South African Journal of Childhood Education
ISSN: (Online) 2223-7682, (Print) 2223-7674

Conceptual and representational confusion: An analysis of three foundation phase teachers' descriptions of how they teach division

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Dates:
Received: 16 July 2019
Accepted: 04 Oct. 2020
Published: 17 Dec. 2020

How to cite this article:
Ramsingh, V., 2020, 'Conceptual and representational confusion: An analysis of three foundation phase teachers' descriptions of how they teach division,' *South African Journal of Childhood Education* 10(1), a796. https://doi.org/10.4102/sajce.v10i1.796

Background: In the foundation phase, division is split along the lines of grouping (quotition) and sharing (partition). Curriculum and Assessment Policy Statements (CAPS) recommends that representations such as drawings, concrete apparatus, and symbols are used to teach all math concepts, including division.

Aim: In this article, the author investigates how three foundation phase teachers describe teaching division and the representations they would use to do this.

Setting: Grade 1 and Grade 2 teachers in one previously disadvantaged urban school were selected for this study.

Methods: Semi-structured interviews were used to gather information about how the teachers say they teach division. They were invited to use or talk about any representations such as actions, apparatus, drawings or writings that they would use during their teaching.

Results: Using a grounded theory approach, analysis of the data showed that all three teachers talked about how they would transform the mathematics in a division word problem into actions, drawings and symbols. However, none of them was able to use their representations to find the answer to the problem or to provide a division number sentence. The representations they used were relegated to end results and served no purpose in solving the mathematical problems.

Conclusion: The results show areas in which teachers need support:

- Their own mathematical knowledge – specifically division in the case of this study
- How to teach division in the foundation phase.

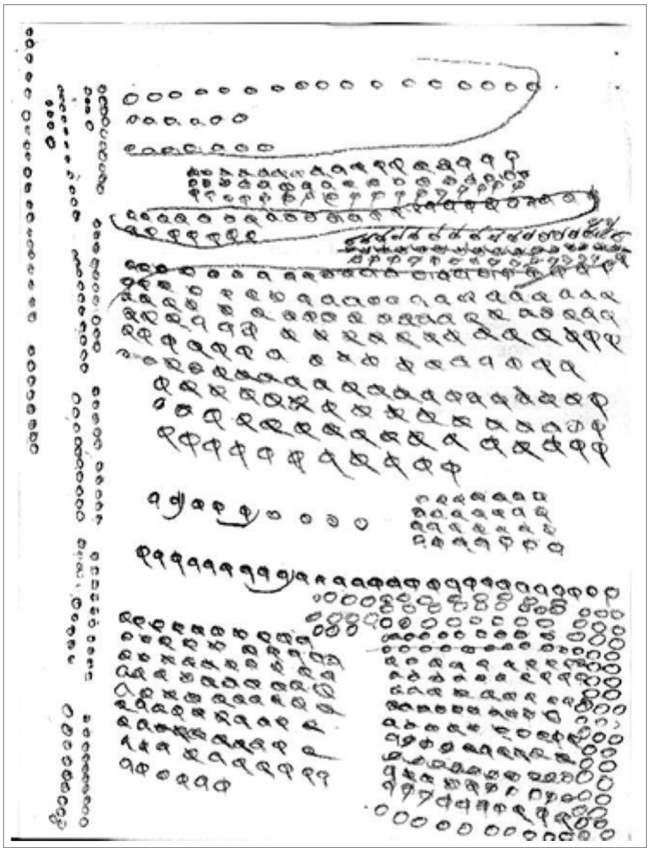
They lack both content knowledge and pedagogical knowledge.

Keywords: foundation phase teachers; representations; transformation; division; grouping; sharing; teacher knowledge; pedagogy.

Introduction and background

Several researchers worldwide have pointed out the difficulties learners experience with division (Harel & Confrey 1994; Lamon 2007; Ma 2010). These difficulties have been linked to a range of sources, including poor teacher understanding and knowledge (Venkat & Spaull 2015), inadequate teaching (Mulligan & Watson 1998; Simon 1993) and incorrect language use (Askew 2008). At a local level, the Annual National Assessments (ANA) Diagnostic Assessment Report for 2014 (Department of Basic Education 2014:9) showed that learners' struggles in working with division continued into the intermediate phase where learners used 'repeated subtraction for … division … even when working with large numbers'. The report found no change in learners' approaches since 26 years earlier when Schollar (2008) revealed that 40% of Grade 5 and 11.5% of Grade 7 learners relied entirely on unit counting. Figure 1 shows a Grade 5 learner's representation for multiplication/division where all numbers were reduced to single tallies.

The Curriculum and Assessment Policy Statements (CAPS) (DBE 2011:286) recommends that learners use representations such as 'drawings and concrete apparatus to show their solutions'. Whilst the intention behind this recommendation is that the representation should progress into becoming a referential base for formal mathematical reasoning, which leads to abstract thinking, overreliance on, for example, unit counting, as seen in Figure 1, does not help the learner to progress to more abstract thinking. The way in which teachers guide learners towards using



Source: Schollar, E., 2008, *Final report: The primary mathematics research project 2004–2007: Towards evidence-based educational development in South Africa*, Eric Schollar & Associates, Johannesburg

FIGURE 1: A grade 5 learner's work on multiplication/division.

more formal transformations of representations is therefore of interest.

In this study, the author investigates how foundation phase teachers say they would use representations to teach division in the foundation phase. Literature on multiplication leading to the two division models used in the foundation phase, namely grouping (quotition) and sharing (partition), the use of representations in teaching and learning mathematics, teacher knowledge and the national curriculum is drawn upon to examine the following key questions:

- What are the three foundation phase teachers' understandings of division?
- What do they say about if and how they use representations to teach division?

Theoretical and conceptual background

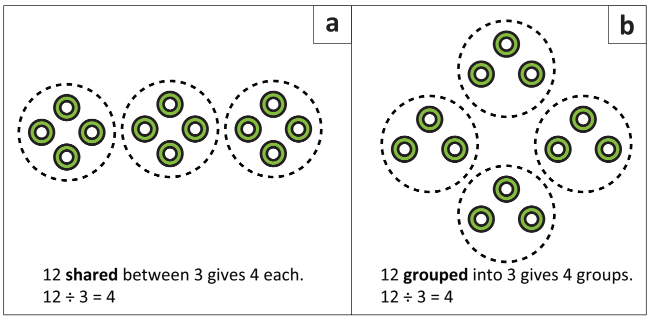
What is division

There is general agreement amongst scholars that the two multiplicative relations of multiplication and its inverse, division, are based on a constant one-to-many and many-to-one relationship between the two quantities (Nunes & Bryant 1996; Park & Nunes 2001; Vergnaud 1983). In other words, in the many-to-one relationship of division, as one value decreases, so too does the other value decreases proportionally. For example, in the relationship of three wheels being used to make one tricycle, three wheels correspond with one tricycle. For every set of three wheels, there will always be only one tricycle. The 3:1 relationship between the wheels and tricycles will always remain constant.

As a mathematical operation, division involves the act of splitting something into equal parts. Division can involve either partition or quotition with each one using *a different solution strategy.* The idea behind partition is one of sharing whilst the idea behind quotition is repeated subtraction (of equal-sized groups). Whether a division problem can be understood in terms of partition or quotition depends on the semantic structure of the problem (Bell, Fischbein, & Greer 1984; Vergnaud 1988).

With reference to partition, 'there is evidence that young children's first successful experiences with one-to-one correspondence come through sharing' (Nunes & Byrant 2009:13), for example, when they share four sweets between themselves and another by giving out one sweet at a time to each person in the typical *one for me, one for you* scenario. In terms of division, the aim of this action that uses a partitative model would be to determine how many objects are in each group if all the objects are shared equally amongst the groups. An example of three children sharing 12 sweets is illustrated in Figure 2a below.

In the quotitive model, in contrast, the 'relation sought is a within measure space relation' (Subramaniam 2019:37), where the aim is to determine how many times or how many groups of a given quantity is contained within a larger quantity. In the question, *I have 12 sweets. I eat 3 sweets a day. How long will the sweets last?* the solution can be arrived at by using the 'building-down' model (Figure 2b) where 3 sweets are repeatedly subtracted from 12 until nothing is left over (Haylock & Cockburn 2013) or the 'building-up' model where 3 sweets are repeatedly added until 12 is reached. Both the 'building-up' and 'building-down' models make use of counting in multiples, in this case, multiples of 3 (Fischbein et al. 1985). What can be challenging for learners is that although the procedures for partition and quotition are different, they 'both are represented by the same abstract symbolisations' (12 ÷ 3 = 4*)*



Source: Department of Basic Education (DBE), 2011, *Curriculum and assessment policy statement: Grades R-3 mathematics*, Department of Basic Education, Pretoria

FIGURE 2: Suggested images that can be used to explain the differences between sharing and grouping.

(Anghileri 1995:11). Understanding the type of problem requires awareness of the messages that are conveyed through the language used. In the partitive situation (Figure 2a), the answer of 4 (group size) is accessed from a drawing showing 3 *groups of 4*, but in the quotative situation (Figure 2b), the answer of 4 (number of groups) is accessed from a drawing showing *4 groups of* 3, espousing the idea that careful understanding of the problem is essential to access the answer correctly.

Askew (2008) reminds us that whilst the arithmetic of division can be used to solve both quotition and partition types of problems, the fundamental understanding of the problem situation rests in how language is used to state and interpret it. He cites an example of a teacher who modelled how to work out the number of boxes that could be filled with 42 cubes if one box held 7 cubes. Whilst the teacher's actions modelled division through quotition (repeated subtraction), her 'running commentary' was of the cubes being 'shared out'. He cautions that Teachers need to appreciate the need for precision and not muddle the two approaches through inappropriate marrying up of words and actions (Askew 2008:23) so that learners are ultimately able to distinguish between when a grouping/repeated subtraction strategy or a sharing strategy would be appropriate.

Based on the understanding that the concepts of division are complex and difficult to teach and learn, the question is what strategies are there that teachers could make use of to make their teaching successful. A wide range of research supports the idea that using representations as tools help to scaffold the learner's development of maths constructs from concrete to abstract. This is discussed below.

Using representational tools to teach division

Representations in mathematics refer to both the internal and external manifestations of mathematical concepts. Internal manifestations are the 'cognitive schemata that are developed by a learner through experience' (Pape & Tchoshanov 2001:119), for example, understanding the meaning of the number *five* in the absence of any sensory stimuli. Seeger (1998:311) explains that external representations 'are structurally equivalent and act in place of something else' and advocates that representations should be used as vehicles for understanding mathematical content in multiple ways. Examples of external representations include word problems, role playing, drawings, symbols and algebraic equations.

Bruner's (1966) model of three levels of engagement proposes that learners should progress from the enactive stage (manipulating concrete materials) through to the iconic stage (creating drawings to represent the concrete materials) and finally the symbolic stage (using mathematical symbols to represent the objects in the situation). Pape and Tchoshanov (2001) elaborate on how this progression takes place as explained below:

> Through the use of analogy, transformation, and simplification, new understandings are built from existing knowledge. … [S]tudents must map or transform the manipulation of the materials onto the symbolic steps of the mathematical operation and simplify these manipulations through the use of the conventional algorithm. This process is successful only to the degree that the concrete material procedures are analogous to procedures with symbols and the degree to which this connection is made explicit for the learner. (Pape & Tchoshanov 2001:123)

For this transformation into the different representations to be of value, learners need to be taught how to construct and interpret them (Greeno & Hall 1997:361-362) and to reason between them (Barmby et al. 2009). It is within the interaction between problem situations and other representations that mathematical understanding develops. As the child gains experience in working with the representational tools, his or her attention increasingly shifts to focusing on the relationships involved. In this way, the representation progresses into becoming a referential base for the formal mathematical reasoning that follows and eventually the learner will no longer need to think about the representational tools to give meaning to the mathematical processes (Gravemeijer & Stephan 2002). It is, however, important to remember Gravemeijer's earlier critique of the use of representations where he explained that:

> [*E*]xternal representations do not come with intrinsic meaning, but that the meaning of external representations is dependent on the knowledge and understanding of the interpreter. (Gravemeijer 1994:8)

In other words, the teacher needs to already have her own mathematical knowledge and understanding in place to make sense of the external representations. To teach the children to connect concrete representations to abstract mathematical concepts, the teacher also needs pedagogical knowledge. Her own skill in moving flexibly forwards and backwards between a range of representations, for example, representing 2 boxes each having 5 sweets as ooooo ooooo or 2 groups of 5 or 5 + 5 or 5 × 2 must form the basis of her teaching.

Using representations as tools to exemplify mathematical situations, the teacher is able to link mathematics to what the child already knows. The teacher uses these representations as tools to organise activities in which everyday life situations are structured in terms of their mathematical relationships. The representational tools focused on in this study are word problems, enactment with concrete apparatus, drawings and symbolic representations.

Word problems are a valuable representational tool for providing contexts that mirror real-life mathematical situations (Greer 1997). In this article, word problems are used to represent the two distinct division models that are investigated. The teacher sets tasks in the form of word problems in which interpretations and solutions depend on understanding what actions are required in that context.

Concrete apparatus such as counters, stones or bottle tops can all be used to represent, for example, crayons or elephants. The counter, stone or bottle top stands in place of the crayon or the elephant in the situation described in the word problem.

Drawings are iconic representations (Bruner 1966). Informal sketches such as tallies to represent crayons or unorganised crisscross sketches that show sharing are examples of drawings. Drawings might include words and/or symbols that the teacher writes to support her explanation.

Finally, mathematical symbols are the numerals and operational signs that stand in place of the drawing, action or object. Pape and Tchoshanov (2001) state that:

> [*S*]ymbols and symbol systems support the cognitive activity by reducing the cognitive load (i.e., by reducing all that the individual must think about to accomplish a task), clarifying the problem space, and revealing immediate implications. (p. 63)

In other words, each of the individual symbols in 8 ÷ 2 = 4 can stand in the place of the words, the apparatus, actions or the drawings that represent the mathematical problem situation. Curriculum and Assessment Policy Statements' position regarding the use of representations is that:

> Solving problems in context [word problems] enables learners to communicate their own thinking orally and in writing through drawings and symbols. (DBE 2011:9)

Mathews (2014:87) observes that when there is coherence between the situation depicted by the word problem, division model (grouping or sharing), the teacher's actions, representation and teacher's talk 'learners seem better able to enact basic procedures independently'. This points to the importance of the teacher's knowledge for teaching.

Teacher knowledge for teaching division

The concept of pedagogical content knowledge (PCK) has its roots in the seminal work of Shulman (1986, 1987). Since then, there has been much research on pinpointing exactly what knowledge is required to be an excellent mathematics teacher. Hill, Rowan and Ball (2005) acknowledge the importance of the *Study of Instructional Improvement*, which was designed to measure elementary teachers' knowledge for teaching mathematics. The developers of the study captured not only the actual mathematics that teachers teach but also the specialised content knowledge (SCK). If the actual mathematics is, for example, *division*, then the SCK would include aspects such as knowledge of how to use diagrams to represent the different division situations or knowing that calculation procedure is appropriate for the problem situation. Hill, Ball and Schilling (2008) further developed their description of SCK to mean the mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods or problems (Hill et al. 2008:377).

To interpret a mathematical situation correctly, teachers of division need to be able to see the 'connections between and amongst concrete situations' (Simon 1993:251).

Whilst the grouping situation of 12 grouped into 3s gives 4 groups and the sharing situation of 12 shared between 3 gives 4 each can be represented by the same mathematical symbols 12 ÷ 3 = 4, teachers need to appreciate that the multiplicative symbolic representation 4 ×__ = 12 is also solved through division. By having access to a wider 'web of knowledge' (Ball, cited in Simon 1993:251) on aspects of multiplication and division, teachers are able to work more flexibly within the mathematical conceptual field.

Moreover, Ball and Bass (2009:11) argue that 'teaching can be more skillful when teachers have mathematical perspective on what lies in all directions, behind as well as ahead'. Teachers therefore need to know not only what learners in their classes need to be taught, they need to know what learners have already been taught and how the skills she is busy imparting will be further developed in the subsequent grades. The CAPS provides guidelines on what and how teachers should teach. Curriculum and Assessment Policy Statements stipulates that for division learners in Grade 1 should (s)olve and explain solutions to practical problems involving equal sharing and grouping with whole numbers up to 5 and with answers that may include remainders (DBE 2011:107).

In Grade 3, the number range extends to 100. Throughout the four grades of the Foundation Phase, learners work with grouping situations where they first discard and later incorporate the remainder in the answer. In sharing situations, learners in all grades begin with discarding the remainder. Then, learners in Grades 2 and 3 progress to sharing – leading to fractions. In Grade 3, repeated subtraction is transformed into the division algorithm where the division sign (÷) is introduced. Curriculum and Assessment Policy Statements warns that before the division sign is introduced, learners must understand that (p)roblems that involve sharing are often about sharing equally and how much each one gets. Problems that involve (grouping) (CAPS refers to sharing here too but the author believes that in this statement *sharing* was meant to be *grouping*) is often about how many groups can be made (DBE 2011:349).

In developing an understanding of the concept of division as sharing or as grouping, CAPS recommends that learners use the number range to determine whether using repeated subtraction would be an appropriate strategy (DBE 2011:465) but does not provide any guidance on how this should be performed. Whilst CAPS' recommendation of using repeated subtraction for division may work well within certain number ranges, the decision regarding whether or not to use repeated subtraction should more importantly be based on the type of division problem that needs to be solved. Repeated subtraction is more clearly connected to whole numbers as in the example *Ben bought 11 chocolate bars. He eats 4 chocolate bars a day. How many days will the chocolate bars last?* (They will last 2 days, and 3 cholcolate bars will be left over).

Curriculum and Assessment Policy Statements' recommendation of using repeated subtraction for the division for smaller numbers creates confusion in *sharing situations*, which can naturally lead to fractions. In the problem situation, 'Share 11 chocolate bars amongst 4 friends so that they all get the same amount of chocolate bar and there is nothing left over' (DBE 2011:317) difficulty will arise in sharing the last 3 chocolate bars when using a repeated subtraction approach.

As argued above, the teaching and learning of division are difficult. Curriculum and Assessment Policy Statement's guidelines regarding what needs to be taught presents its own set of challenges for South African teachers. In considering the local primary mathematics context and what knowledge teachers require to teach primary mathematics well, Alex and Roberts (2019:61-62) adapted Hart's description of what 'mathematical knowledge for teaching' (MKfT) would entail.

According to these scholars, primary teachers must have deep knowledge of the points listed below. The author has added an example for each point to show how it could apply to the teaching of division:

- The mathematical topics at the primary school level that include a robust understanding of why particular concepts and procedures within each topic make sense mathematically (for example, skip counting as a precursor to learning the times tables);
- the future use and further development of this content in previous and subsequent grade levels (for example, how learning to count in multiples of five in Grade 1 can be used for reading time in Grade 2);
- appropriate representations, suitable classroom contexts, alternative approaches and methods (such as might be used by children in solving problems); (for example, why when introducing division, actions and drawings that show sharing are more powerful than the division symbol [÷] in creating conceptual understanding);
- interconnections and interdependence amongst the content and topics, as well as how a new concept can be built upon other existing ideas (for example, how fractions develop from the idea of equal sharing); and
- when the mathematical ideas are developmentally appropriate for children to learn (for example, understanding that procedural fluency in sharing as a model of division needs to be in place before fractions can be understood) (Alex & Roberts 2019:62).

Alex and Roberts' description of mathematical knowledge for teaching helps to highlight the aspects of teachers' knowledge necessary to teach division successfully in the foundation phase.

In the remainder of this article, the author investigates how foundation phase teachers use representations to teach division.

Methodology

The study discussed in this article draws on one component of the data collected as part of broader doctoral research that explores teachers' understandings of multiplicative reasoning before and after an intervention. Data collection began with a pre-intervention interview in which all nine foundation phase teachers in one school were interviewed to establish their understanding of various aspects of multiplicative reasoning. The school had three teachers in each grade from Grade 1 to Grade 3. The data of the Grade 3 teachers have not been used here because this will form part of a future study. During the study, one Grade 1 teacher left the school, and the data of another Grade 1 teacher was incomplete. A Grade 2 teacher asked to be excluded from the study. The data discussed in this article are based on interviews with the three remaining teachers: One teaching Grade 1 and two teaching Grade 2.

The data were gathered through semi-structured interviews in which the author explored how the participants in the study taught division. The teachers were interviewed separately, answering one question at a time in the sequence presented below.

The questions asked were as follows:

- If you had to teach $4 \times 5 =$ ___ how would you go about it?
- If you had to teach $20 \div 5 =$ ___ how would you go about it?
- How would you teach learners to answer this question: *Tom had 20 crayons. He had 5 crayons of each colour. How many colours of crayons does he have?*
- How would you teach learners to answer this question: *Tom has 20 crayons. He puts them into 5 boxes. If each box has the same number of crayons, how many crayons are in each box?*

Before the interview, the author prepared separate cards on which these questions were written. Each teacher was shown a card with the question and asked to explain how she would teach the content on the card to a foundation phase class. She was provided with writing paper and invited to use it if she needed to draw or write anything as she would during her teaching. As each teacher explained what she would do, the author asked probing questions to gain deeper insight when necessary. In cases where a teacher said that she would not teach the given content to the grade she taught, she was asked to show how she imagined she would teach that aspect if she had to teach it at all.

In the school where the study was conducted the Language of Learning and Teaching is English, but this was not the mother tongue of any of the teachers interviewed nor of the learners at the school. The interviews were audio-recorded and transcribed. From these transcripts, the author identified words, phrases, quotations or data segments that related to her four chosen interview questions for this article. Using these, the author iteratively developed categories/themes to classify her data based on the content, concepts or theory embedded in them. What became clear in the classification process was that all the teachers said and showed how they would use representations to teach each of the questions posed to them. It is of interest, however, that the representations they purported to use served no purpose in solving the problems at hand, but instead created further confusion. The two major themes that emerged from the data draws attention to the teachers' own knowledge of division and their PCK. In the rest of the article, these two themes are discussed with examples from the interviews and used to develop recommendations with which the article concludes.

Ethical consideration

Approval to conduct the study was received from the University of the Witwatersrand and Gauteng Department of Education, reference 2016ECE005D (Wits) D2017/016 (GDE), 26 April 2016.

Ethics clearance for these interviews, which were conducted under the umbrella of a larger mathematics education project, was obtained from the university in which this project is located. All participants were informed that their involvement was entirely voluntary and that they could withdraw from the study at any time. In order to protect the identities of the teachers, pseudonyms have been used, and all are referred to as female.

Results

Findings from the study show that whilst all three teachers used a range of representation to illustrate the mathematical problem presented, none of them was able to use their representation to answer the division questions presented. Furthermore, their conceptual knowledge and understanding necessary for teaching division in the foundation phase need further development. The author now presents each teacher's responses to the problems separately.

Participant 1: Fatima, Teacher Grade 2

When the author asked teacher Fatima how she would teach learners to solve Question 3 (grouping situation), she was able to represent the situation in a drawing and verbally hint at the answer. Question 4 (sharing situation), however, presented a greater challenge. Here we see how her interview unfolded.

To explain how she would teach Question 3, Fatima used drawings to show how the situation could be enacted. She stated that she would call a learner to the front, tell the class that he is Tom and then give him 5 crayons of each colour. As she explained, she drew 5 tallies, made a circle around the group of tallies and labelled the group *Red*, *Blue, Yellow, Black* (Figure 3a). She numbered the groups 1, 2, 3, 4 and counted each group up in fives until she reached 20. She concluded that to 'see how many 5s goes into 20 … they will count the groups … 1, 2, 3, 4'. The accompanying number sentence she wrote 5 + 5 + 5 + 5 = 20.

Then moving onto the next problem Participant 1 read the question twice before continuing.

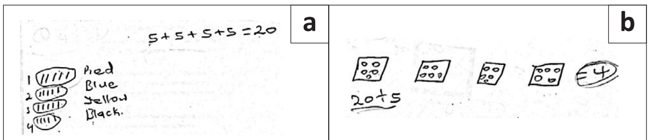
> 'How many crayons … 4, 4, 4 so that they can make twenty. I think I'll give them something to put the crayons inside, and then I'll give them 20 crayons … I'll ask them to put 5 crayons in each box ... So they'll put 5 crayons in the first box, 5 crayons in another box … second box, the third box, and the fourth one, and then they'll count.' (Participant 1, Teacher Grade 2, November 2015)

Having worked out mentally that the answer would be 4, Teacher Fatima proceeded to explain how she would teach learners to solve the problem. In describing and explaining her envisaged teaching and in her drawings for this question, the sharing situation slipped into one of grouping, where 5 boxes were converted into 5 in each box.

She then asked 'Is this a division also?' and after the author confirmed that it was she who drew four squares (boxes), then five small circles in each square (Figure 3b). Finally, she counted the 4 boxes she had drawn and concluded that 'the answer is equals to four' and wrote = 4 after the last box. The author asked Participant 1 for a number sentence. Although her drawings for both questions were similar, the transformation from drawings to symbols (number sentence) for this division problem threw Teacher Fatima off balance.

> 'The number sentence we will say it's division, ne. This one of division … [counting the circles in each box] 5 plus 5 plus 5 plus 5 equals to twenty. The answer should be four ne, because we are looking for this four [she circled the '4' she has written after the 4 squares]. Number sentence in division sign? [laughs] *Yoh!* 2, 4, 5, 2, 4, 5 [counting the circles in each box] … the question is twenty divided by … five, *ne*? The question is twenty divided by five *ne*? (writes 20 ÷ 5)
>
> Twenty … 5 plus 5 plus 5 plus 5 (counting the circles in each square) …' (Participant 1, Teacher Grade 2, November 2015)

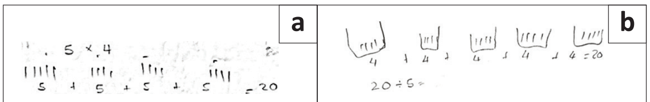
She then began flipping between her drawings for the two questions. She returned to her drawing for Question 3,



a: *Teacher Fatima´s drawing for Question 3* (Tom has 20 crayons. He has 5 crayons of each colour. How many colours of crayons does he have?)

b: *Teacher Fatima´s drawing for Question 4* (Tom has 20 crayons, he puts them into 5 boxes. If each box has the same number of crayons, how many crayons are in each box?)

FIGURE 3: Teacher Fatima's drawings.



a — *Teacher Linda´s drawing for Question 3* (Tom has 20 crayons. He has 5 crayons of each colour. How many colours of crayons does he have?)

b — *Teacher Linda´s drawing for Question 4* (Tom has 20 crayons, and he puts them into 5 boxes. If each box has the same number of crayons, how many crayons are in each box?)

FIGURE 4: Teacher Linda’s drawings.

pointed to the 5 + 5 + 5 + 5 = 20 she had written and resolved ‘I think it will be this one’ but then returned to Question 4, once again pointed to the ‘4’ she has written and uttered ‘but the exact part is this one. We are looking for this. I think the answer should be this one’. She then returned to Question 3, once again pointed at the 5 + 5 + 5 + 5 = 20 she had written and stated. And this one, the author thinks it’s wrong because the answer should be 4. We were looking for 4 because it’s division. We want to know how many 5s gets into 20 … 4 fives. And the answer should be 4 here [pointing to 5 + 5 + 5 + 5 = 20]. So how will you do our number sentence in division sign?

Her inability to read 4 groups of 5 in 5 + 5 + 5 + 5 suggests that for her the drawing and the symbols were not analogous. Looking once again at Question 4, Fatima pointed to the 4 squares with 5 circles in each square, laughed uncomfortably and concluded: ‘I think this one it’s also wrong’. With the symbol 4 not being explicitly visible, Teacher Fatima was unable to produce a division number sentence and instead asked the author ‘*So how will you do our number sentence in division sign?*’

Participant 2: Linda, Teacher Grade 2

Teacher Linda’s actions and drawings for how she would teach Question 3 aligned with the division by grouping situation presented in the question. She drew sets of tallies (Figure 4a) as she explained: ‘So here, … in each group, there will be like five crayons’ (although in reality 3 of the sets had 4 tallies each). Each time as she completed drawing a set of tallies she uttered ‘5 crayons’. She concluded ‘and they [the learners] will say, how many groups are there?’ The author asked Linda what her number sentence would look like and she wrote 5 + 5 + 5 + 5 = 20 below her drawing. Whilst the repeated addition number sentence she wrote echoed her described actions and drawings it did not explicitly provide an answer to the problem situation.

Then to explain how she would teach learners to answer Question 4, Teacher Linda drew 5 boxes and one tally in each box cyclically. As she drew, she explained that she would distribute one crayon into each box until the 20 crayons were finished. She concluded, ‘Then there will be 4 crayons in each box’. The author asked her what her number sentence will look like and this time she wrote 4 + 4 + 4 + 4 + 4 = 20 (Figure 4b). This number sentence too echoed her described actions and drawing but again did not provide an answer to the problem situation.

The author therefore asked her what the answer to the question would be. There was a long silence, so the author offered a prompt.

Researcher: Do you think there might be a division number sentence that could go with those?

Linda: With those two story sums … which one goes with division?

Researcher: Ja, could there be a division number sentence?

Linda: The first one.

Researcher: What would it look like?

Linda: With the first one (Question 3) we are grouping. So we’ll know that at the end it comes to four groups. So it’s the same as … twenty divide[d] by five, which is … we’ll do the … four. Is that correct? (laughs). Sorry, let me check my … okay . (Participant 2, Teacher Grade 2, November 2015)

Participant 2 then shifted her attention from her drawing for Question 3 to her drawing for Question 4.

Linda: I think the division comes with this one where … our answer it’s four … [pointing to the number sentence 4 +4 + 4 + 4 = 20 for Question 4]

Researcher: On second one?

Linda: Ja, the second one. Not the first one (looking at the 5s in the number sentence). (Participant 2, Teacher Grade 2, November 2015)

She then wrote her division number sentence ‘20 ÷ 5 = ’ below the drawing for Question 4. The author pointed to her drawing for Question 3 and asked whether there would be a division number sentence for that as well. To this, she replied ‘*I think it’s the multiplication. Five times four*’ and wrote 5 × 4’ (Participant 2, Teacher Grade 2 November 2015).

For Question 3, Teacher Linda had recognised the relationships between the numbers she has worked with in order to produce a division number sentence. She articulated this relationship by saying ‘They will say how many groups there are’, but when asked to produce a division number sentence and not seeing the symbol 4 anywhere, she did an about turn and pronounced that the number sentence is 5 × 4.

Participant 3: Norma, Teacher Grade 1

Teacher Norma’s story begins at Question 2 where the author asked her how she will teach 20 ÷ 5 =___. To answer the author’s question, she showed how she would teach division by beginning with a drawing, then moved to explaining how this could be performed practically with concrete apparatus. She explained that she would ask learners to ‘*share twenty sweets amongst five children*’ until all 20 sweets were finished. She demonstrated how this would be performed by drawing 20 tallies in a row and five circles below the tallies (Figure 5).

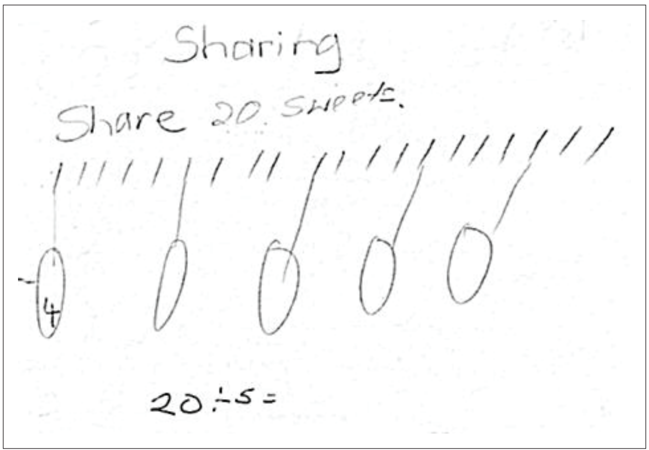


FIGURE 5: Teacher Norma's drawing for share 20 sweets amongst five children.

She then drew lines to connect 5 tallies to one circle each, saying that she will do this '*until they [the tallies] are finished*'. She concluded that to calculate 'how many you have, it is the same as saying, 20 ÷ 5, which will be 4' and wrote '4' in the first circle. Having produced an answer aside from the drawing, the author asked her how she would decide which tally she would join to each circle. She explained that it would be easier to illustrate this practically with the learners where she would call 5 learners to the front of the class. She would then call one more learner to whom she would give a box of 20 marbles (the sweets were now being represented by marbles) to share equally amongst the 5 learners. The learner doing the sharing would distribute one marble at a time to each of the 5 learners cyclically until all the marbles were finished. In the end, she would ask the learners how much each of them had, which would be 'the same as saying, 20 divided by 5 because each one child has got 4, 4, 4, 4, 4'.

The author returned Teacher Norma's attention to the drawing she had completed earlier and asked whether the situation she had just described with the marbles would be represented by a similar drawing to the one she had used for sharing sweets. Teacher Norma sidestepped the author's question about the drawing and once again moved her focus towards producing an answer. She did this by using a grouping strategy that she describes below:

> Then we go back to the counting again in fives. We want to know how many groups of fives we can get from that twenty. Now the number sentence that we are going to write is twenty divided by five [writes 20 ÷ 5 = …] meaning how many groups of five can you get out of twenty … 5, 10, 15, 20 [raises a finger simultaneously as she says each multiple of five]. And I got my 4 [looking at her 4 raised fingers]. So that counting also, the multiples of five, would be able to help the child to work out the division sum ' (Participant 3, Teacher Grade 1, November 2015).

After having demonstrated a sharing situation first in a drawing, then as an enactment, in order to find the answer to the problem Teacher Norma dismissed both these representations and instead resorted to counting in multiples of 5 on her fingers as a solution strategy.

Discussion

Teacher Fatima and Teacher Linda both used drawings to explain how the problem of *How many colours of crayons does he have?* could be represented. What is of interest is that neither of them was able to derive a division number sentence from her drawing. The use of a grouping model for division seeks to establish the number of groups in the situation. In their drawings, each colour was represented by a group of 5 tallies building up to 20 tallies, and both teachers were able to conclude that there were 4 colours. In transforming the tallies into numerals, each group of 5 tallies, already representing 5 crayons, was transformed into the symbolic representation of the numeral '5'. The additive number sentences emerging from Teacher Fatima's and Teacher Linda's drawings was 5 + 5 + 5 + 5 = 20, showing that each group of 5 was created 4 times to reach 20. Whilest both teachers used a 'building-up' model (5 + 5 + 5 + 5 = 20) for division, CAPS suggests that the '[l]earner might use repeated subtraction [20 – 5 – 5 – 5 – 5 = 0] to show how they arrived at an answer' (DBE 2011:225). Although both these symbolic representations of repeated addition and repeated subtraction are acceptable (Fischbein et al. 1985; Haylock & Cockburn 2013), neither of them contains the symbol that correlates with the answer to the problem 'How many colours of crayons does he have?' To demonstrate why 4 is the answer in the symbolic representation, Teacher Fatima and Teacher Linda needed to transform 5 + 5 + 5 + 5 = 20 into 20 ÷ 5 = 4. Their difficulty in reading 4 colours or groups in 5 + 5 + 5 + 5 = 20 led Teacher Fatima to speculate 'I think it's wrong because the answer should be 4' and to Teacher Linda concluding 'I think it's the multiplication … 5 × 4'. For representations to be of any value, with every new transformation, learners 'must connect individual symbols with the objects they represent' (Hiebert in Pape & Tchoshanov 2001:123). In other words, Teacher Linda and Teacher Fatima needed to connect each '5' back to one colour of crayons, so that she could say with confidence that the answer to *How many colours of crayons does he have* is 4, because in their drawings they had already shown 4 groups of 5 crayons, with each group representing one colour.

To explain how she will introduce division through sharing, Teacher Norma used drawings and actions to represent the situation she had described. To determine the answer to the problem, she moved away from her drawing and again from her actions and used counting in multiples of 5 as a solution strategy. When the author drew Teacher Norma's attention to her drawing that carried a sharing model, she did not make the connection between the drawing and her action that followed. Instead, she stated:

> 'Then we go back to the counting again in fives [because] we want to know how many groups of fives we can get from that twenty.' Participant 3, Teacher Grade 1 , November 2015

By disregarding her drawing and her enactment and moving onto counting in multiples Teacher Norma's actions signified that she did not view these representations as tools that help to navigate thinking in the direction of finding a solution.

They were an end in themselves and served no purpose mathematically. For representations to serve as tools, it is essential that clear connections are made between them. For example, when transforming a word problem into a drawing or a numerical symbol, every transformation must connect forwards and back to the objects they represent. By counting in multiples of 5, Teacher Norma paid no attention to the potential of the drawing she had created, or to her enactment, to solve the problem. These were simply seen as add-ons that have no bearing on the solution strategy.

Teacher Norma's drawing and actions of how she would teach 20 ÷ 5 were consistent with sharing. However, by counting in multiples of 5 and raising a finger to keep track of the number of (groups of) 5 she had counted, the calculation strategy she used was analogous with grouping. This brings to the fore a disconnect between the word problem, the action and the drawing on one hand and her calculation strategy on the other hand.

Findings and conclusion

When interviewed all three teachers said that they would use concrete apparatus and drawings to teach and alluded to using some form of enactment in their teaching. Whilst these types of representations may be used in their teaching, the extent to which they are helpful in developing mathematical thinking is questionable. For all of them, the representations are appendages, seen as end results, as things they are expected to do. They already know the answers to the problems, so for them, the representations stand alone; they are not used as tools to serve as cognitive vehicles towards solutions.

In the progressive transformation of mathematical ideas towards abstraction, the problem and its relationship to its solution must remain constantly present and connected. In this way, the focus remains fixed on the problem that is being solved and on how the solution to the problem is read in each transformation. Reading the problem and its solution in each transformation will create an awareness that an answer of 4 (groups) exists in the additive number sentence 5 + 5 + 5 + 5 = 20, but the problem cannot end here. Ending a division problem with a repeated addition number sentence is an incomplete calculation strategy because it does not answer the problem that had been posed. As a final step, at every level of transforming a representation, learners should revisit the problem to ensure that it has been answered. Teachers should explicitly teach this final step to learners as most children tend to leave this step out.

Although this research was conducted on a small scale, the findings that all three teachers interviewed displayed evidence of struggling to use representations to answer partitive and quotition division problems suggests that the issue is likely to be more widespread. Teachers need training not only on how to teach using a range of appropriate representations but also on reading and interpreting problems and their solutions as they are represented in various forms. In other words, the two areas in which they need to be trained are their own understanding of mathematics, in this case division and also on how they teach (their PCK).

Acknowledgements

The author acknowledges Prof. Mike Askew and Prof. Hamsa Venkat for guiding her on formulating the interview questions.

Competing interests

The author has declared that no competing interests exist.

Authors' contributions

The author declares that she is the sole author of this research article.

Funding information

This research received no specific grant from any funding agency in the public, commercial or not-for-profit sectors.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analysed in this study.

Disclaimer

The views and opinions expressed in this article are those of the author and do not necessarily reflect the official policy or position of any affiliated agency of the author.

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