Non Supersymmetric Large N Background for Two Yang - Mills Coupled Matrices

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A thesis submitted to the Faculty of Science,
University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of
Doctor of Philosophy.

Johannesburg, 2010

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not previously in its entirety or in part been submitted for any degree or examination in any other University.

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Abstract

The study of matrix models is of considerable significance. In this thesis we study the large N quantum mechanics of two matrices (X_1, X_2), coupled via a Yang-Mills interaction, in a non supersymmetric setting. Of the two matrices, X_1 is treated exactly, while X_2 is understood as an "impurity" in the background of the first. Considering the ground state wavefunction with no X_2 "impurities", it is observed that this state depends on the eigenvalues of the X_1 matrix, resulting in additional shifts in the calculation of the kinetic term for the X_1 sector. This results in an effective potential which, using the results of collective field theory, is used to obtain the planar large N nonsupersymmetric background. The system is studied in both weak and strong coupling. The strong coupling system is free of infrared divergences.

Acknowledgments

I am greatly indebted to my supervisor, Prof.J.A.P. Rodrigues, for his expert guidance , his invaluable assistance in promoting my knowledge and ability and all the support I have received from him, since I came to this university.

I would also like to thank my husband and my children , for their constant support, and to my sister and parents who believed in me .

Thanks are also due to the National Institute of Theoretical Physics and to the University of Witwatersrand, Johannesburg, for providing the financial support in conducting this research.

Most of all, I am thankful to the Almighty, the one who created it all.

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Introduction

1.1 Background

The origin of string theory was to describe the large number of mesons and hadrons, which interact through strong force. These particles were seen as different oscillation modes of a string. Although it explained well some of the features of hadron spectrum, such as the mass angular momentum relation, this idea was replaced by that of Quantum Chromodynamics (QCD) which is a renormalizable quantum field theory having quarks as the fundamental constituent of matter. This theory has a running coupling constant, which means that at high energies the coupling constant becomes very small (asymptotic freedom), but at low energies, this coupling constant becomes very large (quark confinement). This makes the theory strongly coupled at low energies and it is not easy to perform perturbative calculations. A possible solution of this problem is the large N expansion as suggested by 't Hooft.

Large N Expansion

The first hint of string-gauge duality was given by 't Hooft in 1974 [1], where he proposed that the large N limit of SU(N) gauge theory is equivalent to string theory. This proposal of 't Hooft was intended to overcome the problem of low energy calculations in QCD where the theory becomes strongly coupled and the perturbative methods do not work. QCD which is the candidate for the theory of strong interactions, consists of quarks which come in three colors. Hence the theory is based on SU(3) gauge group. 't Hooft suggested that if instead the gauge group is taken as SU(N), with N being the number of colors, then taking the large N limit such that $\lambda = g_{YM}^2 N$ is held fixed, may lead to a solvable approximation. This is called the 't Hooft limit. This large N expansion is of the form similar to the perturbative expansion of closed strings, thus suggesting the equivalence between gauge theory and string theory. The perturbative expansion of a large N gauge theory in 1/N and $g_{YM}^2 N$ has the form [2],[3]:

$$Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)$$

here $f_g(\lambda)$ is some polynomial and g is the genus or the number of handles in the diagram. The diagram with $g = 0$ is the leading order term in the large N expansion and can be drawn on a plane. These are therefore known as planar diagrams. Each term with $g > 0$ suppresses the leading term by factor of $1/N^2$. This perturbative expansion in gauge theory has form similar to the loop expansion in string theory

$$Z = \sum_{g=0}^{\infty} g_s^{2g-2} Z_g$$

with string coupling g_s equal to 1/N. We therefore see that the large N limit connects gauge theory with string theory. However, this connection is based on perturbative expansion that does not converge. Therefore, it is only indicative and not a rigorous derivation of the equivalence.

AdS/CFT Correspondence

This possible gauge - string theory duality found a concrete realization in the context of the Anti-de-Sitter / Conformal Field theory (AdS/CFT) correspondence, originally proposed by Juan Maldacena [4],[5],[6]. This can be described as the duality between a theory with gravity and one without gravity. This is so because the string theory side of this equivalence, in the AdS background, includes gravitons and thus it describes gravity, and the gauge theory side, which is given by conformal field theory does not contain any particle with spin greater than one and thus is a theory without gravity. The original example of the AdS/CFT correspondence as proposed by Maldacena is the equivalence between type IIB string theory on a ten dimensional space that consists of a five dimensional Anti-de-Sitter space and a five sphere i.e. $AdS_5 \times S^5$, and maximally supersymmetric four dimensional conformal field theory which is $\mathcal{N}$=4 supersymmetric Yang-Mills (SYM) theory. The Anti-de-Sitter space is the maximally symmetric solution of Einstein's equation with negative cosmological constant [7]. The metric of AdS_5 is given as

$$ds^2 = R^2[-dt^2 cosh^2\rho + d\rho^2 + sinh^2\rho d\Omega_3^2]$$

Adding the 5-sphere of radius R, the full metric of $AdS_5 \times S^5$ is

$$ds^2 = R^2[-dt^2 cosh^2\rho + d\rho^2 + sinh^2\rho d\Omega_3^2 + d\psi^2 cos^2\theta + d\theta^2 + sin^2\theta d\Omega'^2_3].$$

$AdS_5 \times S^5$ can be shown to be a solution to the type IIB supergravity equations of motion [8]. The symmetry group of AdS_5 is SO(2,4) and the symmetry group of S^5 is SO(6). Thus $AdS_5 \times S^5$ has an overall symmetry group $SO(2,4) \times SO(6)$ which in complex terms is $SU(2,2 \mid 4)$ [9]. The other side of the equivalence consists of supersymmetric conformal field theory. The conformal field theory is quantum field theory that is invariant under the group of conformal transformations. These transformations preserve the metric up to an over all (in general x dependent) scaling factor $g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x)$, thus preserving angles. Gauge theory, where the coupling constant does not change as a function of the energy scale, is a conformally invariant theory. Now, we add supersymmetry to this conformally invariant gauge theory, to get super conformal field theory. Supersymmetry relates bosons to fermions and contains supercharges. The $\mathcal{N}$=4 super Yang Mills theory consists of 32 supercharges. The field content of this theory includes the complex Weyl fermions, the vector field and six real scalar fields [3]. These six real scalar fields have an SO(6) R-Symmetry. Including the R-Symmetry, the $\mathcal{N}$=4 SYM obeys a global supersymmetry corresponding to the supergroup $SU(2,2 \mid 4)$, which is the same as that of $AdS_5 \times S^5$. This similarity of the supergroups of the two theories is one of the confirmations of the AdS/CFT correspondence.

The AdS/CFT correspondence relates a theory in d+1 dimensions to a theory without gravity in d dimensions. Thus the AdS/CFT correspondence follows the holographic principles [6], [10] which states that all information contained in a volume in d+1 dimensional space can be represented by another theory which lives on the boundary of that volume in d dimensional space. The principle of holography applies to black holes as well stating that the black hole entropy which is the number of degrees of freedom of a black hole can be described using the area of event horizon of the black hole. In the context of AdS/CFT correspondence, the type II string living on $AdS_5 \times S^5$ dual to $\mathcal{N}$=4 SYM, follows this holographic priciple becauce the SYM theory can be thought of as living on the four dimensional boundary of the five dimensional AdS_5 space.

In this correspondence which relates type IIB string theory in ten dimensional $AdS_5 \times S^5$ space time to maximally supersymmetric Yang Mills theory in four dimension, the string model is controlled by two parameters [11]: the string coupling constant g_S and the "effective" string tension R^2/α, where R is the common radius of AdS_5 and S^5 geometries and α is related to the string length. The gauge theory is parameterized by the rank N of the gauge group and the coupling constant g_{YM} or equivalently the 't Hooft coupling $\lambda = g_{YM}^2 N$. According to AdS/CFT correspondence these two sets of parameters are related as

$$\frac{4\pi\lambda}{N} = g_S \qquad\qquad \sqrt{\lambda} = \frac{R^2}{\alpha}.$$

Beside this correspondence of parameters, AdS/CFT also relates the energy eigenstate of $AdS_5 \times S^5$ string with composite gauge theory operators of the form $O_A = \mathrm{Tr}(\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n})$ where ϕ_{i_n} are the elementary fields of $\mathcal{N}$=4 SYM in the adjoint representation of SU(N) i.e. $N \times N$ hermitian matrices. The energy eigenvalue E of a string state with respect to time in global coordinates is conjectured to be equal to the scaling dimension of the dual gauge theory operator. These scaling dimensions are the eigenvalues of dilatation operator acting on the state O_A.

The spectrum on the string side of this duality is known in the low energy limit, corresponding to weakly curved geometries in string units i.e., to the region $\sqrt{\lambda} \gg 1$. While on the gauge side, the theory is understood only in the perturbative regime i.e. $\lambda \ll 1$. Thus AdS/CFT duality is a weak/strong coupling duality. This, on one hand means that it makes the calculations easier in the regions where it was previously difficult like the low energy QCD, while on the other hand this makes any attempt at a derivation of the AdS/CFT conjecture all the more difficult.

BMN Conjecture

The string/gauge map was made considerably more precise by Berenstein, Maldacena and Nastase [12] in 2002, who proposed considering certain limit on both sides of the AdS/CFT duality. The limit on the string theory side was taken by considering a string rotating with large angular momentum on a great circle of the S^5 sphere. In this large J limit with J^2/N held fixed (the BMN limit) the geometry seen by this fastly moving string is a gravitational plane wave. This limit is a particular case of the Penrose limit. Thus, the plane wave or the pp-wave geometry is obtained by taking the Penrose limit of the $AdS_5 \times S^5$ background [13], [14]. These are the maximally supersymmetric solutions to type IIB string theory. This has the advantage that its spectrum is exactly known in the light cone gauge. The corresponding limit applied on the gauge theory leads to the considerations of operators with large R-charge J along with large number of colors N such that the effective quantum loop counting parameter $\lambda' = g_{YM}^2 N/J^2$ and the effective genus counting parameter $g_s' = J^2/N$ are held fixed. This R-charge J is the SO(2) generator of rotation in the plane generated by two of the six Higgs scalars. Choosing of two Higgs scalars from the set of six, corresponds to breaking of SO(6) R symmetry of the gauge theory, which is equivalent to the breaking of SO(6) symmetry of the S_5 sphere by fast moving string along the equator of the S_5 sphere. According to this correspondence [17], the R-charge J of the Yang Mills operator is proportional to the light cone momentum p^+ of the corresponding string state and the operator $\triangle - J$ of the Yang mills theory is proportional to the light cone energy p^- of the same state, where $\triangle$ is the dilatation operator. This can also be stated as an equality between two operators. On the string theory side p^- can be understood as the plane wave light cone string theory Hamiltonian, which on Yang-Mills side is equal to difference between dilatation operator and R-charge operator. Thus the BMN conjecture stated in another way, relates the spectrum of strings which are eigenvalues of light cone Hamiltonian p^- to the spectrum of dilatation operator which is the Hamiltonian of $\mathcal{N} = 4$ gauge theory on $R \times S^3$ restricted to the BMN limit.

We now explain how this spectrum of states matches on either side of the correspondence. On the gauge theory side the operator with lowest value of $\triangle - J = 0$ is the unique single trace operator namely $Tr(Z^J)$ [15], [16] where $Z \equiv (\phi_5 + i\phi_6)$ (where ϕ_5 and ϕ_6 are two of the six Higgs scalars of the $\mathcal{N} = 4, D = 4$ SYM theory and trace is over N color indices). This operator is the chiral primary operator and is associated to the 1/2BPS state whose scaling dimension is exactly equal to J at all value of coupling parameter λ'. It is associated to the vacuum state in the light cone gauge which is unique state with zero light cone energy. In other words we have the correspondence

$$\frac{1}{\sqrt{JN^J}} TrZ^J \longleftrightarrow |0, p^+ >_{l.c.}.$$

Here, $\frac{1}{\sqrt{JN^J}}$ is the normalization constant. On the string theory side, this state corresponds to a ground state supergravity mode with the string momentum $n = 0$. These states generate the flat space spectrum. On the gauge theory side other operators may be generated from the single trace by acting on it with SO(6) supersymmetry lowering operator. This corresponds to the insertion of an "impurity" and summing over all possible positions within $Tr(Z^J)$. This impurity is one of the six Higgs scalar fields, other than the two defining Z. Thus the single impurity operator is given by $\frac{1}{\sqrt{N^{J+1}}} Tr(\phi_i Z^J)$. This operator has the scaling dimension $\triangle = J + 1$. Similarly the two impurity operator which is obtained by acting two distinct lowering operator on $Tr(Z^{J+2})$ yields

$$\frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} Tr(\phi Z^l \psi Z^{J-l})$$

which corresponds to string state $a_0^{\phi^\dagger} a_0^{\psi^\dagger} |0, p^+ >$. Proceeding in this manner, all operators dual to supergravity modes of the gauge theory can be obtained by acting with an appropriate number of lowering operators on the single trace operator i.e. inserting the appropriate number of impurity fields in the "string of Z's". This background Z fields are always assumed to be large in number compared to the number of impurity fields i.e. the system is always assumed to be in "dilute gas" approximation. Although the operator-state correspondence explained here is for scalar fields, but it can be extended to bosonic and fermionic fields as well.

The proposal of BMN extended to the non supergravity modes as well. This corresponds to extending the theory on string side to $n \neq 0$ modes of excitation. The operators consisting of impurities which are the near BPS operators come with position dependent phase term $exp(2\pi inl/J)$ where l is the position of the impurity within the trace. The operator with single impurity is given as

$$\frac{1}{\sqrt{J}}\sum_{l=1}^{J}\frac{1}{\sqrt{JN^{J+1}}}Tr(Z^l\phi Z^{J-l})exp(2\pi inl/J).$$

The operator however vanishes (except for zero momentum $n = 0$) because of the cyclicity of the trace. The corresponding state on the string theory side i.e. ${a_n^i}^\dagger|0,p^+>$ also vanishes because of the constraint that total momentum along the string should be zero. So we can only have more than one oscillator state so that total momentum is zero, and thus corresponds to a physical state. The first non-trivial example of this is the two oscillator state ${a_n^i}^\dagger{a_{-n}^j}^\dagger|0,p^+>$ which corresponds to gauge field theories with two impurities ϕ_i and ϕ_j in the sequence of Z fields summed over the possible position of impurities along with a position dependent phase term . So the operator state correspondence for this case is

$${a_n^i}^\dagger{a_{-n}^j}^\dagger|0,p^+>_{l.c.}\leftrightarrow\frac{1}{\sqrt{JN^{J+1}}}\sum_{l=1}^{J}Tr(\phi_iZ^l\phi_jZ^{J-l})exp(2\pi inl/J).$$

To summarize, each oscillator along the string is associated with one impurity field in the sequence of Z fields on the gauge theory side, with the sum over all possible positions for the insertion of the impurity field and a phase proportional to the momentum. States whose total momentum is not zero along the string correspond to operators that also vanish because of the cyclicity of the trace.

1.2 Motivation: Matrix Models and their Significance

Now that this correspondence between plane wave string theory and BMN gauge theory is established as a limit of the AdS/CFT duality one might consider if other properties of AdS/CFT duality also hold in this limit. One such property of the AdS/CFT duality is the principle of holography. The boundary of the plane wave string theory is a one dimensional light like direction. So, by the principles of holography [17], one might guess that the dual gauge theory living on the boundary of strings on the pp-wave background should be a quantum mechanical model. This quantum mechanical system is the plane wave matrix model [18], which arises by considering $\mathcal{N} = 4, D = 4$ SYM compactified on a three sphere and performing truncation of the resulting Kaluza-Klein spectrum to the lowest lying mode. The Hamiltonian of the SYM theory corresponds to the dilatation operator and the Higgs fields become quantum mechanical matrix coordinates. This matrix theory in the pp-wave background has mass terms, in contrast to the matrix model in the flat Minkowski background. This mass deformation in the pp-wave background makes its energy spectrum discrete, whereas the spectrum in flat background is continuous. The mass parameter of the pp-wave matrix model is related to the four dimensional Yang Mills coupling constant as

$$\{\frac{m}{3}\}^3 = \frac{32\pi^2}{g_{ym}^2}.$$

Thus we see that the pp wave string theory is dual to the matrix quantum mechanics which arises because of the Kaluza-Klein reduction of $\mathcal{N} = 4$ SYM on $R \times S^3$.

The study of quantum mechanics of matrix degrees of freedom is termed as matrix model. This model first appeared in the study of nuclear physics, when studying energy levels of atomic nuclei, and in statistical physics.

In general, the study of matrix model, is of considerable significance and interest, particularly their large N limit. Some of these examples are

- The large N limit of the system of $N \times N$ matrices describes the D0 branes connected by small strings. These D0 branes provide a definition of the M-theory in the light cone frame [19]. Thus, we see that M-theory in the light cone frame, is exactly described by the large N limit of a particular supersymmetric matrix quantum mechanics.
- The correlators in one dimensional matrix model are related to correlators of 1/2 BPS operators in zero coupling limit of $\mathcal{N} = 4$ SYM theory. These correlation functions are used to understand the properties of giant gravitons and related solutions of string theory on $AdS_5 \times S^5$ [20].
- Another useful application in the study of matrix models is the mapping between basis of states made of traces (closed strings) and the eigenvalue of matrices in terms of Schur polynomials [20], [21]
- The plane wave matrix theory is related to $\mathcal{N} = 4$ SYM dilatation operator [18].
- Multi-matrix, multi-trace operators with diagonal free two point functions have also been identified [22], [23].
- The theory of quantum chromodynamics (QCD) has been thought of as reduced to finite number of matrices with quenched momenta [24]. Alternatively they can associated with QCD zero modes on hyperspheres.

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Single Matrix Models

The single hermitian matrix model is the simplest model and was solved by [25], in the large N limit or the planar limit. In this study, the single matrix model was used to obtain the combinatorics of planar diagrams and the ground state energy of a one dimensional oscillator with four vertex interaction which lead to the introduction of fermions. The study of the single matrix model in the fermionic picture describes, on the field theory side, the 1/2 BPS states and their interactions. These states are associated to chiral primary operators with conformal weight $\Delta = J$, where J is a particular U(1) charge in the R-symmetry group. As the value of J changes, its interpretation on the dual quantum gravity changes. For excitation energy is O(1), the dual state is a state of gravitons; if it of $O(\sqrt{N})$ the dual state is a string; If it is O(N) the dual state is a state of giant gravitons and if it is $O(N^2)$ the dual state is an LLM geometry. It was shown by Lin, Lunin and Maldacena (LLM), that a fermionic droplet configuration completely describes 1/2 BPS states [26]. The dynamics of these 1/2 BPS states and their interactions have a simple field theory description in terms of free fermions associated with complex matrix in a harmonic potential. The fermions form a droplet configuration in the phase space. These states can also be thought of as fermions in a magnetic field on the lowest Landau level (Quantum Hall Effect). However, the energy and flux obtained by LLM are exactly reproduced if the free fermion matrix model is replaced by a one dimensional hermitian matrix in a bosonic phase space density description [27]. The study of single hermitian matrix in a bosonic phase space description, gives the energy and flux associated with 1/2 BPS states. These 1/2 BPS states are constructed from a system of two matrices or a complex matrix and performing a truncation to a single hermitian matrix. Specifically, starting with $Z = X_1 + iX_2$, where X_1 and X_2 are the two scalars of the $\mathcal{N} = 4$ SYM theory, and introducing the matrix valued creation and annihilation operators

$$Z = \frac{1}{\sqrt{\omega}}(A + B^\dagger)$$

the 1/2 BPS states correspond to a restriction of the sector with no B excitations. Thus the single hermitian matrix is the matrix describing the dynamics of the A, $A^\dagger$ system i.e.

$$M \equiv \frac{1}{\sqrt{2\omega}}(A + A^\dagger).$$

The supergravity description of these 1/2 BPS states is given in terms of giant gravitons [28]. These giant gravitons are the solutions of D3-branes wrapping a round S^3 in S^5 or in AdS_5, with large angular momentum. The 1/2 BPS excitations of $AdS \times S$ configuration in both type IIB string theory and M-theory, which is the gravity description of 1/2 BPS states was constructed and its energy and flux was found to be in one to one correspondence with those of a general fermionic droplet configuration [26] (referred to as LLM). The 1/2 BPS states in the AdS/CFT correspondence for maximally supersymmetric theories are associated to chiral primary operators with conformal weight $\triangle = J$, where J is the particular U(1) charge in the R- symmetry group. For small excitation energies $J \ll N$, these BPS states correspond to particular gravity modes propagating in the bulk. As the excitation energy increases $J \sim N$, some of the states can be described as branes in AdS or in the internal sphere which are the giant gravitons. Also, these 1/2 BPS states preserve half of the supersymmetries (i.e. 16 of the 32).

The single matrix model is also used in the study of string theory in two dimensions as a description of gravity [29]. When considered in the double scaling limit, the single hermitian matrix model gives the two dimensional theory of quantum gravity. However, if this map of matrix models with string theory is considered for larger number of matrices, then tachyons might appear in the theory.

Two Matrix Models

An extension of the map between 1/2 BPS states and free fermions was studied [27] by considering states associated with a full two matrix problem, as referred to, in section 1.2.1. Here the matrix associated with A, $A^\dagger$ forms the large N background and is treated exactly while the other associated with B, $B^\dagger$ is treated like an impurity creating fluctuation in the large N background generated by the first matrix. The second matrix is treated in a coherent state basis, using creation and annihilation operator. The study of such a system of combined mixed traces leads to a sequence of eigenvalue equations, which are then solved for the case of oscillator potential, which results from the coupling to the curvature of $R \times S^3$. This provides a two dimensional set of eigenstates which can be thought of as an extension to the one dimensional space representing the eigenstates of free fermions. A mapping was proposed associating the eigenstates to the gravity states with either S or AdS radial dependence. This mapping between gravity and matrix model wavefunctions is found to be one to one in contrast to the holographic map, where one of the dimensions is projected out.

The generalization of this approach to include g_{YM} interactions was developed in [30]. By considering a multi local set of states appropriate to the g_{YM} interaction, a full free spectrum for two Hermitian matrices was constructed. In addition to the identification above the direct model of two scalars X_1 and X_2 was also discussed i.e. the following Hamiltonian was considered

$$\hat{H} \equiv \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{\omega^2}{2}\mathrm{Tr}(X_1^2) + \frac{1}{2}\mathrm{Tr}(P_2^2) + \frac{\omega^2}{2}\mathrm{Tr}(X_2^2) - g_{YM}^2 Tr([X_1, X_2])^2. \quad (1.2.1)$$

When the g_{YM} interactions are included, a full string tension corrected BMN type Hamiltonian is obtained [30]. Further properties of the spectrum were studied in [31], where it was shown that the full string tension corrected spectrum depends on two momenta. For a specific value of one of these momenta, the spectrum has the same structure as that of giant magnon bound states. States with arbitrary number of impurities were also considered and their first order (in $g_{YM}^2 N$) spectrum was obtained.

The framework used in [27]-[31] is based on the collective field theory [32], which provides a transition from Yang Mills to string theory description. In this method there is a direct change of variables from the matrices of the U(N) gauge theory to the fields of string theory. The observables in the collective field theory are thus given by loops or traces of matrix products, which are the physical observables describing the dynamics of the theory with large N symmetry. The resulting effective or collective Hamiltonian is built from two interaction terms describing joining and splitting of loops. The spectra and the interactions of these observables should have a gravity/ string field theory interpretation.

1.3 Outline

The approach taken in [27]-[31] is based on a supersymmetric setting, allowing one to consistently neglect normal ordering terms. As a result the planar background is harmonic and g_{YM} independent. In this thesis, a non supersymmtric background, is considered, in an approach where the two matrices are treated asymmetrically, and properties of the large N background are obtained.

The thesis is organised as follows.

Chapter 2 describes the collective field theory technique, as it applies to matrix models. The change of variables to invariant collective fields is described and the Jacobian also wanted with this change of variables is found by requiring Hermiticity. The single matrix case is then discussed in some detail.

Chapter 3 introduces, in a non supersymmetric background, the quantum mechanical Hamiltonian of two hermitian matrices, coupled via Yang - Mills interaction. Here one of the matrices (X_1), is treated exactly and the other (X_2), as impurity in the background of the first matrix. If V is the unitary matrix that diagonalizes X_1, we observe that the commutator of P_1 with $\bar{X}_2 \equiv V^{\dagger} X_2 V$ and $\bar{P}_2$ is non zero and therefore we introduce a canonical transformation for P_1, so that the transformed momenta does commute with $\bar{X}_2$ and $\bar{P}_2$.

In this thesis we are interested in the "state with no $\bar{X}_2$ impurities". Because of the dependence of such states on X_1 degree of freedom, it is seen that the canonically transformed momenta, acting on such states, is not equal to zero. Therefore there is an additional shift in the kinetic term for X_1. This is carefully taken into account in chapter 4, and the shifted kinetic energy is calculated.

In chapter 5 the planar background, in terms of the density of eigenvalues of X_1 matrix, is obtained implicitly through a non-linear integral equation. The equal time correlator for the X_1 matrix, in the weak coupling limit is obtained using the density description and the perturbation theory, to the order λ^2.

Chapter 6 deals with the strong coupling limit ($\lambda \to \infty$), in which we see that the effective potential obtained due to the additional shifts, discussed above, does not contribute. The large N, ground state energy and $< \mathrm{Tr}(X_1)^2 >$ are obtained in this limit and it is seen that these results are free of infrared divergences.

Chapter 7 is reserved for conclusions, where we state the results that we obtained in this thesis.

The Appendix A describes the double index notations of the matrices and Appendix B gives the detailed calculations for the shifted background momenta in original system of coordinates. Appendix C gives the alternate method for the calculation of $< \mathrm{Tr}(X_1)^2 >$, using perturbation theory, to $O(\lambda)$.

Chapter 2

Collective Field Theory

In this thesis we will use the collective field theory in an attempt to obtain an effective Hamiltonian in terms of the density of eigenvalues of a single hermitian matrix. It is therefore useful that this method be revised in this chapter.

Collective field theory represents a systematic formalism for describing the dynamics of invariant observables of the theory. The method consists of a direct change of variables to the invariant observables. This leads to an effective Hamiltonian, and in this new representation, the large N limit is determined by classical stationary points. The large N spectrum is, in general determined by small fluctuations about the stationary collective field. Thus, the Hamiltonian in terms of these new invariant observables describes the full dynamics of the theory.

Consider the Hamiltonian for a multi-matrix system, with hermitian matrices X_i

$$H = -\frac{1}{2}\mathrm{Tr}(\sum_{i=1}^{M}\frac{\partial}{\partial X_i}\frac{\partial}{\partial X_i}) + V(X_i) \tag{2.0.1}$$

$$\equiv -\frac{1}{2}\sum_{i=1}^{M}\sum_{mn}\frac{\partial}{(\partial X_i)_{mn}}\frac{\partial}{(\partial X_i)_{nm}}) + V(X_i) \tag{2.0.2}$$

where the potential $V(X_i)$ is invariant under the unitary transformation

$$X_i \to \mathbf{U}^{\dagger} X_i \mathbf{U}$$

The Hamiltonian is invariant under this symmetry, and one may consider equal time single trace correlators (operators) of the form

$$\mathrm{Tr}(\cdots \prod_{i=1}^{M} X_i^{n_i} \prod_{j=1}^{M} X_j^{m_j} \cdots)$$

In the large N limit, this change of variables from the original variables to invariant loop variables implies a reduction of degree of freedom. For example in the single matrix systems the collective variables corresponds to the eigenvalue basis. These variables are known to be independent in the large N limit, as evidenced in studies of single matrix models and matrix description of lower dimensional strings. For finite N there are constraints, which can be considered after the change of variables. This results in interesting effects related to the stringy exclusion principle.

Consider the change of variable to

$$X_i \to \phi_C \tag{2.0.3}$$

where C is a gauge invariant loop or word index. The kinetic term of Hamiltonian under the above change of variables becomes

$$T = -\frac{1}{2}\mathrm{Tr}(\sum_{i=1}^{M} \frac{\partial}{\partial X_i}\frac{\partial}{\partial X_i}) = -\frac{1}{2}\sum_{C,C'} \Omega(C,C')\frac{\partial}{\partial \phi_C}\frac{\partial}{\partial \phi_{C'}} + \frac{1}{2}\sum_{C} \omega(C)\frac{\partial}{\partial \phi_C} \tag{2.0.4}$$

where

$$\Omega(C,C') = \mathrm{Tr}(\sum_{i=1}^{M} \frac{\partial \phi_C}{\partial X_i}\frac{\partial \phi_{C'}}{\partial X_i})$$

and

$$\omega(C) = -\mathrm{Tr}(\sum_{i=1}^{M} \frac{\partial^2 \phi_C}{\partial X_i \partial X_i})$$

$\Omega(C,C')$ "joins" loops or words and $\omega(C)$ "splits" loops or words. For example, if $\phi_C = \mathrm{Tr}(X_1^J)$ and $\phi_{C'} = \mathrm{Tr}(X_1^{J'})$ then $\Omega = JJ'\mathrm{Tr}(X_1^{J-1}X_1^{J'-1})$. So in general, one may write schematically

$$\Omega(C,C') = \sum \phi_{C+C'}$$

where $C + C'$ is obtained by adding the two words C and C′. Similarly ω can be schematically written as

$$\omega(C) = \sum \omega_{C'}\omega_{C''}$$

which represents all processes of splitting the word C into C′ and C″.

Because of the Jacobian J associated with the change to the new variable ϕ_C, the operator $\frac{\partial}{\partial \phi_C}$, transforms as

$$\frac{\partial}{\partial \phi_C} \to J^{1/2}\frac{\partial}{\partial \phi_C}J^{-1/2} = \frac{\partial}{\partial \phi_C} - \frac{1}{2}\frac{\partial ln J}{\partial \phi_c} \tag{2.0.5}$$

where J is the Jacobian of the transformation.

Under this transformation the kinetic piece of the Hamiltonian becomes

$$\begin{aligned}\omega(C)\partial_C + \Omega(C,C')\partial_C\partial_{C'} \to\ & \omega(C)(\partial_C - \frac{1}{2}\partial_C \ln J) + \Omega(C,C')(\partial_C - \frac{1}{2}\partial_C \ln J) \\ & (\partial_{C'} - \frac{1}{2}\partial_{C'} \ln J) \\ =\ & \omega(C)\partial_C - \frac{1}{2}\omega(C)\partial_C \ln J + \partial_C(\Omega(C,C')\partial_{C'}) - \\ & (\partial_C\Omega(C,C'))\partial_{C'} - \frac{1}{2}\Omega(C,C')\partial_C\partial_{C'} \ln J - \\ & \Omega(C,C')\partial_C \ln J\partial_{C'} + \frac{1}{4}\Omega(C,C')\partial_C \ln J\partial_{C'} \ln J\end{aligned}$$

Using the hermiticity condition, terms linear in ∂_C are zero. This implies that

$$\omega(C) = \Omega(C,C')(\partial_{C'} lnJ) + \partial_{C'}\Omega(C',C) \tag{2.0.6}$$

Therefore, the explicitly hermitian collective field Hamiltonian is

$$H = \frac{1}{2}(\partial_C + \frac{1}{2}\partial_C lnJ)\Omega(C,C')(-\partial_{C'} + \frac{1}{2}\partial_{C'} lnJ) + V \tag{2.0.7}$$

Using equation (2.0.6)

$$\partial_{C'} lnJ = \Omega^{-1}(C,C')\omega(C) - \Omega^{-1}(C,C')\partial_{C'}\Omega(C,C')$$

If we substitute for $\partial_{C'} \ln J$ in equation (2.0.7), then the leading contribution will be

$$H = -\frac{1}{2}\Big(\partial_C\Omega(C,C')\partial_{C'} - \frac{1}{4}\omega(C')\Omega^{-1}(C,C')\omega(C)\Big) + V$$

$$= \frac{1}{2}\Big(\Pi(C)\Omega(C,C')\Pi(C') + \frac{1}{4}\omega(C')\Omega^{-1}(C,C')\omega(C)\Big) + V \tag{2.0.8}$$

where,

$$\Pi(C) = -i\frac{\partial}{\partial\phi(C)}$$

The full Hamiltonian in addition contains counter terms which contribute at loop level. This form is explicitly required by the hermiticity requirement and is

$$\begin{aligned} 2\Delta H = &-\frac{1}{2}\frac{\partial\omega(C)}{\partial\phi_C} + \frac{1}{2}\frac{\partial\Omega(C'',C')}{\partial\phi_{C''}}\Omega^{-1}(C',C)\omega(C) \\ &+ \frac{1}{4}\frac{\partial\Omega(C'',C)}{\partial\phi_{C''}}\Omega^{-1}(C,C')\frac{\partial\Omega(C',C''')}{\partial\phi_{C'''}} \end{aligned}$$

The formalism developed above gives a general description of the collective field theory technique. In the next section we will apply this formalism to the case of single matrix.

2.1 Single Hermitian Matrix

For a single hermitian matrix, one can choose

$$\phi_C \equiv \phi_k \equiv \mathrm{Tr}(e^{ikM})$$

Its Fourier transform is the density of eigenvalues and is called the x representation of the variable. In the x representation, the collective field variable is given as

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx}\phi_k = \sum_i \delta(x - \lambda_i)$$

where λ_i is the eigenvalue of the matrix M. In the x representation $\omega(C)$ and $\Omega(C,C')$ are given as

$$\begin{aligned} \Omega(x,y) &= \partial_x\partial_y(\phi(x)\delta(x-y)) \\ \omega(x) &= 2\partial_x(\phi(x) ⨍ dz\frac{\phi(z)}{x-z}) \end{aligned} \tag{2.1.1}$$

Therefore, the leading order term of the Hamiltonian in (2.0.8) is

$$H = \int dx \int dy - \frac{1}{2}\int dx \partial_x \frac{\partial}{\partial\phi(x)}\phi(x)\partial_x \frac{\partial}{\partial\phi(x)} + \frac{1}{2}\int dx\phi(x)(\fint dy \frac{\phi(z)}{x-y})^2$$
$$+ \int dx\phi(x)V(x) - \mu(\int dx\phi(x) - N) \tag{2.1.2}$$

where the Lagrange multiplier μ enforces the constraint $\int dx\phi(x) = N$. There is an identity

$$\int dx\phi(x)(\fint dy \frac{\phi(y)}{x-y})^2 = \frac{\pi^2}{3}\int dx\phi^3(x)$$

Also, to exhibit the large N dependence, we do the following rescaling

$$x \rightarrow \sqrt{N}x$$
$$\phi(x) \rightarrow \sqrt{N}\phi(x)$$
$$-i\frac{\partial}{\partial\psi(x.0)} \equiv \Pi(x) \rightarrow \frac{1}{N}\Pi(x)$$
$$\mu \rightarrow N\mu$$

Thus, the effective Hamiltonian in the large N limit, which is sufficient for the study of the large N background and fluctuations, is

$$H^0_{eff} = \frac{1}{2N^2}\int dx\partial_x\Pi(x)\phi(x)\partial_x\Pi(x) + N^2(\int dx\frac{\pi^2}{6}\phi^3(x) + \phi(x)(v(x) - \mu)) \tag{2.1.3}$$

The second term of the above equations is of leading order in N. This will therefore generate the background and the first term will generate the fluctuation around this background. These fluctuations can be examined by defining the background ϕ_0 as the result of extremising the second term of (2.1.3) with respect to $\phi(x)$. Thus, we obtain the background as,

$$\pi\phi(x) = \pi\phi_0(x) = \sqrt{2\mu - 2v(x)} \tag{2.1.4}$$

Chapter 3

Non-Supersymmetric Two Matrix Model

The two matrix problem that we are interested in, is described by the quantum mechanical Hamiltoninan

$$\hat{H} \equiv \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{\omega^2}{2}\mathrm{Tr}(X_1^2) + \frac{1}{2}\mathrm{Tr}(P_2^2) + \frac{\omega^2}{2}\mathrm{Tr}(X_2^2) - g_{ym}^2\mathrm{Tr}([X_1, X_2])^2. \quad (3.0.1)$$

Here X_1 and X_2 are the two $N \times N$ hermitian matrices and P_1 and P_2 are their conjugate momenta respectively, such that

$$[X_{ij}^a, P_{mn}^b] = i\delta_{in}\delta_{jm}\delta_{ab} \quad (i, j, m, n = 1 \cdots N)\ (a, b = 1, 2)$$

$$[X_{ij}^a, X_{mn}^b] = [P_{ij}^a, P_{mn}^b] = 0$$

These two matrices are coupled via the standard Yang-Mills interaction.

The Hamiltonian above can be thought of, as associated with two of the six Higgs scalars of bosonic sector of $\mathcal{N} = 4$ SYM, in the leading Kaluza-Klein compactification on $R \times S^3$. The harmonic potential results from the coupling to the curvature of the manifold. In the present study we consider the non-supersymmetric setting.

3.1 Two - Matrix Hamiltonian in Creation/Annihilation Basis

In the two matrix Hamiltonian given in (3.0.1), one of matrices, X_1, is treated in coordinate space and exactly (in the large N limit), and the other, X_2, in a creation/annihilation basis. Letting

$$X_2 \equiv \frac{1}{\sqrt{2w}}(A_2 + A_2^\dagger) \quad P_2 = -i\sqrt{\frac{w}{2}}(A_2 - A_2^\dagger) \tag{3.1.1}$$

the Hamiltonian (3.0.1) takes the form

$$\begin{aligned} \hat{H} &= \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\mathrm{Tr}(X_1^2) + w\mathrm{Tr}(A_2^\dagger A_2) + N^2\frac{w}{2} \\ &- \frac{g_{YM}^2}{2w}\mathrm{Tr}(2[X_1, A_2^\dagger][X_1, A_2] + [X_1, A_2]^2 + [X_1, A_2^\dagger]^2) \\ &+ \frac{g_{YM}^2 N}{w}\mathrm{Tr}(X_1^2) - \frac{g_{YM}^2}{w}(\mathrm{Tr}(X_1))^2 \end{aligned} \tag{3.1.2}$$

Here A_2 and $A_2^\dagger$ are the creation and annihilation operators related to the X_2 matrix such that

$$[(A_2)_{ij}, (A_2^\dagger)_{mn}] = \delta_{im}\delta_{jn} \quad (i, j, m, n = 1 \cdots N)$$

$$[(A_2)_{ij}, (A_2)_{mn}] = [(A_2^\dagger)_{ij}, (A_2^\dagger)_{mn}] = 0$$

As the interaction is quadratic in the oscillators, one can perform a Bogoliubov transformation

$$(V^\dagger A_2 V)_{ij} = \cosh(\phi_{ij})B_{ij} - \sinh(\phi_{ij})B_{ij}^\dagger \tag{3.1.3}$$

with

$$\tanh(2\phi_{ij}) = \frac{\frac{g_{YM}^2}{w}(\lambda_i - \lambda_j)^2}{w + \frac{g_{YM}^2}{w}(\lambda_i - \lambda_j)^2}, \tag{3.1.4}$$

where the λ_i's are the eigenvalues of the matrix X_1 and V is the unitary matrix that diagonalizes X_1. Then (3.1.2) takes the form

$$\hat{H} = \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\mathrm{Tr}(X_1^2) + \sum_{i,j=1}^{N} \sqrt{w^2 + 2g_{YM}^2(\lambda_i - \lambda_j)^2} \quad (B_{ij}^\dagger B_{ji} + \frac{1}{2}). \quad (3.1.5)$$

We would like to obtain an effective Hamiltonian for the X_1 coordinate (or its eigenvalues), which should be able to reproduce expectation values of large N invariant operators, depending only on X_1. In this thesis we will concentrate on the contribution to the large N ground state configuration coming from the zero point energies of the B, $B^\dagger$ oscillators, and we are therefore led to the Hamiltonian:

$$\hat{H}_0 = \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\mathrm{Tr}(X_1^2) + \frac{1}{2}\sum_{i,j=1}^{N} \sqrt{w^2 + 2g_{YM}^2(\lambda_i - \lambda_j)^2} \quad (3.1.6)$$

However, P_1 acts non trivially on the ground state with no B impurities, and therefore in (3.1.6), the Hamiltonian in the X_1 sector has to be corrected. In addition P_1 no longer commutes with B, $B^\dagger$. These issues will be addressed in the sections to follow.

3.2 Diagonalizing One of the Matrices (X_1)

Consider the interaction part of the Hamiltonian (3.0.1)

$$H_{int} = -g_{YM}^2 \mathrm{Tr}([X_1, X_2])^2$$

This can be rewritten as

$$\begin{aligned} H_{int} &= -g_{YM}^2 \mathrm{Tr}[X_1, X_2][X_1, X_2] \\ &= -2g_{YM}^2(\mathrm{Tr}(X_1^2 X_2^2) - \mathrm{Tr}(X_1 X_2 X_1 X_2)) \end{aligned}$$

If we diagonalize matrix X_1 using the unitary transformation $X_1 = V\Lambda V^\dagger$, where V is a unitary matrix, then the interaction piece of the hamiltonian (3.0.1) together with the potential term for matrix X_2 is given as

$$\mathbf{U} = \frac{1}{2}\omega^2 \mathrm{Tr} X_2^2 + 2g_{YM}^2(\mathrm{Tr}(X_1^2 X_2^2) - \mathrm{Tr}(X_1 X_2 X_1 X_2))$$

$$
\begin{aligned}
&= \frac{1}{2}\omega^2 \mathrm{Tr} X_2^2 + 2g_{YM}^2(\lambda_i^2(\bar{X}_2)_{ij}(\bar{X}_2)_{ji} - \lambda_i(\bar{X}_2)_{ij}\lambda_j(\bar{X}_2)_{ij}) \\
&= \frac{1}{2}\omega^2 \mathrm{Tr} X_2^2 + g_{YM}^2(\lambda_i - \lambda_j)^2(\bar{X}_2)_{ij}(\bar{X}_2)_{ji} \\
&= \frac{1}{2}(\omega^2 + 2g_{YM}^2(\lambda_i - \lambda_j)^2)(\bar{X}_2)_{ij}(\bar{X}_2)_{ji} \\
&= \frac{1}{2}\omega_{ij}^2(\lambda)(\bar{X}_2)_{ij}(\bar{X}_2)_{ji}
\end{aligned}
$$

Here λ_i is the eigenvalue of the matrix X_1 and $\bar{X}_2 = V^\dagger X_2 V$. Similarly we can write the momentum for $\bar{X}_2$ as $\bar{P}_2 = V^\dagger P_2 V$. So, the creation and annihilation operators B_{ij}, $B_{ij}^\dagger$ introduced in the previous section are nothing but the creation, annihilation operators associated with the scalar field $(\bar{X}_2)_{ij}$, each with frequency ω_{ij}. We find it easier to work with $(\bar{P}_2)_{ij}$ and $(\bar{X}_2)_{ij}$, instead of B, $B^\dagger$. We write the Hamiltonian sector in terms of "bared" co-ordinates as

$$
H = \frac{1}{2}P_1^A P_{1A} + \frac{1}{2}\omega^2 X_1^A X_{1A} + \frac{1}{2}\bar{P}_2^A \bar{P}_{2A} + \frac{1}{2}\omega_A^2(\lambda)\bar{X}_{2A}\bar{X}_2^A \tag{3.2.1}
$$

Here the indices denote a double index notation i.e. $A = (ij)$. The details for this double index notation are given in Appendix A.

As a result of the definition of $\bar{X}_2$, $\bar{P}_2$, we observe that

$$
[P_1, \bar{X}_2] \neq 0 \qquad\qquad [P_1, \bar{P}_2] \neq 0
$$

In the next section, we derive a canonical transformation which will result in standard commutation relation.

3.3 Canonical transformation

We have seen earlier that the commutators of P_1 with $\bar{P}_2$ and $\bar{X}_2$ are non-zero. These commutators take the form

$$
[(P_1)_A, (\bar{X}_2)^B] = -iF_A{}^{BC}(\bar{X}_2)_C \tag{3.3.1}
$$

$$
[(P_1)_A, (\bar{P}_2)^B] = -iF_A{}^{BC}(\bar{P}_2)_C \tag{3.3.2}
$$

Therefore, we perform a canonical transformation of P_1 to get its commutator with $\bar{P}_2$ and $\bar{X}_2$ equal to zero. This canonical transformation is given as

$$(\bar{P}_1)_A = (P_1)_A + F_A{}^{BC}(\bar{X}_2)_B(\bar{P}_2)_C \tag{3.3.3}$$

It can be shown that the commutator of $(\bar{P}_1)_A$ with $(\bar{P}_2)^B$ and $(\bar{X}_2)^B$ is indeed zero.

$$\begin{aligned}
[(\bar{P}_1)_A, (\bar{P}_2)^B] &= [(P_1)_A + F_A{}^{CD}(\bar{X}_2)_C(\bar{P}_2)_D, (\bar{P}_2)^B] \\
&= -iF_A{}^{BC}(\bar{P}_2)_C + F_A{}^{CD}[(\bar{X}_2)_C, (\bar{P}_2)^B](\bar{P}_2)_D \\
&= -iF_A{}^{BC}(\bar{P}_2)_C + F_A{}^{CD} i\delta_C{}^B(\bar{P}_2)_D \\
&= -iF_A{}^{BC}(\bar{P}_2)_C + iF_A{}^{BC}(\bar{P}_2)_C \\
&= 0
\end{aligned}$$

and,

$$\begin{aligned}
[(\bar{P}_1)_A, (\bar{X}_2)^B] &= [(P_1)_A + F_A{}^{CD}(\bar{X}_2)_C(\bar{P}_2)_D, (\bar{X}_2)^B] \\
&= -iF_A{}^{BC}(\bar{X}_2)_C + F_A{}^{CD}(\bar{X}_2)_C, [(\bar{P}_2)^B, (\bar{X}_2)_D] \\
&= -iF_A{}^{BC}(\bar{X}_2)_C - iF_A{}^{CD}(\bar{X}_2)_C\delta_D{}^B \\
&= -iF_A{}^{BC}(\bar{X}_2)_C - iF_A{}^{CB}(\bar{X}_2)_C \\
&= 0,
\end{aligned}$$

provided $F_A{}^{BC} = -F_A{}^{CB}$. We will show that $F_A{}^{BC}$ is indeed anti-symmetric.

Explicit Form of $F_A{}^{BC}$

Referring to (3.3.1), this can be rewritten as

$$[P_{dc}, (V^\dagger X_2 V)_{pq}] = -iF_{dc,pq,ml}(V^\dagger X_2 V)_{lm} \tag{3.3.4}$$

The left hand side of this equation is simply,

$$-i(\frac{\partial}{\partial X_1})_{cd}(V^\dagger X_2 V)_{pq}$$

This can be calculated using,

$$(\frac{\partial}{\partial X_1})_{cd} = \sum_{ab}\sum_{m\neq b}\frac{V_{am}V^{\dagger}_{mc}V_{db}}{\lambda_b-\lambda_m}\frac{\partial}{\partial V_{ab}} + \sum_b V^{\dagger}_{bc}V_{db}\frac{\partial}{\partial\lambda_b}$$

Thus the left hand side of (3.3.4) is

$$\begin{aligned}
(\frac{\partial}{\partial X_1})_{cd}(V^{\dagger}X_2V)_{pq} &= (\frac{\partial}{\partial X_1})_{cd} V^{\dagger}_{pi}(X_2)_{ij}V_{jq} \\
&= (X_2)_{ij}\frac{V_{db}V^{\dagger}_{mc}V_{am}}{\lambda_b-\lambda_m}\frac{\partial}{\partial V_{ab}}V^{\dagger}_{pi}V_{jq} \\
&= (X_2)_{ij}\frac{V_{db}V^{\dagger}_{mc}}{\lambda_b-\lambda_m}(V^{\dagger}_{pi}V_{jm}\delta_{bq} - V_{jq}\delta_{pm}V^{\dagger}_{bi}) \\
&= \frac{V_{dq}V^{\dagger}_{mc}V^{\dagger}_{pi}V_{jm}}{\lambda_q-\lambda_m} - \frac{V_{db}V^{\dagger}_{pc}V_{jq}V^{\dagger}_{bi}}{\lambda_b-\lambda_p}(X_2)_{ij} \\
&= \frac{(V^{\dagger}X_2V)_{pm}V^{\dagger}_{mc}V_{dq}}{\lambda_q-\lambda_m} - \frac{V^{\dagger}_{pc}V_{db}(V^{\dagger}X_2V)_{bq}}{\lambda_b-\lambda_p} \\
&= \frac{(V^{\dagger}X_2V)_{pb}V^{\dagger}_{bc}V_{dq}}{\lambda_q-\lambda_b} + \frac{V^{\dagger}_{pc}V_{db}(V^{\dagger}X_2V)_{bq}}{\lambda_p-\lambda_b} \\
&= \frac{V^{\dagger}_{bc}V_{dq}(\bar{X}_2)_{pb}}{\lambda_q-\lambda_b} + \frac{V^{\dagger}_{pc}V_{db}(\bar{X}_2)_{bq}}{\lambda_p-\lambda_b} \\
&= \frac{V^{\dagger}_{mc}V_{dq}(\bar{X}_2)_{pm}}{\lambda_q-\lambda_m} + \frac{V^{\dagger}_{pc}V_{dl}(\bar{X}_2)_{lq}}{\lambda_p-\lambda_l} \\
&= (\frac{V^{\dagger}_{mc}V_{dq}\delta_{lp}}{\lambda_q-\lambda_m} + \frac{V^{\dagger}_{pc}V_{dl}\delta_{mq}}{\lambda_p-\lambda_l})(\bar{X}_2)_{lm}
\end{aligned}$$

Comparing this with right hand side of (3.3.4)

$$F_{dc,pq,ml} = \frac{V^{\dagger}_{mc}V_{dq}\delta_{lp}}{\lambda_q-\lambda_m} + \frac{V^{\dagger}_{pc}V_{dl}\delta_{mq}}{\lambda_p-\lambda_l} \tag{3.3.5}$$

This is the expression for ${F_A}^{BC}$. Comparing indices, we have $A = (dc)$, $B = (qp)$ and $C = (lm)$. Thus if we want to write an expression for ${F_A}^{CB}$, we change $(lm) \leftrightarrow (qp)$ i.e. $l \leftrightarrow q$ and $m \leftrightarrow p$. Thus we get

$${F_A}^{CB} = \frac{V^{\dagger}_{pc}V_{dl}\delta_{qm}}{\lambda_l-\lambda_p} + \frac{V^{\dagger}_{mc}V_{dq}\delta_{pl}}{\lambda_m-\lambda_q} \tag{3.3.6}$$

Comparing these two equations, we see that ${F_A}^{BC} = -{F_A}^{CB}$. This proves the anti - symmetry of ${F_A}^{BC}$.

Chapter 4

Effective X_1 Hamiltonian

We proceed by concentrating on the large N configuration of the system for the ground state configuration and correlators without X_2 "impurities". The state with no X_2 "impurity" is given as

$$\Psi_0 \sim \exp(-\frac{1}{2}\omega_{ij}\bar{X}_{2ij}\bar{X}_{2ji}) \tag{4.0.1}$$

The normalization for this "no impurity" state is determined as follows

$$\begin{aligned}
\Psi_0 &= A\exp(-\frac{1}{2}\sum_{i\neq j}\omega_{ij}\bar{X}_{2ij}\bar{X}_{2ji}) \\
\int d\bar{X}_2\Psi_0^2 &= A^2\int d\bar{X}_2\exp(-\sum_{i\neq j}\omega_{ij}\bar{X}_{2ij}\bar{X}_{2ji}) = 1 \\
&\Rightarrow A^2\prod_{i\neq j}\sqrt{\frac{\pi}{\omega_{ij}}} = 1 \\
&\Rightarrow A = \prod_{i\neq j}(\frac{\omega_{ij}}{\pi})^{1/4}
\end{aligned} \tag{4.0.2}$$

The normalized no "impurity" state is thus given as

$$\begin{aligned}
\Psi_0 &= \prod_{i\neq j}(\frac{\omega_{ij}}{\pi})^{1/4}\exp(-\frac{1}{2}\sum_{i\neq j}\omega_{ij}(\bar{X}_2)_{ij}(\bar{X}_2)_{ji}) \\
&= \exp(\frac{1}{4}\sum_A \ln\omega_A - \frac{1}{2}\omega_A(\bar{X}_{2A}))(\bar{X}_2{}^A)
\end{aligned} \tag{4.0.3}$$

The 2 sector acting on this ground state just gives the zero point energy of a harmonic oscillator as is explicitly shown below

$$(\frac{1}{2}\bar{P}_2{}^A\bar{P}_{2A} + \frac{1}{2}\omega_A^2\bar{X}_{2A}\bar{X}_2{}^A)\Psi_0$$

$$= \Big(-\frac{1}{2}(\frac{\partial}{\partial \bar{X}_2})_{ab}(\frac{\partial}{\partial \bar{X}_2})_{ba} + \frac{1}{2}\omega_{ab}^2(\bar{X}_{2ab})(\bar{X}_{2ba})\Big)\exp(-\frac{1}{2}\omega_{ij}(\bar{X}_{2ij})(\bar{X}_{2ji}))) \quad (4.0.4)$$

since,

$$(\frac{\partial}{\partial \bar{X}_2})_{ab}\exp(-\frac{1}{2}\omega_{ij}(\bar{X}_{2ij})(\bar{X}_{2ji})) = -(\omega_{ab}(\bar{X}_2)_{ba})\exp(-\frac{1}{2}\omega_{ij}(\bar{X}_{2ij})(\bar{X}_{2ji}))$$

and,

$$\begin{aligned}&(\frac{\partial}{\partial \bar{X}_2})_{ba}(-\omega_{ab}(\bar{X}_2)_{ba})\exp(-\frac{1}{2}\omega_{ij}(\bar{X}_{2ij})(\bar{X}_{2ji}))\\ &= -(\omega_{ab} - \omega_{ab}(\bar{X}_2)_{ba}\omega_{ba}(\bar{X}_2)_{ab})\exp(-\frac{1}{2}\omega_{ij}(\bar{X}_{2ij})(\bar{X}_{2ji}))\end{aligned}$$

Therefore,

$$(\frac{1}{2}\bar{P}_2{}^A\bar{P}_{2A} + \frac{1}{2}\omega_A^2\bar{X}_{2A}\bar{X}_2{}^A)\Psi_0 = \frac{1}{2}\sum_{ab}\omega_{ab}\Psi_0 \quad (4.0.5)$$

and we obtain equation (3.1.6)

$$\hat{H}_0 = \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{w^2}{2}\mathrm{Tr}(X_1^2) + \frac{1}{2}\sum_{i,j=1}^{N}\sqrt{w^2 + 2g_{YM}^2(\lambda_i - \lambda_j)^2} \quad (4.0.6)$$

Considering an arbitrary wavefunction, written as

$$\psi(\lambda, \bar{X}_2) = f(\lambda)\psi_0(\lambda, \bar{X}_2)$$

we define an effective X_1 Hamiltonian to be

$$H_1^{eff} f(\lambda) \equiv \int \mathrm{d}\bar{X}_2\psi_0 * \star(\lambda, \bar{X}_2)\hat{H}\psi_0(\lambda, \bar{X}_2)f(\lambda) \quad (4.0.7)$$

Given the dependence of ψ_0 on λ, $P_1\psi_0 \neq 0$, and this has to be carefully taken into account.

It will be shown later (4.1.20), that this effective Hamiltonian only depends on the eigenvalues of X_1, resulting in considerable simplification of degrees of freedom.

4.1 Shifted kinetic term

From the canonical transformation in (3.3.3), we have

$$P_{1A} = \bar{P}_{1A} - F_A{}^{BC}\bar{X}_{2B}\bar{P}_{2C} \quad (4.1.1)$$

For simplicity of notation, we drop the bar sign from now on. The kinetic term then becomes

$$\begin{aligned}\frac{1}{2}P_1^A P_{1A}\Psi_0(\lambda, X_2) \rightarrow & \frac{1}{2}(P_1^A - F^{ABC}X_{2B}P_{2C})(P_{1A} - F_A{}^{BC}X_{2B}P_{2C})\Psi_0(\lambda, X_2) \\ = & \frac{1}{2}\Big(\underbrace{P_1^A P_{1A}}_{\text{Term 1}} - \underbrace{[P_1^A, F_A{}^{BC}X_{2B}P_{2C}]}_{\text{Term 2}} - \underbrace{2F_A{}^{BC}X_{2B}P_{2C}P_1^A}_{\text{Term 3}} + \\ & \underbrace{F^{ABC}X_{2B}P_{2C}F_A{}^{BC}X_{2B}P_{2C}}_{\text{Term 4}}\Big)\Psi_0(\lambda, X_2)\end{aligned} \tag{4.1.2}$$

In addition to the shift resulting from the canonical transformation (4.1.1), there is an additional shift that results from the fact that $P_1\Psi_0 \neq 0$.

One has,

$$\begin{aligned}P_{1A}\Psi_0(\lambda, X_2) &= P_{1A}\Big(\exp(\frac{1}{4}\sum_D \ln\omega_D)\exp(-\frac{1}{2}\sum_D \omega_D X_2{}^D X_{2D})\Big) \\ &= \Psi_0(\lambda, X_2)\left(P_{1A} - \frac{i}{4}\sum_D \partial_A \ln\omega_D + \frac{i}{2}\sum_D \partial_A\omega_D X_2{}^D X_{2D}\right) \\ &= \Psi_0(\lambda, X_2)\left(P_{1A} - \frac{i}{4}\sum_D \partial_A \ln\omega_D(1 - 2\omega_D X_2{}^D X_{2D})\right)\end{aligned}$$

Thus,

$$P_{1A}\Psi_0(\lambda, X_2) = \Psi_0(\lambda, X_2)(P_{1A} - i(\Delta Y)_A) \tag{4.1.3}$$

where, the additional shift is given by

$$(\Delta Y)_A = \frac{1}{4}\sum_D \partial_A \ln\omega_D(\delta_D^D - 2\omega_D X_2{}^D X_{2D})$$

Then Term 1 in (4.1.2) is given as

$$\begin{aligned}\frac{1}{2}P_1{}^A P_{1A}\Psi_0(\lambda, X_2) &= \frac{1}{2}\Psi_0(P_1{}^A - i(\Delta Y)^A)(P_{1A} - i(\Delta Y)_A) \\ &= \frac{1}{2}\Psi_0\left(P_1{}^A P_{1A} - i[P_1{}^A, \Delta Y_A] - i2(\Delta Y)_A P_1{}^A - (\Delta Y)^A(\Delta Y)_A\right)\end{aligned} \tag{4.1.4}$$

Taking the ground state expectation value of (4.1.4) gives,

$$\frac{1}{2}\left(P_1{}^A P_{1A} - \overbrace{2i\langle(\Delta Y)_A\rangle P_1{}^A}^{A}\right) - \frac{1}{2}\overbrace{\Big\langle(\Delta Y)^A(\Delta Y)_A\Big\rangle}^{B} - \frac{1}{2}\overbrace{i\Big\langle[P_1{}^A, \Delta Y_A]\Big\rangle}^{C} \tag{4.1.5}$$

where the expectation value of any operator O is given as

$$\left\langle \hat{O}(X_2) \right\rangle \equiv \int \mathrm{d}X_2 \Psi_0^\star(X_2, \lambda) \hat{O}(X_2) \Psi_0(X_2, \lambda)$$

In particular,

$$< X_{2A} X_2^B >= \frac{\delta_A^B}{2\omega_B} \tag{4.1.6}$$

Thus term A in (4.1.5) is given as

$$\begin{aligned}
\langle \Delta Y \rangle_A &= \frac{1}{4} \sum_D \partial_A \ln \omega_D \left\langle (\delta_D^D - 2\omega_D X_2{}^D X_{2D}) \right\rangle \\
&= \frac{1}{4} \sum_D \partial_A \ln \omega_D (\delta_D^D - 2\omega_D \frac{1}{2\omega_D} \delta_D^D) \\
&= 0
\end{aligned} \tag{4.1.7}$$

The vanishing of this term (multiplying P_{1A} linearly) and the vanishing of term 3 in (4.1.2) (as will be shown later), are required for consistency of this method.

Term B of (4.1.5) is given as

$$\begin{aligned}
&-\frac{1}{2} \left\langle (\Delta Y)^A (\Delta Y)_A \right\rangle = \\
&= -\frac{1}{2} \left\langle \left(\frac{1}{4} \partial^A \ln \omega_B (\delta_B^B - 2\omega_B X_2{}^B X_{2B}) \right) \left(\frac{1}{4} \partial_A \ln \omega_B (\delta_B^B - 2\omega_B X_2{}^B X_{2B}) \right) \right\rangle \\
&= -\left\langle \left(\frac{1}{32} \partial^A \ln \omega_B (\delta_B^B - 2\omega_B X_2{}^B X_{2B}) \partial_A \ln \omega_C (\delta_C^C - 2\omega_C X_2{}^C X_{2C}) \right) \right\rangle \\
&= -\frac{1}{32} \left\langle \left(\partial_A \ln \omega_B (\delta_B^B - 2\omega_B X_2{}^B X_{2B}) \partial^A \ln \omega_C \delta_C^C \right.\right. \\
&\qquad \left.\left. -\partial_A \ln \omega_B (\delta_B^B - 2\omega_B X_2{}^B X_{2B}) \partial^A \ln \omega_C 2\omega_C X_2{}^C X_{2C} \right) \right\rangle \\
&= -\frac{1}{32} \Big(-\partial_A \ln \omega_B \delta_B^B \partial^A \ln \omega_C 2\omega_C \frac{\delta_C^C}{2\omega_C} \\
&\qquad + 4\partial_A \ln \omega_B \partial^A \ln \omega_C \omega_B \omega_C \left\langle X_2{}^B X_{2B} X_2{}^C X_{2C} \right\rangle \Big) \\
&= -\frac{1}{32} \Big(-\partial_A \ln \omega_B \partial^A \ln \omega_C \\
&\qquad + 4\omega_B \omega_C \partial_A \ln \omega_B \partial^A \ln \omega_C \Big(\frac{\delta_B^B}{2\omega_B} \frac{\delta_C^C}{2\omega_C} + \\
&\qquad \frac{g_{BC}}{2\omega_B} \frac{g^{BC}}{2\omega_C} + \frac{\delta_B^C}{2\omega_B} \frac{\delta_C^B}{2\omega_C} \Big) \Big) \\
&= -\frac{1}{16} \partial_A \ln \omega_B \partial^A \ln \omega_B
\end{aligned} \tag{4.1.8}$$

Term C in (4.1.5) is given as

$$
\begin{aligned}
& -i\frac{1}{2}\left\langle [P_1{}^A, \Delta Y_A] \right\rangle = \\
& = -i\frac{1}{2}(-i)\partial_1^A \left(\frac{1}{4}\sum_D \partial_A \ln\omega_D (\delta_D^D - 2\omega_D X_2{}^D X_{2D}) \right) \\
& = -\frac{1}{2}\left\{ \frac{1}{4}\sum_D \partial^A\partial_A \ln\omega_D \left\langle (\delta_D^D - 2\omega_D X_2{}^D X_{2D}) \right\rangle + \right. \\
& \left. + \frac{1}{4}\sum_D \partial_A \ln\omega_D (-2\partial^A\omega_D) \left\langle X_2{}^D X_{2D} \right\rangle \right\} \\
& = \frac{1}{8}\partial_A \ln\omega_D \partial^A \ln\omega_D
\end{aligned} \tag{4.1.9}
$$

Now, substituting (4.1.7), (4.1.8) and (4.1.9), for term A, B and C respectively in (4.1.5), gives the term 1 in (4.1.2)

$$
\begin{aligned}
& -\frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B \\
& + \frac{1}{8}\partial_A \ln\omega_D \partial^A \ln\omega_D \\
& = \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B
\end{aligned} \tag{4.1.10}
$$

In order to calculate term 2 in (4.1.2), we note that

$$
P_{2C}\Psi_0 = i\omega_C X_{2C}\Psi_0 \tag{4.1.11}
$$

and hence,

$$
\begin{aligned}
& i\Psi_0 (P_1^A F_A{}^{BC} X_{2B}\omega_C X_{2C}) \\
& = i((P_1^A F_A{}^{BC})\omega_C < X_{2B}X_{2C} > + F_A{}^{BC}(P_1^A\omega_C) < X_{2B}X_{2C} >) \\
& = i((P_1^A F_A{}^{BC})\omega_C \frac{g_{BC}}{2\omega_B} + F_A{}^{BC}(P_1^A\omega_C)\frac{g_{BC}}{2\omega_B}) \\
& = 0 \quad (\text{Since } F_A{}^{BB} = 0)
\end{aligned} \tag{4.1.12}
$$

In order to calculate Term 3, we use (4.1.3) and (4.1.11) to give

$$
-(F_A{}^{BC} X_{2B} P_{2C} P_1^A)\Psi_0 = \Psi_0(-iF_A{}^{BC} X_{2B}\omega_C X_{2C}(P_1{}^A - i(\Delta Y)^A))
$$

$$= (\underbrace{-iF_A{}^{BC}\omega_C < X_{2B}X_{2C} > P_1{}^A}_{A} - \underbrace{F_A{}^{BC}\omega_C X_{2B}X_{2C}(\Delta Y)^A}_{B}) \tag{4.1.13}$$

Term A given in the above expression is calculated as follows

$$\begin{aligned} \text{Term A} &= F_A{}^{BC}\omega_C \frac{\delta_{BC}}{2\omega_C} \\ &= 0 \end{aligned} \tag{4.1.14}$$

and, Term B is given as

$$\begin{aligned} \text{Term B} &= -\frac{1}{4}F_A{}^{BC}\omega_C X_{2B}X_{2C}\partial^A \ln\omega_D(\delta^D_D - 2\omega_D X_2{}^D X_{2D}) \\ &= -\frac{1}{4}\Big(F_A{}^{BC}\omega_C\partial^A \ln\omega_D \langle X_{2B}X_{2C}\rangle + \\ &\quad 2F_A{}^{BC}\omega_C\omega_D\partial^A \ln\omega_D \left\langle X_{2B}X_{2C}X_2{}^D X_{2D}\right\rangle\Big) \\ &= -\frac{1}{2}\left(F_A{}^{BC}\omega_C\omega_D\partial^A \ln\omega_D\left(\frac{g_{BC}}{2\omega_B}\frac{\delta^D_D}{2\omega_D} + \frac{g_{CD}}{2\omega_C}\frac{\delta^D_B}{2\omega_D} + \frac{g_{BD}}{2\omega_D}\frac{\delta^D_C}{2\omega_C}\right)\right) \\ &= -\frac{1}{8}\left(\partial^A \ln\omega_D(F_A{}^{DC}g_{CD} + F_A{}^{BD}g_{BD})\right) \\ &= 0 \end{aligned} \tag{4.1.15}$$

substituting (4.1.14) and (4.1.15) in (4.1.13), we get

$$\text{Term 3} = 0 \tag{4.1.16}$$

Now we are left to calculate term 4 in (4.1.2). Starting with

$$\begin{aligned} &\frac{1}{2}F^{ABC}X_{2B}P_{2C}F_A{}^{DE}X_{2D}P_{2E}\Psi_0(\lambda, X_2) \\ &= \frac{1}{2}F^{ABC}F_A{}^{DE}X_{2B}P_{2C}(X_{2D}i\omega_E X_{2E})\Psi_0 \\ &= \frac{1}{2}\Big(iF^{ABC}F_A{}^{DE}\omega_E X_{2B}((-i\delta_{CD})X_{2E} + X_{2D}(-i\delta_{CE})) \\ &+ iF^{ABC}F_A{}^{DE}\omega_E X_{2B}X_{2D}X_{2E}(i\omega_C X_{2C})\Big)\Psi_0 \\ &= \frac{1}{2}\Big(F^{ABC}F_A{}^{CE}\omega_E X_{2B}X_{2E} + F^{ABC}F_A{}^{DC}\omega_C X_{2B}X_{2D} \\ &- F_A{}^{BC}F^{ADE}\omega_C\omega_E X_{2B}X_{2C}X_{2D}X_{2E}\Big)\Psi_0 \end{aligned}$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}\Big(F^{ABC}F_A{}^{CE}\omega_E < X_{2B}X_{2E} > +F^{ABC}F_A{}^{DC}\omega_C < X_{2B}X_{2D} > \\
&- F_A{}^{BC}F^{ADE}\omega_C\omega_E < X_{2B}X_{2C}X_{2D}X_{2E} > \Big) \\
&= \frac{1}{2}\Big(F^{ABC}F_A{}^{CE}\omega_E\frac{g_{BE}}{2\omega_B} + F^{ABC}F_A{}^{DC}\omega_C\frac{g_{BD}}{2\omega_B} \\
&- F_A{}^{BC}F^{ADE}\omega_C\omega_E\left(\frac{g_{BC}}{2\omega_B}\frac{g_{DE}}{2\omega_E} + \frac{g_{BD}}{2\omega_B}\frac{g_{CE}}{2\omega_C} + \frac{g_{BE}}{2\omega_B}\frac{g_{CD}}{2\omega_C}\right)\Big) \\
&= \frac{1}{2}\Big(\frac{1}{2}(F^{ABC}F_A{}^{CB} + F^{ABC}F_A{}^{BC}\frac{\omega_C}{\omega_B}) - F^{ABC}F_{ACB} + F^{ABC}F_{ABC}\frac{\omega_C}{\omega_B}\Big) \\
&= \frac{1}{2}\Big(-\frac{1}{2}F^{ABC}F_A{}^{BC}(1-\frac{\omega_C}{\omega_B}) + \frac{1}{4}F^{ABC}F_{ABC}\left(1-\frac{\omega_C}{\omega_B}\right)\Big) \\
&= -\frac{1}{8}F^{ABC}F_{ABC}\left(1-\frac{\omega_C}{\omega_B}\right)
\end{aligned}
\tag{4.1.17}
$$

Using (4.1.10), (4.1.12), (4.1.16) and (4.1.17) for term 1, term 2, term 3 and term 4 in (4.1.2) gives the shifted kinetic term as

$$
\frac{1}{2}P_1^A P_{1A} \to \frac{1}{2}P_1^A P_{1A} - \frac{1}{8}\sum_{ABC} F_A{}^{BC}F^A{}_{BC}\left(1-\frac{\omega_C}{\omega_B}\right) + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B \tag{4.1.18}
$$

Using the expression of $F_A{}^{BC}$, we can simplify the above expression, with $A=(cd), B=(pq)$ and $C=(lm)$, $F_A{}^{BC}$ is given as

$$
F_A{}^{BC} = \frac{V_{cp}V^\dagger_{md}}{\lambda_p-\lambda_m}(1-\delta_{pm})\delta_{ql} + \frac{V_{cl}V^\dagger_{qd}}{\lambda_q-\lambda_l}(1-\delta_{ql})\delta_{pm}
$$

Replacing, $c \leftrightarrow d, p \leftrightarrow q, l \leftrightarrow m$, we can get expression for $F^A{}_{BC}$, which is given as

$$
F^A{}_{BC} = \frac{V_{dq}V^\dagger_{lc}}{\lambda_q-\lambda_l}(1-\delta_{ql})\delta_{pm} + \frac{V_{dm}V^\dagger_{pc}}{\lambda_p-\lambda_m}(1-\delta_{pm})\delta_{ql}
$$

Using these two expressions, we find

$$
F_A{}^{BC}F^A{}_{BC} = \frac{(1-\delta_{pm})\delta_{ql}}{(\lambda_p-\lambda_m)^2} + \frac{(1-\delta_{ql})\delta_{pm}}{(\lambda_q-\lambda_l)^2}
$$

Therefore the first term in (4.1.18), can be rewritten as

$$
\begin{aligned}
&-\frac{1}{8}\sum_{ABC} F_A{}^{BC}F^A{}_{BC}\left(1-\frac{\omega_C}{\omega_B}\right) = \\
&-\frac{1}{8}\sum_{pq,lm}\left(\frac{(1-\delta_{pm})\delta_{ql}}{(\lambda_p-\lambda_m)^2} + \frac{(1-\delta_{ql})\delta_{pm}}{(\lambda_q-\lambda_l)^2}\right)\left(1-\frac{\omega_{lm}}{\omega_{pq}}\right) \\
&= -\frac{1}{8}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p-\lambda_m)^2}\left(1-\frac{\omega_{lm}}{\omega_{lp}}\right)
\end{aligned}
$$

$$+\sum_{p,l,q}\frac{(1-\delta_{ql})}{(\lambda_q-\lambda_l)^2}\left(1-\frac{\omega_{lp}}{\omega_{pq}}\right)$$
$$=-\frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p-\lambda_m)^2}\left(1-\frac{\omega_{lm}}{\omega_{lp}}\right)$$

Thus, from (4.1.18), finally we obtain,

$$\frac{1}{2}P_1{}^AP_{1A}\to\frac{1}{2}P_1^AP_{1A}-\frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p-\lambda_m)^2}\left(1-\frac{\omega_{lm}}{\omega_{lp}}\right)+\frac{1}{16}\partial_A\ln\omega_B\partial^A\ln\omega_B \tag{4.1.19}$$

Therefore, we arrive at the expression for H_1^{eff}

$$H_1^{eff}=\frac{1}{2}P_1^AP_{1A}-\frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p-\lambda_m)^2}\left(1-\frac{\omega_{lm}}{\omega_{lp}}\right)+\frac{1}{16}\partial_A\ln\omega_B\partial^A\ln\omega_B+\frac{1}{2}\sum_A\omega_A \tag{4.1.20}$$

It is important to note that the effective potential depends only on the eigenvalues of X_1, and therefore one can use the collective field theory to obtain an effective hamiltonian in terms of the density of these eigenvalues.

This will be done in the next chapter, but in the next section we provide an additional check of our result (4.1.19).

4.2 Shifted Background Momenta in Original System of Coordinates

We show in this section that working with $\bar{X}_2$ is the same as working with the original X_2 coordinate. This is done by rewriting the X_2 frequency that we got in terms of $\bar{X}_2$ again in terms of X_2, as follows:

$$\begin{aligned}\omega_{ij}(\bar{X}_2)_{ij}(\bar{X}_2)_{ji}&=\omega_{ij}(V^\dagger X_2V)_{ij}(V^\dagger X_2V)_{ji}\\&=\omega_{ij}V^\dagger_{ia}(X_2)_{ab}V_{bj}V^\dagger_{jc}(X_2)_{cd}V_{di}\\&=(X_2)_{ab}[V^\dagger_{ia}V_{bj}\omega_{ij}V_{di}V^\dagger_{jc}](X_2)_{cd}\\&=(X_2)_{ab}M_{ab,cd}(X_2)_{cd}\\&=(X_2)^{ba}M_{ab,cd}(X_2)_{cd}\end{aligned}$$

which in the double index notation is

$$X_2^A \omega_A{}^B X_{2B} \qquad , \qquad \omega_A{}^B = M_{ab,cd} \equiv V^\dagger_{ia} V_{bj} \omega_{ij} V_{di} V^\dagger_{jc}$$

where $A = (ab),\ B = (cd)$.

Thus the Hamiltonian in the original system is

$$H = \frac{1}{2}P_1^2 + \frac{1}{2}\omega^2 X_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}(X_2)_{ab} M_{ab,cd} (X_2)_{cd}$$

It has been shown earlier that X_2 sector acting on the ground state with no X_2 impurities is just the zero point energy of simple harmonic oscillator i.e. $\frac{1}{2}M_{ab,cd}$. Thus the hamiltonian acting on the ground state with no X_2 impurity is

$$H_0 = \frac{1}{2}P_1^2 + \frac{1}{2}\omega^2 X_1^2 + \sum_{ab,cd} \frac{1}{2} M_{ab,cd}$$

As before, since this ground state wavefunction has frequency depending in the eigenvalue of X_1, P_1 acting on this wavefunction is not zero. i.e.

$$P_1 \Psi_0(X_1, X_2) \neq 0$$

So the more precise statement of the Hamiltonian in the X_1 sector is

$$H_1^{eff} \equiv \int \mathrm{d}X_2 \Psi_0^*(X_1, X_2) \hat{H} \Psi_0(X_1, X_2) \tag{4.2.1}$$

i.e. We want to integrate over the X_2 degree of freedom. The ground state $\Psi_0(X_1, X_2)$ is of the form

$$\Psi_0(X_1, X_2) = \prod_C \omega_C^{1/4} \exp(-\frac{1}{2} X_2^C \omega_C{}^D X_{2D})$$

Then,

$$\begin{aligned} P_{1A}\Psi_0(X_1, X_2) &= P_{1A}\left(\prod_C \omega_C^{1/4} \exp\left(-\frac{1}{2} X_2^C \omega_C{}^D X_{2D}\right)\right) \\ &= \left(P_{1A} \prod_C \omega_C^{1/4}\right) \exp\left(-\frac{1}{2} X_2^C \omega_C{}^D X_{2D}\right) \\ &\quad + \prod_C \omega_C^{1/4} \left(P_{1A} \exp\left(-\frac{1}{2} X_2^C \omega_C{}^D X_{2D}\right)\right) \end{aligned}$$

$$+\left(\prod_C \omega_C^{1/4} \exp\left(-\frac{1}{2}X_2^C \omega_C{}^D X_{2D}\right)\right) P_{1A}$$

$$= \left(-i\frac{\partial}{\partial X_1^A}\left(\prod_C \omega_C^{1/4}\right)\right)\exp\left(-\frac{1}{2}X_2^C \omega_C{}^D X_{2D}\right)$$

$$+ \prod_C \omega_C^{1/4}\left(-i\frac{\partial}{\partial X_1^A}\exp\left(-\frac{1}{2}X_2^C \omega_C{}^D X_{2D}\right)\right)$$

$$+\left(\prod_C \omega_C^{1/4} \exp\left(-\frac{1}{2}X_2^C \omega_C{}^D X_{2D}\right)\right) P_{1A}$$

$$= \Psi_0(X_1, X_2)\frac{1}{4\omega_C}[P_{1A}, \omega_c]$$

$$+ \Psi_0(X_1, X_2)\left(-\frac{1}{2}X_2^C[P_{1A}, \omega_C{}^D]X_{2D}\right) + \Psi_0(X_1, X_2)P_{1A}$$

$$= \Psi_0(X_1, X_2)\Big(P_{1A} - \frac{1}{2}\Big(X_2^C[P_{1A}, \omega_C{}^D]X_{2D}$$

$$-\frac{1}{2\omega_C}[P_{1A}, \omega_C]\Big)\Big)$$

and,

$$P_1^A P_{1A}\Psi_0(X_1, X_2) = \Psi_0(X_1, X_2)$$

$$\left(P_1^A - \frac{1}{2}\left\{X_2^C[P_1^A, \omega_C{}^D]X_{2D} - \frac{1}{2\omega_C}[P_1^A, \omega_C]\right\}\right)$$

$$\left(P_{1A} - \frac{1}{2}\left\{X_2^C[P_{1A}, \omega_C{}^D]X_{2D} - \frac{1}{2\omega_C}[P_{1A}, \omega_C]\right\}\right)$$

$$= \Psi_0(X_1, X_2)$$

$$\Big[P_1^A P_{1A} - \frac{1}{2}\Big\{X_2^C\left[P_1^A, [P_{1A}, \omega_C{}^D]\right]X_{2D}$$

$$-\frac{1}{2}(P_1^A\frac{1}{\omega_C})[P_{1A}, \omega_C] - \frac{1}{2\omega_C}\left[P_1^A, [P_{1A}, \omega_C]\right]\Big\}$$

$$-\frac{1}{2}\left\{X_2^C[P_{1A}, \omega_C{}^D]X_{2D} - \frac{1}{2\omega_C}[P_{1A}, \omega_C]\right\}P_1^A$$

$$-\frac{1}{2}\left\{X_2^C[P_1^A, \omega_C{}^D]X_{2D} - \frac{1}{2\omega_C}[P_1^A, \omega_C]\right\}P_{1A}$$

$$+\frac{1}{4}\left\{X_2^C[P_1^A, \omega_C{}^D]X_{2D} - \frac{1}{2\omega_C}[P_1^A, \omega_C]\right\}$$

$$\left\{X_2^E[P_{1A}, \omega_E{}^F]X_{2F} - \frac{1}{2\omega_E}[P_{1A}, \omega_E]\right\}\Big]$$

$$= \Psi_0(X_1, X_2)$$

$$\Big[P_1^A P_{1A} - \frac{1}{2}\Big\{X_2^C\left[P_1^A, [P_{1A}, \omega_C{}^D]\right]X_{2D}$$

$$-\frac{1}{2}(P_1^A\frac{1}{\omega_C})[P_{1A},\omega_C]-\frac{1}{2\omega_C}\left[P_1^A,[P_{1A},\omega_C]\right]\Big\}$$
$$-\left\{X_2^C[P_1^A,\omega_C{}^D]X_{2D}-\frac{1}{2\omega_C}[P_1^A,\omega_C]\right\}P_{1A}$$
$$+\frac{1}{4}\left\{X_2^C[P_1^A,\omega_C{}^D]X_{2D}-\frac{1}{2\omega_C}[P_1^A,\omega_C]\right\}$$
$$\left\{X_2^E[P_{1A},\omega_E{}^F]X_{2F}-\frac{1}{2\omega_E}[P_{1A},\omega_E]\right\}\Big]$$

This gives the kinetic piece of the corrected Hamiltonian. In order to get the kinetic piece of (4.2.1) one is left to perform the gaussian integral

$$\left\langle X_{2A}X_2^B\right\rangle_2$$

One obtains,

$$\left\langle X_{2A}X_2^B\right\rangle_2=\frac{1}{2}(\omega^{-1})_A{}^B \tag{4.2.2}$$

Thus, using the above relation the kinetic piece of (4.2.1) can be calculated. In particular, it can be shown that the terms linear in P_{1A} vanishes. In Appendix B we show that,

$$\left\langle\left\{X_2^C[P_1^A,\omega_C{}^D]X_{2D}-\frac{1}{2\omega_C}[P_1^A,\omega_C]\right\}\right\rangle_2=0 \tag{4.2.3}$$

In order to calculate,

$$-\frac{1}{2}\left\langle\left\{X_2^C\left[P_1^A,[P_{1A},\omega_C{}^D]\right]X_{2D}-\frac{1}{2}(P_1^A\frac{1}{\omega_C})[P_{1A},\omega_C]-\frac{1}{2\omega_C}\left[P_1^A,[P_{1A},\omega_C]\right]\right\}\right\rangle_2 \tag{4.2.4}$$

One first requires knowledge of,

$$\left[P_1^A,[P_{1A},\omega_C{}^D]\right]\left\langle X_2^C X_{2D}\right\rangle$$

This equals,

$$-\left(\frac{2}{\omega_{ik}}\frac{\omega_{il}-\omega_{ik}}{(\lambda_k-\lambda_l)^2}+\frac{1}{\omega_{ij}}\frac{1}{\lambda_k-\lambda_p}\frac{\partial\omega_{ij}}{\partial\lambda_k}+\frac{1}{2\omega_{ij}}\frac{\partial^2\omega_{ij}}{\partial{\lambda_k}^2}\right)$$

Also,

$$\frac{1}{2}(P_1^A\frac{1}{\omega_C})[P_{1A},\omega_C]=\frac{1}{2}(-i\frac{\partial}{\partial X_{1A}}\frac{1}{\omega_C})(-i\frac{\partial}{\partial X_{1A}}\omega_C)$$

$$= \frac{1}{2}\sum_p \sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial \omega_{ij}}{\partial \lambda_p}\right)^2$$

and,

$$\left\langle\left\{-\frac{1}{2\omega_C}\left[P_1^A,[P_{1A},\omega_C]\right]\right\}\right\rangle_2 = \frac{1}{\omega_{ij}}\frac{1}{\lambda_k-\lambda_p}\frac{\partial\omega_{ij}}{\partial\lambda_k}+\frac{1}{2\omega_{ij}}\frac{\partial^2\omega_{ij}}{\partial{\lambda_k}^2}$$

Summing these two terms shows that (4.2.4) equals

$$\begin{aligned}
-\frac{1}{2}&\left\langle\left\{X_2^C\left[P_1^A,[P_{1A},{\omega_C}^D]\right]X_{2D}-\frac{1}{2}(P_1^A\frac{1}{\omega_C})[P_{1A},\omega_C]-\frac{1}{2\omega_C}\left[P_1^A,[P_{1A},\omega_C]\right]\right\}\right\rangle_2 \\
&= -\sum_i\sum_{k\neq l}\frac{1}{(\lambda_k-\lambda_l)^2}(1-\frac{\omega_{il}}{\omega_{ik}})+\frac{1}{4}\sum_p\sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\right)^2
\end{aligned} \tag{4.2.5}$$

In addition,

$$\frac{1}{4}\left\{X_2^C[P_1^A,{\omega_C}^D]X_{2D}-\frac{1}{2\omega_C}[P_1^A,\omega_C]\right\}\left\{X_2^E[P_{1A},{\omega_E}^F]X_{2F}-\frac{1}{2\omega_E}[P_{1A},\omega_E]\right\} \tag{4.2.6}$$

$$= \frac{1}{2}\sum_i\sum_{k\neq l}\frac{1}{(\lambda_k-\lambda_l)^2}(1-\frac{\omega_{il}}{\omega_{ik}})-\frac{1}{8}\sum_p\sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\right)^2$$

Using the results of (4.2.5) and (4.2.6), gives the contribution of the corrected kinetic term in (4.2.1) i.e.

$$\begin{aligned}
\frac{1}{2}P_1^AP_{1A} &\to \frac{1}{2}P_1^AP_{1A}+\frac{1}{4}\sum_i\sum_{k\neq l}\frac{1}{\omega_{ik}}\frac{\omega_{il}-\omega_{ik}}{(\lambda_k-\lambda_l)^2}+\frac{1}{16}\sum_p\sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\right)^2 \\
&= \frac{1}{2}P_1^AP_{1A}-\frac{1}{4}\sum_i\sum_{k\neq l}\frac{1}{(\lambda_k-\lambda_l)^2}\left(1-\frac{\omega_{il}}{\omega_{ik}}\right)+\frac{1}{16}\sum_p\sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\right)^2
\end{aligned} \tag{4.2.7}$$

in precise agreement with the result obtained in the previous section.

Chapter 5

Density Description and Weak Coupling Expansion

In this chapter, we will use the collective field theory approach that was discussed in the earlier chapter, to write the effective Hamiltonian in terms of the density of eigenvalues of the background matrix X_1, and hence develop the large N background configuration which is expanded perturbatively to order λ^2.

Considering the effective Hamiltonian, given in the previous chapter (4.1.20), which is rewritten as

$$\begin{aligned} \hat{H}_1^{eff} &= \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{\omega^2}{2}\mathrm{Tr}(X_1^2) + \frac{1}{2}\sum_{i,j=1}^{N} \sqrt{\omega^2 + 2g_{YM}^2(\lambda_i - \lambda_j)^2} \\ &- \frac{1}{4}\sum_{p,l,m} \frac{(1-\delta_{pm})}{(\lambda_p - \lambda_m)^2}\left(1 - \frac{\omega_{lm}}{\omega_{lp}}\right) + \frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B \end{aligned} \tag{5.0.1}$$

In this equation the last two terms represent the change coming from shift in the kinetic term of X_1. Out of these the second term, which is

$$\frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B$$

can be rewritten as,

$$\frac{1}{16}\partial_A \ln\omega_B \partial^A \ln\omega_B = \frac{1}{16}\sum_{ij,cb} \frac{\partial}{\partial X_{1ij}}(\ln\omega_{bc})\frac{\partial}{\partial X_{1ji}}(\ln\omega_{bc})$$

$$
\begin{aligned}
&= \frac{1}{16}\sum_{ij,cb}\Big(\frac{1}{\omega_{bc}}\frac{\partial}{\partial X_{1ij}}\sqrt{\omega^2+2g_{ym}^2(\lambda_b-\lambda_c)^2} \\
&\qquad \frac{1}{\omega_{bc}}\frac{\partial}{\partial X_{1ji}}\sqrt{\omega^2+2g_{ym}^2(\lambda_b-\lambda_c)^2}\Big) \\
&= \frac{1}{16}\sum_{ij,cb}\Big(\frac{1}{\omega_{bc}^2}\sum_k V_{jk}\frac{\partial}{\partial\lambda_k}V^\dagger_{ki}\sqrt{\omega^2+2g_{ym}^2(\lambda_b-\lambda_c)^2} \\
&\qquad \sum_p V_{ip}\frac{\partial}{\partial\lambda_p}V^\dagger_{pj}\sqrt{\omega^2+2g_{ym}^2(\lambda_b-\lambda_c)^2}\Big) \\
&= \frac{1}{16}\sum_{cb}\sum_k\Big(\frac{1}{\omega_{bc}^2}\frac{\partial}{\partial\lambda_k}\sqrt{\omega^2+2g_{ym}^2(\lambda_b-\lambda_c)^2} \\
&\qquad \frac{\partial}{\partial\lambda_k}\sqrt{\omega^2+2g_{ym}^2(\lambda_b-\lambda_c)^2}\Big) \\
&= \frac{1}{8}\sum_{bc}\sum_k\frac{1}{\omega_{bc}^4}(4g_{ym}^2)^2(\lambda_b-\lambda_c)^2\delta_{bk} \\
&= \frac{1}{8}\sum_{bc}\frac{1}{\omega_{bc}^4}(4g_{ym}^2)^2(\lambda_b-\lambda_c)^2
\end{aligned}
$$

Replacing this in (5.0.1), we can rewrite the effective Hamiltonian as

$$
\begin{aligned}
\hat{H}_1^{eff} &= \frac{1}{2}\mathrm{Tr}(P_1^2)+\frac{\omega^2}{2}\mathrm{Tr}(X_1^2)+\frac{1}{2}\sum_{i,j=1}^{N}\sqrt{\omega^2+2g_{YM}^2(\lambda_i-\lambda_j)^2} \\
&-\frac{1}{4}\sum_{p,l,m}\frac{(1-\delta_{pm})}{(\lambda_p-\lambda_m)^2}\left(1-\frac{\omega_{lm}}{\omega_{lp}}\right)+\frac{1}{8}\sum_{bc}\frac{1}{\omega_{bc}^4}(4g_{ym}^2)^2(\lambda_b-\lambda_c)^2
\end{aligned}
\tag{5.0.2}
$$

This equation describes the dynamics of a single hermitian matrix, and the large N background can be described in terms of the density of eigenvalues,

$$
\phi(x)=\sum_i\delta(x-\lambda_i),
$$

as the minimum of the cubic field effective potential

$$
\begin{aligned}
V_{eff} =& \frac{\pi^2}{6}\int dx\phi^3(x)+\frac{\omega^2}{2}\int dx\phi(x)x^2-\mu(\int dx\phi(x)-N) \\
&+\frac{1}{2}\int dx\int dy\sqrt{\omega^2+2g_{YM}^2(x-y)^2}\quad\phi(x)\phi(y) \\
&-\frac{1}{4}\int \mathrm{d}x\int \mathrm{d}y\int \mathrm{d}z\phi(x)\phi(y)\phi(z)\frac{1}{(x-y)^2}\Big(1-\frac{\omega(z,y)}{\omega(z,x)}\Big) \\
&+\frac{1}{8}\int \mathrm{d}x\int \mathrm{d}y\phi(x)\phi(y)\frac{1}{\omega_{xy}^4}(4g_{ym}^2)^2(x-y)^2
\end{aligned}
\tag{5.0.3}
$$

where the Lagrange multiplier μ enforces the constraint $\int dx\phi(x) = N$. To exhibit explicitly the N dependence, we rescale

$$x \to \sqrt{N}x \quad \phi(x) \to \sqrt{N}\phi(x) \quad \mu \to N\mu \tag{5.0.4}$$

This results in

$$\mathrm{Tr}1 = \int dx\phi(x) = N \to \int dx\phi(x) = 1$$

Under the above rescaling, we see that the last term in (5.0.3), is of order N, and therefore is sub leading. Thus we obtain

$$\begin{aligned} V_{eff} &= N^2\Big[\frac{\pi^2}{6}\int dx\phi^3(x) + \frac{\omega^2}{2}\int dx\phi(x)x^2 - \mu(\int dx\phi(x) - 1) \\ &+ \frac{1}{2}\int dx\int dy\sqrt{\omega^2 + 2\lambda(x-y)^2}\phi(x)\phi(y) \\ &- \frac{1}{4}\int dx\int dy\int dz\phi(x)\phi(y)\phi(z)\frac{1}{(x-y)^2}\Big(1 - \frac{\omega(z,y)}{\omega(z,x)}\Big)\Big] \end{aligned} \tag{5.0.5}$$

where $\lambda = g_{YM}^2 N$ is the usual 't Hooft's coupling.
Using the perturbative expansion,

$$\frac{1}{(x-y)^2}\Big(1 - \frac{\omega(z,y)}{\omega(z,x)}\Big) = -\frac{\lambda^2}{2\omega^4}(4z^2 - 4z(x+y) + (x+y)^2) + O(\lambda^3).$$

Therefore, the last term in (5.0.5) can be rewritten as

$$\frac{\lambda^2}{4\omega^4}\Big(3\int dx x^2\phi(x)\Big(\int dx\phi(x)\Big)^2 - 2\int dx\phi(x)\Big(\int dx\phi(x)x\Big)^2\Big)$$

As $N \to \infty$, the large N background configuration minimizes (5.0.5) and it satisfies:

$$\begin{aligned} \pi^2\phi_0^2(x) =& 2\mu - \omega^2x^2 - 2\int dy\sqrt{\omega^2 + 2\lambda(x-y)^2}\phi_0(y) \\ &- \frac{3\lambda^2}{4\omega^4}x^2 - \frac{3\lambda^2}{\omega^4}\int dy y^2\phi_0(y) \end{aligned} \tag{5.0.6}$$

When $\lambda = 0$, (5.0.6) reduces to the well known Wigner distribution:

$$\pi\phi_0(x) = \sqrt{2\mu - 2\omega - \omega^2x^2} = \sqrt{2\omega - \omega^2x^2}, \quad |x| \le x_0 = \sqrt{\frac{2}{\omega}} \tag{5.0.7}$$

with the identification $\mu = 2\omega$ being enforced by the constraint

$$\int_{x_-}^{x_+} \mathrm{d}x\phi_0(x) = 1.$$

Expanding $\sqrt{\omega^2 + 2\lambda(x-y)^2}$, perturbatively in terms of λ, gives

$$\begin{aligned}\sqrt{\omega^2 + 2\lambda(x-y)^2} &= \omega\left(1 + \frac{2\lambda(x-y)^2}{\omega^2}\right)^{1/2} \\ &= \omega + \frac{\lambda}{\omega}(x-y)^2 - \frac{\lambda^2}{2\omega^3}(x-y)^4 + \cdots \\ &= \omega + \frac{\lambda}{\omega}(x^2 + y^2 - 2xy) \\ &- \frac{\lambda^2}{2\omega^3}(x^4 + y^4 - 4x^3y - 4xy^3 + 6x^2y^2) + \cdots\end{aligned}$$

Assuming that the background remains even ($\int \mathrm{d}xx\phi_0 = 0$), (5.0.6), can be rewritten to order λ^2 as

$$\begin{aligned}\pi^2\phi_0^2(x) =& 2\mu - \omega^2x^2 - 2\omega - \frac{2\lambda}{\omega}x^2 - \frac{2\lambda}{\omega}\int \mathrm{d}yy^2\phi_0(y) \\ &+ \frac{\lambda^2}{\omega^3}\left(x^4 + 6x^2\int \mathrm{d}yy^2\phi_0(y) + \int \mathrm{d}yy^4\phi_0(y)\right) \\ &- \frac{3\lambda^2}{2\omega^4}x^2 - \frac{3\lambda^2}{\omega^4}\int \mathrm{d}yy^2\phi_0(y) \qquad (5.0.8) \\ \Rightarrow \phi_0 =& \frac{1}{\pi}\sqrt{\beta - \alpha^2x^2 + \lambda^2\gamma x^4}\end{aligned}$$

where,

$$\begin{aligned}\beta =& 2\mu - 2\omega - \frac{2\lambda}{\omega}\int \mathrm{d}yy^2\phi_0(y) + \frac{\lambda^2}{\omega^3}\int \mathrm{d}yy^4\phi_0(y) - \frac{3\lambda^2}{\omega^4}\int \mathrm{d}yy^2\phi_0(y) \\ \alpha^2 =& \omega^2 + \frac{2\lambda}{\omega} - \frac{6\lambda^2}{\omega^3}\int \mathrm{d}yy^2\phi_0(y) + \frac{3\lambda^2}{2\omega^4} \\ \gamma =& \frac{1}{\omega^3}\end{aligned}$$

To order λ, ϕ_0 has the form

$$\phi_0 = \frac{1}{\pi}\sqrt{\beta - \alpha^2x^2}$$

with β and α now taken only up to order λ. In the above form the background distribution still remains of the Wigner type with suitable adjustments. We have $\beta = 2\alpha$, where $\alpha^2 = \omega^2 + \frac{2\lambda}{\omega}$. Thus,

$$\phi_0 = \frac{1}{\pi}\sqrt{2\alpha - \alpha^2x^2} \qquad (5.0.9)$$

with turning point $X_0 = \sqrt{\frac{2}{\alpha}}$, understood to be expanded to order λ.

To order λ^2, ϕ_0 has the form given in (5.0.8). We now have $\beta = 2\alpha + \lambda^2\Delta\beta$, with the shifted turning point given as $\bar{x} = x_0 + \Delta x$. However we see that their is no need to introduce this shift, since, because of the normalization condition, we have

$$\int \mathrm{d}x\phi_0(x) = 1$$

$$\Rightarrow 1 = \frac{2}{\pi}\int_0^{\bar{x}} \sqrt{\beta - \alpha^2x^2 + \lambda^2\gamma x^4}$$

$$= \frac{2}{\pi}\int_0^{x_0} \sqrt{\beta - \alpha^2x^2 + \lambda^2\gamma x^4} + \frac{2}{\pi}\underbrace{(\Delta x)}_{O(\lambda^2)}\underbrace{\left(\sqrt{\beta - \alpha^2x^2 + \lambda^2\gamma x^4}\right)|_{x=x_0}}_{O(1)\text{ term }=0\ at\ x\ =\ x_0}$$

$$= \frac{2}{\pi}\int_0^{x_0} \sqrt{\beta - \alpha^2x^2 + \lambda^2\gamma x^4}$$

Using the above expression we can calculate $\Delta\beta$

$$1 = \frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} \mathrm{d}x \sqrt{2\alpha + \lambda^2\Delta\beta - \alpha^2x^2 + \lambda^2\gamma x^4}$$

$$= \frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} \mathrm{d}x \sqrt{2\alpha - \alpha^2x^2} \left(1 + \frac{\lambda^2(\Delta\beta + \gamma x^4)}{2\alpha - \alpha^2x^2}\right)^{1/2}$$

$$= \frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} \mathrm{d}x \sqrt{2\alpha - \alpha^2x^2} \left(1 + \frac{\lambda^2}{2}\frac{(\Delta\beta + \gamma x^4)}{2\alpha - \alpha^2x^2}\right)$$

$$= \underbrace{\frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} \mathrm{d}x \sqrt{2\alpha - \alpha^2x^2}}_{=1} + \frac{\lambda^2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} \mathrm{d}x \frac{(\Delta\beta + \gamma x^4)}{\sqrt{2\alpha - \alpha^2x^2}}$$

$$\Rightarrow \int_0^{\sqrt{\frac{2}{\alpha}}} \mathrm{d}x \frac{(\Delta\beta + \gamma x^4)}{\sqrt{2\alpha - \alpha^2x^2}} = 0$$

setting, $x = \sqrt{\frac{2}{\alpha}}\sin\theta$ and $dx = \sqrt{\frac{2}{\alpha}}\cos\theta$, we have,

$$\begin{aligned}
&\int_0^{\pi/2} \Delta\beta + \frac{4\gamma}{\alpha^2}\sin^4\theta d\theta = 0 \\
&\Rightarrow \Delta\beta\frac{\pi}{2} + \frac{4\gamma}{\alpha^2}\int_0^{\pi/2}\sin^4\theta d\theta = 0 \\
&\Delta\beta\frac{\pi}{2} + \frac{\gamma}{\alpha^2}\frac{3\pi}{4} = 0 \\
&\Rightarrow \Delta\beta = \frac{-3\gamma}{2\alpha^2}
\end{aligned} \tag{5.0.10}$$

5.1 Calculation of $\left\langle \mathrm{Tr}(X_1^2)\right\rangle$ Using Density Description

We have,

$$\begin{aligned}
\left\langle \mathrm{Tr}X_1^2\right\rangle &= \int dx x^2\phi_0(x) \\
&= \frac{2}{\pi}\int_0^{\sqrt{2/\alpha}} x^2\sqrt{2\alpha - \alpha^2x^2 + \lambda^2\Delta\beta + \lambda^2\gamma x^4}dx \\
&= \frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} dx x^2\sqrt{2\alpha-\alpha^2x^2}\left(1 + \frac{\lambda^2(\Delta\beta+\gamma x^4)}{2\alpha-\alpha^2x^2}\right)^{1/2} \\
&= \frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} dx x^2\sqrt{2\alpha-\alpha^2x^2}\left(1 + \frac{\lambda^2}{2}\frac{(\Delta\beta+\gamma x^4)}{2\alpha-\alpha^2x^2}\right) \\
&= \underbrace{\frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} dx x^2\sqrt{2\alpha-\alpha^2x^2}}_{\text{Term 1}} + \underbrace{\frac{\lambda^2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} dx\frac{\Delta\beta x^2}{\sqrt{2\alpha-\alpha^2x^2}}}_{\text{Term 2}} + \\
&\quad \underbrace{\frac{\lambda^2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} dx\frac{\gamma x^6}{\sqrt{2\alpha-\alpha^2x^2}}}_{\text{Term 3}}
\end{aligned} \tag{5.1.1}$$

$$\text{Term 1} = \frac{2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}} dx x^2\sqrt{2\alpha-\alpha^2x^2}$$

$$\text{setting } x = \sqrt{\frac{2}{\alpha}}\sin\theta,\ dx = \sqrt{\frac{2}{\alpha}}\cos\theta d\theta$$

$$\text{Term 1} = \frac{2}{\pi}\int_0^{\pi/2}(\frac{2}{\alpha})\sin^2\theta\sqrt{2\alpha}\cos\theta d\theta$$

$$= \frac{2}{\pi\alpha}\int_0^{\pi/2}(1-\cos^2\theta)\mathrm{d}\theta$$
$$= \frac{1}{2\alpha}$$

$$\text{Term 2} = \frac{\Delta\beta\lambda^2}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}}\mathrm{d}x\frac{x^2}{\sqrt{2\alpha-\alpha^2x^2}}$$
$$\text{setting } x = \sqrt{\frac{2}{\alpha}}\sin\theta,\ \mathrm{d}x = \sqrt{\frac{2}{\alpha}}\cos\theta\mathrm{d}\theta$$
$$\text{Term 2} = \frac{\Delta\beta\lambda^2}{\pi}\int_0^{\pi/2}\frac{2}{\alpha^2}\sin^2\theta\mathrm{d}\theta$$
$$= \frac{\lambda^2\Delta\beta}{2\alpha^2}$$
$$= -\frac{3\gamma\lambda^2}{4\alpha^4}$$

$$\text{Term 3} = \frac{\lambda^2\gamma}{\pi}\int_0^{\sqrt{\frac{2}{\alpha}}}\mathrm{d}x\frac{x^6}{\sqrt{2\alpha-\alpha^2x^2}}$$
$$\text{setting } x = \sqrt{\frac{2}{\alpha}}\sin\theta,\ \mathrm{d}x = \sqrt{\frac{2}{\alpha}}\cos\theta\mathrm{d}\theta$$
$$\text{Term 3} = \frac{\lambda^2\gamma}{\pi}\int_0^{\pi/2}(\frac{2}{\alpha})^3\frac{1}{\alpha}\sin^6\theta\mathrm{d}\theta$$
$$= \frac{\lambda^2\gamma}{\pi\alpha^4}\frac{5\pi}{4}$$
$$= \frac{5\lambda^2\gamma}{4\alpha^4}$$

Substituting Term 1, Term 2 and Term 3 in expression (5.1.1), we get

$$\int \mathrm{d}x x^2\phi_0(x) = \frac{1}{2\alpha} - \frac{3\gamma\lambda^2}{4\alpha^4} + \frac{5\lambda^2\gamma}{4\alpha^4}$$
$$= \frac{1}{2\alpha} + \frac{\gamma\lambda^2}{2\alpha^4}$$

Using the following expression for α and γ in the above expression,

$$\alpha^2 = \omega^2 + \frac{2\lambda}{\omega} - \frac{6\lambda^2}{2\omega^4} + \frac{3\lambda^2}{2\omega^4}$$
$$= \omega^2\left[1-\left(\frac{3\lambda^2-4\omega^3\lambda}{2\omega^6}\right)\right]$$
$$\Rightarrow \alpha = \omega\left[1-\left(\frac{3\lambda^2-4\omega^3\lambda}{2\omega^6}\right)\right]^{1/2}$$

$$\text{And } \gamma = \frac{1}{\omega^3} \tag{5.1.2}$$

we get,

$$\begin{aligned}
\int \mathrm{d}x x^2 \phi_0(x) &= \frac{1}{2\omega}\left[1 - \left(\frac{3\lambda^2 - 4\omega^3\lambda}{2\omega^6}\right)\right]^{-1/2} \\
&+ \frac{\lambda^2}{2\omega^3}\frac{1}{\omega^4}\left[1 - \left(\frac{3\lambda^2 - 4\omega^3\lambda}{2\omega^6}\right)\right]^{-2} \\
&= \frac{1}{2\omega}\left[1 + \left(\frac{3\lambda^2 - 4\omega^3\lambda}{4\omega^6}\right) + \frac{3}{8}\frac{4\lambda^2}{\omega^6} + \cdots\right] \\
&+ \frac{\lambda^2}{2\omega^7}\left[1 + 2\left(\frac{3\lambda^2 - 4\omega^3\lambda}{2\omega^6}\right) + \cdots\right] \\
&= \frac{1}{2\omega} - \frac{\lambda}{2\omega^4} + \frac{13\lambda^2}{8\omega^7} + O(\lambda^3) + \cdots
\end{aligned}$$

The final result to $O(\lambda)^2$, is then

$$\left\langle \mathrm{Tr} X_1^2 \right\rangle = \int \mathrm{d}x x^2 \phi_0(x) = \frac{1}{2\omega} - \frac{\lambda}{2\omega^4} + \frac{13\lambda^2}{8\omega^7} \tag{5.1.3}$$

5.2 Calculation of $\left\langle \mathrm{Tr}(X_1^2) \right\rangle$ Using Perturbation theory

In this section, we use standard perturbation theory to calculate $< \mathrm{Tr}(X_1^2) >$ to order λ^2. The Hamiltonian for the two matrix system, with g_{YM} interaction is given as

$$H = \frac{1}{2}\mathrm{Tr}\dot{X_1}^2 + \frac{\omega^2}{2}\mathrm{Tr}X_1{}^2 + \frac{1}{2}\mathrm{Tr}\dot{X_2}^2 + \frac{\omega^2}{2}\mathrm{Tr}X_2{}^2 - g_{YM}{}^2\mathrm{Tr}[X_1, X_2][X_1, X_2] \tag{5.2.1}$$

Using this, we can write the Lagrangian as

$$L = \frac{1}{2}\mathrm{Tr}\dot{X_1}^2 - \frac{\omega^2}{2}\mathrm{Tr}X_1{}^2 + \frac{1}{2}\mathrm{Tr}\dot{X_2}^2 - \frac{\omega^2}{2}\mathrm{Tr}X_2{}^2 + g_{YM}{}^2\mathrm{Tr}[X_1, X_2][X_1, X_2] \tag{5.2.2}$$

Using the path integral formalism, we can write the expression for the expectation value of X_1^2. This is given as

$$\left\langle \mathrm{Tr} X_1^2 \right\rangle = \int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji} e^{iS_0 + iS_{int}}$$

$$= \int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji}e^{iS_0}\sum_{n=0}^{\infty}\frac{(iS_{int})^n}{n!} \tag{5.2.3}$$

where,

$$S_0 = \int \mathrm{d}t\frac{1}{2}\left\{\mathrm{Tr}\dot{X_1}^2 - \omega^2\mathrm{Tr}X_1{}^2 + \mathrm{Tr}\dot{X_2}^2 - \omega^2\mathrm{Tr}X_2{}^2\right\}$$

and

$$S_{int} = \int \mathrm{d}t\frac{\lambda}{N}\mathrm{Tr}[X_1, X_2][X_1, X_2]$$

with $\lambda = g_{YM}{}^2N$

Calculation to O(λ)

Considering $n = 0$ in (5.2.3), we get

$$\left\langle \mathrm{Tr}X_1^2\right\rangle = \int [\mathcal{D}X_1](X_1)_{ij}(t)(X_1)_{ji}(t)e^{iS_0}$$
$$= \Delta(t) \tag{5.2.4}$$

where, $\Delta(t)$ is the free propagator, given as [33]

$$\Delta(t) = i\int\frac{\mathrm{d}E}{2\pi}\frac{e^{-iEt}}{E^2-\omega^2+i\epsilon}$$
$$= \int\frac{\mathrm{d}E}{2\pi}\frac{e^{-iEt}}{2\omega}\left(\frac{1}{E-\omega+i\delta}-\frac{1}{E+\omega-i\delta}\right)$$

When $t > 0$ complete the contour in lower half plane.

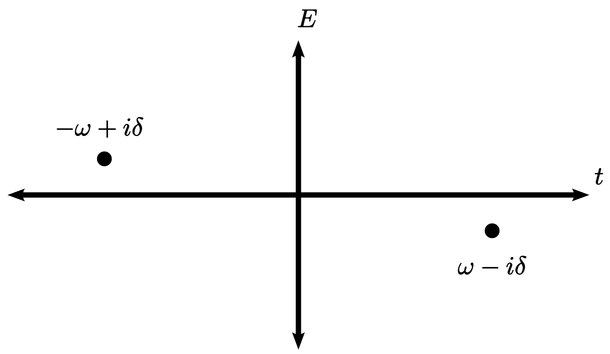

Figure 5.1: Path of integration along the real time axis for $\Delta(t)$.

The path encloses the pole at $E = \omega - i\delta$. When $t < 0$, the contour closes on upper half plane. The path encloses the pole at $E = -\omega + i\delta$.

Therefore, using the Cauchy integral formula, we can write the above expression as

$$\begin{aligned}\Delta(t) &= \frac{i}{2\pi}\frac{1}{2\omega}\left(\theta(t)(-2\pi i)e^{-i\omega t} - \theta(-t)(2\pi i)e^{i\omega t}\right)\\ &= \frac{1}{2\omega}\left(\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right)\end{aligned} \tag{5.2.5}$$

at $t = 0$

$$\Delta(0) = \frac{1}{2\omega} \tag{5.2.6}$$

Now to get the first order correction, put $n = 1$ in (5.2.3)

$$\begin{aligned}\left\langle \mathrm{Tr}X_1^2\right\rangle &= \int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji}e^{iS_0}(iS_{int})\\ &= \frac{i\lambda}{N}\int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji}\int \mathrm{d}t\mathrm{Tr}([X_1,X_2][X_1,X_2])\\ &= \frac{i\lambda}{N}\int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji}\int \mathrm{d}t\mathrm{Tr}(\underbrace{2X_1X_2X_1X_2}_{a} - \underbrace{2X_1^2X_2^2}_{b})\end{aligned} \tag{5.2.7}$$

Term (a) and term (b) corresponds to the following diagrams

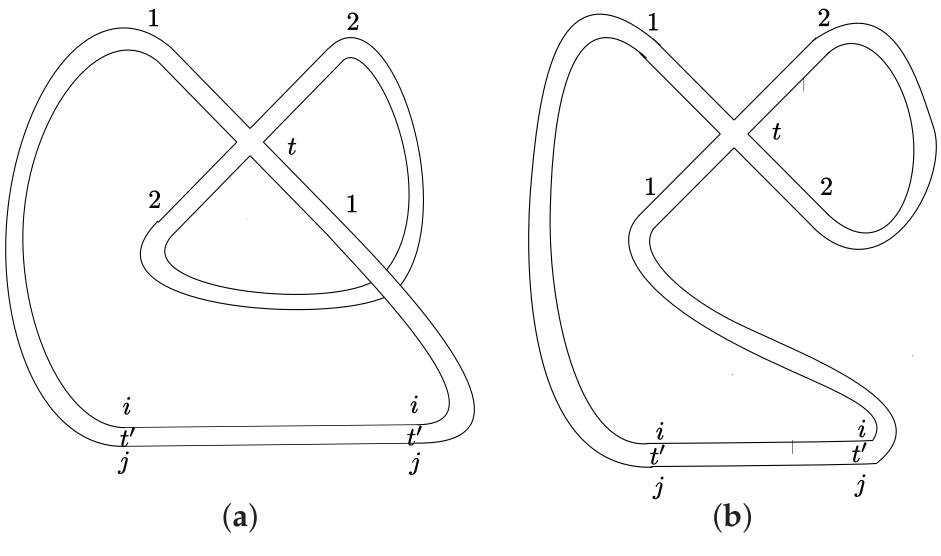

Figure 5.2: Ribbon graph for $O(\lambda)$ terms in the perturbative expansion of $< \mathrm{Tr}(X_1^2) >$.

we see that the diagram (a) is non-planar. Therefore it does not contribute. So, to $O(\lambda)$ only term (b) contributes. Symmetry factor for this diagram is 2.

Hence, to $O(\lambda)$

$$\left\langle \mathrm{Tr} X_1^2 \right\rangle = -4i\lambda\Delta(0)\int \mathrm{d}t[\Delta(t-t')]^2 \tag{5.2.8}$$

To calculate the integral their are two methods.

Method I: Use expression for

$$\Delta(t-t') = i\int \frac{\mathrm{d}E}{2\pi}\frac{e^{-iE(t-t')}}{E^2-\omega^2}$$

Method II: Use expression

$$\Delta(t) = \frac{1}{2\pi}\left(\Theta(t)e^{-i\omega t} + \Theta(-t)e^{i\omega t}\right)$$

Method I

Using the Method I, the integral in (5.2.8) is given as

$$\begin{aligned}
&\int \mathrm{d}t\left(i\int \frac{\mathrm{d}E}{2\pi}\frac{e^{-iEt}}{E^2-\omega^2}\right)\left(i\int \frac{\mathrm{d}E'}{2\pi}\frac{e^{-iE't}}{E'^2-\omega^2}\right)\\
&= -\int \mathrm{d}t\frac{\mathrm{d}E}{2\pi}\frac{\mathrm{d}E'}{2\pi}e^{-i(E+E')t}\frac{1}{(E^2-\omega^2)(E'^2-\omega^2)}
\end{aligned}$$

Using the definition of δ function

$$\delta(E+E') = \int \frac{\mathrm{d}t}{2\pi}e^{-i(E+E')t}$$

the above integral becomes

$$\begin{aligned}
&= -\int \frac{\mathrm{d}E}{2\pi}\frac{\mathrm{d}E'}{2\pi}2\pi\delta(E+E')\frac{1}{(E^2-\omega^2)(E'^2-\omega^2)}\\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{(E^2-\omega^2)^2}
\end{aligned} \tag{5.2.9}$$

The integration has got pole of order 2 at $\pm\omega$

$$\begin{aligned}
\text{The residue at } +\omega &= \lim_{E\to\omega}\frac{d}{dE}\left((E-\omega)^2\frac{1}{(E+\omega)^2(E-\omega)^2}\right)\\
&= \lim_{E\to\omega}\frac{d}{dE}\frac{1}{(E+\omega)^2}\\
&= \lim_{E\to\omega}\frac{-2}{(E+\omega)^3}
\end{aligned}$$

$$= -\frac{1}{4\omega^3}$$

$$\text{The residue at } -\omega = \lim_{E\to-\omega} \frac{d}{dE}\left((E+\omega)^2 \frac{1}{(E+\omega)^2(E-\omega)^2}\right)$$
$$= \lim_{E\to-\omega} \frac{d}{dE}\frac{1}{(E-\omega)^2}$$
$$= \lim_{E\to-\omega} \frac{-2}{(E-\omega)^3}$$
$$= \frac{1}{4\omega^3}$$

Pole at $-\omega$ only lies in the upper half plane. Therefore we only consider that

$$\int \frac{\mathrm{d}E}{(E^2-\omega^2)^2} = 2\pi i \frac{1}{4\omega^3}$$

Substituting in (5.2.9)

$$= -\frac{i}{4\omega^3} = \frac{1}{4i\omega^3}$$

Substituting this in (5.2.8) and $\frac{1}{2\omega}$ for $\Delta(0)$ gives

$$\left\langle \mathrm{Tr}X_1^2 \right\rangle_{O(\lambda)} = \frac{-\lambda}{2\omega^4} \tag{5.2.10}$$

In Appendix C, we give details of the calculation using method II, which is shown to agree with this result. To order λ, the result of $< \mathrm{Tr}(X_1^2) >$ using perturbation theory agrees with the result obtained by the density description method.

Calculation to $O(\lambda^2)$

To calculate $\left\langle \mathrm{Tr}X_1^2 \right\rangle$ to $O(\lambda^2)$, consider $n = 2$ in (5.2.3)

$$\left\langle \mathrm{Tr}X_1^2 \right\rangle = \int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji} e^{iS_0} \frac{(iS_{int})^2}{2!}$$
$$= -\frac{1}{2}\int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji} e^{iS_0} (S_{int})^2$$
$$= -\frac{\lambda^2}{2N^2}\int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji} e^{iS_0} (\int \mathrm{d}t \mathrm{Tr}[X_1, X_2][X_1, X_2])^2$$

$$= -\frac{\lambda^2}{2N^2}\int [\mathcal{D}X_1](X_1)_{ij}(X_1)_{ji}e^{iS_0}\Bigg(\int \mathrm{d}t\int \mathrm{d}t'\mathrm{Tr}(\underbrace{(2X_1X_2X_1X_2)^2}_{\text{term 1}}+$$

$$\underbrace{(2X_1^2X_2^2)^2}_{\text{term 2}}) - \underbrace{8\mathrm{Tr}(X_1X_2X_1X_2)(X_1^2X_2^2)}_{\text{term 3}}\Bigg) \tag{5.2.11}$$

Various possible diagrams for Term 1 are given in (Fig.5.3)

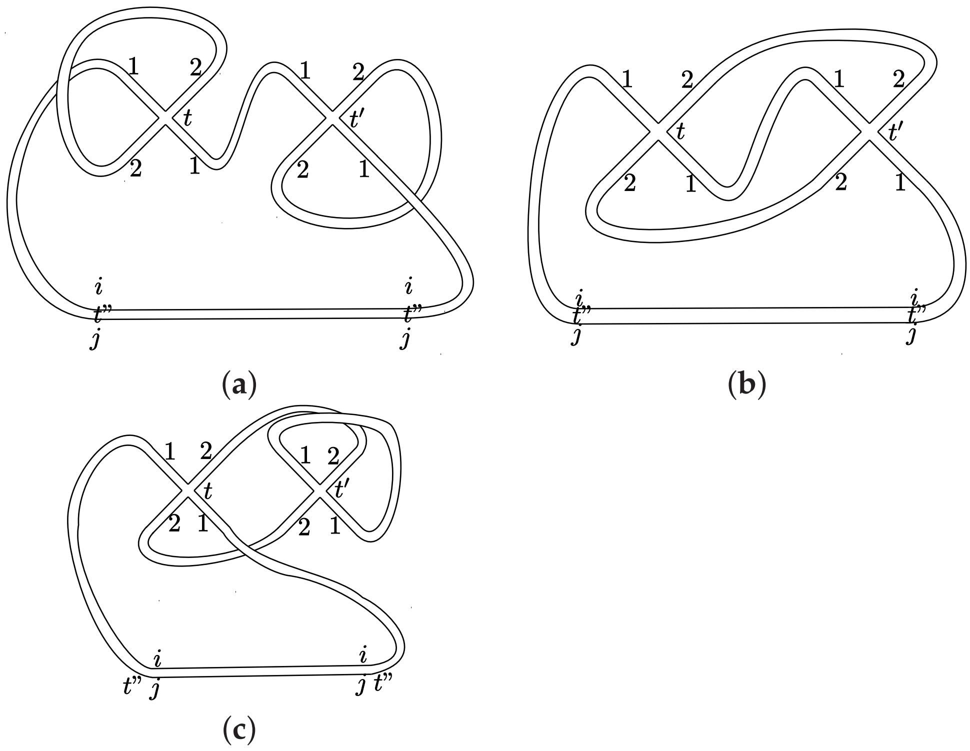

Figure 5.3: Ribbon graph for $O(\lambda^2)$ term with $(2X_1X_2X_1X_2)^2$ interaction

Diagrams (a) and (c) are sub leading. So the only contributing diagram is diagram (b). Various possible diagrams for Term 2 are given in (Fig.5.4) Here, diagram (d) is sub leading and diagrams (e) and (f) contribute. Various possible diagrams of Term 3 are given in (Fig.5.5). However all these diagrams are sub leading. Thus of all the diagrams corresponding to Term $(1),(2)$ and (3), only diagram$(b),(e)$ and (f) contribute.

- Symmetry factor for diagram (b) is 16.

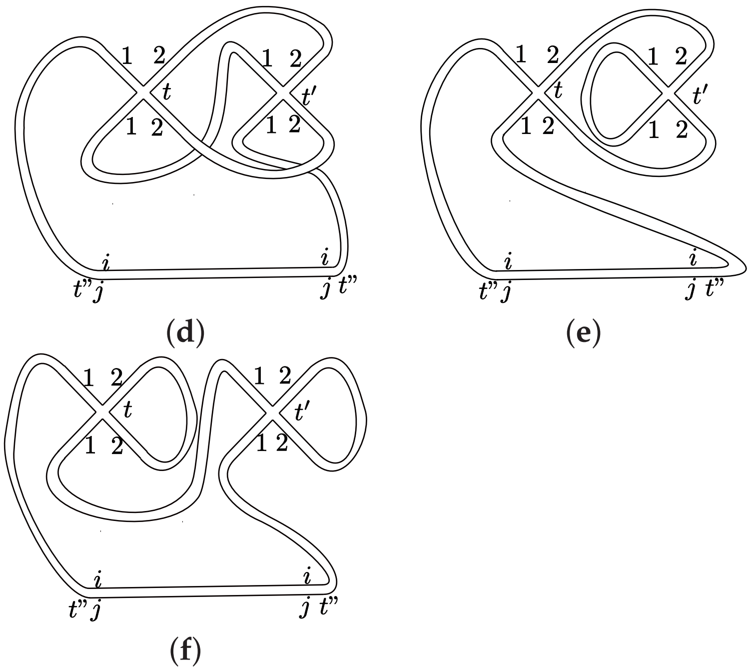

Figure 5.4: Ribbon graph for $O(\lambda^2)$ term with $(2X_1^2X_2^2)^2$ interaction

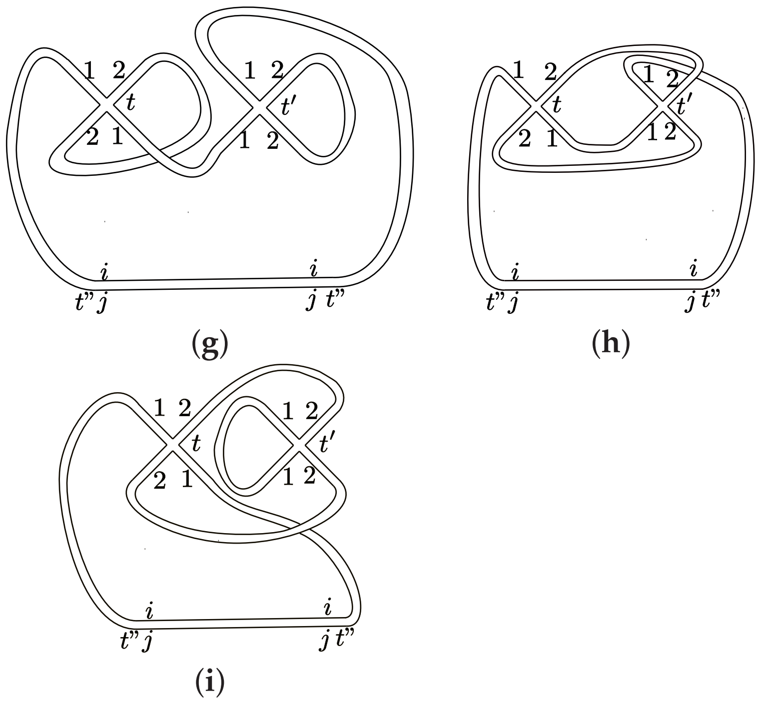

Figure 5.5: Ribbon graph for $O(\lambda^2)$ term with $(X_1X_2X_1X_2)(X_1^2X_2^2)$ interaction

- Symmetry factor for diagram (e) is 8.
- Symmetry factor for diagram (f) is 4.

To $O(\lambda^2)$ term (b), (e) and (f) are given as

$$\text{Term (b)} = -32\lambda^2 \int \mathrm{d}t \int \mathrm{d}t' \Delta(t - t'')\Delta(t' - t'')(\Delta(t - t'))^3 \tag{5.2.12}$$

$$\text{Term (e)} = -16\lambda^2 \Delta(0) \int \mathrm{d}t \int \mathrm{d}t' (\Delta(t - t''))^2 (\Delta(t - t'))^2 \tag{5.2.13}$$

$$\text{Term (f)} = -8\lambda^2 (\Delta(0))^2 \int \mathrm{d}t \int \mathrm{d}t' \Delta(t - t'')\Delta(t' - t'')\Delta(t - t') \tag{5.2.14}$$

This integral can be calculated using the following expression for $\Delta(t)$

$$\Delta(t) = \frac{1}{2\omega}[\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}]$$

Thus in the above integral $(\Delta(t - t'))^3$ can be calculated as follows

$$\begin{aligned}
(\Delta(t - t'))^3 =& \frac{1}{(2\omega)^3}\left(\theta(t - t')e^{-i\omega(t-t')} + \theta(t' - t)e^{i\omega(t-t')}\right)^3 \\
=& \frac{1}{(2\omega)^3}\left(\theta^3(t - t')e^{-3i\omega(t-t')} + \theta^3(t' - t)e^{3i\omega(t-t')}\right. \\
& \left. +3\theta^2(t - t')\theta(t' - t)e^{-i\omega(t-t')} + 3\theta(t - t')\theta^2(t' - t)e^{i\omega(t-t')}\right)
\end{aligned}$$

Using

$$\theta(t)\theta(t)\theta(t) = \theta(t)$$

$$\theta(-t)\theta(-t)\theta(-t) = \theta(-t)$$

$$\theta^2(t)\theta(-t) = \theta(t)\theta^2(-t) = 0$$

$$(\Delta(t - t'))^3 = \frac{1}{(2\omega)^3}\left(\theta(t - t')e^{-3i\omega(t-t')} + \theta(t' - t)e^{3i\omega(t-t')}\right) \tag{5.2.15}$$

Using (5.2.15) in (5.2.12), we get

$$\begin{aligned}
\text{Term (b)} = -\frac{32\lambda^2}{(2\omega)^5} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' & \left(\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right) \\
& \left(\theta(t')e^{-i\omega t'} + \theta(-t')e^{i\omega t'}\right) \\
& \left(\theta(t - t')e^{-3i\omega(t-t')} + \theta(t' - t)e^{3i\omega(t-t')}\right)
\end{aligned}$$

$$= -\frac{32\lambda^2}{(2\omega)^5} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \theta(t)\theta(t')\theta(t - t')e^{-4i\omega t}e^{2i\omega t'}$$

$$+\theta(t)\theta(t')\theta(t'-t)e^{2i\omega t}e^{-4i\omega t'}$$
$$+\theta(t)\theta(-t')\theta(t-t')e^{-4i\omega t}e^{4i\omega t'}$$
$$+\theta(t)\theta(-t')\theta(t'-t)e^{2i\omega t}e^{-2i\omega t'}$$
$$+\theta(-t)\theta(t')\theta(t-t')e^{-2i\omega t}e^{2i\omega t'}$$
$$+\theta(-t)\theta(t')\theta(t'-t)e^{4i\omega t}e^{-4i\omega t'}$$
$$+\theta(-t)\theta(-t')\theta(t-t')e^{-2i\omega t}e^{4i\omega t'}$$
$$+\theta(-t)\theta(-t')\theta(t'-t)e^{4i\omega t}e^{-2i\omega t'}$$

$$= -\frac{32\lambda^2}{(2\omega)^5}\left(\int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t'\theta(t-t')e^{-4i\omega t}e^{2i\omega t'} + \theta(t'-t)e^{2i\omega t}e^{-4i\omega t'}\right.$$
$$+\int_0^\infty \mathrm{d}t \int_{-\infty}^0 \mathrm{d}t' e^{-4i\omega t}e^{4i\omega t'} + \int_{-\infty}^0 \mathrm{d}t \int_0^\infty \mathrm{d}t' e^{4i\omega t}e^{-4i\omega t'}$$
$$\left.+\int_{-\infty}^0 \mathrm{d}t \int_{-\infty}^0 \mathrm{d}t'\theta(t-t')e^{-2i\omega t}e^{4i\omega t'} + \theta(t'-t)e^{4i\omega t}e^{-2i\omega t'}\right)$$

$$= -\frac{32\lambda^2}{(2\omega)^5}\left(\int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}t' e^{-4i\omega t}e^{2i\omega t'} + \int_0^{t'} \mathrm{d}t \int_0^\infty \mathrm{d}t' e^{2i\omega t}e^{-4i\omega t'}\right.$$
$$+\int_0^\infty \mathrm{d}t e^{-4i\omega t} \int_0^\infty \mathrm{d}t' e^{-4i\omega t'} + \int_0^\infty \mathrm{d}t e^{-4i\omega t} \int_0^\infty \mathrm{d}t' e^{-4i\omega t'}$$
$$\left.+\int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t'\theta(t'-t)e^{2i\omega t}e^{-4i\omega t'} + \theta(t-t')e^{-4i\omega t}e^{2i\omega t'}\right)$$

$$= -\frac{32\lambda^2}{(2\omega)^5}\left(\int_0^\infty \mathrm{d}t e^{-4i\omega t}\frac{1}{2i\omega}(e^{2i\omega t}-1)\right.$$
$$+\int_0^\infty \mathrm{d}t' e^{-4i\omega t'}\frac{1}{2i\omega}(e^{2i\omega t'}-1)$$
$$+\frac{1}{4i\omega}\frac{1}{4i\omega} + \frac{1}{4i\omega}\frac{1}{4i\omega}$$
$$\left.+\int_0^{t'} \mathrm{d}t \int_0^\infty \mathrm{d}t' e^{2i\omega t}e^{-4i\omega t'} + \int_0^{t'} \mathrm{d}t \int_0^\infty \mathrm{d}t' e^{-4i\omega t}e^{2i\omega t'}\right)$$

$$= -\frac{32\lambda^2}{(2\omega)^5}\left(\frac{4}{2i\omega}\int_0^\infty \mathrm{d}t(e^{-2i\omega t} - e^{-4i\omega t}) - \frac{2}{16\omega^2}\right)$$

$$
\begin{aligned}
&= -\frac{32\lambda^2}{(2\omega)^5}\left(-\frac{4}{8\omega^2} - \frac{2}{16\omega^2}\right) \\
&= \frac{5\lambda^2}{8\omega^7} \qquad (5.2.16)
\end{aligned}
$$

Similarly, (5.2.13) can be calculated as follows,

$$
\begin{aligned}
\text{Using} \qquad \Delta(t) =& \frac{1}{2\omega}\left(\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right) \\
\text{and} \qquad \Delta(0) =& \frac{1}{2\omega}
\end{aligned}
$$

$$
\begin{aligned}
\text{Term e} =& -\frac{16\lambda^2}{(2\omega)^5}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \left(\theta(t)e^{-2i\omega t} + \theta(-t)e^{2i\omega t}\right) \\
& \left(\theta(t-t')e^{-2i\omega(t-t')} + \theta(t'-t)e^{2i\omega(t-t')}\right) \\
=& -\frac{16\lambda^2}{(2\omega)^5}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \left(\theta(t)\theta(t-t')e^{-4i\omega t}e^{2i\omega t'}\right. \\
& +\theta(t)\theta(t'-t)e^{-2i\omega t'} \\
& +\theta(-t)\theta(t-t')e^{2i\omega t'} \\
& \left.+\theta(-t)\theta(t'-t)e^{4i\omega t}e^{-2i\omega t'}\right) \\
=& -\frac{16\lambda^2}{(2\omega)^5}\left(\int_0^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' e^{-4i\omega t}e^{2i\omega t'} + \int_0^{t'} \mathrm{d}t \int_0^{\infty} \mathrm{d}t' e^{-2i\omega t'}\right. \\
& + \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \theta(t-t')e^{2i\omega t'} \\
& \left.+ \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \theta(t'-t)e^{4i\omega t}e^{-2i\omega t'}\right) \\
=& -\frac{16\lambda^2}{(2\omega)^5}\left(\int_0^{\infty} \mathrm{d}t e^{-4i\omega t}\frac{1}{2i\omega}e^{2i\omega t} + \int_0^{\infty} \mathrm{d}t' t' e^{-2i\omega t'}\right. \\
& \left.+ \int_0^{t'} \mathrm{d}t \int_0^{\infty} \mathrm{d}t' e^{-2i\omega t'} + \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' e^{-4i\omega t}e^{2i\omega t'}\right) \\
=& -\frac{16\lambda^2}{(2\omega)^5}\left(-\frac{1}{\omega^2}\right) \\
=& \frac{\lambda^2}{2\omega^7} \qquad (5.2.17)
\end{aligned}
$$

Also (5.2.14), can be calculated as follows,

$$
\text{Term f} = -\frac{8\lambda^2}{(2\omega)^5}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \left(\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right)
$$

$$
\begin{aligned}
&\left(\theta(t')e^{-i\omega t'} + \theta(-t')e^{i\omega t'}\right) \\
&\left(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{i\omega(t-t')}\right) \\
= &-\frac{8\lambda^2}{(2\omega)^5}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' \Big(\theta(t)\theta(t')\theta(t-t')e^{-2i\omega t} \\
&+\theta(t)\theta(t')\theta(t-t')e^{-2i\omega t'} \\
&+\theta(t)\theta(-t')\theta(t-t')e^{-2i\omega t}e^{2i\omega t'} \\
&+\theta(t)\theta(-t')\theta(t'-t) \\
&+\theta(-t)\theta(t')\theta(t-t') \\
&+\theta(-t)\theta(t')\theta(t'-t)e^{2i\omega t}e^{-2i\omega t'} \\
&+\theta(-t)\theta(-t')\theta(t-t')e^{2i\omega t'} \\
&+\theta(-t)\theta(-t')\theta(t'-t)e^{2i\omega t}\Big) \\
= &-\frac{8\lambda^2}{(2\omega)^5}\Bigg(\int_0^{\infty} \mathrm{d}t \int_0^{t} \mathrm{d}t' e^{-2i\omega t} + \int_0^{t'} \mathrm{d}t \int_0^{\infty} \mathrm{d}t' e^{-2i\omega t'} \\
&+\int_0^{\infty} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t' e^{-2i\omega t}e^{-2i\omega t'} + \int_{-\infty}^{0} \mathrm{d}t \int_0^{\infty} \mathrm{d}t' e^{2i\omega t}e^{-2i\omega t'} \\
&+\int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t' \theta(t-t')e^{2i\omega t'} + \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t' \theta(t'-t)e^{2i\omega t}\Bigg) \\
= &-\frac{8\lambda^2}{(2\omega)^5}\left(2\int_0^{\infty} \mathrm{d}t t e^{-2i\omega t} - \frac{2}{4\omega^2} + 2\int_0^{\infty} \mathrm{d}t t e^{-2i\omega t}\right) \\
= &-\frac{8\lambda^2}{(2\omega)^5}\left(-\frac{6}{4\omega^2}\right) \\
= &\frac{3\lambda^2}{8\omega^7}
\end{aligned}
\tag{5.2.18}
$$

Hence substituting (5.2.16), (5.2.17) and (5.2.18) for term 1 and 2 in (5.2.11) gives,

$$
\begin{aligned}
\left\langle \mathrm{Tr}X_1^2 \right\rangle &= \frac{5\lambda^2}{8\omega^7} + \frac{\lambda^2}{2\omega^7} + \frac{3\lambda^2}{8\omega^7} \\
&= \frac{3\lambda^2}{2\omega^7}
\end{aligned}
\tag{5.2.19}
$$

Thus, the order λ^2 result is not in agreement with the density description result. This discrepancy between the results is perhaps not too surprising, as the effective hamiltonian is defined as

$$\hat{H}^{eff}_{1(0)} \equiv \int \mathrm{d}\bar{X}_2 \psi_0(\lambda, \bar{X}_2)\hat{H}\psi_0(\lambda, \bar{X}_2) \tag{5.2.20}$$

where, $\psi_0(\lambda, \bar{X}_2)$ is the ground state wavefunction with no "impurity" and the subscript "1" in the effective hamiltonian denotes that it is for the X_1 sector. However, higher excited states $\psi_n(\lambda, \bar{X}_2)$ (known to be expressed in terms of Hermite polynomials) have not been included. This would lead to a sequence of effective hamiltonians

$$\hat{H}^{eff}_{1(n)} \equiv \int \mathrm{d}\bar{X}_2 \psi_n(\lambda, \bar{X}_2)\hat{H}\psi_n(\lambda, \bar{X}_2) \tag{5.2.21}$$

which have not been discussed in this thesis.

Chapter 6

Strong Coupling Solution

As discussed at length in chapter 4, the fact that $\psi_0(\lambda, \bar{X}_2)$ depends on X_1 degrees of freedom, results in a shifted P_1 operator and the resultant effective potential terms of equation (4.1.19) or (4.2.7).

Remarkably, as $\lambda \to \infty$ these effective potential terms are sub leading compared to the ground state frequency. Therefore, we expect that the $\lambda \to \infty$ limit ($\lambda \gg \omega^3$) of (5.0.1) is of great relevance to the properties of the strongly coupled system of two matrices [34], and is studied in this chapter

The $\lambda \to \infty$ limit of (5.0.6) takes the form

$$\pi^2 \phi_0^2(x) = 2\mu - 2\sqrt{2\lambda} \int dy |x - y| \phi_0(y) \tag{6.0.1}$$

$$E_0 = N^2 \Big[\frac{\pi^2}{6} \int dx \phi_0^3(x) + \frac{\sqrt{2\lambda}}{2} \int dx \int dy |x - y| \phi_0(x) \phi_0(y) \Big] \tag{6.0.2}$$

Here E_0 is the ground state energy and it is obtained by considering V_{eff} (5.0.5) at ϕ_0 for large λ. In the above expressions, we introduce the following term

$$f(x) = \sqrt{2\lambda} \int dy |x - y| \phi_0(y), \quad \pi^2 \phi_0^2(x) = 2(\mu - f(x)) \tag{6.0.3}$$

which satisfies

$$f(x) = \frac{\sqrt{2\lambda}}{\pi} \int dy|x-y| \sqrt{2(\mu - f(y))}. \qquad (6.0.4)$$

As it was the case in perturbation theory, we assume that $\phi_0(x)$ remains an even, single cut function defined in the interval $[-x_0, x_0]$. To show that this is a consistent ansatz, we note that then:

$$f(x) = \sqrt{2\lambda}\left(|x| \int_{-|x|}^{|x|} \phi_0(y)dy + 2\int_{|x|}^{x_0} \phi_0(y)ydy\right). \qquad (6.0.5)$$

Hence $f(x)$ is also even, establishing the consistency of the ansatz. Using

$$\partial_x^2 |x-y| = 2\delta(x-y),$$

equation (6.0.4) becomes[1]

$$\begin{aligned}\partial_x^2 f(x) =& \frac{\sqrt{2\lambda}}{\pi} \int dy \partial_x^2 |x-y| \sqrt{2(\mu - f(y))} \\ =& \frac{\sqrt{2\lambda}}{\pi} \int dy 2\delta(x-y) \sqrt{2(\mu - f(y))} \\ =& \frac{4\sqrt{\lambda}}{\pi} \sqrt{\mu - f(x)} \end{aligned} \qquad (6.0.6)$$

This can be integrated as follows

$$\begin{aligned}\int \mathrm{d}f(\partial_x f(x)) =& \frac{4\sqrt{\lambda}}{\pi} \int \mathrm{d}f \sqrt{\mu - f(x)} \\ \frac{1}{2}\Big(\partial_x f(x)\Big)^2 =& \frac{4\sqrt{\lambda}}{\pi}(-\frac{2}{3})\Big(\sqrt{\mu - f(x)}\Big)^3 + e \\ \frac{1}{2}(\partial_x f(x))^2 +& \frac{8\sqrt{\lambda}}{3\pi}(\mu - f(x))^{3/2} = e \end{aligned} \qquad (6.0.7)$$

The "energy" constant can be worked out using the condition

$$\text{Let } f(x=0) = f_0 \quad \Rightarrow \partial_x f(0) = 0$$

Therefore (6.0.7) at $x = 0$ becomes

$$0 + \frac{8\sqrt{\lambda}}{3\pi}(\mu - f_0)^{3/2} = e$$

[1] ϕ_0^2 satisfies a very similar equation.

Hence (6.0.7) can be rewritten as

$$\frac{df}{dx} = \frac{4\lambda^{\frac{1}{4}}}{\sqrt{3}\pi}\sqrt{(\mu - f_0)^{\frac{3}{2}} - (\mu - f(x))^{\frac{3}{2}}} \tag{6.0.8}$$

Using the normalization condition for ϕ_0, we get

$$\begin{aligned}
1 = \int_{-x_0}^{x_0} dx\phi_0(x) &= 2\int_0^{x_0} dx\phi_0(x) = 2\int_{f_0}^{\mu} \mathrm{d}f \frac{\phi_0(f)}{\frac{df}{dx}} \\
&= \frac{2\sqrt{2}}{\pi}\frac{\sqrt{3}\pi}{4\lambda^{\frac{1}{4}}}\int_{f_0}^{\mu} \mathrm{d}f \frac{\sqrt{\mu - f(x)}}{\sqrt{(\mu - f_0)^{\frac{3}{2}} - (\mu - f(x))^{\frac{3}{2}}}} \\
&= \frac{2\sqrt{2}}{\pi}\frac{\sqrt{3}\pi}{4\lambda^{\frac{1}{4}}}\frac{4}{3}\sqrt{(\mu - f_0)^{\frac{3}{2}} - (\mu - f(x))^{\frac{3}{2}}}\Big|_{f_0}^{\mu} \\
&= \frac{2\sqrt{2}}{\sqrt{3}\pi}\frac{1}{\lambda^{\frac{1}{4}}}(\mu - f_0)^{\frac{3}{4}} \\
\Rightarrow (\mu - f_0)^{\frac{3}{2}} &= \frac{3\pi}{8}\lambda^{\frac{1}{2}}
\end{aligned}$$

and hence (6.0.8) takes the form:

$$\begin{aligned}
\frac{df}{dx} &= \frac{4\lambda^{\frac{1}{4}}}{\sqrt{3}\pi}(\mu - f_0)^{\frac{3}{4}}\sqrt{1 - \left(\frac{(\mu - f(x))}{(\mu - f_0)}\right)^{\frac{3}{2}}} \\
&= \frac{4\lambda^{\frac{1}{4}}}{\sqrt{3}\pi}\sqrt{\frac{3\pi}{8}}\lambda^{\frac{1}{4}}\sqrt{1 - \left(\frac{(\mu - f(x))}{(\mu - f_0)}\right)^{\frac{3}{2}}} \\
&= \sqrt{2\lambda}\sqrt{1 - \left(\frac{\mu - f(x)}{\mu - f_0}\right)^{\frac{3}{2}}}
\end{aligned} \tag{6.0.9}$$

We will not need to invert (6.0.9) and obtain $f(x)$ explicitly, as all results presented here will be expressed in terms of known definite integrals.

Of particular interest is the large N ground state energy. From (6.0.2) and (6.0.3) this can be written as

$$E_0 = N^2\Big[\frac{\pi^2}{6}\int dx\phi_0^3(x) + \frac{1}{2}\int dx f(x)\phi_0(x)\Big] = N^2\Big[\frac{\mu}{2} - \frac{\pi^2}{12}\int dx\phi_0^3(x)\Big] \tag{6.0.10}$$

One needs to know μ, or f_0, independently. From (6.0.5), one obtains

$$f(0) = f_0 = 2\sqrt{2\lambda}\int_0^{x_0} \mathrm{d}x x\phi_0(x)$$

We therefore need to calculate this integral, which is done as follows

$$\begin{aligned}\int_0^{x_0} \mathrm{d}x x\phi_0(x) &= \int_{f_0}^{\mu} \mathrm{d}f \frac{\phi_0(x)}{\frac{df}{dx}} x(f)\\ &= \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{2\lambda}}\int_{f_0}^{\mu}\mathrm{d}f x(f)\frac{(\mu - f(x))^{\frac{1}{2}}}{\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}}\\ = \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{2\lambda}}&\left\{\left(\frac{4}{3}(\mu-f_0)^{3/2}\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{3/2}}x(f)\right)\Bigg|_{f_0}^{\mu}\right.\\ -\frac{4}{3}(\mu-f_0)^{3/2}&\left.\int_{f_0}^{\mu}\mathrm{d}f\frac{1}{\frac{df}{dx}}\left(\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{3/2}}\right)\right\}\\ &= \frac{x_0}{2} - \frac{1}{2\sqrt{2\lambda}}(\mu - f_0)\end{aligned}$$

Here we have used the expression for $(\mu - f_0)^{3/2}$ and $\frac{df}{dx}$. Now substituting this in expression for f_0 we get

$$f_0 = \sqrt{2\lambda}x_0 - (\mu - f_0) \qquad \mu = \sqrt{2\lambda}x_0.$$

From (6.0.9) one obtains

$$\begin{aligned}\sqrt{2\lambda}x_0 &= \int_{f_0}^{\mu}\mathrm{d}f\frac{1}{\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{3/2}}}\\ &=(\mu-f_0)\int_0^1\frac{dz}{\sqrt{1-(1-z)^{\frac{3}{2}}}} \quad \left(\text{using } \frac{\mu-f(x)}{\mu-f_0} = (1-z) \Rightarrow df = (\mu-f_0)dz\right)\\ &=2(\mu-f_0)\int_0^1\frac{tdt}{\sqrt{1-t^3}} \quad \left(\text{using } (1-z)^{\frac{1}{2}} = t\right)\end{aligned}$$

Also,

$$\begin{aligned}\frac{\pi^2}{12}\int dx\phi_0^3(x) &= \frac{\pi^2}{12}\frac{2\sqrt{2}}{\pi^3}\frac{1}{\sqrt{2\lambda}}\int_{f_0}^{\mu}\mathrm{d}f\frac{1}{\frac{df}{dx}}(\mu-f(x))^{\frac{3}{2}}\\ =&\frac{1}{6}\frac{1}{\pi\sqrt{\lambda}}2\left\{\left(\frac{4}{3}(\mu-f_0)^{3/2}\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{3/2}}(\mu-x(f))\right)\Bigg|_{f_0}^{\mu}\right.\\ &\left.-\frac{4}{3}(\mu-f_0)^{3/2}\int_{f_0}^{\mu}\mathrm{d}f(-1)\left(\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{3/2}}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{6}\int_{f_0}^{\mu} \mathrm{d}f\left(\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{3/2}}\right)\\
&=\frac{1}{6}(\mu-f_0)\int_0^1 \sqrt{1-(1-z)^{\frac{3}{2}}} \quad \left(\text{Let } \frac{\mu-f(x)}{\mu-f_0}=(1-z) \Rightarrow df=(\mu-f_0)dz\right)\\
&=\frac{1}{3}\int_0^1 \mathrm{d}tt\sqrt{1-t^3} \quad \left(\text{Let } (1-z)^{\frac{1}{2}}=t\right)\\
&=\frac{1}{3}\left(\int_0^1 \frac{t}{\sqrt{1-t^3}}\mathrm{d}t-\int_0^1 \frac{t^4}{\sqrt{1-t^3}}\right)
\end{aligned}
$$

Using the property

$$
\int_u^1 \frac{x^m}{\sqrt{1-x^3}}=\frac{2u^{m-2}\sqrt{1-u^3}}{2m-1}+\frac{2(m-2)}{2m-1}\int_u^1 \frac{x^{m-3}}{\sqrt{1-x^3}}\mathrm{d}x
$$

We get

$$
\int_0^1 \frac{x^4}{\sqrt{1-x^3}}\mathrm{d}x=\frac{4}{7}\int_0^1 \frac{x}{\sqrt{1-x^3}}\mathrm{d}x
$$

Thus,

$$
\begin{aligned}
\int_0^1 x\sqrt{1-x^3}\mathrm{d}x=&\int_0^1 \frac{x}{\sqrt{1-x^3}}\mathrm{d}x-\frac{4}{7}\int_0^1 \frac{x}{\sqrt{1-x^3}}\mathrm{d}x\\
=&\frac{3}{7}\int_0^1 \frac{x}{\sqrt{1-x^3}}\mathrm{d}x
\end{aligned}
$$

and

$$
\frac{\pi^2}{12}\int \mathrm{d}x\phi_0^3(x)=\frac{1}{7}(\mu-f_0)\int_0^1 \frac{t}{\sqrt{1-t^3}}\mathrm{d}t
$$

These integrals are tabulated in [35], and are finite. Therefore

$$
\begin{aligned}
E_0=&N^2\left[(\mu-f_0)\int_0^1 \frac{t}{\sqrt{1-t^3}}\mathrm{d}t-\frac{1}{7}(\mu-f_0)\int_0^1 \frac{t}{\sqrt{1-t^3}}\mathrm{d}t\right]\\
=&N^2\Big[\frac{6}{7}\left(\frac{3\pi}{8}\right)^{\frac{2}{3}}\int_0^1 \frac{tdt}{\sqrt{1-t^3}}\,\lambda^{\frac{1}{3}}\Big]\\
=&N^2\Big[\frac{9}{14}\Big(\frac{\sqrt{3}}{4\pi}\Big)^{\frac{1}{3}}\Big(\Gamma\Big(\frac{2}{3}\Big)\Big)^3\,\lambda^{\frac{1}{3}}\Big]
\end{aligned}
\tag{6.0.11}
$$

Similar to the weak coupling case we consider the correlator

$$
< TrX_1^2 >= N^2\int dxx^2\phi_0=2N^2\int_{f_0}^{\mu} x^2(f)\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f
$$

To evaluate the integral, we use integration by part

$$\int_{f_0}^{\mu} x^2(f)\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f = \left(x^2(f)\int\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f\Big|_{f_0}^{\mu}\right) - \int_{f_0}^{\mu}\left(2x\frac{dx}{df}\int\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f\right)\mathrm{d}f \tag{6.0.12}$$

Now we need to evaluate $\int\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f$, which is done as follows

$$\begin{aligned}
\int\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f =& \frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{2\lambda}}\int\frac{\mu-f(x)}{\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}}\mathrm{d}f\\
=&\frac{1}{\pi\sqrt{\lambda}}\int\frac{\mu-f(x)}{\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}}\mathrm{d}f\\
\text{Let, }\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}} =& Z\\
\frac{1}{2}\frac{1}{\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}}\frac{3}{2}\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{1}{2}}\frac{df}{\mu-f_0} =& dz\\
(\mu-f(x))^{\frac{3}{2}}df =& \frac{4}{3}z(\mu-f_0)^{\frac{3}{2}}dz\\
\text{Thus, }\int\frac{\phi_0(f)}{\frac{df}{dx}}\mathrm{d}f =& \frac{1}{\pi\sqrt{\lambda}}\frac{4}{3}(\mu-f_0)^{\frac{3}{2}}\int dz\\
=&\frac{4}{3}\frac{(\mu-f_0)^{\frac{3}{2}}}{\pi\sqrt{\lambda}}\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}
\end{aligned}$$

Substitute in (6.0.12)

$$\begin{aligned}
&= x^2(f)\frac{4}{3}\frac{(\mu-f_0)^{\frac{3}{2}}}{\pi\sqrt{\lambda}}\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}\Big|_{f_0}^{\mu}\\
&-\int 2x(f)\frac{1}{\frac{df}{dx}}\frac{4}{3}\frac{(\mu-f_0)^{\frac{3}{2}}}{\pi\sqrt{\lambda}}\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}\\
&=\frac{4}{3}\frac{(\mu-f_0)^{\frac{3}{2}}}{\pi\sqrt{\lambda}}x_0^2-\frac{4}{3}\frac{(\mu-f_0)^{\frac{3}{2}}}{\pi\sqrt{\lambda}}\int\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}x(f)\frac{1}{\sqrt{1-\left(\frac{\mu-f(x)}{\mu-f_0}\right)^{\frac{3}{2}}}}\mathrm{d}f
\end{aligned}$$

$\left(\text{Calling } \frac{4}{3}\frac{(\mu-f_0)^{\frac{3}{2}}}{\pi\sqrt{\lambda}} = A = \frac{1}{2}\text{By substituting the expression for}(\mu-f_0)^{\frac{3}{2}}\right)$

$$=\frac{x_0^2}{2}-\frac{1}{\sqrt{2\lambda}}\int x(f)\mathrm{d}f$$

$$= \frac{x_0^2}{2} - \frac{1}{\sqrt{2\lambda}} x_0 \mu + \frac{1}{\sqrt{2\lambda}} \int \frac{f}{\frac{df}{dx}} \mathrm{d}f$$

$$\Big(\text{Putting } x_0 = \frac{\mu}{\sqrt{2\lambda}}\Big)$$

$$= -\frac{\mu^2}{4\lambda} + \frac{1}{\sqrt{2\lambda}} \int \frac{f}{\frac{df}{dx}} \mathrm{d}f$$

Thus

$$\begin{aligned} < TrX_1^2 > &= N^2 \Big[-\frac{\mu^2}{2\lambda} + \frac{2}{\sqrt{2\lambda}} \int_{f_0}^{\mu} \frac{f}{\frac{df}{dx}} df \Big] \\ &= 2\, N^2 \left(\frac{3\pi}{8}\right)^{\frac{4}{3}} \lambda^{-\frac{1}{3}} \Big[\Big(\int_0^1 \frac{tdt}{\sqrt{1-t^3}} \Big)^2 - \frac{2}{5} \int_0^1 \frac{dt}{\sqrt{1-t^3}} \Big] \\ &= \frac{N^2}{\pi 2^{\frac{1}{3}} \sqrt{3}} \Big(\frac{3\pi}{8}\Big)^{\frac{4}{3}} \lambda^{-\frac{1}{3}} \Big[\frac{3\sqrt{3}}{\pi} \Big(\Gamma\Big(\frac{2}{3}\Big)\Big)^6 - \frac{2}{5} \Big(\Gamma\Big(\frac{1}{3}\Big)\Big)^3 \Big] \end{aligned} \tag{6.0.13}$$

Summarizing, we have the result for the large N ground state energy and $< \mathrm{Tr}(X_1^2) >$ as follows:

$$\begin{aligned} E_0 &= N^2 \Big[\frac{9}{14} \Big(\frac{\sqrt{3}}{4\pi}\Big)^{\frac{1}{3}} \Big(\Gamma\Big(\frac{2}{3}\Big)\Big)^3 \lambda^{\frac{1}{3}} \Big] \\ < \mathrm{Tr}(X_1^2) > &= \frac{N^2}{\pi 2^{\frac{1}{3}} \sqrt{3}} \Big(\frac{3\pi}{8}\Big)^{\frac{4}{3}} \lambda^{-\frac{1}{3}} \Big[\frac{3\sqrt{3}}{\pi} \Big(\Gamma\Big(\frac{2}{3}\Big)\Big)^6 - \frac{2}{5} \Big(\Gamma\Big(\frac{1}{3}\Big)\Big)^3 \Big] \end{aligned}$$

The strong coupling limit would also correspond to the limit $\omega \to 0$, corresponding to the system of two "massless" matrices (i.e. without the harmonic potential) with a Yang-Mills interaction. However, the perturbation theory for such a system faces the problem of infrared divergences. A remarkable feature of the planar ground state energy and the correlator obtained above is that they are free of infrared divergences and depend only on the appropriate power of λ which is expected from dimensional considerations.

Chapter 7

Conclusions

The aim of this thesis was to study the system of two hermitian matrices, in a harmonic potential, coupled via Yang-Mills interaction, in a nonsupersymmetric setting. These two matrices are two of the six Higgs scalars of the bosonic sector of $\mathcal{N} = 4$ SYM theory, in the leading Kaluza - Klein compactification on $R \times S^3$. The two matrix models have been studied previously as well, [27], [30], [31], where the two matrices were either considered as angular momentum eigenstates or were treated exactly. In all these previous works, a supersymmetric approach was always assumed, because of which the normal ordering terms were consistently neglected. Working in the nonsupersymmetric setting makes us to consider these terms.

The work began by reviewing the collective field theory technique, giving first the general formalism and then applying it to a system of single matrix. This knowledge of the collective field theory was then applied to the system of two hermitian matrices interacting through the Yang - Mills potential. Two approaches were explored. In the first, one matrix was treated exactly and formed the background, while the second matrix was treated in the creation annihilation basis. The resulting Hamiltonian was written (3.1.2), which included normal ordering terms owing to the nonsupersymmetric treatment of the matrices. A Bogoliubov transformation was then introduced and only the ground state configuration coming from the zero point energy of the creation annihilation oscillators, was considered in the resulting Hamiltonian (3.1.6). Due to non trivial commutation relations, it turns out to be easier to consider the scalar field $\bar{X}_2$. A canonical transformation was derived and an effective Hamiltonian acting on the ground state wavefunction with no X_2 "impurity" was introduced. However P_1 acts non trivially on this ground state wavefunction with no "impurities", and therefore the Hamiltonian in the X_1 sector will have additional shifts. This corrected or shifted Hamiltonian is worked out both in the original as well as "bared" system of coordinates, and the result is found to be in exact agreement (4.1.19 and 4.2.7).

An important feature of this effective potential is that it only depends on the eigenvalues of the matrix X_1, and therefore one can use the collective field theory to obtain the large N planar background in terms of the density of these eigenvalues. This has been done and the background has been found to satisfy a self - consistent g_{YM} dependent integral equation. This integral equation has both a weak and a strong coupling expansion.

The weak coupling expansion of the background is described to $O(\lambda^2)$ and the same calculation is also performed perturbatively to $O(\lambda^2)$. In the strong coupling limit ($\lambda \to \infty$), it is seen that the background satisfies a non - linear differential equation, with solution that has also been discussed. The planar ground state energy and examples of correlators have been obtained and these are shown to be finite.

The results obtained in this thesis are possibly of relevance in the study of gauge theory and ADS/CFT correspondence,but in a non supersymmetric background. One can think of the background obtained in this thesis as associated to the non-supersymmetric g_{YM} deformation of the "droplet" description of 1/2 BPS states. The strong coupling background is different from the harmonic background resulting from the supersymmetric arguments.

Another possible relevance is related to the physical interpretation of the eigenvalues as coordinates of a system of D0's.

The strong coupling results are also free of infrared divergences, even in the case of two "massless" matrices (i.e. without the harmonic potential, or in zero curvature limit) with a Yang - Mills interaction.

Clearly, several extensions of the results obtained in this thesis suggest themselves. The most obvious one is a study of the contributions coming from higher excited states in (5.2.21), if these are systematic or if it may be desirable to consider a more symmetric description of the two matrices, as it has been considered for instance in [36].

Multi matrix models are notoriously difficult to study, but it is worthwhile remarking that the methods described in this thesis are straightforwardly generalizable to more than two matrices. The required formalism for such study has been established in this thesis.

Appendices

Appendix A

Double Index Notation

The double index notation is used to describe the indices carried by the matrices and the frequency. It is depicted by letters in bold. The notation is as follows: If A= (ij) and

$$T_A = T_{ij}$$

Then, a raised index is defined as follows

$$T^A = T_{ji} = g^{AB} T_B$$

Also, for instance($A = (ab), B = (cd), C = (ij)$)

$$\omega_{ij} \bar{X}_{2ij} \bar{X}_{2ji} = X_{2ba} V^{\dagger}_{ib} V_{aj} \omega_{ij} V_{di} V^{\dagger}_{jc} X_{2cd} = X_2^A \omega_A{}^B X_{2B}$$

and,

$$\omega_A{}^B = O^{-1}{}_A{}^C \omega_C O_C{}^B$$

where $O_C{}^B$ is the orthogonal matrix given by

$$O_C{}^B = V_{di} V^{\dagger}_{jc} = O^{cd}_{ij}$$

also, since

$$O^{-1} = O^T$$

and

$$(O^T)^B{}_C = O_C{}^B = O_{ij}{}^{cd} = V_{di} V^{\dagger}_{jc}$$

$$\Rightarrow (O^T)_C{}^B = (O^T)_{ij}{}^{cd} = O_{dc}{}^{ji} = V_{id}V^\dagger_{cj}$$
$$\Rightarrow (O^{-1})_{ij}{}^{cd} = V_{id}V^\dagger_{cj}$$

Therefore we have the following result

$$O^{cd}_{ij} = V_{di}V^\dagger_{jc} \qquad\qquad (O^{-1})_{ij}{}^{cd} = V_{id}V^\dagger_{cj} \tag{A.0.1}$$

Appendix B

Calculation of Shifted Kinetic Term in Original System of Coordinates

To calculate the shifted kinetic term in original system of coordinates, we need to calculate (4.2.3), (4.2.4) and (4.2.6). These terms will be referred to as term 1, term 2 and term 3 respectively in the following discussion. Here we discuss the details of calculation of these terms.

B.1 Term 1

The left hand side in (4.2.3) can be rewritten as

$$\begin{aligned} & i[P^{1A}, \omega_C{}^D] < X_2{}^C X_{2D} > -\frac{1}{2\omega_C} i[P^{1A}, \omega_C] \\ & = \frac{1}{2}(\omega^{-1})_D{}^C i[P^{1A}, \omega_C{}^D] - \frac{1}{2\omega_{ij}} \frac{\partial}{\partial X_{ab}} \omega_{ij} \end{aligned} \tag{B.1.1}$$

To calculate this result we need to calculate $i[P^{1A}, \omega_C{}^D]$ which is done as follows

$$i[P^{1A}, \omega_C{}^D] = \frac{\partial}{\partial X_{ab}} \omega_{cd}{}^{ef} \tag{B.1.2}$$

Using the expression for $\frac{\partial}{\partial X_{ab}}$as

$$\frac{\partial}{\partial X_{ab}} = \sum_{k\neq p} \frac{V_{bk}V^{\dagger}{}_{pa}}{\lambda_k - \lambda_p}\hat{R}_{kp} + \sum_k V_{bk}\frac{\partial}{\partial \lambda_k}V^{\dagger}_{ka} \tag{B.1.3}$$

where

$$\hat{R}_{kp}V_{ab} = V_{ap}\delta_{bk}$$

$$\hat{R}_{kp}V^{\dagger}_{ab} = -V^{\dagger}_{kb}\delta_{ap}$$

and

$$\omega_A{}^B = \omega_{ab}{}^{cd} = \sum_{ij} V_{aj}V^{\dagger}_{ib}\omega_{ij}V_{di}V^{\dagger}_{jc} \tag{B.1.4}$$

Using (B.1.3) and (B.1.4) in (B.1.2), we get

$$\begin{aligned}
i[P^{1A}, \omega_C{}^D] &= \sum_{k\neq p} \frac{V_{bk}V^{\dagger}{}_{pa}}{\lambda_k - \lambda_p}\hat{R}_{kp}(V_{cj}V^{\dagger}_{id}\omega_{ij}V_{fi}V^{\dagger}_{je}) + V_{bk}V^{\dagger}_{ka}V_{cj}V^{\dagger}_{id}(\frac{\partial \omega_{ij}}{\partial \lambda_k})V_{fi}V^{\dagger}_{je} \\
&= \sum_{k\neq p} \frac{V_{bk}V^{\dagger}{}_{pa}}{\lambda_k - \lambda_p}(V_{cp}V^{\dagger}_{id}(\omega_{ik} - \omega_{ip})V_{fi}V^{\dagger}_{ke} + V_{cj}V^{\dagger}_{kd}(\omega_{kj} - \omega_{pj})V_{fp}V^{\dagger}_{je}) \\
&+ V_{bk}V^{\dagger}_{ka}V_{cj}V^{\dagger}_{id}(\frac{\partial \omega_{ij}}{\partial \lambda_k})V_{fi}V^{\dagger}_{je} \\
&= O_{kp}{}^{ab}\Big(O^{-1}_{cd}{}^{ip}\frac{\omega_{ik} - \omega_{ip}}{\lambda_k - \lambda_p}O_{ik}{}^{ef} + O^{-1}_{cd}{}^{kj}\frac{\omega_{kj} - \omega_{pj}}{\lambda_k - \lambda_p}O_{pj}{}^{ef}\Big) \\
&+ O_{kk}{}^{ab}O^{-1}_{cd}{}^{ij}(\frac{\partial \omega_{ij}}{\partial \lambda_k})O_{ij}{}^{ef}
\end{aligned} \tag{B.1.5}$$

Using (B.1.5) in (B.1.1) and also,

$$O_{ik}{}^{ef}(\omega^{-1})_{ef}{}^{cd}O^{-1}_{cd}{}^{ip} = O_{ik}{}^{C}O^{-1}_{C}{}^{D}(\omega^{-1})_D O_D{}^B O^{-1}_{B}{}^{ip} = \delta^D_{ik}(\omega^{-1})_D\delta^{ip}_D$$

But this term is 0 because $p \neq k$ Similarly

$$O_{pi}{}^{ef}(\omega^{-1})_{ef}{}^{cd}O^{-1}_{cd}{}^{ki} = \delta^C_{pi}\omega^{-1})_C\delta^{ki}_C$$

which is also equal to 0. Thus (B.1.1) is left with

$$\begin{aligned}
&\frac{1}{2}O_{kk}{}^{ab}\frac{\partial \omega_{ij}}{\partial \lambda_k}O_{ij}{}^{ef}(\omega^{-1})_{ef}{}^{cd}O^{-1}_{cd}{}^{ij} - \frac{1}{2\omega_{ij}}O_{kk}{}^{ab}\frac{\partial \omega_{ij}}{\partial \lambda_k} \\
&= \frac{1}{2}O_{kk}{}^{ab}\sum_{ij}\frac{1}{\omega_{ij}}\frac{\partial \omega_{ij}}{\partial \lambda_k} - \frac{1}{2\omega_{ij}}O_{kk}{}^{ab}\frac{\partial \omega_{ij}}{\partial \lambda_k} \\
&= 0
\end{aligned} \tag{B.1.6}$$

Thus we have shown that term 1 is equal to 0.

B.2 Term 2

Term 2, which is (4.2.4), consists of three terms, which we can call term-A, term-B and term-C. These are

$$\begin{aligned}\text{Term A} &= [P^{1^A}, [P^{1A}, \omega_C{}^D]] < X_2{}^C X_{2D} > \\ \text{Term B} &= \frac{1}{2}(P_1^A \frac{1}{\omega_C})[P_{1A}, \omega_C] \\ \text{Term C} &= \left\langle \left\{ -\frac{1}{2\omega_C} \left[P_1^A, [P_{1A}, \omega_C] \right] \right\} \right\rangle_2 \end{aligned} \tag{B.2.1}$$

The details of calculations of these terms are given below.

Term A

$$\begin{aligned}\text{Term A} &= -\frac{1}{2}(\omega^{-1})^C{}_D[iP^{1^A}, [iP^{1A}, \omega_C{}^D]] \\ &= -\frac{1}{2}(O_D{}^L \omega^{-1}{}_L (O^{-1})_L{}^C)[iP^{1^A}, [iP^{1A}, \omega_C{}^D]] \\ &= -\frac{1}{2}(V_{el_2} V^\dagger_{l_1 f} \omega^{-1}_{l_1 l_2} V_{dl_1} V^\dagger_{l_2 c})[iP^{1^A}, [iP^{1A}, \omega_C{}^D]] \end{aligned} \tag{B.2.2}$$

Now we need to calculate the commutator $[iP^{1^A}, [iP^{1A}, \omega_C{}^D]]$, for which we need (B.1.5), which is rewritten as

$$\begin{aligned}[iP_{1A}, \omega_C{}^D] =& V_{ak} V^\dagger_{pb} V_{cp} V^\dagger_{id} \Big(\frac{\omega_{ik} - \omega_{ip}}{\lambda_k - \lambda_p}\Big) V_{fi} V^\dagger_{ke} \\ &+ V_{ak} V^\dagger_{pb} V_{ci} V^\dagger_{kd} \Big(\frac{\omega_{ik} - \omega_{ip}}{\lambda_k - \lambda_p}\Big) V_{fp} V^\dagger_{ie} \\ &+ V_{ak} V^\dagger_{kb} V_{cj} V^\dagger_{id} \Big(\frac{\partial \omega_{ij}}{\partial \lambda_k}\Big) V_{fi} V^\dagger_{je} \end{aligned} \tag{B.2.3}$$

Using this we can write

$$\begin{aligned}[iP^{1^A}, [iP^{1A}, \omega_C{}^D]] =& \frac{V_{bl} V^\dagger_{ma}}{\lambda_l - \lambda_m} \hat{R}_{lm} \Bigg(V_{ak} V^\dagger_{pb} V_{cp} V^\dagger_{id} \Big(\frac{\omega_{ik} - \omega_{ip}}{\lambda_k - \lambda_p}\Big) V_{fi} V^\dagger_{ke} \\ &+ V_{ak} V^\dagger_{pb} V_{ci} V^\dagger_{kd} \Big(\frac{\omega_{ik} - \omega_{ip}}{\lambda_k - \lambda_p}\Big) V_{fp} V^\dagger_{ie} \\ &+ V_{ak} V^\dagger_{kb} V_{cj} V^\dagger_{id} \Big(\frac{\partial \omega_{ij}}{\partial \lambda_k}\Big) V_{fi} V^\dagger_{je} \Bigg) \end{aligned}$$

$$+ V_{bl}V^{\dagger}_{la}\frac{\partial}{\partial\lambda_l}\Bigg(V_{ak}V^{\dagger}_{pb}V_{cp}V^{\dagger}_{id}\Big(\frac{\omega_{ik}-\omega_{ip}}{\lambda_k-\lambda_p}\Big)V_{fi}V^{\dagger}_{ke}$$
$$+V_{ak}V^{\dagger}_{pb}V_{ci}V^{\dagger}_{kd}\Big(\frac{\omega_{ik}-\omega_{ip}}{\lambda_k-\lambda_p}\Big)V_{fp}V^{\dagger}_{ie}$$
$$+V_{ak}V^{\dagger}_{kb}V_{cj}V^{\dagger}_{id}\Big(\frac{\partial\omega_{ij}}{\partial\lambda_k}\Big)V_{fi}V^{\dagger}_{je}\Bigg) \quad \text{(B.2.4)}$$

This calculated and substituted in (B.2.2) gives

$$\text{TermA} = -\left(\frac{2}{\omega_{ik}}\frac{\omega_{il}-\omega_{ik}}{(\lambda_k-\lambda_l)^2} + \frac{1}{\omega_{ij}}\frac{1}{\lambda_k-\lambda_p}\frac{\partial\omega_{ij}}{\partial\lambda_k} + \frac{1}{2\omega_{ij}}\frac{\partial^2\omega_{ij}}{\partial{\lambda_k}^2}\right) \quad \text{(B.2.5)}$$

Term B

$$\begin{aligned}
\text{Term B} &= \frac{1}{2}(P^A_1\frac{1}{\omega_C})[P_{1A},\omega_C] \\
&= \frac{1}{2}(-i\frac{\partial}{\partial X_{1A}}\frac{1}{\omega_C})(-i\frac{\partial}{\partial X^A_1}\omega_C) \\
&= -\frac{1}{2}(-\frac{1}{\omega_C^2}\frac{\partial}{\partial X_{1A}}\omega_C)(\frac{\partial}{\partial X^A_1}\omega_C) \\
&= \frac{1}{2}\frac{1}{\omega_{ij}^2}\sum_p(V_{bp}V^{\dagger}_{pa}\frac{\partial}{\partial\lambda_p}\omega_{ij})\sum_p(V_{ap}V^{\dagger}_{pb}\frac{\partial}{\partial\lambda_p}\omega_{ij}) \\
&= \frac{1}{2}\sum_p\sum_{ij}\Big(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\Big)^2
\end{aligned} \quad \text{(B.2.6)}$$

Term C

In order to calculate term C we have to calculate $[P^{1A},[P_{1A},\omega_C]]$, which is given below

$$\begin{aligned}
-[P^{1A},[P_{1A},\omega_C]] &= [iP^{1A},[iP_{1A},\omega_C]] \\
\text{Now,}[P_{1A},\omega_C] &= \frac{\partial}{\partial X^{ab}}\omega_{ij} \\
&= \frac{\partial}{\partial X_{ba}}\omega_{ij} \\
&= V_{al}V^{\dagger}_{lb}\frac{\partial\omega_{ij}}{\partial\lambda_l} \\
\text{Therefore, }[iP^{1A},[iP_{1A},\omega_C]] &= \frac{\partial}{\partial X_{ba}}(V_{al}V^{\dagger}_{lb}\frac{\partial\omega_{ij}}{\partial\lambda_l})
\end{aligned}$$

$$= \frac{V_{bk}V^{\dagger}_{pa}}{\lambda_k - \lambda_p}\hat{R}_{kp}(V_{al}V^{\dagger}_{lb}\frac{\partial\omega_{ij}}{\partial\lambda_l})$$
$$+ V_{bk}V^{\dagger}_{ka}\frac{\partial}{\partial\lambda_k}(V_{al}V^{\dagger}_{lb}\frac{\partial\omega_{ij}}{\partial\lambda_l})$$
$$= \frac{2}{\lambda_k - \lambda_p}\frac{\partial\omega ij}{\partial\lambda_k} + \frac{\partial^2\omega_{ij}}{\partial\lambda_k^2} \tag{B.2.7}$$

Therefore, using this expression, we get term C as

$$\text{Term C} = \frac{1}{\omega_{ij}}\frac{1}{\lambda_k - \lambda_p}\frac{\partial\omega_{ij}}{\partial\lambda_k} + \frac{1}{2\omega_{ij}}\frac{\partial^2\omega_{ij}}{\partial{\lambda_k}^2} \tag{B.2.8}$$

Sum of these three terms gives Term 3 as,

$$\text{Term 3} = -\sum_i\sum_{k\neq l}\frac{1}{(\lambda_k - \lambda_l)^2}(1 - \frac{\omega_{il}}{\omega_{ik}}) + \frac{1}{4}\sum_p\sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\right)^2 \tag{B.2.9}$$

B.3 Term 3

In term 3, which is (4.2.6), the product of second term in first bracket with the second bracket is zero using the results derived for term 1. Thus this equation can be rewritten as

$$-\frac{1}{4}\Big([iP_{1A},\omega_C{}^D][iP^{1A},\omega_E{}^F] < X_2^C X_{2D} X_2^E X_{2F} >$$
$$-\frac{1}{2\omega_E}[iP_{1A},\omega_C{}^D][iP^{1A},\omega_E] < X_2^C X_{2D} > \Big)$$
$$= -\frac{1}{4}\Big([iP_{1A},\omega_C{}^D][iP^{1A},\omega_E{}^F](< X_2^C X_{2D} >< X_2^E X_{2F} >$$
$$+ < X_2^C X_2^E >< X_{2D}X_{2F} > + < X_2^C X_{2F} >< X_2^E X_{2D} >)$$
$$-\frac{1}{2\omega_E}[iP_{1A},\omega_C{}^D][iP^{1A},\omega_E] < X_2^C X_{2D} > \Big) \tag{B.3.1}$$

In the above expression the following term is equal to zero, using the result of term 1.

$$[iP_{1A},\omega_C{}^D] < X_2{}^C X_{2D} > \Big([iP^{1A},\omega_E{}^F] < X_2{}^E X_{2F} > -\frac{1}{2\omega_E}[iP^{1A},\omega_E]\Big) = 0$$

Thus (B.3.1)reduces to

$$-\frac{1}{16}\Big(\underbrace{[iP_{1A},\omega_C{}^D][iP^{1A},\omega_E{}^F](\omega^{-1})^{CE}(\omega^{-1})_{DF}}_{\text{Term A}}$$

$$+ \underbrace{[iP_{1A}, \omega_C{}^D][iP^{1A}, \omega_E{}^F](\omega^{-1})^C{}_F(\omega^{-1})_D{}^E}_{\text{Term B}}\Big) \tag{B.3.2}$$

Solving for term A which is given as

$$\begin{aligned}
&-\frac{1}{16}[iP_{1A}, \omega^{CD}][iP^{1A}, \omega_{EF}](O^{-1})_C{}^L(\omega^{-1})_L(O)_L{}^E(O^{-1})_D{}^M(\omega^{-1})_M(O)_M{}^F \\
&= -\frac{1}{16}\Big([iP_{1A}, \omega^{CD}](O^{-1})_C{}^L(O^{-1})_D{}^M[iP^{1A}, \omega_{EF}](O)_L{}^E(O)_M{}^F\Big)(\omega^{-1})_L(\omega^{-1})_M
\end{aligned} \tag{B.3.3}$$

Substituting the expression for $[iP_{1A}, \omega^{CD}] = [iP_{1A}, \omega_C{}^D]$ with $C = (cd) \to (dc)$ and is given in (B.1.5), we get

$$\begin{aligned}
[iP_{1A}, \omega^{CD}](O^{-1})_C{}^L(O^{-1})_D{}^M &= O_{m_2 l_1, ab}\Big(\frac{\omega_{m_1 m_2} - \omega_{m_1 l_1}}{\lambda_{m_2} - \lambda_{l_1}}\Big)\delta_{l_2 m_1} \\
&+ O_{m_1 l_2, ab}\Big(\frac{\omega_{m_2 l_2} - \omega_{m_2 m_1}}{\lambda_{l_2} - \lambda_{m_1}}\Big)\delta_{l_1 m_2} + O_{kk, ab}\delta_{l_2 m_1}\delta_{l_1 m_2}\frac{\partial \omega_{l_1 l_2}}{\partial \lambda_k}
\end{aligned} \tag{B.3.4}$$

Similarly,

$$\begin{aligned}
[iP^{1A}, \omega_{EF}](O)_L{}^E(O)_M{}^F &= O^{ab}_{m_2 l_1}\Big(\frac{\omega_{m_1 m_2} - \omega_{l_1 l_2}}{\lambda_{m_1} - \lambda_{l_2}}\Big)\delta_{l_1 m_2} \\
&+ O^{ab}_{m_2 l_1}\Big(\frac{\omega_{l_2 l_1} - \omega_{m_2 m_1}}{\lambda_{l_1} - \lambda_{m_2}}\Big)\delta_{l_2 m_1} + O^{ab}_{pp}\delta_{l_1 m_2}\delta_{l_2 m_1}\frac{\partial \omega_{l_1 l_2}}{\partial \lambda_p}
\end{aligned} \tag{B.3.5}$$

substituting (B.3.4) and (B.3.5) in (B.3.3), we get result for term A as

$$\begin{aligned}
\text{Term A} = &\sum_{l_2}\sum_{l_1 \neq m_2}\Big(\frac{\omega_{l_2 m_2} - \omega_{l_1 l_2}}{\lambda_{m_2} - \lambda_{l_1}}\Big)^2\frac{1}{\omega_{l_1 l_2}}\frac{1}{\omega_{l_2 m_2}} \\
&+ \sum_{l_1}\sum_{l_2 \neq m_1}\Big(\frac{\omega_{l_1 m_1} - \omega_{l_1 l_2}}{\lambda_{m_1} - \lambda_{l_2}}\Big)^2\frac{1}{\omega_{l_1 l_2}}\frac{1}{\omega_{l_1 m_1}} \\
&+ \sum_{k}\sum_{l_1 l_2}\Big(\frac{1}{\omega_{l_1 l_2}}\frac{\partial \omega_{l_1 l_2}}{\partial \lambda_k}\Big)^2
\end{aligned} \tag{B.3.6}$$

Similarly,

$$\begin{aligned}
\text{Term B} = &\sum_{l_1}\sum_{l_2 \neq m_2}\Big(\frac{\omega_{l_1 m_2} - \omega_{l_1 l_2}}{\lambda_{m_2} - \lambda_{l_2}}\Big)^2\frac{1}{\omega_{l_1 l_2}}\frac{1}{\omega_{l_1 m_2}} \\
&+ \sum_{l_2}\sum_{l_1 \neq m_1}\Big(\frac{\omega_{l_2 m_1} - \omega_{l_1 l_2}}{\lambda_{m_1} - \lambda_{l_1}}\Big)^2\frac{1}{\omega_{l_1 l_2}}\frac{1}{\omega_{l_2 m_1}} \\
&+ \sum_{p}\sum_{l_1 l_2}\Big(\frac{1}{\omega_{l_1 l_2}}\frac{\partial \omega_{l_1 l_2}}{\partial \lambda_p}\Big)^2
\end{aligned} \tag{B.3.7}$$

Substituting, (B.3.6) and (B.3.7) in (B.3.2), and after simplification gives,

$$\text{Term 3} = \frac{1}{2}\sum_{i}\sum_{k\neq l}\frac{1}{(\lambda_k-\lambda_l)^2}(1-\frac{\omega_{il}}{\omega_{ik}}) - \frac{1}{8}\sum_{p}\sum_{ij}\left(\frac{1}{\omega_{ij}}\frac{\partial\omega_{ij}}{\partial\lambda_p}\right)^2 \tag{B.3.8}$$

Appendix C

Method II for Calculation of $< \mathrm{Tr}(X_1^2) >$ to $O(\lambda)$

Method II

Substituting

$$\Delta(t) = \frac{1}{2\omega}\left(\Theta(t)e^{-i\omega t} + \Theta(-t)e^{i\omega t}\right)$$

and using

$$\Theta(t)\Theta(t) = \Theta(t)$$

$$\Theta(-t)\Theta(t) = 0$$

$$\Theta(-t)\Theta(-t) = \Theta(-t)$$

$$\begin{aligned}\int \mathrm{d}t[\Delta(t)]^2 =& \frac{1}{4\omega^2}\int_{-\infty}^{\infty} \mathrm{d}t\left(\Theta(t)e^{-2i\omega t} + \Theta(-t)e^{2i\omega t}\right) \\ =& \frac{1}{4\omega^2}\Big(\int_0^{\infty} \mathrm{d}t\left(\Theta(t)e^{-2i\omega t} + \Theta(-t)e^{2i\omega t}\right) \\ &+ \int_{-\infty}^{0} \mathrm{d}t\left(\Theta(t)e^{-2i\omega t} + \Theta(-t)e^{2i\omega t}\right)\Big)\end{aligned}$$

In 0 to ∞ limit $t > 0$. So $\Theta(t) = 1$ and $\Theta(-t) = 0$. So in 0 to ∞ limit

$$\int_0^{\infty} \mathrm{d}te^{-2i\omega t} = \frac{1}{2i\omega}$$

and, in $-\infty$ to 0 limit $t < 0$. So $\Theta(t) = 0$ and $\Theta(-t) = 1$. So

$$\int_0^\infty \mathrm{d}t e^{-2i\omega t} = \frac{1}{2i\omega}$$

$$\int \mathrm{d}t[\Delta(t)]^2 = \frac{1}{4i\omega^3} \tag{C.0.1}$$

Substituting (C.0.1) in (5.2.8) and putting $\frac{1}{2\omega}$ for $\Delta(0)$ gives

$$\left\langle \mathrm{Tr} X_1^2 \right\rangle_{O(\lambda)} = \frac{-\lambda}{2\omega^4} \tag{C.0.2}$$

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