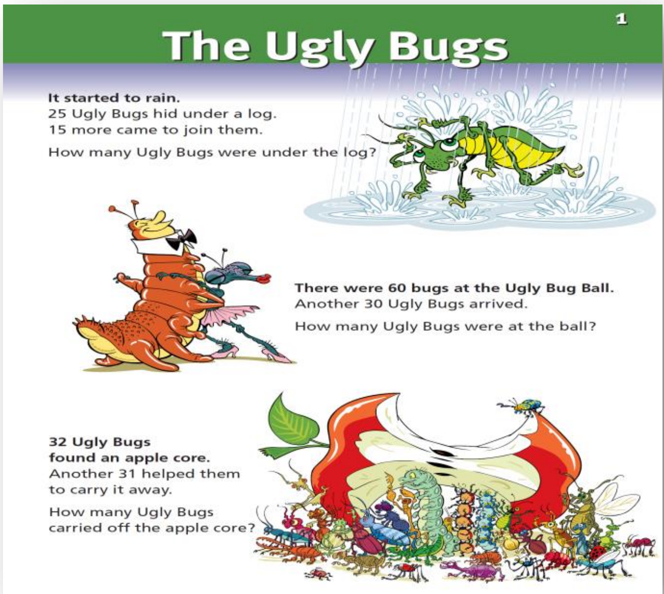


Figure 3: An example of the introductory contextualized problems

Each of the problems would then be discussed with a keen focus on how different learners interpreted each problem and the strategies they used to solve them:

> the emphasis here [was] on direct teaching, not teaching directively. In other words, the role of the teacher [was] to support the children in their struggle to make sense of the problems rather than to explicitly direct them to use particular methods (Askew, 2004, p7)

In supporting the learners' explanations in my role as the teacher, it is advised by the Big Book that part of my role was to

- ✓ Ask clarifying questions
- ✓ Set up models, pictures and/or diagrams; and to
- ✓ Introduce notation that will make learners approaches understood by other learners

Linking-up the problems

For the next 15 minutes the whole class discussion would then be focused on bringing out the character of the three problems that the learners had just solved, as well as to draw learners' attention to the 'common mathematical structure underlying the problems' (Askew, 2004, p7). This was achieved through a classification of the problems provided by the Big Book.

It is worth mentioning here that, although the focus of my study was on models and strategies, I thought it prudent to have this classification availed to learners in order for the discussions (interactions) on the different problems for addition and subtraction to be richer, as well as to make learners aware of the existence of a many-to-one mapping between certain action phrases and their synonyms on the one hand, and the operations of addition and subtraction on the other (Anghileri, 2006). During discussion it sufficed simply to highlight words and phrases that are indexical of the operation to be used in a given problem. In the delayed-post-test, although learners were asked to classify each question, there were no marks allocated for this.

The table below explicates upon this classification of word problems that places addition and subtraction problems into three categories: *change*; *combining* and *separating*; and *comparison*, as summarized from Askew (2004, years 3 and 4):

Root situation	***Change***		***Combining*** and ***Separating***		***Comparison***
Specification	*Increase*	*Decrease*	*Combining*	*Separating*	*Comparing*
A description of the situation.	Situations where there is an initial quantity and this is increased or decreased in some way.		Situation where two sets are put together to create a new set that did not previously exist	The reverse of combining, where a single set is split into two.	Situations where nothing actually changes or is combined but where two sets are compared.
Example	*I have 10c in my purse and put another 5c. How much is in my purse now?*	*I have 10 jelly beans and eat 6. How many jelly beans do I have left?*	*Gran gave me 10c and granddad gave me 5c. How much do I have altogether?*	*I have 10 books in a pile and put 6 on the shelf. How many books are left in the pile?*	*I have 10c, Penny has 5c. How much more do I have?* *I have 5c, Mark has 10c. How much more do I need to have the same as Mark?*
Calculation implication	Change 10 by ***adding*** 5.	Change 10 by ***taking away*** 6.	***Combine*** 10 and 5.	***Separate*** the 10 into 6 and . . .	What is the ***difference*** between 10 and 5?

Table 8: Classification of the different addition and subtraction problems

Follow-up problems

Following the linking-up of the problems in a whole class discussion, the learners would then be afforded an opportunity to work more independently on the ideas just discussed in class. This was achieved through an exercise provided to learners in the form of a worksheet whose problems had the same underlying structure as the three introductory problems that had just been discussed. This individual engagement with the worksheet formed the basis of the sharing session in pairs that constituted the second part of the follow-up phase.

Wrap-up

Eventually, the class is reconvened into a whole class discussion for further exploration of the nature of the problems in the worksheet and consolidation of the key ideas for the remaining 15 minutes.

In Appendix 1, I provide a summary detailing the specifics of the 4 sections for each of the six intervention lessons. This allows the reader to interpret learner outcomes with a sense of the sequence of lesson content.

4.3.2 The test

In order to make the test encompassing of the different kinds of word problems that learners can encounter in Grade 4, the test was compiled following the classification of word problems in table 8 above. There were ten questions, with a distribution of the three problem types across the first eight questions. The last two questions were addition and subtraction number sentences presented as bald number sentences written in symbolic form, for example (16 +12) and (17 – 14) respectively, and the first eight were word problems. Tabulated below are the ten test items. Details on the mark allocation follow:

Problem	*Most sophisticated strategy possible*	*Maximum mark allocation*
1. I have 10 jelly beans and I eat 6. How many jelly beans do I have now?	Recalled fact (bonds of 10)	4
2. Romeo had R12 in his pocket. Juliet gave him R20. How much does Romeo have now?	N10 or 10s	4
3. 5 turtles swam alongside 9 dolphins. How many turtles and dolphins were swimming altogether?	A10 or N10C	5
4. On Thursday Chris walked 17 km, on Friday he walked 10 km, and on Saturday he walked 10 km. How many km did Chris walk in the three days?	N10 or 10s	4
5. 26 polar bears got into a snowball fight. 19 ran away. How many polar bears were left fighting?	A10 or N10C	5
6. Joe has 13 books in his school bag. He puts 6 of them on the shelf. How many books are left in his school bag?	A10 or N10C	5
7. Jonathan ate 11 Easter-eggs during the Easter-holidays. In order to reach his goal of 25 Easter-eggs, how many more Easter eggs does he need to eat?	CA or A10 or N10C	5
8. Carol weighs 36 kg. Her friend Camille weighs 50 kg. How much heavier is Camille than Carol?	CA or A10 or N10C	5
9. 89 +67	A10 or N10C	5
10. 92 – 87	CA or N10C	5

Table 9: The ten problems in the test

On a paradigm continuum that has positivism and constructivism on the extreme poles, the analysis of a research study falls into one of two broad categories: the qualitative or the quantitative. The focus of my research is not in whether the learners provide the correct answer to addition and subtraction tasks, but more importantly, in the quality of the answer according to the types of models and strategies they use to arrive at the answer. As a result, I have also allocated marks for the model and for the strategy used in the solution based on the hierarchy established in the theoretical framework. The motivation for allocating marks to the model and strategy used is that a focus on models and strategies is essentially a focus on the approach adopted by the learner, and not merely a concern with whether the leaner produced the correct answer or not. Using the marking scheme tabulated below was an attempt at applying an initial holistic assessment of the extent of efficiency of the learners' use of models and strategies. That is to say, given that some approaches are more efficient than others, I saw it prudent to allocate marks to models and strategies according to their sophistication as opposed to simply giving a tick or a cross for the correct or incorrect answer respectively:

Model	***Strategy***	***Mark allocation for setting up the model***	***Mark allocation for the use of the appropriate strategy***	***Mark allocation for the answer***
Empty Number Line	*Connecting arc*	2	2	1
	N10C	2	2	1
	A10	2	2	1
	N10	2	1	1
Vertical (Column)	*Column addition (or subtraction) algorithm*	1	1 (or 2 marks if the learner added-on for a subtraction problem, e.g. 11+14=25 as opposed to 25 – 11=14)	1
Horizontal Number Sentence	*1010*	1	1 (or 2 marks if the learner added-on for a subtraction problem, e.g. 11+14=25 as opposed to 25 – 11=14)	1
Tallies	*Count-all*	1	1	1
Pictorial	*Count-all*	1	1	1

Table 10: Distribution of marks for models and strategies

4.3.3 Analytical processes

The choice of the empty number line was not an arbitrary one, as previously discussed; it is the one most commensurate with the philosophy of RME and its advocacy for making the learning experience as organic as possible. More importantly, as mentioned in the previous chapter, solving the task by concurrently making jumps on the number line facilitates learners' cognitive involvement with their own actions. For a learner to be entitled to the 2 marks allocated to the use of the empty number line, they had to set it up completely and correctly. This meant that the jumps and the landing places had to correlate; if any of the numbers was incorrect, the learner did not get the marks. Also, the empty number line is the one most amenable to scaffolding the most sophisticated strategies, identified in the table above as N10, A10 , N10C and connecting arc. Consequently, I have prized the use of these strategies above all others, as can be seen in the mark allocation. No marks were allocated for word sentences.

I have preferred this way of assessing as a deliberate move away from erstwhile mechanistic measurement theories that advocated for assessments to be "objective" whilst preferring "formula-based methods" to solving mathematical problems (Shepard, 2000). If an assessment is to have models and strategies as its focal point, then it makes sense for it to attach appropriate significance to the models and strategies used during the solution process as these are the mental tools used by the learner during the solution process. Part of the motivation for allocating marks to models and strategies according to their usefulness is to add impetus towards legitimizing the use of the most sophisticated (effective and efficient) of strategies. As Graven et al., (2013) have suggested it may be a necessary step towards establishing a focus on number sense across foundation and intermediate phase mathematics.

Beginning with the tests I focused specifically on areas which learners found difficult to answer in the pre-test to see how they handled the same tasks in the post-test. The next move was to look at the solutions in learners' exercise books with an eye on whether or not there was a change in their use of strategies and models during the intervention. I then selected episodes in which I found significant shifts and these will make up part of my report. One indicator of growth was the competent use of the empty number line as a model to solve erstwhile challenging problems. The shifts in the learners' use of models were tracked through the solutions of tasks in their exercise books as well as between their post and delayed-post-test scripts. It made sense to me to compare the post and the delayed-post-test for shifts because both were written after the intervention when the empty number line was already part of their repertoire of models. In general, learners' ability to solve problems across a broader range of problem situations served as a marker of success.

As already mentioned in the first chapter of the report, I first sought to get a sense of the models and strategies that the learners in the chosen Grade 4 class initially used to solve different types of addition and subtraction problems, as well as to assess their fluencies where these models and strategies were concerned. I am now in a position to provide an answer to this research question following an analysis of the scripts of the learners in the pretest. The enactment of the intervention that followed the administering of the pretest was meant as a way of moving learners on from the use of counting strategies (such as count-all as evidenced by the use of tallies) towards the use of calculating strategies (such as bridging through ten and compensation) as encouraged by the use of a specific model in the empty number line and the strategies which it supports in *N10C, A10C* and *the connecting arc;* a model and strategies that are in sync with my theoretical framework.

The intervention was enacted in accordance with the guidelines provided by the Beam Big Book Project which advances an approach that, instead of encouraging strategies like looking for 'keywords' in word problems, would rather have learners meeting any new problem by thinking about what else they have seen that is similar whilst working towards using appropriate language to describe the type of problem at hand (Askew, 2004). For having the use of the empty number line as a way towards scaffolding learners' appreciation of what is similar across apparently disparate situations in contextualized word problems, it became a viable option for a short term in-depth investigation like this is. For purposes of this study, six one hour long sessions constituted the contact time of the intervention, and my findings regarding my second research question will be answered on a lesson to lesson basis over the six lessons that constituted the intervention. Ultimately, I sought to discover the influence that such an intervention can have on learners' use of models and strategies, so at the end of the intervention a posttest and delayed post-test were administered to assess learners' use of models and strategies.

In a nutshell, then, learners' solutions in the pre-test – which they wrote on the 28th of February 2013 – provided answers to the types of models and strategies that were in learners' repertoire for solving addition and subtraction problems prior to the intervention. Once these were identified, the intervention was then designed and implemented in my capacity as researcher and teacher respectively. At the end of the process of the six-lesson intervention – after a period of 10 weeks – the learners had to sit for the post-test (on the 13th of June 2013) where they were encouraged to solve the same addition and subtraction problems that were in the pre-test with the use of the models and strategies with which they were most comfortable.

As mentioned earlier, for a learner to be entitled to the two marks allocated for setting-up the empty number line model, regardless of which (relevant) strategy the learner was supporting with its use, every aspect of the empty number line had to be in place. That means that if a learner chose to make a forward jump of ten, for instance, they had to land on a number that was ten more than the number they started with or else they would not get even a single mark. A further two marks were allocated for the use of either one of the strategies of *N10C, A10C* and *the connecting arc* as the most sophisticated of the strategies in the hierarchy established in my framework for reasons elaborated on earlier. Where a learner used the *N10* or *10s* strategy – that is with the subtrahend split merely into tens and units – only one mark was allocated for the use of that strategy. Similarly, where a learner approached a subtraction problem additively with the use of the column model, they were accorded two marks under strategy as opposed to the one mark allocated for the use of the standard algorithm for column addition/subtraction. Important to note is that the idea was not to penalize the use of the traditional methods but to credit the use of the emergent ones as they were the deemed to have a longer application span.

So the use of the column model and its appropriate strategy could yield a maximum of four marks, whilst the use of empty number line model and one of its most sophisticated strategies could avail a maximum of five marks (when we include the mark for the correct answer in each case). In other words, whilst I do acknowledge empirically, that efficient and effective use of alternative models is acceptable, the differential marking allocation is theoretically driven. After all, this is in line with my hypothesis as I started this study, that learners who have mastered the use of the empty number line as a model for addition and subtraction are better positioned for more seamless transitions between phases given its longevity in application, and hence the prizing of its use over all others.

4.4. Ethical considerations

Under the auspices of the broader project, informed consent was sought and granted by the principals of the schools, the teachers and the parents on behalf of the learners involved in the project. Herein the right to privacy of the learners was assured. Also, the participants were guaranteed protection from harm, and the researchers took the necessary precautions to guard against treating participants as objects. On my part I have approached the principal, teacher, parents and learners in the Grade 4 class I worked with, sharing information on the study and gaining informed consent regarding the intervention identified in my study. As in the broader study, pseudonyms are used within the findings and analysis chapter that follows.

4.5. Reliability, Objectivity and Limitations

Since the intervention was offered to the entire class, the design can be viewed as intervening on a single group. One limitation of this pre and post-test single-group design is that it does not provide a sufficient basis of comparison for us to unambiguously ascribe the shifts to the intervention, and hence no generalisations will hold as this is intended as an exploration on the uptake of the empty number line as a model. This may have the effect of rendering the *internal validity* (Welman, Kruger and Mitchell, 2005) of my conclusions suspect. Ideally, there should be a control group against which we can compare the results of the experiment group. Given that my study was exploratory in its design – aimed to look at the feasibility of promoting the use of the empty number line, the lack of a control group did not centrally impact on my gaining useful insights from the study.

The abovementioned limitations notwithstanding, seeing that a valid test is always reliable, (Welman, Kruger and Mitchell, 2005) it will be sufficient for me that the results meet the validity requirement. From the six (1 hour) lessons that constituted the intervention of this study and the three tests written (pre, post and delayed-post-test), I was able to gather sufficient data to enable me to paint a picture of the models and strategies used by this particular class of Grade 4 learners before, during and after the intervention. The analysis of my findings is framed by the theoretical framework in the chapter before this one. This framed picture is uncovered in the findings and analysis chapter which follows.

Chapter 5: Findings and Analysis

5.1 Introduction and Rationale

The purpose for writing this chapter is to present the findings of my investigation by way of answering the research questions that inform my study into learners' use of models and strategies for solving different types of addition and subtraction problems.

The test (which was the same for all three sittings: that is pre, post and delayed-post-test) was designed following an understanding of the different classes of word problems as outlined in the previous two chapters. There were ten questions in total, with the classes of *change, combine/separate* and *compare* represented by at least two questions each, and presented in no particular order. To supplement these eight context problems, the last two questions were given as the numerical expressions (89+67) and (92 – 87) without any further instructions.

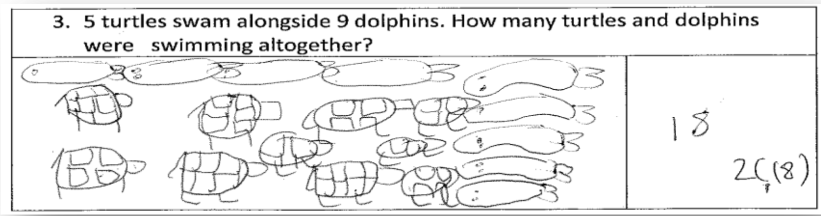
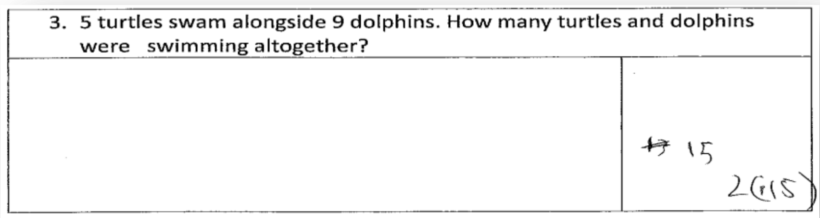
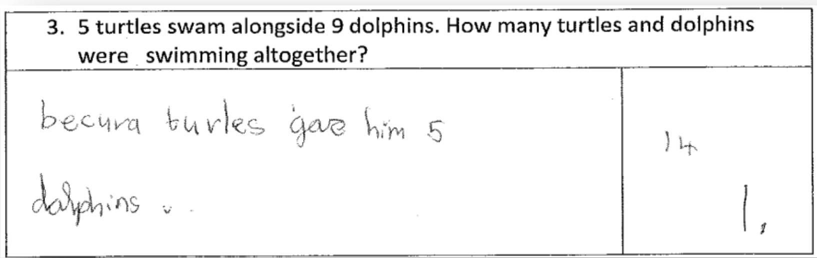
Following this inclusion of word problems and 'bald' number problems, a general trend that I observed was that there was a stark difference in learners' choice of model between the word problems and the numerical problems across learners' workings in all three tests. I therefore partitioned my commentary between the two formats in order to address the overarching trends within each format in each of the three tests. To this end I began by grouping the scripts according to the model that dominated in each learner's workings in the pre-test. This predominance is informed by the model that the learner used for the majority of the questions tackled. I then use this distribution of models in learners' scripts in the pre-test to answer my first research question. Herein I found that the model of choice was used in no less than five questions with other (often different) models used for the remaining questions on the test. By grouping learners on the basis of their predominant model of choice in the pre-test I was able to map out the different growth paths of each of the groups of learners beginning in the model that dominated their working in the pre-test.

I then outline the key findings in the intervention as a way into answering my second research question, with a keen interest in the sorts of models and strategies advanced by this intervention and the way in which learners were using these during the process of intervention.

Ultimately, I group the questions in the test into their respective classes in order to comment broadly on the effect of the intervention on learners' use of models and strategies where the solving of the different types of addition and subtraction problems presented to them in the test was concerned. By looking across between the pre-test and the delayed-post-test I discuss the gains that the intervention had on learners' use of models and strategies. Part of the idea here is to provide a sense of how the findings in this study agree with or deviate from what has been reported in the RME literature. I conclude this chapter with an overall summary of my observations.

5.2.1 What models and strategies did these learners initially use to solve different types of addition and subtraction problems and how did they perform?

In order to keep the presentation compact, I use codes instead of the full names of the models. In the table below I provide an example for each of the models found to be used by learners in the pre-test:

Model	***Example***
Pictorial model (**PCT**)	
No Model (**NM**)	
Word Sentence **(WS)**	

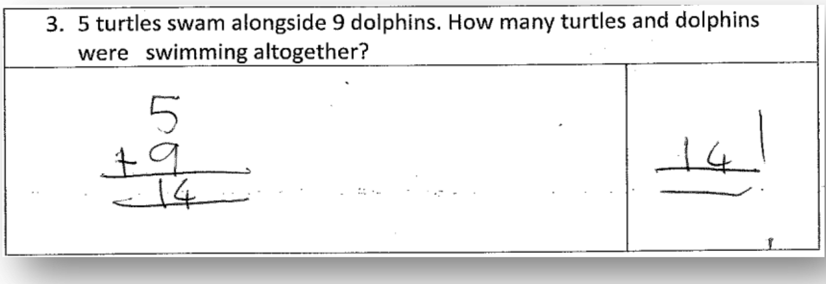
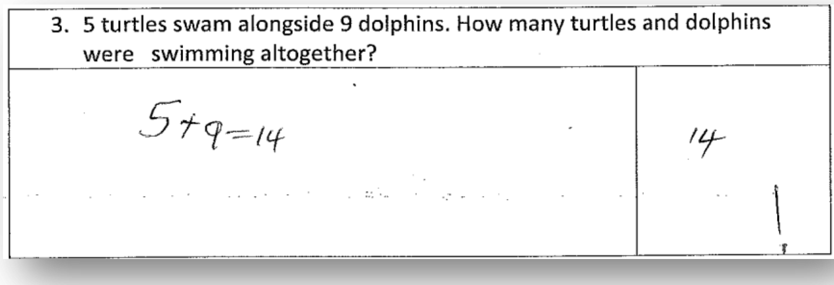
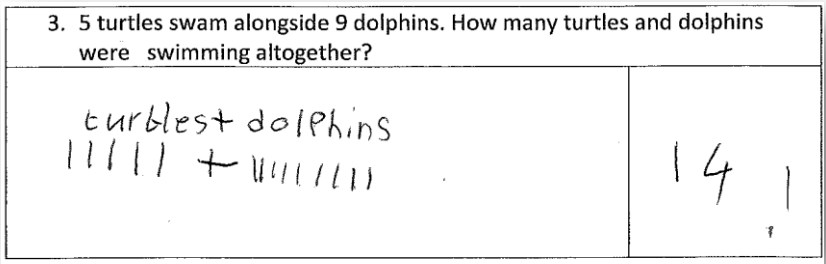
Column Model (**CM**)	 3. 5 turtles swam alongside 9 dolphins. How many turtles and dolphins were swimming altogether? 5 +9 14 14
Horizontal Number Sentence (**HNS**)	 3. 5 turtles swam alongside 9 dolphins. How many turtles and dolphins were swimming altogether? 5+9=14 14
Tallies (**TL**)	 3. 5 turtles swam alongside 9 dolphins. How many turtles and dolphins were swimming altogether? turbles+ dolphins 11111 + 111111111 14

Table 11: The examples of the models used in the pre-test

The table that follows provides an overall distribution of the models used by the learners in this Grade 4 class when tackling the problems in the pre-test. It is worth noting that in some cases learners used two models in one solution. With the assumption that if the first model was sufficient there would be no need for the second, only the second model was counted in such cases. Consequently, only one model was counted for each solution and so the totals agree with the number of learners in the class. Each cell records the number of learners who successfully used the model listed in that row, over the total number of learners who used that model for each one of the ten questions in the pre-test. A cell is blank if no learners used that model for that particular question:

PRE TEST		***WORD PROBLEMS***								***NUMERICAL PROBLEMS***	
Number of learners MATHEMATIZING to arrive at the correct	***MODEL***	***Q1***	***Q2***	***Q3***	***Q4***	***Q5***	***Q6***	***Q7***	***Q8***	***Q9***	***Q10***
	TL	19/19	25/25	19/20	16/18	15/19	17/19	11/19	9/13	2/10	1/5
	HNS	7/9	3/5	5/7	8/9	2/6	4/7	1/5	2/4	2/6	0/7
	CM	1/2	2/3	2/3	5/5	5/8	5/6	6/9	0/8	9/20	6/19
	NM	8/8	3/5	3/6	3/6	2/4	5/6	4/5	0/11	3/3	2/8
	WS	1/2	2/2	2/2	1/2	0/2	1/2	1/2	0/4	0/1	0/1
	PCT			2/2		0/1					
	Totals	**36/40**	**35/40**	**33/40**	**33/40**	**24/40**	**32/40**	**23/40**	**11/40**	**16/40**	**9/40**

Table 12: The distribution of models used in the pre-test

Three features are noteworthy from this table. The first one is that the pictorial model was used in only two questions by at most three learners. The second issue that stands out is that in spite of the inclusion of two-digit numbers in nine out of the ten questions in the test, the tally model was by far the favored model, particularly for the first eight questions. As can be seen in table 12 above, twenty-five out of forty learners in the class opted for the use of tallies in question 2. Predictably, the table shows that the tally model become increasingly error-prone as the number range gets larger. This echoes and confirms Schollar’s (2008) findings from Grades 5 and 6 in South Africa.

An interesting counter to this finding though is that, thirdly, a lot of learners switched to the use of the column model for the numerical questions, with twenty and nineteen learners attempting questions 9 and 10 with its use respectively, suggesting a shift from the use of informal strategies for the word problems to the formal algorithm when the problem is presented as a number sentence. The results indicate a high proportion of incorrect answers using the column model across all questions (apart from Question 4 which involved the calculation 17 + 10 + 10), again reflecting many of the arguments that I have detailed in the literature review.

With the tendency for learners to switch models based on the format in which the problem was posed, I used the above table to group learners according to the model that was predominant in their workings in the pre-test. Tabulated below are the models that learners used in the pre-test, along with the number of learners who used each model for the majority of the questions in the pre-test. Five models proved their dominance and so learners were working in five predominant categories:

Model	Number of learners PREDOMINANTLY using the model
○ **Pictorial model (PCT)**	**0**
○ **No Model (NM)**	**5**
○ **Word Sentence (WS)**	**2**
○ **Column Models (CM)**	**9**
○ **Horizontal Number Sentence (HNS)**	**7**
○ **Tallies (TL)**	**17**

Table 13: The distribution of the predominant use of models in the pre-test

Pictorial model (PCT)

The first bullet point speaks to the fact that, although the pictorial model was used by some learners for some of the questions in the test, it was not the predominant model for any of the learners concerned, and hence the zero in the second column.

No model (NM)

The second bullet point refers to any script for which the solution was blank in the section set aside for workings. That is, scripts wherein only the answer is provided for the given question. The trend herein was that these learners provided answers-only for questions 1 to 8, and then reverted to tallies and/or the column model for the last two questions. As alluded to earlier, a possible reason for the use of the column methods in the last two questions is the format in which the problem was presented as commented on earlier: as numerical problems (89+67) and (92 – 87). There were five learners in this category, and not one of them showed their working for the first four questions of the test. The table below provides a distribution of the predominance of blank spaces in the scripts of the learners concerned, and it uses the same coding for models as the table above. For correct answers, only the code is annotated in the relevant cell to indicate the model used by the learner. Where a code follows a number in brackets, this number indicates the incorrect answer provided by the learner for that specific problem; these cells are shaded for ease of reference:

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
NM	NM	NM	NM	CM/TL	TL	NM	(86)CM	CM	CM
NM	NM	NM	NM	NM	NM	NM	(50)NM	(416)CM	(15)NM
NM	(31)NM	(15)NM	(20)NM	(26)NM	NM	(0)NM	(75)NM	(1656)CM/TL	(15)TL
NM	(31)NM	(15)NM	(27)NM	(26)NM	NM	NM	(75)TL	(147)TL	(29)NM
NM	NM	(19)NM	NM	NM	NM	NM	(24)NM	NM	NM

Table 14: The NM group

As it can be gleaned from the distribution above only one of the five learners provided answers-only for all ten problems in the test with the other four changing models from question 5 onwards. In particular, for the numerical questions they reverted to tallies and the column model, with interesting results. For instance, the two learners who used the column method for question 9 (89+67) there is a sense in which things fell apart in the process of applying its accompanying strategy in the column algorithm. For both learners the problem was not with the setting up of the model (horizontal mathematization) which is fairly trivial, but in applying the seemingly 'arbitrary rule' of having to 'carry' the ten in the 16 that resulted from adding the units 9 and 7 (vertical mathematization). For question 10 (92 – 87) it appears that 'borrowing' was also a bridge too far at this stage, as the two learners are still subtracting the smaller from the bigger of the two units. For two learners in this group the answer to question 10 is 15.

Whilst on the one hand, it may be safe to say that only one amongst the five learners in this group demonstrated sufficient facility with both dimensions of mathematizing where the column method is concerned having solved the last two questions correctly, on the other, the blank spaces left by the other learner who also provided correct answers to questions 9 and 10 say little about what her thought processes were when she was tackling these problems. As a result, she did not get the marks allocated for the use of models and strategies in the marking scheme informed by the analytical framework used in this study.

In general then, a key observation is that learners using this model seemed to only have recourse to either tally or column models as alternatives, with both predominantly associated with incorrect responses. In fact, save for question 1 where an answer-only response was regarded as a recalled fact of the bonds of ten and so yielded four marks, all other correct answer-only responses yielded only one mark. The learners in this group got 20, 12, 7, 8 and

11 out of 47 respectively. These marks place the average performance of this group at 24.68%, which is well below the class average of 42.66% in the pre-test.

Word Sentence (WS)

The two learners in this category wrote out word sentences in an attempt to illuminate their working. One learner used sentences such as "I count them", "I add them" and "I youed my brane (sic)", and the other one used sentences such as "Camille is heavier than Carol because Carol is 36kg", "because Chris walked three times" and "because Juliet gave him R20". No marks were allocated in the test for the use of word sentences because according to my framing the use of a WS does not demonstrate sufficient cognitive engagement with the activity to be deemed as horizontal mathematization.

Pre-test									
Q1 (10-6)	*Q2 (12 + 20)*	*Q3 (5 + 9)*	*Q4 (17 + 10 + 10)*	*Q5 (26 – 19)*	*Q6 (13 – 6)*	*Q7 (25 – 11)*	*Q8 (50 – 36)*	*Q9 (89 + 67)*	*Q10 (92 – 87)*
WS	WS	WS	WS	(0)WS	WS	WS	(Camilla Is the heavier)WS	(151)TL	(0)WS
(3)WS	WS	WS	(30)WS	(10)WS	WS	(20)WS	(50)WS	(150)WS	(15)CM

Table 15: The WS group in the pre-test

Nevertheless, a mark was allocated for vertical mathematization provided the action is explicated as requiring addition or subtraction in the word sentence. Consequently the two learners in this group obtained 8 and 13 respectively, placing their average at 22.34%, which is lower than that of the NM group (24.68%), and even lower than the class average of 42.66%. One of the two learners provided 15 as the answer to question 10.

Column Models (CM)

For nine out of the forty learners in this class the column model was the predominant one across their workings. This is the standard model that learners are taught at the beginning of Grade 4, meant as support for the use of the column addition/subtraction algorithm as a strategy. As can be seen in how they tackled question 2, although the learners' initially attack the problem using the column model, many reverted to the use of tallies as the need arose.

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
CM/TL	CM/TL	CM/TL	CM	CM/TL	CM	CM	(50)NM	CM	CM
HNS	CM/TL	CM	HNS	CM	CM	CM	(50)HNS	TL	(15)CM
RCL	CM/TL	NM	CM	(13)CM	(9)NM	TL	(86)CM	CM	(15)CM
TL	CM/TL	TL	CM	CM	TL	CM	(50)WS	(146)CM	CM
CM/TL	CM/TL	CM	CM/TL	CM	CM	CM	(50)CM	CM	(10)CM
HNS	CM/TL	(13)CM	CM	(13)CM	CM	CM	(50)CM	(146)CM	(15)CM
HNS	CM	HNS	HNS	CM	NM	CM	(50)NM	(136)CM	CM
(16)CM	CM/TL	CM/TL	HNS	(8)TL	CM	CM	(4)CM	(166)CM	(15)CM
TL	CM/TL	TL	CM	CM	CM	(36)CM	(96)CM	CM	CM

Table 16: The CM group in the pre-test

Keke and Sasha (in rows 1 and 5) are two of the learners in this group. They tackled each and every single one of the ten questions using this approach, accompanied (in the first three to four questions) by the use of tally marks:

Figure 4: Column addition and Column subtraction (Keke's work)

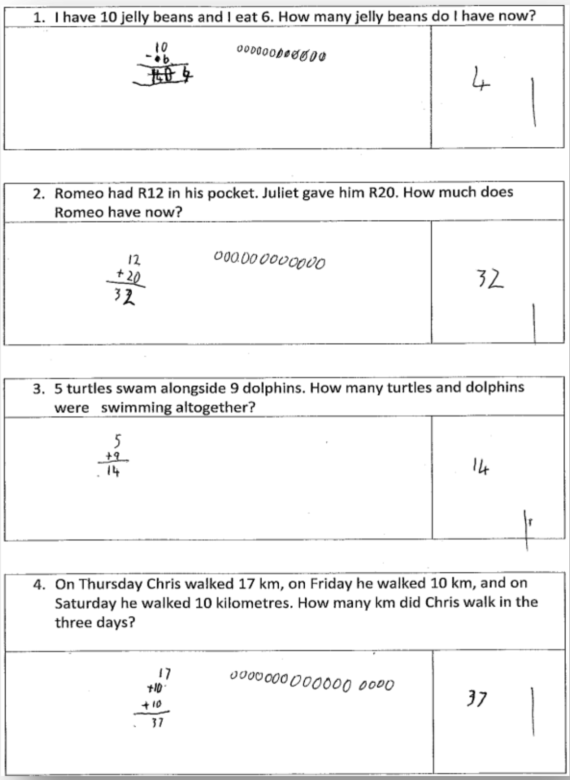
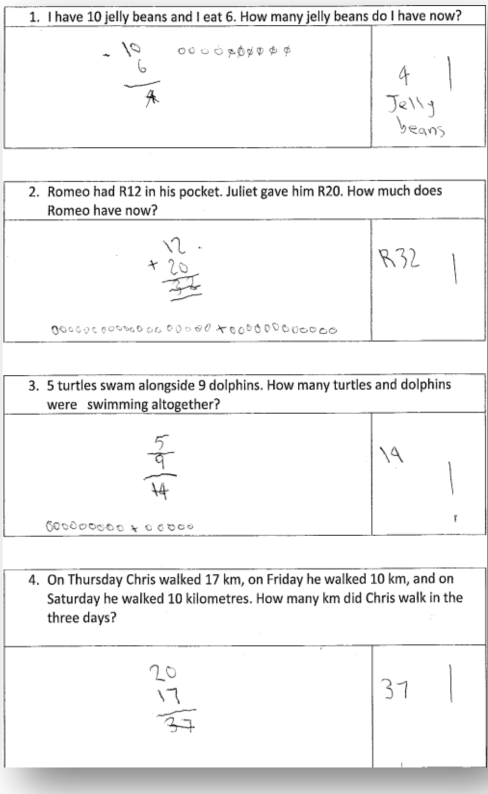


Figure 5: Column addition and Column subtraction (Sasha's work)



This group of learners experienced the same type of difficulties as the NM and WS groups in that, although they got a mark for the setting-up of the model, they experienced challenges when having to apply the column addition/subtraction algorithm. Moreover, like the NM and WS groups, none of the learners in this group got the *compare* problem (question 8) correctly. To reiterate, not one of the sixteen learners in the three groups of NM (5), WS (2) and CM (9) got question 8 correctly. As can be seen in table 12 above, this question proved to be the second most challenging question for this class, with the most difficult question being question 10; only eleven learners solved question 8 correctly and only nine successfully tackled question 10 in the pre-test. Four of the learners in this group used the column model to arrive at an answer of 15 for question 10.

In contrast to the NM and WS groups, however, this group has a higher average at 49.17%, which is about double the average that the either of the two groups obtained, and more than six percent above the class average for the pre-test. With questions 2, 3 and 4 which questions directly implicating addition as contrasted from questions 5, 6, 7 and 8 which implicated subtraction, the shading in table 16 above also indicates that the column model was used more successfully in the earlier questions on the pre-test as compared to the shading across the NM and WS models.

Horizontal number sentence (HNS)

For seven learners in this class the horizontal number sentence (HNS) model was the predominant model in the pre-test as seen from the bulk of questions they tackled using it.

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
TL/HNS	TL/HNS	TL/HNS	TL/HNS	TL	HNS	*HNS*	(50 more than 36)WS	(157)HNS	(15)HNS
PCT/HNS	PCT/HNS	PCT	HNS	(13)TL/HNS	(5)HNS	TL	TL	HNS	(15)HNS
HNS	HNS	HNS	HNS	(13)HNS	HNS	(25)HNS	HNS	(126)HNS	(15)HNS
HNS	(30)CM	HNS	HNS	HNS	HNS	(24)HNS	(50)NM	(1565)CM	(15)NM
CM	CM	HNS	HNS	HNS	HNS	(25)HNS	HNS	(136)CM	(15)CM
(5)HNS	(24)HNS	(10)HNS	(110)HNS	(31)HNS	(16)HNS	(36)HNS	(80)HNS	(31)TL	(71)TL
(5)HNS	(8)HNS	(80)HNS	(2)NM	(30)HNS	(11)HNS	(40)TL	(60)NM	(81)HNS	(80)TL/HNS

Table 17: The HNS group

The second in the list of the seven learners in this group used the HNS model alongside pictorial representations within the working of an individual question:

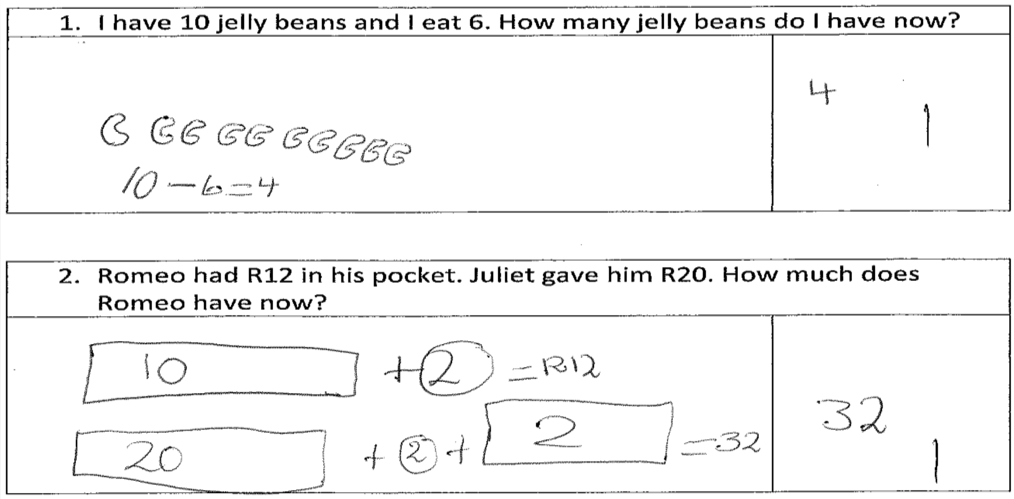


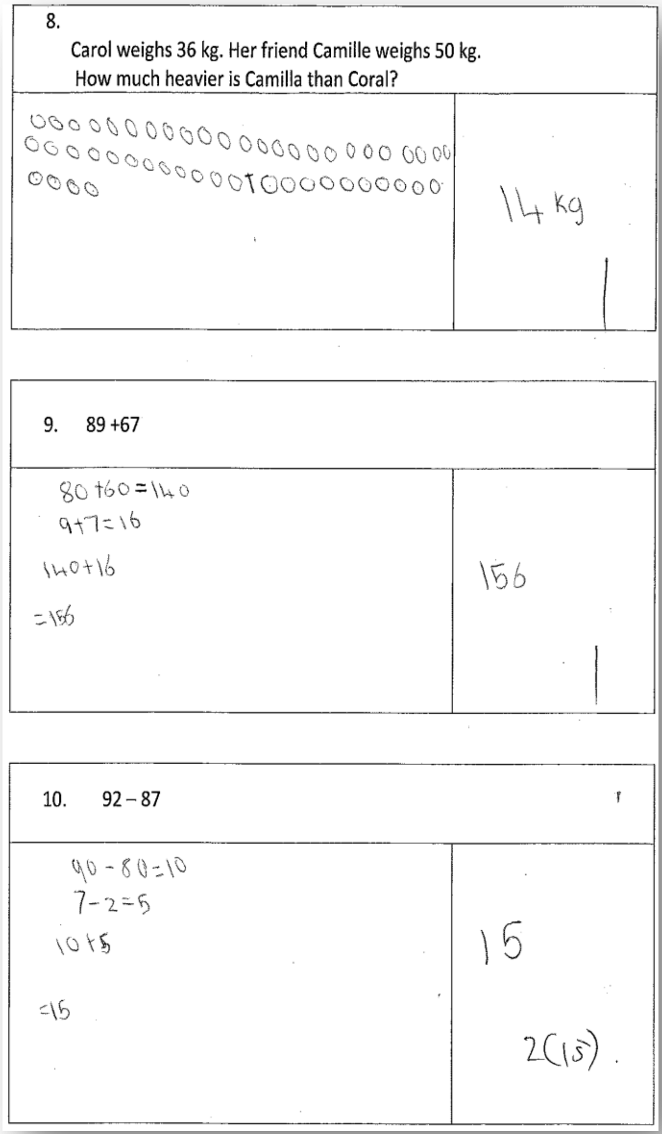
Figure 6: A Pictorial model accompanied by a Horizontal Number Sentence

For the last two learners in table 17 above the HNS model did not yield a single correct answer. Ironically, however, two of the learners in this group were able to solve question 8 correctly using the HNS model, with a third one using tallies successfully. Two other learners used the HNS model for each and every question, except for questions 9 and 10 where one learner reverted to using tallies and the other used the CM for questions 2 and 9.

The main strategy used for this model was the decomposition (1010) strategy, and as can be seen in the work of Sheba in figure 7 below, the strategy often resulted in incorrect answers when applied to subtraction problems – reflecting a point that has been made in the literature about why N10 is preferred at least initially. Five out of the seven learners in this group provided 15 as the answer to question 10:

Figure 7:

The HNS model and the accompanying decomposition (1010) strategy (Sheba's work)



Albeit lower than the class average, the average performance of the learners in this group was 39.51% which is the third highest of all the groups after the CM and the TL groups.

Tallies (TL)

This is by far the biggest of all the categories with no less than seventeen of the forty learners making predominant use of tallies to solve the problems in the pre-test. While nine of the eleven learners that managed to solve question 8 correctly did so with the use of tallies, this is also a reflection of the relatively 'time generous' format of my pre-test; it does not take away from the overall inefficiency of the model. As a result, performance-wise the TL group is only a percent below the CM group with an average group mark of 48.19%, and over five percent above the class average.

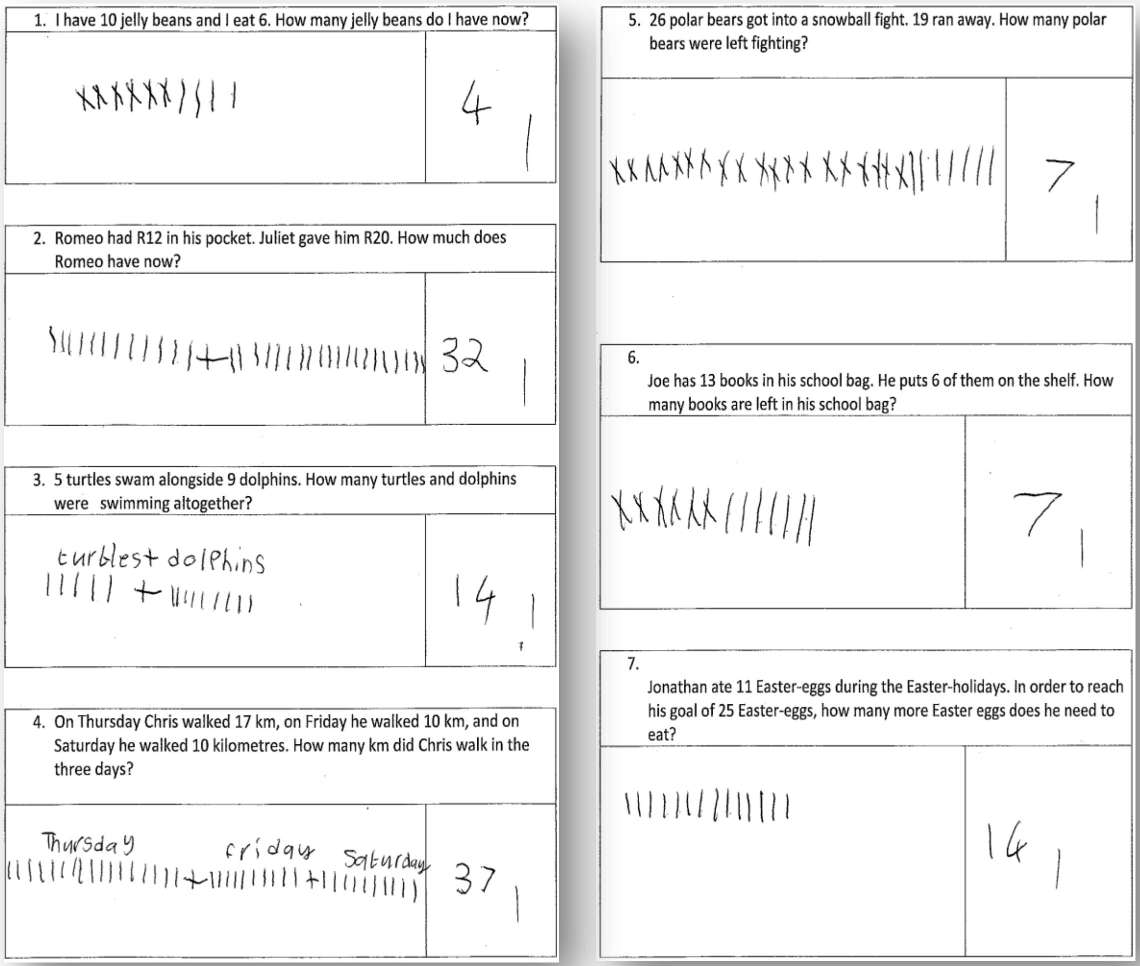
Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
TL	TL	PCT	TL	(3)PCT	TL	TL	TL	(163)CM	(14)TL
TL	TL	TL	TL	TL	(6)TL	(36)TL	TL	CM	(15)CM
TL	TL	TL	TL	TL	TL	TL	TL	HNS	(15)HNS
TL	TL	TL	TL	(8)TL	TL	TL	TL	NM	NM
TL	TL	TL	TL	TL	TL	(17)TL	TL	2(1416)HNS	(15)HNS
CM/TL	TL	TL	TL	TL	TL	TL	(66)TL	CM	(15)CM
TL	TL	TL	(40)TL	TL	TL	TL	TL	2(166)TL/CM	(10)TL/CM
TL	TL	TL	TL	TL	TL	(25)TL	(26)CM	CM/TL	(1)TL/HNS
TL	TL	TL	TL	TL	TL	TL	TL	NM	(15)NM
TL	TL	TL	TL	TL	TL	(23)TL	(50)TL	CM	(15)CM
TL	TL	(13)TL	TL	(9)TL	TL	TL	(60)NM	(130)TL	(48)NM
TL	TL	TL	TL	TL	TL	TL	TL	CM	CM
TL	TL	TL	(36)TL	TL	TL	(37)TL	(50)NM	TL	TL
HNS/TL	HNS/TL	TL	TL	(45)CM	(12)CM/TL	(36)TL/CM	(86)CM	(15)CM/TL	(14)CM
NM	CM/TL	TL	TL	TL	TL	TL	(50)NM	(210)CM	(15)CM
NM	TL	TL	TL	TL	TL	(33)TL	(26)NM	(1415)TL	(-17)TL
TL	TL	TL	TL	(6)TL	TL	(26)TL	(39)TL	(116)TL	(86)NM

Table 18: The TL group

Two of the seventeen learners in the table above – rows 4 and 9 – used tallies exclusively for questions 1 to 8, and showed no model for questions 9 and 10. The other fifteen learners switched to a different model in CM or HNS for the last two questions to arrive at answers that are unsurprisingly similar to those provided by learners in the CM and HNS groups. In particular, only three of the seventeen learners in this group provided the correct answer for the last question, with seven of seventeen providing 15 as the answer.

The finding that the majority of the problems were tackled using tallies corroborates what research has observed regarding the tendency for learners to remain highly dependent on concrete strategies at grade 2 (Venkat, 2011) and grade 3 (Ensor et al., 2009). According to the finding in this study, this overreliance on the use of tallies appears to extend into Grade 4, as evidenced by the work of Tosh in figure 7 below:

Figure 8: Tallies (Tosh's work)



The seven learners in this group who provided 15 as the answer to the last question brings to nineteen the total number of learners who subtracted the smaller from the larger across all the groups. This means that 47.50% of the learners in the class – almost half the learners in the class – tackled the last question by subtracting the smaller from the larger digit in each column to provide 15 as the answer – suggesting digit based breakdowns of number by place value without appropriate links to the quantities represented by the digits in two digit numbers (Thompson, 1997). The table below attests to the fact that exactly a quarter of the learners in this class used the column model to arrive at this erroneous result.

Model used to arrive at an answer of 15	Number of learners providing 15 as the answer to question 10
PCT	0
NM	3
WS	0
CM	10
HNS	5
TL	1
Total	19

Table 19: The culpability of each model in producing 15 as the answer to question 10.

To summarise then, from the point of view of the model used to tackle the majority of the problems in the pre-test, the learners in this Grade 4 class had a preference for the use of tallies (17) followed by the column model (9), closely followed by the use of the horizontal number sentence (7), the use of no model (5), and word sentences (2) respectively. Also, the class as a whole exhibits the same type of challenges observed by other researchers in similar studies; namely:

- an overreliance on tally counting (Ensor et al., 2009; Venkat, 2011);
- difficulties with tackling problems of the compare type (Carpenter et al., 1999); and
- the tendency to subtract the smaller from the larger whenever the unit of the subtrahend is larger than that of the minuend (Kilpatrick et al., 2001) often associated with a lack of understanding of quantities underlying the digits in double digit and larger numbers (Thompson, 1997)

In general then, less formal ways of working (in the use of tallies) were preferred over the more formal types (in the column model) as seen from the number of problems solved with their use in the pre-test. The following table records the proportions of learners who were successful within each class of problems across all the models used:

Format	Word problems (n=8)			Number sentence problems (n=2)	
Class of problem	Change problems	Combine/Separate problems	Compare problems	89 + 67	92 – 87
Questions representing this class of problems in test	(Q1 and Q2)	(Q3 to Q6)	(Q7 and Q8)	(Q9)	(Q10)
Success rate	88.75%	74.14%	42.50%	40%	22.50%

Table 20: The percentage of learners who were successful in tackling each class of problems

5.2.2 In an intervention focused on improving the learners' performance on addition and subtraction problems, what sorts of models and strategies are advanced? What sorts of models and strategies are learners using during the process of the intervention?

The intervention that was enacted was based on the Big Book Project compiled by Askew (2004). The overarching objective of the programme is to enable learners to use and apply a variety of mental strategies for different addition and subtraction problems depending on the numbers implicated in the problem. It advanced the empty number line as a model that can support the use of more sophisticated strategies than the ones demonstrated in the solution methods of learners in the pre-test.

According to the intervention programme the empty number line model can be introduced during the whole class discussion of the first lesson, immediately after the learners have attempted the three problems that reveal the theme of the unit. The problems in this first unit are fairly straightforward from the point of view of their structure and the numbers implicated in them. As a result, I did not introduce the empty number line in this session. In the second lesson the learners were allowed to use a model of their choice for the introductory problems, but they had to use the empty number line for the follow-up problems. From the third session onwards it was clear to the learners that a big part of the engagement was to develop fluency in the use of the empty number line. Although they still asked whether they could use alternatives, they soon understood that they had to use the number line as much as possible in all of their workings, and they did.

In this section report on the sorts of models and strategies learners were using during the process of the intervention. To reiterate, save for the two numerical questions, all the problems that learners had to solve were set in contexts that were sufficiently "real" to them. The tasks were presented as word problems and each lesson had a different theme while preserving the same basic form. This was done so that learners could quickly get used to what was expected of them in the lesson so that they concentrate on thinking about the mathematics. There were four parts to each of the six lessons, namely:

- Solving the Big Book problems
- Linking up the problems
- Follow-up problems
- Wrap-up

Lesson 1_Ugly bugs_7th March 2013

In this lesson the column and horizontal number sentence methods were prevalent, with a few learners using tallies. Whilst the standard addition algorithm was used by learners who were comfortable with its use, the HNS model and decomposition strategy were preferred for vertical and horizontal methods. For instance, for the first question

> *It started to rain. 25 ugly bugs hid under a log. 15 more came to join them. How many ugly bugs were under the log?*

this is what one of the learners offered as a way of solving it during the whole class discussion:

$$10 + 20 + (5 + 5) = 40$$

Another learner suggested the following for tackling the second problem:

$$\begin{array}{r} 60 \\ \underline{30} \\ 0 \\ \underline{90} \\ \underline{90} \end{array}$$

In her explanation she mentioned that they first added the units to get zero, and then added the tens to get 90. They then had to add the two results to get the final answer of 90. We agreed to call this the modified column method for showing the partial sums for the tens and the units. Similarly, for the third problem which required the adding of 32 and 31, when asked how he did it, Tebo said:

> *I added the two and the one, and then I added the three and the three*

That is

(2 + 1) + (30 + 30)

3 + 60

63

Three different decomposition-based strategies emerged during learners working on the introductory tasks:

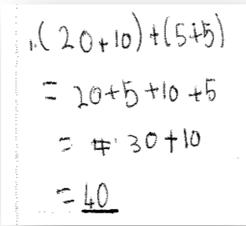
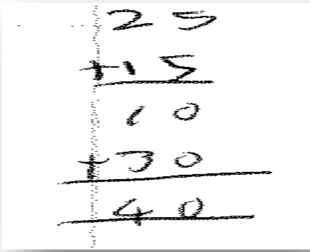
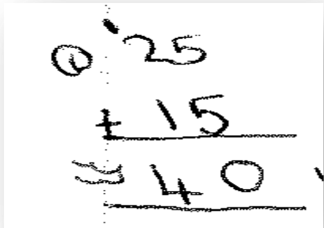
<table>
<tr><td>• the horizontal number sentence method (Sheba)</td><td>• the ‘modified’ column method (Tosh)</td><td>• the standard column algorithm method (Sasha)</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="3">Contextual problem: It started to rain. 25 Ugly Bugs hid under a log. 15 more came to join them. How many Ugly Bugs were under the log?</td></tr>
</table>

Table 21: Three different decomposition methods

Lesson 2_Collections_14th March 2013

According to the design of the programme of intervention, the objective of the second lesson was to further reinforce learners’ appreciation of the change-increase problem. For purposes of my study, it was an opportunity to introduce the empty number line as a relational model that can support strategies that are more efficient when faced with addition and subtraction problems. An analysis of the distribution of learners’ preferences revealed that learners fell into three main categories. These three categories could be described in terms of three broad clusters of phenomena which I have coded as follows:

- Clinging to column (CtC)
- Leaning on Column (LoC)
- Moving into empty number line (MiENL)

Descriptions of these three categories are seen in the analysis that follows the overview categorization of learners based on their model use in Lesson 2 in table 22 below:

CtC	LoC	MiENL
8 learners	25 learners	7 learners

Table 22: Distribution of preferences of models by learners during lesson 2

In the first category (CtC) were learners who had not yet begun to use the empty number line and were using the column method exclusively. There were eight learners in this category, two of whom got the first three questions correctly. In the second category (LoC) were learners who were using the column method first, followed by the empty number line with inconsistent results. There were 25 learners in this category, many of whom did not get to the third question, as can be seen in the work of Tosh:

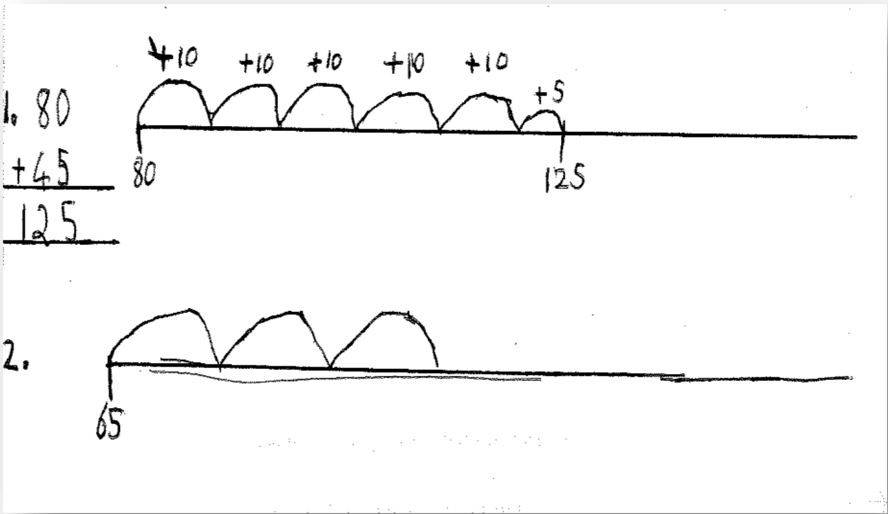


Figure 9: Tosh's tackling of the follow-up questions for lesson 2

Judging from the work of Tosh – who got 30 in the pre-test with the exclusive use of tallies – it was not surprising that for many of the learners in this category, the initial encounter with the empty number line was an uneasy one.

In the third category (MiENL), of the seven learners who demonstrated an inclination towards a preference for the empty number line, only one got through the first two questions, three attempted the first three questions, and, of the three that remained, only one (Sheba) attempted the first four questions with interesting workings. Firstly, in question 2 where learners are required to add 55 to 65, Sheba starts at 65, makes four jumps of 20, 20, 10 and 5 and lands on 85, 100, 115 and 120 respectively. Although the jumps add up to 55 and the last value she lands on is the correct one, her partial sums are incorrect. Secondly, and similarly, in question 3 there is a disconnection between the jumps and the landing positions in that the partial sums are not written below the line. Judging by the fact that she used the CM for the last two questions she attempted, it is probably safe to assume that for questions 2 and 3 she first calculated the result using the CM, and then only presented an illustration of her working on the ENL as required.

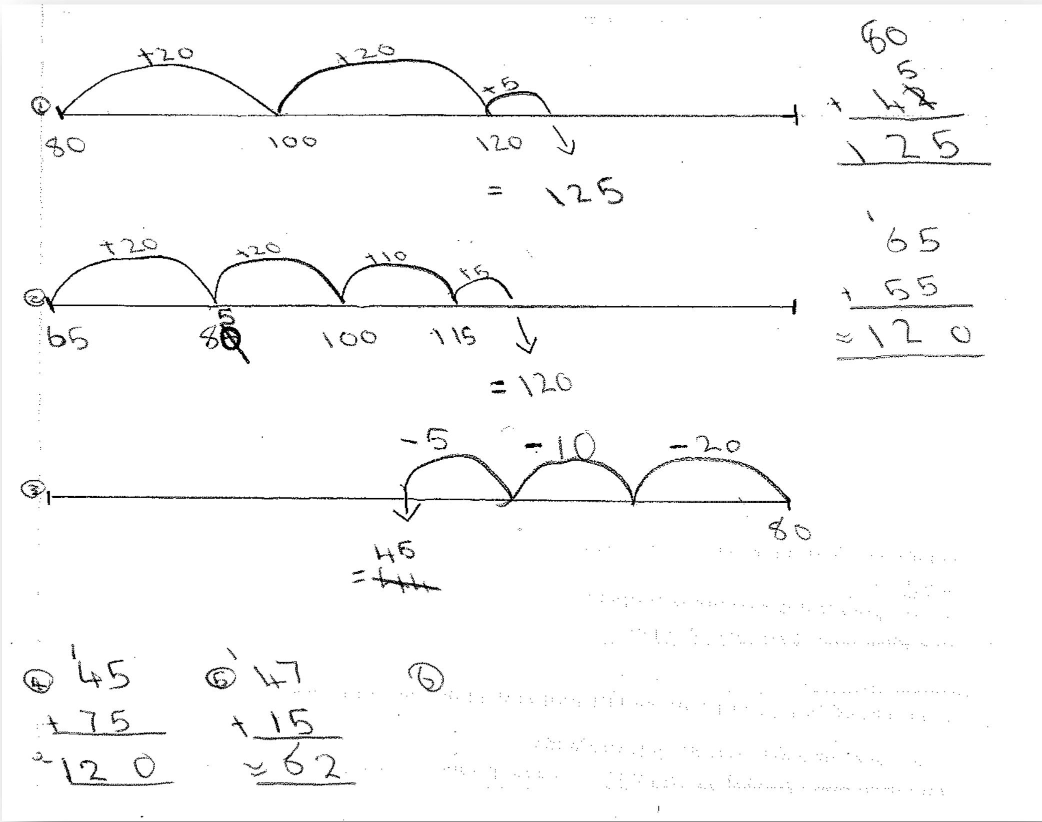
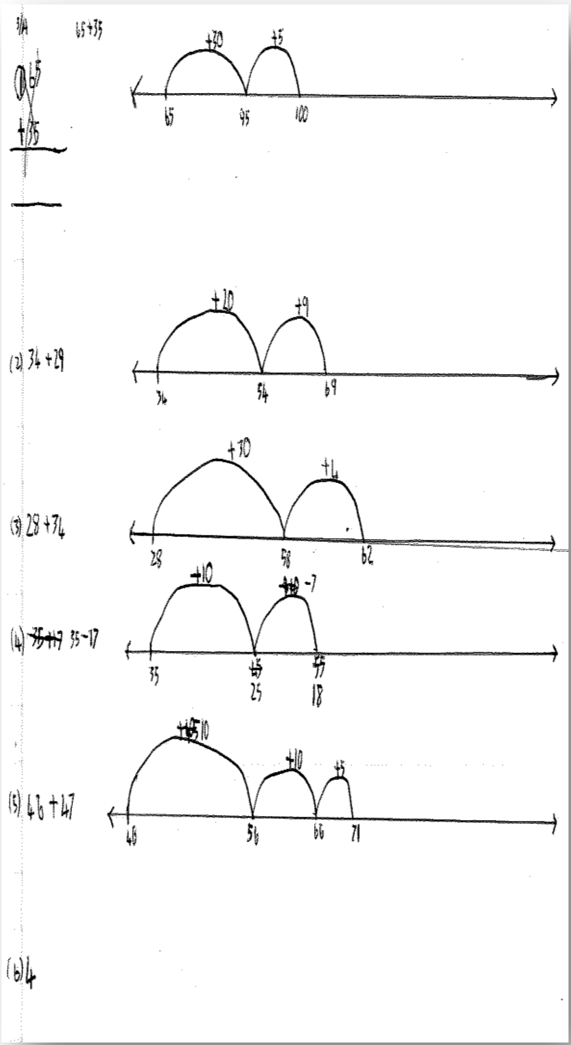
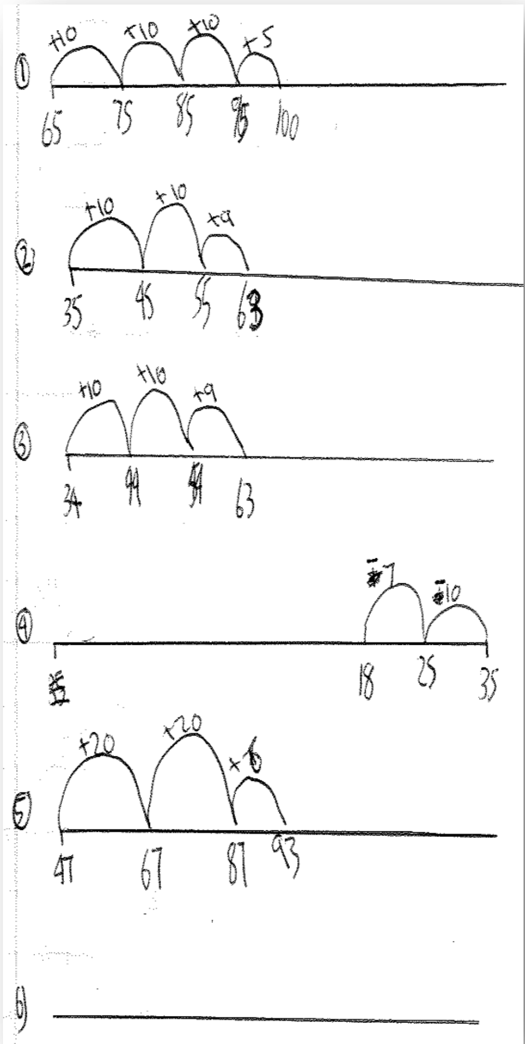


Figure 10: Sheba's tackling of the follow-up questions for lesson 2

Contrary to the observed trend in the LoC category, the learners in the MiENL category either used the empty number line exclusively, or started with the empty number line and then confirmed their answer with the column method.

Lesson 3_Champion Grannies_11th April 2013

Unlike in the previous lesson where learners could use the method of their choice, once the introductory problems were tackled and discussed, the learners were expected to use the empty number line for the follow-up problems. It was clear from the workings of the majority of the learners in the class that the setting-up of the model (horizontal mathematization) still had room for improvement. For instance, as can be seen in figure 11 below, although the jumps that Keke made were to the left for the subtraction question, his number line had bigger values to the left. As a result, for the single jump of ten he appeared to be adding whilst for the single jump of seven he appeared to be subtracting.

Figure 11: Keke's tackling of the follow-up questions for lesson 3	***Figure 12: Sasha's tackling of the follow- up questions for lesson 3***
	

Similarly, although Sasha was able to mathematize both horizontally and vertically for question 4 of the follow-up problems, she stumbled in question 3: having set up the model to count-on from the larger of the two numbers to be added – a more sophisticated strategy than counting-on from the first of the two numbers (Carpenter et al, 1999; Thompson, 1997), she added-on 29 instead of 28 as required.

Lesson 4_On the Shelf_25th April 2013

In this lesson, although all the learners heeded the call to try and make the number of jumps as few as possible, the majority amongst the twelve who were now in the MiENL category stayed with making a single jump for the tens and another for the units, keeping them at the N10 level in their use of strategies. A few of the learners made the effort to apply the knowledge of the compensation strategy during this lesson, a move towards the use of more sophisticated calculating strategies. Whilst Sheba did so successfully, Tosh still needed more practice with the reversing of direction required for compensation on the empty number line model, as can be seen in his unsuccessful attempt at tackling (26+21) below:

Figure 13: Sheba's work showing evidence of the use of the compensation strategy

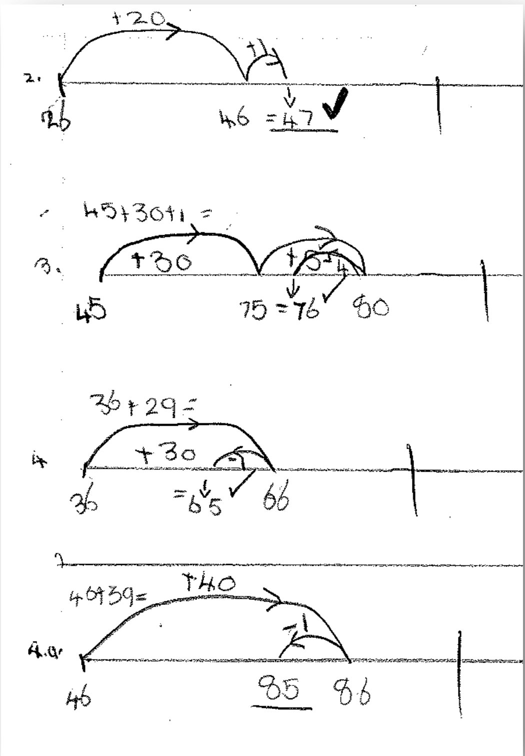
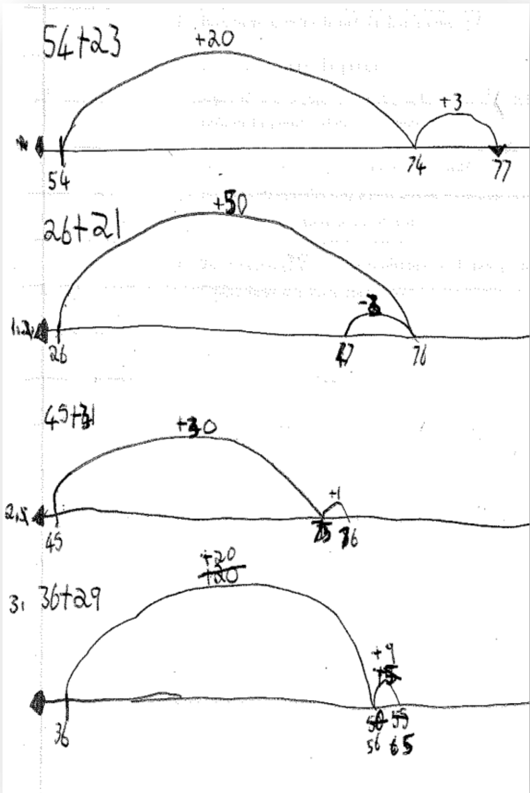


Figure 14: Tosh's work showing evidence of the use of the compensation strategy



Where this lesson was concerned I noted that the twelve learners who were now in the MiENL category were well on their way towards using the empty number line satisfactorily for many of the questions in the follow-up tasks. Although only Sheba used the compensation strategy successfully, I did observe that the majority of the learners were beginning to

- ✓ Make a single jump for a multiple of ten instead of several jumps of ten that amount to the tens in the addend/ subtrahend, and
- ✓ Used the friendly numbers (in multiple of five) to ease the burden of calculating

On the other end there were nine learners whose setting-up the empty number line model was still incomplete. Some of their key challenges resided in their inability to

- ❖ Annotate the change amount in the jump and the partial sum that results
- ❖ Have the numbers on the number line increasing to the right, and consequently
- ❖ Connect a forward jump with addition and a backward jump with subtraction

In the third and last group were nineteen learners who were making more than two jumps for a multiple of ten, and yet were beginning to make backward jumps for subtraction problems. The majority in this group tended to make jumps of ten – and in some cases jumps of five – and hence the need for them to make several jumps for a multiple of ten. In general, it was evident that a few more lessons were needed to afford the learners the opportunity to engage with this relational model a little more.

Lesson 5_Robyn the Girl Wonder_9th May 2013

The big idea for this lesson was to consolidate subtraction problems by affording learners the opportunity to work with these in flexible ways. Interesting to note was that, on the one extreme, a handful of learners were setting-up the model completely and correctly, whilst on the other extreme, many more learners were either missing some numbers on the line (Kamo), or the arcs that indicate the jumps do not appear to start where the last one left off but rather somewhere in the middle of the previous jump (Tebo). In other words, many learners were still unable to mathematize horizontally.

Also, whilst for the handful of learners only two jumps were executed to produce an answer, many still required as many as four. In the middle some learners (like Sasha) had to split the tens into two jumps and so they made a total of three jumps to get to an answer. Captured in the figures below are the typical ways in which the learners in this class were working during this lesson:

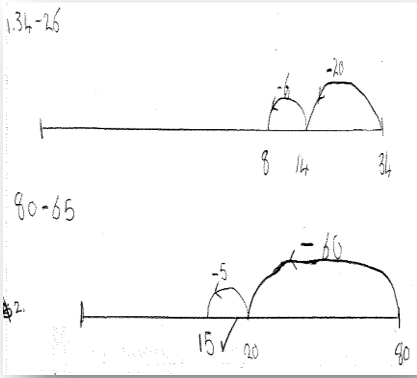
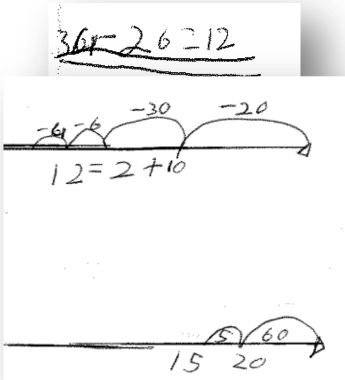
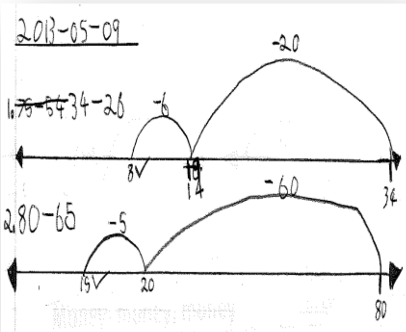
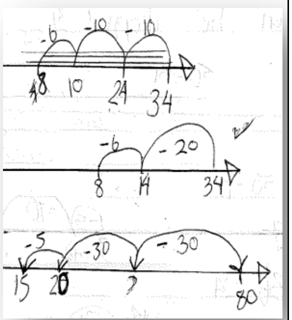
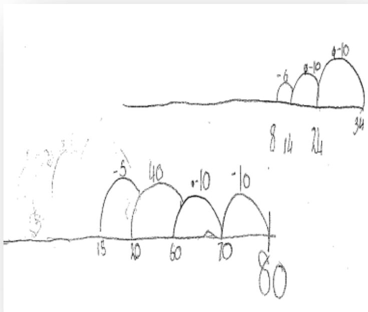
Lesson 5		
MiENL	**LoC**	**CtC**
Sheba 	Keke (absent on the day)	Kamo 
Tosh 	Sasha 	Tebo 

Table 23: Ways of working with the ENL model by learners during lesson 5

Lesson 6_Money Money Money_23rd May 2013

The big idea that I tried to push in this session was the inverse relationship between addition and subtraction. So after recapping on the different classes of problem and the actions associated with them, I presented learners with a challenge: to solve (73 – 69). It was interesting to hear one learner provide 4 as the answer and when asked during whole class discussion how he got the answer responded that

If you take 70 – 60 you get 10. Then if you add this 10 to the 3 and subtract 9 you get 4.

I found this to be an innovative way of getting around the pitfalls of a decomposition approach; one that avoids the 'false reversal' (Klein, Beishuizen and Treffers, 1998) that is always lurking when the approach is applied to a subtraction problem. The strategy used by this learner is what the authors above referred to as the 10s strategy (in table 6 in chapter 3 above).

Whilst most of the learners did not make use of the inverse relationship between addition and subtraction, they still demonstrated a growing facility with the use of the empty number line model which, given the prescription to use it, resulted in a blurring of the lines that separated the three categories of CtC, LoC and MiENL. A few of the learners did, however, make use of the inverse relationship, and amongst them are the two learners whose working is captured in the figures below:

Figure 15: Sheba's working in lesson 6

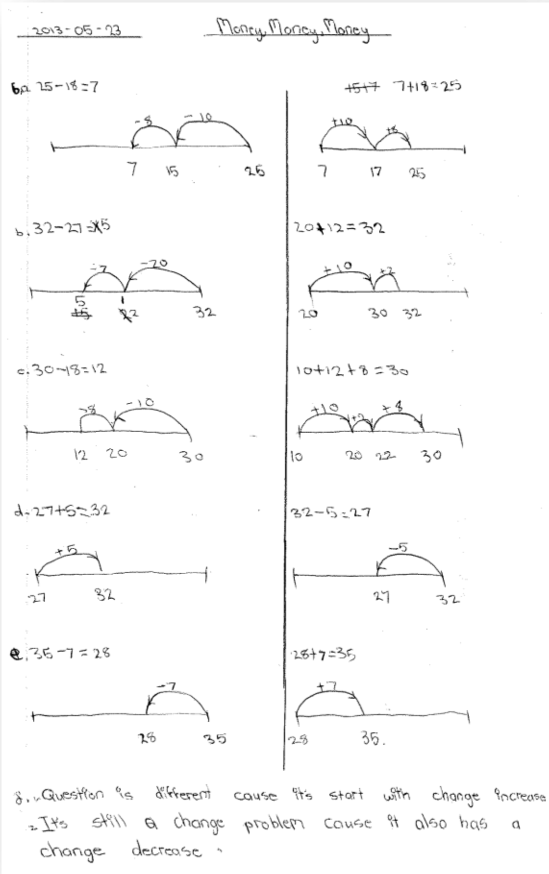
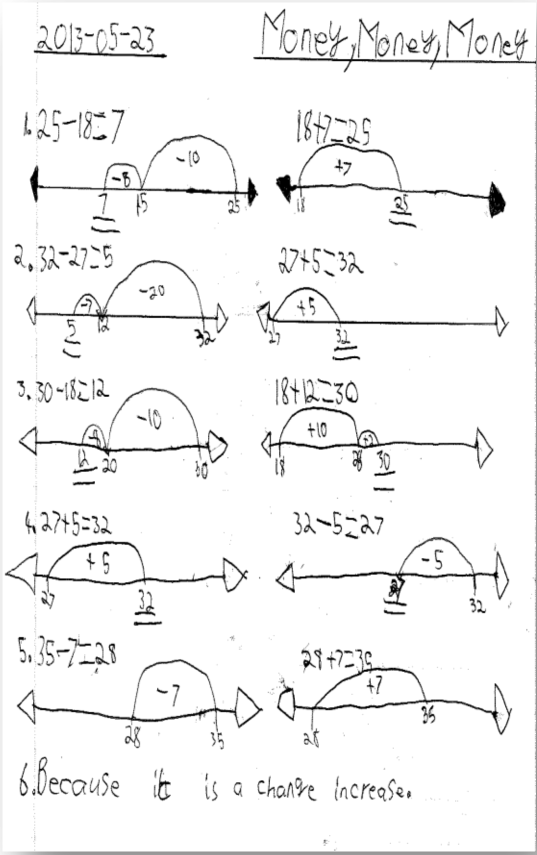


Figure 16: Tosh's working in lesson 6



These two learners were making clear their appreciation of the inverse relationship between addition and subtraction by approaching each problem in two different ways. Tosh's improvement was particularly interesting when tracked across all six lessons; from being able to do no more than two problems using the empty number line in lesson 2 to completing the worksheet correctly in lesson 6. Like the rest of the learners in the class, however, these two learners did not readily default to the most sophisticated strategy in compensation. This notwithstanding, these learners have demonstrated that the empty number line can be useful even to learners who in the current dispensation could be deemed as already excellent.

5.2.3 What effect, if any, has this intervention had on learners' use of models and strategies for solving different types of addition and subtraction problems?

The two tables below provide an overall distribution of the models and strategies used by the learners in this Grade 4 class for the problems in the pre-test and the post-test respectively, with the first one being a reprint of table 12 above for ease of reference. Again, only one model was counted for each solution and so the totals agree with the number of learners in the class. Each cell records the number of learners who successfully used the model listed in that row, over the total number of learners who used that model for each one of the ten questions in the pre, post and delayed-post-test.

PRE TEST		***WORD PROBLEMS***								***NUMERICAL PROBLEMS***	
Number of learners MATHEMATIZING to arrive at the correct	***MODEL***	***Q1***	***Q2***	***Q3***	***Q4***	***Q5***	***Q6***	***Q7***	***Q8***	***Q9***	***Q10***
	TL	19/19	25/25	19/20	16/18	15/19	17/19	11/19	9/13	2/10	1/5
	HNS	7/9	3/5	5/7	8/9	2/6	4/7	1/5	2/4	2/6	0/7
	CM	1/2	2/3	2/3	5/5	5/8	5/6	6/9	0/8	9/20	6/19
	NM	8/8	3/5	3/6	3/6	2/4	5/6	4/5	0/11	3/3	2/8
	WS	1/2	2/2	2/2	1/2	0/2	1/2	1/2	0/4	0/1	0/1
	PCT			2/2		0/1					
	Totals	**36/40**	**35/40**	**33/40**	**33/40**	**24/40**	**32/40**	**23/40**	**11/40**	**16/40**	**9/40**

Table 12: The distribution of models used in the pre-test

Looking across from the pre to the post-test one observes that fewer learners were making use of the TL model in the post-test; from double to single digits.

POST TEST		***WORD PROBLEMS***								***NUMERICAL PROBLEMS***	
Number of learners MATHEMATIZING to arrive at the correct solution	*MODEL*	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
	TL	7/7	0/1	11/11	3/4	5/5	9/9	6/9	3/5	1/4	0/2
	HNS	7/7	9/10	6/7	6/6	2/7	5/7	2/6	0/4	1/5	1/5
	CM	20/20	26/26	18/19	26/28	13/25	15/21	10/22	7/27	14/30	10/32
	NM	2/2	2/2	1/2	1/2	0/2	1/2	0/2	0/2	0/1	0/1
	WS							0/1	1/2		
	PCT	2/2									
	ENL	2/2	1/1	1/1		1/1	1/1				
	Totals	**40/40**	**38/40**	**37/40**	**36/40**	**21/40**	**31/40**	**18/40**	**11/40**	**16/40**	**11/40**

Table 24: The distribution of models used in the post-test

Interestingly, given that the intervention emphasized the number line model, very small numbers of learners took up this model voluntarily in the post-test. Instead, whilst the dichotomy between the models used for the word problems and those used for the numerical problems remains, many more learners used the CM for the word problems, from single to double digits and with much higher rates of success on some questions than seen in the pre-test. This shift towards the predominant use of the column model can be seen across each of the groups identified above as being the TL, HNS, CM, WS, and NM groups.

No model (NM)

The learners who did not show any model in the pre-test were more prepared to show their workings in the post-test, as can be seen in the tables below:

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
NM	NM	NM	NM	CM/TL	TL	NM	(86)CM	CM	CM
NM	NM	NM	NM	NM	NM	NM	(50)NM	(416)CM	(15)NM
NM	(31)NM	(15)NM	(20)NM	(26)NM	NM	(0)NM	(75)NM	(1656)CM/TL	(15)TL
NM	(31)NM	(15)NM	(27)NM	(26)NM	NM	NM	(75)TL	(147)TL	(29)NM
NM	NM	(19)NM	NM	NM	NM	NM	(24)NM	NM	NM

Table 14: The NM group in the pre-test

Post-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
CM	*CM*	*CM*	*CM*	*CM*	*CM*	*CM*	*CM*	*CM*	*CM*
NM	*NM*	*NM*	*NM*	*(45)NM*	*(19)NM*	*(36)NM*	*(86)NM*	*(21)CM*	*(15)NM*
TL	*CM*	*TL*	*CM*	*(45)CM*	*(19)CM*	*(36)CM*	*(86)CM*	*(21)TL*	*(15)CM/TL*
TL/CM	*TL/CM*	*CM*	*CM*	*(6)CM*	*CM*	*CM*	*(86)CM*	*(152)CM/TL*	*(15)CM*
TL	*TL/HNS*	*TL*	*TL*	*TL*	*TL*	*TL*	*(86)HNS*	*TL*	*(3)HNS*

Table 25: The NM group in the post-test

Remembering that only the second model was counted in cases where more than one model was used, one observes a clear predominance of the CM on the part of the fourth learner in the table above. Given that in the pre-test only the first three learners used the CM, only one – the last one in table 25 above – did not use the CM at all in the post-test, but instead moved towards a predominant use of tallies. The learner whose choice of models is recorded in the second row was still reluctant to show his working except for question 9 where he used the CM. So, of the five learners in this group, one has shifted towards the use of tallies and two of them have shifted towards the use of the CM.

The learners in this group are getting fewer questions incorrectly, particularly the first four questions. Also, the errors are more consistent in the post-test, with 86 being the incorrect answer provided by all four learners who got question 8 incorrectly. That is, they all added instead of subtracting the numbers in this *compare* problem. At the same time, three of the four learners with incorrect answers for question 10 are still subtracting the smaller from the bigger of the two units for the numbers in this problem to arrive at an answer of 15. These mishaps notwithstanding, this group's performance improved from an average mark of 24.68% in the pre-test to 43.83% in the post-test, which is a substantive improvement.

Word Sentence (WS)

The learners who used word sentences as a way of showing their workings in the pre-test were more prepared to show more formal workings in the post-test, as can be seen in the tables below:

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
WS	WS	WS	WS	(0)WS	WS	WS	(Camilla Is the heavier)WS	(151)TL	(0)WS
(3)WS	WS	WS	(30)WS	(10)WS	WS	(20)WS	(50)WS	(150)WS	(15)CM

Table 15: The WS group in the pre-test

Post-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
CM	*CM*	*CM*	*CM*	*CM*	*(17)CM*	*CM*	*CM*	*(166)CM*	*CM*
PCT	*HNS*	*(15)HNS*	*HNS*	*HNS*	*HNS*	*(5)WS*	*(50)WS*	*(300)HNS*	*(15)HNS*

Table 26: The WS group in the post-test

Whilst one learner shifted to the exclusive use of the CM and was able to correctly answer question 8, the other moved to the predominant use of the HNS but answered question 8 in exactly the same way as in the pre-test. While the use of the CM helped the first learner to arrive at the correct answer to question 19 this time around, the use of the HNS by the other learner did not yield positive results; again he provided 15 as the answer. With this shift toward the use of more formal models the overall performance of this group has more than doubled from an average of 22.34% in the pre-test to 52.13% in the post-test.

Column Models (CM)

As can be seen in how they worked with question 2 in the pre-test, eight of the nine learners in this group tended to revert to the use of tallies alongside their use if the CM in question 2. In the post-test this need did not arise, suggesting that they are more facile with the use of the CM. As a result, there is less of a dichotomy in this group's use of models between the word problems and the numerical problems in the post-test. Nevertheless, all the learners used the CM for the numerical problems.

All nine learners are getting the first three questions correctly in the post-test, with only one using the CM successfully for question 8. Four of the nine learners used the CM exclusively in the post-test:

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
CM/TL	CM/TL	CM/TL	CM	CM/TL	CM	CM	(50)NM	CM	CM
HNS	CM/TL	CM	HNS	CM	CM	CM	(50)HNS	TL	(15)CM
NM	CM/TL	NM	CM	(13)CM	(9)NM	TL	(86)CM	CM	(15)CM
TL	CM/TL	TL	CM	CM	TL	CM	(50)WS	(146)CM	CM
CM/TL	CM/TL	CM	CM/TL	CM	CM	CM	(50)CM	CM	(10)CM
HNS	CM/TL	(13)CM	CM	(13)CM	CM	CM	(50)CM	(146)CM	(15)CM
HNS	CM	HNS	HNS	CM	NM	CM	(50)NM	(136)CM	CM
(16)CM	CM/TL	CM/TL	HNS	(8)TL	CM	CM	(4)CM	(166)CM	(15)CM
TL	CM/TL	TL	CM	CM	CM	(36)CM	(96)CM	CM	CM

Table 16: The CM group in the pre-test

Post-test									
Q1 (10-6)	*Q2 (12 + 20)*	*Q3 (5 + 9)*	*Q4 (17 + 10 + 10)*	*Q5 (26 – 19)*	*Q6 (13 – 6)*	*Q7 (25 – 11)*	*Q8 (50 – 36)*	*Q9 (89 + 67)*	*Q10 (92 – 87)*
CM	CM	CM	CM	CM	CM	(25)CM	(86)CM	(146)CM	CM
CM	CM	CM	CM	HNS	HNS	CM	CM	CM	CM
HNS	CM	TL	CM	CM	TL	HNS	(15)TL	(153)CM	CM
TL	CM	TL	CM	CM	TL	(25)CM	(50)CM	(155)CM	CM
CM	CM	CM	CM	(13)CM	(10)CM	(00)CM	(83)CM	CM	(10)CM
CM	CM	CM	CM	(06)CM	CM	(36)CM	(86)CM	CM	(16)CM
CM	CM	CM	(27)CM	CM	CM	(36)CM	(86)CM	(136)CM	(17)CM
CM	CM	CM	CM	(13)CM	(13)CM	TL	(86)CM	CM	(15)CM
CM	CM	TL	CM	CM	CM	(36)CM	(86)CM	CM	CM

Table 27: The CM group in the post-test

Like the learners in the NM group, the common error for question 8 was that learners added when they were supposed to subtract in the post-test, with 86 being the incorrect answer provided by four of the eight learners who got question 8 incorrectly. This is a shift from the pre-test where 50 was the incorrect answer provided by six of the nine learners in this group. In the pre-test they were picking the bigger of the two numbers in the problem – 50 and 36 – and in the post-test the challenge was with which operation to choose between addition and subtraction.

Where question 10 is concerned, one more learner was able to arrive at the correct answer with the use of the CM in the post-test as compared to the pre-test. With only one learner subtracting the smaller from the bigger digit for question 10, there is less consistency amongst the incorrect answers in the post-test; namely: 10, 16, 17 and 15. Overall, there is better use of the CM as there are fewer incorrect answers provided by this group of learners in the post-test. This has translated in an increase in the group performance from an average of 49.17% in the pre-test to 51.54% in the post-test.

Horizontal number sentence (HNS)

For the seven learners for whom the horizontal number sentence (HNS) model was the predominant model in the pre-test, the switch to the use of the CM in the post-test yielded some positive results. This is particularly true for question 9 where the three learners who answered correctly used the CM in the post-test as compared to only one who used the HNS successfully in the pre-test. On the downside, however, compared to the one instance of the unsuccessful use of the CM for question 10 in the pre-test, five of the seven learners in this group provided 15 as the answer to question 10.

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
TL/HNS	TL/HNS	TL/HNS	TL/HNS	TL	HNS	*HNS*	(50 more than 36)WS	(157)HNS	(15)HNS
PCT/HNS	PCT/HNS	PCT	HNS	(13)TL/HNS	(5)HNS	TL	TL	HNS	(15)HNS
HNS	HNS	HNS	HNS	(13)HNS	HNS	(25)HNS	HNS	(126)HNS	(15)HNS
HNS	(30)CM	HNS	HNS	HNS	HNS	(24)HNS	(50)NM	(1565)CM	(15)NM
CM	CM	HNS	HNS	HNS	HNS	(25)HNS	HNS	(136)CM	(15)CM
(5)HNS	(24)HNS	(10)HNS	(110)HNS	(31)HNS	(16)HNS	(36)HNS	(80)HNS	(31)TL	(71)TL
(5)HNS	(8)HNS	(80)HNS	(2)NM	(30)HNS	(11)HNS	(40)TL	(60)NM	(81)HNS	(80)TL/HNS

Table 17: The HNS group in the pre-test

One learner (in the second-last row) clung to the exclusive use of the HNS and did not switch to the use of a different model for the numerical problems in the post-test, whilst another (row 4) showed no model except to use the CM in the first and the last questions:

Post-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
TL/CM	*CM*	*CM*	*CM*	*(13)CM*	*CM*	*CM*	*WS*	*(30)CM*	*(15)CM*
TL/HNS	*HNS*	*HNS*	*HNS*	*(13)HNS*	*HNS*	*(25)HNS*	*(50)HNS*	*CM*	*(15)CM*
ENL	*ENL*	*ENL*	*CM*	*ENL*	*ENL*	*CM*	*CM*	*CM*	*(15)CM*
CM	*NM*	*(14)NM*	*(27)NM*	*(6)NM*	*NM*	*(36)NM*	*(86)NM*	*(13)NM*	*(15)CM*
CM	*CM*	*CM*	*CM*	*(13)CM*	*CM*	*CM*	*(44)CM*	*CM*	*(15)CM*
(5)HNS	*HNS*	*HNS*	*HNS*	*(315)HNS*	*(73)HNS*	*(36)HNS*	*(86)HNS*	*(1416)HNS*	*(8171)HNS*
HNS	*(30)HNS*	*(6)HNS*	*(20)CM*	*(30)CM*	*(56)CM*	*(36)CM*	*(60)CM*	*(81)CM*	*(81)CM*

Table 28: The HNS group in the post-test

The two learners whose use of the HNS model did not yield a single correct answer in the pre-test were now able to provide four correct answers between them in the first four questions of the post-test. Consequently, the group provided only six incorrect answers in the post-test for the first four questions of the test as compared to the nine incorrect answers in the pre-test.

With the responses to question 8 remaining a mixed bag of incorrect answers, the answer of 86 is now amongst these. One learner (whose models are captured in row 2) had used tallies to produce the correct answer in the pre-test. She now used the HNS to provide 50 as the answer. The overall effect on the marks for the group is that of a marginal increase of just over a percent in performance from an average of 39.51% in the pre-test to 40.73% in the post-test.

Tallies (TL)

With the overarching shift being from the use of tallies to the use of the CM, it is not surprising that a lot of the learners that were making predominant use of tallies switched to the use of the CM in the post-test. Although only two learners have switched completely to the use of the CM, the majority have shifted towards a predominant use of the CM. In fact, the two learners who do not meet this criterion used tallies and the HNS exclusively; their use of models is recorded in the fifth-last and the second-last rows of table 29 below.

Pre-test									
Q1 (10-6)	***Q2 (12 + 20)***	***Q3 (5 + 9)***	***Q4 (17 + 10 + 10)***	***Q5 (26 – 19)***	***Q6 (13 – 6)***	***Q7 (25 – 11)***	***Q8 (50 – 36)***	***Q9 (89 + 67)***	***Q10 (92 – 87)***
TL	TL	PCT	TL	(3)PCT	TL	TL	TL	(163)CM	(14)TL
TL	TL	TL	TL	TL	(6)TL	(36)TL	TL	CM	(15)CM
TL	TL	TL	TL	TL	TL	TL	TL	HNS	(15)HNS
TL	TL	TL	TL	(8)TL	TL	TL	TL	NM	NM
TL	TL	TL	TL	TL	TL	(17)TL	TL	2(1416)HNS	(15)HNS
CM/TL	TL	TL	TL	TL	TL	TL	(66)TL	CM	(15)CM
TL	TL	TL	(40)TL	TL	TL	TL	TL	2(166)TL/CM	(10)TL/CM
TL	TL	TL	TL	TL	TL	(25)TL	(26)CM	CM/TL	(1)TL/HNS
TL	TL	TL	TL	TL	TL	TL	TL	NM	(15)NM
TL	TL	TL	TL	TL	TL	(23)TL	(50)TL	CM	(15)CM
TL	TL	(13)TL	TL	(9)TL	TL	TL	(60)NM	(130)TL	(48)NM
TL	TL	TL	TL	TL	TL	TL	TL	CM	CM
TL	TL	TL	(36)TL	TL	TL	(37)TL	(50)NM	TL	TL
HNS/TL	HNS/TL	TL	TL	(45)CM	(12)CM/TL	(36)TL/CM	(86)CM	(15)CM/TL	(14)CM
NM	CM/TL	TL	TL	TL	TL	TL	(50)NM	(210)CM	(15)CM
NM	TL	TL	TL	TL	TL	(33)TL	(26)NM	(1415)TL	(-17)TL
TL	TL	TL	TL	(6)TL	TL	(26)TL	(39)TL	(116)TL	(86)NM

Table 18: The TL group in the pre-test

Post-test

Q1 (10-6)	**Q2 (12 + 20)**	**Q3 (5 + 9)**	**Q4 (17 + 10 + 10)**	**Q5 (26 – 19)**	**Q6 (13 – 6)**	**Q7 (25 – 11)**	**Q8 (50 – 36)**	**Q9 (89 + 67)**	**Q10 (92 – 87)**
CM	CM	CM	CM	CM	(19)CM	CM	(19)CM	(346)CM	2(15)CM
CM	CM	TL	CM	(13)CM	TL	(13)TL	(15)TL	CM	2(15)CM
TL	TL/HNS	TL	CM	CM	CM	CM	CM	(22)CM	CM
PCT	CM	CM	CM	(13)CM	CM	TL	TL	CM	2(15)CM
TL	CM	TL	CM	TL	TL	(36)CM	(86)CM	CM	2(15)CM
CM	CM	CM	CM	CM	CM	CM	(80)CM	CM	CM
TL	CM	TL	TL	TL	TL	CM/TL	(19)CM	(22)CM	2(10)CM
ENL	TL/HNS	TL	(27)TL	(13)TL/HNS	TL/HNS	TL/HNS	TL/CM	(22)TL/HNS	2(15)TL/CM
TL/HNS	TL/HNS	TL/HNS	TL/CM	TL/CM	TL/CM	(36)TL/CM	TL/CM	HNS	HNS
CM	CM	(15)CM	CM	(16)CM	TL	CM/TL	(86)CM	(717)CM	2(15)CM
NM	CM	CM	CM	TL	TL	(15)TL	(86)CM	(889)CM	2(10)CM
CM	CM	CM	CM	CM	CM	TL	TL	CM	CM
HNS	HNS	HNS	HNS	(45)HNS	2(19)HNS	(37)HNS	(96)HNS	(91)HNS	2(89)HNS
CM	CM	CM	CM	(7)CM	2(102)CM	(36)CM	(86)CM	CM	2(25)CM
(16)HNS	HNS	HNS	HNS	(6)HNS	HNS	(36)HNS	(86)CM	(102)CM	2(18)CM
TL	2(33)TL	TL	TL	TL	TL	(16)TL	TL	(415)TL	2(51)TL
CM	CM	CM	HNS	CM	CM	(36)CM	(86)CM	2(146)CM	2(15)CM

Table 29: The TL group in the post-test

This gravitation towards the use of the CM on the part of these learners resulted in a breaking of the pattern observed earlier where the use of models followed the format of the problems. Instead, we see learners using the CM as early as the first question in the post-test. Be that as it may, except for the two learners who used the TL and the HNS models (referred to above), and another who used the empty number line for the first question, all other members of this group used the CM method to answer the numerical questions.

Also important to note is the fact that two more learners are getting question 8 incorrectly in the post-test as compared to the pre-test, with 86 being the incorrect answer provided by no less than six of the learners in the post-test compared to only one in the pre-test. Furthermore, exactly the same number of learners (seven) provided the incorrect answer of 15 with the use of the CM for question 10. Effectively, this translated into another marginal increase in performance of just over a percent from an average of 48.19% in the pre-test to 49.44% in the post-test.

To summarise then, the general shift from the pre-test to the post-test was away from the use of the less formal models in word sentences and tallies towards the use of the more formal column model. Ensor et al (2009) suggest that these 'specialization' moves are important within mathematical progression, with the increases in performance backing their view. This resulted in somewhat of a breaking of the pattern observed in the pre-test where the informal models were used for the word problems and the more formal ones were evident in the solutions for the numerical problems as a lot of learners used the column model for many more word problem in the post-test.

Furthermore, a total of 17 learners subtracted the smaller from the larger across all the groups to produce the incorrect answer of 15 for question 10:

Model used to arrive at an answer of 15	Number of learners providing 15 as the answer to question 10
PCT	0
NM	1
WS	0
CM	14
HNS	1
TL	1
Total	17

Table 30: The culpability of each model in producing 15 as the answer to question 10 in the post-test.

Although there were nineteen such learners in the pre-test, with each of the groups gravitating to the use of the column model in the post-test, four more are using the column model incorrectly as compared to the ten in the pre-test. As a result, over a third of the learners in this class used the column model to arrive at this erroneous result. Save for the *compare* class of problems for which the percentage of learners who were successful dropped by 6.25%, the table that follows records a higher average success rate of the forty learners within each class of problems across all the models used in the post-test relative to the pre-test.

Format	Word problems (n=8)			Number sentence problems (n=2)	
Class of problem	Change problems	Combine/Separate problems	Compare problems	89 + 67	92 – 87
Questions representing this class of problems in test	(Q1 and Q2)	(Q3 to Q6)	(Q7 and Q8)	(Q9)	(Q10)
Success rate	97.5%	78.13%	36.25%	40%	27.50%

Table 31: The percentage of learners who were successful in tackling each class of problems

The empty number line (ENL)

In the post-test only two learners used the empty number line to solve the word problems, with the one learner using it only once in question 1, and the other learner using it for the first five questions. As already mentioned in the previous chapter, this necessitated the need for a delayed-post-test where the use of the empty number line was prescribed. This imposition notwithstanding, learners used the ENL mainly for the word problems and switched to a different model (the column model predominantly) for the numerical problems. The distribution of the use of models is captured in the table below:

DELAYED POST TEST		***WORD PROBLEMS***								***NUMERICAL PROBLEMS***	
Number of learners MATHEMATIZING to arrive at the correct solution	*MODEL*	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
	TL								0/1	0/1	0/3
	HNS					0/1	0/1	0/1	0/1	0/1	
	CM								0/2	12/23	9/22
	NM	0/1	1/1	0/1	0/1	0/1	0/1	0/1	0/1	1/2	0/2
	WS										
	PCT										
	ENL	35/39	25/39	36/39	28/39	23/38	30/38	15/38	15/35	4/13	5/13
	Totals	**35/40**	**26/40**	**36/40**	**28/40**	**23/40**	**30/40**	**15/40**	**15/40**	**17/40**	**14/40**

Table 32: The distribution of the models used in the delayed-post-test

A quick glance at the table above reveals that one learner resorted to tallies from question 8 onwards, another reverted to the HNS from question 5 onwards, and a third one showed no workings for any of the questions in the post-test. Twenty-three learners used the CM for question 9 and twenty-two used it for question 10 with the result that about half of them solved the two problems correctly in each of the two cases.

With the ENL used mainly for the word problems in the delayed-post-test, the table below provides a breakdown of the models used to produce 15 as the answer to question 10:

Model used to arrive at an answer of 15	**Number of learners providing 15 as the answer to question 10**
ENL	3
PCT	0
NM	1
WS	0
CM	6
HNS	0
TL	2
Total	12

Table 33: The culpability of each model in producing 15 as the answer to question 10 in the delayed-post-test.

On the one hand, of the thirteen learners who used the empty number line to solve the last problem in the delayed-post-test only five produced the correct answer. On the other hand, fifteen of the forty learners in the class produced the correct answer for question 8. Compared to the eleven in the pre and the post-test, this is a ten percent increase in success rate where this *compare* problem is concerned. When both compare questions are considered, the result is only five percent lower than that in the pre-test:

Format	*Word problems (n=8)*			*Number sentence problems (n=2)*	
Class of problem	Change problems	Combine/Separate problems	Compare problems	89 + 67	92 – 87
Questions representing this class of problems in test	(Q1 and Q2)	(Q3 to Q6)	(Q7 and Q8)	(Q9)	(Q10)
Success rate	76.25%	73.13%	37.50%	42.50%	35%

Table 34: The percentage of learners who were successful in tackling each class of problems

Following the marking scheme in my analytical framework in a class of forty learners, the average percentage for the entire class in the pre-test came to 42.66%. Looking across to the delayed-post-test one observes an average percentage of 55.59%, implying an improvement of 12.93%. With the post-test sitting at 47.82%, this translates into quite a substantial average improvement from pre-test (42.66%) to post-test (47.82%) to delayed-post-test (55.59%), with the interesting feature that the increase is bigger post-test to delayed post-test, and associated with a much greater degree of imposition of a specific model – the empty number line.

Despite the short nature of the intervention, these results suggest that, broadly speaking, improvements can be achieved in relatively short time frames, and importantly, that these improvements can be retained beyond their immediate coverage in class. It is a finding that tends to work against the common complaint that children 'forget' things soon after teaching.

Chapter 6: Discussion

6.1 In relation to literature

The reintroduction of the concept of mind into theories of learning saw the use of the word mental in RME literature. The subsequent adoption of elements of the realistic approach to learning and teaching by the education system in South Africa meant that researchers and practitioners of mathematics education had to make sense of this word. Consequently, in this study I have posited the use of models and strategies as evidence of an active process of mental construction and sense making (Shepard, 2000).

Observations of different ways of working with models and strategies by Carpenter et al. (1999) paved the way towards establishment of a hierarchy of strategies informed by the level of their sophistication. This hierarchy formed the basis of my theoretical and analytical framework in which I related each strategy to an appropriate model. Driven by the conviction that as learners graduate into the Intermediate Phase they must be exposed to calculating strategies over and above the counting strategies encountered in the Foundation Phase, I established the empty number line as the most amenable to supporting the most sophisticated of calculating strategies such as compensation. As a result, the empty number line became the most prized of models in my framework.

The intervention enacted during my study promoted the use of the empty number line within a realistic approach. Whilst the ideal was to make use of the two heuristics of guided reinvention through progressive mathematization and emergent models, the reality on the ground was that, given the need to 'reach certain given educational goals' and to 'plan instructional activities in advance' (Gravemeijer and Doorman, 1999, p.124), elements of a top-down instruction were inevitable.

6.2 Implications for the learning and teaching of additive relations

This study has shown that it is possible for learners to retain some learning even in a short term intervention such as this one based on 6 lessons. More importantly, it was found in this study that learners struggle with the arbitrary rules of 'borrowing' and 'carrying' that accompany the use of the column model when learners have to apply the column addition/subtraction algorithm that serves as its strategy. Furthermore, given its linear character the use of a relational model such as the empty number line can improve learners' appreciation of additive relations.

On the one hand, the anti-didactical approach of launching learners into the use of the formal column addition/subtraction method proved to be detrimental in that only nine of the twenty-two learners who used the column subtraction method for the last question in the test arrived at the correct answer. Amongst the incorrect responses six of them were 15. On the other hand, the empty number line has been confirmed in this study as an innovative mental tool that can be used to circumvent the pitfalls of a decomposition approach. It is a model that avoids the 'false reversal' (Klein, Beishuizen and Treffers, 1998) that is always lurking when the decomposition approaches are applied to a subtraction problem.

With the use of the empty number line 35% of the learners in this study were able to successfully mathematize to produce the correct answer. When compared with the 22.50% and 27.50% of learners who were successful in the pre-test and post-test respectively, this is a noteworthy result. There is a sense in which working with the empty number line also appeared to feed in more generally towards increased success with the use of more abstract models like the column model. So the moves forward, while not linear in relation to the forward use of the empty number line, may be more broadly useful at the levels of shifts towards more efficient models, and by extension, strategies. This shift is a welcome one especially in the face of the prevalence of the column model which, as noted earlier, is pushed by teachers as a model of choice as soon as learners enter Grade 4. It appears that the recommendation to have learners 'develop more efficient techniques for calculations, including using columns' (DBE, 2011b, p.13) as the number range for doing calculations increases is taken as a prescription to push the standard methods as the way to solving (often de-contextualized) problems from the very start of Grade 4. The admonition that 'these techniques should only be introduced and encouraged once learners have an adequate sense of place value and understanding of the properties of numbers and operations' (DBE, 2011b, p.13) appears not to be heeded.

Finally, with literature identifying the *compare* class of problems as being the most difficult to work with where learners are concerned, this study reveals that a bigger proportion of learners mathematized to arrive at the correct answer when using the empty number line as compared to other models. In the delayed-post-test all fifteen of the learners who used the empty number line to answer the compare problem in question 8 did so successfully. This is substantive considering that only eleven were successful in both the pre-test and the delayed-post-test.

6.3 Reflections and recommendations

In order to limit having to infer the strategy used by the learners, it would have been progressive to establish a well-structured interview which would serve the purpose of illuminating the strategies used by learners. This would require interviewing each learner on their working for every single question in the test. Given the short-term and the pilot nature of this study, it is recommended that such an investigation be unfolded over a longer period. Given the tentative gains made over six hours of contact, the results indicate that if the learners were to be immersed for a longer period of time the gains could be substantive. In particular, it would appear that attention to mathematizing and models in classroom can produce relatively quick improvements in performance on Grade 4 addition and subtraction.

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APPENDIX 1

The intervention prescribes four stages to each lesson, namely:

- Solving the Big Book problems
- Linking up the problems
- Follow-up problems
- Wrap-up

Lesson 1_Ugly bugs_7th March 2013

Stage	Discussion/ Tasks	
Solving the Big Book problems	The theme for the day (Ugly Bugs) is introduced and the meaning of words such as *bugs* and *logs* (and their connection) are discussed after the class has read through the first problem. Learners then attempt the introductory problems using the model of their choice.	Three contextualised *change increase* problems serve as introductory tasks.
Linking up the problems	The methods used by learners are collected on the board and discussed. Some have tally methods but partition and column methods prevail. Looking across the three problems learners' attention is drawn to the fact that in each case an initial amount with which we started was increased and hence the problems were all of the change increase class.	

Follow-up problems	Six contextualised problems serve as follow-up tasks. All except questions 4 and 6 are change increase problems, with question 4 being a *change decrease* problem, and question 6 asking which one of the questions before it is not an increase question.
Wrap-up	The first two questions are discussed. Two methods emerge for solving them. We agree to label the first one the column method and the second one the modified column method. Question 4 is discussed and identified as involving a different operation to the other questions: a minus operation is implied by the word 'ran off'.

Lesson 2_Collections_14th March 2013

Stage	Discussion/ Tasks	
○ Solving the Big Book problems	The theme for the day (Collections) is introduced and the meaning of words such as *collections* and *autographs* (and their connection) are discussed after the class has read through the first problem. Learners then attempt the introductory problems using the model of their choice, followed by the use of the empty number line.	Three contextualised *change increase* problems serve as introductory tasks.
○ Linking up the problems	The methods used by learners are collected on the board and discussed. Learners beginning to use the empty number line, so the setting-up is discussed. Looking across the three problems learners' attention is drawn to the fact that in each case an initial amount was increased and hence the problems were all of the *change increase* class.	

○ Follow-up problems	Six contextualised problems serve as follow-up tasks. All except questions 3 and 6 are *change increase* problems, with question 3 being a *change decrease* problem, and question 6 having learners make up an addition change problem beginning with the phrase 'Biddy planted 35 bean seeds . . .'
○ Wrap-up	Learners encouraged to look across the problems to see what is similar and what is different between them. The direction of the jump on the empty number line is discussed. Special attention is given to question 3 which is different from the rest as it is a take-away question, implying a backward jump. It is labelled as a *change decrease* problem as the initial amount was reduced by some amount.

Lesson 3_Champion Grannies_11th April 2013

Stage	Discussion/ Tasks	
○ Solving the Big Book problems	A drill-and-practice worksheet with fourteen partially drawn empty number lines is provided to learners to complete. The theme for the day (Champion Grannies) is then introduced and the meaning of words such as *slalom* and *cones* (and their connection to the contextualised problems) are discussed after the class has read through the introductory problems. Learners then attempt the introductory problems using the empty number line.	Three contextualised *combine* problems serve as introductory tasks.
○ Linking up the problems	The different ways with which learners are working with the empty number line are discussed. This is the second lesson in which the learners are using the empty number line, so its setting-up is elaborated upon. Looking across the three problems learners' attention is drawn to the fact that in each case there are two initially distinct sets which are subsequently brought together and hence that these are *combine* problems.	

○ Follow-up problems	Six contextualised problems serve as follow-up tasks. All except questions 4 and 6 are *combine* problems, with question 4 being a *separate* problem, and question 6 having learners look back over the five questions that precede it to identify the one question that is not a the *combine* problem and to explain why they think that it is not a *combine* question.
○ Wrap-up	Learners are encouraged to look across the problems to see what is similar and what is different between them. The similarity of this set of tasks is compared to the set of tasks encountered in the first two lessons. The direction of the jump on the empty number line is discussed. Special attention is given to question 4 which is different from the rest as it is a take-away question, implying a backward jump. It is labelled as a *separate* problem as the initial set was reduced by some subset.

Lesson 4_On the Shelf_25th April 2013

Stage	Discussion/ Tasks	
○ Solving the Big Book problems	As a warm-up exercise that is designed to alert learners to the friendly numbers of 5 and 10, the multiples of five are collected on the board as provided by the learners with a view to 'bridging through ten'.. The theme for the day is then introduced and the meaning of words such as *library* and *jumble sale* (and their connection to the theme for the day) are discussed after the class has read through the introductory problems. Learners then attempt the introductory problems using the empty number line.	Three contextualised *combine* problems serve as introductory tasks.
○ Linking up the problems	The different ways with which learners are working with the empty number line are discussed. This is the third lesson in which the learners are using the empty number line, so its setting-up and the use of the relevant strategy is discussed. In particular, the calculation strategy of *compensation* is introduced for the first time during this lesson. Looking across the three problems learners' attention is drawn to the fact that in each case there are two initially distinct sets which are subsequently whose total needs to be calculated. It is highlighted that the problems can all be classified as *combine* problems.	

○ Follow-up problems	Six contextualised problems serve as follow-up tasks. All except questions 5 and 6 are *combine* problems, with question 5 being a *change increase* problem, and question 6 having learners look at question 5 and to identify it as either a *change* or a *combine* problem and to explain their choice.
○ Wrap-up	Learners are encouraged to look across the problems to see what is similar between them so as to highlight the defining feature of all combine problems. The similarity of this set of tasks is compared to the set of tasks encountered in the Ugly Bugs lesson. The direction of the jump on the empty number line is discussed. Special attention is given to the strategies of *bridging through ten* and *compensation* as efficient ways of solving the given problems.

Lesson 5_Robyn the Girl Wonder_9th May 2013

<table>
<tr><th>Stage</th><th colspan="2">Discussion/
Tasks</th></tr>
<tr><td>○ Solving the Big Book problems</td><td>First the different classes of problems are recapped.

The theme for the day is then introduced and the notion of a superhero is discussed with several suggested by the learners including Spiderman.

The class is then instructed to read through the introductory problems and to attempt them using the empty number line. In the process of reading the problems the meaning of words such as sprouts is unpacked.</td><td>Three contextualised change decrease problems serve as introductory tasks.</td></tr>
<tr><td>○ Linking up the problems</td><td colspan="2">The different ways with which learners are working with the empty number line are discussed. This is the first lesson in which the majority of the questions involved subtraction. Again, the calculation strategy of compensation is reinforced.

Looking across the three problems learners' attention is drawn to the fact that in each case there was an initial amount which was reduced. It is highlighted that the problems can all be classified as change decrease problems. Learners were encouraged to keep the number of jumps to an absolute minimum by jumping in multiples of ten as opposed to making several jumps of ten.</td></tr>
</table>

○ Follow-up problems	Six contextualised problems serve as follow-up tasks with all except question 6 being change problems. Question 6 has learners finish off a change problem beginning with 'The amazing Rex was given 50 bones…'
○ Wrap-up	Learners are encouraged to look across the problems to see what is similar between them so as to highlight the defining feature of all *change decrease* problems. The character of this set of tasks is related to the set of tasks encountered in the Ugly Bugs lesson as a way of emphasising the backward jump associated with all subtraction questions on the empty number line. Again, learners' attention was drawn to the fact that the number of jumps could be minimized by jumping in multiples of ten as opposed to making several jumps of ten.

Lesson 6_Money Money Money_23rd May 2013

Stage	Discussion/ Tasks	
○ Solving the Big Book problems	The theme for the day is introduced and the different currencies that are used around the world are discussed. This is done with the help of an animation character with which learners are familiar: *Mook* (who travels around the world by bicycle). The class then has to read through the introductory problems and to attempt them using the empty number line.	Three contextualised *change decrease* problems serve as introductory tasks.
○ Linking up the problems	The different ways with which learners are working with the empty number line are discussed. This is the fifth lesson in which the learners are using the empty number line, so its setting-up and the use of the relevant strategies is discussed. In particular, the *commutativity* of addition is discussed for the first time during this lesson: a subtraction question can be approached additively. Looking across the three problems learners' attention is drawn to the fact that money can be made or it can be spent, implying *change increase* or *a decrease*, and hence addition and subtraction respectively.	

○ Follow-up problems	Six contextualised problems serve as follow-up tasks. All except questions 4 and 6 are *change decrease* problems, with question 4 being a *change increase* problem, and question 6 having learners look at question 4 and to identify why it is still a change problem.
○ Wrap-up	Learners are encouraged to look across the problems to see what is similar between them so as to highlight what sets *change increase* problems apart from *change decrease* problems. The character of this set of tasks is related to the set of tasks encountered in the Ugly Bugs lesson as a way of emphasising the forwards jump associated with change increase problems on the one hand, and the backward jumps associated with all subtraction questions on the empty number line on the other. Learners' attention was drawn to the fact there are words in the problem that connect with the operation involved; like 'bought' and 'spent' for decrease, for instance