TEACHER PROFESSIONAL NOTICING AND MATHEMATICAL KNOWLEDGE FOR TEACHING DIRECTED NUMBERS AT GRADE 8

A research report submitted to the Faculty of Humanities, School of Education, University of the Witwatersrand, Johannesburg, in partial fulfilment of the requirements for the degree of Master of Education

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ABSTRACT

This study investigated teachers' professional noticing skills and mathematical knowledge for teaching (MKT) directed numbers at grade 8. The focus was on the professional noticing skills and MKT that teachers displayed in their teaching as well as how their level of professional noticing and MKT can be explained. A qualitative case study design was employed for this study. The study drew its theories and analytical tools from Thomas, Jong, Fisher and Schack's (2017) professional noticing and MKT framework. This framework informed this study in formulation of research questions, data analysis and presentation of findings. Data for this study was a secondary data whose primary source was the project Mathematics Performance Lag Advance Programme (MPLAP). The lesson transcripts of two teachers involved in the MPLAP project were the sources of data for this study. Deductive content analysis was conducted on the data to provide answers regarding teachers' professional noticing skills and MKT as well as explaining each teacher's level of professional noticing and MKT.

The findings of this study revealed the differences in teachers' professional noticing skills and MKT. The difference was in the way teachers implemented their instruction, how they adjusted tasks during the teaching and learning process and whether their instructional practices linked to learners' mathematical learning progression. Relying on chorused responses from learners seemed to affect teachers' professional noticing skills and MKT.

DEDICATION

This study is dedicated to my late father, Mr. Sekekane Michael Liau. I know how much he would appreciate my achievement and above all living in his footsteps as a teacher.

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ACRONYMS AND ABBREVIATIONS

CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
LoLT	Language of Learning and Teaching
TIMSS	Trends in International Mathematics and Science Study
MPLAP	Mathematics Performance Lag Advance Programme
PCK	Pedagogical Content Knowledge
MKT	Mathematical Knowledge for Teaching
CCK	Common Content Knowledge
HCK	Horizon Content Knowledge
SCK	Specialized Content Knowledge
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In recent years, research in mathematics education has argued for teaching that is responsive to learners' mathematical contributions in their learning process. Such teaching includes among others, teacher's knowledge of "mathematics for teaching (MKT)" (Ball, Thames, & Phelps, 2008) and skills to engage in professional noticing (Jacobs, Lamb & Philipp, 2010) of learners' thinking. So, this study explores teacher professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8.

In this chapter, I introduce the proposed study by starting with the background of the study, the reasons for focusing at directed numbers, the problem statement, the purpose of the study, the research questions, the significance of the study and lastly the rationale for the study.

1.2 Background

It has been emphasized that to teach a subject matter, a thorough knowledge of that subject and the expertise on how to effectively teach it to learners are required (Shulman, 1986; Nesher, 1987). This amalgam of the knowledge of content and how to teach it continues to be indispensable in the teaching and learning situations. For instance, Ball (2003) emphasized that teachers should know beyond how to multiply 0.3 by 0.7 to obtain 0.21. They should as well be able to explain and justify this algorithm to students. Hill and Ball (2009, p.67) also argued that "good teachers know both content and how to get it across to their students". A similar argument is also raised by Brodie (2014, August 8) who states that,

> teachers need a strong grounding in the content knowledge that they need to teach, … In addition, teachers need pedagogical content knowledge, which enables them to represent disciplinary knowledge to learners and interact with learners' understandings of the knowledge. (p.39)

These arguments stress on the need for teachers to be equipped with the content knowledge and skills on how to teach it effectively to learners.

However, it appears that it is not always the case that teachers may have an in-depth knowledge of the content knowledge and relevant skills. It may happen that a teacher masters the content he or she is supposed to teach but lacks the skills on how to get it across to the learners. For example, a grade 7 teacher (Ms. González) in Hill and Ball's (2009) work is a clear indication of such teacher. Ms. González had the task for her class to use a chip representation model to work out $-1 - (-3)$(see Appendix C). She had requested her learners to attempt to the question first on their own. The "Black chips were used to represent positive numbers and red chips for representing negative numbers" (Hill & Ball, 2009, p. 68). The following was noted by Hill and Ball (2009) as responses by learners to the task given;

> a student ventures that the answer is -1; another proposes that the answer is 5; and a third argues for an answer of -2. Many more note that matching a black with red leaves four reds, or a result of -4. (p.68)

The following actions are described by Hill and Ball (2009) as taken by Ms. González against the learners' responses. Firstly, Ms. González did not attend to the learners' responses to establish their underlying thinking instead she checked for the answer in the textbook to realize it was given as 2. By so doing it shows that Ms. González relied on the final or given answers from the book rather than working with the responses given by her learners. It is also alleged that she modeled the situation herself and successfully obtained $-1 - (-3)$ but struggled to explain to learners how the representation (red chips and black chip) could be used to obtain the answer. She instead used her own understanding of how to carry out the operation to demonstrate to learners how to perform the operation to get the answer 2 (Hill & Ball, 2009).

This scenario of Ms. González shows that she was clear on how to perform the basic operations on directed numbers. She however lacked expertise on how to meaningfully teach the concept to the learners which included the use of the chip model for demonstrating how the operations could be carried out. Ms. González lacked a skill that is very crucial for teachers which differentiates her from any person knowing the subject but not for the purpose of teaching it. She failed to engage with the learners' responses which would enable her to establish what errors or misconceptions they might be having (Ball et al., 2008). By engaging with learners' errors this was going to help her improve her instruction as well as to assist them to understand the concept better (Borasi, 1994; Makonye & Khanyile, 2015; Nesher, 1987; Smith, DiSessa & Roschelle, 1993). Again, failure to engage with learners' responses characterize Ms. González as lacking an important teaching skill which is to notice learners' thinking or reasoning about mathematics (Barnhart & van Es, 2015; Jacobs et al., 2010; Thomas, Jong, Fisher & Schack, 2017; Wessels, 2018). The teacher must notice what learners are thinking as well as their reasoning so that he or she can decide as to how to move them forward in their learning. Lastly, by working out the answer using her own method without making use of the representation, Ms. González is depriving her learners the opportunity the representation would have in mediating the learning of the operation (Ball et al., 2008).

While Ms. González's situation may seem unique to her only, however this study believes such challenges are faced by many teachers. Teachers may have good knowledge of content they are to teach but lack the teaching skills to best support learners in their learning process. In this reality, much will be needed to support teachers to improve their pedagogical skills. This may be by attending to their professional noticing skills (Jacobs et al., 2010) and MKT. Professional noticing as described by Jacobs et al. (2010) involves teacher's skills such as attending and interpreting learner's mathematical understanding and thus deciding how to advance their learning based on their understanding. MKT is described by Ball et al. (2008) as a theoretical framework detailing all the "mathematical knowledge necessary for teaching mathematics" (p. 398). This may include knowing mathematical terminologies, notations used, correct use of notations and terminologies, and those skills necessary to support students to effectively learn the subject. So, this study intends to investigate the teachers' professional noticing skills and their mathematical knowledge for teaching directed numbers at senior phase.

1.3 Why directed numbers?

McDonald (2011) defines directed numbers as numbers that have "both direction and magnitude or size" (p.2). The magnitude is determined by the number while the direction by using the signs; positive and negative. When one sign is used to specify a certain direction, the other will denote the opposite direction. For example, "if right is used as positive then left is treated as negative" (McDonald, 2011, p. 2). This can also be generalized to other opposite directions or situations such as; East and West, South and North, before and after, above and below. Rowland (1982) notes that if the numbers are limited to negative and positive whole numbers, those are called integers. So, in this study the terms directed numbers and integers are used with a similar meaning of referring to negative and positive whole numbers including zero.

Directed numbers have a significant number of practical uses. Makonye and Fakude (2016) state that they can be used for representing temperatures and also in book-keeping where "amounts owed are often represented by red numbers or a number in parenthesis as an alternative notation to represent negative numbers" (p.2). Makwarise (2016) asserts that "the relevance of directed numbers also extends to the business sector where profit is considered as having a positive impact and loss as having a negative impact in the company's working faculties" (p.13). In the field of technology, integers have become applicable in network coding purposes (Gennaro, Katz, Krawczyk, & Rabin, 2010).

In school mathematics, directed numbers appear to be foundational for most mathematics topics in the curriculum. Makwarise (2016) states that "a student who fails to master the concept of directed numbers will find it difficult to understand [other] mathematical aspects" (p.13). In Peled and Carraher's (2008) work on *signed numbers and algebraic thinking*, they concluded among other things that signed numbers "contribute to the understanding of algebraic concepts" (p.325). For instance, they argued that "the study of functions is more complete, and meaningful with the extension of the number line to include negative numbers promoting the habit of checking different cases" (Peled, & Carraher, 2008, p. 325).

However, understanding directed numbers and operations on directed numbers may be a struggle for some learners (see for example Peled, & Carraher, 2008; Vlassis, 2008; Soga, 2017). In particular, Soga (2017) associates learners' difficulties regarding their understanding of integers with "the incorrect methods of teaching integers, learners' errors when working with integers or the language of learning and teaching (LoLT) used" (Soga, 2017, p.1).

1.3.1 Directed numbers in the CAPS curriculum

Negative numbers in the 'Curriculum and Assessment Policy' (CAPS) are first introduced at senior phase beginning from grade 7 (See appendix D for the breakdown of how the topic is taught from grade 7 to grade 9). At grade 7 learners begin the study of integers with a focus on counting them forward and backwards as well as recognizing, ordering and comparing them. Since this is the first time learners encounter the signed numbers they have to understand how they fit into the counting system they are already familiar with. They ought to be able to compare them, particularly in terms of size and hence be able to order them. The CAPS further stipulates that learners should learn or be taught how to calculate integers. At grade 7 they are to deal with only addition and subtraction while from grade 8 they are to perform all the four operations on integers (DBE, 2011). While learners have already been introduced to the four operations on positive whole numbers at the earlier phases, at this phase they are to extend their understanding to include the negative signed numbers.

Furthermore, CAPS requires that at all grades (7, 8 & 9), what has been covered or dealt with be used in solving contextualized problems. That is, learners are expected to use or apply what they have learned in solving problems in context (DBE, 2011). This encourages learners to realise the practicality of what they are learning in contexts familiar to them and thus enhancing their understanding of the topic. Also, this can help learners realise the importance of integers in their own life situation.

Just as the research of Peled and Carraher (2008) and Makwarise (2016) has argued; directed numbers are foundational to other topics in mathematics curriculum, particularly algebra, the sequencing of directed numbers and algebra in CAPS conforms to that. In all the three grades directed numbers are always taught first before algebra. This shows that CAPS also recognises the importance of integers in teaching and learning other topics in the curriculum.

Taken together, the aforementioned importance of directed numbers in practical situations as well as in the mathematics curriculum, specifically in CAPS curriculum, and the struggles learners face in understanding directed numbers, motivated me to focus my study in this area at senior phase grade 8. The choice of grade 8 is motivated by the fact that it is at the early stage of introduction to the signed numbers. Makonye and Fakude (2016) emphasize that if directed numbers are not attended well at grade 8, that might result in learners' poor performance on mathematics and other mathematics related subjects.

1.4 Problem statement

The practical importance of directed numbers as well as their importance in the school curriculum has been highlighted in the previous section. However, students do not find this concept of directed numbers easy to understand (see for example, Bofferding, 2010; Stephan & Akyüz, 2012). The same challenge is also alluded in studies focusing on teaching and learning of directed numbers in South Africa (see for example, Makonye & Fakude, 2016; Soga, 2017). Makonye and Fakude

(2016) studied grade 8 learners' errors and misconception on adding and subtracting directed numbers. Their findings showed that 83.3% of a sample of 35 grade 8 students that were studied had misconceptions in the learning of integers. It was found that though some learners answered the tasks given correctly in their books, when interviewed it was revealed that they had misconceptions. For instance, it is reported that learners believed that -9 is greater than -3 which was concluded after one learner was asked about this and he or she indicated "9 is more than 3, so −9 is more than −3" (Makonye & Fakude, 2016, p.8). This is a clear indication of a misconception in learners' understanding of intergers. Soga (2017) identified learners' understanding of zero as the smallest number as one misconception that was very common. According to Soga (2017), this kind of thinking makes it hard for learners to accept that positive numbers are greater than negative numbers.

The challenges with understanding directed numbers are also likely to affect learners' understanding of other topics such as algebra. Bush and Karp (2013) argue that "conceptual understanding and procedural fluency with integers are widely seen as important to student success in algebra" (p.618). Similarly, Peled and Carraher (2008) assert that the understanding of signed numbers; or more specifically the directed numbers play a role in learners understanding of algebra. According to Peled and Carraher (2008) the converse is also true. They argue that while the understanding of signed numbers is a base for understanding algebra but also algebra "provides a helpful context for introducing signed numbers" (Peled & Carraher, 2008, p.325). That is the understanding of the two topics provides a mutual support for each other. These claims show that if learners fail to understand one of these topics they are likely to do the same with the other one. In particular, this study is much concerned with failure on directed numbers which might impact negatively on algebra understanding.

The concern with failure on directed numbers thereby having the same impact on algebra is justified by the learners' challenges as reported in Vlassis's (2008) work. Learners had challenges when they were to check their solutions to the equations in which a subtraction sign was to be followed by a negative sign. When dealing with the equation $4 - x = 5$ (Vlassis, 2008), learners had challenges when checking their solution by substituting $x = -1$ in the equation since it led to $4 - -1 = 5$. These learners were unable to differentiate the functions of a minus sign thereby being confused as to how $4 - -1$ can give 5 (Vlassis, 2008; Bofferding, 2010).

Furthermore, a lack of understanding of directed numbers on algebra understanding can be argued to be the cause of a challenge raised by the grade 12 examiners in the end of year examinations reports (see for example DBE, 2016; DBE, 2017; DBE, 2018). The diagnostic reports argue for learners' poor performance on algebra as caused by their lack of fundamental concepts. The examiners state that:

> The algebraic skills of the candidates are poor. Most candidates lacked fundamental and basic mathematical competencies which could have been acquired in the lower grades. This becomes an impediment to candidates answering complex questions correctly (DBE, 2018, p.133).

The fundamental and basic mathematical competencies that students lack as stated in the reports can be considered inclusive of knowledge of directed numbers argued by Peled, and Carraher (2008), Bush and Karp's (2013) as well as Vlassis (2008). The examiners in the reports highlight the need for teachers to "emphasise the importance of the sign of the middle term in factorising quadratic equations" (DBE, 2018, p.135). The problem with the sign in the middle term may be due to an insufficient understanding of directed numbers. Thus, this may suffice to argue that the fundamental mathematics competencies the examiners are referring to, are inclusive of directed numbers. Taken together, the findings of the diagnostic reports and the studies by Makonye and Fakude (2016) and Soga (2017) indicate the problem with understanding directed numbers for South African learners.

In the light of the discussion presented above, the study argues that there is a problem regarding learners' understanding of directed numbers which might impede their learning of algebra as one example of topics that directed numbers are prerequisite to. This problem has a possibility of affecting learners' future opportunities as Moses and Cobb (2001) assert that algebra is a gate keeper subject or gateway for future opportunities. Again, learners' failure to understand directed numbers may create learning deficits (Taylor, Muller, & Vinjevold, 2003) as they progress to other topics within the school curricular. The consequence of this as Spaull and Kotze (2015) argued is that such learners experiencing the learning deficits will likely fall behind relative to the curriculum and thereby experiencing challenges in their learning process.

In this study, the teachers' mathematical knowledge for teaching and how they notice their learners' mathematical thinking are regarded as factors that contribute to the challenges of learners discussed above. The case of Ms. González described above has shown that indeed for some teachers this may be the case. Teachers may lack the mathematical knowledge for teaching and skills to notice their learners' mathematical thinking especially in the teaching of directed numbers. Thus, they may fail to "identify and act upon salient mathematical actions of children" (Thomas et al., 2017, p. 5) during their teaching which would help in maximizing learners' understanding of the topic while at the same time being responsive to learners' mathematical productions.

1.5 Purpose of the study

The purpose of this study was to investigate the teachers' professional noticing skills and their mathematical knowledge for teaching directed numbers at grade 8. Thomas et al.'s (2017) framework on professional noticing and mathematical knowledge for teaching was used to explore teachers' practices aimed at supporting learners in the classroom teaching and learning space with an intention to explain their level of professional noticing and mathematical knowledge for teaching directed numbers. To achieve this, an interpretive paradigm was chosen to inform the study with a qualitative case study employed as an approach to study the cases of two teachers (T3 and T4) from two different schools in the inner city of Johannesburg. Because of the challenges presented by the covid-19 pandemic, secondary data in the form of lesson transcripts of the lessons taught by the teachers was used. Thus, the study adopted a document analysis approach for analysing data.

1.6 Research questions

The study was conducted to answer the following research questions;

1. What professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8 are displayed by the teachers?
2. How can the levels of the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 be explained?

1.7 Significance of study

Lai and Tessol (2016) point out that research in education has three importances: to enrich knowledge, to promote school practice, as well as to inform policy debates. They further explain that educational research has the ability to expand on the existing knowledge around the subject and in turn help in broadening educator's pedagogical strategies. This could be understood to mean that if research can establish, for example, the challenges around the teaching and learning of a particular topic such knowledge could be used by educators to improve their pedagogical strategies. Consequently, with improved pedagogical strategies students' learning or achievement can also be expected to improve. This matter is supported by the work of Pournara, Hodgen, Adler & Pillay (2015) on grade 10 algebra, functions and geometry. Their research revealed that by attending to teachers' mathematical knowledge for teaching grade 10 algebra, functions and geometry there was a slight gain in learners' attainment.

For the reasons stated above, the study is significant because it intends to investigate teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8. It is hoped that the findings of this study will contribute in expanding on the knowledge around the teaching of directed numbers. This will in turn help educators to improve on their teaching strategies so that the challenges faced by learners with regard to directed numbers are dealt with.

1.8 Rationale

The poor mathematics performance of learners in South Africa has been revealed by studies such as the cross-national study 'Trends in International Mathematics and Science Study' (TIMSS). TIMSS can be regarded as a benchmark to gauge performance of learners across participating countries. Throughout the years that TIMSS has been administered (1995 to 2015), South Africa has been ranked low compared to other participating countries (see for example Spaull, & Kotze (2015); Reddy, Juan, Isdale, & Fongwa, 2019). Drawing from the TIMSS's results of 2011 on grade 9 learners, Spaull, and Kotze (2015) argued that learners' low performance was not because the tests administered were difficult but it was the result of learners falling behind relative to the curriculum. They claimed that learners who are experiencing this lag are the majority of South African learners who happen to be from low socioeconomic status (Spaull & Kotze, 2015). Their findings indicated that at:

> Grade 3, children in Quintiles 1-3 are already three years' worth of learning behind their Quintile 5 peers and that this gap grows as they progress through school to the extent that by Grade 9 they are four years' worth of learning behind their Quintile 5 peers. (Spaull, & Kotze, 2015, p.26).

It becomes crucial therefore that this learning gaps experienced from disadvantaged communities are also addressed to afford such learners similar opportunities to their peers from middle to high socioeconomic status. One way this can be achieved is by investigating teachers' MKT as well as their professional noticing skills when teaching topics that are problematic to learners such as directed numbers. Teachers' MKT and professional noticing skills are some of the knowledge and skills that this study finds a need to be investigated with a hope that the knowledge established can be used in improving the quality of instruction and learners' achievement. In their work, Hill et al. (2008) found that mathematical knowledge for teaching in particular, does correlate with quality of mathematical instruction. Other researchers (e.g. Hill, Rowan & Ball, 2005; Pournara et al., 2015) observed that teachers' mathematical knowledge for teaching also has a significant impact on learners' mathematical achievement. Thomas et al. (2017) argued that professional noticing is a responsive instructional strategy that helps a teacher to support each learner's needs as they emerge during their learning process.

So, in light of the findings given above the study has a motivation to investigate teachers' mathematical knowledge for teaching and how they professionally notice their students' mathematical thinking in the teaching of directed numbers at grade 8.

1.9 Study Outline

This study comprises of six chapters. Chapter one introduces the study by describing the background, the problem to be investigated, the need to focus on directed numbers, the purpose of the study, its significance and rationale.

Chapter two reviews literature on the development and understanding of directed numbers, difficulties learners experience with directed numbers, some common errors and misconceptions associated with directed numbers, mathematical knowledge for teaching, MKT and the quality of instruction, MKT and student achievement, and professional noticing of learners' mathematical thinking together with its impact on learners' learning. Thomas et al.'s (2017) professional noticing and mathematical knowledge for teaching theoretical framework which underpins this study is also discussed in chapter two.

Chapter three discusses the choice and justification of the methodology for this study. Data used for this study is secondary data; therefore, chapter three also describes in detail the source of this data and details surrounding its collection. It further explains issues of ethical consideration for conducting the study as well as reliability and validity of the findings. How data will be analysed including its limitations are also explained in this chapter.

Chapter four presents analysis of data, research findings and the discussion. Lastly is chapter five which is about the conclusion and recommendation made by this study

1.10 Conclusion

This chapter has described why this study is carried out. It began with a description of the background highlighting the importance of MKT most especially on teachers' need to adhere to learners' work and professional noticing of students' mathematical productions. The need to focus on directed numbers at senior phase was also described. The chapter has also presented the problem that it intends to investigate as well as the research question that it intends to answer. The purpose, significance as well as rationale of the study have also been described. The chapter has also presented the outline of the study.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 Introduction

In this chapter, I review the literature related to directed numbers. In particular, I review literature on the development and understanding of directed numbers to help understand why they are introduced at grade 7 in CAPS. The difficulties learners experience when learning directed numbers as well as the common errors and misconceptions students are likely to make as they study directed numbers are reviewed from the literature to help understand what challenges teachers are likely to face in their teaching. The review of literature on teaching models or strategies for directed numbers is also done. Also, the review on MKT and the quality of instruction, MKT and student achievement, professional noticing of learners' mathematical thinking and professional noticing of learners' mathematical thinking and its impact on learners' learning is done. The chapter concludes by discussing Thomas et al.'s (2017) professional noticing and mathematical knowledge for teaching theoretical framework which underpins this study.

2.2 Development and understanding of directed numbers

As indicated earlier, in the South African curriculum CAPS, learners are first introduced to negative whole numbers at grade 7 (DBE, 2011). Throughout their entire schooling prior to grade 7, they are working with only positive whole numbers. The extension to include negative whole numbers introduces learners to a new system of numbers which are integers (Rowland, 1982). Bofferding (2014) states that learners can make sense of the negative numbers by considering them as directed magnitudes (McDonald, 2011). Learners build the idea of magnitude from working with positive whole numbers, so for them to interpret the negative numbers is if the attached negative is understood to specify the direction. By considering the sign negative and also positive as specifying the direction, the whole set of negative and positive numbers including zero is now called directed numbers McDonald (2011).

As Bofferding (2014) stated, research findings on number and operations seem to differ on the grade level students start to understand the negative numbers. Bofferding (2010) conducted a research on second grade learners to determine their understanding of the changing meaning of the minus sign and also their understanding about adding and subtracting negative numbers. The findings indicated that learners at that grade were able to: 1) give an insightful and complicated reasoning about negative numbers, 2) form ideas about multiple roles of a minus sign, lastly; 3) relate operations on negative numbers - with their own prior knowledge on addition and subtraction of positive whole numbers. The study even further suggested that the negative numbers be introduced earlier in the curricula.

In the study on ‘obstacles and affordances for integer reasoning’, Bishop et al. (2014) found the grades 1 to 4 learners who participated in the study could correctly solve 3-5 by counting backwards on a number line. Of particular interest is a grade 1 learner (Lucy) in Bishop et al.’s (2014) work who used a number line to work out 3-5. She counted five steps backward from 3 to come to a number less than zero that she represented as “02” instead of -2 (p.48). The learner proved to understand the existence of numbers less than zero and decided to use her own notation to denote them. In another study conducted by Whitacre et al. (2012) it found that grade 1, 3 and 5 learners were able to reason about opposite context magnitude with sophistication increasing from younger to older. They further argued that children “have intuitions that can potentially support their later reasoning about integers in sophisticated ways” (Whitacre et al., 2012, p.365). Bishop (2011) also indicated that they “found that six- and seven-year-olds were quite ready to grapple with negative numbers and displayed a wide range of understanding” (p. 356). The findings further suggest that “given the right tools and opportunities young children can reason in powerful ways about negative number” (Bishop, 2011, p. 358).

While the introduction of directed numbers begins at senior phase in the CAPS, the review of literature points out that at any grade level learners can start to work with negative numbers. It clearly emphasizes that what is important are the supporting tools and opportunities afforded to learners to grapple with the concept. Thus, this highlights the importance of the teaching that is responsive to learners’ thinking to best support them in their learning.

2.3 Difficulties learners experience with directed numbers

The extension of positive whole numbers to include negative whole numbers poses several challenges to learners (Bishop et al.,2014; Stephan & Akyüz, 2012). Bishop et al. (2014) discusses four cognitive obstacles that they claim to be the source of these students’ difficulties regarding integers. The first obstacle pertains to a view of numbers in terms of magnitude only. Learners’ first experience with numbers is for numeration and counting (Bishop et al., 2014) and thus when negative numbers are introduced; student struggle to understand what they mean in terms of magnitude. The second cognitive obstacle concerns recognition of numbers as representing tangible or concrete situation. Consequently, considering negative numbers as those numbers less than zero (nothing) creates a conflict on what tangible or concrete situations do they represent (Bishop et al., 2014). Bishop et al. (2014) state that: the third cognitive obstacle relates to conceiving subtraction as taking away or removing (Gallardo & Rojano, 1994; Vlassis, 2008). Learners find it hard to understand how taking away a bigger number from a small number e.g. 3-7 or removing a number from nothing say 0-4, is a meaningful operation to do (Bishop et al., 2014; Stephan & Akyüz, 2012). Instead, when they are to work out such numbers, they would rather reverse the operation to 7-3 or 4-0 which would then be easy for them to work out (Bofferding, 2010; Stephan & Akyüz, 2012). The last obstacle relates to learners understanding of addition as yielding large numbers and subtraction resulting in small number. So, with an introduction of negative numbers, learners get challenged with the fact that now addition can result in a smaller number while subtraction can result in a bigger number (Bishop et al., 2014; Bofferding, 2010; Bolyard & Moyer-Packenham, 2012; Carraher, & Schliemann, 2002).

Vlassis (2008) found out that learners had difficulties understanding the function of a negative sign. These functions are identified by Gallardo and Rojano (1994) as unary, binary and symmetric. Drawing on Gallardo and Rojano's (1994) work, Vlassis (2008) explains the unary function of a minus sign as to form a negative number by attaching it to the number, while the binary function "relates to a minus sign as an operational sign" (p.561) and the symmetric function which considers a minus sign as an operator for taking away the opposite. Vlassis (2008) observed that when students were to solve the equation $-6x = 24$, they had difficulties of understanding the "unary function of the sign attached to the solution -4" (p. 569). Learners were also observed to fail to distinguish the different functions of a negative sign in the case where the signs appeared to follow each other e.g. $4 - -5$. They were unable to realize the binary function of the first minus sign as opposed to the unary function of the second minus sign (Bofferding, 2010; Vlassis, 2008). According to Bofferding (2010) students consider the negative signs as resembling the binary function only, that is they are all subtraction signs.

2.4 Errors and misconceptions on directed numbers

Some challenges which learners encounter working with directed numbers concern the errors and misconceptions they have. An error is defined by Goswami (2018) as "a result of carelessness, misrepresentation of symbols, a lack of knowledge of that particular area, or a task that is far too demanding of the child's current level" (p.49). As Nesher (1987) pointed out learners make errors as part of their learning process. As such, they should not be seen as "undesirable" (Makonye & Hantibi, 2014, p.1565) because they are a reflection of their participation in the learning situation.

Five common errors were identified by Makonye and Hantibi (2014) to prevail in students' written work. Such errors included "systematic errors, careless errors, transformation errors, process skills errors and encoding errors" (Makonye & Hantibi, 2014, p.1569). A *systematic error* is one where a certain incorrect response appears to constantly repeat for a particular algorithm (Cox, 1975; Makonye & Hantibi, 2014). The systematic error cannot be generalized on a single learner response, it has to be an error observed as common for a particular algorithm and for a number of students. For instance, 2 and 3 seemed to be the common responses for the question 27-29 in Makonye and Hantibi's (2014), this lead them to conclude that student had a systematic error.

Careless error on the other hand is one that occurs even when the student knows how to get the correct answer at the time he is giving the incorrect answer and can be expected to give a correct answer to the same question should it be asked later (Makonye & Hantibi, 2014). This kind of error students commit unconsciously. A student in Makonye and Hantibi (2014) obtained the answer 3 subtracting 27 from 29 but later came to realize the error committed and corrected the response to get the correct answer 2. A *transformation error* is described by Kristayulita (2017) as one that concerns "failure to identify the correct operation or a sequence of operations even when the learner understood what the question required" (p.20). This can be observed when a learner gets -6 or 6 as the answer to $(-2) \times (-4)$. The learner is adding instead of multiplying (Makonye & Hantibi, 2014) which indicates that he or she failed to recognize the correct operation. In other cases, a learner may know what operation to use but fail to understand the procedure that will lead him or her to the correct answer and this kind of error is called *Process skills error* (Kristayulita, 2017; Makonye & Hantibi, 2014). For example, when working out $14 \div (-2)$ a learner gave 7 not -7 (Makonye & Hantibi, 2014). In this case a learner is aware of the operation to use but is confused of the sign to use. *Encoding error* is defined in Kristayulita's (2017, p.20) work as occurring when the student “correctly solves the problem but fails to present the solution in an appropriate form or using an appropriate notation”.

Other types of errors learners make when working with directed numbers are discussed by Makonye and Fakude (2016). Drawing on Schoenfeld and Kilpatrick’s (2008) five strands of mathematical proficiency, they categorized learner errors as procedural errors, strategic errors and logic errors. Procedural errors “are about failure of the learners to manipulate the signs in front of the numbers in conjunction with the sign in operation, which is either a plus or a minus” (Makonye & Fakude, 2016, p.7). Strategic errors involved such errors as those arising from failure of students to “formulate the setup of the numbers, and represented them correctly” while logical errors are committed when “the learner could not display his or her capacity to think logically about the relationship among concepts and situation” (Makonye & Fakude, 2016, p.5).

Makonye and Hantibi (2014) emphasize that errors committed by learners are due to misconceptions that they may be having regarding a particular mathematical concept. A misconception is defined as the underlying faulty thinking or understanding that may yield a series of errors (Nesher, 1987). Misconceptions are a reflection of what a learner knows about the concept and as such they can be productive in the teaching and learning situation (Borasi, 1994; Goswami, 2018; Makonye, & Khanyile, 2015; Nesher, 1987; Smith et al., 1993).

On the whole, Makonye and Hantibi (2014) found the following misconceptions learners had about operations on directed numbers: the misconception of overgeneralization, the misconceptions of interference, and the misconception of meaning. Learners are considered to have a misconception of overgeneralization if a rule that applies for a certain situation is considered to apply even for other situations where it doesn’t hold true. Makonye and Hantibi (2014) found that learners overgeneralized the operation of addition even for situations where they were to multiply e.g. “a learner can say 2x3 =5” (p.1565). they also found that when working out 27-29 learners obtained 2 which implied that they overgeneralized the commutative property of addition to subtraction (Makonye & Hantibi, 2014). The misconception of interference was observed where learners gave the answer -8 when working out $-2 \times -2 \times -4$ (Makonye & Hantibi, 2014). This shows that learners’ understanding of addition interfered with the new skill of multiplying directed numbers. The misconception of meaning was observed where learners got $-3a$ or 5 instead of leaving the answer as $-a+4$ (Makonye & Hantibi, 2014). In this case learners understand the final answer to have one term so they considered $-a+4$ not representing the final answer and needed to be completed.

2.5 Teaching models for directed numbers

The introduction of negative numbers in the school curricula poses a lot of challenges for learners as the review of literature indicates. To assist students against these challenges, researchers developed teaching and learning models as well as designing real-world contexts that they could use to teach directed numbers meaningfully (Stephan & Akyüz, 2012). Stephan and Akyüz (2012) categorized the models into; Neutralization models which use a principle of cancellation of integers and a number line which uses position and movements to mediate understanding of the signed numbers.

2.5.1 Neutralization models

Neutralization models involve the use of the concept of opposite units that neutralize or cancel each other. For instance, a negative and a positive charge. The combination of these two charges results in a neutral charge or they can be said to cancel each other's charge. Examples of the neutralization models include; the use of different coloured chips such as the red and black chips used by Ms. González in Hill and Ball's (2009) work (see Appendix C), helium-filled balloons and sandbags by Janvier (1985) cited in Schwarz, Kohn and Resnick (1994), money technique (Chang, 1985), positively and negatively charged particles (Battista, 1983) etc.

2.5.1.1 The use of different coloured chips

The idea of using coloured chips to help learners understand operations on directed numbers is found in several studies (see for example Battista, 1983; Kohn, 1978; Hill & Ball, 2009). Kohn (1978) used the red and yellow colours for the chips. The red chips represented positive numbers while the negative numbers were represented by the yellow chips. Battista (1983) on the other hand used white coloured chips for the positive charges and red chips for the negative charges. Ms. González in Hill and Ball's (2009) research used the red chips for negative numbers and black chips for positive numbers. From these three researches it is evident that no particular colours are the only ones to be used or designated for a particular charge or sign of number. The underlying concept in all these researches is that opposites add up to zero. That is a pair of different colours always yields a zero.

Kohn (1978) used the chips to explain how addition of integers can be done. This can be done by summing up or bringing together the different colours representing the numbers to be added (see figure 2.5.1.1).

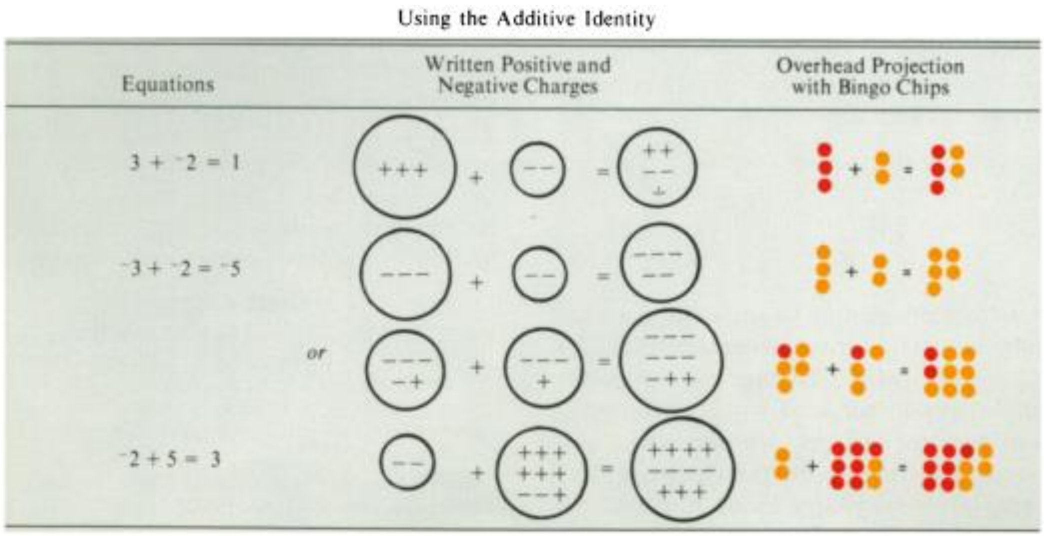


Figure 2.5.1.1: Using coloured chips to add integers. Source: Kohn (1978, p.735)

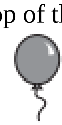
After combining the chips, pairs with different colours cancel out and the answer will be that of the remaining chips. For instance, in figure 2.5.1.1 a positive 3 and a negative 2 are added by taking three red chips and joining them with two yellow chips (Kohn, 1978). In the results two pairs with different colours cancel out leaving one red chip, which means the answer is negative 1.

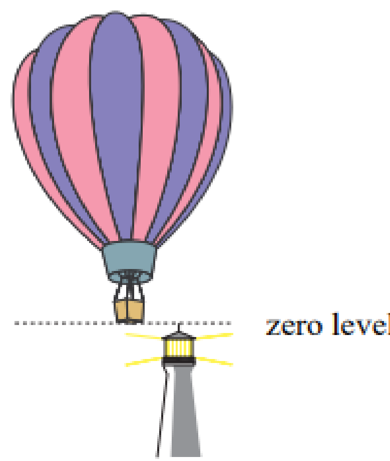
In the case of subtraction, the difference is that the action of 'take away' is used instead of joining (Battista, 1978; Hill & Ball, 2009). Considering the case where a greater number is to be taken away from the small number such as "5 – 8" (Stephan & Akyüz, 2012, p.431), Kohn (1978) explains that zero pairs will have to be added so that the subtrahend can be taken away from the minuend. What remains after taking away the subtrahend is the answer.

Kohn (1978) also explained how multiplication of integers can be done. According to Kohn (1978) multiplying a positive number by a positive number as well as multiplying a negative number by a positive number can be done using the idea that multiplication is repeated addition. For instance, 3×-2 can be expressed as $3 \times -2 = -2 + -2 + -2$ and then added the way negative numbers are added (Kohn, 1978). The difference lies with multiplying two negative numbers. As Kohn (1978) explains, one has to start first with the number of zeros that will allow "for the removal of negative charges the required number of times" (p.736). For example, to multiply -3 by -2, one can add six zeros or more and then remove -2 three times and the remaining chips will be equal to the answer.

2.5.1.2 The use of helium-filled balloons and sandbags

Janvier (1985) cited in Schwarz et al. (1994) explained how helium-filled balloons and sandbags can be used to add and subtract directed numbers. Attaching helium balloons on the hot air balloon moves it up while sandbags move it down. A similar model is also explained online and cited verbatim.

> Suppose that it is a calm day and hot air balloon is tethered at the top of a tower, which is considered to be the zero level. If you hook a sandbag  on your balloon, it will go down, say one unit below the top of the tower. On the other hand, if you hook a helium balloon  on the balloon, it will rise, say one unit above the top of the tower.



> Now assume that the hot air balloon is right at the top of the tower, having equal number of helium balloons and sandbags hooked on it. [Table 2.5.1.2 explains how the operations can be carried out]

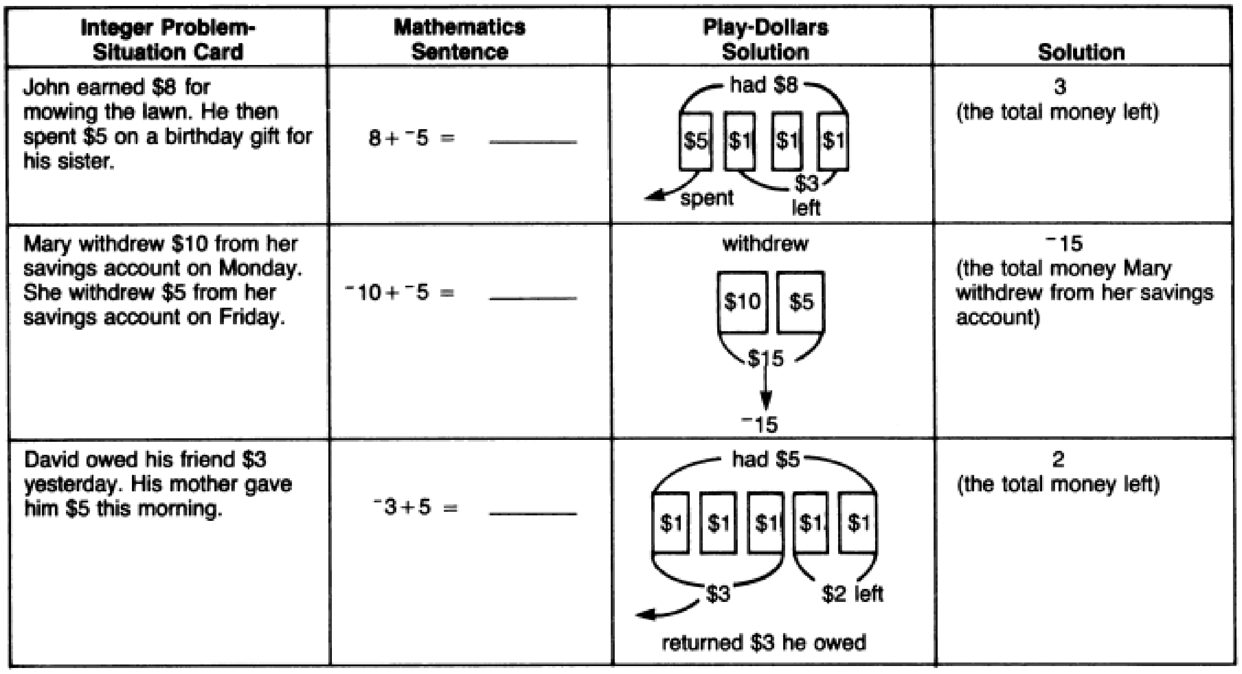
Table 2.5.1.2 Explanation of how addition and subtraction can be understood using hot air balloon

Case	Content	Mathematical expression
1	Suppose that you hook on 3 helium balloons and then 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
2	Suppose that you hook on 3 helium balloons and then 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
3	Suppose that you hook on 3 sandbags and then 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
4	Suppose that you hook on 3 sandbags and then 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
5	Suppose that you hook on 3 helium balloons and then take off 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )
6	Suppose that you hook on 3 helium balloons and then take off 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )
7	Suppose that you hook on 3 sandbags and take off 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )
8	Suppose that you hook on 3 sandbags and then take off 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )

Retrieved from: https://cd1.edb.hkedcity.net/cd/maths/en/ref_res/material/ld_e/Dirno.pdf

2.5.1.3 The money technique

A money technique for operating with integers in classroom was used by Chang (1985) among other researchers. Underlying his technique was that positive numbers should be represented by money earned or deposited in a savings account while money that was spent or withdrawn from the savings account represented negative numbers (Chang, 1985). To use this technique, play money and problem-situation cards are used. A learner has to pick a problem-situation card and thereafter write a mathematics sentence the problem situation represents. The learner then solves the problem using play money. The figure below illustrates how the technique works.



Integer Problem-Situation Card	Mathematics Sentence	Play-Dollars Solution	Solution
John earned $8 for mowing the lawn. He then spent $5 on a birthday gift for his sister.	$8 + {}^{-}5 =$ ______	had $8; $5 $1 $1 $1; spent; $3 left	3 (the total money left)
Mary withdrew $10 from her savings account on Monday. She withdrew $5 from her savings account on Friday.	${}^{-}10 + {}^{-}5 =$ ______	withdrew; $10 $5; $15; ${}^{-}15$	${}^{-}15$ (the total money Mary withdrew from her savings account)
David owed his friend $3 yesterday. His mother gave him $5 this morning.	${}^{-}3 + 5 =$ ______	had $5; $1 $1 $1 $1 $1; $3; $2 left; returned $3 he owed	2 (the total money left)

Figure 2.5.1.3 Money technique for adding integers. Source: Chang (1985, p.14)

Stephan and Akyüz (2012) also used the money technique though it was blended with a vertical number line. Their technique involved the use of net worth, asserts and debts to help learners understand operations on directed numbers. For Stephan and Akyüz (2012), asserts (what one owns) represented positive numbers and debts (what one owes) represented negative numbers. So if a learner was to work out "$-5 - (-10)$" (p.460), -5 would be interpreted as a net worth and then -10 as debt and thus the learner would be required to take away a debt of 10 from a net worth of -5 (Stephan & Akyüz, 2012).

Other researchers who used a money technique to carry out operation on directed numbers are Lestari, Putri and Hartono (2015). Their approach involved playing a chips card game where blue chips card represented amount of money and red chips card represented amount of debt (Lestari et al., 2015). To play this game, a learner would be required to pick two cards and then determine his or her result by adding what is on the two cards. Other rules of the game and how the winner is determined are described by Lestari et al. (2015) as follows:

- When getting blue chips on the first card and blue chip on the second card, students will add them by adding a lot of blue chips on the first card and a lot of blue chip on second card. Then the students determine the result as a positive number.

- When getting red chip on first card and red chip on second card, students will add them by adding a lot of red chips on first card and a lot of red chip on second card. Then the students determine the result as a negative number.
- When getting blue chips on first card and red chip on second card, students will cancel each different colours pair. Then the students determine the result as a positive number if the remaining chips is blue and negative if the remaining chips is red.
- Students can determine that the player who has a greater than others as the winner of the game. (p.20)

2.5.1.4 The use of positive and negative charged particles

The use of positive charges and negative charges works similarly with the coloured chips model for carrying out operations on directed numbers. Battista (1983) explains how the four operations can be performed using the charged particles. “Addition is represented by a "joining" action [while] subtraction is represented by a "take away" action” (p.27). As for multiplication and division, they involve the use of repeated addition and subtraction (Battista, 1983). Supposing that one has to perform the following operation “$(+5) - (-3)$” (Battista, 1983, p. 28), he explains that one has to begin with a jar containing five positive charges. The next step would be to remove three negative charges from the jar but since there are no negative charges then one needs to add three pairs of positive and negative charges to the jar (Battista, 1983). The addition of the three pairs does not change the total charge of five positives in the jar since the pairs cancel out to a zero charge. Now the three negative charges can be removed from the jar leaving the jar with eight positive charges which is the answer to $(+5) - (-3)$ (Battista, 1983).

Another model using positive and negative charges is described by Gilderdale and Kiddle (2011). The model also uses the concept of cancellation or neutralization of opposite charges. Gilderdale and Kiddle’s (2011) model works by partitioning the charges so that different sums can be made (see the figure below).

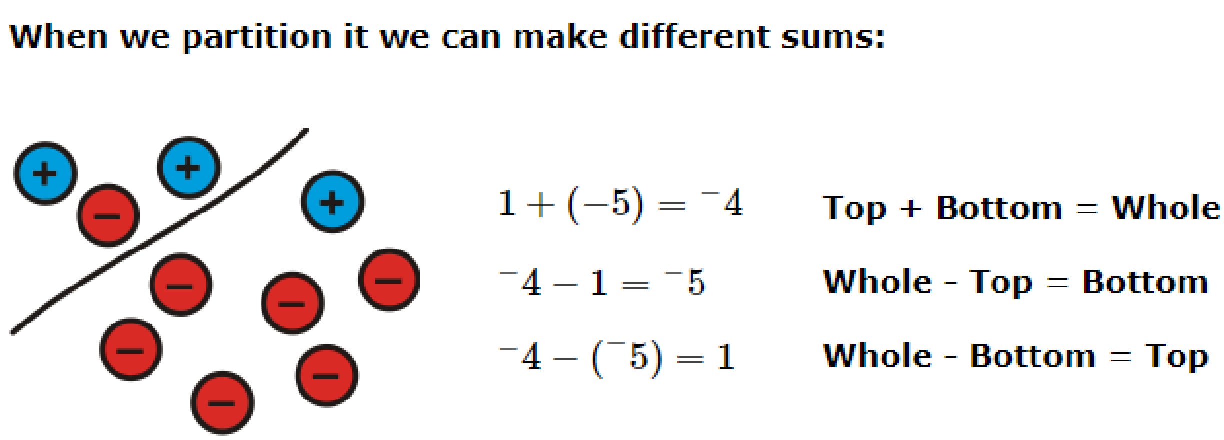


Figure 2.5.1.4 negative and positive charges for adding and subtracting integers: Source: Gilderdale and Kiddle (2011). Retrieved from https://nrich.maths.org/5947

2.5.2 The use of number line

The use of number line to help learners understand directed numbers and operations on directed numbers is supported by CAPS curriculum as well as a number of researchers (see for example Chang, 1985; McDonald, 2011; Thompson and Dreyfus, 1988). According to CAPS teachers should use "structured, semi-structured or empty number lines" (DBE, 2011, p.67) to help learners in counting backward or forward the directed numbers. It was indicated earlier that the introduction of negative numbers to learners poses some challenges some originating from learners' perception of numbers in terms of magnitude (Bishop et al.,2014). So they struggle to understand what negative numbers mean in terms of magnitude. Now the use of a number line for counting forward and backward may help learners realise the existence of numbers less than zero. Also, Thompson and Dreyfus (1988) used this concept of movement on a number line with an animated turtle that moved along a horizontal number line. The turtle moved according to the following rules:

- [inserting a positive] number, [t]he turtle walks number steps in its current direction.
- [inserting a negative] number, [t]he turtle turns around, walks number, and then turns back around.
- [Pressing] START.AT number, [p]uts the turtle at the position named number. This is where the turtle will begin its next itinerary. (Thompson and Dreyfus (1988, p.117)

Chang (1985) and McDonald (2011) used a drawn number line to assist learners understand and perform operations on signed numbers. For both Chang (1985) and McDonald (2011) movement to the right on a number line indicates a positive direction while to the left is a negative direction. McDonald (2011) explained how the number "$-2 + -3$" (p.5) can be worked out. The number '-2' is considered as the starting position on the number line where the mover should be placed. The mover is then "moved 3 places to the left of -2" (McDonald, 2011, p.5). Where the mover stops (which is at -5) is the solution for $-2 + -3$ (see figure 2.5.2a below for an illustration)

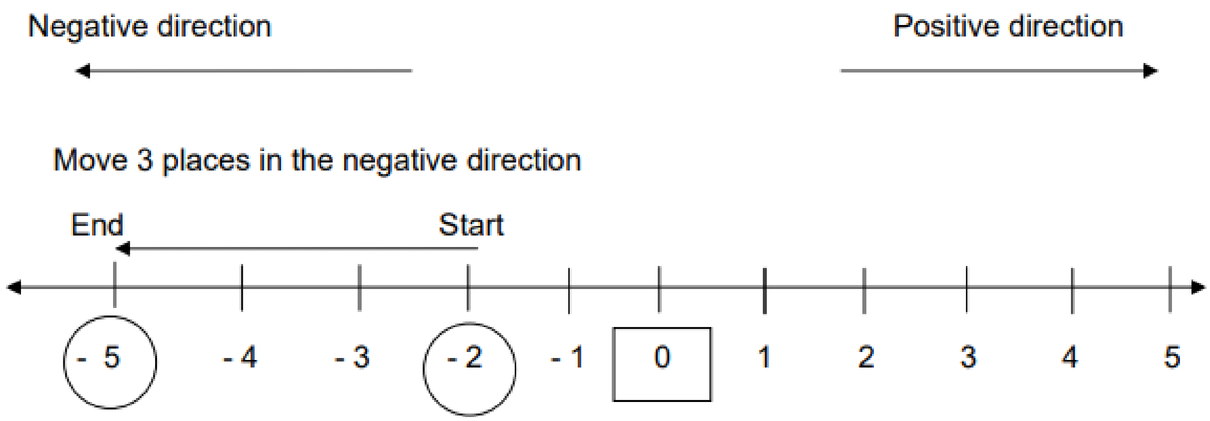


Figure 2.5.2a Adding directed numbers on a number line. Source: McDonald (2011, p.5)

Whitman (1992) used a Froggy Frolic game to show how multiplication of integers can be done on a number line. To play this game the following are the requirements:

> Froggy Frolic game board, frog markers, and a deck of hop cards. Each hop card has two parts: the frog half and the hop half. The hop half tells you two things: (1) the direction the frog must face to move, where "+" means face Shaka Swamp [right] and "-" means face Pupule Pond [left], and (2) the number of hops the frog must make. The frog half also tells you two things: (1) whether the frog is forward-hopping (hops in the direction it is facing) or backward-hopping (hops in the reverse of the direction it is facing) and (2) the number of spaces it travels in each hop. Whitman (1992, p.37)

As Whitman (1992) explains, if a learner picks up the hop card as shown in figure 2.5.2b, the numbers on the card can be translated to the following mathematics sentence: ${}^{+}2 \times {}^{+}3$. To work this out the learner has to mark the origin (or position 0) on the number line as the starting position of the frog then follow the steps indicated above to get the answer (see figure 2.5.2b below).

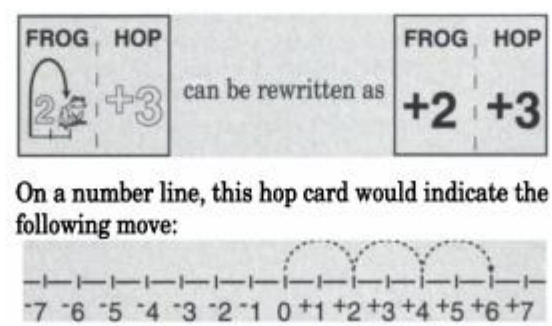


Figure 2.5.2b Multiplying directed numbers. Source: Whitman (1992, p.51)

2.6 Mathematical knowledge for teaching

Mathematical knowledge for teaching (MKT) is the refinement of Lee Shulman's idea of pedagogical content knowledge (PCK) (Speer, King & Howell, 2015). Initially, Shulman had proposed that teachers needed apart from the content knowledge and the general pedagogical knowledge, an overlapping kind of knowledge needed for transforming the subject matter knowledge into a form that can be easily understood by the learners which he called PCK (Shulman, 1986). To conceptualize the idea of PCK in the teaching of mathematics, Ball and her colleagues introduced the idea of MKT. This idea was intended to address such questions as; "how teachers need to know [mathematics] content … what else teachers need to know about mathematics and how and where teachers might use such mathematical knowledge in practice" (Ball et al., 2008, p.395).

According to Ball et al. (2008), MKT is a practice-based theory aimed at understanding all aspects involved in the teaching of mathematics. It articulates the mathematical knowledge needed to carry out the teaching of mathematics. It consists of six domains; common content knowledge (CCK), horizon content knowledge (HCK), specialized content knowledge (SCK), knowledge of content and students (KCS), the knowledge of content and teaching (KCT) and Knowledge of content and curriculum. These domains are subdivisions of the subject matter knowledge and the pedagogical content knowledge suggested by Shulman (see figure 2 below).

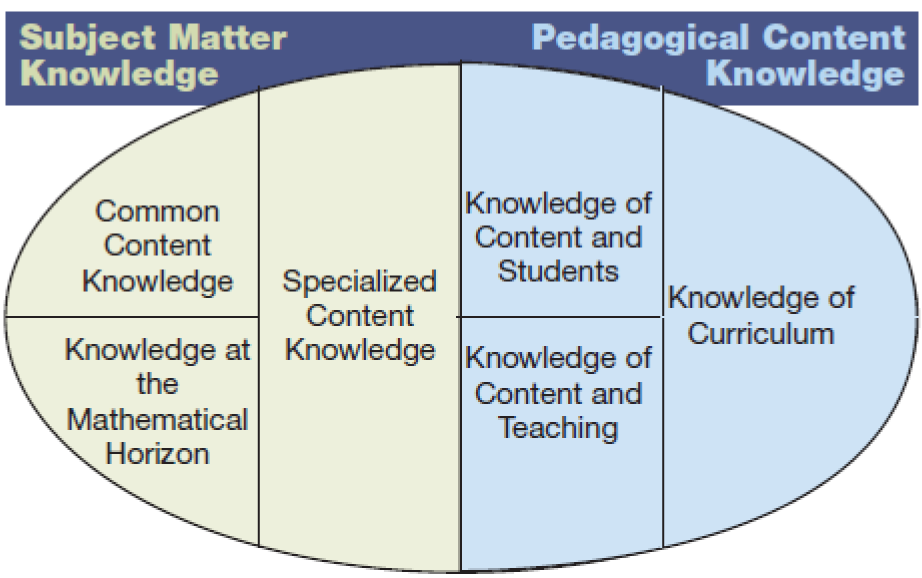


Figure 2.6: Mathematical knowledge for teaching framework. Source: Ball et al. (2008, p.403)

As illustrated in the figure, the subject matter knowledge comprises of the horizon content knowledge, common content knowledge and specialized content knowledge. The HCK defines teachers' "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p.403). For example, in CAPS curriculum the sequencing of subtopics for the topic of integers has been done. This involves the sequencing of subtopics to be covered for each grade and even across other grades where integers are taught. So teachers' awareness of this sequence defines their HCK. The CCK is defined as the "mathematical knowledge and skills used in settings other than teaching" (Ball et al., 2008, p.399). Hill (2010) defines it as knowledge that both nonteaching and teaching mathematically literate adults possess. It enables teachers to know whether an answer provided by student is correct, whether the definition of a concept or object is correct and as well as how to carry out procedures correctly (Hill, & Ball, 2009; Ball et al., 2008). For instance, being able to workout (+5)- (-3) correctly defines the CCK, it is knowledge that any mathematically literate individual must have not only teachers.

Teachers unlike any other mathematically literate adults - need knowledge specific or unique to teaching: the specialised content knowledge (Ball et al., 2008). It involves:

> Knowing mathematical explanations for common rules or procedures; constructing and/or linking nonsymbolic representations of mathematical subject matter; interpreting, understanding, and responding to nonstandard mathematical methods and solutions; deploying mathematical definitions or proofs in accurate yet also grade-level-appropriate ways; and diagnosing errors in student work. (Hill, 2010, p.521)

Ipek (2018) states that SCK in the case of addition and subtraction of integers involves teachers' knowledge of the use, explanation and justification of representations like the number line, coloured counters and real world contexts.

On the pedagogical content knowledge, the domain *knowledge of content and students* comprises of both "knowing about the students and knowing about the mathematics" (Ball et al., 2008, p.401). They argue that "teachers must anticipate what students are likely to think and what they will find confusing" (Ball et al., 2008, p.401) as they design lessons. This will enable them to design stimulating lessons. One of the stimulating factors for the lesson may be a desirable choice of examples that are interesting and motivating to learners (Ball et al., 2008; Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006). Teachers need to be aware of their learners so that they are able to choose for them examples that will motivate them and thus ensure that they are engaged in their learning. Moreover, teachers with KCS are able to hear and interpret their learners' incomplete thinking as they continue to engage with the tasks given (Ball et al., 2008).

The other domain of the pedagogical content knowledge is *knowledge of content and teaching*. It is described as an amalgam of "knowing about teaching and knowing about mathematics" (Ball et al., 2008, p.401). For instance, in the design of a mathematics instruction teachers must be aware of the instructional advantages or disadvantages a particular representation of a specific mathematical idea will have during classroom discussion with learners (Ball et al., 2008). That is, in the use of representations such as a number line, coloured counter or real world contexts (Ipek, 2018) for teaching addition and subtraction of integers teachers must know advantages and disadvantages of each over the other. Also, during instruction, teachers need to be able to decide on students' responses to select for highlighting and moving mathematics learning forward or when to pause for clarification (Hill, & Ball, 2009; Ball et al., 2008).

The third domain of the pedagogical content knowledge is the knowledge of content and curriculum (KCC). This is the curricular knowledge that Shulman described as;

> represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances. (Shulman, 1986, p.10)

Purevdorj (2019) also asserts that KCC "includes knowledge of articulating the strands of the curriculum, knowing students' prior and after knowledge in the curriculum, and determining learning goals about a particular topic for a particular activity" (p.36).

2.6.1 Mathematical knowledge for teaching and the quality of instruction

Hill et al. (2008) highlighted that teacher's knowledge causes students' learning through instruction. By Eisner (1964), instruction is defined as "those activities that are consciously planned and executed by the teacher which are intended to move pupils toward the attainment of the educational objectives held by the teacher" (p.118). While there may be other definitions for instruction, this study shall consider the one given by Eisner (1964). According to the definition, the teacher draws the lesson objectives for his learners and then uses his expertise to design the instructional activities that will best support learners to achieve the set learning objectives.

Sogunro (2017) argues that an instruction that is capable of meeting students' learning needs, learning styles, interests, expectations, and is well aligned to standards and is adequately delivered, should be regarded as a 'quality instruction'. With a focus on mathematics teaching and learning, Hill et al. (2008) claim that quality instruction is "a composite of several dimensions that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables" (Hill et al., 2008, p.431). In their work, they found that "teachers' mathematical knowledge for teaching plays a role in instruction" (Hill et al., 2008, p.498). A powerful link seemed to occur between MKT and mathematical quality of instruction. Teachers with well-developed MKT "held many examples of rich mathematics, teacher skill in responding to students, and other aspects of skilled mathematics instruction, from choosing examples wisely to ensuring equitable opportunities to learn" (Hill et al., 2008, p.497). Hatirasu and Erbas (2017) working on MKT the function concept noted also that teachers' MKT influenced the quality of their instruction. Specifically, they argued that teachers' "SCK and KCS of the function concept seemed to influence their instructional practices" (Hatirasu, & Erbas, 2017, p.719). Their conclusion was based on the students' learning outcomes or achievement (Hatirasu, & Erbas, 2017).

2.6.2 Mathematical knowledge for teaching and students' achievement

As proposed by Ball et al. (2008) MKT represents a framework for studying or understanding all the tasks required of teachers to effectively teach or support students in their learning of mathematics. For instance: knowing the mathematics well, being able to engage with learners' work in order to make sense of their mathematics, and choosing representations that are capable of supporting or enhancing students understanding, were identified by Ball et al. (2008) as teacher's tasks required to support students in their learning. Other researchers have identified tasks or strategies useful in supporting students in their learning, this include: teachers' intellectual use of examples (Zaslavsky, & Zodik, 2007; Bills, & Bills, 2005; Warshauer, 2015), use of analogies, asking questions or revoicing students' solutions (Warshauer, 2015), attempting to learners' errors and misconception (Borasi, 1992; Makonye & Khanyile, 2015) etc. In essence, all these tasks or strategies may be understood to define the teacher's MKT. Now critical is the question of whether the teacher's MKT and students' learning outcomes are interrelated.

In their work, Hatisaru and Erbas (2017) found that there were "some interactions between the teachers' MKT and students' learning outcomes" (p.719). The "learning experiences and opportunities that the teachers had provided to the students as representing their knowledge seemed to influence student learning outcomes" (Hatisaru, & Erbas, 2017, p.719). In the work of Pournara, Hodgen, Adler, and Pillay (2015) to determine whether improving teachers' mathematical knowledge can contribute to learning gains for learners, they observed a slight gain in learners' attainment when teachers' mathematical knowledge was improved. Similarly, other studies (see for example Shechtman, Roschelle, Haertel, & Knudsen, 2010; Hill, Rowan, & Ball, 2005) revealed that there was small link between students' achievement and teachers' MKT. Shechtman et al., (2010) found that out of the grades they were working with only in the Seventh-Grade Experiment Year One a link between teachers' MKT and student achievement was observed. They highlighted that "teachers with higher MKT had students who learned more cognitively demanding mathematics" (Shechtman et al., 2010, p.348). So, indeed as these studies suggest with an improved teachers' MKT learners' achievement is improved though not very significantly.

2.7 Professional noticing

Mason (2011, p.35) wrote that "we do not notice, either because we are not attuned or sensitized or because our attention is directed and occupied elsewhere".

As Mason (2011) points out, noticing is an intentional process that requires the observer to be attentive of what he intends to notice. He calls it a discipline that involves such techniques as;

> (a) preparing to notice in the moment, that is, to have come to mind appropriately; (b) post-paring by reflecting on the recent past to select what one wants to notice or be sensitized to particularly; in order (c) to pare, that is, notice in the moment and so be enabled to act freshly rather than habitually. (Mason, 2011, p.48)

In the teaching and learning of mathematics, a teacher needs to be prepared as to what important aspects during the learning process must he take into consideration, and when they occur, how must he act so that he is able to freshly support the students. Noticing enables a teacher to note those important details of students' strategies which Mason (2011) calls 'account of'. It also enables the teacher to make sense of the students' strategies to inform further decision which is the 'accounting for' according to Mason (2011). In this way, noticing can be seen as entailing "attending to student mathematical reasoning and making sense of this information to inform teaching decisions and teacher moves" (Wessels, 2018, p.733).

Jacobs et al. (2010) suggested that noticing in particular ways is a skill that needs to be developed as part of expertise in a profession, hence the term professional noticing. Thomas et al. (2017) define professional noticing as "a skill teachers use to identify and act upon salient mathematical actions of children" (p. 5). Jacobs et al. (2010) used the construct "professional noticing of children's mathematical thinking" to represent the teachers' reflection-in-action (Schön, 1987) or in-the-moment decision making about learners' thinking regarding the mathematics they are learning. Wessels (2018) used the construct 'professional noticing of students mathematical reasoning' to study how the pre-service teachers' professional noticing developed over time with a focus on how they reflected about students' mathematical reasoning.

According to Jacobs et al. (2010) professional noticing involves three interrelated teacher's skills which are: attending, interpreting/analysing and deciding how to respond on the basis of children's understandings. Attending to salient mathematical details of students' strategies requires the teacher to confront the learners' strategies so that they can figure out and understand what they are doing at the moment they are doing it. Barnhart and van Es (2015) put emphasis on that; the teacher should highlight the student thinking on the basis of the information collected about such student.

Interpreting or analysing involves the teacher's "coordination of the observed strategies with current theory of mathematical development" (Thomas et al., 2017, p.6). For Barnhart and van Es (2015) teachers' interpretation or analysis of student's strategies must reflect the extent to which the learners' strategies align with what is known about children's mathematical development. The third category or skill, deciding how to respond, concerns how teachers use what they have learned or gathered as students' ideas from the specific situation to inform their teaching and whether their reasoning coincides with research on children's mathematical development (Jacobs et al., 2010; Barnhart & van Es, 2015).

Some researchers (see for example Jacobs et al., 2010; Wessels, 2018; Van Es and Sherin (2008; Barnhart and van Es (2015) studied how prospective and practicing teachers professionally notice their students' mathematical thinking. In their work on how prospective and practicing teachers' professional noticing their children's mathematical thinking, Jacobs et al. (2010) found that the teaching experience for the teachers linked with how they noticed the salient mathematical productions of their learners. That is, experienced teachers notice their learners' thinking better than the novice teachers do. They also found that with a support through professional development, teachers' skills on how they professionally notice their students' strategies improved (Jacobs et al., 2010). Wessels (2018) also found that pre-service teachers improved their skills on how they professionally notice students' mathematical productions when professional development was offered through lesson study. Van Es and Sherin (2008) used video club to help teachers learn how to professionally notice students' mathematical thinking. Their results indicated that some teachers who participated in the video club improved in the way they noticed and interpreted students' mathematical thinking while others were still at the initial stage of learning to notice.

Barnhart and van Es (2015) used a video-based course that was designed to develop the pre-service teachers' professional noticing skills to compare how participants improved in their skills to notice compared to those who did participate. Their results indicated that the participants of the course showed an improved level in the noticing skills compared to their counterparts who did not participate in the course. Schack, Fisher, Thomas, Eisenhardt, Tassell and Yoder (2013) also used video excerpts as part of the modules they had designed to develop the professional noticing skills for prospective elementary school teachers. Similar to Barnhart and van Es (2015), their results showed that the prospective elementary school teachers demonstrated a significant growth in how they enacted the three components of professional noticing (attending, interpreting and deciding).

2.7.1 Professional noticing of learners' mathematical thinking and its impact on learners' learning

In their concluding remarks Jacobs et al. (2010, p.191) wrote; "evidence from our cross-sectional study indicates that the construct of professional noticing of children's mathematical thinking merits attention from teachers, professional developers, and researchers working toward the vision of successful classrooms". What Jacobs and his colleagues are indicating here is that professional noticing is important for the success of the classroom. And one characteristic of a successful classroom could be the learners' achievement or their success in learning with understanding what is intended for them to learn. Choppin (2011) noted in his study about 'The Impact of Professional Noticing on Teachers' Adaptations of Challenging Tasks' that teachers who "attended to and interpreted details of student thinking" (p.192) as they engaged with the challenging tasks were able to use what they noticed to inform their adaptation of the next task thereby enhancing "students' opportunities to learn with understanding" (p.194). Choppin's (2011) work shows that indeed if teachers professionally notice their learners' thinking as they engage with tasks, they can use what they have gathered to help learners to learn with understanding.

Other researchers (see for example Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Fennema, Carpenter, Franke, Levi, Jacobs & Empson, 1996) had a similar argument that if teachers use knowledge of their children's mathematics thinking in their classroom teaching, that will help them improve their instruction and thus enhance learners' achievement. Jacobs, Franke, Carpenter, Levi and Battey (2007) conducted an experimental study that involved 180 teachers and 3735 students. It is reported that about half of the teachers who were part of the study participated in a professional development project that focused on helping teachers develop an understanding of students' algebraic reasoning. Their study revealed among others "evidence that professional development focused on students' mathematical thinking is productive for teachers and their students" (Jacobs et al., 2007, p.282). They further state that teachers who participated in the professional development project were able to "generate a significantly greater number of strategies as well as more strategies that reflected the use of relational thinking" (p.283). The same was true with their students, they were also able to "generate strategies reflecting relational thinking, … and demonstrate[d] a significantly stronger understanding of the equal sign than did students in nonparticipating teachers' classes" (p.283). This provides further evidence that if teachers notice or have knowledge of their learners' mathematical thinking, which has a positive impact on the learners' learning.

The reviewed literature for this study indicates that MKT links to student achievement and also the same is with professional noticing of children's mathematical thinking. These two independent frameworks prove to be fruitful in the teaching of mathematics. That is, knowledge of the teacher

(MKT) and how he or she executes his or her practice during teaching (professional noticing) are important in advancing learners in their learning of mathematics. Of interest is, can these two frameworks be linked into a single framework? Can the resulting framework be used to analyse the teachers' practices? Can it also have a positive impact in learners' learning? In the following section I present the theoretical framework of this study which draws from the two frameworks (MKT and professional noticing) discussed above.

2.8 Theoretical and analytical framework

2.8.1 Professional noticing and mathematical knowledge for teaching

Mathematical knowledge for teaching describes the knowledge base required of the teacher to effectively carry out the teaching of mathematics (Ball et al., 2008) while professional noticing describes the teacher's practice during the teaching and learning process (Jacobs et al., 2010).

Though these two constructs appear distinct, literature shows that there is a link existing between them. Dick (2017) studied the relationship between professional noticing and MKT, specifically the specialized content knowledge. He focused on how the elementary pre-service interns' MKT development and the way they professionally noticed their students' mathematical thinking related. Dick's (2017) work indicated that there is indeed a relationship between the interns' SCK and their noticing of students' mathematical thinking. In his words he wrote that "as their (the elementary pre-service interns) SCK increased, they engaged more with professionally noticing their students' mathematical thinking" (Dick, 2017, p.354). Ribeiro, Badillo, Sánchez-Matamoros, Montes, and Gamboa (2017) studied a primary prospective teacher's (Carla) MKT as revealed in her practice and her noticing skills from her analysis of the video episode of her own practice. As one of their findings they noted the following;

> Focusing on her analysis of the students' reasoning (only the last six minutes of the video), the fact that she can differentiate various aspects of understanding from different students reveals Carla's (advanced) level of professional competency of noticing according to Sherin et al.'s (2009) criteria. Such can be linked with her KCS" (Ribeiro et al., 2017, p.3380)

This observation by Ribeiro and colleagues shows that there is a link between professional noticing and other aspects of MKT just like Dick (2017) had observed. Wessels (2018) also noted that the mathematical knowledge for teaching is one factor that can influence noticing. Likewise, Thomas et al. (2017) also claimed that "professional noticing allows for theoretically locating and analyzing responsive instructional practices while MKT provides a framework for considering and investigating the varied knowledge-types required for rich mathematics teaching" (p.10). As Thomas and his colleagues suggest each of the constructs mutually support each other in terms of detailing or describing what each entail. More importantly, Thomas and colleagues drew on Jacobs et al.'s (2010) framework for professional noticing along with the theoretical framework for MKT by Ball et al (2008) to show how they are related and hence developed a framework for assessing teachers' noticing and knowledge (see figure 2.1).

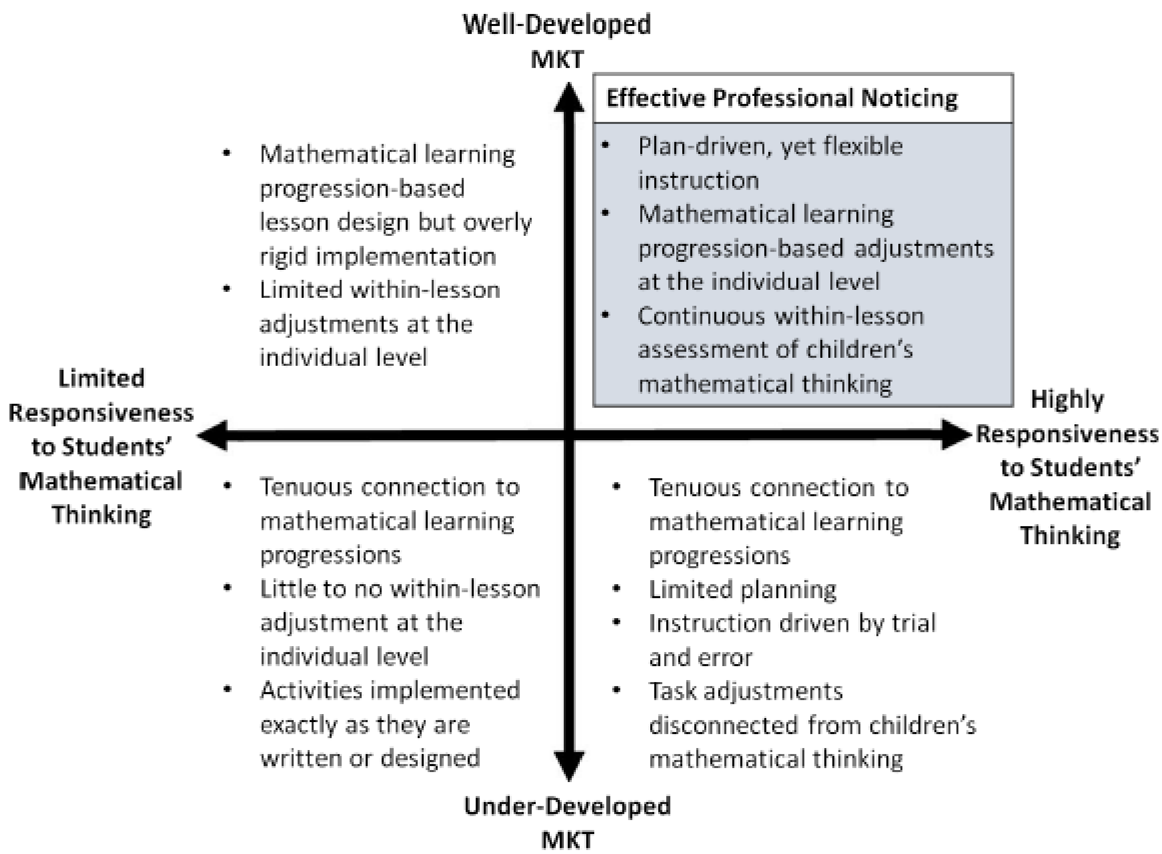


Figure 2.8.1: Professional noticing and Mathematical knowledge for teaching framework. Source: Thomas et al. (2017, p.14)

In this case the teacher's professional noticing skills are measured in terms of his or her responsiveness to students' mathematical thinking (Thomas et al., 2017). Furthermore, Thomas et al. (2017) warns that the interpretation of the framework should not be in terms of the dichotomies such well-developed MKT against under-developed MKT or highly responsive practices as opposed to limited responsive practices. They argue that teachers' MKT or their responsive practices cannot be measured on only those extremes because they exist on a continuum (Thomas et al., 2017). However, they argue that "it is useful to present the outcomes in this manner [using the extremes] to examine the instructional implications at each quadrant of the plane" (Thomas et al., 2017, p.14). The quadrants are summarized as follows:

Quadrant I – *Highly responsiveness to students' mathematical thinking/ well-developed MKT*

According to the framework and also what Thomas and colleagues stated "effective professional noticing occurs at the intersection of developed MKT and a high level of responsiveness to the mathematical activities of students" (Thomas et al., 2017, p.14). This quadrant is characterized by plan-driven instructional practices of the teacher which are at the same time flexible (Thomas et al., 2017) to change depending on what the teacher notices from his or her students. The teacher plans his or her instruction using his or her knowledge of how children's mathematics develops (Jacobs et al., 2010). Like Choppin (2011) indicated that the adaptation of the next challenging task is informed by what teachers notice as they attend and interpret details of student thinking, Thomas et al. (2017) also highlights that progression in mathematics learning in this quadrant is based on adjustments that are informed by attending to individual learners mathematical thinking. In other words, the lesson continues based on the outcomes of the teacher's assessment of learners' mathematical thinking (Thomas et al., 2017).

Quadrant II - *Limited responsiveness to students' mathematical thinking/ well-developed MKT*

The teacher's instruction is categorized as belonging to this quadrant if children's mathematical learning is aligned to their mathematical development. That is, the planned lesson coincides with how children's mathematical learning progresses as outlined in the research on "children's mathematical development" (Jacobs et al., 2010). However, the lesson designed is implemented rigidly with limited adjustments based on individual learner's mathematical thinking (Thomas et al., 2017). In this regard the enactment of the planned instruction is rigid with no room for adjustments on the bases of each individual learner's mathematical thinking. That is, even though teacher may exhibit well-developed MKT but his or her instruction is not responsive to learners' mathematical thinking. He or she either does not notice learners' mathematical thinking or does not use what he or she notices to inform his or her instruction. The implementation of the designed lesson goes as planned with no room for alterations.

Quadrant III - *Limited responsiveness to students' mathematical thinking/ under-developed MKT*
The instructional practices of the teacher are classified in this category if the teacher's lesson design shows little or no connections to how learners' mathematics develops. In other words, the teacher's planned instruction shows a "tenuous connection to mathematical learning progressions" (Thomas et al., 2017, p. 14). Also, during teaching he or she shows "little or no … adjustments at the individual level" (Thomas et al., 2017, p. 14). That is, the teacher's sequence of instructional practices is minimally or not informed by learners' mathematical thinking at all. He or she implements the activities "exactly as they are written or designed" (Thomas et al., 2017, p. 14).

Quadrant IV - *Highly responsiveness to students' mathematical thinking/ under-developed MKT*
This quadrant defines teacher's instructional practices that reflect a "tenuous connection to mathematical learning progressions" (Thomas et al., 2017, p. 14). That is, they reflect little or no connections to how learners' mathematics develops. Thus, instruction is based on trial and error (Thomas et al., 2017). Whatever task or activity the teacher thinks useful to use during the teaching he or she uses it and Thomas et al. (2017) assert that even the "task adjustments [made are] disconnected from the children's mathematical thinking" (p.14).

Key to the categorization of the teacher's professional noticing and mathematical knowledge for teaching as depicted from the framework are; implementation of instruction (whether it is plan-driven or shows limited planning), task adjustment and mathematical learning progression.

2.8.1.1 Implementation of instruction

This aspect of the framework deals with issues of whether the teachers' instructional practices as implemented during classroom teaching are presented as planned or resemble a planned instruction. Vassileva and Wasson (1996) highlight two phases in instructional planning which are content planning and delivery planning. Content planning is about all the planning that happens prior to the delivery of the instruction and involves generation, ordering and section of content goals to focus on during instruction (Vassileva & Wasson, 1996). Delivery planning is about "the process of optimal selecting and sequencing of the tutorial interactions focused on a given content" (Vassileva & Wasson, 1996, p.7). The delivery plan represents what is actually enacted during implementation of instruction. When an instruction is implemented in the way it is planned without any alterations, Thomas et al. (2017) refer to it as a plan-driven instruction that is rigid, but if changes are incorporated depending on what is the current thinking of individual students then it is plan-driven and flexible. Flexibility in this sense can be interpreted in terms of how far the teacher adjusts his or her planned activities in response to individual learners' thinking so that he or she is able to move such student to the desired performance.

2.8.1.2 Task adjustment

The use of tasks in the teaching and learning process is one way of assisting learners achieve the instructional goals. They help in keeping learners focused on the mathematics idea (Stein, Grover & Henningsen, 1996). Tasks are also useful in keeping learners engaged (Smith and Stein, 2011) and thus enabling them develop a deeper understanding of the concepts. Tasks differ in the level of thinking they require from student to successfully complete them (cognitive demand) (Smith & Stein, 2011). Some tasks may be challenging for students while others not. Challenging tasks are believed to increase students' opportunities to learn with understanding (Choppin, 2011). Choppin (2011) further notes that it is the responsibility of the teacher to maintain the level of the task even for the successive ones so that learners can benefit from them. To maintain the level of the task, teachers need to scaffold student's thinking (Simith & Stein, 2011; Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche & Walker, 2015). This can be by interrogating learners through questioning to determine their thinking. In the case where they are struggling to cope with the task, then the teacher can use prompts that can support their learning (Sullivan et al., 2015) or if they are keeping up with the task then use prompts to extend their thinking (Sullivan et al., 2015). This shows that to maintain the level of the task, teachers need to interpret their learners' mathematical thinking as they engage with the task and thus build on the information they gathered either in assisting learners towards completion of the task or in deciding on the next task (Choppin, 2011). This has a bearing that adjusting tasks during instruction should be informed by the teachers' interpretation of their learners' mathematical thinking. This enables teachers to know their learners' current thinking and how they can move them forward in their learning.

2.8.1.3 Mathematical learning progression

Mathematical learning progression or just learning progression can be understood as "a sequence along which students can move incrementally from novice to more expert performance" (Haritage, 2008, p.3) in a particular knowledge domain. That is, it is a developmental process involving students moving from what they currently know step by step to the desired performance. Haritage (2008) warns that during teaching teachers should be aware or "have in mind a continuum of how learning develops in any particular knowledge domain" (p.1) so that they are able to facilitate a smooth progress in learners' learning. Jacobs et al. (2010) emphasize that teachers can ensure students' mathematical learning progression if they know about the research on children's mathematical development. Key in the children's mathematical development is the idea that "later knowledge often builds on earlier knowledge" (Bailey, Watts, Littlefield & Geary, 2014, p.1). in particular, in the teaching of directed numbers learners cannot be taught operation on directed numbers before they could even understand what directed numbers are. Knowledge should build from simple concepts to more abstract concepts. Thus, the teacher needs to be aware of these developmental traits; in order to enhance students' learning of directed numbers. Understanding of students' mathematical thinking (Varol and Farran, 2006) is also critical for teachers to help students move step by step in their learning.

Accordingly, as Haritage (2008) argues, engaging in formative assessments can help the teacher be aware of students' current performance and device, based on the student current performance, how he or she can move the student towards the desired performance. Engaging formative assessment in the teaching and learning progress enables the teacher to determine the learners' position towards acquiring the intended learning goal or the gap between their current understanding and what they are expected to understand. Such knowledge will in turn help the teacher find ways on how to close that gap.

2.9 Conclusion

The review of literature revealed that students at any grade level can be introduced to directed numbers. It pointed out that, what is most important are the support and opportunities afforded to learners to best understand the concept directed numbers. The challenges, common errors together with the misconceptions learners are likely to make when learning directed numbers at senior phase; were also revealed from the literature. It also revealed different teaching models that could be used to support learners in their learning of directed numbers. A further review of the literature revealed that there is a link between MKT and quality instruction, MKT and students' achievement, professional noticing and teaching experience, professional noticing and professional development and that professional noticing of learners' mathematical thinking has an impact on learners' learning. Finally, the theoretical framework for this study which draws from MKT framework and professional noticing of children's mathematical thinking framework was also discussed.

CHAPTER 3

RESEARCH DESIGN AND METHODOLOGY

3.1 Introduction

This chapter discusses the research methodology that is intended to guide the study together with the justification for the choice. It begins with the discussion of the research paradigm and the choice of an interpretive paradigm. Why a qualitative research approach and a qualitative case study are chosen is also discussed and justified. The choice of a secondary data intended for answering the research questions is also justified. How data for this study was analysed from the source is explained. The chapter further discuss the issues relating to the trustworthiness of the study to ensure its quality as well as the limitation of the study. It concludes by discussing the ethical issues to be considered while conducting the study.

3.2 Research paradigm

A paradigm is defined by Opie (2004, p.18) as a “basic set of beliefs that guides action”. Creswell and Poth (2016) use the word worldview to also refer to the paradigm. This shows that a paradigm is similar to beliefs that shape how a researcher views the world or beliefs held by the researcher which help him or her as he or she carry out his or her research. In other words, a paradigm can be understood as a lens through which researchers view the world. We can also view a paradigm as a tool that helps the researcher establish a position or stance as he or she embarks in a research. Scott and Morrison (2006) assert that in the field of education research different paradigms have been developed which can be grouped into; positivism/empiricism, phenomenology, critical theory and postmodernism. According to Opie (2004) the main two paradigms that have influenced educational research are positivism and interpretivism. This study is located within an interpretive paradigm.

3.3 Interpretive paradigm

What differentiates the research paradigms are their underlying philosophical assumptions (Creswell & Poth, 2016). Creswell and Poth (2016, p.16) write “philosophical assumptions consist of a stance toward the nature of reality (ontology), how the researcher knows what she or he knows (epistemology), the role of values in the research (axiology), the language of research (rhetoric), and the methods used in the process (methodology)”. Ontologically an interpretive paradigm holds that “reality is subjective and can differ considering different individuals” (Alharahsheh & Pius, 2020, p.42). This can be interpreted to mean that each person interprets the world or an event in their own way and therefore if multiple people are to interpret an event, there can be multiple perspectives about that same event. That is, different people attach or may have different interpretations and meanings about the same event. Thus, this study is located within the interpretive paradigm because it intends to understand how teachers professionally notice their learners’ mathematical thinking as well as their mathematical knowledge for teaching directed numbers in grade 8. The study intends to understand the teachers’ own practice in the context of their own teaching which may be unique for individual teacher.

In terms of how the researcher knows what he or she knows (epistemology), interpretivism holds that a researcher should be “as close as possible to the participants being studied” (Creswell & Poth, 2016, p.18). This gives the researcher an opportunity to understand or observe reality “from inside through the direct experience of the people” (Mack, 2010, p.8) or to understand reality in the lived experiences of the participants. In this way the researcher gets to know what he knows by exploring in depth the individual experiences (Mack, 2010). So, an interpretive paradigm is useful for this study as it intends to explore the teachers’ practices in depth in their own place of work. Creswell and Poth (2016) assert that these contexts are important as they help the researcher in “understanding what the participants are saying” (p.18). This study uses secondary data and as such it means the researcher did not work closely with the participants. However, the lesson transcripts which are the source of data for this study are representative of the actual experiences of the teachers’ practices. So, they help the researcher to understand the lived experiences of the teachers.

Alharahsheh and Pius (2020) claim that based on the qualities that an interpretive paradigm permits the researcher to have or do, “qualitative methods are most suited methods to gain the deep insights based on a specific context” (p.43). This study therefore adopts qualitative research methods to understand the teachers’ instructional practices.

3.4 Qualitative research approach

In conducting a research or study, plans and procedures that guide the study are very crucial in order to help the researcher establish his stance or position or conduct the research systematically. These plans and procedures represent the research approach of a study. This study is guided by the qualitative research approach. This approach suits the current study as it aims at describing or explaining teachers' professional noticing and mathematical knowledge for teaching directed numbers. Alharahsheh and Pius (2020) assert that qualitative research approach is more concerned with the meanings and processes which cannot be examined using numbers. Furthermore, they write; "qualitative research aims to provide specific understanding to a phenomenon based on the ones experiencing it with less generalization [and also] to attain deep understanding of a specific case with in depth exploratory studies to enable finding quality responses throughout the research" (Alharahsheh & Pius, 2020, p.40). So, as indicated above the qualitative research approach is employed to explore and provide detailed description of teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8. The idea is not to generalize the findings from the teachers studied to other teachers but to understand in depth their professional noticing and MKT the way they experience it in their own teaching.

Creswell and Poth (2016) identified features of qualitative research approach that distinguished it from other research approaches. Among those features are the following;

- Qualitative researchers tend to collect data in the field at the site where participants' experience the issue or problem under study.
- The qualitative researchers collect data themselves through examining documents, observing behaviour, and interviewing participants.
- Qualitative researchers typically gather multiple forms of data, such as interviews, observations, and documents, rather than rely on a single data source.
- Qualitative researchers build their patterns, categories, and themes from the "bottom-up," by organizing the data into increasingly more abstract units of information.
- In the entire qualitative research process, the researchers keep a focus on learning the meaning that the participants hold about the problem or issue, not the meaning that the researchers bring to the research or writers from the literature. (pp.37-39)

A qualitative research approach is suitable for this study as it intends to understand the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 in their work as teachers or in the context of their teaching practice. Qualitative research approach again suits this study as it helps in "embracing the idea of multiple realities" (Creswell & Poth, 2016, p.16). For instance, this study needs to understand teachers' instructional practices as experienced by the teachers not what the researcher expects. The study also uses a qualitative approach as it allows for the use of documents as sources of data and also enables the researcher to make his own interpretations of the data in conjunction with those of the participants (Creswell, & Poth, 2016). Another reason for using qualitative research approach resonates with Creswell and Poth's (2016) argument that "we conduct qualitative research because we want to understand the contexts or settings in which participants in a study address a problem or issue" (p.40). This study followed two grade 8 mathematics teachers from two different high schools in Johannesburg hence a qualitative case study research design was used.

3.5 Qualitative case study

To explore in depth teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 a qualitative case study research was used. Creswell and Poth (2016) define a case study research as "a qualitative approach in which the investigator explores a bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information (e.g., observations, interviews, audiovisual material, and documents and reports), and reports a case description and case-based themes" (p.73). In this study two teachers from two schools in Inner city Johannesburg are selected to study in-depth their professional noticing skills and MKT. From Creswell and Poth's (2016) distinction of the case studies, this study follows or employs a multiple case study approach. For Creswell and Poth (2016) a multiple case study consists of more than one case selected to illustrate a specific issue or concern.

Furthermore, in Creswell and Poth's (2016) definition of a case study research as given above, emphasis is made on exploring the case over a period of time and using multiple sources of data. However, for this study that is its limitation because due to the covid-19 pandemic access to the field or teachers' work place was denied and thus only documents could serve as sources of data. As such, relying on documents only restricted the study to gain access to multiple sources of data. Creswell and Poth (2016) also points to a need to provide a description of the case studied. This also is a limitation of this study. This is so because while using documents as sources of data, the study can only rely on the description as given in the documents.

3.6 Data collection or selection

Creswell and Poth (2016, p.75) points out that "data collection in case study research is typically extensive, drawing on multiple sources of information, such as observations, interviews, documents, and audiovisual materials". This study draws its data from documents (in the form of lesson transcripts). This data is secondary as the researcher was not part of its collection, the design of the study that led to its collection and also not part of the organisation of the data results (Davis-

Kean, Jager & Maslowsky, 2015). During the course of this study the researcher had to resort to the use of secondary data as the source of data for this study because access to the empirical sites was no longer possible due to the covid-19 pandemic and the lockdowns imposed.

As Davis-Kean et al. (2015) point out, this type of data is accessible if data from primary sources was collected for the community of researchers or if it is made available through data archives for use by other researchers. The secondary data for this study was a primary data for the project Mathematics Performance Lag Advance Programme (MPLAP) a project run by my supervisor funded by National Research Foundation (NRF). This project focused on grade 8 teachers' professional development on teaching directed numbers. It began in 2019 and was expected to last for two years, meaning it was to end in 2020. The researcher became part of this project in 2020 as a research assistant. In 2019 data was collected for this project but in 2020 because of the covid-19 pandemic collection of data was disrupted. Therefore, the researcher had to seek permission to use data that was collected in 2019. The MPLAP data had not been published in any platform, so granting to use it for this study was sought from the collector.

The MPLAP was a teaching intervention programme that was aimed at bridging learner performance in mathematics in order to help learners learn grade 8 mathematics, specifically directed numbers, successfully. Two schools in the Inner City Johannesburg participated in the project. A quasi-experimental research design was followed where two groups; the experimental and control groups were involved. Teachers in the experimental group underwent a continued professional development on MKT, in particular the SCK and KCS domains on directed numbers and the use of researched models for teaching directed numbers. Since data for MPLAP was collected on the teaching of directed numbers at grade 8, this was found useful to help the researcher in answering the research questions for the current study as it is also about the teaching of directed numbers at grade 8. However, the study intends to focus only on the data from the two teachers who taught the control groups. The reason being that the study doesn't aim to test the effectiveness of the intervention rather to understand the teachers' instructional practices in relation to how they professionally notice and their MKT as they teach directed numbers.

Collection of data for the control groups was through lesson observations. One lesson per teacher was observed. Altogether, two lessons in total provided data for this study. While the lessons may be seen as a small sample to use, Marshall (1996) argues that "an appropriate sample size for a qualitative study is one that adequately answers the research question [or questions]". The observed lessons were transcribed verbatim.

3.7 Research Site

Bowen (2009) claims that documents can be the source of data about the context within which research participants operate. In this case the information provided by the MPLAP provides the context within which the teachers who are participants of this study operated. The MPLAP project was conducted at two different high schools in the Inner City of Johannesburg as mentioned above. For the MPLAP project two grade 8 classes per school were chosen: one was the experimental group and the other one was the control group. As indicated above this study focused on the control groups from those two schools hence it shares the same research sites as the MPLAP.

The next section presents how data will be analysed from the source.

3.8 Data analysis

Data for this study is a secondary data in the form of transcripts of lessons observed during primary data collection. As such the sources of this data are documents. Bowen (2009) indicates that documents contain text and images which were recorded without the intervention of the researcher. In this case also, the researcher did not take part in transcribing data from the observed lessons. To analyse this kind of data Bowen (2009) asserts that document analysis can be employed. According to Bowen (2009) document analysis may combine some elements of thematic analysis and content analysis. In this study I use content analysis to analyse the lesson transcripts. Content analysis is defined by Bowen (2009) as "the process of organising information into categories related to the central questions of the research" (p. 32). Forman and Damschroder (2007) use the phrase 'qualitative content analysis' to refer to content analysis. They posit that qualitative content analysis "examines data that is the product of open-ended data collection techniques aimed at detail and depth, rather than measurement" (Forman & Damschroder, 2007, p.48). That is, it is used in analysing data which is in the form of texts. Similarly, this study seeks to explore in depth the teachers' professional noticing and mathematical knowledge for teaching directed numbers in grade 8 as captured in the lesson transcripts which are in the form of texts.

Furthermore, Forman and Damschroder (2007) emphasize that to succeed in analysing data using qualitative content analysis, the researcher should develop the coding schemes. Codes can be useful in “reorganize[ing] data in a way that facilitates interpretation and enables the researcher to organize and retrieve data by categories that are analytically useful to the study, thereby aiding interpretation” (Forman & Damschroder, 2007, p.48). Since large amounts of data can be collected in qualitative research, so analysis of such data may present challenges and thus using codes can help in simplifying the process. This study uses deductive codes which “exist priori and are identified or constructed from theoretical frameworks” (Forman & Damschroder, 2007, p.48). Thomas et al.’s (2017) theoretical framework on professional noticing and mathematical knowledge for teaching will be used to construct the coding schemes to analyse the transcripts of the lessons observed.

Analysis of the teachers’ instructional practices as captured in the lesson transcripts will be done with an attempt to determine the professional noticing skills and mathematical knowledge for teaching displayed by the teachers as they teach directed numbers in grade 8. The guiding question for this analysis is the first research question of the study:

What professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8 are displayed by the teachers?

The categories; nature of instruction, tasks adjustment and mathematics learning progression were used to explore teachers’ professional noticing skills and mathematical knowledge from their instructional practices. The intention was to display the teachers’ skills and knowledge as outlined by Thomas et al. (2017) in their theoretical framework. Table 3.1 below gives the summary of each category. The descriptors or indicators for each category were developed based on the descriptions provided from Thomas et al.’s (2017) theoretical framework.

Table 3.1 *Guidelines for investigating teachers' professional noticing skills and mathematical knowledge for teaching*

Category	Sub-Category	Descriptors
Implementation of instruction	Plan-driven and flexible	- All activities enacted in class link to what is planned in the teacher's lesson plan. - Coherence in the manner that activities or tasks are presented (building from one another) - Activities during classroom interactions depending on individual learner's thinking
	Plan-driven and rigid	- All activities enacted in class link to what is planned in the teacher's lesson plan. - Coherence in the manner that activities or tasks are presented (building from one another) - Activities are not adjusted in response to learners' mathematical thinking. They are implemented as they are from the plan.
	Limited planning	- Teacher makes in-the-moment decisions like developing which tasks, activities etc. to use which are not informed by learners' thinking. - Instruction is driven by trial and error
Tasks adjustment	Adjusted at individual level	Teacher attends and interpret student understanding or thinking as evidenced in their responses. The teacher, based on his or her interpretation of the learner's thinking find the alternative tasks to use to advance the learner's understanding.
	Limited adjustment at individual level	- The teacher does not attempt to interpret the individual learner's thinking and continues implementing tasks as planned. - In the case he or she has attempted, he or she does not adjust the tasks in response to the learner's thinking rather sticks to the plan or little adjustment is made.
Mathematics learning progression	Connected to individual learner's level	Teacher attends to individual learner's understanding and advances their understanding step by step to the desired understanding.
	Tenuous connections	Teacher's instructional practices are not connected to or helping individual learner advance from their current thinking to the desired thinking or understanding.

Furthermore, the results of the analysis of teacher's instructional practices will be used to help in explaining the level of each teacher's professional noticing skills and mathematical knowledge for teaching using Thomas et al.'s (2017) theoretical framework. The guiding question for this part is the second research question of the study:

How can the levels of the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 be explained?

The categories used to analyse the teachers' instructional practices in this stage include are presented in table 3.2 below. The table gives the summary of each level as well as the descriptors or indicators for each category as described in Thomas et al.'s (2017) theoretical framework.

Table 3.2 *Guidelines for investigating the level of teachers' professional noticing skills and mathematical knowledge for teaching* (Thomas et al., 2017)

Level	Descriptors
Highly responsiveness to students' mathematical thinking and well-developed MKT	- Plan-driven, yet flexible instruction - Mathematical learning progression-based adjustments at the individual level - Continuous within-lesson assessment of children's mathematical thinking
Limited responsiveness to students' mathematical thinking and well-developed MKT	- Mathematical learning progression-based lesson design but overly rigid implementation - Limited within-lesson adjustments at the individual level
Limited responsiveness to students' mathematical thinking and under-developed MKT	- Tenuous connection to mathematical learning progressions - Little to no within-lesson adjustment at the individual level - Activities implemented exactly as they are written or designed
Highly responsiveness to students' mathematical thinking and under-developed MKT	- Tenuous connection to mathematical learning progressions - Limited planning - Instructions driven by trial and error - Task adjustments disconnected from children's mathematical thinking

3.9 Trustworthiness

To ensure quality and sound qualitative research, the issue of trustworthiness of the research must be addressed (Lincoln & Guba, 1986). According to Lincoln and Guba (1986) the criteria for trustworthiness in qualitative research involves the four components; dependability, credibility, transferability and confirmability. Creswell and Poth (2016) state that *credibility* addresses the question "Are the results an accurate interpretation of the participants' meaning?" (p.206). Anney (2014) lists the following as some of the credibility strategies that the researcher can incorporate; prolonged engagement in field or research site, use of peer debriefing, triangulation, member checks, negative case analysis and persistent observation. While some of these strategies were not possible to ensure under the covid-19 pandemic but the researcher was able to ensure others like the use of peer debriefing. Assistance from colleagues to read and comment on the findings of the study helped the researcher to revisit the data and check whether his interpretations align with the theoretical framework. Apart from that, to ensure that the results were the correct interpretations of the participants' meaning, the transcribed lessons were quoted verbatim. This ensured that the sources of interpretations were revealed.

Another strategy for ensuring trustworthiness is *transferability*. Anney (2014) emphasize that transferability refers to "the degree to which the results of qualitative research can be transferred to other contexts with other respondents" (p.278). Some of the issues that can help in deciding on transferability are given by Shenton (2004) as;

a) the number of organisations taking part in the study and where they are based;

b) any restrictions in the type of people who contributed data;

c) the number of participants involved in the fieldwork;

d) the data collection methods that were employed;

e) the number and length of the data collection sessions;

f) the time period over which the data was collected. (p.70)

Except for (e) which is about the number and length of data collection sessions, other issues are dealt with in the methodology section. This will thus make it easy for the readers of this study to make informed decisions on whether the findings of this study are transferable.

Confirmability is another strategy identified by Lincoln and Guba (1986) as criteria for trustworthiness. Conformability relates to the extent to which the researcher' biasness or interests affect his or her interpretations (Baxter & Eyles, 1997). For Anney (2014) conformability can be referred to as "the degree to which the results of an inquiry could be confirmed or corroborated by other researchers" (p.279). If the researcher's motivations are divorced from their interpretation, it will be easy for the results of such a study to be confirmed by other researchers. While it cannot be guaranteed that the lesson transcripts are true representations of what the participants said (audiotapes could not be accessed to confirm the lesson transcripts) but as a secondary data user my biasness is removed from the design of the lesson transcripts. In this way the results of this study can be corroborated.

The other strategy that Lincoln and Guba (1986) identified in ensuring trustworthiness of the research report is *dependability*. Bitsch (2005) writes; "dependability refers to the stability of findings over time. Dependability answers the question whether research results would be the same, were the study replicated with the same or similar participants in a similar context" (p.86). Data collected for this study is secondary data available in documents which guaranties availability in an unaltered form after any specified time. This ensures that should the study be replicated the same results could be obtained as the data used would be the same. Moreover, Shenton (2004) views the issue of dependability in the case whereby the same study is to be repeated by future researchers. He suggests that the processes underpinning the study should be reported in detail. In this regard, this study has ensured dependability since the necessary details for all the processes within this study have been given.

3.10 Ethical considerations

To ensure that the study adheres to ethical concerns, an ethical clearance to conduct a study was sought from the Human Research Ethics Committee (Non- Medical) of the University of the Witwatersrand. This study uses secondary data, as such there was no conduct with the primary sources of data. This meant that the ethical issues concerning the collection of data from the participants, specifically gaining consent from the participants before collecting data, was not an issue for this study. However, in analysing and reporting the findings of the study anonymity and confidentiality of the participants had to be addressed. In the lesson transcripts of the primary data; pseudonyms such as T3 and T4 were used to hide the identity of the teachers while L1, L2, L3 etc. were used to represent different learners who participated in the classroom discussions. Moreover, Lnrs was used for responses given by learners in groups. The same pseudonyms will be used to make sure that anonymity of participants is observed. Similarly, for the schools the pseudonyms; Schools A and School B were used and this study also will use the same names.

3.11 Conclusion

The chapter has outlined and justified the research paradigm (interpretivism paradigm) chosen for this study. It further elaborated on why a qualitative research approach and a qualitative case study were chosen to underpin this study. The choice of a secondary data intended for answering the research questions was explained and justified. Data analysis process for this study was also explained. The chapter further discussed the issues relating to the trustworthiness of the study to ensure its quality and also discussed the limitation of the study. It concluded by discussing the ethical issues to be considered while conducting the study.

CHAPTER 4

DATA ANALYSIS, FINDINGS AND DISCUSSION

4.4.11 Introduction

This study was undertaken to investigate the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8. Two research questions that the study sought to answer were:

- What professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8 are displayed by the teachers?
- How can the levels of the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 be explained?

To seek answers to these questions, in this chapter, I engaged with the lesson transcripts of both T3 and T4's lessons. The first attempt was to analyse the teachers' instructional practices to understand their professional noticing skills and mathematical knowledge for teaching directed numbers using the categories; implementation of instruction, adjustment of tasks and mathematical learning progression. Secondly I use the displayed instructional practices of the teachers from the data analysis in an attempt to explain their levels of professional noticing and mathematical knowledge for teaching directed numbers at grade 8. The categories for the levels are;

- Highly responsiveness to students' mathematical thinking and well-developed MKT
- Limited responsiveness to students' mathematical thinking and well-developed MKT
- Limited responsiveness to students' mathematical thinking and under-developed MKT
- Highly responsiveness to students' mathematical thinking and under-developed MKT

Analysis of each of the lessons of T3 and T4 began with the summary of those lessons. This was done by segmenting each of the different activities that happened during the course of the teaching and learning process, as well as the record of the time during which they occurred. Excerpts demonstrating each of the categories for exploring the teachers' professional noticing and mathematical knowledge for teaching were selected from the lesson transcripts and analysed. The chapter ends with a discussion of finding from the analysis of data.

4.4.12 Analysis of T3's instructional practices

This section provides an analysis of T3's instructional practices with an attempt to provide answers about his professional noticing skills and mathematical knowledge for teaching directed numbers. The main events of his lesson are summarized in table 4.2.1 below. Next, I engage with the excerpts selected from segments of the lesson that demonstrate each of the three categories used for understanding the teacher's professional noticing skills and mathematical knowledge for teaching.

4.2.1 T3's lesson

The lesson focused on addition and subtraction of directed numbers. The lesson began with the teacher introducing learners to the addition and subtraction of directed numbers. After that the teacher and learners focused on how they can add directed numbers using a number line as well as doing it conceptually. The lesson went on to explore how to distinguish subtraction expressions and also how to subtract directed numbers using a number line. The table below represents the summary of the lesson.

Table 4.2.1 *Summary of T3's lesson focusing on addition and subtraction of directed numbers*

Segment	Time	Activity
1	**00:00 – 07:15**	**Introduction (Adding directed numbers using a number line)**
2	**07:37 – 13:30**	**Adding directed numbers using a number line**
3	**13:30 – 14:47**	**Using a conceptual way of adding negative 11 and negative 15:** $\mathbf{-11 + (-15)}$
4	**15:20 – 21:17**	**Distinguishing subtraction expressions.**
5	**21:50 – 31:43**	**Subtraction of directed numbers using a number line.**
End of lesson		

4.2.2 T3's implementation of instruction

Vassileva and Wasson (1996) concur with the need to have a planned instruction before its implementation phase. Such planning includes among other things, the selection of tasks, design of activities etc. that will enable the teacher in advancing learners to the desired goal. Ordering of the selected tasks, activities etc. is also important for a coherent lesson so that learners can develop concepts in an integrated manner and thus be able to realise the connectedness in the concepts. However, the translation of that plan in to practice (implementation of instruction) may not be the same for individual teachers. For instance, during classroom interactions it may happen that some learners do not cope with tasks or activities planned for the instruction as the teacher would anticipate. In this case, the difference becomes; how responsive the teacher's instruction can be to the learners' thinking which is a reflection of their professional noticing skills and MKT. The following segment of the class together with the respective excerpt are selected to provide answers about how T3 implemented his instruction

Segment 1: Lesson Introduction

T3 began his class by allowing learners connect or recall what they did in the previous lesson about directed numbers. He presented a task ($(-11)+(-5)$), read it out for the learners to ensure that they all understood it and then invited them to work it out. In the selected excerpt below, I focus on two moments; in turn 1 where the teacher wants to establish if learners recall what they did and thus give them a problem to work on; in turns 2 – 13 where learners are giving their solutions to the problem and the teacher's reaction to the individual learner's solutions.

Excerpt 1: Working out $(-11)+(-5)$

Turns	Time-intervals	Transcript	
1	00:00	T3	A big question is to see if you really understood what we did yesterday. Right, now let's try this question: (pause). Can you answer this: $(-11)+(-5)$. Negative eleven plus negative five. How do we answer that? Try it and then if you confident enough you can come here in front then show us how you got the answer. (A Learner was then given time to work out the given problem on the chalkboard)
2	02:11	L1	(learner writes incorrect answer on the board)
3	03:20	T3	Alright, now do you agree with that?

4	03:22	Lnrs	No (chorus response)
5	03:30	L2	$(-11) + (-5) = -6$ (another learner suggesting a different solution on the chalkboard)
6	06:15	T3	Do you want to explain to us? Anyone to explain this? Anyone who can explain this? (Teacher pointing at another learner). Yes, you want to explain it?
7	06:32	L3	Yes sir
8	06:35	T 3	Try
9	06:50	L3	Sir, I think she forgot the negative at 5.Positive times negative is equal to negative $(+ \times - = -)$ then the answer is negative 16(-16).That is $(-11) + (-5) = -11 - 5 = -16)$
10	07:05	T3	Do we agree with this answer?
11	07:07	Lnrs	Yes sir (chorus response)
12	07:10	T3	Ok, and any other explanations?
13	07:15	L4	My explanation is negative 11 plus negative 5 is negative 16. She said positive times negative which is equals to negative. Then she puts a negative (to be $-11 - 5 = -16$)(T3 accept the method used of solving the problem)

In turn 1 lines 1-2 of this excerpt, T3 introduces the lesson by saying to the learners "*a big question is to see if you really understood what we did yesterday*". He then wrote the problem $(-11) + (-5)$ (turn 1, line 4) for learners to work out so as to establish if they still recall what they did in the previous lesson. T3's action to help learners recall and connect what they had previously dealt with in their prior lesson shows that the current lesson hinges on the prior lesson. Thus, this is evidence of a planned instruction as this connection can only be made if the teacher has engaged in a plan of how he can sequence and order the mathematical ideas (Vassileva & Wasson, 1996) as well as how he can make sure that he helps learners establish a clear connection between the concepts in his teaching. So, in this incidence, it can be argued that the way T3 has begun implementing his instruction shows that it is plan-driven (Thomas et al., 2017).

Now looking at turn 2, the learner (L1) offered an incorrect response to the problem $(-11) + (-5)$ but T3 in turn 3 turns to the whole class to agree or disagree with the given response. This he does without attempting to find out what might be the underlying thinking of the learner in his or her solution. In turn 5 another learner (L2) tries the problem and gives the response as $(-11) + (-5) = -6$. T3 in this incidence again, instead of seeking an explanation from L2 in order to determine his or her thinking, he seeks explanations from any member of the class (turn 6). L2 might be having a difficulty in understanding the functions of the negative signs involved as Vlassis (2008) stated as a challenge for learners. But T3 did not bother to find out what was L2's thinking. This shows T3's lack of attendance to his learners' mathematical understanding or thinking. Another learner (L3) is given chance to explain L2's response and also offer his or her alternative response (turn 6-7). L3 notes that L2 missed the negative at 5 and offers the correct response as $(-5) = -11 - 5 = -16)$ (turn 9). To this, T3 again asks the class to confirm whether the answer given is correct and even as the other learners have agreed with the solution he further asked for another explanation (turns 10 – 13). In this regard, looking at how T3 handled the learners' responses which represented their own understanding or thinking, one could say he really denied himself chance to get a feeling of his learners' thinking. In other words, the way he implemented the instruction was not responsive to his learners' thinking. This in turn let him in maintaining the same problem thorough out and not opting for some that would at least be workable for L1 and L2. Thus, T3's instruction lacked flexibility as it was never adjusted to assist individual learners who seemed to struggle to also advance to the desired understanding. In this way I argue that though T3' instruction was driven by plan but it was implemented rigidly. That is no room was allowed to deviate from the problem $(-11) + (-5)$ to other problems depending on individual learner's struggles with it. T3 had no intentions to understand his learners' thinking which would lead him in altering his planned exercise or problem to see if other problems could help the struggling learners.

4.2.3 T3's adjustment of tasks

The use of tasks in the teaching and learning of mathematics has been argued to help in keeping learners focused on the mathematics idea (Stein et al., 1996) and as well as keeping learners engaged (Smith and Stein, 2011). Tasks differ in the level of thinking they demand from learners; some are challenging while others are not. Choppin (2011) emphasized that challenging tasks can be beneficial to learners especially if the teacher maintains the level of the task learners are given to work with. As Choppin (2011) noted, teachers can be able to maintain the level of the tasks learners are working with if they are able to interpret their learners' mathematical thinking. Thus, this will allow them to build on the information they gathered so that they are able to assist learners towards completion of the task or in deciding on the next task. Segment 4 of the lesson and its respective excerpt (excerpt 2) below are selected to provide answers about T3's adjustment of tasks during classroom interactions.

Segment 4: Distinguishing subtraction expressions.

At this stage of the lesson T3 wanted to move learners to where they can be able to differentiate between subtraction expressions. He began by writing on the chalkboard the expressions; a) $+8 - -5$, $b)$ $8 - 5$ and c) $-3 - 9$. T3 then ordered learners to spot out the difference in the expressions. Excerpt 2 below is about the interactions that took place between the teacher and the learner (L6) as the learner attempted to identify the differences.

Excerpt 2: Distinguishing between $+8 - -5$, $8 - 5$ and $-3 - 9$

28	16:29	T3	Alright, so this won't take about few minutes. So let's say we have this example: a) $+8 - -5$, *and we have another one* $b)$ $8 - 5$ and we have another one c) $-3 - 9$.Now let's look at the three problems. Can we identify the difference among the three? Like what is the different? Compare the first one and the second one (teacher ask a learner to compare)
29	17:27	L6	Sir , the first one is 8 subtract negative 5 and the second one is positive 8 subtract 5
30	17:43	T3	Ok, do read out by spotting a difference I am not saying that you should read out what is written here. Can you try and identify the difference among the three?
31	18:00	L6	Number (a) if you subtracting a whole number by a negative number. You are subtracting two positive numbers
32	18:15	T3	Which one? What are we subtracting in number (a) ,on (a)
33	18:17	L6	On (b) as well
34	18:18	T3	(b) We are subtracting positive numbers, ah?
35	18.26	L6	At (a) we are subtracting ah positive number with a negative number (teacher also agreeing with learner)

36	18:30	T3	So what you saying is that on (b) we are subtracting two positive numbers and then on (a) we are subtracting a negative number from a positive number. What about (c) now do we agree with what he is saying here?

Excerpt 2 is an example of where the teacher (T3) adjusted the task based on individual learner's understanding or thinking. In turn 1 the teacher gave learners three subtraction expressions; a) $+8 - -5$, $b)$ $8 - 5$ and c) $-3 - 9$. T3 gave learners a task of comparing the expressions to identify their difference, specifically in turn 28, lines 6-7, he states that they should start by comparing only the first two expressions. A learner (L6) is chosen to respond to the task and in turn 29 L6 responds by indicating that '*the first one is 8 subtract negative 5 and the second one is positive 8 subtract 5'*. T3 appear not satisfied by L6's response as he states that he is not expecting him to read out what is written (turn 30). Still on turn 30, T3 follows L6's response with a question this time asking him to compare the three expressions. In this instance it is evident that T3 is attending to L6's thinking as it is displayed in his response and he is willing to use that information obtained to advance L6's thinking to the desired thinking without lowering the demand of the task. I say he is not lowering the task but maintaining its demand because instead of comparing the two, this time T3 requests L6 to compare three expressions. This shows that the tasks are adjusted with respect to what T3 has learned from L6's response. However, the adjustment of tasks was not outside what was already planned by T3 (no new tasks incorporated), learners were already required to distinguish between the three expressions but the teacher split the task in to two; first to compare the first two expressions and second to compare the three expressions. This adjustment of task did not lower the thinking demand they required from the learners. Smith and Stein (2011) and Sullivan et al. (2015) argue that to maintain the level of the task, the teacher can scaffold the learner's thinking. The teacher can do this by engaging the learner with questions that will assist him to correct his thinking. In turns 31 – 36, T3 appears to be following up on L6's responses with questions intended to assist L6 to advance to the desired response which he does in turn 35. And by so doing, T3 is scaffolding L6's thinking in a manner to assist him get to the desired outcome. I therefore argue that T3's adjustment of the tasks is limited but of utmost importance is that it was connected to the individual learner's thinking and at the same time not lowering the thinking demand.

4.2.4 Mathematical learning progression in T3's lesson

As it was indicated earlier mathematical learning progression involves a sequence or steps in a particular knowledge domain that learners can move incrementally from novice to more expert performance (Haritage, 2008). It was also further noted that one of the key features of mathematical learning progression is that expert performance builds from novice performance or as Bailey et al. (2014) put it "later knowledge often builds on earlier knowledge" (p.1). Similarly, we can say that abstract concepts build from simple concepts. In a teaching and learning situation this suggests that the teacher has to make sure that his activities build from simple knowledge to more abstract knowledge. For Thomas et al. (2017), this should also be done as a response to an individual learner's thinking. That is, the teacher should interpret the learner's thinking to understand their current understanding and thereafter advance the learner's understanding step by step to the desired learning goal.

To understand or provide answers about mathematical learning progression in T3's lesson I chose segment 5 of his lesson and its respective excerpt (excerpt 3).

Segment 5: Subtraction of directed numbers using a number line.

In this section of the lesson T3 wanted learners to use a number line to subtract the expressions (a) $+8 - -5$, b) $8 - 5$ and c) $-3 - 9$) which they were using in the previous activity where they were required to distinguish between the subtraction expressions. T3 invited learners to work out (a) $+8 - -5$ using the number line. The first student invited to demonstrate how the answer could be obtained found the answer to be -3. Excerpt 3 below captures the discussions that took place thereafter.

Excerpt 3: Using number line to work out $+8 - -5$

51	23:43	T3	Right ok, ok is the answer negative three?
52	23:52	Lnrs	No (chorus response)
53	24:00	T3	Ok you come and help (pointing at the other learner)
54	24:05	L7	Positive 3
55	24:06	T3	Positive 3? (other learner say no while others say yes)

56	20:12	Lnrs	No (chorus response)
57	20:13	T3	Ok , can you come and show us using a number line and show us how you are getting positive 3
58	24:30	L7	(Learner demonstrating)
59	25:32	T3	Alright ok , can you please explain to us how you get the answer
60	25:40	L7	(using a number line) I move 8 times to the right then move 5 times to the left: $+8-(-5)=+3$ (Teacher re-voicing what the learner is saying)
61	26:01	T3	So you move 8 times to the right and then 5 times to the left. Ok do you agree with that (teacher asking the whole class and pointing at one learner to help others).

This section of T3's lesson was dedicated to helping learners use a number line to subtract directed numbers. And as research suggests (Haritage, 2008; Bailey et al., 2014) we would expect to see how T3 moves learners through a sequence of steps that build from each other to the desired knowledge. Specifically, we would want to see how he leads learners in a sequence of steps relating to the use of a number line to help them perform the operation successfully. However, the discussions portrayed in excerpt 3 show no clear developmental path to guide learners in to understanding how the number line can be used to perform the operation. Evidence of this begins from turn 51. At this turn the learner who had been selected to perform the operation gave an incorrect answer -3 and T3 turns to the whole class to confirm the answer to which they say it is not correct. Instead of building on the learner's current understanding which happens to be flawed, he turns to a different learner to request that he helps the class on how to perform the operation (turn 53). The learner (L7) gives the answer as positive 3 (turn 54) and unlike in the first instance, this time T3 requests L7 to explain using the number line (turns 57-59). At this point T3 is trying to make sense of the learner's thinking or understanding in as far as the use of a number line is concerned.

In turn 60 L7 explains the procedure saying, *I move 8 times to the right then move 5 times to the left*. T3 re-voiced L7's statement and then asked the rest of the class whether they agree (turn 61). After they indicated that they disagree, T3 then moved to another learner requesting her to help the class on how to perform the operation using a number line (turn 61). In this regard T3 is aware of L7's thinking and understanding on how to use the number line and he realizes that it is not the correct thinking but instead of helping him he moves to another learner. I therefore argue that T3's instructional practices are not showing evidence of mathematical learning progression that is connected to or helping individual learner advance from their current thinking to the desired thinking or understanding.

4.2.5 Summary of T3's professional noticing skills and MKT

The analysis of T3's instructional practices from the selected excerpts revealed that his instruction lacked flexibility. Regardless of whether a learner struggled with the exercise or problem used to mediate understanding a certain concept, no adjustments were made to assist the individual learner to advance to the desired understanding. Rather T3 continued with the same problem and this portrayed rigidity in the way he implemented his instruction. This rigidity in the implementation of instruction may have been influenced by his reliance on chorused responses from learners. Such chorused responses did not allow him a chance to understand each individual learner's thinking or understanding

Moreover, the instructional practices of T3 revealed his limited adjustment of the tasks at the individual learner's level. T3 was able to interrogate learners through questions so as to give them chance to express their thinking in relation to the task they were doing. However, as indicated earlier, little adjustment to the task the learners were engaged with was done and this could not allow T3 opportunity to advance individual learner's thinking to the desired understanding. But, even though T3's adjustment of tasks was limited he however made sure that the thinking level they required from learners was not decreased.

The analysis has further shown that T3's instructional practices lacked connection to the mathematical learning progression specifically in relation to the learners' development of the concept of subtracting directed numbers using a number line. T3 could not assist individual learner to advance from their current understanding step by step to the desired understanding. And in this case also, his reliance on chorused responses and not following up on what the learner has expressed as his or her understanding denied him chance to move each individual learner to the expert knowledge.

4.2.5.1 Explaining T3's level of professional noticing and MKT

To provide an answer to the second research question which is about an explanation of teacher's (T3) level of professional noticing and mathematical knowledge for teaching directed numbers at grade 8, I look at what the analysis of the data has revealed about his instructional practices. In terms of implementation of instruction, the analysis shows that T3 *implemented his instruction rigidly*, that is, not deviating from his plan. On task adjustment, T3's instructional practices revealed *limited adjustment of the tasks at the individual learner's level*. And lastly on the issue of mathematical learning progression, T3's instructional practices *lacked connection to the mathematical learning progression* of learners, particularly in learning about subtraction of directed numbers using a number line as explained in the analysis.

Using the four quadrants of categorising the level of teacher's professional noticing and mathematical knowledge for teaching as depicted in Thomas et al.'s (2017) framework, T3 may be argued to have *limited responsiveness to students' mathematical thinking and under-developed MKT* in relation to the teaching of directed numbers at grade 8. This quadrant is characterized by teacher instructional practices which show tenuous connection to mathematical learning progressions, have little to no within-lesson adjustment at the individual level and activities that are implemented exactly as they are written or designed (see table 3.2 above).

4.4.13 Analysis of T4's instructional practices

In this section, an analysis of T4's instructional practice is provided with a focus on providing answers about her professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8. Just as in the case of T3, the main events of T4's lesson are summarized in a table form (see table 4.3.1). In the following sections, I engage with the excerpts selected from segments or moments within the lesson that demonstrate each of the three categories used for understanding the teacher's professional noticing skills and mathematical knowledge for teaching.

4.3.1 T4's lesson

The lesson was on addition and subtraction of directed numbers. The lesson began with T4 reviewing learners' understanding about integers. After that they focused on comparing and arranging directed numbers in the order of their sizes. Thereafter they began working on how they can add directed numbers. T4 introduced learners to the golden rules they can use in adding directed numbers and thereafter they continued on addition and subtraction of directed numbers. The table below represents the summary of the lesson.

Table 4.3.1 *Summary of T4's lesson on addition and subtraction of directed numbers*

Segment	Time	Activity
1	**00:00 – 04:55**	**Introduction of the lesson on integers**
2	**05:00 – 10:37**	**Comparing directed numbers.**
3	**11:18 – 14:53**	**Addition and subtraction of directed numbers**
4	**15:16 – 15:43**	**Teaching about the golden rules**
5	**15:45 – 24:01**	**Addition and subtraction of directed numbers.**
End of lesson		

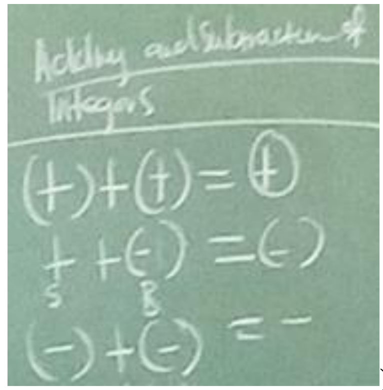
4.3.2 T4's implementation of instruction

To answer the question about T4's professional noticing skills and MKT as displayed in her instructional practices I first look at how she implemented her instruction. Two segments (segment 1 and segment 3) and their respective excerpts are selected to provide such answers. The segments are not presented in the order they happen in class but in a way that they help in understanding how T4 implements her instruction. Thus, I begin with the analysis of segment 3 in the following section.

Segment 3: Addition and subtraction of directed numbers (integers)

In this segment of the lesson, T3 wanted to make it easy for learners to add directed numbers by first showing them the rules they can follow to help them decide on the sign of the answer. She presents the rules on the chalkboard and through questioning wanted to see if the learners could follow how the rules work. She also used numbers, instead of just asking about the combinations of signs, to check whether learners can see the applicability of the rules. Excerpt 4 below gives details of what transpired during the interactions.

Excerpt 4: Using rules for addition of directed numbers

29	11:18	T4	(T4 writes on the chalkboard: ) So now we are going to look at adding and subtracting of integers , adding and subtracting of integers (teacher writes on the board).Ok if you add positive integers your answer will be
30	11:52	Lnrs	Positive (chorus)
31	12:00	T4	Your answer will be positive. But then, if you are adding smaller positive number plus adding a negative number
32	12:06	Lnrs	Negative (chorus)
33	12:07	T4	If you are adding a smaller positive number and your negative number is bigger and you are adding a positive and a negative what will be your answer?
34	12:20	L7	Positive
35	12:26	L8	Negative
36	12:28	T4	Why negative?
37	12:30	T4	Yes these two signs will make a negative and our answer always the sign of the biggest number .If you have negative plus negative what do you think our answer will be
38	12:54	Lnrs	Positive (chorus response)
39	12:55	Lnrs	Negative (chorus response from other group of learners)
40	12:57	Lnrs	Positive (other learners giving chorus response)
41	12:58	Lnrs	Negative (other group giving chorus response)

42	13:00	T4	Ok, because we are arguing about it .ok let's say we are given negative 2 plus negative 3 : $-2+(-3)$.I want you to give the answer
43	13:18	L9	1
44	13:20	T4	$-2+(-3)$ I want you to calculate and give me the answer there. Let's give others a chance
45	13:30	L10	-5
46	13:32	T4	Your answer will be :  Your answer is negative because you are adding negatives and your answer is negative 5

T4's presentation of activities shows evidence of a planned instruction though it is a limited one and it also shows some little flexibility from the planned instruction. Vassileva and Wasson (1996) talk about the importance of selecting and sequencing of activities, such that they complement each other to assist the learner to understand the concept, as an illustration of a planned instruction. Looking at how T4 implemented her instruction we can realise a sequence that is helping learners to finally be able to add directed numbers. In turn 29, T4 begins by presenting the rules on adding directed numbers. She then follows up that with tasks for the learners to help them use those rules (turns 29 – 41). It should be observed that throughout those turns T4 is just using statements to help learners understand how the rules work, she is not using the actual numbers. In turn 42, after she realised that learners are struggling, she switched to using the actual numbers to illustrate how the rules work, specifically in turn 42 line 2, she used the example '$-2+(-3)$'. Also, her explanation of the final answer in turn 46 is consistent and make use of the addition rules she presented to learners to be familiar with. Her explanation was; *your answer is negative because you are adding negatives and your answer is negative 5.* Adding negatives was also included in the list of rules for adding directed numbers. At this point we can realise that T4 is not implementing her instruction rigidly as planned. Evidence is in the use of trial and error (Thomas et al., 2017) as she chooses the tasks to support learners to understand the addition rules. This can be seen from the following segment and its respective excerpt.

Segment 1: Introduction of the lesson on integers.

Earlier in this segment T4 had asked learners to give out the example of positive and negative numbers. Now, in excerpt 5 below he is trying to see if learners can compare the directed numbers in terms of which one is bigger and which is smaller.

<u>**Excerpt 5: Comparing directed numbers in terms of size**</u>

14	03:45	T4	Positive 100 and negative 100 (teacher re-voices). So these our set of positive numbers and negative numbers including zero ok. So if we are given temperature: 80 degrees Celsius. Let's say 21degrees Celsius, zero degrees Celsius and negative 1 degree Celsius. So which one is the hottest day or which one is the coldest temperature?
15	04:35	L5	Negative 1

In this instance, T4's chosen example appears not to have been planned and carefully thought of rather she was using trial and error to see what will work. This is evidenced in turn 14 lines 3-5 where she said "*So if we are given temperature: 80 degrees Celsius. Let's say 21degrees Celsius, zero degrees Celsius and negative 1 degree Celsius"*. T4 wanted to make an example of temperatures in a certain day. Now, because she had not planned and thought careful about that example, she begins with the temperature of 80 degrees Celsius. Realising that it is not practical to have such a temperature, she then changed and began with 21 degrees Celsius. So, in this way it can be argued that the way T4 implemented her instruction showed some little flexibility which is seen in the way she used trial and error to choose tasks to be used. This trial and error could also be an indication of a limited planning from T4.

4.3.3 T4's adjustment of tasks during instruction

The research of Choppin (2011) has emphasized the importance of adjusting tasks based on the teacher's interpretation of individual learner's understanding. It has shown that teachers are able to adapt subsequent tasks that are of the same cognitive demand as the preceding ones if they attend and interpret their children's thinking. Similarly, Thomas et al. (2017) have argued that a teacher whose instruction is responsive to learner's mathematical thinking, his or her adjustment of tasks is informed by his or her interpretation of the individual learner's thinking. In this section I attempt to analyse T4's lesson to understand how she adjusted the tasks in her class. Segment 3 of the lesson and its respective excerpts (excerpt 4 above and except 6 below) are chosen to provide the answers about T4's adjustment of tasks in her teaching.

Segment 3: Addition and subtraction of directed numbers (integers)

In this segment of the lesson, T3 wanted to make it easy for learners to add directed numbers by first showing them the rules they can follow to help them decide on the sign of the answer. She presented the rules on the chalkboard and through questioning, she checked whether learners could follow how the rules work. She also used numbers, instead of just asking about the combinations of signs, to check whether learners can see the applicability of the rules in adding the directed numbers (see excerpt 5 above). In excerpt 6 below, T4 is further adjusting her tasks this time to clarify to learners how subtraction of a negative number can finally reduce to addition of that number.

<u>Excerpt 6: subtracting a negative number</u>

47	13:40	Lnrs	Oooh, negative 5 (chorus response)
48	13:47	T4	Ok, so if you have minus and negative what do you think your answer will be?
49	13:50	Lnrs	Positive (chorus response)
50		T4	(teacher not commenting on the learners response continues to ask questions) What do you think your answer will be
51	14:02	Lnrs	Negative (chorus response)
52	14:01	T4	Ok, let's say we have $-2-(-3)$ your answer will be
53	14:12	Lnrs	Positive 1, negative 1, positive 5, negative 1, minus 1 (learners gives different answers showing confusion). It's one. positive 1
54	14:35	T4	No we have to work it out and tell me the true reflection. It's what , so basically $-2+3$ (teacher pointing at consecutive negative on the expression) because these two the match together to create a positive ,so its $-2+3$
55	14:47	L11	-5
56	14:48	T4	No, $-2+3$ will give you?
57	14:52	L11	Positive 1
58	14:53	T4	Yes, it will give a positive. So you answer is a positive 1. Ok can I rub this one off

As will be shown below, T4 is able to adjust her tasks during instruction and at the same time she is able to maintain the thinking demand of the tasks. However, the way she adjusted her tasks is not based on individual learner's thinking, she relies on chorused responses from the learners. In excerpt 4 turn 29, T4 writes the addition rules on the chalkboard and follows that with a question for learners to state what will be the answer when two positive integers are added. A chorused response is given by learners suggesting the answer as positive. In the same excerpt but now in turn 32, T4 accepts the answer and then changes the task to that of adding a small positive with a [large] negative. T4 repeats the same steps in turns 33 where he accepts a chorused response and then moves to a new task. This occurs even at turns 37 – 40 of excerpt 4. This way of banking on chorused responses denies T4 a chance to experience individual learners' understanding which might help her to make informed decisions as she adjusts her tasks.

From turn 42 T4 changes to the use of numbers after realising the confusion from learners when using statements, example is the task; $-2+(-3)$. Evidence of T4 attending to learners' responses is present in this turn though she is still working with the whole class not attending to individual learners. From excerpt 6 turn 48, T4 now oscillates between using word problems and then clarifying with the use of numbers. For instance, in turn 48 T4 says "ok, so if you have minus and negative what do you think your answer will be?" and in turn 52 she asks "ok, let's say we have $-2-(-3)$ your answer will be?". In this second statement T4 is using numbers to help learners visualize what the task is like and also to see if they can manage to work it out. T4 is adjusting the tasks to help learners to understand how the operation is done and at the same time she is not lowering the thinking level required by the tasks. The use of numbers in this case can just be understood as a way of scaffolding learners (Snith & Stein, 2011). But still, T4 is working on the chorused responses than individual learners. Where she tried to work with individual learner is in turns 55-57. But again, her response to the answer given by the learner is not aimed at understanding or interpreting the learner's thinking rather it is judgmental. The learner (L11) says the answer is "-5" in turn 55 but her response is "No, $-2+3$ will give you?" (in turn 56). The 'no' given by T4 means she is evaluating or judging L11's response not allowing her to explain her thinking. Thus, one can argue that T4's adjustment of tasks is disconnected from learners' mathematical thinking.

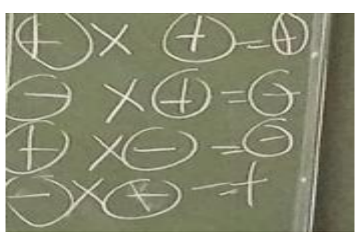
4.3.4 Mathematical learning progression in T4’s lesson

Mathematical learning progression involves knowing how learners’ mathematical knowledge in a certain domain builds. In their work, Spaull and Kotze (2015) indicated that mathematics is one of the subjects with a strong vertical structure and whose concepts are integrated. Meaning the concepts have a strong relationship and they are such that they build from one another with the more abstract concepts building from the simple ones. This being the case, the teacher needs to be aware of this hierarchal nature of concepts within a topic or across topics in order to assist learners as they learn these concepts. In this section I analyse T4’s lesson presentation to provide answers about how she addresses the issue of mathematical learning progression in her class. Segment 4 and its respective excerpt (excerpt 4) are chosen to provide such answers.

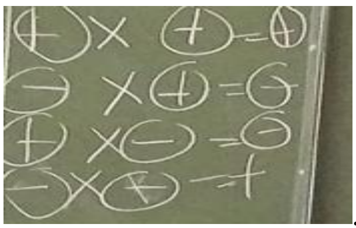
Segment 4: Teaching about the golden rules

In this part of the lesson T4 introduced learners to the golden rules that she believed would assist learners in handling the issue of signs as they work with the signed numbers. In particular, T4 wanted these rules to help learners as they added and subtracted directed numbers. Excerpt 7 below shows that T4 wrote the rules on the chalkboard and thereafter engaged learners through questions to see if they could apply them.

<u>Excerpt 7: Using the golden rules when subtracting directed numbers</u>

59	15:16	T4	(The teacher writes the four golden rules that she taught earlier and ask she questions)  So what if you have a positive and a negative?
60	15:20	L12	Negatively
61	15:22	T4	What if you have positive minus a negative. Sorry a positive minus a positive. A positive minus a positive, what will be your answer?

62	15:32	Lnrs	Positive (chorus response)
63	15:34	T4	Your answer will be what
64	15:35	Lnrs	Positive (chorus response)
65	15:40	T4	Ok let's say $5 - 3$ that gives us a what?
66	15:43	Lnrs	Positive 2 (chorus response)

In turn 59, T4 presents the following rules which he refers to as golden rules . When looking at these rules, they are about multiplication of signed numbers, showing what kind of result to expect in each case. After presenting these rules she asks "So what if you have a positive and a negative?" (in turn 59). This question may be ambiguous because it doesn't state what is happening with positive and negative. Assuming it was relating to multiplying positive number by negative number, one learner could still interpret it as adding or subtracting a negative number from a positive number because the topic of the day is about addition and subtraction of directed numbers. The result of this may be to lead learners into developing misconceptions in as far as adding and subtracting directed numbers is concerned. In turn 60, a learner (L12) gave the answer as 'negatively'. T4 doesn't comment about the response given by L12 which might even imply that she did not notice that L12 said negatively not negative. She instead went on to ask a question: *A positive minus a positive, what will be your answe*r? (in turn 61). This question again is not answerable because depending on the numbers used, the answers will be different in terms of their signs which seem to have been her focus. Learners on the other hand give the answer as 'positive' (in turn 62). Now, the questions that one would ask are; do the learners say the answer is positive because they literally consider subtracting a positive number from a positive number always yielding a positive number or is it because they are looking at the golden rules and then deducing from them that if when two positive numbers are multiplied the answer is positive then even when they are subtracted the answer will still be positive? T4 seems to be agreeing with her learners looking at the example she makes which results in a positive number (in turn 65). The problem with this is that for some learners they might generalize that a positive number subtracted from a positive number gives a positive number hence developing a misconception. T4 completed this section by giving learners exercises to practice on (see appendix A).

The analysis above shows that T4's approach to teaching learners how to add and subtract directed numbers using the golden rules doesn't connect with how learners should develop those concepts. T4 was unable to engage in steps that are interrelated and building from simple to the desired level of understanding looking at how he began from multiplication (golden rules) to how learners can subtract the numbers. So, as Thomas et al. (2017) would suggest, T4's instruction involves little to no connection to mathematical learning progression of learners on addition and subtraction of directed numbers.

4.3.5 Summary of T4's professional noticing skills and MKT

The following categories informed the analysis of T4's instructional practices; how she implemented her instruction, how she adjusted tasks used in the teaching as well as how her instructional practices related to how learners should learn the knowledge domain they are taught (mathematical learning progression). The analysis revealed that T4's implementation of instruction showed limited planning. The way she sequenced the activities used, was such that they complemented each other which showed evidence of planned instruction. However, her trial and error in selecting some of the tasks to use is the one that led to a conclusion that her implementation of instruction showed limited planning. And the same trial and error she engaged can also be considered evidence of her response to learners' understanding and thus portraying her instruction as flexible.

Furthermore, T4 adjusted tasks in her teaching but such adjustment of tasks appeared not to be based on individual learner's thinking or understanding. Similar to T3, T4 relied on chorused responses from learners and such chorused responses were the one that informed how she adjusted her tasks. But like Thomas et al. (2017) indicated, to show responsiveness in terms of task adjustment, the teacher must interpret each individual learner's understanding and such interpretation be used to decide on the next task that will help advance learners understanding. So, acting on the bases of the chorused responses means T4 was not even aware of the current understanding of her individual learners. This may also suggest that the way she adjusted her tasks was disconnected to the learners' mathematical thinking. In one occasion where there was no chorused response but an individual learner gave a response, T4 judged or evaluated the learner's response without delving much in to the response to understand what might be the learner's understanding. Thus, this shows that the way T4 adjusted her tasks was not in response to what she interpreted about individual learner's understanding.

Also revealed by the analysis is that T4's instruction involved little to no connection to mathematical learning progression of learners on addition and subtraction of directed numbers. As it was shown from the analysis, T4's use of multiplication rules for directed numbers when teaching about addition and subtraction showed no link on how they can assist learners understand the operations easily. This could mean that T4 was not aware of how learners develop the concept of addition and subtraction of directed numbers or how she was not able to organize her activities step by step in such a way that they would assist learners develop the concepts easily.

4.3.6 Explaining T4's level of professional noticing and MKT

The analysis of T4's instructional practices revealed among others that the way she implemented her instruction involved use of *trial and error* in selecting some of the tasks for her learners. Her in-the-moment design of tasks led to some not working as she would have wanted to and thus forcing her to trying those that could work. This use of trial and error in deciding which tasks to use was argued to show little flexibility in the way T4 implemented her instruction as well as a *limited plan* of instruction by T4.

Apart from how she implemented her instruction, the way T4 adjusted her tasks was such that it appeared *disconnected to the learners' mathematical thinking*. For instance, in a situation where she evaluated her learner's response without interpreting what could be the learner's understanding, implied that whatever the task she would next decide on would be disconnected to what the learner was thinking. Also, relying on chorused responses did not allow T4 to adjust her tasks as informed by the individual learners' thinking.

Lastly, T4's instructional practices showed *little to no connection to mathematical learning progression* of learners on addition and subtraction of directed numbers. As explained above, her use of multiplication rules for directed numbers when teaching about addition and subtraction showed no link on how they can assist learners understand the operations easily. So, such an approach implied that T4 might not be aware of how the development of the concepts of addition and subtraction of directed numbers could be simplified for learners.

The features of T4's instructional practices as described above explains her level of professional noticing and MKT as *highly responsiveness to students' mathematical thinking and under-developed MKT* (Thomas et al., 2017). This is as a result of her instructional practices that portray the following features: tenuous connection to mathematical learning progressions, limited planning, instruction driven by trial and error and task adjustments disconnected from children's mathematical thinking (Thomas et al., 2017, p.14).

4.4 Discussion

This study set out to investigate the teachers' professional noticing skills and their mathematical knowledge for teaching directed numbers at grade 8. It employed Thomas et al.'s (2017) framework on professional noticing and MKT to explore teachers' instructional practices in order to display/reveal the teachers' professional noticing skills and MKT. This was also to help in explaining the teachers' level of professional noticing and mathematical knowledge for teaching directed numbers.

The research questions the study sort to answer pertained to teachers' professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8. This involved determining what noticing skills and MKT are revealed in the teachers' instructional practices. Also, to be answered was what explanations can the revealed noticing skills and MKT of teachers provide about their level of professional noticing and mathematical knowledge for teaching directed numbers at grade 8.

The study answered the questions by analysing documents of the observed lesson transcripts from the primary source. In this section, the findings from the analysis of data are discussed in relation to the literature on the teaching of directed numbers and the theoretical framework (teachers' professional noticing and MKT) of the study. The emerging themes from the data analysis are also discussed.

4.4.1 Errors and misconceptions in teaching and learning of directed numbers

Nesher (1987) explained that misconceptions represent an incorrect understanding or thinking of the learner and that such understanding is a true reflection of what the learner knows about the concept. The literature has pointed to some common misconceptions learners have with regard to understanding directed numbers and operations on directed numbers. While these misconceptions were not found to be committed by learners in the data analysis but the analysis has shown that teachers can contribute to learners developing these incorrect understandings. In this case the use of ambiguous language by the teacher has been identified to be the factor that can possibly contribute in learners developing a faulty understanding. As explained above, the use of ambiguous statements like 'what if you have a positive and a negative' by T4 when not explained what operation we are focusing on may lead to learners developing incorrect understanding. Possibly learners may end up with a misconception of overgeneralization (Makonye & Hantibi, 2014). As Makonye and Hantibi (2014) explained this misconception, it happens when learners think a rule that applies in a certain situation can also be applied on a situation that it doesn't hold. I say this kind of use of language by the teacher can possibly lead to this misconception because for learners who understood the teacher to mean that when you have a negative and a positive the answer will always be negative they are likely to use that understanding for any operation. That statement (what if you have a positive and a negative) when not qualified about the operation referred to, is not providing a mathematically correct procedure and thus can result in a misconceived understating from learners.

4.4.2 Use of teaching models for directed numbers

The use of teaching models (neutralization models and number line) has been argued to assist learners understand directed numbers by a number of researchers (Battista, 1983; Chang, 1985; McDonald, 2011; Stephan & Akyüz, 2012; Thompson & Dreyfus, 1988). The analysis of data has reflected the use of these models, specifically the use of a number line, by the teachers to help learners in their learning of directed numbers. However, the same data has also shown an insufficient use of the number line to guide learners. It was evident from one learner who volunteered to explain how the number line could be used to add directed numbers that he was not quite sure of how to use it. Instead of explaining to that learner and even to the rest of the class how the number line can be used, T3 instead overlooked that. So, he denied the learners the opportunity the model would afford to them; which is, enabling them to understand the directed numbers better as the literature suggests (see for example Stephan & Akyüz, 2012).

4.4.3 Teachers' professional noticing skills and MKT

The analysis of data in the preceding sections has shown a variation in the teachers' professional noticing skills and MKT. This was reflected in the way they implemented their instructions, adjusted their tasks during instruction and how their teaching aligned to the learners' mathematical learning progression. These three themes were derived from Thomas et al.'s (2017) framework and were regarded as the defining features of the different levels of teachers' professional noticing and MKT.

4.4.3.1 Implementation of instruction

The way teachers implemented their instruction was differentiated on the following aspects: whether it was plan-driven/showed limited planning and whether it was flexible/rigid (Thomas et al., 2017). Planning in this instance involves "the process of optimal selecting and sequencing of the tutorial interactions focused on a given content" (Vassileva & Wasson, 1996, p.7). Similarly, it can also be regarded as involving the selection and sequencing of exercises or examples to be used to initiate interactions or focus learners on the content to be covered during teaching and learning. How this plan is executed, Thomas et al.'s (2017) framework shows that effective professional noticers do not follow it rigidly but they become flexible enough to change in respect of their learners' contributions. The analysis of data for the two teachers involved in this study has shown some unconvincing flexibility in the way the teachers implemented their instruction. This was evidenced in the way they engaged with learners' contributions in their teaching. Whether a learner struggled with the exercise or problem used to mediate understanding of a concept, little or no adjustment to the initial plan was done in order to cater for the struggling individual learner.

This rigid implementation of instruction shows lack of engagement of students in their learning and denies the teacher an opportunity to realise whether his/her learners have a misconceived understanding of the concept they are dealing with. An explanation by Nesher (1987) shows that misconceptions are a faulty understanding or thinking of the learner and that such understanding is a true reflection of what the learner knows about the concept. So, by not engaging with the students' contributions may possibly leave them with their misconceived understanding and thus adding to the challenges they have in terms of understanding the signed numbers and operations on them as alluded to by different researchers (Bishop et al., 2014; Bofferding, 2010; Bolyard & Moyer-Packenham, 2012; Carraher, & Schliemann, 2002). For Ball et al. (2008), failure to engage with learners' strategies denies the teacher an opportunity to understand or identify and act on their non-standard approaches. This shows lack of specialised content knowledge as described by Ball et al., 2008.

4.4.3.2 Task adjustment

This aspect deals with how teachers adjust their tasks or activities during their classroom teaching. Of major concern in adjusting tasks is whether the adjustment is informed by what the teacher has interpreted as the learners' current thinking (Choppin, 2011; Thomas et al., 2017). According to Thomas et al. (2017) teachers who are highly responsive to students' mathematical thinking and have well developed MKT adjust their tasks based on individual learner's current thinking. That is teachers need to continually assess learners to determine their understanding and use that to decide on the next task to use. The analysis of data has shown prevalence of teachers' limited adjustment of the tasks at the individual learner's level. It also revealed that where adjustments were done, they appeared disconnected to the individual learner's mathematical thinking. Observed from analysing of data was that the teachers did not engage much with learners' responses to the tasks they were given. When a learner gave an incorrect answer to the task, more often the teacher would pass to the next learner to respond. If the learner gets the answer correct then the teacher would move to the next task. So, clearly the choice of this second task isn't in response to what was the underlying incorrect thinking of the learner. As such the teacher is just moving with those who seem to be understanding by the correct responses they give. The result of this is to leave learners behind with the misconceptions or difficulties they might be having. Vlassis (2008) argued that sometimes learners face difficulties in understanding the functions of negative signs in a problem. This situation could be argued from a leaner who gave the answer as negative 6 from working out (-11) + (-5) in T3's lesson. Instead of attending to the learner's solution and thus interpreting what might be his/her underlying thinking, T3 moved to the other learner who gave the correct response. There after T3 moved to the next task. As stated above this kind of adjustment of tasks is disconnected to the learners' thinking.

4.4.3.3 Mathematical learning progression

Mathematical learning progression has been described by Haritage (2008) to involve a step by step developmental path for learners from novice to more expert performance in a particular domain. In mathematics, for example, topics are hierarchal in nature (Spaull, & Kotze, 2015), that is more abstract concepts build from simple concepts. So, in teaching any topic in mathematics, teachers must be aware of its hierarchal nature and be sure which concepts need to be treated prior to others. This will enable a smooth transition of learners from novice expertise to the desired level. Haritage (2008) argue that this can be possible if teachers know the current thinking of their learners and where they want to take them to. She indicates further that by engaging formative assessment in their teaching, teachers can be able to diagnose their learners current understanding and then use their knowledge of learners' mathematical learning progression to advance them to the desired performance. The analysis of data has indicated that for both teachers their instructional practices showed little to no connections to learners' mathematical learning progression. Evidence was in the use of multiplication rules for directed number while teaching about addition and subtraction of directed numbers for T4 and T3's inadequate use of a number line to develop the concept of subtracting directed numbers step by step.

4.5 Issues emerging from the analysis

Two issues of concern emerged from the analysis of data above: teachers' reliance on chorused responses and teachers as possible sources of misconceptions for learners.

4.5.1 Reliance on chorused responses.

Thomas et al.'s (2017) framework for professional noticing and MKT emphasizes much on an individualized approach to teaching. That is, emphasis is put on how teachers respond to individual learner's mathematical contributions in class or how teachers' instruction is shaped by individual learner's thinking or understanding. In short, the framework stresses on how the teacher works with individual learners in his or her class for the benefit of the learners. However, the analysis of data has shown that the teachers involved in this study preferred to work with chorused responses from learners. Little attempt was made to attend and interpret individual learner's responses. Since chorused responses, when correct, may deceive the teacher to think that all learners understand what's being taught, teachers are at risk of leaving students behind in their instruction. This may have been a possible reason for the rigidity in the implementation of their instruction which was observed however this may require research to prove.

4.5.2 Teachers as possible sources of misconceptions

While the literature has not dealt much with the causes of learners' misconceptions, the analysis of has revealed a possible cause of misconceptions for learners. As explained in details in one of the sections above, the teachers' use of language was identified as a possible source for learners' misconceptions. I have explained that ambiguity in statements like 'what if you have a positive and a negative' when not explained what operation we are focusing on may lead to learners developing incorrect understanding. So, if learners are left with these unclear statements, they are likely to see them holding for any operation which will be a misconceived understanding

4.6 Conclusion

This chapter has looked at T3 and T4's instructional practices. Thomas et al.'s (2017) framework helped in providing tools to assist in getting an insight in to each teacher's instructional practices. The framework also enabled me to decide on the level of each teacher's professional noticing and mathematical knowledge for teaching directed numbers.

The data from the lesson transcripts showed some differences and similarities in T3 and T4's instructional practices. In terms of how they implemented their instruction, the data has shown that T3 stuck to his plan and was never deviated by learners' responses while T4 at some instances changed from his plan and use trial and error to find suitable tasks that would advance learners' understanding. What was similar for T3 and T4 is their reliance on chorused responses, not paying much attention to individual learners' responses.

How T3 and T4 adjusted their tasks and how their instructional practices linked to mathematical learning progression in terms of addition and subtraction of directed numbers were looked at. In this regard, data has shown some similarity in T3 and T4 in that their adjustment of tasks was not informed by individual learner's thinking which could have due to the fact that they relied most of the time on chorused responses. As for mathematical learning progression, their instructional practices showed little to no connection to learners' mathematical learning progression. Based on these categories the analysis helped in explaining T3's level of professional noticing and MKT as *limited responsiveness to students' mathematical thinking and under-developed MKT* while T4 as *highly responsiveness to students' mathematical thinking and under-developed MKT.*

In the next chapter I present the concluding remarks and recommendations of the study.

CHAPTER 5

SUMMARY OF FINDINGS AND CONCLUSION

5.1 Introduction

The case, discussed at the beginning of this study, of Ms. González (a grade 7 teacher) as reported in Hill and Ball's (2009) work, motivated me to focus this study in this area of professional noticing and mathematical knowledge for teaching. The way Ms. González overlooked her learners' contributions and instead relied on what was offered in the textbook made me to inquire what valuable information might she be missing in the process. This case of Ms. González was used in Hill and Ball (2009) to illustrate the situation where a teacher lacked MKT. But while searching the literature I could also relate it to a lack of a skill of noticing her learners' mathematical thinking. This then motivated me to focus this study on the two concepts (professional noticing and MKT) specifically in the teaching of directed numbers which the literature had revealed as challenging to learners. The aim was to explore teachers' instructional practices in relation to these two concepts. This study sought to answer the questions;

- What professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8 are displayed by the teachers?
- How can the levels of the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 be explained?

In this chapter I present the summary of the findings from chapter 4 and thus draw conclusions and recommendations based on the findings.

5.2 Summary of findings

Categorization of teacher's professional noticing and mathematical knowledge for teaching is in the following four levels: Highly responsiveness to students' mathematical thinking and well-developed MKT, Limited responsiveness to students' mathematical thinking and well-developed MKT, Limited responsiveness to students' mathematical thinking and under-developed MKT and Highly responsiveness to students' mathematical thinking and under-developed MKT (Thomas et al., 2017). As illustrated by Thomas et al.'s (2017) framework, determination of each level is in terms of how the teacher implements his instruction, how he adjusts the tasks used in his teaching and how his activities intended to advance learners' understanding of the concept relate to the way learners develop such concept. Because these activities were not described in details in Thomas et al.'s (2017) work, literature was searched to provide deeper understanding for each activity. This enabled construction of descriptors for each activity and thus made it easier for me to analyse the teachers' instructional practices in order to answer the first research question.

5.2.1 Research question 1

The first research question of this study was;

What professional noticing skills and mathematical knowledge for teaching directed numbers at grade 8 are displayed by the teachers?

A focus on how the teachers implemented their instruction was used as a way of exploring their professional noticing skills and MKT. The findings revealed that T3 and T4 had some differences in the way they implemented their instruction. The idea here is not to compare who is the best between the two teachers but to understand how each implemented his or her own instruction. Notably is that the findings showed evidence of planned instruction for both T3 and T4. The difference, however, concerned how each teacher was flexible enough to deviate from his/her plan in response to the learners' mathematical thinking or understanding. The findings have pointed out that T3 implemented his instruction rigidly without deviating from his plan or changing some of his planned activities or tasks to accommodate the new thinking raised by the learners. Relying on chorused responses could have also contributed in T3's rigid implementation of instruction because he missed the opportunity to get different understanding from learners. Not to deny that chorused responses may still be advantageous in the teaching and learning situation but in most cases they contribute in masking different views from individual learners. T4's implementation of instruction showed some little flexibility or deviation from the plan. T4 like T3 also relied on chorused responses but the difference is that she was able to deviate from her planned activities in response to what the chorused response suggests of learners' understanding. Engaging in trial and error to find the task or activity that would be best suitable for learners showed her willingness to respond to learners' thinking.

The second aspect that was a focus for exploring the teachers' professional noticing skills and MKT, was how they adjusted their tasks during the teaching and learning process. Similarities and some differences were observed in how T3 and T4 adjusted their tasks during their lessons. As it was indicated that both T3 and T4 seemed to rely on chorused responses, they both had limited adjustment of tasks at the level of individual learners. This finding seems to support Choppin's (2011) research which indicated that by engaging or interpreting the learners' understanding or thinking enables the teacher to adjust his or her tasks such that they help in advancing the learners' understanding. So, working with the chorused responses masked individual learner's understanding and as such denied T3 and T4 opportunity to engage with each learner's understanding. Not understanding what each learner is thinking meant that both T3 and T4 would go on with their planned tasks without adjusting them to cater for the different thoughts raised by learners. And this way of adjusting tasks shows a disconnection to the individual learner's mathematical thinking. This is similar to a situation where the teacher evaluates or judges the learner's response without attempting to interpret what might be the learner's understanding. Such a situation also shows that the adjustment of the tasks will be disconnected from what learners are thinking. Judging or evaluating learners' responses was observed in only T4's lesson and the findings indicated that her task adjustment was disconnected from learners' mathematical thinking.

The third aspect focused on how the teacher's activities intended to advance learners' understanding of the concept related to the way learners develop such concept (mathematical learning progression). The findings in this regard have shown that both T3 and T4's instructional practices showed little to no connection to learners' mathematical learning progression. Although both teachers were not working on the same concept but their approaches showed that they are either not aware of how learners develop understanding of the concepts they were treating or that they were unable to present their activities sequentially such that they build from one another. Jacobs et al. (2010) emphasized that it is a need for teachers to be aware or understand the research on children's mathematical development. This will enable the teacher to know how each knowledge domain, such as directed numbers and operations on directed numbers, is developed and thus be in the position to organize his or her instruction in such a way that it makes it easier for learners to understand.

5.2.2 Research question 2

The second research question was;

How can the levels of the teachers' professional noticing and mathematical knowledge for teaching directed numbers at grade 8 be explained?

The analysis presented in chapter 4 has shown that T3 and T4's levels of professional noticing and mathematical knowledge for teaching directed numbers at grade 8 differed. The intention here was not to answer the question of what might be contributing to the difference. The aim was just to see what the instructional practices of each teacher locate him or her in the four quadrants of Thomas et al.'s (2011) framework. The findings categorized T3's level of professional noticing and MKT in quadrant III which explains his level as *limited responsiveness to students' mathematical thinking and under-developed MKT.* This quadrant is characterized by such teacher's instructional practices that include implementing instruction rigidly or exactly as it is planned not allowing for alterations that may be brought by learners' mathematical productions in class. Not adjusting tasks in response to individual learner's thinking and tenuous connection to mathematical learning progression are also characteristics of teacher's instructional practices in this quadrant. The findings have thus pointed T3's instructional practices to this level. This is the level that characterizes teaching that doesn't make use or accommodate and hence build on learners own strategies. Ball et al. (2008) indicate that if teachers can engage with learners' strategies they can be in the position to realise and resolve their non-standard approaches or errors and misconceptions that they have. But for T3 that would not be possible as the findings show that he wasn't engaging much with the learners' mathematical productions.

T4 on the other hand had his instructional practices categorising her professional noticing and MKT as *highly responsiveness to students' mathematical thinking and under-developed MKT.* While other aspects of her teaching were similar to that of T3, the findings however showed that T4 was able to change tasks in response to her learners understanding. Her in-the-moment decision to develop a task to address the challenge learners faced showed her responsiveness to the learner's thinking. Evidence that this was in-the-moment decision stems from the way she had to try out figures to see which ones worked.

These categories of T3 and T4's level of professional noticing and mathematical knowledge for teaching directed numbers - may not be items that define their practices always. Those levels may differ or stay the same when they teach different topics or even for different concepts within the same topic. So the findings here cannot be taken to be portraying the status of T3 and T4's level of professional noticing and MKT rather this is what their instructional practices on that particular topic had to reveal. While these findings cannot be generalized on T3 and T4's daily practices as well as to other teachers teaching the same topic at the same grade but they help to provide an insight into different teachers' instructional practices.

5.3 Limitations of study

This study has a number of limitations of which some were due to the restrictions that came as a result of the prevailing covid-19 pandemic. For instance, a qualitative research study relies on multiple sources of data (Creswell & Poth, 2016). This was however not possible since access to the empirical worlds was denied. Thus, alternative forms of obtaining data that could help in answering the research questions or addressing the problem investigated had to be sought. So, this study ended up with one source of data (secondary data in the form of lesson transcripts) and this has a bearing on some issues relating to the trustworthiness of the study. Some tools (such as teachers' lesson plans) that could be useful in understanding in depth the instructional practices of the teacher could not be accessed and this means yet another limitation.

Again, this problem with accessing the empirical world also affected this study in way that the detailed description of the cases selected for this study was not possible. This is a requirement for the case study research approach (Scott & Morrison, 2006). So, I had to rely on the description provided by the primary data collector and since his study followed a quasi-experimental research design approach, little was written about the teachers who were in the control class and whose data was used for this study.

The fact that this study followed a qualitative case study approach, its findings cannot be generalized to a population of grade 8 teachers teaching directed numbers rather they can just provide an insight to the different instructional practices.

5.4 Recommendations

The findings of this study have shown that when teachers rely too much on chorused responses, that reliance; compromises their ability to be responsive to their learners' mathematical productions. These sung responses overshadow productive thinking that would emerge from individual learners which could be used be used as a base for furthering learners' understanding of the topic they are dealing with. This would lead to a flexible instruction that accommodates each individual learner in the learning process. While it cannot be ruled out that chorused responses can also be of some importance in the teaching and learning process but this study recommends a decreased use of it in order to enable teachers to professionally notice their learners' mathematical thinking as well as to develop the MKT.

The teacher being judgmental about the learners' responses, that is, judging the correctness of the response without giving learners chance to explain their thinking behind the responses was also reflected in the findings. This hinders a teacher from delving deeper in to the learner's thinking in order to establish their own understanding. Such understanding could be useful in shaping the teacher's instructional practices as well. So, teachers should rather refrain from being evaluative instead scaffold learners' thinking (Smith & Stein, 2011) so that they can understand their thinking and thus use that knowledge to inform their instruction.

Jacobs et al. (2010) indicated that the ability of teachers to professionally notice their learners' mathematical thinking is tied or linked to the professional development they receive particularly if it is "focused on children's mathematical thinking" (p.191). Thus, this study recommends professional development for teachers that is focused on the area of professional noticing and MKT.

In terms of future research, this study recommends research that is focussed on establishing the links between teachers' professional noticing skills and MKT and learners' achievement. Separately these two frameworks have been shown to be linked to learners' achievement and as such it would be of interest to test the combined framework whether it produces similar results or not.

5.5 Conclusion

This chapter has presented the summary of findings of the study. It also provided answers to the research questions that guided this study. Thomas et al.'s (2017) theoretical framework has been successful in categorising the level of teachers' professional noticing and mathematical knowledge for teaching specifically in the teaching of directed numbers at grade 8. The framework helped in illuminating each teacher's instructional practices that made it possible to decide on their levels of professional noticing and MKT.

This chapter has also highlighted the limitations of this study, citing challenges relating to: access of the empirical sites for data collection, generalizability of findings etc. Furthermore, recommendations based on research findings as well as recommendations for future research have also been made.

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APPENDIX A

Exercises in T4's lesson

1. Given the temperatures $0℃$, $-5℃, 71℃, -21℃, 15℃, 29℃, 50℃$, arrange them in;

(a) ascending order

(b) descending order

2. Work out

(a) $5 + (-3)$

(b) $10 — (-15)$

(c) $-11 - (-9)$

(d) $6 - (-4)$

APPENDIX B

WITS SCHOOL OF EDUCATION

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

SCHOOL OF EDUCATION ETHICS COMMITTEE

CONSTITUTED UNDER THE UNIVERSITY HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

CLEARANCE CERTIFICATE **PROTOCOL NUMBER: 2020ECE163M**

PROJECT TITLE Teacher professional noticing and mathematical knowledge for teaching directed numbers at grade 8

INVESTIGATOR **NEO THOMAS LIAU**

SCHOOL/DEPARTMENT OF INVESTIGATOR WITS SCHOOL OF EDUCATION

DATE CONSIDERED 12 October 2020

DECISION OF THE COMMITTEE Approved unconditionally

EXPIRY DATE Date of submission of the project report

ISSUE DATE OF CERTIFICATE: 20 October 2020 **CHAIRPERSON:** ______________
(Dr. Paul Goldschagg)

cc: Dr Judah Makonye

DECLARATION OF INVESTIGATOR

To be completed in duplicate and **ONE COPY** returned to the Chairperson of the School/Department ethics committee.

I fully understand the conditions under which I am are authorized to carry out the abovementioned research and I guarantee to ensure compliance with these conditions. Should any departure to be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee.

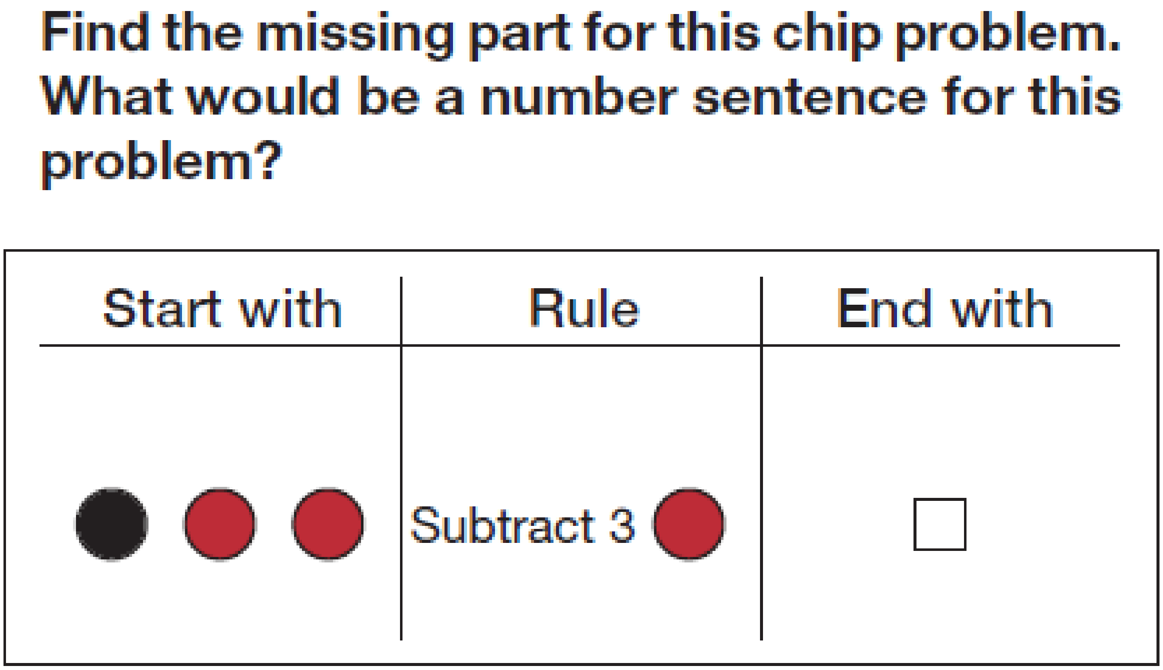
____
Signature

22 / 10 / 2020
Date

PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES

APPENDIX C

A chip representation model to work out $-1 - (-3)$. Source: Hill and Ball (2009, p.68)



APPENDIX D

Specific integer focus for each grade at the Senior Phase. Source: DBE (2011, p.16)

TOPICS	GRADE 7	GRADE 8	GRADE 9
1.3 **Integers**	**Counting, ordering and comparing integers** • Count forwards and backwards in integers for any interval • Recognize, order and compare integers	**Counting, ordering and comparing integers** • Revise: - counting forwards and backwards in integers for any interval - recognizing, ordering and comparing integers	
	Calculations with integers • Add and subtract with integers	**Calculations with integers** • Revise addition and subtraction with integers • Multiply and divide with integers • Perform calculations involving all four operations with integers • Perform calculations involving all four operations with numbers that involve the squares, cubes, square roots and cube roots of integers	**Calculations with integers** • Revise: - perform calculations involving all four operations with integers - perform calculations involving all four operations with numbers that involve the squares, cubes, square roots and cube roots of integers
	Properties of integers • Recognise and use commutative and associative properties of addition and multiplication for integers	**Properties of integers** • Recognise and use commutative, associative and distributive properties of addition and multiplication for integers • Recognize and use additive and multiplicative inverses for integers	**Properties of integers** • Revise: - Commutative, associative and distributive properties of addition and multiplication for integers - additive and multiplicative inverses for integers
	Solving problems Solve problems in contexts involving addition and subtraction with integers	**Solving problems** Solve problems in contexts involving multiple operations with integers	**Solving problems** Solve problems in contexts involving multiple operations with integers