Topology and its Applications 341 (2024) 108731

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

Cyclic sequences of periodic sums in $\beta\mathbb{N}$

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ARTICLE INFO

Article history:
Received 13 August 2023
Received in revised form 19 September 2023
Accepted 24 September 2023
Available online 28 September 2023

Keywords:
Stone-Čech compactification
Idempotent
Right cancelable ultrafilter
Cyclic semigroup
Ramsey theory

ABSTRACT

Let $m \geq 2$ and $n \geq 2$ and define $\nu : \omega \to \{0, \ldots, m-1\}$ by $\nu(k) \equiv k \pmod m$. We show that there is a sequence $p_0, \ldots, p_{m-1}$ in $\beta\mathbb{N}$ such that all sums $\sum_{j=i}^{k} p_{\nu(j)}$, where $i \in \{0, \ldots, m-1\}$ and $k \in \{i, \ldots, mn-1\}$ for each i, are distinct and $\sum_{j=i}^{mn} p_{\nu(j)} = \sum_{j=i}^{mn-m} p_{\nu(j)}$ for each i. As a consequence we derive a new Ramsey theoretic result: there is a partition $\{A_{i,k} : i \in \{0, \ldots, m-1\}$ and $k \in \{i, \ldots, mn-1\}$ for each $i\}$ of $\mathbb{N}$ such that, whenever for each (i,k), $\mathcal{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathcal{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod m$ for each $t < s$, one has $x_{j_0} + \ldots + x_{j_s} \in B_{i_0,k_0}$, where $i_0 = \nu(j_0)$ and k_0 is $i_0 + s$ if $i_0 + s \leq mn - 1$ and $mn - m + \nu(i_0 + s - mn)$ otherwise.

1. Introduction

The addition of the discrete semigroup $\mathbb{N}$ of natural numbers extends to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that for each $a \in \mathbb{N}$, the left translation $\lambda_a : \beta\mathbb{N} \ni x \mapsto a + x \in \beta\mathbb{N}$ is continuous, and for each $q \in \beta\mathbb{N}$, the right translation $\rho_q : \beta\mathbb{N} \ni x \mapsto x + q \in \beta\mathbb{N}$ is continuous.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. For every $A \subseteq \mathbb{N}$, $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ and $A^* = \overline{A} \setminus A$. The subsets $\overline{A}$, where $A \subseteq \mathbb{N}$, form a base for the topology of $\beta\mathbb{N}$, and $\overline{A}$ is the closure of A. For $p, q \in \beta\mathbb{N}$, the ultrafilter $p + q$ has a base consisting of subsets of the form $\bigcup_{x \in A}(x + B_x)$, where $A \in p$ and for each $x \in A$, $B_x \in q$.

Being a compact Hausdorff right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. Every right (left) ideal of $\beta\mathbb{N}$ contains a minimal right (left) ideal, the intersection of a minimal right ideal and a minimal left ideal

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https://doi.org/10.1016/j.topol.2023.108731
0166-8641/

is a group, and the idempotents in a minimal right (left) ideal form a right (left) zero semigroup, that is, $x + y = y$ $(x + y = x)$ for all x, y.

The semigroup $\beta\mathbb{N}$ has important applications to Ramsey theory and to topological dynamics. The first application to Ramsey theory was the proof of Hindman's theorem: whenever $\mathbb{N}$ is finitely colored, there is an infinite subset all of whose sums are monochrome. An elementary introduction to $\beta\mathbb{N}$ can be found in [4].

In 1979, E. van Douwen asked (in [3], published much later) whether there are topological and algebraic copies of $\beta\mathbb{N}$ contained in $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. This question was answered in the negative by D. Strauss in [6], where it was in fact established that continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$ have finite images. It follows that if $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ is a continuous homomorphism, then $\varphi(\beta\mathbb{N})$ is a finite cyclic semigroup generated by $p = \varphi(1)$. That is, there are $l \geq 1$ and $1 \leq m \leq l$ called the *order* and the *period* of p (and of the cyclic semigroup) such that all $ip = \underbrace{p + \ldots + p}_{i}$, where $i \in \{1, \ldots, l\}$, are distinct and $(l+1)p = (l+1-m)p$. Conversely, every element $p \in \mathbb{N}^*$ of finite order determines a continuous homomorphism $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ by $\varphi(1) = p$. In 1996, Y. Zelenyuk proved that $\beta\mathbb{N}$ contains no nontrivial finite groups (see [4, Theorem 7.17]). Since the periodic part of a cyclic semigroup is a group, it follows that if $p \in \beta\mathbb{N}$ is an element of order l, then $(l + 1)p = lp$, that is, p has period 1.

As distinguished from finite groups, $\beta\mathbb{N}$ does contain bands (semigroups of idempotents): for example, left zero semigroups, right zero semigroups, chains of idempotents (with respect to the order $x \leq y$ if and only if $x + y = y + x = x$), and even rectangular bands (direct products of a left zero semigroup and a right zero semigroup). To ask whether $\beta\mathbb{N}$ contains a finite semigroup distinct from bands is the same as asking whether $\beta\mathbb{N}$ contains an element of order 2 which is the same as asking whether there exists a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ [4, Question 10.19]. If the answer to this question is positive, then there is a subset A of $\mathbb{N}$ such that, whenever A is finitely colored, there is an infinite subset in the complement of A, all of whose sums two or more terms at a time are monochrome [2, Corollary 3.5].

The question whether $\beta\mathbb{N}$ contains an element of order 2 was solved in the affirmative in [7, Theorem 1]. In [8], some further finite semigroups in $\beta\mathbb{N}$ consisting of idempotents and elements of order 2 were constructed, in particular null semigroups ($x + y = 0$ for all x, y), and a connection of finite semigroups with Ramsey theory was discussed, see also [1]. In [10], it was shown that for every $m \geq 1$, the direct product of the m-element null semigroup and the rectangular band $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$ (that the rectangular band $2^{\mathfrak{c}} \times 2^{\mathfrak{c}}$ has copies in $\beta\mathbb{N}$ was established in [5]). The question whether $\beta\mathbb{N}$ contains an element of finite order > 2 was solved in the affirmative in [9, Theorem 3].

Let $l \geq m \geq 1$. Define $\nu = \nu_m : \omega \to \{0, \ldots, m-1\}$ by $\nu(k) \equiv k \pmod{m}$. We say that a sequence $(x_k)_{k=0}^{\infty}$ in some set is *cyclic of order* l *and period* m if the elements $x_0, \ldots, x_{l-1}$ are distinct and for every $k \geq l$, $x_k = x_{l-m+\nu(k-l)}$. Given a sequence $p_0, \ldots, p_{m-1}$ in an additive semigroup, the *periodic sums with initial term* p_i are sums of the form $\sum_{j=i}^{i+k} p_{\nu(j)}$, and $(\sum_{j=i}^{i+k} p_{\nu(j)})_{k=0}^{\infty}$ is the *sequence of periodic sums with initial term* p_i. The sequence $(\sum_{j=i}^{i+k} p_{\nu(j)})_{k=0}^{\infty}$ is cyclic of order l and period m if and only if the sums $\sum_{j=i}^{i+k} p_{\nu(j)}$, where $k \in \{0, \ldots, l-1\}$, are distinct and $\sum_{j=i}^{i+l} p_{\nu(j)} = \sum_{j=i}^{i+l-m} p_{\nu(j)}$. If $(\sum_{j=i}^{i+k} p_{\nu(j)})_{k=0}^{\infty}$ is cyclic of order l and period m and $s = [\frac{l}{m}]$ (that is, $sm \in \{l + 1 - m, \ldots, l\}$), then

$$(s+1) \sum_{j=i}^{i+m-1} p_{\nu(j)} = \sum_{j=i}^{i+(s+1)m-1} p_{\nu(j)} = \sum_{j=i}^{i+sm-1} p_{\nu(j)} = s \sum_{j=i}^{i+m-1} p_{\nu(j)},$$

so $\sum_{j=i}^{i+m-1} p_{\nu(j)}$ is an element of order s and period 1 and $s \sum_{j=i}^{i+m-1} p_{\nu(j)}$ is an idempotent in the periodic part of the sequence, the same as for the cyclic semigroup of order l and period m.

In this paper we combine and modify constructions in [10] and [9] and show that for each $m \geq 2$ and each $n \geq 2$, there is a sequence $p_0, \ldots, p_{m-1}$ in $\beta\mathbb{N}$ such that for each $i \in \{0, \ldots, m-1\}$, the sequence of periodic sums with initial term p_i is cyclic of order $mn-i$ and period m and all these sequences are pairwise disjoint. In particular, $p_0+p_1+\ldots+p_{m-1}$ is an element of order n and for each $i\in\{1,\ldots,m-1\}$, $p_i+\ldots+p_{m-1}+p_0+\ldots+p_{i-1}$ an element of order $n-1$.

As a consequence we derive a new Ramsey theoretic result. We show that for each $m\geq 2$ and each $n\geq 2$, there is a partition $\{A_{i,k}: i\in\{0,\ldots,m-1\}$ and $k\in\{i,\ldots,mn-1\}$ for each $i\}$ of $\mathbb{N}$ such that, whenever for each (i,k), $\mathcal{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k}\in\mathcal{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that for every finite sequence $j_0<\ldots<j_s$ such that $j_{t+1}\equiv j_t+1 \pmod m$ for each $t<s$, one has $x_{j_0}+\ldots+x_{j_s}\in B_{i_0,k_0}$, where $i_0=\nu(j_0)$ and k_0 is i_0+s if $i_0+s\leq mn-1$ and $mn-m+\nu(i_0+s-mn)$ otherwise.

2. Construction

Let $m\geq 2$ and $n\geq 2$. Let W_m denote the set of words $w=i_0\ldots i_k$ over $\{0,1,\ldots,m-1\}$ such that $k\geq 0$ and $i_{s+1}=\nu(i_s+1)$ for each $s\leq k-1$. For each $i\in\{0,\ldots,m-1\}$ and $k\geq 0$, let $w(i,k)$ denote the word $i_0\ldots i_k$ in W_m with $i_0=i$. For example, for $m=3$, $w(1,5)=120120$. Let $W_{m,n}$ denote the subset of W_m consisting of words $w(i,k)$, where $i\in\{0,\ldots,m-1\}$ and $k\leq mn-1-i$ for each i. For each $i\in\{0,\ldots,m-1\}$, let $W_m(i)$ denote the subset of W_m consisting of words $w(i,k)$, where $k\geq 0$, and $W_{m,n}(i)=W_{m,n}\cap W_m(i)$. Define a mapping $\pi=\pi_{m,n}:W_m\to W_{m,n}$ as follows. For each $w\in W_{m,n}$, $\pi(w)=w$, and for each $i\in\{0,\ldots,m-1\}$ and each $k\geq mn-i$,

$$\pi(w(i,k))=w(i,mn-m-i+\nu(k-(mn-i))).$$

Notice that the last letter in $w(i,mn-i-1)$ is $m-1$, so for $k\geq mn-i$,

$$w(i,k)=w(i,mn-m-i-1)w(0,m-1)w(0,k-(mn-i)) \text{ and}$$
$$\pi(w(i,k))=w(i,mn-m-i-1)w(0,\nu(k-(mn-i))).$$

Consequently, for $w=i_0\ldots i_k$ in $W_m(i)\setminus W_{m,n}$, $\pi(w)$ is the word in

$$\{w(i,mn-m-i),\ldots,w(i,mn-i-1)\}$$

which is an initial subword of w and whose last letter is i_k, equivalently which is a terminal subword of w and whose first letter is i. It then follows that $\pi(vw)=\pi(v\pi(w))=\pi(\pi(v)w)=\pi(\pi(v)\pi(w))$ whenever $vw\in W_m$, and clearly $\pi^2=\pi$.

To see for example that $\pi(vw)=\pi(v\pi(w))$, let $v=i_0\ldots i_k$ and $w=j_0\ldots j_s$. If $w\in W_{m,n}$, $\pi(w)=w$, so $\pi(v\pi(w))=\pi(vw)$. Let $w\in W_m\setminus W_{m,n}$. Then $\pi(w)$ is the word in $\{w(j_0,mn-m-j_0),\ldots,w(j_0,mn-j_0-1)\}$ which is an initial subword of w and whose last letter is j_s. Write $\pi(w)=w(j_0,s_0)$. Then $mn-m-j_0\leq s_0\leq mn-j_0-1$ and $v\pi(w)=w(i_0,s_0+k+1)$. Notice that $k+1\geq j_0-i_0$, so $s_0+k+1\geq mn-m-j_0+j_0-i_0=mn-m-i_0$. Consequently, $\pi(v\pi(w))$ is the word in $\{w(i_0,mn-m-i_0),\ldots,w(i_0,mn-i_0-1)\}$ which is an initial subword of $v\pi(w)$, and so of vw, and whose last letter is j_s, the same as for $\pi(vw)$. Hence, $\pi(v\pi(w))=\pi(vw)$.

Given a sequence $p_0,\ldots,p_{m-1}$ in an additive semigroup and a word $w=i_0\ldots i_k$ in W_m, let p_w denote the sequence $p_{i_0},\ldots,p_{i_k}$ and $\sum p_w=p_{i_0}+\ldots+p_{i_k}$. Then to say that the sequence $(\sum_{j=i}^{i+k}p_{\nu(j)})_{k=0}^{\infty}$ of periodic sums with initial term p_i is cyclic of order $mn-i$ and period m is the same as saying that the elements $\sum p_w$, where $w\in W_{m,n}(i)$, are distinct and for every $k\geq mn-i$, $\sum p_{w(i,k)}=\sum p_{\pi(w(i,k))}$.

Let $w_0=w(0,m-1)$, so $\sum p_{w_0}=p_0+\ldots+p_{m-1}$.

Lemma 2.1. *Let $p_0,\ldots,p_{m-1}$ be a sequence in an additive semigroup and suppose that $p_{m-1}+(n-1)\sum p_{w_0}+p_0=p_{m-1}+(n-2)\sum p_{w_0}+p_0$. Then for every $w\in W_m$, $\sum p_w=\sum p_{\pi(w)}$.*

Proof. We first notice for every $k \geq n-1$, $p_{m-1} + k\sum p_{w_0} + p_0 = p_{m-1} + (n-2)\sum p_{w_0} + p_0$. Indeed, for $k = n-1$, this is our supposition, and then by induction for $k \geq n$, $p_{m-1} + k\sum p_{w_0} + p_0 = p_{m-1} + p_0 + \ldots + p_{m-2} + p_{m-1} + (k-1)\sum p_{w_0} + p_0 = p_{m-1} + p_0 + \ldots + p_{m-2} + p_{m-1} + (n-2)\sum p_{w_0} + p_0 = p_{m-1} + (n-1)\sum p_{w_0} + p_0 = p_{m-1} + (n-2)\sum p_{w_0} + p_0$.

Now let $w \in W_m \setminus W_{m,n}$ and pick $i \in \{0, \ldots, m-1\}$ such that $w \in W_m(i)$. Then w can be written as $i \ldots (m-1)(w_0)^k 0 \ldots j$ for some $k \geq n-1$ and $j \in \{0, \ldots, m-1\}$, and $\pi(w)$ is the word obtained from w by replacing the subword $(w_0)^k$ with $(w_0)^{n-2}$. Then $\sum p_w = p_i + \ldots + p_{m-1} + k\sum p_{w_0} + p_0 + \ldots + p_j$ and $\sum p_{\pi(w)} = p_i + \ldots + p_{m-1} + (n-2)\sum p_{w_0} + p_0 + \ldots + p_j$. Since $p_{m-1} + k\sum p_{w_0} + p_0 = p_{m-1} + (n-2)\sum p_{w_0} + p_0$, one has $\sum p_w = \sum p_{\pi(w)}$. $\square$

The subsemigroup $\mathbb{H}$ of $\beta\mathbb{N}$ is defined by

$$\mathbb{H} = \bigcap_{j=0}^{\infty} \overline{2^j\mathbb{N}}.$$

The next theorem tells us that for each $m \geq 2$ and each $n \geq 2$, there is a sequence $p_0, \ldots, p_{m-1}$ in $\mathbb{H}$ such that for each $i \in \{0, \ldots, m-1\}$, the sequence $(\sum_{j=i}^{i+k} p_{\nu(j)})_{k=0}^{\infty}$ of periodic sums with initial term p_i is cyclic of order $mn - i$ and period m and all these sequences are pairwise disjoint.

Theorem 2.2. *Let $m \geq 2$ and $n \geq 2$. There is a sequence $p_0, \ldots, p_{m-1}$ in $\mathbb{H}$ such that all elements $\sum p_w$, where $w \in W_{m,n}$, are distinct and for every $w \in W_m$, $\sum p_w = \sum p_{\pi(w)}$.*

Proof. For every $x \in \mathbb{N}$, supp x is a unique finite nonempty subset of $\omega = \mathbb{N} \cup \{0\}$ such that

$$x = \sum_{j \in \mathrm{supp}\, x} 2^j.$$

Pick an increasing sequence $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = \omega$ of subsets of ω such that $I_k \setminus I_{k-1}$ is infinite for each $k \in \{0, 1, \ldots, n\}$ (with $I_{-1} = \emptyset$). Define a function h from $\mathbb{N}$ onto the decreasing chain $0 > 1 > \ldots > n$ of idempotents (with the operation $k * j = \max\{k, j\}$) by

$$h(x) = \min\{k \leq n : \mathrm{supp}\, x \subseteq I_k\} = \max\{k \leq n : (\mathrm{supp}\, x) \cap (I_k \setminus I_{k-1}) \neq \emptyset\}$$

and let the same letter h denote its continuous extension $\beta\mathbb{N} \to \{0, 1, \ldots, n\}$. If $x, y \in \mathbb{N}$ and $\max \mathrm{supp}\, x < \min \mathrm{supp}\, y$, then $h(x+y) = h(x) * h(y)$. It then follows (see [4, Theorem 4.21]) that for any $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, one has $h(u+v) = h(u) * h(v)$, in particular, the restriction of h to $\mathbb{H}$ is a homomorphism. For each $k \in \{0, 1, \ldots, n\}$, let

$$T_k = h^{-1}(\{0, 1, \ldots, k\}) \cap \mathbb{H}.$$

Then $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n = \mathbb{H}$ is an increasing sequence of closed subsemigroups of $\mathbb{H}$ such that $h(K(T_k)) = \{k\}$ for each $k \leq n$, and so $T_k \cap \overline{K(T_{k+1})} = \emptyset$ for each $k < n$ and $K(T_n) = K(\beta\mathbb{N}) \cap T_n$ [8, Lemma 3.1], in particular, all $K(T_0), K(T_1), \ldots, K(T_n)$ are pairwise disjoint. Moreover, $h(K(\beta\mathbb{N})) = \{n\}$, and so $T_{n-1} \cap \overline{K(\beta\mathbb{N})} = \emptyset$.

To see this, let $u \in K(\beta\mathbb{N})$. Then $u + \beta\mathbb{N}$ is the minimal right ideal of $\beta\mathbb{N}$ containing u and $\beta\mathbb{N} + u$ the minimal left ideal containing u. Let v be the identity of the group $(u + \beta\mathbb{N}) \cap (\beta\mathbb{N} + u)$. Then $u = u + v$ and $v \in K(\mathbb{H})$, so $h(u) = h(u+v) = h(u) * h(v) = h(u) * n = n$.

For each $k \in \{0, 1, \ldots, n\}$, let

$$X_k = \{x \in \mathbb{N} : (\mathrm{supp}\, x) \cap (I_k \setminus I_{k-1}) \neq \emptyset\}.$$

Notice that for any $v \in \overline{X_k} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $u + v \in \overline{X_k}$, and for any $v \in \overline{X_k}$ and $w \in \mathbb{H}$, $v + w \in \overline{X_k}$.

Define $\phi_k : X_k \to \omega$ by

$$\phi_k(x) = \max((\operatorname{supp} x) \cap (I_k \setminus I_{k-1}))$$

and let the same letter ϕ_k denote its continuous extension $\overline{X_k} \to \beta\omega$. Notice that $\{2^j : j \in I_k \setminus I_{k-1}\} \subseteq X_k$ and, since $\phi_k(2^j) = j$, ϕ_k homeomorphically maps $\overline{\{2^j : j \in I_k \setminus I_{k-1}\}}$ onto $\overline{I_k \setminus I_{k-1}}$. If $x \in \mathbb{N}$, $y \in X_k$ and $\max\operatorname{supp} x < \min\operatorname{supp} y$, then $x + y \in X_k$ and $\phi_k(x + y) = \phi_k(y)$. And if $y \in X_k$, $z \in \mathbb{N} \setminus X_k$ and $\max\operatorname{supp} y < \min\operatorname{supp} z$, then $\phi_k(y + z) = \phi_k(y)$. It then follows that for any $v \in \overline{X_k} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $\phi_k(u + v) = \phi_k(v)$, and for any $v \in \overline{X_k}$ and $w \in \mathbb{H} \setminus \overline{X_k}$, $\phi_k(v + w) = \phi_k(v)$.

To see for example the first statement, we first note that for any $x \in \mathbb{N}$ and $v \in \overline{X_k} \cap \mathbb{H}$, $\phi_k(x+v) = \phi_k(v)$ because the continuous functions $\phi_k \circ \lambda_x$ and ϕ_k agree on $X_k \cap 2^j\mathbb{N}$, where $j = (\max\operatorname{supp} x) + 1$. Then for any $v \in \overline{X_k} \cap \mathbb{H}$ and $u \in \beta\mathbb{N}$, $\phi_k(u+v) = \phi_k(v)$ because the continuous function $\phi_k \circ \rho_v$ is constantly equal to $\phi_k(v)$ on $\mathbb{N}$.

Notice that $K(T_k) \subseteq \overline{X_k} \cap \mathbb{H}$ and $T_{k-1} \subseteq \mathbb{H} \setminus \overline{X_k}$ (with $T_{-1} = \emptyset$).

We shall construct

(i) a chain $e_0 > e_1 > \ldots > e_n$ of idempotents with $e_k \in K(T_k)$,

(ii) for each $k \in \{0, 1, \ldots, n\}$, a left zero semigroup $\{e_{k,i} : i \in \{0, \ldots, m-1\}\} \subseteq K(T_k)$ such that $e_{k,0} = e_k$ and $e_{k,i} = e_{0,i} + e_k$ for all $i \in \{0, \ldots, m-1\}$, and

(iii) for each $k \in \{1, \ldots, n-1\}$, a right zero semigroup $\{e_k(j) : j \in \omega\} \subseteq K(T_k)$ such that $e_k(0) = e_k$, $e_k(j) < e_{k-1}$ for all $j \in \omega$, and ϕ_k is injective on $\{e_k(j) : j \in \omega\}$.

Notice that (i) and (ii) imply that

$$e_{k,i} + e_{l,s} = e_{k*l,i}$$

for all $k, l \in \{0, 1, \ldots, n\}$ and $i, s \in \{0, \ldots, m-1\}\}$.

Indeed,

$$\begin{aligned} e_{k,i} + e_{l,s} &= e_{0,i} + e_k + e_{0,s} + e_l = e_{0,i} + (e_k + e_0) + e_{0,s} + e_l \\ &= e_{0,i} + e_k + (e_0 + e_{0,s}) + e_l = e_{0,i} + e_k + e_0 + e_l \\ &= e_{0,i} + e_{k*l} = e_{k*l,i}. \end{aligned}$$

The construction goes by induction on $k \in \{0, 1, \ldots, n\}$.

For $k = 0$, pick an injective sequence $\{r_{0,i} : i \in \{0, \ldots, m-1\}\}$ in $\{2^j : j \in I_0\}^*$.

Lemma 2.3. *$(r_{0,i} + T_n) \cap (r_{0,s} + T_n) = \emptyset$ if $i \neq s$.*

Proof. Consider the function $\mathbb{N} \ni x \mapsto \min\operatorname{supp} x \in \omega$ and let θ denote its continuous extension $\beta\mathbb{N} \to \beta\omega$. If $x, y \in \mathbb{N}$ and $\max\operatorname{supp} x < \min\operatorname{supp} y$, then $\theta(x + y) = \theta(x)$. It then follows that for any $u \in \beta\mathbb{N}$ and $v \in \mathbb{H}$, $\theta(u + v) = \theta(u)$. Consequently, $\theta(r_{0,i} + T_n) = \{\theta(r_{0,i})\}$ and $\theta(r_{0,s} + T_n) = \{\theta(r_{0,s})\}$. Since $\theta(2^j) = j$, $\theta(r_{0,i}) \neq \theta(r_{0,s})$, so $(r_{0,i} + T_n) \cap (r_{0,s} + T_n) = \emptyset$. □

For each $i \in \{0, \ldots, m-1\}$, choose a minimal right ideal $R_{0,i}$ of T_0 contained in $r_{0,i} + T_0$. Pick a minimal left ideal L_0 of T_0, and for each $i \in \{0, \ldots, m-1\}$, let $e_{0,i}$ be the identity of the group $R_{0,i} \cap L_0$. By Lemma 2.3, $e_{0,i} \neq e_{0,s}$ if $i \neq s$. Put $e_0 = e_{0,0}$.

For $k = 1$, choose a minimal right ideal R_1 of T_1 contained in $e_0 + T_1$. Pick an injective sequence $(r_{1,j})_{j=0}^{\infty}$ in $\{2^j : j \in I_1 \setminus I_0\}^*$, and for every $j \in \omega$, choose a minimal left ideal $L_{1,j}$ of T_1 contained in $T_1 + r_{1,j} + e_0$. For every $j \in \omega$, let $e_1(j)$ be the identity of the group $R_1 \cap L_{1,j}$. Then $\phi_1(e_{1,j}) = \phi_1(r_{1,j} + e_0) = \phi_1(r_{1,j})$.

Since $e_1(j) \in e_0 + T_1$, one has $e_0 + e_1(j) = e_1(j)$, and since $e_1(j) \in T_1 + r_{1,j} + e_0$, one has $e_1(j) + e_0 = e_1(j)$, so $e_1(j) < e_0$. Put $e_1 = e_1(0)$. For each $i \in \{0, \ldots, m-1\}$, put $e_{1,i} = e_{0,i} + e_1$. Then $e_{1,i} + e_{1,s} = e_{0,i} + e_1 + e_{0,s} + e_1 = e_{0,i} + (e_1 + e_0) + e_{0,s} + e_1 = e_{0,i} + e_1 + (e_0 + e_{0,s}) + e_1 = e_{0,i} + e_1 + e_0 + e_1 = e_{0,i} + e_1 = e_{1,i}$, so $\{e_{1,i} : i \in \{0, \ldots, m-1\}\}$ is a left zero semigroup (in $K(T_1)$). Since $e_{1,i} = e_{0,i} + e_1 \in r_{0,i} + T_0 + e_1 \in r_{0,i} + T_1$, by Lemma 2.3, $e_{1,i} \neq e_{1,s}$ if $i \neq s$.

For $k \in \{2, \ldots, n-1\}$ (for $n \geq 3$), choose a minimal right ideal R_k of T_k contained in $e_{k-1} + T_k$. Pick an injective sequence $(r_{k,j})_{j=0}^{\infty}$ in $\{2^j : j \in I_k \setminus I_{k-1}\}^*$, and for every $j \in \omega$, choose a minimal left ideal $L_{k,j}$ of T_k contained in $T_k + r_{k,j} + e_{k-1}$, and let $e_k(j)$ be the identity of the group $R_k \cap L_{k,j}$. Then $\phi_k(e_k(j)) = \phi_k(r_{k,j} + e_0) = \phi_k(r_{k,j})$ and $e_k(j) < e_{k-1}$ for all j. Put $e_k = e_k(0)$. For each $i \in \{0, \ldots, m-1\}$, put $e_{k,i} = e_{0,i} + e_k$. Then $\{e_{k,i} : i \in \{0, \ldots, m-1\}\}$ a left zero semigroup and $e_{k,i} \neq e_{k,s}$ if $i \neq s$.

For $k = n$, pick a minimal right ideal R_n of T_n contained in $e_{n-1} + T_n$ and a minimal left ideal L_n of T_n contained in $T_n + e_{n-1}$ and let e_n be the identity of the group $R_n \cap L_n$. For each $i \in \{0, \ldots, m-1\}$, put $e_{n,i} = e_{0,i} + e_n$.

Now for each $i \in \{1, \ldots, m-1\}$, let

$$D_{n-1,i} = \{e_{n,i} + e_{n-1}(j) : j \in \mathbb{N}\}$$

and pick $q_{n-1,i} \in \overline{D_{n-1,i}} \setminus D_{n-1,i}$. Then by induction on $k \in \{n-2, \ldots, 1\}$, let

$$D_{k,1} = \{e_{k+1,1} + q_{k+1,1} + e_k(j) : j \in \mathbb{N}\},$$

and for each $i \in \{2, \ldots, m-1\}$,

$$D_{k,i} = \{e_{k,i} + q_{k+1,i} + e_k(j) : j \in \mathbb{N}\},$$

and pick $q_{k,i} \in \overline{D_{k,i}} \setminus D_{k,i}$ for each $i \in \{1, \ldots, m-1\}$.

Since $e_{n,i} \in K(\beta\mathbb{N})$ and $\overline{K(\beta\mathbb{N})}$ is an ideal of $\beta\mathbb{N}$ [4, Theorem 4.44], it follows inductively that for each $k \in \{n-1, \ldots, 1\}$, $D_{k,i} \subseteq \overline{K(\beta\mathbb{N})}$ and $q_{k,i} \in \overline{K(\beta\mathbb{N})}$.

For each $s \in \{0, 1, \ldots, n\}$, $e_{n,i} = e_{s,i} + e_{n,i}$ and $e_{s,i} \in \overline{X_s}$, so $e_{n,i} \in \overline{X_s}$. It then follows inductively that for each $k \in \{n-1, \ldots, 1\}$, $D_{k,i} \subseteq \overline{X_s} \cap \mathbb{H}$ and $q_{k,i} \in \overline{X_s} \cap \mathbb{H}$. Notice that for each $k \in \{n-1, \ldots, 2\}$ (for $n \geq 3$), ϕ_k is injective on $D_{k,i}$ (because $\phi_{n-1}(e_{n,i} + e_{n-1}(j)) = \phi_{n-1}(e_{n-1}(j))$ and $\phi_k(e_{k+1,i} + q_{k+1,i} + e_k(j)) = \phi_k(e_k(j))$), and ϕ_1 is injective on $D_{1,i}$ ($\phi_1(e_{2,i} + e_1(j)) = \phi_1(e_1(j))$ for $n = 2$ and $\phi_1(e_{2,i} + q_{2,i} + e_1(j)) = \phi_1(e_1(j))$ for $n \geq 3$).

An ultrafilter $q \in \mathbb{N}^*$ is *right cancelable (in $\beta\mathbb{N}$)* if the right translation of $\beta\mathbb{N}$ by q is injective. An ultrafilter $q \in \mathbb{N}^*$ is right cancelable if and only if $q \notin \mathbb{N}^* + q$ [4, Theorem 8.18].

Lemma 2.4. *Let $k \in \{0, 1, \ldots, n\}$, let D be a countable subset of $\overline{X_k} \cap \mathbb{H}$, and suppose that ϕ_k is injective on D. Then every $q \in \overline{D} \setminus D$ is right cancelable.*

Proof. This is [9, Lemma 5]. □

It follows from Lemma 2.4 that all $q_{k,i}$, where $k \in \{1, \ldots, n-1\}$ and $i \in \{1, \ldots, m-1\}$, are right cancelable.

The next lemma gives us relations between $q_{k,i}$ and $e_{l,s}$.

Lemma 2.5. *Let $k \in \{1, \ldots, n-1\}$, $i \in \{1, \ldots, m-1\}$, and $s \in \{0, \ldots, m-1\}$. Then*

(1) $q_{n-1,i} + e_{n-1,s} = e_{n,i}$,
(2) $q_{k,i} + e_{k-1,0} = q_{k,i}$,
(3) *for $k \leq n-2$ and $i \geq 2$, $q_{k,i} + e_{k,s} = e_{k,i} + q_{k+1,i}$, and $q_{k,1} + e_{k,s} = e_{k+1,1} + q_{k+1,1}$.*

Proof. (1) Since $(e_{n,i} + e_{n-1}(j)) + e_{n-1,s} = e_{n,i} + (e_{n-1}(j) + e_{n-2,0}) + e_{n-1,s} = e_{n,i} + e_{n-1}(j) + (e_{n-2,0} + e_{n-1,s}) = e_{n,i} + e_{n-1}(j) + e_{n-1,0} = e_{n,i} + e_{n-1,0} = e_{n,i}$, it follows that $q_{n-1,i} + e_{n-1,s} = e_{n,i}$ (because $\rho_{e_{n-1,s}}$ is constantly equal to $e_{n,i}$ on $D_{n-1,i}$ and so $\rho_{e_{n-1,s}}(q_{n-1,i}) = e_{n,i}$).

(2) Every element in $D_{k,i}$ is of the form $u + e_k(j)$ (u depends only on k and i) and $(u + e_k(j)) + e_{k-1,0} = u + (e_k(j) + e_{k-1,0}) = u + e_k(j)$.

(3) For $i \geq 2$, $(e_{k,i} + q_{k+1,i} + e_k(j)) + e_{k,s} = e_{k,i} + q_{k+1,i} + (e_k(j) + e_{k-1,0}) + e_{k,s} = e_{k,i} + q_{k+1,i} + e_k(j) + (e_{k-1,0} + e_{k,s}) = e_{k,i} + q_{k+1,i} + e_k(j) + e_{k,0} = e_{k,i} + q_{k+1,i} + e_{k,0} = e_{k,i} + q_{k+1,i}$, and $e_{k+1,1} + q_{k+1,1} + e_k(j) + e_{k,s} = e_{k+1,1} + q_{k+1,1} + (e_k(j) + e_{k-1,0}) + e_{k,s} = e_{k+1,1} + q_{k+1,1} + e_k(j) + (e_{k-1,0} + e_{k,s}) = e_{k+1,1} + q_{k+1,1} + e_k(j) + e_{k,0} = e_{k+1,1} + q_{k+1,1} + e_{k,0} = e_{k+1,1} + q_{k+1,1}$. □

From Lemma 2.5 we obtain that for any $k \in \{1, \dots, n-1\}$, $i \in \{1, \dots, m-1\}$, $l \in \{0, \dots, n\}$, and $s \in \{0, 1, \dots, m-1\}$,

$$q_{k,i} + e_{l,s} = \begin{cases} q_{k,i} & \text{if } l < k \\ e_{l,i} + q_{l+1,i} & \text{if } k \leq l \leq n-2, i \geq 2 \\ e_{l+1,1} + q_{l+1,1} & \text{if } k \leq l \leq n-2, i = 1 \\ e_{n,i} & \text{if } l \geq n-1. \end{cases}$$

Indeed, if $l < k$, then $q_{k,i} + e_{l,s} = (q_{k,i} + e_{k-1,0}) + e_{l,s} = q_{k,i} + (e_{k-1,0} + e_{l,s}) = q_{k,i} + e_{k-1,0} = q_{k,i}$, and if $l = k$, then

$$q_{k,i} + e_{k,s} = \begin{cases} e_{k,i} + q_{k+1,i} & \text{if } k \leq n-2, i \geq 2 \\ e_{k+1,1} + q_{k+1,1} & \text{if } k \leq n-2, i = 1 \\ e_{n,i} & \text{if } k = n-1 \end{cases}$$

Let $l > k$ and suppose that the formula holds for l replaced with $l-1$. Write $q_{k,i} + e_{l,s} = q_{k,i} + (e_{l-1,s} + e_{l,s}) = (q_{k,i} + e_{l-1,s}) + e_{l,s}$. Let $i \geq 2$. If $l \leq n-2$, $q_{k,i} + e_{l,s} = (q_{k,i} + e_{l-1,s}) + e_{l,s} = e_{l-1,i} + q_{l,i} + e_{l,s} = e_{l-1,i} + e_{l,i} + q_{l+1,i} = e_{l,i} + q_{l+1,i}$. For $l = n-1$, $q_{k,i} + e_{n-1,s} = (q_{k,i} + e_{n-2,s}) + e_{n-1,s} = e_{n-2,i} + q_{n-1,i} + e_{n-1,s} = e_{n-2,i} + e_{n,i} = e_{n,i}$. For $l = n$, $q_{k,i} + e_{n,s} = (q_{k,i} + e_{n-1,s}) + e_{n,s} = e_{n,i} + e_{n,s} = e_{n,i}$. The case $i = 1$ is similar.

It then follows that for $l \geq k$

$$q_{k,i} + \ldots + q_{k,m-1} + e_{l,s} = \begin{cases} e_{l,i} + q_{l+1,i} + \ldots + q_{l+1,m-1} & \text{if } l \leq n-2, i \geq 2 \\ e_{l+1,i} + q_{l+1,i} + \ldots + q_{l+1,m-1} & \text{if } l \leq n-2, i = 1 \\ e_{n,i} & \text{if } l \geq n-1. \end{cases}$$

Now take the sequence $e_{1,0}, q_{1,1}, \dots, q_{1,m-1}$ as $p_0, p_1, \dots, p_{m-1}$. To show that it is as required, we first write each $\sum p_w$, where $w \in W_{m,n}$, in a canonical expression using the formula above. We shall see that all those expressions are distinct and $p_{m-1} + (n-1)\sum p_{w_0} + p_0 = p_{m-1} + (n-2)\sum p_{w_0} + p_0$. We then show that distinct expressions represent distinct elements which finishes the proof.

For words containing no 0 or in which only first letter is 0, the expressions are unchanged, that is, $q_{1,i} + \ldots + q_{1,s}$ or $e_{1,0}$ or $e_{1,0} + q_{1,1} + \ldots + q_{1,s}$. We call such words and expressions trivial. Obviously, all trivial expressions are distinct.

Consider the sums $\sum p_w$ for nontrivial words $w \in W_{m,n}(0)$. The words can be written as $(w_0)^k$, where $k \in \{2, \dots, n\}$, or $(w_0)^k 0 \ldots s$, where $k \in \{1, \dots, n-1\}$ and $s \in \{0, \dots, m-2\}$. Computing we obtain that for $k \leq n-1$,

$$k \sum p_{w_0} = e_{k,0} + q_{k,1} + \ldots + q_{k,m-1} + \ldots + q_{1,1} + \ldots + q_{1,m-1}$$

and

$$n\sum p_{w_0}=e_{n,0}+q_{n-1,1}+\ldots+q_{n-1,m-1}+\ldots+q_{1,1}+\ldots+q_{1,m-1},$$

for $k\le n-2$,

$$k\sum p_{w_0}+p_0+\ldots+p_s$$
$$=e_{k+1,0}+q_{k+1,1}+\ldots+q_{k+1,m-1}+\ldots+q_{2,1}+\ldots+q_{2,m-1}+q_{1,1}+\ldots+q_{1,s}$$

(for $s=0$ meaning that

$$k\sum p_{w_0}+p_0=e_{k+1,0}+q_{k+1,1}+\ldots+q_{k+1,m-1}+\ldots+q_{2,1}+\ldots+q_{2,m-1})$$

and

$$(n-1)\sum p_{w_0}+p_0+\ldots+p_s$$
$$=e_{n,0}+q_{n-1,1}+\ldots+q_{n-1,m-1}+\ldots+q_{2,1}+\ldots+q_{2,m-1}+q_{1,1}+\ldots+q_{1,s}.$$

To see for example the first one, by induction, for $k>1$,

$$k\sum p_{w_0}=p_0+\ldots+p_{m-1}+(k-1)\sum p_{w_0}$$
$$=e_{1,0}+q_{1,1}+\ldots+q_{1,m-1}+e_{k-1,0}$$
$$+q_{k-1,1}+\ldots+q_{k-1,m-1}+\ldots+q_{1,1}+\ldots+q_{1,m-1}$$
$$=e_{1,0}+e_{k,0}+q_{k,1}+\ldots+q_{k,m-1}+$$
$$+q_{k-1,1}+\ldots+q_{k-1,m-1}+\ldots+q_{1,1}+\ldots+q_{1,m-1}$$
$$=e_{k,0}+q_{k,1}+\ldots+q_{k,m-1}+\ldots+q_{1,1}+\ldots+q_{1,m-1}.$$

It is easy to see that all these expressions are distinct, and in each of them the first element is $e_{k,0}$ for some $k\ge 2$, so it is distinct from all trivial expressions.

Next consider the sums $\sum p_w$ for nontrivial words $w\in W_{m,n}(1)$. The words can be written as $1\ldots(m-1)(w_0)^k$, where $k\in\{1,\ldots,n-1\}$, or $1\ldots(m-1)(w_0)^k0\ldots s$, where $k\in\{0,\ldots,n-2\}$ and $s\in\{0,\ldots,m-2\}$. Computing we obtain that for $k\le n-2$,

$$p_1+\ldots+p_{m-1}+k\sum p_{w_0}=e_{k+1,1}+q_{k+1,1}+\ldots+q_{k+1,m-1}+\ldots+q_{1,1}+\ldots+q_{1,m-1}$$

and

$$p_1+\ldots+p_{m-1}+(n-1)\sum p_{w_0}=e_{n,1}+q_{n-1,1}+\ldots+q_{n-1,m-1}+\ldots+q_{1,1}+\ldots+q_{1,m-1},$$

for $k\le n-3$,

$$p_1+\ldots+p_{m-1}+k\sum p_{w_0}+p_0+\ldots+p_s$$
$$=e_{k+2,1}+q_{k+2,1}+\ldots+q_{k+2,m-1}+\ldots+q_{2,1}+\ldots+q_{2,m-1}+q_{1,1}+\ldots+q_{1,s}$$

(that is,

$$p_1 + \ldots + p_{m-1} + k \sum p_{w_0} + p_0 = e_{k+2,1} + q_{k+2,1} + \ldots + q_{k+2,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1})$$

and

$$p_1 + \ldots + p_{m-1} + (n-2) \sum p_{w_0} + p_0 + \ldots + p_s$$
$$= e_{n,1} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1} + q_{1,1} + \ldots + q_{1,s}.$$

All these expressions are distinct, and in each of them the first element is $e_{k,1}$ for some $k \geq 2$, so it is distinct from all trivial expressions.

We also obtain that

$$p_{m-1} + (n-1) \sum p_{w_0} + p_0$$
$$= q_{1,m-1} + e_{n,0} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1}$$
$$= e_{n,m-1} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1}$$

and

$$p_{m-1} + (n-2) \sum p_{w_0} + p_0$$
$$= q_{1,m-1} + e_{n-1,0} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1}$$
$$= e_{n,m-1} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1},$$

so

$$p_{m-1} + (n-1) \sum p_{w_0} + p_0 = p_{m-1} + (n-2) \sum p_{w_0} + p_0.$$

Now consider the sums $\sum p_w$ for nontrivial words $w \in W_{m,n}(i)$, where $i \geq 2$. The words can be written as $i \ldots (m-1)(w_0)^k$, where $k \in \{1, \ldots, n-1\}$, or $i \ldots (m-1)(w_0)^k 01 \ldots s$, where $k \in \{0, 1, \ldots, n-2\}$ and $s \in \{0, 1, \ldots, m-2\}$. Computing we obtain that for $k \leq n-2$,

$$p_i + \ldots + p_{m-1} + k \sum p_{w_0}$$
$$= e_{k,i} + q_{k+1,i} + \ldots + q_{k+1,m-1} + q_{k,1} + \ldots + q_{k,m-1} + \ldots + q_{1,1} + \ldots + q_{1,m-1}$$

and

$$p_i + \ldots + p_{m-1} + (n-1) \sum p_{w_0}$$
$$= e_{n,i} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{1,1} + \ldots + q_{1,m-1},$$

for $k \leq n-3$,

$$p_i + \ldots + p_{m-1} + k \sum p_{w_0} + p_0 + \ldots + p_s = e_{k+1,i} + q_{k+2,i} + \ldots + q_{k+2,m-1}$$
$$+ q_{k+1,1} + \ldots + q_{k+1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1} + q_{1,1} + \ldots + q_{1,s}$$

and

$$p_i + \ldots + p_{m-1} + (n-2) \sum p_{w_0} + p_0 + \ldots + p_s$$
$$= e_{n,i} + q_{n-1,1} + \ldots + q_{n-1,m-1} + \ldots + q_{2,1} + \ldots + q_{2,m-1} + q_{1,1} + \ldots + q_{1,s}.$$

All these expressions are distinct. In the expression for $p_i + \ldots + p_{m-1} + p_0 + \ldots + p_s$, where $s \in \{0, \ldots, m-1\}$, the first element is $e_{1,i}$ and the second $q_{2,i}$ and in all other expressions the first element is $e_{k,i}$ for some $k \geq 2$, so each of these expressions is distinct from all trivial expressions.

Since for each $i \in \{0, \ldots, m-1\}$ and for each nontrivial $w \in W_{m,n}(i)$ the first element in the expression for $\sum p_w$ is $e_{k,i}$ for some k, it follows that all expressions for $\sum p_w$, where $w \in W_{m,n}$, are distinct.

Finally, we show that all these expressions represent distinct elements.

Assume on the contrary that some two distinct expressions represent the same element. Then canceling the equality by $q_{k,i}$-s we arrive at one of the following cases:

(1) $u + q_{k,i} = v + q_{t,j}$ for some $u, v \in \beta\mathbb{N}$ and $(k, i) \neq (t, j)$,
(2) $u + q_{k,i} = q_{t,j}$ for some $u \in \beta\mathbb{N}$,
(3) $u + q_{k,i} = e_{l,s}$ for some $u \in \beta\mathbb{N}$,
(4) $e_{k,i} = e_{l,s}$ with $(k, i) \neq (l, s)$.

The last one is obviously impossible.

In (1), we have that $\phi_k(q_{k,i}) = \phi_k(u + q_{k,i}) = \phi_k(v + q_{t,j}) = \phi_k(q_{t,j})$. If $k = t$, then $i \neq j$ and $\phi_k(q_{k,i}) = \phi_k(q_{k,j})$, a contradiction. If $k \neq t$, say $k < t$, then $\phi_k(q_{t,j}) = \phi_k(q_{t,j} + e_{k,0}) = \phi_k(e_{k,0})$ and $\phi_k(q_{k,i}) \neq \phi_k(e_{k,0})$, again a contradiction.

In (2), since $q_{t,j}$ is right cancelable, one has $t \neq k$. Suppose $k < t$. Then $\phi_k(q_{k,i}) = \phi_k(q_{t,j})$. But $\phi_k(q_{t,j}) = \phi_k(e_{k,0})$ and $\phi_k(q_{k,i}) \neq \phi_k(e_{k,0})$, a contradiction. The case $t < k$ is essentially the same, since applying ϕ_t to $q_{t,j} = u + q_{k,i}$ gives us $\phi_t(q_{t,j}) = \phi_t(q_{k,i})$.

In (3), since $q_{k,i} \in \overline{K(\beta\mathbb{N})}$, $e_{t,s} \in T_{n-1}$ for $t \leq n-1$, and $T_{n-1} \cap \overline{K(\beta\mathbb{N})} = \emptyset$, one has $l = n$. Then $\phi_k(q_{k,i}) = \phi_k(e_{n,s})$. But $\phi_k(e_{n,s}) = \phi_k(e_{n,s} + e_{k,0}) = \phi_k(e_{k,0})$ and $\phi_k(q_{k,i}) \neq \phi_k(e_{k,0})$, a contradiction. □

3. Ramsey theoretic consequence

Theorem 3.1. *Let $m \geq 2$ and $n \geq 2$. Let $p_0, \ldots, p_{m-1}$ be a sequence in $\mathbb{H}$ guaranteed by Theorem 2.2 and for each $w \in W_{m,n}$, let $(A_w(j))_{j=0}^{\infty}$ be a sequence of members of the ultrafilter $\sum p_w$. There is a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in A_{p_{\nu(j)}}(j) \cap 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$ with $v = \nu(j_0) \ldots \nu(j_s) \in W_m$, one has $x_{j_0} + \ldots + x_{j_s} \in A_{\pi(v)}(j_0)$.*

Proof. We construct inductively a sequence $(x_j)_{j=0}^{\infty}$ satisfying for every j the following conditions in addition to $x_j \in 2^j\mathbb{N}$:

for each finite sequence $j_0 < \ldots < j_s = j$ with $v = \nu(j_0) \ldots \nu(j_s) \in W_m$,

$$x_{j_0} + \ldots + x_{j_s} \in A_{\pi(v)}(j_0)$$

and for each $w \in W_{m,n}(\nu(j+1))$,

$$x_{j_0} + \ldots + x_{j_s} + \sum p_w \in \overline{A_{\pi(vw)}(j_0)}.$$

To define x_0, for each $w \in W_{m,n}(1)$, choose $P(w) \in p_0$ such that $P(w) + \sum p_w \subseteq \overline{A_{\pi(0w)}(0)}$. We can do this because $p_0 + \sum p_w = p_{\pi(0w)}$ and the right translation by $\sum p_w$ is continuous. Pick

$$x_0 \in A_0(0) \cap \bigcap_{w \in W_{m,n}(1)} P(w).$$

Then $x_0 \in A_0(0)$ and for each $w \in W_{m,n}(1)$, $x_0 + \sum p_w \in P(w) + \sum p_w \subseteq \overline{A_{\pi(0w)}(0)}$, so x_0 is as required.

Fix $j \geq 0$ and suppose that we have defined $x_0, \ldots, x_j$ as required. To define x_{j+1}, let F be the set of all sequences $j_0 < \ldots < j_s \leq j$ with $\nu(j_0)\ldots\nu(j_s) \in W_m$ and $\nu(j_s) = \nu(j)$ and let $i = \nu(j+1)$ and $r = \nu(j+2)$. For each $w \in W_{m,n}(r)$, choose $B(w) \in p_i$ such that $B(w) + \sum p_w \subseteq \overline{A_{\pi(iw)}(j+1)}$. Then for each $(j_0, \ldots, j_s) \in F$, choose $C(j_0, \ldots, j_s) \in p_i$ such that $x_{j_0} + \ldots + x_{j_s} + C(j_0, \ldots, j_s) \subseteq A_{\pi(vi)}(j_0)$, where $v = \nu(j_0)\ldots\nu(j_s)$, and for each $w \in W_{m,n}(r)$, choose $D(j_0, \ldots, j_s, w) \in p_i$ such that $x_{j_0}+\ldots+x_{j_s}+D(j_0, \ldots, j_s, w)+\sum p_w \subseteq \overline{A_{\pi(viw)}(j_0)}$. We can do the first because by the inductive hypothesis $x_{j_0} + \ldots + x_{j_s} + p_i \in \overline{A_{\pi(vi)}(j_0)}$ and λ_x, where $x = x_{j_0} + \ldots + x_{j_s}$, is continuous, and the second because $p_i + \sum p_w = \sum p_{\pi(iw)}$ and by the inductive hypothesis $x_{j_0} + \ldots + x_{j_s} + p_{\pi(iw)} \in \overline{A_{\pi(v\pi(iw))}(j_0)} = \overline{A_{\pi(viw)}(j_0)}$ (since $\pi(viw) = \pi(v\pi(iw))$) and λ_x and ρ_p, where $p = \sum p_w$, are continuous. Pick

$$x_{j+1} \in 2^{j+1}\mathbb{N} \cap A_i(j+1) \cap \bigcap_{w\in W_{m,n}(r)} B(w) \cap \bigcap_{(j_0,\ldots,j_s)\in F} (C(j_0, \ldots, j_s) \cap \bigcap_{w\in W_{m,n}(r)} D(j_0, \ldots, j_s, w))$$

(all those sets are members of p_i).

To see that x_{j+1} is as required, let any $j_0 < \ldots < j_s = j+1$ with $\nu(j_0)\ldots\nu(j_s) \in W_m$ be given. If $s = 0$, then $x_{j+1} \in A_i(j+1)$ and for each $w \in W_{m,n}(r)$, $x_{j+1} + \sum p_w \in B(w) + \sum p_w \subseteq \overline{A_{\pi(iw)}(j+1)}$. If $s \geq 1$, then

$$x_{j_0} + \ldots + x_{j_s} \in x_{j_0} + \ldots + x_{j_{s-1}} + C(j_0, \ldots, j_{s-1}) \subseteq A_{\pi(vi)}(j_0),$$

where $v = \nu(j_0)\ldots\nu(j_{s-1})$, and for each $w \in W_{m,n}(r)$,

$$x_{j_0} + \ldots + x_{j_s} + \sum p_w \in x_{j_0} + \ldots + x_{j_{s-1}} + D(x_{j_0}, \ldots, x_{j_{s-1},w}) + \sum p_w \subseteq \overline{A_{\pi(viw)}(j_0)}. \quad \square$$

Now from Theorem 2.2 and Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let $m \geq 2$ and $n \geq 2$. There is a partition*

$$\{A_{i,k} : i \in \{0, \ldots, m-1\} \text{ and } k \in \{i, \ldots, mn-1\} \text{ for each } i\}$$

of $\mathbb{N}$ such that, whenever for each (i, k), $\mathcal{B}_{i,k}$ is a finite partition of $A_{i,k}$, there exist $B_{i,k} \in \mathcal{B}_{i,k}$ and a sequence $(x_j)_{j=0}^{\infty}$ such that $x_j \in B_{\nu(j),\nu(j)} \cap 2^j\mathbb{N}$ and for every finite sequence $j_0 < \ldots < j_s$ such that $j_{t+1} \equiv j_t + 1 \pmod{m}$ for each $t < s$, if $i_0 = \nu(j_0)$ and

$$k_0 = \begin{cases} i_0 + s & \text{if } i_0 + s \leq mn - 1 \\ mn - m + \nu(i_0 + s - mn) & \text{otherwise,} \end{cases}$$

then $x_{j_0} + \ldots + x_{j_s} \in B_{i_0,k_0}$.

Proof. Let $p_0, \ldots, p_{m-1}$ be a sequence guaranteed by Theorem 2.2. Choose a partition $\{A_w : w \in W_{m,n}\}$ of $\mathbb{N}$ such that A_w is a member of the ultrafilter $\sum p_w$, and for each $i \in \{0, \ldots, m-1\}$ and each $k \in \{i, \ldots, mn-1\}$, put $A_{i,k} = A_{w(i,k-i)}$. To see that the partition $\{A_{i,k} : i \in \{0, \ldots, m-1\}$ and $k \in \{i, \ldots, mn-1\}$ for each $i\}$ is as required, for each (i, k), let $\mathcal{B}_{i,k}$ be a finite partition of $A_{i,k}$. Pick $B_{i,k} \in \mathcal{B}_{i,k}$ which is a member of the ultrafilter $\sum p_{w(i,k-i)}$, and for every $j \geq 0$, put $A_{w(i,k-i)}(j) = B_{i,k}$. Let $(x_j)_{j=0}^{\infty}$ be a sequence guaranteed by Theorem 3.1 and let a finite sequence $j_0 < \ldots < j_s$ with $v = \nu(j_0)\ldots\nu(j_s) \in W_m$ be given. Then $x_{j_0} + \ldots + x_{j_s} \in A_{\pi(v)}(j_0)$. Let $i_0 = \nu(j_0)$. Then $v = w(i_0, s)$ and $\pi(v) = w(i_0, s_0)$, where s_0 is s if $s \leq mn - 1 - i_0$ and $mn - m - i_0 + \nu(s - mn + i_0)$ otherwise. Let $k_0 = i_0 + s_0$. Then k_0 is $i_0 + s$ if $i_0 + s \leq mn - 1$ and $mn - m + \nu(i_0 + s - mn)$ otherwise, and $A_{\pi(v)}(j_0) = B_{i_0,k_0}$. $\square$

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