Ultrafilters and Semigroup Algebras

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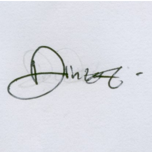
Abstract

The operation defined on a discrete semigroup S can be extended to the Stone-Čech compactification βS of S so that for all $a \in S$, the left translation $\beta S \ni x \mapsto ax \in \beta S$ is continuous, and for all $q \in \beta S$, the right translation $\beta S \ni x \mapsto xq \in \beta S$ is continuous. Because any compact right topological semigroup, βS contains a smallest two-sided ideal $K(\beta S)$ which is a completely simple semigroup. We give an exposition of some basic results related to the semigroup βS and to the semigroup algebra $\ell^1(\beta S)$. In particular, we review the result that $\ell^1(\beta \mathbb{N})$ is semisimple if and only if $\ell^1(K(\beta \mathbb{N}))$ is semisimple. We also review the reduction of the question whether $\ell^1(K(\beta \mathbb{N}))$ is semisimple to a question about $K(\beta \mathbb{N})$.

Declaration

I, the undersigned, hereby declare that the work contained in this dissertation is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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Contents

1. Introduction

The Stone-Čech compactification was presented by M. Stone and E. Čech, independently, in 1937. It has a number of applications in modern analysis [11], one which we study is in the field of topological algebra, in particular, semigroups and Banach algebras [14].

In point-set topology, the Stone-Čech compactification is a technique used for constructing a universal map from a topological space X to a compact Hausdorff space βX. The Stone-Čech compactification βX of a topological space X is the largest compact Hausdorff space in the family $C(X)$, of all maps from X to a compact Hausdorff space that factors uniquely through βX, [11].

The operation defined on a discrete semigroup $(S, \cdot)$ can be extended naturally, also , denoted by $\cdot$, to the Stone-Čech compactification βS of S so that for all $a \in S$, the left translation

$$\beta S \ni x \mapsto ax \in \beta S$$

is continuous, and for all $q \in \beta S$, the right translation

$$\beta S \ni x \mapsto xq \in \beta S$$

is continuous.

We choose the points of βS to be the ultrafilters on S and identify the principal ultrafilters with the points of S. The topology of βS is generated by collecting as a base the subsets of the form

$$\widehat{A} = \{p \in \beta S : \ A \in p\}$$

where $A \subset S$. For each $p, q \in \beta S$, the ultrafilter pq has a base consisting of subsets of the form

$$\bigcup \{xB_x : x \in A\}$$

where $A \in p$ and $B_x \in q$.

Being a compact right topological semigroup, βS has a smallest two-sided ideal $K(\beta S)$ which is a completely simple semigroup. The structure of a completely simple semigroup is given by the Rees-Suschkewitsch theorem (3.2.27). In particular, it is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals, the intersection of a minimal right ideal and a minimal left ideal is a group, and all these groups are isomorphic. As is well known, $K(\beta \mathbb{N})$ consists of 2^{2^ω} minimal right ideals and 2^{2^ω} minimal left ideals, and its structure group contains the free group on 2^{2^ω} generators.

In [6], the study of the semigroup algebra $\ell^1(\beta S)$ was initiated. In particular, the question was raised whether $\ell^1(\beta \mathbb{N})$ is semisimple. In [8] it was shown that $\ell^1(\beta \mathbb{N})$ is semisimple if and only if $\ell^1(K(\beta \mathbb{N}))$ is semisimple. Also they reduced the question of whether $\ell^1(K(\beta \mathbb{N}))$ is semisimple to a question about $K(\beta \mathbb{N})$.

In [14], they state that the study of the semigroup βS has interested several mathematicians since its inception in the late 1950s. The dense bibliography given at the end of [14], shows a large number of research papers are devoted to the semigroup properties of βS. Furthermore, several reasons exists for an interest in the algebra of βS, one being the fact that it is intrinsically interesting as being a natural extension of S which plays a special role among semigroup compactifications of S. It is the largest possible compactification of this kind: If T is a compact right topological semigroup, φ is a continuous homomorphism from S to T, φ is dense in T, and $\lambda_{\varphi(s)}$ is continuous for each $s \in S$, then T is a quotient of βS, [14]. Other interesting properties includes that βS has applications to combinatorial number theory and to topological dynamics.

As an example of the applications of βS is to the part of combinatorial number theory known as *Ramsey theory* where the algebraic properties of βS have proved useful. Although there is no universally accepted definition of Ramsey theory [14], we look at a brief idea of what it might be. Ramsey theory has played an important role in mathematics over the last century, its methods has reached various fields such as algebra, combinatorics, set theory, logic, geometry and analysis. It's embedded in an old decision problem whereby it is concerned with the preservation of properties under set partitions, so that is how the definition of Ramsey theory might sound or reworded as a study of unavoidable regularity in large sets [14]. What is studied is this, given a certain set V that has a property T, is it ever true that whenever V is partitioned into finitely many subsets, one of the subsets must also have the property T? To have a clear understanding of the discussion to this problem and the research around it see [14]. Now then in [14], it is stated that the results in Ramsey Theory have what is termed 'twin beauty'. There exists simple statements easy for almost anyone to understand though not easy to prove. Also, Ramsey Theory has been widely applied from its early inceptions.

According to [14], one of the most important examples of an application of the algebraic structure of βS to Ramsey Theory is provided by the *Finite Sums Theorem*. This theorem says that whenever $\mathbb{N}$ is partitioned into finitely many classes (or in the common terminology in Ramsey Theory, is *finitely colored*), there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty})$ contained in one class (or *monochrome*). (Here $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : \ F \ \text{ is a finite nonempty subset of } \ \mathbb{N}\}$.) This theorem had been an open problem for some decades, even though several mathematicians (including Hilbert) had worked on it. Although it was initially proved without using $\beta\mathbb{N}$, the first proof given was of enormous complexity [14].

An amazingly simple proof of the Finite Sums Theorem using the algebraic structure of $\beta\mathbb{N}$ was provided by F. Galvin and S. Glazer in 1975 [14]. Ever since then, numerous strong combinatorial results have been obtained using the algebraic structure of βS , where S is an arbitrary discrete semigroup. During the process, the knowledge of the algebra of βS has been obtained and in more detail [14].

In [14], they state that the semigroup βS, in particular $\beta\mathbb{N}$, is interesting both for its own sake and for its applications to combinatorial number theory, to topological dynamics, topological groups, and most importantly in Banach algebras (which in this dissertation we exploit this application). Indeed, the theory of Banach algebras is a huge field in functional analysis with several subfields and applications to diverse fields of analysis and the rest of mathematics (in particular, semigroups theory). We have that in [6], it is explained that the questions asked about Banach algebras are often resolved by inspecting the properties of this semigroup, and perhaps at times requiring new results about it. We wish to emphasize that the semigroup $\beta\mathbb{N}$ is a deep, subtle, and significant mathematical object, with a distinguished history and about which there are challenging open questions (as we have seen from the above example in Ramsey theory) [6].

As already stated in the literature, the properties of the algebra $\ell^1(X)$, defined as a set of all sequences $\{a(n)\}$ over a linear space X such that $\sum |a(n)| < \infty$, are that it is very natural and transparent. One key application of this algebra is in Fourier transforms, as we will see when defining the Gel'fand transform. There are several reasons why Banach algebras are studied [5], such as they cover many examples; have an abstract approach that leads to clear, quick proofs and gives new hindsights; blend algebra and analysis; and have beautiful results on intrinsic structures (such as $\beta\mathbb{N}$, for example).

Filters play a significant role in the study of the Stone-Čech compactification constructed from a discrete semigroup. This is so, because the Stone-Čech compactification constructed from a discrete semigroup S can be thought of as the spectrum of the algebra $\mathcal{B}(S)$ or the as the set of all ultrafilters on S [6, 14]. It is for this important fact that we review as much as possible the notion of filters.

In brief, we have tried to incorporate some material from topological semigroups and Banach algebra theory. This way, we will get some sense from the questions that are raised from the Banach algebra theory about βS.

The aim of this dissertation is to give an exposition of some basic results related to the semigroup βS and to the semigroup algebra $\ell^1(\beta S)$.

The dissertation is organized as follows.

In Chapter (2), we introduce filters and then go on to define ultrafilters and then see how all this give rise to topological spaces. Filters were introduced in 1937 by H. Cartan and others and subsequently were used ever since. There is an equivalent notion to filters called *nets* which were developed in 1921 by E. H. Moore, H. L. Smith and others [11]. Both notions are very important as each is adequate for topology in the very sense that sequences are not, because it was acknowledged that sequences are inadequate to describe topologies in general [19]. Nets resemble sequences strongly, and are much handier to use in discussions of continuity of functions, and algebraic operations; whilst filters are preferable in dealing with compactness and completeness [19]. So both these theories are necessary. However we chose to focus on filters because of the fact that we will be dealing with compactness, in particular we construct the Stone-Čech compactification using ultrafilters for a discrete topological spaces which we exploit its properties in the last stages of the chapter.

In Chapter (3), we give the definition of a semigroup and expose the notion in order to make use of its properties most important one being *idempotents* as they are important to us throughout this dissertation. After doing this we topologise the semigroup and define the right topological semigroup. We shall be concerned with certain compact right topological semigroups. Of fundamental importance is the result which states there exists an idempotent in a compact right topological semigroup, see Theorem (3.3.5). Then lastly we expose the Stone-Čech compactification of a discrete semigroup. Also, as any compact Hausdorff right topological semigroup, βS has idempotents, and the smallest two-sided ideal $K(\beta S)$ is provided.

In Chapter (4), we give all the necessary background from Banach algebras and abstract their properties to more general semigroup algebras in the chapter that follow. In Chapter (2), we focused primarily on topological spaces and the continuous functions carried by them. In Chapter (3), we focused on rather algebraic ideas and their topological counterparts. In Chapter (4) and the next chapter, this ideas, topological and algebraic, are united by a single notion, that of a Banach algebra. This notion of Banach algebras is interesting for it establishes a broad field of study in which various mathematical concepts meet significantly [5]. Our main task in this chapter is to get to a point where we understand what is meant about a Banach algebra being semi-simple. For this to happen, we give an adequate definition of the *Jacobson radical* of a Banach algebra. Note that the term 'Jacobson radical' is used because, in pure algebra theory, there are several different definitions of radicals [1].

In Chapter (5), we determine the radical of the semigroup algebra $\ell^1(\beta S)$ based on the the semigroup βS, the Stone-Čech compactification of a cancellative, countable, commutative semigroup. A semigroup algebra, is a Banach algebra with continuous multiplication. As such semigroup algebras combine aspects of both topological semigroups and Banach algebras. Our brief study of semigroup algebras, elaborates the effects of continuity on this combined structure. Lastly we show how the results concerning the semisimplicity $\ell^1(\beta\mathbb{N})$ are entertained.

2. Ultrafilters and The Stone-Čech Compactification

In this chapter, we introduce some basic concepts and notations that are relevant to the study of ultrafilters, and look at some examples. In section 2, we look at how ultrafilters give rise to topological spaces and thus get to see why they are needed and useful. In section 3, we introduce the Stone-Čech compactification for a discrete topological space and finally plan out where this compactification fills our needs.

2.1 Filters and Ultrafilters

2.1.1 Definition. [19] Let X be a nonempty set. A nonempty collection $\mathcal{F}$ of subsets of X is called a *filter* on X if it satisfies the following conditions

(F1) $\emptyset \notin \mathcal{F}$;

(F2) if $F, G \in \mathcal{F},$ then $F \cap G \in \mathcal{F}$; and

(F3) if $F \in \mathcal{F}$ and $G \supset F$ then $G \in \mathcal{F}$.

2.1.2 Remark. (i) Condition (F1) states that no element of $\mathcal{F}$ is empty;

(ii) Since $\mathcal{F}$ is nonempty, it contains one subset, say F, of X and because $X \supset X$, it follows from (F3) that $X \in \mathcal{F}$.

(iii) Condition (F2) implies that $F_1, \ldots, F_n \in \mathcal{F}$, then $F_1 \cap \ldots \cap F_n \in \mathcal{F}$. Then according to (F1), $F_1 \cap \cdots \cap F_n \neq \emptyset$. Any set which satisfies this condition, we say, it has the *finite intersection property* (FIP), and so $\mathcal{F}$ has the FIP.

(iv) Since X is the intersection of the empty subset of $\mathcal{F}$, it follows from (ii) and (iii) that $\mathcal{F}$ is closed for finite intersections.

(v) Observe from (F1) that the power set of X, $\mathcal{P}(X)$, is not a filter on X, as $\emptyset \in \mathcal{P}(X)$. Thus any filter on X has to be a proper subset of $\mathcal{P}(X)$.

(vi) If $F \in \mathcal{F}$ then $X \setminus F \notin \mathcal{F}$. to see this, if both F and $X \setminus F$ belong to $\mathcal{F}$, then by (F2), $F \cap X \setminus F = \emptyset \in \mathcal{F}$ which contradicts (F1).

Filters by intuition are a family of large subsets of X.

2.1.3 Definition. A filter $\mathcal{F}$ on X is said to be *free* if $\bigcap_{F \in \mathcal{F}} F = \emptyset$ and *fixed* if $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

2.1.4 Example. [16, 19]

(i) By remark (2.1.2)(ii), there is no filter on the empty set $\emptyset$.

(ii) Let $X = \{a, b, c\}$ and let $\mathcal{F}_1 = \{X\}$; $\mathcal{F}_2 = \{\{a, b\}, X\}$; $\mathcal{F}_3 = \{\{b, c\}, X\}$; $\mathcal{F}_4 = \{\{c, a\}, X\}$; $\mathcal{F}_5 = \{\{a\}, \{a, b\}, \{a, c\}, X\}$; $\mathcal{F}_6 = \{\{b\}, \{a, b\}, \{b, c\}, X\}$; $\mathcal{F}_7 = \{\{c\}, \{c, a\}, \{b, c\}, X\}$; $\mathcal{F}_8 = \{\{a, b\}, \{b, c\}, X\}$; $\mathcal{F}_9 = \{\{a\}, \{b\}, \{a, b\}, X\}$; $\mathcal{F}_{10} = \{\emptyset, \{a\}, \{a, b\}, X\}$.
Now, $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6$ and $\mathcal{F}_7$ are filters on X. Whereas, $\mathcal{F}_8, \mathcal{F}_9$ and $\mathcal{F}_{10}$ are not filters.

(iii) Let X be a nonempty set. Then the set $\{X\}$ is always a filter on X called the *indiscrete filter*.

(iv) Let X be a nonempty set and let $x \in X$. Let $\mathcal{F}_x = \{F \subset X \ : \ x \in F\}$. Then $\mathcal{F}_x$ is a filter on X called the *discrete filter at* x. The discrete filter is also called the *principal filter generated by* x. We show that $\mathcal{F}_x$ is a filter.

We have that $\mathcal{F}_x$ is nonempty since $\{x\} \in \mathcal{F}_x$. Firstly, we have $x \in \mathcal{F}_x \ \forall F \in \mathcal{F}_x$, and so no element of $\mathcal{F}_x$ is empty and it follows that $\emptyset \notin \mathcal{F}_x$. If $F \in \mathcal{F}_x$ and $G \supset F$. Then $x \in F$, and so $x \in G$. Hence $G \in \mathcal{F}_x$. Lastly, if $F \in \mathcal{F}_x$ and $G \in \mathcal{F}_x$. Then $x \in F$ and $x \in G$, it follows that $x \in F \cap G$; hence $F \cap G \in \mathcal{F}_x$.

(v) Let X be a non-empty set and let S be a nonempty subset of X. Then the collection $\mathcal{F} = \{F \subset X \ : \ F \supset S\}$ is a filter on X that is called the *principal filter generated by* S.

We have $\mathcal{F}$ is nonempty since $S \in \mathcal{F}$. Furthermore, since S is nonempty and every element of $\mathcal{F}$ contains S it follows that each element of $\mathcal{F}$ is nonempty. Hence $\emptyset \notin \mathcal{F}$. Next, if $F \in \mathcal{F}$ and $G \supset F$. Then $S \subset F$, and so $S \subset G$. Hence $G \in \mathcal{F}$. Lastly, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$. Then $S \subset F$ and $S \subset G$, it follows that $S \subset F \cap G$; hence $F \cap G \in \mathcal{F}$.

(vi) Let X be an infinite set. Then $\mathcal{F} = \{F \subset X \ : X \setminus F \text{ is finite}\}$ is a filter on X called the *cofinite* filter. In particular, if $X = \mathbb{N}$, the cofinite filter on $\mathbb{N}$ is called the *Fréchet filter*.

We have $\mathcal{F}$ is nonempty, since $X \setminus X = \emptyset$ is finite, and so $X \in \mathcal{F}$. Furthermore, since X is infinite and $X \setminus F$ is finite, it follows that F is an infinite set and so no element of $\mathcal{F}$ is nonempty. Hence $\emptyset \notin \mathcal{F}$. Next, if $F \in \mathcal{F}$ and $G \supset F$. Then since $X \setminus F$ is finite, $X \setminus G$ must be finite. Hence $G \in \mathcal{F}$. Lastly, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$. Then $X \setminus F$ is finite and $X \setminus G$ is finite. Now by De-Morgan law we have

$$X \setminus (F \cap G) = (X \setminus F) \cup (X \setminus G).$$

Since $X \setminus F$ and $X \setminus G$ are finite, it follows that $X \setminus (F \cap G)$ is finite. Hence $F \cap G \in \mathcal{F}$.

(vii) The cofinite filter $\mathcal{F}$ is a free filter.

For any $x \in X$, let $F = X \setminus \{x\}$. Then $F \in \mathcal{F}$, since $X \setminus F = \{x\}$ is finite. But $x \notin F$, and so $x \notin \bigcap_{F \in \mathcal{F}} F$. Hence $\bigcap_{F \in \mathcal{F}} F = \emptyset$.

2.1.5 Definition. [16, 9] Let $\mathcal{F}$ and $\mathcal{U}$ be two filters on X. Then $\mathcal{F}$ is said to be *finer* (or stronger or larger) than $\mathcal{U}$, written $\mathcal{F} \supset \mathcal{U}$ (alternatively we say $\mathcal{U}$ is *coarser* (or weaker or smaller) than $\mathcal{F}$). Moreover, if $\mathcal{F} \neq \mathcal{U}$ also, then $\mathcal{F}$ is said to be *proper* (or *strictly finer*) than $\mathcal{U}$ (or $\mathcal{U}$ *strictly coarser* than $\mathcal{F}$).
The filters $\mathcal{F}$ and $\mathcal{U}$ are said to be *comparable* if and only if $\mathcal{F}$ is finer than $\mathcal{U}$ or $\mathcal{U}$ is finer than $\mathcal{F}$. The set of all filters on X is *directed* by the relation $\leq$ defined by $\mathcal{F} \leq \mathcal{U}$ if and only if $\mathcal{F}$ is coarser than $\mathcal{U}$.

2.1.6 Example. Look at the filters defined as in Example (2.1.4)(ii). Now $\mathcal{F}_1$ is weaker than all the filters on X. The filter $\mathcal{F}_5$ is stronger than $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_4$; $\mathcal{F}_6$ is stronger than $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$; $\mathcal{F}_7$ is stronger than $\mathcal{F}_3$ and $\mathcal{F}_4$; lastly, $\mathcal{F}_6$ and $\mathcal{F}_7$ aren't comparable etc.

Note that if X is an arbitrary nonempty set, it follows that $\{X\}$ is the weakest filter on X. Otherwise stated $\{X\}$ is the smallest element of the ordered set of all filters on X. But there is no largest filter on X, if X has more than one element. We have that among the collection satisfying (F2) and (F3), the power set $\mathcal{P}(X)$ is the largest and it (and only it) is excluded by (F1).

2.1.7 Example. Let X be an infinite set and let $\mathcal{F}$ be a filter on X such that $\bigcap_{F\in\mathcal{F}} F = \emptyset$. Then $\mathcal{F}$ is finer than the cofinite filter on X. In other words, every free filter is finer than the cofinite filter.

Let $\mathcal{C}$ be the cofinite filter on X. We already know that $\mathcal{C}$ is a free filter, see Example (2.1.4)(vii). So, suppose that $\mathcal{F} \not\supset \mathcal{C}$. Then there is $C \in \mathcal{C}$ such that $C \notin \mathcal{F}$. Then $X \setminus C$ is finite. Let

$$X \setminus C = \{x_1, x_2, \ldots, x_n\}.$$

Now $\bigcap_{F\in\mathcal{F}} F = \emptyset$. Therefore, there is $F_i \in \mathcal{F} \quad (i = 1, \ldots, n)$ such that $x_i \notin F_i$. Since $\mathcal{F}$ is a filter, $G = \bigcap\{F_i : 1 \le i \le n\} \in \mathcal{F}$. This G will not contain any element of $\{x_1, x_2, \ldots, x_n\}$. Hence $X \setminus G \supset X \setminus C$ and consequently $G \subset C$. Since $\mathcal{F}$ is a filter $C \subset G$ imply $C \in \mathcal{F}$. This is a contradiction according to our supposition. Hence $\mathcal{F}$ is finer than the cofinite filter.

2.1.8 Theorem. *[19] Let $\{\mathcal{F}_\alpha : \alpha \in \Lambda\}$ be any nonempty collection of filters on a nonempty set X. Then the set $\mathcal{F} = \bigcap\{\mathcal{F}_\alpha : \alpha \in \Lambda\}$ is also a filter on X*

Proof. We have that $\mathcal{F}$ is nonempty since X belongs to every $\mathcal{F}_\alpha$, hence $X \in \mathcal{F}$. Furthermore, since $\emptyset \notin \mathcal{F}_\alpha \ \forall \alpha \in \Lambda$, it follows that $\emptyset \notin \mathcal{F}$. If $F \in \mathcal{F}$ and $G \supset F$. Then $F \in \mathcal{F}_\alpha \ \forall\alpha \in \Lambda$. Since each $\mathcal{F}_\alpha$ is a filter, $G \in \mathcal{F}_\alpha \ \forall\alpha \in \Lambda$ and so $G \in \mathcal{F}$. Lastly, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$. Then $F \in \mathcal{F}_\alpha$ and $G \in \mathcal{F}_\alpha \quad \forall\alpha \in \Lambda$, it follows that $F \cap G \in \mathcal{F}_\alpha \ \forall\alpha \in \Lambda$. Hence $F \cap G \in \mathcal{F}$. □

2.1.9 Remark. (i) the union of two filters on a set need not be a filter. For example, in Example (2.1.4)(ii), $\mathcal{F}_2$ and $\mathcal{F}_3$ are filters whereas $\mathcal{F}_2 \cup \mathcal{F}_3$ is not a filter.

(ii) The intersection of all filters on X is the filter $\{X\}$ which is the weakest filter on X.

From the definition of a filter, we are able to specify it by determining its entire collection $\mathcal{F}$ of subsets of X. But this could prove difficult to do, in other words, it is not necessary to specify all of its sets, because whenever condition (iii) above in Definition (2.1.1) holds, then we have some elements in $\mathcal{F}$. That is if we specify instead smaller subsets of X and then we can define a filter in terms of them. To do this, let $\mathcal{A}$ be a collection of subsets of a set X. In the following result we state the necessary and sufficient conditions $\mathcal{A}$ must satisfy in order that we may obtain a filter containing $\mathcal{A}$ on X.

2.1.10 Theorem. *[16] Let $\mathcal{A}$ be a nonempty collection of subsets of a set X, such that every intersection of finitely many elements of $\mathcal{A}$ is nonempty. Then there is a filter on X, that contains $\mathcal{A}$ if and only if $\mathcal{A}$ has the FIP. In particular, this is the smallest filter containing $\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ has the FIP. To show that there exists a filter on X containing $\mathcal{A}$. Let $\mathcal{B} = \{B : \ B$ is the intersection of a finite subcollection of $\mathcal{A}\}$. Since $\mathcal{A}$ has the FIP, no element of $\mathcal{B}$ is empty, and so $\emptyset \notin \mathcal{B}$. Now define $\mathcal{F} = \{F : \ F$ contains an element of $\mathcal{B}\}$. We have $\mathcal{F} \supset \mathcal{A}$ since every element of $\mathcal{A}$ belongs to $\mathcal{B}$. We show that $\mathcal{F}$ is a filter on X.
Since $\emptyset \notin \mathcal{B}$ and every element of $\mathcal{F}$ contains some element of $\mathcal{B}$, we have that $\emptyset \notin \mathcal{F}$. If $F \in \mathcal{F}$ and $G \supset F$. Then F contains an element of $\mathcal{B}$, and so G must contain an element of $\mathcal{B}$. Hence $G \in \mathcal{F}$. Lastly, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$. Then $F \supset A$ and $G \supset B$ where A and B are some elements of $\mathcal{B}$. Since A and B are some finite intersections of elements of $\mathcal{A}$, $A \cap B$ is also a finite intersection of elements of $\mathcal{A}$. Hence $A \cap B \in \mathcal{A}$. Also $F \cap G \supset A \cap B$. Thus $F \cap G$ contains an element of $\mathcal{B}$, it follows that $F \cap G \in \mathcal{F}$.
For the other direction, let $\mathcal{F}$ be a filter on X that contains $\mathcal{A}$. Then $\mathcal{F}$ also contains the collection $\mathcal{B}$ of finite intersections of elements of $\mathcal{A}$. Since $\mathcal{F}$ is a filter, $\emptyset \notin \mathcal{B}$. Hence $\mathcal{A}$ has the FIP.

To see that $\mathcal{F}$ is the coarsest filter. we have that if $\mathcal{F}'$ is any filter containing $\mathcal{A}$, then $\mathcal{F}'$ must contain all finite intersections of elements of $\mathcal{A}$ and their supersets and thus $\mathcal{F}' \supset \mathcal{F}$. □

2.1.11 Definition. [16] The filter $\mathcal{F}$ defined in the theorem above is said to be *generated* by $\mathcal{A}$ and $\mathcal{A}$ is said to be a *subbase* of $\mathcal{F}$.

Note that it is necessary and sufficient for $\mathcal{A}$ to have the FIP for it to be a subbase.

2.1.12 Definition. [19] A collection $\mathcal{B}$ of subsets of a set X is called a *filter base* in X if

(i) $\emptyset \notin \mathcal{B}$;

(ii) $\mathcal{B} \neq \emptyset$;

(iii) $A \in \mathcal{B},\ B \in \mathcal{B}$ imply that $\mathcal{B}$ contains a set C with $C \subset A \cap B$.

Hence a filter is a filterbase. A filter $\mathcal{F}$ is generated by a filterbase $\mathcal{B}$ in the obvious way,

$$\mathcal{F} = \{A : A \supset B \text{ for some } B \in \mathcal{B}\}.$$

2.1.13 Example. [19] Let Y be a proper subset of a set X. Let $\mathcal{F}$ be a filter on Y. Then $\mathcal{F}$ is not a filter on X, as $X \notin \mathcal{F}$, but $\mathcal{F}$ is a filterbase on X. Yes indeed if $\mathcal{B}$ is a filterbase on Y, then $\mathcal{B}$ is also a filterbase on X. This example and Definition (2.1.11), make filterbases much easier to work with or use than filters.

Families with the finite intersection property behave like filters, see [20]; in fact, if $\mathcal{E}$ is such a family and $\mathcal{F}$ is the collection of all sets with the FIP from $\mathcal{E}$ then $\mathcal{F}$ is a filterbase, so every family $\mathcal{E}$ with the FIP generates a filter as laid out by Definition (2.1.11). Conversely, every filter is a family with the FIP.

Although, we're not necessarily interested on filterbases as our interest lies beyond filters, for detailed study and further properties of filterbases [see 20, 18].

Following Definition (2.1.11) every family $\mathcal{A} \subset \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) with the FIP generates a filter $\langle \mathcal{A} \rangle$ on X by

$$\langle \mathcal{A} \rangle = \{A \subset X : A \supset \bigcap \mathcal{B} \text{ for some finite } \mathcal{B} \subset \mathcal{A}\}.$$

Thus $\langle \mathcal{A} \rangle$ is a filterbase.

From Examples (2.1.4) we see that the filters generated by some collection $\mathcal{A} \subset \mathcal{P}(X)$ are for (iv) and (v) respectively $\mathcal{A} = \{x\}$ and $\{F\}$. The cofinite filter in (vi) is generated by the family $\mathcal{A} = \{X \setminus \{x\} : x \in X\}$.

2.1.14 Definition. [11] Let $\mathcal{F}$ be a filter on a set X. Then a subcollection $\mathcal{B}$ of $\mathcal{F}$ is called a *base* of $\mathcal{F}$ if every element of $\mathcal{F}$ contains an element of $\mathcal{B}$.

2.1.15 Remark. [11]

(i) A collection $\mathcal{B}$ of subsets of a set X is a filterbase on X if and only if $\mathcal{B}$ is a base of some filter on X.

((ii) If $\mathcal{A}$ is a subbase of a filter $\mathcal{F}$, then the family $\mathcal{B}$ of all finite intersections of elements of $\mathcal{A}$ is a base for $\mathcal{F}$.

2.1.16 Definition. [16] A filter $\mathcal{U}$ is an *ultrafilter* if $\mathcal{U}$ has no proper finer filter.

In other words, for every filter $\mathcal{F} \subset \mathcal{G}$, we have that $\mathcal{F} = \mathcal{G}$ (that is, $\mathcal{F}$ is a maximal filter). It follows that an ultrafilter on X is a maximal element of the collection of all filters on X partially ordered by the inclusion relation $\subset$.

2.1.17 Example. [14]

(i) From Example (2.1.4) (i) we have that the discrete filter is an ultrafilter. Such ultrafilters are called *principal*. Ultrafilters which are not principal are called *nonprincipal*. To see that D_x is an ultrafilter, let $\mathcal{U}$ be a filter with $D_x \subset \mathcal{U}$. Let $U \in \mathcal{U}$. Choose $F = \{x\} \in D_x \subset \mathcal{U}$ and so $F \cap U \neq \emptyset$ and so $x \in U$. Thus $U \in D_x$; hence $\mathcal{U} \subset D_x$.

(ii) The Fréchet filter on an infinite set X is not an ultrafilter, since the set X can be partitioned into two infinite, disjoint sets, and none of these sets are cofinite. This is easily seen from the following criterion of ultrafilters.

The following properties of an ultrafilter are important and we shall make use of them frequently.

2.1.18 Theorem. *[14, 21, 16]*
Let $\mathcal{F} \subset \mathcal{P}(X)$. Then the following statements are equivalent:

(a) $\mathcal{F}$ is an ultrafilter;

(b) For every subset A of X, we have that if $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$, then $A \in \mathcal{F}$;

(c) If F and G are subsets of X such that $F \cup G \in \mathcal{F}$ then either $F \in \mathcal{F}$ or $G \in \mathcal{F}$;

(d) For every subset F of X, either $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$.

Proof. (a) $\Longrightarrow$ (b) Suppose $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. Then

$$\{F \cap A : F \in \mathcal{F}\}$$

is a collection of nonempty sets closed under the FIP, because $\mathcal{F}$ is closed under the FIP. Therefore,

$$\mathcal{U} = \{C \subset X : C \supset F \cap A \text{ for some } F \in \mathcal{F}\}$$

is a filter by Definition (2.1.11). It follows that, $\mathcal{F} \subset \mathcal{U}$ by definition of $\mathcal{U}$ and $A \in \mathcal{U}$ because $A \supset F \cap A$. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{F} = \mathcal{U}$. Hence, $A \in \mathcal{F}$.
(b) $\Longrightarrow$ (c) Suppose there is $H_1, H_2 \in \mathcal{F}$ such that $H_1 \cap F = \emptyset$ and $H_2 \cap G = \emptyset$, where $F \cup G \in \mathcal{F}$. Then $(H_1 \cap H_2) \cap (F \cup G) = \emptyset$. This is a contradiction as $(H_1 \cap H_2)$ and $(F \cup G)$ belongs to a filter $\mathcal{F}$. So, without loss of generality, suppose $H \cap F \neq \emptyset$, for all $H \in \mathcal{F}$. Hence, $F \in \mathcal{F}$.
(c) $\Longrightarrow$ (d) Since $F \cup (X \setminus F) = X$ then $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$.
(d) $\Longrightarrow$ (a) Suppose that for every subset F of X, either $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$. Suppose $\mathcal{F}$ is not an ultrafilter. Then there exists a filter $\mathcal{F}_1$ such that $\mathcal{F} \subsetneq \mathcal{F}_1$. So there is a subset F of X such that $F \in \mathcal{F}_1$ but $F \notin \mathcal{F}$. By hypothesis, $X \setminus F \in \mathcal{F}$. This implies that $X \setminus F \in \mathcal{F}_1$. So both F and $X \setminus A$ are both elements of $\mathcal{F}_1$ and $F \cap (X \setminus F) = \emptyset$. This is a contradiction to $\mathcal{F}_1$ being a filter, hence $\mathcal{F}$ is an ultrafilter. □

2.1.19 Corollary. [17] Let $\mathcal{U}$ be an ultrafilter on X. If $A_1 \cup \ldots \cup A_k \in \mathcal{U}$, then $A_k \in \mathcal{U}$ for some $1 \leq k \leq n, n \in \mathbb{N}$.

Proof. Suppose $A_1 \cup \ldots \cup A_k \in \mathcal{U}$, then by (d) of Theorem (2.1.18)
$X \setminus (A_1 \cup \cdots \cup A_k) = (X \setminus A_1) \cap \cdots \cap (X \setminus A_k) \notin \mathcal{U}$. So it follows that $\forall k \leq n$ $(X \setminus A_k) \notin \mathcal{U}$, thus $A_k \in \mathcal{U}$ for some $k \leq n$. □

The following theorem tells us that the principal ultrafilters are the only ones which its members can be explicitly defined. There are none others which can be defined within Zermelo-Fraenkel set theory.

We recall that if $x \in X$ the set $\{A \subset X \ : \ x \in A\}$ is the principal ultrafilter generated by x.

2.1.20 Theorem. *[14] Let X be a set and let $\mathcal{F}$ be an ultrafilter on X. Then the following statements are equivalent:*

(a) $\mathcal{F}$ is a principal ultrafilter.

(b) There is some finite set $F \in \mathcal{P}(X)$ such that $F \in \mathcal{F}$.

(c) The Fréchet filter is not contained in $\mathcal{F}$.

(d) $\bigcap \mathcal{F} \neq \emptyset$.

(e) There is some $x \in X$ such that $\bigcap \mathcal{F} = \{x\}$.

Proof. (a) $\implies$ (b) Choose $x \in X$ such that $\mathcal{F} = \{A \subset X \ : \ x \in A\}$. Let $F = \{x\}$.
(b) $\implies$ (c) Let F be a finite subset of X and $F \in \mathcal{F}$ then $X \setminus F \notin \mathcal{F}$.
(c) $\implies$ (d) Choose $A \subset X$ such that $X \setminus A$ is finite and $A \notin \mathcal{F}$. Let $F = X \setminus A$. Then $F \in \mathcal{F}$ and $F = \bigcup \{\{x\} : x \in F\}$ so by Corollary (2.1.19) we can choose some $x \in F$ such that $\{x\} \in \mathcal{F}$. Then for each $B \in \mathcal{F}$, $B \cap \{x\} \neq \emptyset$, so $x \in \bigcap \mathcal{F}$.
(d) $\implies$ (e) Suppose that $\bigcap \mathcal{F} \neq \emptyset$ and choose $x \in \bigcap \mathcal{F}$. Then $X \setminus \{x\} \notin \mathcal{F}$ so $\{x\} \in \mathcal{F}$. Thus $\bigcap \mathcal{F} \subset \{x\}$.
(e) $\implies$ (a) Choose $x \in X$ such that $\bigcap \mathcal{F} = \{x\}$. For each $A \in \mathcal{F}$, $A \cap \{x\} \neq \emptyset$ so $x \in A$. Thus $\mathcal{F} \subset \{A \subset X : \ x \in A\}$. □

Following Theorem (2.1.20), we have the following definition

By part (c) of the previous theorem every filter which contains the Fréchet filter is free, that is, nonprincipal. Following the paragraph before Theorem (2.1.20), that there are no other ultrafilters which their members can be explicitly defined. However, we have that the axiom of choice (which is equivalent to Zorn's lemma) produces abundantly a set of nonprincipal ultrafilters on any infinite set.

The next theorem states that any filter can be extended to an ultrafilter and it uses the axiom of choice to which is a fact that it cannot be proved in Zermelo-Fraenkel set theory alone.

What we define now is a special form of Zorn's lemma. Let $\mathcal{C}$ be a collection of sets. A *chain* in $\mathcal{C}$ is a collection $\{C_i : i \in I\}$ of elements of $\mathcal{C}$ such that for all $i, j \in I$, we have that either $F_i \subset F_j$ or $F_j \subset F_i$.

2.1.21 Lemma. (Zorn)[19] Every nonempty partially ordered set includes a maximal chain.

2.1.22 Theorem. *(Ultrafilter Theorem)[14] Every filter on a set X can be extended to an ultrafilter on X.*

The proof uses Zorn's lemma, so to apply it, we need the following fact.

2.1.23 Lemma. [19] The union of a chain of filters is a filter.

Proof. Let $\mathcal{C}$ be a chain of filters, we show that $\bigcup\mathcal{C}$ is a filter. First, $\emptyset \notin \bigcup\mathcal{C}$ since the empty set, $\emptyset$, does not belong to any elements of $\mathcal{C}$; $X \in \bigcup\mathcal{C}$ since X belongs to any (in fact all) elements of $\mathcal{C}$. Thus condition (i) of Definition (2.1.1) is satisfied. For condition (ii), let $F, G \in \bigcup\mathcal{C}$. Then $F \in \mathcal{F}_1$ and $G \in \mathcal{F}_2$ for some $\mathcal{F}_1,\ \mathcal{F}_2 \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $\mathcal{F}_1 \subset \mathcal{F}_2$ or $\mathcal{F}_2 \subset \mathcal{F}_1$; say the former without loss of generality. Then $F \in \mathcal{F}_2,\ G \in \mathcal{F}_2$ and so $F \cap G \in \mathcal{F}_2$. Hence $F \cap G \in \bigcup\mathcal{C}$. Lastly for condition (iii). Let $F \in \bigcup\mathcal{C}$ and $F \subset G$. Then $F \in \mathcal{F}$ for some $\mathcal{F} \in \mathcal{C}$, Thus $G \in \mathcal{F}$ and so $G \in \bigcup\mathcal{C}$. Therefore, $\bigcup\mathcal{C}$ is a filter. □

Proof. of Theorem (2.1.22)

Let $P = \{\mathcal{F} \subset \mathcal{P}(X) : \mathcal{F} \subset \mathcal{F}' \text{ and } \mathcal{F} \text{ is a filter}\}$. As $\mathcal{F} \in P$ then P is nonempty and is partially ordered under set containment. Let $\mathcal{C}$ be a chain in P then by Lemma (2.1.23) $\bigcup\mathcal{C}$ is a filter on X and contains $\mathcal{F}$ as each chain does. So $\bigcup\mathcal{C} \in P$, also it is an upper bound for the chain $\mathcal{C}$ and so we have shown that every chain in P has an upper bound in P. Therefore by Zorn's lemma P has a maximal element say $\mathcal{U}$. We claim $\mathcal{U}$ is an ultrafilter on X, that is, $\mathcal{U}$ is also maximal in the set of all filters on X. Let $\mathcal{V}$ be a filter on X such that $\mathcal{U} \subset \mathcal{V}$, and so $\mathcal{V} \supset \mathcal{F}$ since $\mathcal{F} \subset \mathcal{U}$ and so $\mathcal{V} \in P$. But $\mathcal{U}$ is maximal in P and it follows that $\mathcal{U} = \mathcal{V}$. Hence $\mathcal{U}$ is an ultrafilter on X. □

2.1.24 Corollary. [14] If X is an infinite set, then there exists a free ultrafilter on X.

Proof. Consider an ultrafilter on X which contains the cofinite filter on X. □

2.1.25 Corollary. [16] If $\mathcal{B}$ is a nonempty set of subsets of X and $\mathcal{B}$ has the the FIP, then there is an ultrafilter $\mathcal{F}$ on X satisfying $\mathcal{B} \subseteq \mathcal{F}$.

Proof. Let $P = \{\mathcal{C} : \mathcal{C} \text{ is a collection of subsets of } X \text{ with the FIP such that } \mathcal{C} \supset \mathcal{A}\}$. Then P is nonempty since $\mathcal{A} \in P$. Also P is partially ordered by the inclusion relation $\subset$. Furthermore it is easy to see that every linear ordered subset of P has an upper bound. hence by Zorn's lemma we can choose a maximal element $\mathcal{F}$ of P. We claim $\mathcal{F}$ is a filter on X.
We have that $\mathcal{F}$ is nonempty since $\mathcal{F} \supset \mathcal{A}$ and $\mathcal{A}$ is nonempty. Firstly, since $\mathcal{F}$ has the FIP, no element of $\mathcal{F}$ can be empty and it follows that $\emptyset \notin \mathcal{F}$. If $F \in \mathcal{F}$ and $G \supset F$. Because $\mathcal{F}$ has the FIP, it follows that $\{G\} \cup \mathcal{F}$ also has the FIP. Also $\{G\} \cup \mathcal{F} \supset \mathcal{F}$. Hence $\{G\} \cup \mathcal{F} \in P$. By maximality of $\mathcal{F}$, $G \in \mathcal{F}$. Lastly, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$. Since $\mathcal{F}$ has the FIP, it follows that $\{F \cap G\} \cup \mathcal{F}$ also has the FIP. Also $\{F \cap G\} \cup \mathcal{F} \supset \mathcal{F}$. Hence $\{F \cap G\} \cup \mathcal{F} \in P$. By maximality of $\mathcal{F}$, $F \cap G \in \mathcal{F}$. Therefore, $\mathcal{F}$ is an ultrafilter as required. □

2.1.26 Proposition. [16] Let $f : X \to Y$ and let $\mathcal{F}$ be a filter on X. Then $\bar{f}(\mathcal{F}) = \{f(F) :\ F \in \mathcal{F}\}$ is a filter on Y if and only if f is surjective.

Proof. Suppose $\bar{f}(\mathcal{F})$is a filter on Y. Then clearly f is surjective, for let $Y \in f(\mathcal{F})$ then we have that $Y = f(X)$ for some $X \in \mathcal{F}$. That is the range of f is the whole of Y.

Conversely, suppose f is surjective. As $\mathcal{F}$ is a filter, we have $\emptyset \notin f(\mathcal{F})$ and since f is surjective, $Y \in f(\mathcal{F})$. Now let $A, B \in f(\mathcal{F})$. Then we can choose $F_1, F_2 \in \mathcal{F}$ such that $A = f(F_1)$, $B = f(F_2)$, so $f^{-1}(A) = F_1$, $f^{-1}(B) = F_2$. Since $F_1, F_2 \in \mathcal{F}$, we have that $f^{-1}(A) \cap f^{-1}(B) = F_1 \cap F_2 \in \mathcal{F}$, it follows that $A \cap B = f(F_1 \cap F_2) \in f(\mathcal{F})$. Lastly, let $A \in f(\mathcal{F})$ and $A \subseteq B$. Then $A = f(F_1)$ for some $F_1 \in \mathcal{F}$, so it follows that $f^{-1}(A) = F_1$ and $F_1 = f^{-1}(A) \subseteq f^{-1}(B)$, thus $F_1 \subseteq f^{-1}(B)$. Hence, $f^{-1}(B) \in \mathcal{F}$ and so $f^{-1}(B) = F_2$ for some $F_2 \in \mathcal{F}$. Therefore, $B = f(F_2) \in f(\mathcal{F})$. □

Observe that from Proposition (2.1.26), we could have written the map f as

$$\bar{f}(\mathcal{F}) = \{F \subseteq Y : f^{-1}(F) \in \mathcal{F}\}$$

and still reach the desired conclusion that it is a filter on the set Y.

2.1.27 Theorem. *[17] (Ultrafilter Theorem) Let $\mathcal{F}$ be an ultrafilter on a set X. For every map $f : X \to Y$ the filter $\bar{f}(\mathcal{F})$ is an ultrafilter on the set Y if and only if f is surjective.*

Proof. If $Y = Y_1 \cup Y_2$, then $X = f^{-1}(Y_1) \cup f^{-1}(Y_2)$. By Theorem (2.1.20), either $f^{-1}(Y_1) \in \mathcal{F}$ or $f^{-1}(Y_2) \in \mathcal{F}$. To be definitive, suppose that $F = f^{-1}(Y_1)$ and $F \in \mathcal{F}$. Then $f(F) = Y_1$ and $f(F) = f(\mathcal{F})$. Thus, $Y_1 \in f(\mathcal{F})$ as required. □

2.2 Ultrafilters on Topological Spaces

2.2.1 Definition. [19] A *topology* $\mathcal{T}$ on a set X is a collection of subsets of X satisfying:

(i) $\emptyset,\ X \in \mathcal{T}$;

(ii) The union of any subcollection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$;

(iii) The intersection of any finite subcollection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets* in X. (Whenever $\mathcal{T}$ is a topology in X, we will just say X is a topological space). A subset $F \subset X$ is said to be a *closed set* if the compliment, $X \setminus F$ is an open set. A set that is both open and closed is said to be a *clopen set*.

A set might be both open and closed, or it might be neither. In particular, both $\emptyset$ and X are both open and closed. The collection of closed sets has the following properties, which are dual to the properties of the open sets can be deduced by de Morgan's laws. Both X and $\emptyset$ are closed, finite union of closed sets are closed, and arbitrary intersections of closed sets are closed.

2.2.2 Example. [19, 16] The following are examples and non-examples of topological spaces.

(i) Let $\mathcal{F}$ be a filter on X, since $\emptyset \notin \mathcal{F}$ then $\mathcal{F}$ is not a topology on X . Similarly, any topology $\mathcal{T}$ on X is not a filter on X, because $\emptyset \in \mathcal{T}$.

(ii) For any filter $\mathcal{F}$ on X, $\mathcal{T} = \mathcal{F} \cup \{\emptyset\}$ is a topology on X.

(iii) For any filter $\mathcal{F}$ on X, $\mathcal{T} \setminus \{\emptyset\}$ is not a filter on X. Indeed, consider $\mathcal{T}$ as the Euclidean topology on $\mathbb{R}$, and so $\mathcal{B} = \mathcal{T} \setminus \{\emptyset\}$ is not a filter because the open interval $(0, 1)$ is an element of $\mathcal{B}$, whereas the closed interval $[0, 1]$ doesn't belong to $\mathcal{B}$, and so $\mathcal{B}$ does not satisfy Definition (2.1.1)(iii).

(iv) Let X be an arbitrary set, $\mathcal{T} = \mathcal{P}(X)$, the power set of X is a topology in X called the *discrete topology* and $\mathcal{T} = \{\emptyset, X\}$ is a topology in X called the *indiscrete topology*.

From this examples, we note that a nonempty set X can have many different topologies. The same as filters, the collection of all topological spaces on X is partially ordered by set inclusion. So, if $\mathcal{T}' \subset \mathcal{T}$, then it is said that $\mathcal{T}'$ is *weaker* or *coarser* than $\mathcal{T}$, and that $\mathcal{T}$ is *stronger* or *finer* than $\mathcal{T}'$.

It is easy to show that the intersection of a collection of topologies on a set is again a topology. If $\mathcal{A}$ is an arbitrary nonempty family of subsets of a set X, then there is a smallest topology that includes $\mathcal{A}$. It is the intersection of all topologies that contains $\mathcal{A}$. (The discrete topology always includes $\mathcal{A}$.) This topology is said to be the *topology generated by* $\mathcal{A}$ and consists precisely of $\emptyset$, X and the collection of all sets which are finite intersection of sets from $\mathcal{A}$.

2.2.3 Definition. [19] A *base* for a topology $\mathcal{T}$ is a subcollection $\mathcal{B}$ of $\mathcal{T}$ such that each $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$. Every base $\mathcal{B}$ satisfies the following conditions:

(BO1) For each $x \in X$ there is $U \in \mathcal{B}$ such that $x \in U$,

(BO2) For each $U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$ there is $U \in \mathcal{B}$ satisfying $x \in U \subset U_1 \cap U_2$.

Then we equivalently define the basis as follows: $\mathcal{B}$ is a base for $\mathcal{T}$ if for every $x \in X$ and every open set U containing x, there is a basis open set $B \in \mathcal{B}$ such that $x \in B \subset U$.
Conversely, if $\mathcal{B}$ is a family of sets that is closed under finite intersections and $\bigcup \mathcal{B} = X$, then the family $\mathcal{T}$ of all unions of members of $\mathcal{B}$ is a topology for which $\mathcal{B}$ is a base.

A subcollection $\mathcal{S}$ of a topology $\mathcal{T}$ is a subbase for $\mathcal{T}$ if the collection of all the finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{T}$. Observe, if $\emptyset$ and X belong to a subset of $\mathcal{S}$, then $\mathcal{S}$ is a subbase for the topology it generates.

If Y is a subset of a topological space X, then it is easy to show that the collection $\mathcal{T}_Y$ of subsets of Y, defined by

$$\mathcal{T}_Y = \{Y \cap U \mid \ U \in \mathcal{T}\}$$

is a topology on Y, called the *subspace topology*. When $Y \subset X$ is equipped with the subspace topology, then Y is the *subspace* of X.

2.2.4 Definition. [19] A subset of a topological space is:

(i) a $\mathcal{G}_\delta$*-set*, if it is a countable intersection of open sets.

(ii) an $\mathcal{F}_\sigma$*-set*, if it is a countable union of closed sets.

This example $(0,1] = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \bigcap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right)$ shows that a set can simultaneously be a $\mathcal{G}_\delta$- and an $\mathcal{F}_\sigma$-set [19].

2.2.5 Definition. [19] Let X be a topological space and $A \subset X$ an arbitrary subset of X.

(a) The *interior* of A, denoted $\operatorname{Int} A$, is the largest (with respect to set inclusion) open set in X which is contained in A. (It is the union of all open subsets of A.)

(b) The *closure* of A, denoted $c\ell\, A$, is the smallest closed set in X which contains A. (It is the intersection of all closed sets including A.)

It is easy to show that a set A is open whenever $A = \text{Int}\, A$, and a set B is closed whenever $B = c\ell\, B$. Also, the closure and interior are related by the following formulae, $\text{Int}(X \setminus A) = X \setminus c\ell\, A$ and $c\ell(X \setminus A) = X \setminus \text{Int}\, A$.

2.2.6 Definition. [19] Let X be a topological space and $A \subset X$ an arbitrary subset of X. A *neighbourhood* (written nbd) of a point $x \in X$ is any open subset of X which contains x in its interior. That is, a set $N \subset X$ is a neighbourhood of x if there is an open set $G \in X$ such that $x \in G$ and $G \subset N$.

The collection of all the neighbourhoods of the point $x \in X$ is called the *neighbourhood system*, at the point x and is denoted by $\mathcal{N}_x$. The set "N" as a neighbourhood of x is written "nbd $N(x)$". It follows that $\mathcal{N}_x$ satisfies:

(i) $X \in \mathcal{N}_x$.

(ii) For each $G \in \mathcal{N}_x$, $x \in G$ (so $\emptyset \notin \mathcal{N}_x$).

(iii) If $G, H \in \mathcal{N}_x$, then $G \cap H \in \mathcal{N}_x$.

(iv) If $G \in \mathcal{N}_x$ and $G \subset H$, then $H \in \mathcal{N}_x$.

Thus we see that the neighbourhood system $\mathcal{N}_x$ is a canonical example of a filter (and a good reason why filters are important in topology). The neighbourhood system $\mathcal{N}_x$, is then called the *neighbourhood filter*, at the point x.

The set $c\ell\, A$ always exists and, the point x is a *point of closure* of the set A so it is the set of all points $x \in X$ such that every neighbourhood of x intersects A, i.e.

$$\forall N \in \mathcal{N}_x (N \cap A \neq \emptyset).$$

2.2.7 Definition. [19] Let X be a topological space and $A \subset X$ an arbitrary subset of X. The subset A of X is *dense* in X if $c\ell\, A = X$. Put differently, a set A is dense if and only if for every nonempty open subset of X it contains a point in A. Furthermore, if A is dense in X and x belongs to X, then every neighbourhood of x contains a point that belongs in A. A set N is *nowhere dense* if its closure has empty interior. A topological space is *separable* if it includes a countable dense subset.

2.2.8 Definition. [19] Let X and Y be topological spaces. and $A \subset X$ an arbitrary subset of X.

(a) A function $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.
This definition is global, locally we define continuity as follows. A function $f : X \to Y$ is *continuous at the point* $x \in X$, if for each nbd $V(f(x))$ there exists a nbd $U(x)$ such that $f(U) \subset V$. The definition for both local and global continuity are equivalent.

(b) Let $f : X \to Y$ be a bijection. If both the map f and the inverse mapping $f^{-1} : Y \to X$ are continuous , then the map f is called a *homeomorphism*. The spaces X and Y are said to be *homeomorphic*, denoted $X \cong Y$ if there exists a homeomorphism from X to Y.

A homeormorphism sets up a bijective correspondence between the open sets in X and those in Y, via $U \leftrightarrow f(U)$. In other words a homeomorphism is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open if and only if U is open. With this, we can now say that if Z is the image set $f(X)$, as a subspace of Y; then the function $f' : X \to Z$ reached by restricting the domain of f is bijective. If f' is a homeomorphism of X with Z, we say that the map $f : X \to Y$ is an *embedding*, of X in Y.

Following the definitions of closure and continuity, we have the following:

2.2.9 Remark. [19]

(i) The properties of continuity.

(1) (Composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, so also is $g \circ f : X \to Z$.

(2) (Restriction of domain) If $f : X \to Y$ is continuous and A is a subspace of X, then $f \mid A : A \to Y$ is continuous.

(3) (Restriction or extension of range) If $f : X \to Y$ is continuous and Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the the range of f is continuous. If Z is a space that has Y as a subspace, then the function $h : X \to Z$ obtained by extending the the range of f is continuous.

(ii) The following are characterisation of continuity,

(4) f is continuous.

(5) For every subset A of X, one has $f(c\ell A) \subset c\ell f(A)$.

(6) For every closed set C of Y, the set $f^{-1}(C)$ is closed in X.

2.2.10 Definition. [9] Let X be a topological space and let $\mathcal{F}$ be a filter in X.

(a) $\mathcal{F}$ *converges* to x, written $\mathcal{F} \to x$, if: $\forall\ N \in \mathcal{N}_x\ \exists\ F \in \mathcal{F}\ (F \subset N)$. In other words, $N \in \mathcal{F}$.

(b) $\mathcal{F}$ *accumulates* at x, written $\mathcal{F} \succ x$, if: $\forall\ N \in \mathcal{N}_x\ \forall\ F \in \mathcal{F}\ (F \cap N \neq \emptyset)$.

It follows from this definition that $\mathcal{F} \succ x$ if and only if $x \in \bigcap\{c\ell\, F\ :\ F \in \mathcal{F}\}$; this alternate way of defining accumulation points will be used most often.

2.2.11 Example. [19]

i) Let X be a topological space and $x \in X$. Then $\mathcal{N}_x \to x$. This is clear, since $\mathcal{N}_x$ is a filter and contains itself.

(ii) A sequence (x_n) in a topological space X *converges* to $x \in X$, denoted $x_n \to x$, if, for each neighbourhood N of x, $x_n \in N$ eventually, that is there is an $n_0 \in \mathbb{R}$ such that $n \geq n_0$ implies $x_n \in N$. Let $\mathcal{F} = \{A \subset X \mid x_n \in A \text{ eventually}\}$. Then $\mathcal{F}$ becomes a filter and $x_n \to x$ iff $\mathcal{F} \to x$. This follows because the set $\mathcal{F}$ is closed under supersets and has the FIP and so it is a filter. On the other hand, suppose $x_n \to x$, then for each nbd $N(x)$, $x_n \in N(x)$ eventually; hence $N(x) \in \mathcal{F}$. This shows that $\mathcal{N}_x \subset \mathcal{F}$ and so $\mathcal{F} \to x$. Conversely, suppose $\mathcal{F} \to x$, then for each nbd $N(x)$, it follows that, $N(x) \in \mathcal{F}$ by hypothesis. Thus $x_n \in N(x)$ eventually; hence $x_n \to x$.

From the definition of sequences of part (ii) of the previous example, we can say equivalently that the filter of cofinite subsets on $\{x_1, x_2, \ldots\}$ converges to x where $x_1, x_2, \ldots$ are distinct. The reason why the ordinary definition is insufficient for our purposes is that, for the most important topological spaces we can't describe them and the following theorem fails if we replace "filter" by "sequence".

We recall that if the filter $\mathcal{F}$ is finer than $\mathcal{U}$ then the following holds

$$\forall\, U \in \mathcal{U}\, \exists\, F \in \mathcal{F} : F \subset U.$$

The subset $\subset$ ordering has the following properties:

2.2.12 Remark. [9]

(i) If $\mathcal{U} \subset \mathcal{F}$, then every element of $\mathcal{F}$ meets every element of $\mathcal{U}$.

(ii) $\mathcal{F} \to x$ if and only if $\mathcal{N}_x \subset \mathcal{F}$.

Proof. (i) Suppose $\exists\, U \in \mathcal{U}\, \exists\, F \in \mathcal{F}$ such that $U \cap F = \emptyset$. Since $\mathcal{U} \subset \mathcal{F}$, then for this U we can choose an element $F_0 \in \mathcal{F}$ such that $F_0 \subset U$, and so $F_0 \cap F = \emptyset$ which contradicts the fact that $\mathcal{F}$ is a filter.

(ii) In comparison to Definition (2.2.10)(a), we have that for any $N \in \mathcal{N}_x$, there is $F \in \mathcal{F}$ such that $F \subset N$ and so $N \in \mathcal{F}$.

□

2.2.13 Theorem. *[9] Let $\mathcal{U}$ be an ultrafilter on X. Then $\mathcal{U} \succ x$ if and only if $\mathcal{U} \to x$.*

Proof. We only show the implication $\mathcal{U} \succ x \Rightarrow \mathcal{U} \to x$ as the other direction follows from Remark (2.2.12). Since $\mathcal{N}_x$ is a filter and so can be extended to an ultrafilter $\mathcal{U}$, we have that if given $N(x)$ then there is a $U \subset N$ or $U \subset N^c$. Since $\mathcal{U} \succ x$, then $\forall\, U \in \mathcal{U}\, (U \cap N \neq \emptyset)$, and so the case $U \subset N^c$ cannot occur; hence $\mathcal{U} \to x$. □

2.2.14 Theorem. *[19], [20] Let X be a topological space and $A \subset X$. A point a is in $c\ell A$ if and only if there is a filter in A which converges to a.*

Proof. Suppose we can choose a filter in A which converges to a. Then $\mathcal{N}_a \subset \mathcal{F}$, and so $N(a) \cap A \neq \emptyset$ for some $F \in \mathcal{F}$, thus $a \in c\ell A$.
Conversely, let $a \in c\ell A$ and so $N \cap A \neq \emptyset$ for each $N \in \mathcal{N}_a$. Then the set $\mathcal{F} = \{N \cap A \;:\; N \in \mathcal{N}_a\}$ has the FIP and so a filter according to Definition (2.1.11) and $\mathcal{F} \to x$ because for each $N \in \mathcal{N}_a$, $N \cap A \in \mathcal{F}$ and $N \cap A \subset N$. □

2.2.15 Theorem. *[19] For any topological spaces X and Y, let $x \in X$. Let f be a mapping of X onto Y. Then for each filter $\mathcal{F}$ in X converging to x, we have that $f(\mathcal{F}) \to f(x)$ if and only if f is continuous at x. The same conclusion holds with $\mathcal{F}$ an ultrafilter.*

Proof. Suppose f is continuous at x and let $\mathcal{F}$ be a filter in X which converges to x. Let the nbd $N(f(x))$ be in Y. Then since f is continuous, $f^{-1}(N)$ is a nbd of x in X. Then $f^{-1}(N) \in \mathcal{F}$ and so $f(f^{-1}(N)) \in f(\mathcal{F})$. But $N \supset f(f^{-1}(N))$ and so $f(\mathcal{F}) \to f(x)$.
Conversely, suppose the given condition is satisfied. Let B be a closed set in Y and let $A = f^{-1}(B)$. We show that $c\ell A \subset A$. Let $x \in c\ell A$, then there is a filter $\mathcal{F}$ satisfying $\mathcal{F} \to x$, and so $f(\mathcal{F}) \to f(x)$. Then $B \supset f(A)$, and so $x \in A$. Therefore, f is continuous. □

2.2.16 Corollary. [9] Let $f : X \to Y$. Then f is continuous at $x \in X$ if and only if the nbd filter $f(\mathcal{N}_x)$ converges to $f(x)$.

Proof. The definition of continuity at x says

$$\forall\, V(f(x))\ \exists\, U(x)\ (f(U) \subset V).$$

So this is the exact definition that the nbd filter $f(\mathcal{N}_x)$ converges to $f(x)$. □

2.2.17 Definition. [19] Let X be a topological space.

(a) X is a *Hausdorff space* if for each distinct point $x, y \in X$ there are nbds $U(x)$ and $V(y)$ respectively such that $U(x) \cap V(y) = \emptyset$.

(b) X is *regular* if K is a closed subset of X and $x \in X$ does not belong to K, then there is disjoint open sets G and H such that $K \subset G$ and $x \in H$.

(c) X is *completely regular* if K is a closed subset of X and $x \in X$ does not belong to K, then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f[K] = \{1\}$.

(d) A collection $\mathcal{A} = \{U_\alpha\ : \alpha \in J\}$ of open subsets of X is an *open cover* of X if $X = \bigcup_{\alpha \in J} U_\alpha$ where J is some index. The space X is *compact* if every open covering of X contains a finite subcollection that also covers X.

(e) X is *locally compact* if every point of X has a compact neighbourhood.

(f) A locally compact space is *σ-compact* if it can be expressed as the union of at most countably many compact spaces.

Whenever we consider a topological space X and a subspace Y of X we will interpret the notion *open cover of Y* as a cover of Y by sets open in X.

The definition of regularity has the following equivalences:

2.2.18 Remark. [9] The following are equivalent:

(i) X is regular.

(ii) For each $x \in X$ and a nbd N of x, there exists a nbd U of x with $x \in U \subset c\ell\, U \subset N$.

(iii) For each $x \in X$ and a closed set A not containing x, there is a nbd U of x with $c\ell\, U \cap A = \emptyset$.

The following are convergence properties of filters.

2.2.19 Theorem. *[19] The space X is Hausdorff if and only if each convergent filter in X converges to at most one point.*

Proof. Suppose X is Hausdorff and $\mathcal{F}$ a filter on X which is convergent to two distinct points x and y. Choose two disjoint nbd $U(x), V(y)$ in $\mathcal{F}$ respectively. Then $U(x) \cap V(y) = \emptyset$, but this contradicts the fact that $\mathcal{F}$ is a filter.
Conversely, suppose that each filter on X is convergent to at most one point but the space X is not Hausdorff. Choose two distinct points $x, y \in X$ such that for each nbd $U(x), V(y)$, $U(x) \cap V(y) \neq \emptyset$.

So the collection $\{U \cap V,\ U \in \mathcal{N}_x,\ V \in \mathcal{N}_y\}$ has the FIP and so can be extended to a filter $\mathcal{F}$. It follows that, $\mathcal{N}_x \cup \mathcal{N}_y \subset \mathcal{F}$ and so the filter $\mathcal{F}$ converges to two distinct points x and y, a contradiction with the supposition. □

The following theorem will be important later when dealing with extending functions on subsets of X to X.

2.2.20 Theorem. *[9] For any pair f, g of continuous mappings of a space X into a Hausdorff space Y, the set $\{x \in X : f(x) = g(x)\}$ is closed in X.*

Proof. Let $B = \{x \in X : f(x) = g(x)\}$ and let $\mathcal{F}$ be a filter and lies in B converging to y. Now by Theorem (2.2.15) $f(\mathcal{F}) \to f(y)$ and $g(\mathcal{F}) \to g(y)$. Since $f(\mathcal{F}) = g(\mathcal{F})$, we deduce that $f(y) = g(y)$. Hence $y \in B$ and B is closed. □

2.2.21 Corollary. [9] If f, g are continuous mappings of a space X into a Hausdorff space Y, and f and g agree on a dense set D in X, then $f = g$.

Proof. By Theorem (2.2.15), $D = \{x \in X : f(x) = g(x)\}$ is closed. Since D is assumed dense in X, we have $X = c\ell D = D$, i.e., $f(x) = g(x)$ for all $x \in X$. □

The class of weak topologies is by far one of the interesting notions to study.

2.2.22 Definition. [19] Let X be a nonempty set, and let $\{(Y_i, \mathcal{T}_i)\}_{i \in I}$ be a family of topological spaces and for each $i \in I$ let $f_i : X \to Y_i$ be a function. The *weak topology* on X generated by the collection of functions $\{f_i\}_{i \in I}$ is the weakest topology on X that makes all the functions f_i continuous. It is the topology generated by the collection of sets

$$\{f_i^{-1}(U) :\ i \in I \text{ and } U \in \mathcal{T}_i\}.$$

A subbase for this topology is of the form

$$\{f_i^{-1}(U) :\ i \in I \text{ and } U \in \mathcal{S}_i\},$$

where $\mathcal{S}_i$ is a subbase for $\mathcal{T}_i$.

A base for the weak topology is a collection of sets of the form $\bigcap_{k=1}^{n} f_{i_k}^{-1}(U_{i_k})$, where each $U_{i_k} \in \mathcal{T}_{i_k}$.

A very special case is the weak topology generated by a family of real functions. For a family $\mathcal{G}$ of real functions on X, the weak topology generated by $\mathcal{G}$ is denoted $\sigma(X,\ \mathcal{G})$. It follows that a subbase for $\sigma(X,\ \mathcal{G})$ can be found by taking all sets of the form

$$U(f, x, \varepsilon) = \{y \in X :\ |f(y) - f(x)| < \varepsilon\},$$

where $f \in \mathcal{G},\ x \in X$ and $\varepsilon > 0$.

It is said that the collection $\mathcal{G}$ of real functions on X, *separates points* if for each distinct points $x,\ y \in X$, there is a function $f_\alpha \in \mathcal{G}$ such that $f_\alpha(x) \neq f_\alpha(y)$.

2.2.23 Proposition. [19] Let X be a Hausdorff space. Then the weak topology $\sigma(X,\ \mathcal{F})$ is Hausdorff.

Proof. Assume that $x,\ y \in X$ are distinct. Then choose $i \in I$ such that such that $f_i(x) \neq f_i(y)$. Since X is Hausdorff, we can choose elements $G \in \mathcal{N}_x,\quad H \in \mathcal{N}_y$ satisfying $f_i(x) \in G,\ f_i(y) \in H$ and $G \cap H = \emptyset$. But according to the $\sigma(X,\ \mathcal{F})$-topology the sets $f_i^{-1}(G)$ and $f_i^{-1}(H)$ are open with $x \in f_i^{-1}(G),\ y \in f_i^{-1}(H)$ and $f_i^{-1}(G) \cap f_i^{-1}(H) = \emptyset$. □

We would like to get a better understanding of the $\sigma(X,\ \mathcal{F})$-topology as it plays a huge role in our study of Banach algebras later on the thesis.

2.2.24 Definition (University lecture notes)**.** Let Λ be a set of indexes and for every $\alpha \in \Lambda$, let X_α be a topological space. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ denote the *Cartesian product* of the sets X_α, that is, the set of all functions $(x_\alpha) : \Lambda \ni \alpha \mapsto X_\alpha$. The *product topology* on X is defined by taking as a base the sets of the form $\prod_{\alpha\in\Lambda} U_\alpha$, where each $U_\alpha \subset X_\alpha$ is open and $\{\alpha \in \Lambda : U_\alpha \neq X_\alpha\}$ is finite. The set X endowed with the product topology is called the *product of spaces* X_α and is denoted by $\prod_{\alpha\in\Lambda} X_\alpha$.

The map $\pi_\beta : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$, defined by $\pi_\alpha(x) = x_\beta$, is called the *projection map* of $\prod_{\alpha\in\Lambda} X_\alpha$ on X_β, or precisely, the *βth projection map*.

Note that if $X_\alpha = X$ for each $\alpha \in \Lambda$, then $\prod_{\alpha\in\Lambda} X_\alpha$ is just the set X^Λ of all functions from Λ to X.

Also notice that the $\prod_{\alpha\in\Lambda} U_\alpha$, where $U_\alpha = X_\alpha$, except for all but finitely many $\alpha's$, (that is, where $\{\alpha \in \Lambda : U_\alpha \neq X_\alpha\}$ is finite), can be written as

$$\prod_{\alpha\in\Lambda} U_\alpha = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}).$$

Thus the product topology is that topology which has for a subbase the collection $\{\pi_\alpha^{-1}(U_\alpha) \mid \alpha \in \Lambda,\ U_\alpha \text{ open in } X_\alpha\}$.

Observe that the βth projection map is continuous and open, but need not be closed [20].

2.2.25 Proposition. [20] The product topology is the weakest topology on $\prod_{\alpha\in\Lambda} X_\alpha$ such that each projection map π_β is continuous.

Proof. Suppose that $\mathcal{U}$ is a topology on $\prod_{\alpha\in\Lambda} X_\alpha$ for which each projection is continuous, then for each β, if U_β is open in X_β, $\pi_\beta^{-1}(U_\beta) \in \mathcal{U}$. Since these sets form a subbase for the product topology we can conclude that the product topology is contained in $\mathcal{U}$ as required. □

By Proposition (2.2.25), the product topology is the $\sigma(\prod_{\alpha\in\Lambda} X_\alpha,\ \mathcal{G})$-topology, where $\mathcal{G}$ is the family of all projection functions $\{\pi_\alpha : \alpha \in \Lambda\}$. In other words, the product topology can be characterised as the weakest topology on $\prod_{\alpha\in\Lambda} X_\alpha$ to which all projection functions (π_α) are continuous.

2.2.26 Definition. [20] If, for each $\alpha \in \Lambda, f_\alpha : X \to X_\alpha$, then the evaluation map $e : X \to \prod_{\alpha\in\Lambda} X_\alpha$ induced by the collection $\{f_\alpha \mid \alpha \in \Lambda\}$ is defined as follows: for each $x \in X$, $[e(x)]_\alpha = f_\alpha(x)$. That is, for $x \in X$, $e(x)$ is the point in $\prod_{\alpha\in\Lambda} X_\alpha$ whose αth coordinate is $f_\alpha(x)$ for each $\alpha \in \Lambda$.

2.2.27 Theorem. *[20] For each $\alpha \in \Lambda$, let $f_\alpha : X \to X_\alpha$. Then the evaluation map $e : X \to \prod_{\alpha\in\Lambda} X_\alpha$ is an embedding if and only if X has the weak topology given by the functions f_α and the collection $\{f_\alpha \mid \alpha \in \Lambda\}$ separates points in X.*

Theorem (2.2.27) forms the essential core of many constructions meant to deal with questions like: if given a space X and a property $\mathcal{P}$ of spaces, can X be embedded in a larger space say Y having property $\mathcal{P}$? The best known example is given in Section (2.3) which is the Stone-Čech compactification βX of a product space X and is a typical use of this theorem [20].

2.2.28 Theorem. *[20] A filter $\mathcal{F}$ converges to x in $\prod_{\alpha\in\Lambda} X_\alpha$ if and only if $\pi_\alpha(\mathcal{F}) \to \pi_\alpha(x)$ in X_α.*

Proof. Suppose $\mathcal{F} \to x$ in $\prod X_\alpha$ and let N be a nbd of $\pi_\alpha(x)$. Then since π_α is continuous, $\pi_\alpha^{-1}(N) \in \mathcal{N}_x$. So $\pi_\alpha^{-1}(N)$ is the nbd of x and so $\pi_\alpha^{-1}(N) \in \mathcal{F}$ so $N \in \pi_\alpha(\mathcal{F})$. Thus $\pi_\alpha(\mathcal{F}) \to \pi_\alpha(x)$ in X_α.

Conversely, suppose $\pi_\alpha(\mathcal{F}) \to \pi_\alpha(x)$, for each α. Let $\bigcap_{k=1}^n \pi_{\alpha_k}^{-1}(N_k)$ be a nbd of x in $\prod X_\alpha$. Then N_k is a nbd of $\pi_{\alpha_k}(x)$, for each k. So $N_k \in \pi_{\alpha_k}(\mathcal{F})$, for each k, and hence $F_k \subset \pi_{\alpha_k}^{-1}(N_k)$, for some $F_k \in \mathcal{F}$. Then $\bigcap_{k=1}^n F_k \in \mathcal{F}$ since $\mathcal{F}$ has FIP and $\bigcap_{k=1}^n F_k \subset \bigcap_{k=1}^n \pi_{\alpha_k}^{-1}(N_k)$, so $\bigcap_{k=1}^n \pi_{\alpha_k}^{-1}(N_k) \in \mathcal{F}$. Thus $\mathcal{F} \to x$. □

The definition of compactness has several equivalent formulations:

2.2.29 Theorem. *[20] For a topological space X, the following statements are equivalent:*

(i) X is compact.

(ii) Each collection $\mathcal{A}$ of closed subsets of X with the FIP has nonempty intersection.

(iii) Each filter in X has at least one accumulation point.

(iv) Each ultrafilter in X converges.

Proof. (i) implies (ii). Assume that X is compact, and let $\mathcal{A}$ be a collection of closed subset of X. If $\bigcap_{A\in\mathcal{A}} A = \emptyset$, then $X = \bigcup_{A\in\mathcal{A}} A^c$, thus $\{A^c : A \in \mathcal{A}\}$ is an open cover of X. Therefore we can choose $A_1^c, \dots, A_m^c \in \mathcal{A}$ satisfying $X = \bigcup_{i=1}^n A_i^c$. Thus $\bigcap_{i=1}^n A_i = \emptyset$, and so $\mathcal{A}$ does not have the FIP. Hence if $\mathcal{A}$ has the FIP then $\bigcap_{A\in\mathcal{A}} A \neq \emptyset$.

(ii) implies (iii). Let $\mathcal{F}$ be a filter in X. Then $\mathcal{F}$ has the FIP. So $\{c\ell\, F \mid F \in \mathcal{F}\}$ is a collection of closed sets with the FIP, and so there is a point $x \in \bigcap\{c\ell\, F \mid F \in \mathcal{F}\}$. Then $\mathcal{F}$ has x as an accumulation point.

(iii) implies (iv). Let $\mathcal{U}$ be an ultrafilter on X. Since $\mathcal{U} \succ x$, and $\mathcal{U}$ is maximal and so $\mathcal{U} \to x$.

(iv) implies (i). Suppose that every ultrafilter on X is convergent and let $\mathcal{O}$ be an open cover of X. Suppose, on the contrary, that $\mathcal{O}$ has no finite subcover. Then

$$\mathcal{F} = \{A \subset X \ : \ X \setminus \bigcup \mathcal{V} \subset A \text{ for some finite } \mathcal{V} \subset \mathcal{O}\}$$

is a filter on X by Definition (2.1.11). Extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. Now let $x \in X$ and choose $O \in \mathcal{O}$ such that $x \in O$. Since $X\setminus \in \mathcal{F} \subset \mathcal{U}$, $\mathcal{U}$ does not converge to x. Therefore, $\mathcal{U}$ is not convergent, which is a contradiction. □

The following are properties of compactness.

2.2.30 Remark. [19]

(a) Continuous image of a compact set is compact.

(b) A compact subset A of a Hausdorff space X is closed in X.

(c) A subspace of compact space is compact if and only it is closed.

2.2.31 Theorem. *[20]* (**Tychonoff**) *Then product $\prod_{\alpha\in\Lambda} X_\alpha$ is compact if and only if each X_α is compact.*

Proof. If $\prod_\alpha X_\alpha$ is compact, then since each projection $\pi_\beta : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$ is continuous surjection, we have by (a) of Remark (2.2.30) that each X_α is compact.
Conversely, suppose that each X_α is compact, and let $\mathcal{F}$ be an ultrafilter in $\prod_{\alpha\in\Lambda} X_\alpha$. Since the image of an ultrafilter is an ultrafilter by Theorem (2.1.20), and consequently, since X_α is compact, each $\pi_\alpha(\mathcal{F})$ is convergent to some $x_\alpha \in X_\alpha$. According to Theorem (2.2.29), we have that $\mathcal{F}$ converges to the point $x = \{x_\alpha\} \in \prod_{\alpha\in\Lambda} X_\alpha$, and so $\prod_{\alpha\in\Lambda} X_\alpha$ is compact. □

Tychonoff theorem is the most important result in general (nongeometric) topology. It forms part in the development of a wealth of theorems within topology and other applications of topology to other fields [20]. The construction of the Stone-Čech compactification βX of any product space X depends on it, the proof of compactness of the maximal ideal space of a Banach algebra requires it and hence it is central to the development of the Gelfand representation theorem [20].

2.3 βD-Ultrafilters and The Stone-Čech Compactification of a Discrete Space

In this section we define a topology on the set of all ultrafilters on a set D, denoted as βD. this topology is, in fact, compact and Hausdorff. The space βD can be thought, in a usual way, as a compactification of D with the discrete topology, and as such it enjoys a universal property which explains its central importance. We will establish this property and some other properties of the resulting space.

2.3.1 The Topological Space βD.

2.3.2 Definition. [14] Let D be a discrete topological space.

a) $\beta D = \{p : p \text{ is an ultrafilter on } D\}$.

b) Given $A \subset D$, $\widehat{A} = \{p \in \beta D : A \in p\}$.
The collection
$$\mathcal{B} = \{\widehat{A} \ : \ A \subset D\},$$
consisting of all sets $\widehat{A}$ is called the *base* of βD.

In the upcoming subsection, we shall make clear the reason for the notation βD. We shall from now on use lower case letters to denote ultrafilters on D, since we shall be taking ultrafilters as points in a topological space.

2.3.3 Definition. [14] Let D be a set and let $a \in D$. Then $e(a) = \{A \subset D : a \in A\}$.

Therefore for each $a \in D$, $e(a)$ is the principal ultrafilter corresponding to a.

2.3.4 Lemma. [14] Let D be a set and let $A, B \subset D$.

(a) $(\widehat{A \cap B}) = \widehat{A} \cap \widehat{B}$;

(b) $(\widehat{A \cup B}) = \widehat{A} \cup \widehat{B}$;

(c) $(\widehat{D \setminus A}) = \beta D \setminus \widehat{A}$;

(d) $\widehat{B} = \emptyset$ if and only if $B = \emptyset$;

(e) $\widehat{B} = \beta D$ if and only if $B = D$;

(f) $\widehat{B} = \widehat{A}$ whenever $B = A$.

Proof. This proof makes an extensive use of Theorem (2.1.18).

(a) $p \in (\widehat{A \cap B})$ if and only if $A \cap B \in p$ if and only if $A \in p$ and $B \in p$ if and only if $p \in \widehat{A} \cap \widehat{B}$.

(b) $p \in (\widehat{A \cup B})$ if and only if $A \cup B \in p$ if and only if $A \in p$ or $B \in p$ if and only if $p \in \widehat{A} \cup \widehat{B}$.

(c) $p \in \widehat{D \setminus A}$ if and only if $D \setminus A \in p$ if and only if $A \notin p$ if and only if $p \in \beta D \setminus \widehat{A}$.

(d) By Theorem (2.1.18)(b), $B \in p$ if and only if $B \neq \emptyset$ and so $\widehat{B} \neq \emptyset$.

(e) Assume $B = D$ and let $p \in \beta D$. Then $D \in p$, so $B \in p$ hence $\beta D \subset \widehat{B}$.
Conversely, suppose $\widehat{B} = \beta D$. Then for any $p \in \widehat{B}$, $B \in p$. So choose $D \in p$ such that $D \subset B$.
Alternative: Note we could make extensive use of (d) as follows
$D \setminus B = \emptyset$ if and only if $\widehat{D \setminus B} = \emptyset$ if and only if $\beta D \setminus \widehat{B} = \emptyset$ if and only if $\beta D = \widehat{B}$.

(f) Suppose $B = A$ then $\widehat{B} = \{p \in \beta D : B \in p\} = \{p \in \beta D : A \in p\} = \widehat{A}$.

□

We observe that Lemma (2.3.4) shows that the sets of the form $\widehat{A}$ are closed under finite intersections, unions and by complements. Consequently by closure under finite intersections, there is a unique topology on βD which has $\mathcal{B} = \{\widehat{A} : A \subset D\}$ as an open base. Observe that also this sets of the form $\widehat{A}$ are a base for closed sets because of (c). The following theorem describes some of the basic topological properties of βD.

2.3.5 Theorem. *[14] Let D be any set.*

(a) βD is a compact Hausdorff space.

(b) The sets of the form $\widehat{A}$ are the clopen (i.e., closed and open) subsets of βD.

(c) For every $A \subset D$, $\widehat{A} = c\ell_{\beta D}\, e[A]$.

(d) If $A \subset D$ and $p \in \beta D$ are arbitrary, then $p \in c\ell_{\beta D}\, e[A]$ if and only if $A \in p$.

(e) The function e is one-to-one and $e[D]$ is a dense subset of $A \subset D$ whose points form the isolated points of βD.

(f) If O is an arbitrary open subset of βD, then $c\ell_{\beta D}\, O$ is also open.

Proof. (a) Suppose that p and q are distinct points in βD. Being maximal, p cannot be contained in q, so choose some $A \in p \setminus q$. Then $D \setminus A \in q$, and so $\widehat{A}$ and $\widehat{D \setminus A}$ are disjoint open subsets of βD containing p and q respectively. Hence βD is Hausdorff.
To show that βD is compact, we consider a collection $\mathcal{A}$ of sets of the form $\widehat{A}$ with the FIP and will show that $\mathcal{A}$ has nonempty intersection. Let $\mathcal{B} = \{A \subset D : \widehat{A} \in \mathcal{A}\}$. If $\mathcal{F}$ is a finite nonempty subset of $\mathcal{B}$, then there is some $p \in \bigcap_{A \in \mathcal{F}} \widehat{A}$ and so $\bigcap \mathcal{F} \in p$ and hence $\bigcap \mathcal{F} \neq \emptyset$. In other words, $\mathcal{B}$ has the FIP, and according to Corollary (2.1.25), choose $q \in \beta D$ satisfying $\mathcal{B} \subset q$. So $q \in \bigcap \mathcal{A}$.

(b) By the remark before this theorem, we pointed out that each set $\widehat{A}$ was closed as well as open. Suppose that C is any clopen subet of βD. Let $\mathcal{A} = \{\widehat{A} : A \subset D$ and $\widehat{A} \subset C\}$. Because C is open, it follows that $\mathcal{A}$ is an open cover of C. Because C is closed, it is compact by (a) and so choose a finite sub collection $\mathcal{F}$ of $\mathcal{P}(D)$ satisfying $C = \bigcup_{A \in \mathcal{F}} \widehat{A}$. According to Lemma (2.3.4)(b), we can conclude $C = \widehat{\bigcup \mathcal{F}}$.

(c) For each $a \in A$, $e(a) \in \widehat{A}$ and hence $c\ell_{\beta D}\, e[A] \subset \widehat{A}$. For the inclusion, let $p \in \widehat{A}$. Whenever $\widehat{B}$ is a nbd of p, it follows that $A \in p$ and $B \in p$, hence $A \cap B \neq \emptyset$. Choose $a \in A \cap B$. Because $e(a) \in e[A] \cap \widehat{B}$, $e[A] \cap \widehat{B} \neq \emptyset$ and therefore $p \in c\ell_{\beta D}\, e[A]$.

(d) According to (c) and the definition of $\widehat{A}$, we have that $p \in c\ell_{\beta D}\, e[A]$ if and only if $p \in \widehat{A}$ if and only if $A \in p$.

(e) If $a, b \in D$ are distinct, then $D \setminus \{a\} \in e(a) \setminus e(b)$ and it follows that $e(a) \neq e(b)$.
According to Lemma (2.3.4)(d), $A \neq \emptyset$ whenever $\widehat{A}$ is a nonempty open subset of βD. We have that any $a \in A$ satisfies $e(a) \in e[D] \cap \widehat{A}$ and so $e[D] \cap \widehat{A} \neq \emptyset$. Thus $e[D]$ is dense in βD.
For any $a \in D$, $e(a)$ is isolated in βD because $\widehat{\{a\}}$ is an open subset of βD, then $\{p\} \cap e[D] \neq \emptyset$ and so $p \in e[D]$.

(f) If $O = \emptyset$, then we can conclude that $c\ell_{\beta D}\, O$ is open and so suppose $O \neq \emptyset$. Let $A = e^{-1}[O]$.
We claim that $O \subset c\ell_{\beta D}\, e[D]$ is also open. So let $p \in O$ and let $\widehat{B}$ be a nbd of p. Then $O \cap \widehat{B}$ is a nonempty open set and so by (e), $O \cap \widehat{B} \cap e[D] \neq \emptyset$. Then choose $p \in B$ with $e(b) \in O$, and so $e(b) \in \widehat{B} \cap e[A]$ and it follows that $\widehat{B} \cap e[A] \neq \emptyset$.
Next $e[A] \subset O$ and whence $O \subset c\ell_{\beta D}\, e[A] \subset c\ell_{\beta D}\, O$. Therefore, $c\ell_{\beta D}\, O = c\ell_{\beta D}\, e[A] = \widehat{A}$ (according to (c)), and so $c\ell_{\beta D}\, O$ is open in βD.

□

We next establish a characterization of the closed subsets of βD which will be useful later.

2.3.6 Definition. [14] Let D be a set and let $\mathcal{A}$ be a filter on D. Then $\widehat{\mathcal{A}} = \{p \in \beta D : \mathcal{A} \subset p\}$.

2.3.7 Theorem. *[14] Let D be a set.*

a) If $\mathcal{A}$ is a filter on D, then $\widehat{\mathcal{A}}$ is a closed subset of βD.

b) If $\emptyset \neq A \subset \beta D$ and $\mathcal{A} = \bigcap A$, then $\mathcal{A}$ is a filter on D and $\widehat{\mathcal{A}} = c\ell A$.

Proof. (a) Let $p \in \beta D \setminus \widehat{\mathcal{A}}$. Choose $B \in \mathcal{A} \setminus p$. Then $\widehat{D \setminus B}$ is a nbd of p which does not meet $\widehat{\mathcal{A}}$.

(b) $\mathcal{A}$ is the intersection of filters and so is a filter. Furthermore, for each $p \in A$, $\mathcal{A} \subset p$ and so $A \subset \widehat{\mathcal{A}}$ and hence by (a), $c\ell$A $\subset \mathcal{A}$. To see the reverse inclusion, let $p \in \widehat{\mathcal{A}}$ and let $B \in p$. Suppose that $\widehat{B} \cap A = \emptyset$. Then for each $q \in A$, $D \setminus B \in q$ and so $D \setminus B \in \mathcal{A} \subset p$, a contradiction.

□

It is customary to identify a principal ultrafilter $e(x)$ with the point x, and we shall adopt that practice ourselves after we have proved that βD is the Stone-Čech compactification of the discrete space D in the next section. Once this is done, we can write the following definition as $A^* = \widehat{A} \setminus A$. For the moment, we shall continue to maintain the distinction between x and $e(x)$.

2.3.8 Definition. [14] Let D be a set and let $A \subset D$. Then $A^* = A \setminus e[A]$.

The following theorem is simple but useful. Once $e(x)$ is identified with x the conclusion becomes "$U \cap D \in p$".

2.3.9 Theorem. *[14] Let $p \in \beta D$ and let O be a subset of βD. If O is a neighbourhood of p in βD, then $e^{-1}[O] \in p$.*

Proof. If O is a nbd of p, choose an open set $\widehat{A}$ of βD for which $p \in \widehat{A} \subset O$. Thus $A \in p$ and so $e^{-1}[O] \in p$, because $A \subset e^{-1}[O]$. □

We call a space *zero dimensional* if and only if its collection of clopen sets forms a basis.

2.3.10 Theorem. *[14] Let X be a zero dimensional space and let Y be a compact subset of X. The clopen subsets of Y are the sets of the form $C \cap Y$ where C is clopen in X. In particular, if D is an infinite set, then the nonempty clopen subsets of D^* are the sets of the form A^* where A is an infinite subset of D.*

2.3.11 Corollary. [14] Let S be an infinite discrete space. Then

$$\{A^* : A \text{ is an infinite subset of } S\}$$

is a basis for the topology of S^*.

2.3.12 Stone-Čech Compactification. In this section we show that βD is the Stone-Čech compactification of the discrete space D. Recall that by an *embedding* of a topological space X into a topological space Z, one means a function $\varphi : X \to Z$ which defines a homeomorphism from X onto $\varphi[X]$.

We remind the reader that we are assuming that all hypothesized topological spaces are Hausdorff.

2.3.13 Definition. [14] Let X be an arbitrary topological space. The pair (φ, C) is a *compactification* of X satisfying C is a compact space, φ is an embedding of X into C, and $\varphi[X]$ is dense in C.

Any completely regular space X has a largest compactification called its Stone-Čech compactification.

2.3.14 Definition. [14] Let X be an arbitrary completely regular topological space. A pair (φ, Z) is a Stone-Čech compactification of X satisfying

(a) Z is compact,

(b) φ is an embedding into Z,

(c) $\varphi[X]$ is a dense subset of Z, and

(d) for any compact space Y and any continuous mapping $f : X \to Y$ there exists a continuous function $g : Z \to Y$ satisfying $g \circ \varphi = f$.

One by custom refers to *the* Stone-Čech compactification of a space X rather than *a* Stone-Čech compactification of X. This is clear by the following remark.

2.3.15 Remark. Let X be a completely regular space and let (φ, Z) and (τ, W) be Stone-Čech compactifications of X respectively. Then there is a homeomorphism of $\gamma : Z \to W$ such that $\gamma \circ \varphi = \tau$.

2.3.16 Theorem. *[14] Let D be an arbitrary discrete space. Then $(e, \beta D)$ is the Stone-Čech compactification of X.*

Proof. The conditions (a), (b), and (c) of Definition (2.3.14) hold by Theorem (2.3.5). So we verify condition (d).
Let Y be a given compact space and let $f : D \to Y$. Let $\mathcal{U}_p = \{c\ell_Y f[A] : A \in p\}$ for each $p \in \beta D$. Then $\mathcal{U}_p$ has the FIP and so has a nonempty intersection for each $p \in \beta D$. Choose $g(p) \in \bigcap \mathcal{U}_p$. Then we will show that $g \circ e = f$ and that g is continuous.
To see the first assertion, let x belong to D. Then $\{x\} \in e(x)$ and so $g(e(x)) \in c\ell_Y f[\{x\}] = c\ell_Y[\{f(x)\}] = \{f(x)\}$ and so $g \circ e = f$.
We show that g is continuous, let $p \in \beta D$ and let O be a nbd of $g(p)$ in Y. Because Y is regular, choose a nbd N of $g(p)$ with $c\ell_Y N \subset O$ and let $A = f^{-1}[N]$. We claim $A \in p$ and so assume that $D \setminus A \in p$. Then $g(p) \in c\ell_Y f[D \setminus A]$ and N is a nbd of $g(p)$ and it follows that $N \cap f[D \setminus A] \neq \emptyset$, contradiction as we have that $A = f^{-1}[N]$. So $\widehat{A}$ is a nbd of p. We show that $g[\widehat{A}] \subset O$, and so let $q \in \widehat{A}$ and suppose that $g(q) \notin O$. Then $Y \setminus c\ell_Y N$ is a nbd of $g(q)$ and $g(q) \in c\ell_Y f[A]$ and so $(Y \setminus c\ell_Y N) \cap f[A] \neq \emptyset$, this also contradicts that $A = f^{-1}[N]$. □

It is common practice in dealing with βD to identify the points of D with the principal ultrafilters generated by those points, and we shall adopt this practice from this point on. Only rarely will it be necessary to remind the reader that when we write s we sometimes mean $e(s)$.

Once we have identified $s \in S$ with $e(s) \in \beta D$, we shall suppose that $D \subseteq \beta D$ and shall write $D^* = \beta D \setminus D$, rather than $D^* = \beta D \setminus e[D]$. Further, with this identification, $\widehat{A} = c\ell_{\beta D} A$ for every $A \in \mathcal{P}(D)$, by Theorem (2.3.5). So the notations $\widehat{A}$ and $c\ell A$ become interchangeable.

2.3.17 Theorem. *[14](Stone-Čech Compactification-restated). Let D be an infinite discrete space. Then*

a) βD is a compact space,

b) $D \subset \beta D$,

c) *D is a dense subset of βD, and*

d) *for any any compact space Y and any mapping $f : D \to Y$ there is a continuous mapping $g : \beta D \to Y$ satisfying $g_{|D} = f$.*

2.3.18 More on topology of βD. If f is a continuous function from a completely regular space X into a compact space Y, then we shall often denote $\widetilde{f}$ to be the continuous function from βX to Y which extend f, although in some cases we may use the same notation for a function and its extension. (Notice that there can be only one continuous extension, since any two extensions agree on a dense subspace.)

Let D be a discrete space, let Y be a compact space, and let $f : D \to Y$. Then $\widetilde{f}$ is the continuous function from βD to Y such that $\widetilde{f}_{|D} = f$.

If $f : X \to Y$ is a continuous function between completely regular spaces, it has a continuous extension $f^{\beta} : \beta X \to \beta Y$.

2.3.19 Lemma. [14] Let D and E be discrete spaces and let $f : D \to E \subset \beta E$. For each $p \in \beta D$, $\widetilde{f}(p) = \{A \subset E : f^{-1}[A] \in p\}$. It follows that, if $A \in p$, then $f[A] \in \widetilde{f}(p)$; and if $B \in \widetilde{f}(p)$, then $f^{-1}[B] \in p$.

Proof. By Theorem (2.1.20), $\widetilde{f}$ is an ultrafilter on E. For each $p \in \beta D$, let $g(p) = \{A \subset E : f^{-1}[A] \in p\}$. Now, given $x \in D$ we have $g(x) = \{A \subset E : f(x) \in A\}$. Recall that we identify x with $e(x)$ and $f(x)$ with $e(f(x))$, we have that $g(x) = f(x)$. To see that g is continuous, let $\widehat{A}$ be an open set in βE. then $g^{-1}[\widehat{A}] = \widehat{f^{-1}[A]}$. Since g is a continuous extension of f, we have that $\widetilde{f} = g$. □

We establish some results about the Stone-Čech compactification of a semigroup which will usually be cancellative and countable.

2.3.20 Definition. [14] A topological space is *extremally disconnected* if the closure of every open subset is open.

In Theorem (2.3.5)(f), it was shown that βD is an extremally disconnected space. Since we're interested in the space D^*, the reader is cautioned that D^* is not extremally disconnected [14].

We shall off-times use the following results.

2.3.21 Theorem. *[14] Let D be an arbitrary discrete space and let G and H be σ-compact subsets of βD. Whenever $G \cap c\ell\, H = c\ell\, G \cap H = \emptyset$, $c\ell\, G \cap c\ell\, H = \emptyset$.*

Proof. See [14, Theorem 3.40]. □

2.4 Uniform Limits via Ultrafilters

We introduce the notion of p-limits. We would like $p\text{-}\lim_{s \in D} x_s = y$ to mean that x_s is "often" "close to" y. Where closeness is determined by neighbourhoods of y while "often" is determined by members of p.

As we shall note, the notion is as versatile as the notion of filter convergence, and has two substantial advantages [14]: (1) in a compact space a p-limit always converges and (2) it provides a "uniform" way of taking limits, in opposition to randomly choosing from among many possible limits points of a convergent filter.

2.4.1 Definition. [14] Let D be an arbitrary discrete space, let $p \in \beta D$, let $\langle x_s \rangle_{s\in D}$ be an indexed collection in an arbitrary topological space X, and let $y \in X$. Then $p\text{-}\lim\limits_{s\in D} x_s = y$ if and only if for every neighbourhood O of y, $\{s \in D \ : \ x_s \in O\} \in p$.

Since we identified $\widehat{A}$ with βA for $A \subset D$. It follows that, if $A \in p$ and x_s is defined for $s \in A$, we write $p\text{-}\lim\limits_{s\in A} x_s$ without worrying about possible values of x_s for $s \in D \setminus A$.

From a general concept of limit in a topological space, we shall show that p-limits and limits coincide for function defined on βD.

2.4.2 Definition. [14] Assume that X and Y are arbitrary topological spaces, with $A \subset X$ and with $f : A \to Y$. Let $x \in c\ell_X A$ and $y \in Y$. Define $\lim\limits_{a\to x} f(a) = y$ if and only if, for every neighbour -hood N of y, there is a neighbourhood O of x such that $f(A \cap O) \subset N$.

The limit $\lim\limits_{a\to x} f(a)$, is unique, if it exists.

2.4.3 Theorem. *[14] Let D be an arbitrary discrete space, let Y be a topological space, and let $p \in \beta D$ and $y \in Y$. Whenever $A \in p$ and $f : A \to Y$, $p\text{-}\lim\limits_{a\in A} f(a) = y$ if and only if $\lim\limits_{a\to x} f(a) = y$.*

Proof. See [14, Theorem 3.46]. □

2.4.4 Theorem. *[14] Let D be an arbitrary discrete space, let $p \in \beta D$, and let $\langle x_s \rangle_{s\in D}$ be an indexed collection in an arbitrary topological space X.*

(a) *If $p\text{-}\lim\limits_{s\in D} x_s$ exists, then it is unique.*

(b) *If X is a compact space, then $p\text{-}\lim\limits_{s\in D} x_s$ exists.*

Proof. See T[14, Theorem 3.48]. □

2.4.5 Theorem. *[14] Let D be an arbitrary discrete space, let $p \in \beta D$, and let X and Y be an arbitrary topological spaces, let $\langle x_s \rangle_{s\in D}$ be an indexed collection X, and let $f : X \to Y$. If f is continuous and $p\text{-}\lim\limits_{s\in D} x_s$ exists, then $p\text{-}\lim\limits_{s\in D} f(x_s) = f(p\text{-}\lim\limits_{s\in D} x_s)$.*

Proof. See [14, Theorem 3.49]. □

2.4.6 Corollary. [14] Let D be a discrete space, let X be a compact space, let $f : D \to X$, and let $\widetilde{f} : \beta D \to X$ be its continuous extension. Then for all $p \in \beta D$, $\widetilde{f}(p) = p\text{-}\lim\limits_{s\in D} f(x_s)$.

Proof. See [14, Corollary 3.49.1]. □

We now assert the affirmation that the notion of p-limit sis as versatile as the notion convergence of filters by proving some of the standard results about filter convergence in the context of p-limits.

2.4.7 Lemma. [14] Let $\mathcal{A}$ be a collection of closed subsets of the topological space X with the FIP. Let $D = \{\bigcap \mathcal{F} : \mathcal{F}$ is a finite nonempty subset of $\mathcal{A}\}$. For each $A \in D$ let $\mathcal{B}_A = \{B \in D : B \subset A\}$. Then $\{\mathcal{B}_A : A \in D\}$ has the finite intersection property.

2.4.8 Theorem. *[14] Let X be a topological space. Then X is a compact space iff whenever $\langle x_s \rangle_{s \in D}$ is an indexed family in X and p is an ultrafilter on D, $p\text{-}\lim_{s \in D} x_s$ exists.*

Proof. See [14, Theorem 3.52]. □

2.4.9 Theorem. *[14] Let X be a topological space, let $A \subset X$ and let $y \in Y$. Then $y \in c\ell A$ if and only if there exists an indexed family $\langle x_s \rangle_{s \in D}$ in A such that $p\text{-}\lim_{s \in D} x_s = y$.*

Proof. See [14, Theorem 3.53]. □

2.4.10 Theorem. *[14] Let X and Y be topological spaces, and let $f : X \to Y$. Then f is continuous if and only if whenever $\langle x_s \rangle_{s \in D}$ is an indexed family in X, $p \in \beta D$, and $p\text{-}\lim_{s \in D} x_s$ exists, one has $p\text{-}\lim_{s \in D} f(x_s) = f(p\text{-}\lim_{s \in D} x_s)$.*

Proof. See [14, Theorem 3.54]. □

3. Semigroups

This chapter introduces what we will be working mostly in this study, which is semigroups. We will give the definition then with examples of semigroups and also give some basic results on semigroups.

The most basic concept we will focus on is that of an idempotent element which we will look at through out this work. There are several examples of semigroups having no idempotent elements. The main result in section (3.3), however, is that every compact right topological semigroup has idempotent elements. This result will be applied repeatedly to the *Stone-Čech compactification* βS of a discrete semigroup S. It is a fact responsible for many applications of this theory, like combinatorial respectively dynamical theories.

3.1 Algebraic Theory

3.1.1 Definition. [2] A *semigroup* is a pair $(S, \cdot)$, where S is a nonempty set and $\cdot$ is a binary associative operation on S defined by $(s, t) \to s \cdot t : S \times S \to S$. Associativity means that

$$r \cdot (s \cdot t) \;=\; (r \cdot s) \cdot t \quad (r, s, t \in S).$$

A semigroup with only one element is called *trivial*.

Semigroups thus form one of the most basic types of algebraic structure. The definition of a semigroup can be weakened or strengthened, this can be done by removing the associative property or by adding an identity and inverses. Structures satisfying the weak condition are called groupoids and those satisfying the strong condition are called groups. There are more semigroups than groups. For example, the are 5 essentially different groups with 8 elements, but there are $3\ 684\ 030\ 417$ different (non- isomorphic) semigroups with 8 elements.

The operation on S will usually be called *multiplication*, and $s \cdot t$ will be called the product of s and t. Other notations other than $\cdot$ are $+$, $\circ$ and $*$, the choice depends on the context. We shall generally denote the semigroup $(S, \cdot)$ and the product of s and t by juxtaposition, as just S and st. That is drop the space symbol for multiplication, unless otherwise a need arises to pick the symbol. If $s \in S$ and $n \in \mathbb{N}$, we shall write s^n for $s \cdot s \cdot s \cdots s$.

Since the multiplication operation is associative, there's no ambiguity in the product $s_1 s_2 \ldots s_n$ (where each $s_i \in S$): the product is the same regardless of how we insert the brackets. This follows easily by induction on $n \in \mathbb{N}$.

3.1.2 Example. [14] The following are semigroups:

(a) $(\mathbb{N}, +)$.

(b) $(\mathbb{N}, \cdot)$.

(c) $(\mathbb{R}, +)$.

(d) $(\mathbb{R}, \cdot)$.

(e) $(\mathbb{R} \setminus \{0\}, \cdot)$.

(f) $(\mathbb{R}^+, +)$.

(g) $(\mathbb{R}^+, \cdot)$.

(h) $(\mathbb{N}, \vee)$, where $s \vee t = \max\{s, t\}$.

(i) $(\mathbb{N}, \wedge)$, where $s \wedge t = \min\{s, t\}$.

(j) $(\mathbb{R}, \wedge)$.

(k) $(S, \cdot)$, where S is any nonempty set and $s \cdot t = t$ $(s, t \in S)$.

(l) $(S, \cdot)$, where S is any nonempty set and $a \in S$ and $s \cdot t = a$ $(s, t \in S)$.

(m) $(X^X, \circ)$, where $X^X = \{f \mid f \ : \ X \to X\}$, and $\circ$ represents the composition of functions.

(n) $(\mathcal{P}_f(A), \cup)$, where for a given set A, $\mathcal{P}_f(A) = \{F : \emptyset \neq F \subset A$ and F is finite$\}$.

(o) $(\mathcal{P}(A), \cup)$, where $\mathcal{P}(A)$ is the power set of A.

(p) $(\mathcal{P}(A), \cap)$, where $\mathcal{P}(A)$ is the power set of A.

The semigroups of examples (k) and (l) above are called respectively *right zero* and *left zero* semigroups.

3.1.3 Definition. [14] Let $(S, *)$ and $(T, \cdot)$ be semigroups. The following holds.

(a) The function $\gamma : S \to T$ is called a *homomorphism* from S into T if it satisfies $\gamma(s * t) = \gamma(s) \cdot \gamma(t) \quad (s, t \in S)$.

(b) An *epimorphism* is a homomorphism mapping S into T which is surjective. A *monomorphism* is an injective homomorphism mapping S to T.

(c) An *isomorphism* mapping S into T is a bijective homomorphism.

(d) The semigroups S, T are said to be *isomorphic* if and only if there exists an isomorphism mapping S into T. If S and T are isomorphic we write $S \simeq T$.

(e) An *anti-homomorphism* from S to T is a function $\gamma : S \to T$ satisfying $\gamma(s * t) = \gamma(t) \cdot \gamma(s) \quad (s, t \in S)$.

(f) An *anti-isomorphism* from S to T is an anti-homomorphism from S to T which is a bijection.

(g) The semigroups S, T are said to be *anti-isomorphic* if and only if there exists anti-isomorphism from S to T.

It follows with ease that the composition of two homomorphisms, it exists, is also a homomorphism.

3.1.4 Definition. [14] For a semigroup S.

(a) Let e be an element of S. If $es = s$ $(s \in S)$ the element e is a *left identity*. If $se = s$ $(s \in S)$ the element e is a *right identity*. If $es = se = s$ $(s \in S)$ the element e is a *two-sided identity* or simply an *identity*. A semigroup that contains an identity element is called a *monoid*.

(b) Let o be an element of S. If $o = os$ $(s \in S)$ the element o is a *left zero*. If $so = o$ $(s \in S)$ the element o is a *right zero*. If $os = so = o$ $(s \in S)$ the element o is a *two-sided zero* or simply a *zero*. If S has a zero and $st = o$ $(s, t \in S)$, we call S a *null semigroup*.

Observe that in a left zero semigroup, every element is a right identity and in a right zero semigroup, every element is a left identity. Also, if every member S is a left zero, then S is called a left zero semigroup. For right zero semigroup is defined analogously see Example (3.1.2) (k) and (l).

3.1.5 Proposition. [14] If e is a left identity (respectively, left zero) and f is a right identity (respectively, right zero) of a semigroup S, then $e = f$.

Proof. Since e is a left identity of S, $e = f$. Since f is a right identity of S, $e = f$. Hence, $e = ef = f$. □

Suppose a semigroup S lacks an identity, then one may be adjoined. This is done by adjoining e to S such that $es = se = s$ $(s \in S \cup \{e\})$; if the original product is retained for pairs from S, then $S \cup \{e\}$ is a semigroup with identity e.
Similarly if S lacks a zero, we may adjoin a symbol o to S and extend multiplication on S to $S \cup \{o\}$ such that $os = so = o$ $(s \in S \cup \{o\})$; again if the original product is retained for pairs from S, then $S \cup \{o\}$ is a semigroup with zero o. For any semigroup S, define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup \{e\} & \text{otherwise;} \end{cases}$$
$$S^o = \begin{cases} S & \text{if } S \text{ has a zero} \\ S \cup \{o\} & \text{otherwise.} \end{cases}$$

The semigroups S^1 and S^o are called, respectively, the *semigroup obtained from adjoining an identity to S if necessary* and the *semigroup obtained by adjoining a zero to S if necessary*.

3.1.6 Proposition. [14]

(a) Suppose S is a semigroup with identity e. Then e is also the identity of S^o.

(b) Suppose S is a semigroup with zero o. Then o is also a zero of S^1.

Proof. We prove (a) as (b) follows analogously.
If S contains a zero, then $S^o = S$ and we're done. Otherwise $S^o = S \cup \{o\}$. Then for each $s \in S$, $es = se = s$ since e is an identity of S, and $oe = eo = o$ by the definition of S^o. Hence e is an identity for S^o. □

3.1.7 Definition. [14] Let $S := \prod\{S_i \ : \ i \in I\}$ denote the Cartesian product of a collection of semigroups. Under coordinatewise multiplication

$$(s_i)(t_i) = (s_i t_i) \quad ((s_i), (t_i) \in S)$$

S is a semigroup, called the *direct product* of the collection $\{S_i \ : \ i \in I\}$. Observe that coordinatewise multiplication is the unique multiplication for which the projection mappings $\pi_i : S \to S_i$ are homomorphisms. If each S_i has an identity e, then the *canonical injection* $\omega_i : S_i \to S$ is defined by $\pi_j(\omega_i(s)) = 1$ if $j \neq i$ and $\pi_i(\omega_i(s)) = s$. It follows that, ω_i is an isomorphism of S_i onto $\omega_i(S_i)$.

3.1.8 Definition. [14] Let S be a semigroup, let $s \in S$. If there exists an element $s' \in S$ such that $ss' = e$. Then s' is a *right inverse* for s, and s is *right invertible*. Similarly, if there exists $s'' \in S$ such that $s''s = e$. Then s'' is a *left inverse* for s, so s is *left invertible*. If s is left and right invertible, then s is *invertible*.

3.1.9 Definition. [14] A *group* is a nonempty set S such that

(a) S is a semigroup and

(b) There is an element $e \in S$ such that

  (i) e is a left identity for S and

  (ii) For each $s \in S$ there exists $t \in S$ such that t is a left inverse for s.

3.1.10 Theorem. *[14] Let S be a semigroup. The following statements are equivalent.*

(a) S is a group.

(b) There is a two sided identity e for S with the property that for each $s \in S$ there is some $t \in S$ such that t is a (two sided) inverse for s.

(c) There is a left identity for S and given any left identity e for S and any $s \in S$ there is some $t \in S$ such that t is a left inverse for s.

(d) There is a right identity for S such that for each $s \in S$ there is some $t \in S$ such that t is a right inverse for s.

(e) There is a right identity for S and given any right identity e for S and any $s \in S$ there is some $t \in S$ such that t is a right inverse for s.

Proof. (a) implies (b). Choose e as given by the definition of a group above. We show first that any element has an inverse. So let $s \in S$ be given and let t be a left inverse for s. Let u be a left inverse for t. Then $st = e(st) = (ut)(st) = u(t(st)) = u((ts)t) = u(et) = ut = e$, so t is a right inverse for s, hence a two-sided inverse for s as required.
Next we show that e is a right identity for S, so let $s \in S$ be given. Choose an inverse t for s. Then $se = s(ts) = (st)s = es = s$.

(b) implies (c). Choose e as given by the hypothesis. Given any left identity f for S we have by Proposition (3.1.6) that $e = f$ and so every element of S has a left identity f.

(b) implies (c). This is precisely the definition of a group and that associativity is inherited from S.

The implication (d) implies (b), (b) implies (e), and (e) implies (d) follow by left-right switches. □

Recall that in a right zero semigroup every element is a left identity and if given any left identity e and $s \in S$, then e is a right inverse for S. According to [14], this is essentially the only example of this process. Theorem (3.2.7) shows that any semigroup with a left identity e such that every element has a right inverse is the Cartesian product of a group with a right zero semigroup. In particular, it follows that if a semigroup has a unique left identity e and every element has a right inverse, then the semigroup is a group.

A familiar example of a semigroup that is not a group is the set of natural numbers $(\mathbb{N}, +)$ under the operation of addition. This is not a group since it does not contain an identity. However, in the semigroup $(\mathbb{N}, \vee)$, 1 is the unique identity and the only element with an inverse.

3.1.11 Definition. [14] Let S be a semigroup.

(a) S is *commutative* if and only if $st = ts \quad (s, t \in S)$.

(b) The *centre* of S is the set $Z(S) = \{s \in S \ : \ \text{for all } t \in S, st = ts\}$.

(c) Given $s \in S$, $\lambda_s \ : \ S \to S$ is defined by $\lambda_s(t) = st$.

(d) Given $s \in S$, $\rho_s \ : \ S \to S$ is defined by $\rho_s(t) = ts$.

(e) $L(S) = \{\lambda_s \ : \ s \in S\}$.

(f) $R(S) = \{\rho_s \ : \ s \in S\}$.

Note that, for a semigroup S, $(L(S), \circ)$ and $(R(S), \circ)$ are semigroups.

3.1.12 Notation. For subsets A, B of the semigroup S and for each element t in S define

$$At = \lambda_t(A), \qquad tA = \rho_t(A)$$
$$At^{-1} = \lambda_t^{-1}(A), \quad t^{-1}A = \rho_t^{-1}(A).$$

and

$$AB = \bigcup_{t \in B} At = \bigcup_{t \in A} tB = \{st : \ s \in A, t \in B\}.$$

If $A_1, A_2, \ldots, A_n$ are subsets of S, define $A_1A_2 \cdots A_n$ inductively by $A_1A_2 \cdots A_n = (A_1A_2 \cdots A_{n-1})A_n$. If each $A_i = A$, we write A^n for $A_1A_2 \cdots A_n$. Finally, if S is a group, define

$$A^{-1} = \{s^{-1} : s \in A\}.$$

Standard examples of commutative semigroups include [2] $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Q}^+$ under ordinary addition or ordinary multiplication. Moreover, $(\mathbb{R} \setminus \{0\}, \ \cdot)$, $(\mathbb{Q} \setminus \{0\}, \ \cdot)$, $\mathbb{C} \setminus \{0\}, \ \cdot)$ are commutative groups. A very important example of a noncommutative semigroup is the set $M(n, \mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ under matrix multiplication $(n \geq 2)$.

3.1.13 Definition. [14] Let S be a semigroup.

(a) S is *nilpotent* if it contains a zero and there exists some $n \in \mathbb{N}$ such that $S^n = \{0\}$.

(b) S is a *nilsemigroup* if it contains a zero and for every $s \in S$, there exists some $n \in \mathbb{N}$ such that $s^n = \{0 = \{f \mid f \ : \ X \to X\}\}$.

3.1.14 Definition. [14] Let S be a semigroup.

(a) We say $s \in S$ is *right cancellable* if and only if whenever $t, u \in S$ and $ts = us$, one has $t = u$.

(b) We say $s \in S$ is *left cancellable* if and only if whenever $t, u \in S$ and $st = su$, one has $t = u$.

(c) Every $s \in S$ is right cancellable if and only if S is *right cancellative*.

(d) Every $s \in S$ is left cancellable if and only if S is *left cancellative*.

(e) If S is left and right cancellative then it is *cancellative*.

Note, a non-trivial semigroup with zero cannot be cancellative.

3.1.15 Theorem. *[14] Let S be a semigroup.*

(a) The function $\lambda : S \to L(S)$ is a homomorphism onto $L(S)$.

(b) The function $\rho : S \to R(S)$ is an anti-homomorphism onto $R(S)$.

(c) If S is right cancellative, then S and $L(S)$ are isomorphic.

(d) If S is left cancellative, then S and $R(S)$ are anti-isomorphic.

Proof. We prove (a) and (c) as the proofs of (b) and (d) are left-right switches.

(a) For any $s, t \in S$ define $\lambda : S \to L(S)$ by $\lambda(s) = \lambda_s$ where $\lambda_s : S \to S$ is defined by $\lambda_s(t) = st$ [similarly for ρ]. Then given any u in S one has $(\lambda_s \circ \lambda_t)(u) = \lambda_s(tu) = s(tu) = (st)u = \lambda_{st}(u)$ so $(\lambda_s \circ \lambda_t) = \lambda_{st}$.

(c) Define λ as a mapping $s \to \lambda_s$. Then from (a) λ is a homomorphism onto $L(S)$. We show that λ is injective. Let $s_1, s_2 \in S$ be such that $\lambda(s_1) = \lambda(s_2)$, then for any $t \in S$

$$\begin{aligned} &\lambda_{s_1}(t) = \lambda_{s_2}(t) \\ &\Rightarrow s_1 t = s_2 t \\ &\Rightarrow s_1 = s_2 \quad \text{since } S \text{ is right cancellative.} \end{aligned}$$

□

3.1.16 Definition. [14] let S be a semigroup.

(a) An element s of S is an *idempotent* if $S^2 = ss = s$.

(b) $E(S) = \{s \in S \ : \ s \text{ is an idempotent}\}$. If $E(S) = S$, then S is called an *idempotent semigroup* or a *band*. A commutative idempotent semigroup is called a *semilattice*.

(c) Q is a *subsemigroup* of S if and only if $QQ \subset Q$, that is $Q \subset S$ and Q is a semigroup under the same operations in S.

(d) Q is a *subgroup* of S if and only if $Q \subset S$ and Q is a group under the same operations in S.

(e) Let $e \in E(S)$. Then $H(e) = \bigcup\{G \ : \ G \text{ is a subgroup of } S \text{ and } e \in G\}$.

3.1.17 Example. [2]

(a) Let $U = \{0, \ldots, k\}$ for some $k > 0$. Define an operation $\wedge$ on U by $m \wedge n = \min\{m, n\}$. The operation $\wedge$ is associative, so $(U, \wedge)$ is a semigroup. Observe that $0 \wedge m = m \wedge 0 = 0$ and $m \wedge k = k \wedge m = m$ for all $m \in U$. Hence U has as zero o and identity k. Furthermore, $m \wedge m = m$ $(m \in U)$, so every element of U is an idempotent. Finally, $m \wedge n = n \wedge m = m$ $(m, n \in U)$ and so U is commutative.

(b) Similarly we define the semigroup operation $\wedge$ on $\omega = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ by $m \bigtriangleup n = \min\{m, n\}$. Then $(\omega, \wedge)$ has a zero but no identity. It is commutative and all its elements are idempotents.

(c) Consider the set of all 2×2 integer matrices:

$$M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, \ b, \ c, \ d \ \in \mathbb{Z} \right\}.$$

With the usual matrix multiplication, $M_2(\mathbb{Z})$ is a semigroup with identity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and zero $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. From elementary algebra we have $M_2(\mathbb{Z})$ is not commutative, that is not all of its elements are idempotents. Since $M_2(\mathbb{Z})$ contains a zero it is not cancellative. The set of 2×2-matrices can be generalized to $n \times n$-matrices with coefficients in $M(\mathbb{Z})$.

(d) Now let V be the set of all 2×2 integer matrices with non-zero determinant. Again, V is a semigroup with identity. Let $P, Q, R \in V$. Suppose $RP = RQ$. Since $\det R \neq 0$, the matrix R has a inverse $R^{-1} \in M_n(\mathbb{Q})$. [Observe that $R^{-1} \notin V$ whenever $\det R \neq \pm 1$, so V is not group]. So $R^{-1}RP = R^{-1}RQ$ and so $P = Q$. Hence V is left-cancellative. Similarly, it is right-cancellative and therefore cancellative.

(e) Let L and R be a left and right zero semigroups respectively. Let $B = L \times R$. This semigroup is an $|L| \times |R|$ *rectangular band*, or simply a *rectangular band*. For $(\ell_1, r_1), (\ell_2, r_2) \in B$, we have

$$(\ell_1, r_1)(\ell_2, r_2) = (\ell_1\ell_2, r_1r_2) = (\ell_1, r_2),$$

since ℓ_1 is a left zero and r_2 is a right zero. Thus every element of B is an idempotent, since $(\ell, r)(\ell, r) = (\ell, r)$ for all $(\ell, r) \in B$. Furthermore, for any $(\ell_1, r_1), (\ell_2, r_2) \in B$ we have

$$(\ell_1, r_1)(\ell_2, r_2)(\ell_1, r_1) = (\ell_1\ell_2\ell_1, r_1r_2r_1) = (\ell_1, r_1)$$
$$(\ell_2, r_2)(\ell_1, r_1)(\ell_2, r_2) = (\ell_2\ell_1\ell_2, r_2r_1r_2) = (\ell_2, r_2).$$

Hence (ℓ_2, r_2) is an inverse of (ℓ_1, r_1). Thus every element is an inverse of every element.

(f) Let X be any set. Then the idempotents of $(X^X, \circ)$ are the functions $f \in X^X = \{f \mid f \ : \ X \to X\}$ with the property that $f(x) = x$ $(x \in f[X])$.

3.1.18 Proposition. [14]

(a) Let G be a group with identity e. Then $E(G) = \{e\}$.

(b) The set of invertible elements of a semigroup with identity forms a subgroup.

Proof. (a) Suppose $f \in E(G)$. Then $ff = f = fe$. Multiplying on the left by the inverse of f, we get $f = e$.

(b) Let T be a set of invertible elements of a semigroup S. Note that $T \neq \emptyset$ since $e \in T$. Let $s, t \in T$. Then since s, t are invertible we have that $(st)^{-1}st = t^{-1}s^{-1}st = t^{-1}t = e$ and $st(st)^{-1} = stt^{-1}s^{-1} = ss^{-1} = e$. Hence st is invertible, and so lies in T. Hence T is a subsemigroup of S. Moreover, $e \in T$ is also an identity for T and so T is a subsemigroup with identity of S. Hence, all elements of T are invertible and so T is a subgroup of S.

□

3.1.19 Definition. [2, 14] Let T be a nonempty subset of a semigroup S. The intersection of all subsemi- groups of S that contain T is called the *subsemigroup generated by* T, and the elements of T are called *generators*. The subsemigroup generated by T will be denoted by $\langle T\rangle$. If $S = \langle T\rangle$, we say S is *generated by* T. A semigroup generated by a single element is said to be *cyclic*. If there's a finite generating set for S, then S is said to be *finitely generated*.

The subsemigroup generated by T is concretely realized as the set of all products $s_1 s_2 \cdots s_n$, where $n \in \mathbb{N}$ and $s_i \in T,\ 1 \le i \le n$, see the proposition below. It is clearly the smallest subsemigroup of S containing T.

3.1.20 Proposition. [2] Let $\emptyset \ne T \subset S$. Then $\langle T\rangle = \{s_1 s_2 \cdots s_n \ :\ n \in \mathbb{N},\ s_i \in T,\ 1 \le i \le n\}$.

Proof. Let $U = \{s_1 s_2 \cdots s_n \ :\ s_i \in T\}$. Then U is closed under multiplication and so a subsemigroup of S. Furthermore, $T \subset U$. Hence U is a subsemigroup that contains T, and so $\langle T\rangle \subset T$. Since $T \subset \langle T\rangle$ and $\langle T\rangle$ is closed under multiplication, $U \subset \langle T\rangle$. Therefore $\langle T\rangle = U$. □

Following Proposition (3.1.18) the statement "$e \in G$" in the definition of $H(e)$ is synonymous with "e is the identity of G". It is possible for $H(e)$ to equal $\{e\}$, but $H(e)$ is never empty. The next result affirms that $H(e)$ is such a large subgroup containing e [14].

3.1.21 Theorem. *[2, 14] If S is a semigroup with $e \in E(S)$. Then $H(e)$ is the largest subgroup of S containing the identity e.*

Proof. We show that $H(e)$ is a group and since $se = es = s$ for all $s \in H(e)$, e is an identity for $H(e)$ and $H(e)$ contains every group with with e as identity. From this it is sufficient to show that $H(e)$ is closed. So let $s, t \in H(e)$ and pick subgroups G_1 and G_2 of S with $e \in G_1 \cap G_2$ and $s \in G_1$ and $t \in G_2$. Let G be a subsemigroup of S generated by $H(e)$. Then $st \in G$ and $e \in G$ so it suffices to show that G is a group. For this we show that every element of G has a left inverse then invoke Theorem (3.1.10). Let $s \in G$. Then $s = s_1 s_2 \cdots s_n$, where $n \in \mathbb{N}$ and $s_i \in H(e), i = 1, \ldots, n$. For each i choose $t_i \in H(e)$ such that $s_i t_i = t_i s_i = e$, and set $t := t_n \cdots t_2 t_1$. Then $st = ts = e$, which shows that G is a group. Therefore, $H(e) = G$. □

The groups $H(e)$ are referred to as maximal groups. Surely, given any group $G \subset S$, G has an identity and $G \subset H(e)$.

3.1.22 Lemma. [14] Let S be a semigroup, let $e \in E(S)$, and let $x \in S$. Then the following statements are equivalent:

(a) $s \in H(e)$.

(b) $es = s$ and there is some $t \in S$ such that $et = t$ and $st = ts = e$.

(c) $s = se$ and there is some $t \in S$ such that $te = t$ and $st = ts = e$.

Proof. We show the equivalence of (a) and (b); the equivalence of (a) and (c) then follows by a right-left switch. The fact that (a) implies (b) follows from the definition of $H(e)$ as every member of $H(e)$ is a subgroup.
(b) implies (a). Let $G = \{s \in S \ :\ es = s$ and there is some $t \in S$ such that $et = t$ and $st = ts = e\}$. It suffices to show that G is a group with identity e. To demonstrate closure, let $s, u \in G$. Then

$eus = us$. Choose t and v in S such that $et = t$, $ev = v$, $st = ts = e$ and $uv = vu = e$. Then $etv = tv$ and $tvus = tes = ts = e = uv = uev = ustv$.
Obviously, e is a left identity for G so it suffices to prove that each member of G is left invertible in G. Let $s \in G$ and choose $t \in S$ satisfying $et = t$ and $st = ts = e$. Note that surely t does satisfy the requirements to be in G. □

3.1.23 Definition. [14] Let S be a semigroup.

(a) Whenever $\emptyset \neq M \subseteq S$ and $SM \subseteq M$, we say that M is a *left ideal* of S.

(b) Whenever $\emptyset \neq N \subseteq S$ and $NS \subseteq N$, we say that N is a *right ideal* of S.

(c) We say that T is an *ideal* of S if and only if T is left ideal and right ideal of S; that is, if $ST \cup TS \subseteq T$.

Every ideal, whether left, right, or two-sided, is a subsemigroup. Any ideal T of S which satisfies $T \neq S$ is called a *proper* ideal of S. Of importance in this thesis are the notion of minimal right and left ideals with respect to inclusion.

3.1.24 Definition. [14] Let S be a semigroup.

(a) We say that the left ideal M of S is *minimal* if and only if M is a left ideal of S and whenever K is a left ideal of S and $K \subseteq M$ then $K = M$.

(b) We say that the right ideal N of S is *minimal* if and only if N is a right ideal of S and whenever K is a right ideal of S and $K \subseteq N$ then $K = N$.

(c) If S is a minimal left ideal of S then S is *left simple*.

(d) If S is a minimal right ideal of S then S is *right simple*.

(e) If the only ideal of S is S then S is *simple*.

We do not define a minimal ideal. Therefore, it follows from Lemma (3.1.25) below that there is at most one minimal two sided ideal. Thus we use the term "smallest" to refer to an ideal which does not properly contain another ideal.

Note that S is left simple if and only if it has no proper left ideals. On a similar note we observe that S is right simple if and only if it has no proper right ideals. Whenever there's a theorem about left ideals, there's a corresponding theorem about right ideals. We shall not usually state both results.

3.1.25 Lemma. [14] Let S be a semigroup.

(a) Let M_1 and M_2 be left ideals of S. Then $M_1 \cap M_2$ is a left ideal of S if and only if $M_1 \cap M_2 \neq \emptyset$.

(b) Let M and N be left and right ideals of S respectively. Then $M \cap N \neq \emptyset$.

Proof. (a) Suppose $M_1 \cap M_2$ is a left ideal of S. Suppose $M_1 \cap M_2 \neq \emptyset$ follows by definition. Conversely choose $m \in M_1 \cap M_2$ and let $s \in S$. Then $m \in M_1$ and $m \in M_2$. Since, M_1 and M_2 are left ideals of S it follows that $sm \in M_1$ and $sm \in M_2$, and so $sm \in M_1 \cap M_2$, hence $M_1 \cap M_2$ is a left ideal of S.

(b) Let $m \in M$ and $n \in N$. Then $nm \in M$ because $m \in M$ and $nm \in N$ because $n \in N$.

□

3.1.26 Lemma. [14] Let S be a semigroup.

(a) For any $s \in S$, sS is a right ideal, Ss is a left ideal and SsS is an ideal.

(b) Suppose $e \in E(S)$, then e is a left identity for eS, a right identity for Se, and an identity for eSe.

Proof. (a) This is immediate since $(sS)S = s(SS) \subset sS$ and sS is a right ideal. Similarly, $S(Ss) = (SS)s \subset Ss$ and sS is a left ideal. Lastly, $S(SsS)S = (SS)s(SS) \subset SsS$ and Fix S to be a semigroup so SsS is an ideal.

(b) Let $e \in E(S)$. We show that e is a right identity for Se and likewise the others follow. Let $e \in Se$ and choose $t \in S$ such that $s = te$. Then $se = tee = te = s$.

□

3.1.27 Theorem. *[14] Let S be a semigroup.*

(a) Suppose S is left simple with $e \in E(S)$, then e is a right identity for S.

(b) Suppose M is a left ideal of S with $s \in M$, then $Ss \subseteq M$.

(c) Fix $\emptyset \neq M \subseteq S$. Then M is a minimal left ideal of S if and only if $Ss = M$ for each $s \in M$.

Proof. (a) By Lemma (3.1.26)(a), Se is a left ideal of S, so $Se = S$ by minimality of S and so Lemma (3.1.26)(b) applies.

(b) Since $s \in M$, $s \in S$ so by definition $Ss \subset SM \subset M$.

(c) Suppose L is minimal. We have Ss is a left ideal according to Lemma (3.1.26)(a) and so by (b) it follows that $Ss \subset M$, so $Ss = M$.
Conversely, suppose $M = Ss$ for each $s \in M$. Then M is a left ideal. For any K a left ideal of S with $K \subset M$, choose $s \in K$. So by (b), $Ss \subset K$, whence $K \subset M = Ss \subset K$.

□

3.1.28 Definition. [14] Let S be a semigroup.

(a) The *principal ideal generated by* s is the smallest ideal of S which contains $s \in S$.

(b) The *principal left ideal generated by* s is the smallest left ideal of S which contains $s \in S$.

(c) The *principal right ideal generated by* s is the smallest right ideal of S which contains $s \in S$.

The following result ensures that the above objects exists.

3.1.29 Theorem. *[14] Let S be a semigroup with $s \in S$.*

(a) *The principal ideal generated by s is $S^1sS^1 = SsS \cup sS \cup Ss \cup \{s\}$.*

(b) *Suppose S have an identity, then the principal ideal generated by s is SsS.*

(c) *The principal left ideal generated by s is $S^1s = Ss \cup \{s\}$ and the principal right ideal generated by s is $sS^1 = sS \cup \{s\}$.*

Proof. Recall that for a semigroup without identity we have that

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup \{e\} & \text{otherwise.} \end{cases}$$

From this we have that

$$\begin{aligned} S^1A &= \{sa \mid \ s \in S^1, a \in A\}, \\ &= \{sa \mid \ s \in S \cup \{e\}, a \in A\}, \\ &= \{sa \mid \ s \in S, a \in A\} \cup \{ea \mid \ a \in A\} \\ &= SA \cup A. \end{aligned}$$

In fact, if $A = \{a\}$ then $S^1a = Sa \cup \{a\}$. So by duality, $aS^1 = aS \cup \{a\}$ and likewise $S^1aS^1 = SaS \cup aS \cup Sa \cup \{a\}$.

To establish (a), it is clear that $SsS \cup sS \cup Ss \cup \{s\}$ is a two-sided ideal containing s. Let T be a two-sided ideal containing s. We show that $SsS \cup sS \cup Ss \cup \{s\} \subset T$. It suffices to show that $SsS \subset T$. Let $s_1ss_2 \in SsS$. Since T is a left ideal, $ST \subset T$ and so $s_1s \in T$. Also, since T is a right ideal, $TS \subset T$ and so $(s_1s)s_2 \in T$. Therefore, $s_1ss_2 \in T$.

To establish (b), let e be an identity of S. Then $S^1sS^1 = SsS \cup sS \cup Ss = SsS$ because $S^1s = Ss \Leftrightarrow s \in Ss$. Also, the following is true if for each $s \in S$, $S^1s = Ss$ if S has either, an identity or an idempotent.

To establish (c), we show that S^1s is a principal left ideal generated by s, then by duality sS^1 is a principal right ideal generated by s.
We have $s = se$ and so $s \in S^1s$ and $S(S^1s) = (SS^1)s \subset S^1s$ and so S^1s is a left ideal containing s. Let M be a left ideal containing s then $S^1s \subset S^1M = SM \cup M \subset M$. Hence S^1s is the smallest ideal containing s. □

3.1.30 Example. [14]

(a) Consider the semigroup $(\mathbb{N}, +)$. Let $n \in \mathbb{N}$ and let $T_n = \{m \in \mathbb{N} : m \geq n\}$. Then T_n is an ideal of $\mathbb{N}$; indeed, $T_n = M(n) = N(n) = K(n)$. Where $M(n), N(n)$, and $K(n)$ are, respectively, the principal left ideal generated by n, principal right ideal generated by n, and principal ideal generated by n. This semigroup $(\mathbb{N}, +)$ has no minimal ideals.

(b) Let S be a right zero semigroup, and let Q be a subset of S. Then $SQ = Q$ since $xy = y$ for any $x \in S$ and $y \in Q$. So T is a left ideal of S. On the other hand $QS = S$ and so Q a right ideal if and only if $Q = S$.

(c) Let G be a group and T a subset of G. For any $x \in G$ and $y \in T$, we have $x = xy^{-1}y$ belongs to Gy; hence $Gy = G$. So T is a left ideal if and only if $T = G$; similarly T is a right ideal if and only if $T = G$. So the only left ideal and right ideal of G is G itself.

(d) Similar to (b) the semigroup spaces $(\mathcal{P}(A), \cup)$ and $(\mathcal{P}(A), \cap)$ have as their (two-sided) ideal the space $\mathcal{P}(A)$, where $\mathcal{P}(A)$ is the power set of A.

3.2 Minimal Left Ideals and their Idempotents

Having defined already minimal left and minimal right ideals, we now exploit another notion related to them which is that of minimal idempotents. Later we shall see that important consequences follow from the existence of minimal left (right) ideals, particularly those with idempotents. This is very important for us, because as we shall see when we study compact right topological semigroups.

3.2.1 Definition. [14] Let S be a semigroup and let $e, f \in E(S)$.

(a) $e \leq_M f$ if and only if $e = ef$.

(b) $e \leq_N f$ if and only if $e = fe$.

(c) $e \leq f$ if and only if $e = ef = fe$.

3.2.2 Remark. [14] Let S be a semigroup. The relations $\leq_M, \leq_N,$ and $\leq$ are transitive and reflexive on $E(S)$. In addition, $\leq$ is antisymmetric making it a partial order on $E(S)$.

When it is said that a point e is minimal with respect to a (not necessarily antisymmetric) relation $\preceq$ on a set B, it is meant that if $f \in B$ and $f \preceq e$, then $e \preceq f$ (so if $\preceq$ *is* antisymmetric, then it follows that $e = f$).

3.2.3 Theorem. *[14] Let S be a semigroup with $e \in E(S)$. The following are equivalent.*

(a) The element e is minimal with respect to $\leq$.

(b) The element e is minimal with respect to $\leq_M$.

(c) The element e is minimal with respect to $\leq_N$.

Proof. We show the equivalence of (a) and (c). The equivalence of (a) and (b) follow by left-right switch (or say duality).
(c) imply (a). Suppose e is minimal with respect to $\leq_M$ and let $f \leq e$. Then $f = ef$, so $f \leq_M e$ then we can conclude that $e \leq_M f$ by minimality of e. Hence $e = ef = f$.
(a) imply (c). Suppose e is minimal with respect to $\leq$ and let $f \leq_M e$. Let $g = ef$. Then $gg = efef = eff = ef = g$ and so $g \in E(S)$. Also, $g = ef = efe$ and so $eg = eefe = efe = g = efee = ge$. Therefore $g \leq e$ and so $g = e$ by minimality of e. Hence, $e = fe$ so $e \leq_M f$. □

3.2.4 Definition. Let S be a semigroup. Then e is a *minimal idempotent* if and only if $e \in E(S)$ and e is minimal with respect to any (hence all) of the orders $\leq_M, \leq_N,$ or $\leq$.

We note that the notions of "minimal left (right) ideals" and "minimal idempotents" are closely related. The "left" version of the following theorem has a dual "right" statement and thereby the proof is the mirror image of that of the other. Henceforth, we shall record only one of the left/right statements and refer to the other as its "dual".

3.2.5 Theorem. *[14] Let S be a semigroup with $e \in E(S)$.*

(a) Let e be a member of some minimal left ideal, so e is a minimal idempotent.

(b) Suppose S is simple and e is minimal. Then Se is a minimal left ideal.

(c) *If every left ideal of S contains an idempotent and e is minimal, then Se is a minimal left ideal.*

(d) *If S is simple or every left ideal of S has an idempotent then the following statements are equivalent.*

(i) *e is minimal.*

(ii) *e is a member of some minimal left ideal of S.*

(iii) *Se is a minimal left ideal of S.*

Proof. (a) Let M be a minimal left ideal with $e \in M$. Then $M = Se$ by Theorem (3.1.27)(c). Fix $f \in E(S)$ satisfying $f \leq e$. Then $f = fe \in Se$, so $f \in M$ and thus by Theorem (3.1.27) (c) $M = Sf$, so $e \in Sf$, it follows by Lemma (3.1.26)(b), $e = ef$, hence $e = ef = f$.

(b) Let M be a left ideal with $M \subset Se$. We show that $Se \subset M$. Choose $s \in L$. Then $s \in Se$ so by Lemma (3.1.26)(b), $s = se$. Since S is simple and $e \in S$, by Theorem (3.1.29)(b) $S = SsS$, so we can choose $u, v \in S$ with $e = usv$. Let $t = eve$ and $r = eu$. Then $rst = euseve = eusve$ $= eee = e$ and $et = eeve = eve = t$. Let $f = trs$. Then $ff = trstrs = t(rst)rs = ters$ $= teeus = teus = trs = f$, and so $f \in E(S)$. Also, $fe = trse = trs = f$ and$ef = etrs =$ $trs = f$ and so $f \leq e$ so $f = e$. Henceforth, $Se = Sf = Strs \subset Ss \subset M$.

(c) Let M be a left ideal with $M \subset Se$. We show that $e \in M$ (so that $Se \subset M$ and whence $Se = M$). Choose an idempotent $t \in M$, and let $f = et \in SM$. So $f \in M$. Since $t \in Se$, $t = se$ by Lemma (3.1.26)(b). Hence $f = et = ete$. Thus, $ff = etet = ett = et = f$ and so $f \in E(S)$. Next $ef = eete = ete = f$ and so $f \leq e$ and so $f = e$ and hence $e \in M$.

(d) This is the equivalence of (a), (b), and (c).

□

The following are several characterizations of a group.

3.2.6 Theorem. *[14] Let S be a semigroup. The following are equivalent.*

(a) *S is simple, cancellative, and $E(S) \neq \emptyset$.*

(b) *S is right and left simple respectively.*

(c) *For each a and b in S, the equations $as = b$ and $ta = b$ have solutions s, t in S.*

(d) *S is a group.*

Proof. Since (a) implies (b). Choose an idempotent $e \in S$. So e is a (two-sided) identity. To see this, let $s \in S$. Then $es = ees$ and so by cancellation $s = es$. Likewise, $s = se$. Next we show that S is right simple. Let N be a right ideal of S. Since S is simple, $SN = S$, so we can choose $s \in S$ and $t \in N$ such that $e = st$. Then $sts = es = s = se$ and so cancelling on the left we get $ts = e$. Thus $e \in N$ and so $S = NS \subset N$. Consequently S is right simple, likewise S is left simple.
(b) implies (c). Let $a, b \in S$. Then $Sa = S$, there is some $t \in S$ such that $ta = b$. Similarly, since $aS = S$, there is some $s \in S$ such that $as = b$.

(c) implies (d). Choose $a \in S$ and $e \in S$ such that $ea = a$. We have that e is a left identity for for S. To see this, let $b \in S$. We show that $eb = b$. Choose some $t \in S$ such that $at = b$. Then
$eb = eat = at = b$.
Next we show that every element of S is left invertible. For any $s \in S$ there is some $t \in S$ such that $ts = e$ hence every element of S is left invertible.

(d) implies (a). For any $s \in S$ let $t, u \in S$ be such that $st = tu$. Let w be a left inverse of s. Then $wst = wsu \Rightarrow et = eu$ and so $t = u$. Thus S is left cancellative, likewise right cancellative. Therefore, cancellative. Next we have that $E(S) \neq \emptyset$, because $e \in S$ is a (two- sided) identity. Lastly, to see that S is simple. Fix I to be an ideal of S and choose $s \in I$. Fix t be the inverse of s. Then $ts \in I$ because I is an ideal and so $I = S$. □

As noted earlier and we now have enough machinery to show that any semigroup with a left identity e such that every element has a right inverse e must be isomorphic to the Cartesian product of a group with a right zero semigroup.

3.2.7 Theorem. *[14] Let S be a semigroup with e as a left identity for S satisfying for all $x \in S$ there exists $y \in S$ with $xy = e$. Let $Y = E(S)$ and let $G = Se$. Then Y is a right zero semigroup, G a group and $G \times Y \simeq GY = S$.*

Proof. We show first that:

$$\text{For all } s \in Y \text{ and for all } t \in S, \quad st = s. \tag{3.2.1}$$

To prove (3.2.1), let $s \in Y$ and $t \in S$ be given. Choose $u \in S$ such that $su = e$. Then $se = ssu = su = e$. Therefore $st = s(et) = (se)t = et = t$.
From (3.2.1) it follows that for all $s, t \in Y$, $st = t$, and $Y \neq \emptyset$ because $e \in Y$, to see that Y is a right zero semigroup, it is sufficient to show that Y is closed and thereby a semigroup. But this easily follows from (3.2.1) since, given $s, t \in Y$, one have $st = t \in Y$.
Next we prove that $G = Se$ is a group. By Lemma (3.1.26)(b), e is a right identity for G. Now by supposition all element of S are right invertible *in* S. So all elements of G are right invertible *in* S. By Theorem (3.1.10), we need only show that all elements of G are right invertible *in* G. To see this, let $s \in G$ be given, choose $t \in S$ satisfying $st = e$. So $te \in G$ and $ste = ee = e$, so te is as required.
Since $GG = SeSe \subset SSSe \subset Se = G$, we have that, G is closed, it follows that G is a group.
Now define $\varphi : G \times Y \to S$ by $\varphi(g, y) = gy$. We prove that φ is a homomorphism, let
$(g_1, y_1), (g_2, y_2) \in G \times Y$. Then

$$\begin{aligned}
\varphi(g_1, y_1)\varphi(g_2, y_2) &= g_1 y_1 g_2 y_2 \\
&= g_1 g_2 y_2 \qquad \text{(by (3.2.1))} \\
&= g_1 g_2 y_1 y_2 \qquad \text{(by (3.2.1))} \\
&= \varphi(g_1 g_2, y_1 y_2).
\end{aligned}$$

To see that φ is surjective, let $s \in S$ be given. Then $se \in Se = G$, and so there exist $t \in Se$ such that $t(se) = (se)t = e$. We claim that $ts \in Y = E(S)$. Indeed,

$$\begin{aligned}
tsts &= tsets \qquad (\text{since } s \in G,\ et = t) \\
&= tes \\
&= ts \qquad (\text{since } s \in G,\ te = t).
\end{aligned}$$

Thus $(se, ts) \in G \times Y$ and $\varphi(se, ts) = sets = es = s$. Since φ is onto S, we have proved that $S = GY$.
Lastly to see that φ is injective, fix $(g, y) \in G \times Y$ and $s = \varphi(g, y)$. We prove that $g = se$ and $y = ts$ for t as the (unique) inverse of se in Se. Now $s = gy$, so

$$\begin{aligned} se &= gye \\ &= ge \qquad (\text{by } (3.2.1)\ ye = e) \\ &= g \qquad (\text{since } g \in Se). \end{aligned}$$

Also

$$\begin{aligned} ts &= tgy \\ &= tgyey \qquad (ye \in Y \text{ so by } (3.2.1)\ yey = y) \\ &= tsey \\ &= ey \\ &= y. \end{aligned}$$

□

We know (by Theorem (3.1.10)) that the existence of a left identity e for a semigroup S such that every element of S has a right inverse e does not serve to make S a group. A right zero semigroup is the standard example. Theorem (3.2.7) tells us that is essentially the only example.

In the upcoming results we shall see that many consequences follow from the existence of minimal left (or right) ideals, especially those with idempotents. This is very important, as we shall see in the next section when discussing the notion of compact right topological semigroups.

We first establish an easy consequence of Theorem (3.2.7).

3.2.8 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal M of S which contains an idempotent e. Then $M = XG \simeq X \times YG$, where M is the (left zero) semigroup of idempotents of M, and $G = eM = eSe$ becomes a group. All the maximal groups in M are isomorphic to G.*

Proof. For any $s \in M$, Ms is a left ideal of S and $Ms \subset M$ and so $Ms = M$ and hence there is some $t \in M$ such that $ts = e$. By Lemma (3.1.26)(b), e is a right identity for $Me = M$. Thus the duality of Theorem (3.2.7) applies with M replacing S. It can be shown that the maximal groups of $X \times G$ are the sets of the form $\{x\} \times G$. □

3.2.9 Lemma. [14] Let S be a semigroup. Suppose M is a left ideal of S, and let T be a left ideal of M.

(a) For all $t \in T$, Mt is a left ideal of S and $Mt \subset T$.

(b) If M is a minimal left ideal of S, then $T = M$. (So minimal left ideals are left simple.)

(c) If T is a minimal left ideal of M, then T is a left ideal of S.

Proof. (a) Mt is a left ideal because $S(Mt) = (SM)t \subset Mt$. Lastly, $Mt \subset MT \subset T$.

(b) Choose any $t \in T$. By (a), Mt is a left ideal of S and $Mt \subset T \subset M$ and so $Mt = M$and so $T = M$.

(c) We prove that T is a left ideal of S, choose any $t \in T$. By (a), Mt is a left ideal of S, and so Mt is a left ideal of M. Since $Lt \subset T$, $Mt = T$ by supposition. Whence, $ST = S(Mt) = (SM)t \subset Mt \;\; = T$. Now to see that T is minimal in S, let K be a left ideal of S with $K \subset T$. So K is a left ideal of M, and so $K = M$ by supposition.

□

3.2.10 Lemma. [14] Let S be a semigroup. Suppose I is an ideal of S, and let M be a minimal left ideal of S. Then $M \subset I$.

Proof. By Lemma (3.1.25) $M \cap I \neq \emptyset$ and since $M \cap I \subset M$, $M \cap I = M$ by supposition. Consequently, $M \subset I$. □

We now have that all minimal left ideals of a semigroup are intimately connected with each other.

3.2.11 Theorem. *Let S be a semigroup. Suppose M is a minimal left ideal of S, and $T \subset S$. Then T is a minimal left ideal of S iff there exists $a \in S$ such that $T = Ma$.*

Proof. Choose $a \in T$. Then $SMa \subset Ma$ and $Ma \subset ST \subset T$, so Ma is a left ideal of S contained in T, so $Ma = T$.
Conversely. Because $SMa \subset Ma$, Ma is a left ideal of S. Suppose that K is a left ideal of $S$and $J \;\subset La$. Let $B = \{s \in L : sa \in J\}$. Then $B \subset M$ and $B \neq \emptyset$, because if $t \in K \subset T = Ma$ then we can choose $s \in M$ such that $t = sa$ and so $t \in B$. We claim that B is a left ideal of S, then let $s \in B$ and let $t \in S$. So $sa \in K$, since K is a left ideal of S, and so $tsa \in K$ and, since $s \in M$, $ts \in M$, so $ts \in B$ as required. Thus B is a left ideal of S contained in M, and so $B = M$, so $Ma \subset K$ and so $Ma = K$. □

3.2.12 Corollary. [14] Let S be a semigroup. Suppose S contain a minimal left ideal, then every left ideal of S contains a minimal left ideal.

Proof. Let M be a minimal left ideal of S and let K be a left ideal of S. Choose $a \in K$. Then according to Theorem (3.2.11), Ma is a minimal left ideal which is contained in K. □

3.2.13 Theorem. *[14] Let S be a semigroup with $e \in E(S)$. Statements (a) through (f) are equivalent and imply statement (g). If either S is simple or every left ideal of S has an idempotent, so all statements are equivalent.*

(a) Se is a minimal left ideal.

(b) Se is left simple.

(c) eSe is a group.

(d) $eSe = H(e)$.

(e) eS is a minimal right ideal.

(f) eS is right simple.

(g) e is a minimal idempotent.

Proof. By Theorem (3.2.5)(a), we have that (a) implies (g) and by Theorem (3.2.5)(d), if either S is simple or every left ideal of S has an idempotent, then (g) implies (a). We will show that

$$\begin{array}{ccc} (a) & \Rightarrow & (b) \\ \Uparrow & & \Downarrow \\ (d) & \Leftarrow & (c) \end{array}$$

from which

$$\begin{array}{ccc} (e) & \Rightarrow & (f) \\ \Uparrow & & \Downarrow \\ (d) & \Leftarrow & (c) \end{array}$$

follows by duality, as in, left-right switch and the fact that (c) and (d) are two sided statements.
(a) implies (b). Choose $t \in Se$, then St is a left ideal of Se and so it follows from Lemma (3.2.9)(b) that Se is left simple.
(b) implies (c). Let $s = exe$ and $t = eye$ in eSe be given for some $x, y \in S$ then $st = (exe)(eye) = exeeye = exeye \ = exye \in eSe$ and so eSe is closed. By Lemma (3.1.26) e is a two sided identity for eSe. Let $s = \ exe \in eSe$ be given. It follows that $s \in Se$ so Ss is a left ideal of Se and accordingly $Ss = Se$, because Se is left simple. Hence $e \in Ss$, so choose $t \in S$ such that $e = ts$. Then $ete \in eSe$ and $etes = ets = ee = e$ and so s has a left inverse $e \in eSe$.
(c) implies (d). Since eSe is a group and $e \in eSe$, it follows that eSe is contained in $H(e)$. Then again, by Theorem (3.1.21), e is the identity of $H(e)$ so if given $s \in H(e)$, it follows that $s = ese \in eSe$, and so $H(e) \subset eSe$.
(d) implies (a). Let M be a left ideal of S with $M \subset Se$ and choose $t \in M$. Then $t \in Se$ so $et \in eSe$. Choose $s \in eSe$ such that $s(et) = e$. Then $st = (se)t = s(et) = e$ and so $e \in M$ and so $Se \subset SM \subset M$. □

We see that in a semigroup $(\mathbb{N}, \cdot)$, 1 is the only idempotent, and is consequently minimal, while $\mathbb{N}1$ is not a minimal ideal. Thus Theorem (3.2.13)(g) does not in general imply the other statements of Theorem (3.2.13). In contrast to ring theory where a ring may have many minimal ideals, a semigroup can have at most one minimal two-sided ideal.

3.2.14 Lemma. [14] Let S be a semigroup and let K be an ideal of S. If K is minimal in $\{J \ : \ J \text{ is an ideal of } S\}$ where I is an ideal of S, then $K \subset I$.

Proof. By Lemma (3.1.25)(b), $K \cap I \neq \emptyset$, so $K \cap I$ is an ideal contained in K, so $K \cap I = K$, since K is minimal. □

According to [14], the term "minimal ideal" is widely used in the literature. As by Lemma (3.2.14), there can be at most one minimal ideal in a semigroup, the terminology "smallest ideal" will be used.

3.2.15 Definition. Let S be a semigroup. If S contains a smallest ideal, then $K(S)$ is that smallest ideal.

3.2.16 Theorem. *[14] Let S be a semigroup. If S contains a minimal left ideal, so $K(S)$ exists and $K(S) = \bigcup\{M \ : \ M$ is a minimal left ideal of $S\}$.*

Proof. Let $I = \bigcup\{M : M$ is a minimal left ideal of $S\}$. By Lemma (3.2.10), if K is any ideal of S, then $I \subset K$, so it suffices to prove that I is an ideal of S. We know that $I \neq \emptyset$ by assumption, so let $s \in I$ and $t \in S$. Choose a minimal left ideal of S such that $s \in M$. then $st \in M \subset I$. Also, by Theorem (3.2.11), Mt is a minimal left ideal of S and so $Mt \subset I$ while $ts \in Mt$. □

So for we see that a simple condition guarantees the existence of $K(S)$. However, many common semigroups do not have smallest ideal. For example it is true for both $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$.

3.2.17 Lemma. [14] Let S be a semigroup.

(a) Let M be a left ideal of S. Then M is minimal if and only if $Ms = M$ for each $s \in M$.

(b) Let I be an ideal of S. Then I is the smallest ideal if and only if $IsI = I$ for every $s \in I$.

Proof. (a) If M is minimal and $s \in M$, then Ms is a left ideal of S and $Ms \subset M$ and so $Ms = M$. Conversely, suppose that $Ms = M$ for each $s \in M$ and let K be a left ideal of S with $K \subset M$. Choose $s \in K$. Then $Ms = M \subset MK \subset K \subset M$.

(b) Let I be the smallest ideal and let $s \in I$. Since I is the smallest ideal of S, then $IsI \subset I$ and so $IsI = I$.
Conversely, suppose that $IsI = I$ for each $s \in I$ and let K be a left ideal of S with $K \subset IsI$. Choose $s \in K$. Then $I = IsI \subset IKI \subset KI \subset K \subset I$. Thus, $K = I$.

□

3.2.18 Theorem. *[14] Let S be a semigroup. If M is a minimal left ideal of S and N is a minimal right ideal of S, then $K(S) = MN$.*

Proof. MN is an ideal since $S(MN)S = (SM)(NS) \subset MN$. We employ Lemma (3.2.17) to show that $K(S) = MN$. So, let $s \in MN$. Then $MNsL$ is a left ideal of S which is contained in M so $MNsM = M$ and so $MNsMN = MN$. □

3.2.19 Theorem. *[14] Let S be a semigroup and assume that $K(S)$ exists with $e \in E(S)$. The following are equivalent and are implied by any of the equivalent statements (a) through (f) of Theorem (3.2.13).*

(h) $e \in K(S)$.

(i) $K(S) = SeS$.

Proof. By Theorem (3.2.16), it follows that Theorem (3.2.13)(a) implies (h).
(h) implies (i). Since SeS is an ideal, we have $K(S) \subset SeS$. Since $e \in K(S)$, we have that $SeS \subset K(S)$.
(i) implies (h). We have $e = eee \in SeS = K(S)$. □

From here onwards, we presents several results which have as hypothesis "Let S be a semigroup and suppose that there is a minimal left ideal of S which contains an idempotent". These are important to us as we shall see when we study compact right topological semigroups. Thereby the following is a legitimate definition.

A semigroup S is called *completely simple* if it is simple and there is a minimal left ideal of S which contains an idempotent. Whence the semigroups with these statements are completely simple.

3.2.20 Theorem. *[14] Let S be a semigroup and suppose that there exist a minimal left ideal of S which contains an idempotent. Then every minimal left ideal has an idempotent.*

Proof. Let M be a minimal left ideal with an idempotent e and let K be a minimal left ideal. According to Theorem (3.2.11), choose $s \in S$ satisfying $K = Ms$. According to Theorem (3.2.8), $eL = eSe$ is a group, so let $t = ete$ be the inverse of ese in eSe. So $ts \in Ms = K$ and $tsts = (te)s(et)s = \ t(ese)ts = ets = ts$. □

From the following result we see that the right left version follows from the left.

3.2.21 Lemma. [14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent. Then there is a minimal right ideal of S which has an idempotent.

Proof. Choose a minimal left ideal M of S and an idempotent $e \in M$. According to Theorem (3.1.27)(c) Se is a minimal left ideal of S, so by Theorem (3.2.13), eS is a minimal right ideal of S and e is an idempotent of eS. □

3.2.22 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent. Let $Q \subset S$.*

(a) Q is a minimal left ideal of S iff there is some $e \in E(K(S))$ such that $Q = Se$.

(b) Q is a minimal right ideal of S iff there is some $e \in E(K(S))$ such that $Q = eS$.

Proof. Choose a minimal left ideal M of S and an idempotent $f \in M$.

(a) Suppose Q is a minimal left ideal of S. Since Sf is a left ideal contained in M, $Sf = M$. Therefore by Theorem (3.2.13), fSf is a group. Choose $a \in Q$. Then $faf \in fSf$, so choose $y \in fSf$ such that $y(faf) = f$. Then

$$\begin{aligned} yaya &= (yf)a(fy)a \\ &= (yfaf)ya \\ &= fya \\ &= ya. \end{aligned}$$

Hence, ya is an idempotent. Also $ya \in Q$ when $Q \subset K(S)$ by Theorem (3.2.16) and so $ya \in E(K(S))$. Lastly, Sya is a left ideal contained in Q, so $Q = Sya$.
Conversely, because $e \in K(S)$, choose by Theorem (3.2.16) a minimal left ideal K of S with $e \in K$. Then $Se = K$ by Theorem (3.1.27)(c).

(b) As a result of Lemma (3.2.21) this follows by duality.

□

3.2.23 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent with $e \in E(S)$. the following are equivalent:*

(a) Se is a minimal left ideal.

(b) Se is left simple.

(c) eSe is a group.

(d) $eSe = H(e)$.

(e) eS is a minimal right ideal.

(f) eS is right simple.

(g) e is a minimal idempotent.

(h) $e \in K(S)$.

(i) $K(S) = SeS$.

Proof. By Corollary (3.2.12) and Theorem (3.2.20) every left ideal contains an idempotent and then by Theorem (3.2.13) statements (a) through (g) are equivalent. By Theorem (3.2.16) and Theorem (3.2.19), we need only show that (h) imply (a). But this follows from Theorem (3.2.22). □

3.2.24 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent, and let e be an idempotent of S. There is a minimal idempotent f of S such that $f \leq e$.*

Proof. Let M be a minimal left ideal of S with an idempotent g, and let e be an idempotent in S. Then Se is a left ideal and thus by Corollary (3.2.12) and Theorem (3.2.20) $M \subset Se$. It follows that $g \in Se$ and so $ge = g$ by Lemma (3.1.26). Let $f = eg$. Then $ff = egeg = egg = eg = f$ and so f is an idempotent. Since M is a left ideal, $f \in M$ and so $M = Sf$ and so by Theorem (3.2.23) f is a minimal idempotent. Lastly $fe = ege = eg = f$ and $ef = eeg = eg = f$ and so $f \leq e$. □

3.2.25 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which has an idempotent. Given any minimal left ideal M of S and any minimal right ideal N of S, there exists an idempotent $e \in N \cap M$ such that $N \cap M = NM = eSe$ and eSe is a group.*

Proof. Let M and N be as given by the supposition. Choose by Theorem (3.2.22) an idempotent $f \in K(S)$ such that $M = Sf$. By Theorem (3.2.13), fSf is a group. Choose $a \in N$ and let t be the inverse of faf in fSf. Then $t \in Sf = M$ and so by Lemma (3.1.25) $at \in N \cap M$. By Theorem (3.2.16) $at \in K(S)$. Also

$$\begin{aligned} atat &= a(tf)a(ft) \\ &= a(tfaf)t \\ &= aft \\ &= at. \end{aligned}$$

Let $e = at$. Then $eSe \subset St \subset M$ and $eSe \subset aS \subset N$ and so $eSe \subset N \cap M$. We show the reverse inclusion, let $b \in N \cap M$. By Theorem (3.1.27) $M = Se$ and $N = eS$ and so by Lemma (3.1.25), $b = eb = be$. Thus $b = eb = ebe \in eSe$.
Next we have that $NM = eSSe \subset eSe \subset NM$, and so $NM = eSe$.
As we saw, $e \in K(S)$, so by Theorem (3.2.23) eSe is a group. □

The next result guarantees the existence of a minimal left ideal with an idempotent.

3.2.26 Lemma. [14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent. Let M be a left ideal of S and let N be a right ideal of S.

(a) There exists a minimal left ideal of M which contains an idempotent.

(b) There exists a minimal left ideal of $M \cap N$ which contains an idempotent. In fact if K is a minimal left ideal of S with $K \subset M$, then $N \cap K$ is a minimal left ideal of $N \cap M$ which contains an idempotent.

(c) There is a minimal left ideal of N which contains an idempotent.

Proof. Choose by Corollary (3.2.12) a minimal left ideal N of S such that $K \subset M$.

(a) If I is a left ideal of M with $I \subset K$, then I is a left ideal of K, so by Lemma (3.2.9)(b), $I = K$. By Theorem (3.2.20), K contains an idempotent.

(b) By supposition, N is a right ideal and so by duality to Corollary (3.2.12), we can choose a minimal right ideal R of N such that $R \subset N$. By Lemma (3.2.21) R contains an idempotent. Choose by Theorem (3.2.25) an idempotent $e \in R \cap K$. We show that $N \cap K$ is a minimal left ideal $N \cap M$. Let I be a left ideal of $N \cap M$ with $I \subset N \cap K$. We now show that $N \cap K \subset I$, let $u \in N \cap K$. Then choose $s \in I$. Now $e \in K$ and $K = Ks$ by Lemma (3.2.17)(a), we can choose $t \in K$ such that $e = ts$. Then $e = ee = ets$. Now $e \in N$ and $t \in M$ and so $ets \in (N \cap M)s \subset I$ and so $e \in I$. Now $u \in K = Ke$ and so $u = ue$. Also $u \in N \cap M$ and so $u \in (N \cap M)e \subset I$ as required.

(c) By (b) $N \cap K$ is a minimal left ideal of $N \cap M$. Also $N \cap M$ is a left ideal of N. So by Lemma (3.2.9)(c), $N \cap K$ is a minimal left ideal of N.

□

We now begin our study in some detail, the structure of a particular semigroup. Our motto would be that this allows us to analyse the structure of the smallest ideal of any semigroup that has a minimal left ideal with an idempotent.

3.2.27 Theorem. *[14] For any left zero semigroup X, and any right zero semigroup Y with a group G. Let e be the two-sided identity of G, fix $u \in X$ and $v \in Y$ and let $[\ ,\ ] : Y \times X \to G$ be a mapping satisfying $[y, u] = [v, x] = e$ for each $y \in Y$ and each $x \in X$. Let $S = X \times G \times Y$ and define the operation $\cdot$ on S as $(x, g, y) \cdot (x', g', y') = (x, g[y, x']g', y')$. So S is a simple semigroup and each of the following holds:*

(a) For each $(x, y) \in X \times Y, (x, [y, x]^{-1}, y)$ is an idempotent (where the inverse is taken in G) and all idempotents are of this form. In particular, the idempotents in $X \times G \times \{v\}$ are of the form (x, e, v) and the idempotents in $\{u\} \times G \times Y$ are of the form (u, e, y).

(b) For each $y \in Y$, $X \times G \times \{y\}$ is a minimal left ideal of S and each minimal left ideal of S are of this form.

(c) *For each $x \in X$, $\{x\} \times G \times Y$ is a minimal right ideal of S and each minimal right ideal of S is of this form.*

(d) *For each $(x, y) \in X \times Y, \{x\} \times G \times \{y\}$ is a maximal group in S and each maximal group in S is of this form.*

(e) *The minimal left ideal $X \times G \times \{v\}$ is the direct product of X, G, and $\{y\}$ and the minimal right ideal $\{u\} \times G \times Y$ is the direct product of $\{u\}$, G, and Y.*

(f) *Each maximal group in S is isomorphic to G.*

(g) *Each minimal left ideal of S is isomorphic to $X \times G$ and each minimal right ideal of S is isomorphic to $G \times Y$.*

Proof. See [14, Theorem 1.63]. □

Observe that in Theorem (3.2.29), the set S is the cartesian product of X, G, and Y, but is not the direct product unless $[y, x] = e$ for every $(y, x) \in Y \times X$.
Note that as a consequence of Theorem (3.2.29)(g) we have that for any $y \in Y$, $X \times G \times \{y\}$ $\simeq X \times G$. However, there is no transparent isomorphism unless $[y, x] = e$ for all $x \in X$, such as when $y = v$.
Theorem (3.2.29) spelt out in detail the structure of $X \times G \times Y$. We see now that this is in fact the structure of the smallest ideal of any semigroup which has a minimal left ideal with an idempotent.

3.2.28 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent. Let N be a minimal right ideal of S, let M be a minimal left ideal of S, let $X = E(M)$, let $Y = E(N)$, and let $G = NM$. Define an operation on $X \times G \times Y$ as $(x, g, y) \cdot (x', g', y') = (x, gyx'g', y')$. So $X \times G \times Y$ satisfies the conclusions of Theorem (3.2.29) (where $[y, x] = yx$) and ($K(S) = S = X \times G \times Y$). In particular:*

(a) *The minimal right ideals of S partition $K(S)$ and the minimal left ideals of S partition $K(S)$.*

(b) *The maximal groups in $K(S)$ partition $K(S)$.*

(c) *All minimal right ideals of S are isomorphic and all minimal left ideals of S are isomorphic.*

(d) *All maximal groups in $K(S)$ are isomorphic.*

Proof. According to Lemma (3.2.21), S has a minimal right ideal with idempotent and so N exists and thus by the preceding result Theorem (3.2.20), N has an idempotent. By Theorem (3.2.25) we know that NM is a group and by Theorem (3.2.8), we know that X is a left zero semigroup and Y is a right zero semigroup. Let e be the identity of $NM = N \cap M$ and let $u = v = e$. If $y \in Y$ then, because Y is a right zero semigroup, it follows that $[y, u] = yu = u = e$. On the other hand, given $x \in X$, $[v, x] = e$. Therefore the hypothesis of Theorem (3.2.27) are satisfied.
Define $\varphi : X \times G \times Y \to S$ by $\varphi(x, g, y) = xgy$. We claim φ is an isomorphism onto $K(S)$. To see this, let (x, g, y), $(x', g', y') \in X \times G \times Y$. Then

$$\begin{aligned}\varphi((x, g, y) \cdot (x', g', y')) &= \varphi(x, gyx'g', y') \\ &= xgyx'g'y' \\ &= (xgy)(x'g'y') \\ &= \varphi(x, g, y)\varphi(x', g', y').\end{aligned}$$

Thus, φ is a homomorphism. By Theorem (3.2.8), we have that $M = XG$ and $N = GY$. According to Theorem (3.2.18), $K(S) = MN = XGGY = XGY = \varphi[X \times G \times Y \to S]$. Hence it suffices only to establish an inverse for φ.
For each $t \in K(S)$, let $\gamma(t)$ be the inverse of ete in $G = eSe$. Then $t\gamma(t) = t\gamma(t)e \in Se = M$ and

$$\begin{aligned} t\gamma(t)t\gamma(t) &= t\gamma(t)etet\gamma(t) \\ &= te\gamma(t) \\ &= t\gamma(t), \end{aligned}$$

and so $t\gamma(t) \in X$. Similarly, $\gamma(t)t \in Y$.
Define $\eta : K(S) \to X \times G \times Y$ by $\eta(t) = (t\gamma(t), ete, \gamma(t)t)$. We show that $\eta = \varphi^{-1}$. Let $(x, g, y) \in X \times G \times Y$. Then

$$\eta(\varphi(x, g, y)) = (xgy\gamma(xgy), exgye, \gamma(xgy)xgy).$$

Then

$$\begin{aligned} xgy\gamma(xgy) &= xxgy\gamma(xgy) && (\text{since } x = xx) \\ &= xexgye\gamma(xgy) && (\text{since } x = xe \text{ and } \gamma(xgy) = e\gamma(xgy)) \\ &= xe \\ &= x. \end{aligned}$$

Similarly $\gamma(xgy)xgy = y$. Since X and Y are left and right zero semigroups respectively, $exgye = ege = g$. Thus we have $\eta = \varphi^{-1}$ as required. □

Theorem (3.2.28) is known in the literature as The Structure Theorem and it is due to A. Suschkewitsch in the case of finite semigroups and to D. Rees in the general case [14]. Theorem (3.2.28) together with Corollary(3.2.12), Theorem (3.2.16), Theorem (3.2.19) and Theorem (3.2.25) are known as the major structure theorems [6].

The following theorem enables us to identify the smallest ideal of many semigroups that arise in topological applications.

3.2.29 Theorem. *[14] Let S be a semigroup which contains a minimal left ideal with an idempotent. Let Q be a subsemigroup of S which also contains a minimal left ideal with an idempotent and suppose that $Q \cap K(S) \neq \emptyset$. Then the following statements hold:*

(1) $K(Q) = K(S) \cap Q$.

(2) The minimal left ideals of Q are precisely the nonempty of the form $Q \cap M$, where M is the minimal left ideal of S.

(3) The minimal right ideals of Q are precisely the nonempty of the form $Q \cap N$, where N is the minimal right ideal of S.

(4) If Q is an ideal of S, then $KQ) = K(S)$.

Proof. (1) By Theorem (3.2.16), $K(Q)$ exists and so, since $K(S) \cap Q$ is an ideal of Q, $K(Q) \subset K(S) \cap Q$ by Lemma (3.2.10). To see the reverse inclusion, let $s \in K(S) \cap Q$. Then Qs is a left ideal of Q and so by Corollary (3.2.12) and Theorem (3.2.20), Qs contains a minimal left ideal Qe of Q for some idempotent $e \in Q$. Now $s \in K(S)$ and so by Theorem (3.2.16) choose a minimal left ideal M of S with $s \in M$. Then $M = Ss$ and $e \in Qs \subset Ss$ and so $M = Se$ and so $e \in Se$ and so by Lemma (3.1.26), $s = se \in Qe \subset K(Q)$.

(2) Let K be a minimal left ideal of Q. Suppose that M is a minimal left ideal of S and that $Q \cap M \neq \emptyset$. By Corollary (3.2.12) $K \subset Q \cap M$ and that by Theorem (3.2.20), there is an idempotent $e \in K$. Now by Theorem (3.1.27)(b) Se is a left ideal of S and Qe is a left ideal of Q and so $Se = M$ and $Qe = K$. Thus $Q \cap M = Q \cap Se$.
We claim that $Q \cap Se = Qe$ for any idempotent $e \in Q$ and consequently $Q \cap M = Q \cap Se = Qe = K$. Indeed, let $s \in Q \cap Se$. Then $s = se \in Qe$, and so $Q \cap Se \subset Qe$. For the reverse inclusion, $s \in Qe$, then $s \in Q$ and $s \in Se$ since Q is a subsemigroup of S. Thus $Qe \subset Q \cap Se$.
Now let K be a minimal left ideal of Q. Then K contains an idempotent $e \in K(Q)$, since $e \in K(S)$ by (1), Se is a minimal left ideal of S by Theorem (3.2.23) and $Q \cap Se = Qe = K$.

(3) This follows by duality from (2).

(4) If Q is an ideal of S, then $K(S) \subset Q$ by Lemma (3.2.10). So $K(Q) = Q \cap K(S) = K(S)$.

□

As we now know from the Structure Theorem (Theorem (3.2.28)) that maximal groups in the smallest ideal are isomorphic. So it will be convenient for us later on to know an explicit isomorphism between them.

3.2.30 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent. Fix $e, f \in E(K(S))$. Let g be the inverse of e in eSe, so the mapping $\varphi : eSe \to fSf$ defined as $\varphi(x) = fxgf$ is an isomorphism.*

Proof. We first show that φ is a homomorphism. Fix $s, t \in eSe$. So

$$\begin{aligned}\varphi(s)\varphi(t) &= fsgfftgf \\ &= fsgftgf \\ &= fsgefetgf \quad \text{(since } ge = g\text{, and } et = t) \\ &= fsetgf \\ &= fstgf \\ &= \varphi(st).\end{aligned}$$

Next we show that φ is one-to-one, let s be in the kernel of φ. Then

$$\begin{aligned}fsgf &= f \\ efsgfe &= efe \\ efesgefe &= efe \quad \text{(since } es = s\text{, and } ge = g) \\ efese &= efe \\ efes &= efee \\ s &= e \quad \text{(left cancellation in } eSe).\end{aligned}$$

So $\ker \varphi = \{e\}$ and so φ is one-to-one.

Lastly, we show that φ is surjective to fSf, let $t \in fSf$ and let h be the inverses of fgf and let k be the inverses of fef respectively in fSf. So $ekthe \in eSe$ and

$$\begin{aligned}\varphi(ekthe) &= fekthegf \\ &= fefkthgf \quad (\text{since } fk = f, \text{ and } eg = g) \\ &= fthgf \quad (\text{since } fefk = f) \\ &= fthfgf \quad (\text{since } h = hf) \\ &= fyf \quad (\text{since } hfgf = f) \\ &= t.\end{aligned}$$

□

We conclude the Section with a theorem characterizing arbitrary elements of $K(S)$.

3.2.31 Theorem. *[14] Let S be a semigroup and suppose that there exists a minimal left ideal of S which contains an idempotent. Fix $s \in S$. The following are equivalent.*

(a) $s \in K(S)$.

(b) For each $t \in S$, $S \in Sts$.

(c) For each $t \in S$, $S \in stS$.

(d) For each $t \in S$, $S \in stS \cap Sts$.

Proof. (a) implies (d). Choose by Theorem (3.2.16) and Lemma (3.2.21) a minimal left ideal M of S and a minimal right ideal N of S with $s \in M \cap N$. Let $t \in S$. Then $st \in N$, hence stS is a right ideal contained in N then $stS = N$. On the other hand $Sts = M$.
The fact that (d) implies (b) and (d) implies (c) follow easily by supposition of (d).
(b) implies (a). Choose $t \in K(S)$. Then $s \in Sts \subset K(S)$.
(c) implies (a) follow similarly. □

3.3 Compact Right Topological Semigroups

From the previous sections we saw some examples of semigroups which do not possess idempotents, like, $(\mathbb{N}, +)$. In this section, however, as we promised that from a very special class of semigroups, the compact right topological ones, do possess idempotents.

For the following definition see [14]

3.3.1 Definition. [14]

(a) The triple $(S, \cdot, \mathcal{T})$ is a *right topological semigroup* where $(S, \cdot)$ is a semigroup, $(S, \mathcal{T})$ is a topological space, and for all $s \in S$, $\rho_s : S \to S$ is continuous.

(b) The triple $(S, \cdot, \mathcal{T})$ is a *left topological semigroup* where $(S, \cdot)$ is a semigroup, $(S, \mathcal{T})$ is a topological space, and for all $s \in S$, $\lambda_s : S \to S$ is continuous.

(c) A right topological semigroup which is also a left topological semigroup is *semitopological semigroup*.

(d) The triple $(S, \cdot, \mathcal{T})$ is a *topological semigroup* where $(S, \cdot)$ is a semigroup, $(S, \mathcal{T})$ is a topological space, and $\cdot : S \to S$ is continuous.

(e) The triple $(S, \cdot, \mathcal{T})$ is a *topological group* if $(S, \cdot)$ is a group, $(S, \mathcal{T})$ is a topological space, $\cdot : S \to S$ is continuous, and the inversion $s \to s^{-1} : S \to S$ is continuous.

We did not include any the separation axioms in the definitions above, we will only assume throughout the rest of this section that all hypothesized topological spaces are Hausdorff except where clearly stated otherwise. In a right topological, semigroup we say that the operation "$\cdot$" is "right continuous". We shall customarily not mention either the operation or the topology and say something like "let S be a right topological semigroup" [14].

Note that trivially each topological group is a topological semigroup, each topological semigroup is a semitopological semigroup and each semitopological semigroup is both a left and right topological semigroup [14].

This leads to the example that any semigroup which is not a group provides an example of a topological semigroup which is not a topological group simply by providing it with the discrete topology. Also, there is a semitopological semigroup which is not a topological semigroup [14].

3.3.2 Theorem. *[14] Let $(X, \mathcal{T})$ be any topological space and let $\mathcal{V}$ be the product topology on X^X.*

(a) $(X^X, \circ, \mathcal{V})$ is a right topological semigroup.

(b) For each $f \in X^X$ and $\lambda_f : X \to X$, then λ_f is continuous if and only if f is continuous.

3.3.3 Definition. [14] Let S be a right topological semigroup. The *topological center* of S is the set

$$\Lambda(S) = \{x \in S \ : \ \lambda_x \text{ is continuous}\}.$$

Thus a right topological semigroup S is a semitopological semigroup if and only if $\Lambda(S) = S$. Note that trivially the algebraic center of a right topological semigroup is contained in its topological center.

3.3.4 Example. [2] The first three examples of semigroups have a compact Hausdorff topology on them. In fact, they are topological semigroups.

(a) The unit circle in the complex plane

$$C = \{z \in \mathbb{C} : |z| = 1\}$$

is a compact subset of $\mathbb{C}$ and closed under complex multiplication, thus a topological semigroup. In particular, a topological group.

(b) The unit disk in $\mathbb{C}$

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$$

is a compact topological semigroup, under complex multiplication.

(c) Every finite semigroup, with the discrete topology, is a compact topological semigroup.

(d) With respect to the usual topology, $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are topological groups under addition and topological semigroups under multiplication. Also, under multiplication the circle group $\mathbb{T}$ is a compact topological group and the unit disk $\mathbb{D}$ is a compact topological semigroup.

(e) The set $M(n, \mathbb{C})$ of $n \times n$ matrices with complex entries is a topological semigroup with respect to matrix multiplication and the usual topology. The subgroup $GL(n, \mathbb{C})$ of nonsingular matrices is a topological group, and the subgroup is $U(n)$ of unitary matrices is a compact topological group.

(f) Let S be a locally compact, noncompact, Hausdorff, right topological semigroup and let $S_\infty = S \cup \{\infty\}$ be the one-point compactification of S. Recall that the topology of S_∞ consists of the open subset of S together with complements in S_∞ of compact subsets of S. Then S_∞ has a natural semigroup structure, namely the one that makes S a subsemigroup and ∞ a zero. (From now on, we will assume that S_∞ has this structure.) The following criterion is a necessary and sufficient condition on S for S_∞ to be a right topological semigroup:

$$\text{For each } s \in S \text{ and compact subset } K \text{ of } S,\ Ks^{-1} \text{ is compact in } S. \tag{3.3.1}$$

Of course, S_∞ is right topological if and only if for each $s \in S$ the mapping $\rho_s : S_\infty \to S_\infty$ is continuous at ∞, if and only if $\rho^{-1}(S_\infty \setminus K)$ is an open neighbourhood of ∞ for each $s \in S$ and compact $K \subset S$. The latter condition is equivalent to 3.3.1 since $Ks^{-1} = S_\infty \setminus \rho^{-1}(S_\infty \setminus K)$. Observe that 3.3.1 is trivially satisfied if S is a group. On the other hand, if, for example, S has a right zero, then 3.3.1 fails to hold and S_∞ is not right topological.
If S is discrete, then condition 3.3.1 may be replaced by the following simpler criterion:

$$\{t\}s^{-1} \text{ is finite for each } \ s, t \in S. \tag{3.3.2}$$

(g) Let $S := \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$ be topologized so that S is homeomorphic to a compact interval of real numbers. Extend addition from $\mathbb{R}$ to S by the rules

$$r + t = t + r = s + t = t \quad (r \in \mathbb{R}, s, t \in \{-\infty, \infty\}).$$

then S is a right topological semigroup with $\Lambda(S) = \mathbb{R}$.

(h) Let $S = (\mathbb{R}, +)$ with the topology for which a basis of neighbourhoods for x consists of the sets $\{y : x \leq y < x + 1/n\}$, $n \in \mathbb{N}$. Then S is a topological semigroup, but inversion is not continuous.

(i) Let $S = (\mathbb{R}, +)$ with the topology for which the open sets are the complements of finite sets. Then S is a semitopological group with continuous inversion. Also S is not Hausdorff and it is not a topological semigroup.

For more examples see [2].

We shall be concerned throughout the rest of the thesis with certain kinds of compact right topological semigroups. Of fundamental importance is the following theorem.

3.3.5 Theorem. *[14] Let S be a compact right topological semigroup. Then $E(S) \neq \emptyset$.*

Proof. Let $\mathcal{U} = \{Q \subset S : Q \neq \emptyset, Q \text{ is compact, and } Q \cdot Q \subset Q\}$. That is, $\mathcal{U}$ is the set of compact subsemigroups of S ordered by inclusion. We prove that $\mathcal{U}$ contains a minimal element using Zorn's Lemma. Because $S \in \mathcal{U}$, $\mathcal{U} \neq \emptyset$. Let $\mathcal{B}$ be a linearly ordered set in $\mathcal{U}$. So $\mathcal{B}$ is a collection of closed subsets of the compact space S which has the FIP, then $\bigcap \mathcal{B} \neq \emptyset$ and $\bigcap \mathcal{B}$ is compact and a semigroup by definition of $\mathcal{U}$. Hence, $\bigcap \mathcal{B} \in \mathcal{U}$, and so pick a minimal element A of $\mathcal{U}$.
Choose $s \in A$. We shall show $ss = s$. We begin by showing that $As = A$. Let $B = As$. Then $B \neq \emptyset$ since $A \neq \emptyset$, and since $B = \rho_s[A]$, B is the continuous image of a compact space, whence compact. Next $BB = AsAs \subset AAAs \subset As = B$. Therefore $B \in \mathcal{U}$. Since, $B = As \subset AA \subset A$, $B = A$, as A is minimal.
Let $C = \{t \in A \ : \ ts = s\}$. Since $s \in A = As$, it follows that $C \neq \emptyset$. Note, $C = A \cap \rho^{-1}[\{s\}]$, and so C is closed and hence compact. For any $r, t \in C$, we have that $tr \in AA \subset A$ and $trs = ts = s$ and so $tr \in C$. Therefore $C \in \mathcal{U}$. Because $C \subset A$ and A is minimal, it follows that $C = A$ then $s \in C$ and so $ss = s$ as required. □

In Section (3.2) there were some results which had as part of their suppositions "Let S be a semigroup and assume there is a minimal left ideal of S which has an idempotent." Due to the following corollary, we are able to incorporate all of these results.

3.3.6 Corollary. [14] Let S be a compact right topological semigroup. Then each left ideal of S has a minimal left ideal. Minimal left ideals are closed, and each minimal left ideal contains an idempotent.

Proof. Let M be a left ideal of S and let $\mathcal{U} = \{Q \subset S : Q \text{ is a closed left ideal of } S \text{ and } Q \subset M\}$. By applying Zorn's lemma to $\mathcal{U}$, we get a left ideal K minimal among all closed left ideals contained in M. Let I be any left ideal of S contained in K. If $s \in I$ then $Ss \subset I \subset K$ and since Ss is a closed left ideal of S (Ss is closed because $Ss = \rho_s[S]$), $Ss = K$, hence $K = I$. Therefore K is a minimal left ideal. Since K is closed, Theorem (3.3.5) implies K contains an idempotent. □

We now derive some results of Corollary (3.3.6).

3.3.7 Theorem. *[14] Let S be a compact right topological semigroup.*

a) Each right ideal of S contains a minimal right ideal which contains an idempotent.

b) Let $Q \subseteq S$. Then Q is a minimal left ideal of S if and only if there exists $e \in E(K(S))$ satisfying $Q = Se$.

c) Let $Q \subseteq S$. Then Q is a minimal right ideal of S if and only if there exists $e \in E(K(S))$ satisfying $Q = eS$.

d) Given any minimal left ideal M of S and any minimal right ideal N of S, there exists an idempotent $e \in N \cap M$ satisfying $N \cap M = eSe$ and eSe is a group.

Proof. (a) This follows from Corollary (3.3.6), Lemma (3.2.21), Corollary (3.2.12), and Theorem (3.2.20).

(b) and (c) follows from Corollary (3.3.6) and Theorem (3.2.22).

(d) This follows Corollary (3.3.6) and Theorem (3.2.25).

□

3.3.8 Theorem. *[14] Let S be a compact right topological semigroup. Then S contains a smallest two sided ideal $K(S)$ which is the union of all minimal left ideals of S and also the union of all minimal right ideals of S. Each of $\{Se : e \in E(K(S))\}$, $\{eS : e \in E(K(S))\}$, and $\{eSe : e \in E(K(S))\}$ partitions $K(S)$.*

Proof. This follows Corollary (3.3.6) and Theorem (3.2.22), (3.2.25), and (3.2.28). □

3.3.9 Theorem. *[14] Let S be a compact right topological semigroup with let $e \in E(S)$. The following are equivalent:*

(a) Se is a minimal left ideal.

(b) Se is left simple.

(c) eSe is a group.

(d) $eSe = H(e)$.

(e) eS is a minimal right ideal.

(f) eS is right simple.

(g) e is a minimal idempotent.

(h) $e \in K(S)$.

(i) $K(S) = SeS$.

Proof. This follows Corollary (3.3.6) and Theorem (3.2.23). □

3.3.10 Theorem. *[14] Let S be a compact right topological semigroup. For $s \in S$, the following are equivalent.*

(a) $s \in K(S)$.

(b) For each $t \in S$, $s \in Sts$.

(c) For each $t \in S$, $s \in stS$.

(d) For each $t \in S$, $s \in stS \cap Sts$.

Proof. This follows Corollary (3.3.6) and Theorem (3.2.31). □

The few results that we have presented thus far have had purely algebraic conclusions. We at once incur a result with both topological and algebraic conclusions. We say two topological spaces which are also semigroups are *topologically and algebraically isomorphic* if there is a function from one of them onto the other which is both an isomorphism and a homeomorphism.

3.3.11 Theorem. *[14] Let S be a compact right topological semigroup.*

a) All the maximal subgroups of $K(S)$ are (algebraically) isomorphic.

b) *Maximal subgroups of $K(S)$ which are contained in the same minimal right ideal are topologically and algebraically isomorphic. In fact, if N is a minimal right ideal of S and $e, f \in E(N)$, then the restriction of ρ_p to eSe is an isomorphism and a homeomorphism onto fSf.*

c) *All minimal left ideals of S are homeomorphic. In fact, if M and M' are minimal left ideals of S and $r \in M'$, then $\rho_{r|M}$ is a homeomorphism from M onto M'.*

Proof. (a) This follows from Corollary (3.3.6) and Theorem (3.2.30).

(b) Let N be a minimal right ideal of S and let $e, f \in E(N)$. So eS and fS are right ideals contained in N and so $N = eS = fS$. Then by Lemma (3.1.26), $ef = f$ and $fe = e$. If $s \in eSe$, then $\rho_f(s) = sf = esf = fesf = fsf$ and $\rho_e(\rho_f(s)) = sfe = se = x$. If $r \in fSf$, then $\rho_f(ere) = efref = frf = r$. Since ρ_f and ρ_e are continuous, we have that the restriction of ρ_f to eSe is a homeomorphism from eSe onto fSf. We have that ρ_f is a homomorphism, let $s, t \in eSe$. Then $\rho_f(st) = setf = sftf = \rho_f(s)\rho_f(t)$.

(c) Let M and M' be minimal left ideals of S and let $r \in M'$. By Theorem (3.3.7)(b), choose $e \in E(K(S))$ satisfying $M = Se$. Then $\rho_{r|M}$ is a continuous function from Se into $Sr = M'$ and $\rho_r[Se] = M'$ because Ser is a left ideal of S which lies in M'. We now show that ρ_r is one-to-one on Se. Let g be the inverse of ere in eSe, we show that for $s \in Se$, $\rho_g(\rho_r(s)) = s$, and so let $s \in Se$ be given. Then

$$\begin{aligned} srg &= sereg \quad (s = se \text{ and } g = eg) \\ &= se \\ &= s. \end{aligned}$$

Since $\rho_{r|M}$ is one-to-one and continuous and M is compact, $\rho_{r|LM}$ is a homeomorphism.

□

3.3.12 Theorem. *[14] Let S be a compact right topological semigroup and let $e \in E(S)$. There is a $\leq_R$-maximal idempotent f in S with $e \leq_R f$.*

Proof. Let $A = \{s \in E(S) \ : \ e \leq_R f\}$. Then $A \neq \emptyset$, because $e \in A$ by supposition. Let C be a $\leq_R$-chain in A. Then $\{c\ell\{r \in C \ : \ s \leq_R r\} : \ s \in C\}$ is a collection of closed subsets of S with the FIP, and so $\bigcap_{s \in C} c\ell\{r \in C \ : \ s \leq_R r\} \neq \emptyset$. Since S is Hausdorff, $\bigcap_{s \in C} c\ell\{r \in C \ : \ s \leq_R r\} \subset \{t \in S : \text{ for all } s \in C, ts = s\}$. Therefore, $\{t \in S : \text{ for all } s \in C, ts = s\}$ is a compact subsemigroup of S and hence by Theorem (3.3.5) there is an idempotent q such that for all $s \in C$, $qs = s$. This q is an upper bound for C, and so A has a maximal element. □

3.3.13 Lemma. [14] Let S and Q be compact right topological semigroups, let D be a dense subsemigroup of S such that $D \subseteq \Lambda(S)$, and let η be a continuous function from S to Q such that

1) $\eta[D] \subseteq \Lambda(Q)$ and

2) $\eta_{|D}$ is a homomorphism.

Then η is a homomorphism.

Proof. For each $d \in D$, $\eta \circ \lambda_d$ and $\lambda_{\eta(d)} \circ \eta$ are continuous functions agreeing on the dense subset D of S. Therefore for all $d \in D$ and all $t \in S$, $\eta(dt) = \eta(d)\eta(t)$. Hence, for all $t \in S$, $\eta \circ \rho_t$ and $\rho_{\eta(t)} \circ \eta$ are continuous functions agreeing on the dense subset of S and so for all s and t in S, $\eta(st) = \eta(s)\eta(t)$. □

We analyse for a brief moment the closures of right ideals, left ideals, and maximal subgroups of right topological semigroups. We also take into consideration their Cartesian product.

3.3.14 Theorem. *[14] Let S be a compact right topological semigroup with N a right ideal of S. Then $c\ell\, N$ is a right ideal of S.*

Proof. The continuity of the mapping ρ_s for $s \in S$ implies that $c\ell(N)S \subset c\ell(NS) \subset c\ell\, N$.

□

3.3.15 Theorem. *[14] Let S be a compact right topological semigroup and suppose that $\Lambda(S)$ is dense in S. Let M be a left ideal of S. Then $c\ell\, M$ is a left ideal of S.*

Proof. Let $t \in c\ell\, M$ and let $s \in S$. We show that $st \in c\ell\, M$. Let O be an open neighbourhood of st, choose a neighbourhood V of s such that $Vt = \rho_t[V] \subset O$ and choose $r \in \Lambda(S) \cap V$. Then $rt = \lambda(t) \in O$ and so choose a neighbourhood W of t such that $rW \subset O$. Then $w \in W \cap M$. Then, since M is a left ideal, $rw \in rW \cap M \subset O \cap M$ and so $rw \in O \cap M$. Therefore, $st \in c\ell\, M$. □

3.3.16 Lemma. [14] Let S be a compact right topological semigroup. The following are equivalent.

a) The (algebraic) center of S is dense in S.

b) There is a dense commutative subset A of S with $A \subseteq \Lambda(S)$.

3.3.17 Theorem. *[14] Let S be a compact right topological semigroup with dense center.*

a) If N is a right ideal of S, then $c\ell\ N$ is a two sided ideal of S.

b) If $e \in E(K(S))$, then $c\ell(eSe) = Se$.

Proof. Let A be the center of S.

(a) According to Theorem (3.3.14), $c\ell\, N$ is a right ideal of S. We show that $S \cdot (c\ell\, N) \subset c\ell\, N$. Let $t \in c\ell\, N$ be given, then given any $s \in A$, we have $\rho_t(s) = st = ts \in (c\ell\, N)s \subset c\ell\, N$ (by Lemma (3.3.16)). Thus $\rho_t[A] \subset c\ell\, N$, and so since A is dense $\rho_t[S] = \rho_t[c\ell\, A] \subset c\ell\, \rho_t[A] \subset c\ell\, N$. Thus $St \subset c\ell\, N$ as required.

(b) Since Se is closed (being the continuous image of a compact space), $c\ell(eSe) \subset Se$. Then again, since ρ_e is continuous, $Se = (c\ell\, A)e = c\ell(Ae) = c\ell(eAe) \subset c\ell(eSe)$.

□

3.3.18 Theorem. *[14] Let S be a compact right topological semigroup with dense center. Assume S has some minimal right ideal N which is closed. Then $N = K(S)$ and all maximal subgroups of $K(S)$ are closed and pairwise algebraically and topologically isomorphic.*

Proof. According to Theorem (3.3.17), $c\ell\, N$ is an ideal and then $K(S) \subset c\ell\, N = N \subset K(S)$. Given $e \in E(K(S))$, $H(e) = eSe = N \cap Se$ by Theorems (3.3.9) and (3.3.7) and so $H(e)$ is closed. Any two maximal subgroups of $K(S)$ contained in the same minimal right ideal are algebraically and isomorphic by Theorem (3.3.11). □

3.3.19 Theorem. *[14] Let S be a compact right topological semigroup with dense center. The following are equivalent.*

a) $K(S)$ is a minimal right ideal of S.

b) All the maximal subgroups of $K(S)$ are closed.

c) There is some maximal subgroup of $K(S)$ which is closed.

Proof. (a) implies (b). Let $e \in E(K(S))$. Then according to Theorem (3.3.9) eS is a minimal right ideal and Se a minimal left ideal, so according to Theorem (3.3.7)(d), $eSe = eS \cap Se$. Because $K(S)$ is a minimal right ideal $eS = K(S)$. Because $Se \subset K(S)$, $eSe = eS \cap Se = Se$, and Se is closed according to Corollary (3.3.6).

(b) implies (c). Follows.

(c) implies (a). Choose $e \in E(K(S))$ satisfying eSe is closed. Let $N = eS$. According to Theorem (3.3.17)(b), $c\ell(eSe) = Se$. So $Se = eSe \subset eS = N$. Immediately any other minimal right ideal of S would be disjoint from N and so would miss Se, which can not be by Lemma (3.1.25)(b). Therefore N is the only minimal right ideal of $K(S)$, which happens to be the union of all minimal right ideals, so $K(S) = N$. □

3.4 The Stone-Čech Compactification of a Discrete Semigroup

In this section, we assume that $(S, \cdot)$ is a semigroup, that also possess the discrete topology. As already discussed, we will consider S as a (dense) subset of its Stone-Čech Compactification βS, that is, $S \subset \beta S$. Using properties of βS, we will extend the operation of multiplication of a discrete semigroup to its Stone-Čech Compactification which will make βS a right topological semigroup with S contained in its topological center.

Without identification allpof the point of s of S with the principal ultrafilter $e(s)$, conclusion (a) and (c) of Theorem (3.4.1) would sound as follows [14]:

(a) The embedding e is a homomorphism.

(c) For all, $s \in S$, $\lambda_{e(s)}$ is continuous.

3.4.1 Theorem. *[14] Let S be a discrete space and let $\cdot$ be an operation defined on S. There exists a unique binary operation $* : \beta S \times \beta S \to \beta S$ such that the following conditions holds:*

(a) For each $s, t \in S$, $s * t = s \cdot t$.

(b) For every $q \in \beta S$, the mapping $\rho_q : \beta S \to \beta S$ is continuous, where $\rho_q(p) = p * q$.

(c) For every $s \in S$, the mapping $\lambda_s : \beta S \to \beta S$ is continuous, where $\lambda_s(q) = s * q$.

Proof. We prove uniqueness and existence at the same time. We first define $*$ on $S \times \beta S$. Given any $s \in S$, define $\ell_s : S \to S \subset \beta S$ by $\ell_s(t) = s \cdot t$. So according to Theorem(2.3.17), there is a continuous map $\lambda_s : \beta S \to \beta S$ satisfying $\lambda_{s|S} = \ell_s$. If $s \in S$ and $q \in \beta S$, we define $s* = \lambda_s(q)$. So (c) is true and so does (a), because λ_s extends ℓ_s. Moreover, the extension λ_s is unique because by Corollary (2.2.16), functions that agree on a dense subspace are the same. So this is the definition of $*$ satisfying (a) and (c).
Immediately we extend $*$ throughout $\beta S \times \beta S$. Given $q \in \beta S$, define $r_q : S \to \beta S$ by $r_q(s) = s * q$. Then there is a continuous function $\rho_q : \beta S \to \beta S$ such that $\rho_{q|S} = r_q$. For $p \in \beta S \setminus S$, we define $p * q = \rho_q(p)$ and observe that if $s \in S$, $\rho_q(s) = r_q(s) = s * q$. Then for all, $p \in \beta S, \rho_q(p) = p * q$. It follows that (b) is true. By uniqueness again of continuous extensions, this is the only possible definition which satisfies the conditions. □

It is customary to denote the operation on βS by the same symbol as that used for the operation on S and we shall adopt that practice. There are exceptions. If we are working with the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$, we shall *not* speak about the semigroup $(\beta\mathcal{P}_f(\mathbb{N}), \cup)$.

We shall see that $(\beta S, \cdot)$ is a semigroup whenever $(S, \cdot)$ is a semigroup. First we analyse the operations on βS. The operation on βS has a characterization in terms of limits. The statements in the following remark follow immediately from the fact that λ_s is continuous for every $s \in S$ and $\rho_q \in \beta S$. (Here the elements s and t represent members of S.)

3.4.2 Remark. [14] Let $\cdot$ be an operation on a discrete space S.

(a) Let $s \in S$ and $q \in \beta S$, then $s \cdot q = \lim\limits_{t \to q} s \cdot t$.

(b) Let $p, q \in \beta S$, then $p \cdot q = \lim\limits_{s \to p}(\lim\limits_{t \to q} s \cdot t)$.

The following modification of Remark (3.4.2)(b) will often be useful.

3.4.3 Remark. [14] Let $\cdot$ be an operation on a discrete space S, let $p, q \in \beta S$, let $P \in p$ and $Q \in q$. So $p \cdot q = \underset{s \in P}{p\text{-lim}}(\underset{t \in Q}{q\text{-lim}}\ s \cdot t)$.

We recall that if $f : X \to Y$ is a continuous function and $\lim\limits_{s \to p} f(x_s)$ and $\lim\limits_{s \to p} x_s$ exist, then $\lim\limits_{s \to p} f(x_s) = f(\lim\limits_{s \to p} x_s)$, by Theorem (2.4.5). We shall regularly use this fact without mention.

3.4.4 Theorem. *[14] Let $(S, \cdot)$ be a semigroup. Then operation on βS which is extended is associative.*

Proof. Let $p, q, r \in \beta S$. We look at $\lim\limits_{a \to p} \lim\limits_{b \to q} \lim\limits_{c \to r} (a \cdot b) \cdot c$, where a, b and c denote the elements of S.

We have:

$$\begin{aligned}
\lim_{a\to p}\lim_{b\to q}\lim_{c\to r}(a\cdot b)\cdot c &= \lim_{a\to p}\lim_{b\to q}\lim_{c\to r}\lambda_{a\cdot b}(c)\\
&= \lim_{a\to p}\lim_{b\to q}\lambda_{a\cdot b}(r) \quad \text{(because } \lambda_{a\cdot b} \text{ is continuous)}\\
&= \lim_{a\to p}\lim_{b\to q}(a\cdot b)\cdot r\\
&= \lim_{a\to p}\lim_{b\to q}(\lambda_a(b))\cdot r\\
&= \lim_{a\to p}\lim_{b\to q}\rho_r(\lambda_a(b))\\
&= \lim_{a\to p}\lim_{b\to q}(\rho_r\circ\lambda_a)(b)\\
&= \lim_{a\to p}(\rho_r\circ\lambda_a)(q) \quad \text{(because } \rho_r\circ\lambda_a \text{ is continuous)}\\
&= \lim_{a\to p}(a\cdot q)\cdot r\\
&= \lim_{a\to p}\rho_r(\rho_q(a))\\
&= \lim_{a\to p}(\rho_r\circ\rho_q)(a)\\
&= (\rho_r\circ\rho_q)(p) \quad \text{(because } \rho_r\circ\rho_q \text{ is continuous)}\\
&= (p\cdot q)\cdot r.
\end{aligned}$$

Also

$$\begin{aligned}
\lim_{a\to p}\lim_{b\to q}\lim_{c\to r}a\cdot(b\cdot c) &= \lim_{a\to p}\lim_{b\to q}\lim_{c\to r}a\cdot(\lambda_b(c))\\
&= \lim_{a\to p}\lim_{b\to q}\lim_{c\to r}(\lambda_a\circ\lambda_b)(c)\\
&= \lim_{a\to p}\lim_{b\to q}(\lambda_a\circ\lambda_b)(r) \quad \text{(because } \lambda_a\circ\lambda_b \text{ is continuous)}\\
&= \lim_{a\to p}\lim_{b\to q}a\cdot(b\cdot r)\\
&= \lim_{a\to p}\lim_{b\to q}\lambda_a(\rho_r(b))\\
&= \lim_{a\to p}\lim_{b\to q}(\lambda_a\circ\rho_r)(b))\\
&= \lim_{a\to p}(\lambda_a\circ\rho_r)(q) \quad \text{(because } \lambda_a\circ\rho_r \text{ is continuous)}\\
&= \lim_{a\to p}a\cdot(q\cdot r)\\
&= \lim_{a\to p}\rho_{q\cdot r}(a)\\
&= \rho_{q\cdot r}(p) \quad \text{(because } \rho_{q\cdot r} \text{ is continuous)}\\
&= p\cdot(q\cdot r).
\end{aligned}$$

So $(p\cdot q)\cdot r = p\cdot(q\cdot r)$. □

As a result of Theorem (3.4.1) and (3.4.4), we see that $(\beta S,\cdot)$ is a compact right topological semigroup, so that all of the results of Section (3.3) apply. In addition, we shall see that $(\beta S,\cdot)$ is maximal among the semigroup compactification of S in a sense similar to that in which βS is maximal among topological compactifications.

3.4.5 Definition. Let S be a semigroup which is also a topological space. A pair (φ, T) is a semigroup compactification of S where Q is a compact right topological semigroup, $\varphi : S \to Q$ is a continuous homomorphism, $\varphi[S] \subset \Lambda(Q)$, and $\varphi[S]$ is dense in Q.

3.4.6 Theorem. *[14] Let S be a semigroup, and let $\iota : S \to \beta S$ be the inclusion map.*

(a) $(\iota, \beta S)$ is a semigroup compactification of S.

(b) If T is a compact right topological semigroup and $\varphi : S \to T$ is a continuous homomorphism with $\varphi[S] \subset \Lambda(T)$ (in particular, if (φ, T) is a semigroup compactification of S), there is a continuous homomorphism $\eta : \beta S \to T$ such that $\eta_{|S} = \varphi$.

Proof. (a) This follows from Theorems (2.3.17), (3.4.1), and (3.4.4).

(b) Let $*$ be the operation on T. By Theorem (2.3.17) choose a continuous function $\eta : \beta S \to T$ such that $\eta_{|S} = \varphi$. We only need to show that η is a homomorphism. By Theorems (2.3.17) and (3.4.1), S is dense in βS and $S \subset \beta S$, and so by Lemma (3.3.13), η is a homomorphism.

□

Since the points of βS are ultrafilters, we want to know which subsets of S are members of $p \cdot q$.

3.4.7 Definition. Let $(S, \cdot)$ be a semigroup, $s \in S$ and $A \subset S$.

(a) $s^{-1}A = \{t \in S : st \in A\}$.

(b) $As^{-1} = \{t \in S : ts \in A\}$.

Observe that $s^{-1}A$ is simply an alternative notation for $\lambda_s^{-1}[A]$, and its use does not imply that s has an inverse in A. If s does have an inverse s^{-1} in A, then $\lambda_s^{-1}[A] = \{s^{-1}a : a \in A\}$.

We shall often work with semigroups where the operation is denoted by $+$, and so we introduce the appropriate notation.

3.4.8 Definition. [14] Let $(S, +)$ be a semigroup, $s \in S$ and $A \subset S$.

(a) $-s + A = \{t \in S : s + t \in A\}$.

(b) $A - s = \{t \in S : t + s \in A\}$.

3.4.9 Theorem. *[14] Let $(S, \cdot)$ be a semigroup with $A \subset S$.*

(a) For each $s \in S$ and $q \in \beta S$, $A \in s \cdot q$ iff $s^{-1}A \in q$.

(b) For each $p, q \in \beta S$, $A \in p \cdot q$ iff $\{s \in S : s^{-1}A \in q\} \in p$.

Proof. (a) Let $A \in sq$. Then $\widehat{A}$ is a neighbourhood of $\lambda_s(q)$ and so choose $B \in q$ such that $\lambda_s[\widehat{B}] \subset \widehat{A}$. Since $B \subset S \setminus A$, $s^{-1}A \in q$.
Conversely, let $s^{-1}A \in q$ and suppose that $A \notin sq$. Then $S \setminus A \in sq$ and so, by by the already established necessity, $s^{-1}(S \setminus A) \in q$. This is a contradiction since $s^{-1}A \cap s^{-1}(S \setminus A) = \emptyset$.

(b) Let $A \in pq$. Then $\widehat{A}$ is a neighbourhood of $\rho_q(p)$ and so choose $D \in q$ such that $\rho_q[\widehat{D}] \subset \widehat{A}$. Then for every $s \in D$, $A \in sq$ and so by (a) $s^{-1}A \in q$. Hence, $\{s \in S : s^{-1}A \in q\} \in p$. Conversely, let $\{s \in S : s^{-1}A \in q\} \in p$ and suppose that $A \notin pq$. Then $S \setminus A \in pq$ and so, by by the already established necessity, $\{s \in S : s^{-1}(S \setminus A) \in q\} \in p$. But $s^{-1}A \cap s^{-1}(S \setminus A) = \emptyset$ for each $s \in S$. It follows that

$$\{s \in S : s^{-1}A \in q\} \cap \{s \in S : s^{-1}(S \setminus A) \in q\} = \emptyset,$$

which is a contradiction.

□

The following notion will be of significant interest to us in applications involving idempotents in βS.

3.4.10 Definition. [14] Let $(S, \cdot)$ be a semigroup, let $A \subset S$ and let $p \in \beta S$. then $A^\star(p) = \{s \in S : s^{-1}A \in p\}$.

We shall oft-times write $A^\star$ rather than $A^\star(p)$.

3.4.11 Lemma. [14] Let $(S, \cdot)$ be a semigroup, let $p \cdot p = p \in \beta S$, and let $A \in p$. For each $s \in A^\star(p), s^{-1}(A^\star(p)) \in p$.

Proof. Let $s \in A^\star(p)$, and let $B = s^{-1}A$. Then $B \in p$ and, since p is an idempotent, $B^\star(p) \in p$. We claim that $B^\star(p) \subset s^{-1}(A^\star(p))$. To see this, let $t \in B^\star(p)$. Then $t \in B$ and so $st \in A$. Also $t^{-1}B \in p$. That is, $(st)^{-1}A \in p$. Since $st \in A$ and $(st)^{-1}A \in p$, we have that $st \in A^\star(p)$ as required. □

Lemma (3.4.11) is an immediate consequence of Theorem (3.4.9)(b), but much stronger. To see this, one has that a point p of βS is an idempotent iff for every $A \subset S, A^\star(p) \in p$. Of course, whenever $A \in p$ and $s \in A^\star(p)$, $s^{-1}A \in p$ [14].

3.4.12 Theorem. *[14] Let $(S, \cdot)$ be a semigroup, let $p, q \in \beta S$ and let $A \subset S$. Then $A \in p \cdot q$ if and only if there exists $B \in p$ and an indexed family $\langle C_s \rangle_{s \in B}$ in q such that $\bigcup_{s \in B} sC_s \subset A$.*

3.4.13 Lemma. [14] Let $(S, \cdot)$ be a semigroup, let $s \in S$, let $q \in \beta S$ with $A \subset S$.

(a) Whenever $A \in q$, $sA \in s \cdot q$.

(b) Let S be left cancellative and $sA \in s \cdot q$ then $A \in q$.

Proof. (a) We have $A \subset s^{-1}(sA)$, so by Theorem (3.4.9)(a) $sA \in s \cdot q$.

(b) Since S is cancellative, $s^{-1}(sA) = A$ and so by (a), $A \in q$.

□

The following results will oft-times be useful to us later.

3.4.14 Theorem. *[14] Fix S to be a discrete semigroup, let (φ, Q) be a semigroup compactification of S with $L \subset R \subset S$. Suppose that B is a subsemigroup of S.*

(a) $c\ell(\varphi[B])$ *is a subsemigroup of* Q.

(b) If L *is a left ideal of* R*, then* $c\ell(\varphi[L])$ *is a left ideal of* $c\ell(\varphi[R])$.

(c) If L *is a right ideal of* R*, then* $c\ell(\varphi[L])$ *is a right ideal of* $c\ell(\varphi[R])$.

Proof. (a) This is a consequence of [14, Exercise 2.3.2].

(b) Suppose that L is a left ideal of R. Let $s \in c\ell\,\varphi[R]$ and $t \in c\ell\,\varphi[L]$. Then $st = \lim_{\varphi(u)\to s} \lim_{\varphi(v)\to t} \varphi(u)\varphi(v)$, where u and v are elements of R and L respectively. If $u \in R$ and $v \in A$, we have $\varphi(u)\varphi(v) = \varphi(uv) \in \varphi[L]$ and so $st \in c\ell\,\varphi[L]$.

(c) The proof follows similarly to (b).

□

3.4.15 Corollary. [14] Fix S to be a subsemigroup of the discrete semigroup Q. Then $c\ell\, S$ is a subsemigroup of βQ. Whenever S is a right or left ideal of Q, then $c\ell\, S$ is respectively a right or left ideal of βQ.

3.4.16 Example. [14] Recall that if $S \subset T$, we have identified βS with the subset $\widehat{S}$ of βT. We have that if $(S,\cdot)$ is a subsemigroup of $(T,\cdot)$ and $p, q \in \beta S$, then the product $p \cdot q$ is the same whether it is computed in βS or in βT. That is, if $r = \{A \subset S : \{x \in S : x^{-1}A \in q\} \in p\}$ (the product $p \cdot q$ computed in βS), then $\{B \subset T : B \cap S \in r\} = \{A \subset T : \{x \in T : x^{-1}A \in q\} \in p\}$ (the product $p \cdot q$ computed in βT). (The notation $x^{-1}A$ is ambiguous. In the first case it should be $\{y \in S : xy \in A\}$ and in the second case it should be $\{y \in T : xy \in A\}$.)

3.4.17 Commutativity in βS. We look into some elementary results about commutativity in βS here, later we shall present more deep results when we want to understand better the space $\beta\mathbb{N}$.

3.4.18 Theorem. *[14] If* $(S,\cdot)$ *is a semigroup, then* S *is contained in the center of* $(\beta S,\cdot)$.

3.4.19 Theorem. *[14] If* S *be a discrete commutative semigroup. Then the topological center of* βS *coincides with algebraic centre.*

If $(S,\cdot)$ happens to be a commutative semigroup and $(\beta S,\cdot)$ commutative then it follows equivalent that $(\beta S,\cdot)$ is a left topological and semitopological semigroup see [14, Theorem 4.25].

3.4.20 Lemma. [14] Let $(S,\cdot)$ be a semigroup, let $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ be sequence in S and let $p, q \in \beta S$. If $\{\{x_n : n > k\} : k \in \mathbb{N}\} \subset q$ and $\{y_k : k \in \mathbb{N}\} \in p$, so $\{y_k \cdot x_n : k, n \in \mathbb{N} \text{ and } k < n\} \in p \cdot q$.

We are now in the process of being able to characterize when βS is commutative.

3.4.21 Theorem. *[14] Let* $(S,\cdot)$ *be a semigroup. Then* $(\beta S,\cdot)$ *is not commutative if and only if there exists sequences* $\langle x_n\rangle_{n=1}^{\infty}$ *and* $\langle y_n\rangle_{n=1}^{\infty}$ *such that*

$$\{y_k \cdot x_n : k, n \in \mathbb{N} \text{ and } k < n\} \cap \{x_k \cdot y_n : k, n \in \mathbb{N} \text{ and } k < n\} = \emptyset.$$

As a consequence of Theorem (3.4.24), it follows that neither $(\beta\mathbb{N}, +)$ nor $(\beta\mathbb{N},\cdot)$ is commutative.

3.4.22 Example. [14]

(a) If S is a left zero semigroup, so is βS.

(b) If S is a right zero semigroup, so is βS. Because of the lack of symmetry in the definition of βS, this is not "dual" to part (a) above.

3.4.23 The semigroup βS. For many reasons in the literature, the semigroup $S^* = \beta S \setminus S$ is the one most researchers are interested in. For in the first place, it is the algebra S^* that is the "new" material to study. In the second place, it turns out that the structure of S^* provides most of the combinatorial applications that are a large part of the motivation for studying this subject [14].

3.4.24 Theorem. *[14] Let S be a semigroup. Then S^* is a subsemigroup of βS iff for all $L \in \mathcal{P}_f(S)$ and for all infinite subsets R of S there is $G \in \mathcal{P}_f(R)$ such that $\bigcap_{x\in F} x^{-1}L$ is finite.*

3.4.25 Corollary. [14] Let S be a semigroup. Whenever S is either right or left cancellative so S^* is a subsemigroup of βS.

We take a look at simple conditions characterizing when S^* is a left ideal of βS.

3.4.26 Definition. [14] Let S be a semigroup with $A \subset S$.

(a) A is a *left solution* iff there is u and v in S such that $A = \{x \in S : ux = v\}$.

(b) A is a *right solution* iff there is u and v in S such that $A = \{x \in S : xu = v\}$.

(c) S is *weakly left cancellative* iff every left solution set in S is finite.

(d) S is *weakly right cancellative* iff every right solution set in S is finite.

(c) S is *weakly cancellative* iff S is weakly left cancellative and weakly right cancellative.

A left cancellative semigroup is weakly left cancellable. On the other hand the semigroup $(\mathbb{N}, \vee)$ is weakly left (right) cancellative but is very far from being cancellative [14]. When it is said that a function is *finite-to-one*, it is meant that for each s in the range of f, $f^{-1}[\{s\}]$ is finite. Thus a semigroup S is weakly left cancellative if and only if for each $s \in S$, λ_s is finite-to-one

3.4.27 Theorem. *[14] Whenever S is an infinite semigroup, S^* is a left ideal of βS iff S is weakly left cancellative.*

The characterization of S^* as a right ideal is considerably more complicated. [14, Exercise 4.3.7] shows that weakly right cancellativity of S is not sufficient for S^* to be a right ideal of βS. Unfortunately the exercise uses what we call *free semigroups* which we do not study here.

3.4.28 Theorem. *[14] Let S be an infinite semigroup. The following are equivalent:*

(a) A right ideal of βS is S^*.

(b) For any finite subset A of S, each sequence $\langle z_n\rangle_{n=1}^{\infty}$ on S, and any injective sequence $\langle x_n\rangle_{n=1}^{\infty}$ on S, there exists $n < m$ in $\mathbb{N}$ satisfying $x_n \cdot z_m \notin A$.

(c) *For any $a \in S$, any sequence $\langle z_n \rangle_{n=1}^{\infty}$ on S, and each injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ on S, there exists $n < m$ in $\mathbb{N}$ satisfying $x_n \cdot z_m \neq a$.*

3.4.29 Corollary. [14] Let S be an infinite semigroup.

(a) Whenever S is left cancellative, S^* is a left ideal of βS.

(b) Whenever S is right cancellative, S^* is a right ideal of βS.

3.4.30 Corollary. [14] Let S be an infinite semigroup. Whenever S^* is a right ideal of βS, S is weakly right cancellative.

3.4.31 Corollary. [14] Let S be an infinite semigroup. If S is weakly left cancellative and for all but finitely many $t \in S$, λ_t is finite-to-one, then S^* is a right ideal.

It happens that the situation with respect to S^* as a two sided ideal is considerably simpler than the situation with respect to S^* as a right ideal.

3.4.32 Theorem. *[14] Let S be an infinite semigroup. Then S^* is an ideal of βS if and only if S is left cancellative and weakly right right cancellative.*

This following criterion is of importance, since it is oft-times easier to work with S^* than with βS.

3.4.33 Theorem. *[14] Let S be an infinite semigroup. Whenever S^* is an ideal of βS, the minimal left ideals, minimal right ideals, and the smallest ideals of S^* and of βS are equal.*

3.4.34 Example. [14] $(\mathbb{N}^*, +)$ and $(-\mathbb{N}^*, +)$ are both left ideals of $(\beta\mathbb{Z}, +)$. Here $-\mathbb{N}^* = \{-p : p \in \mathbb{N}^*\}$ and $-p$ is the ultrafilter on $\mathbb{Z}$ generated by $\{-A : A \in p\}$.

In this subsection, we establish some results about the Stone-Cěch compactification of a semigroup which will usually be cancellative and countable.

We take from [8], the following definition of a specific subsemigroup $\mathbb{H}$ of $(\mathbb{N}^*, +)$.
Let $n \in \mathbb{N}$. Then $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is a group with respect to addition mod n, so that there is a quotient map $q_n : \mathbb{Z} \to \mathbb{Z}_n$ which is a group homomorphism; the map q_n extends to a semigroup homomorphism $q_n : \beta\mathbb{Z} \to \mathbb{Z}_n$. The subset $\mathbb{H}$ of $\mathbb{N}^*$ is defined to be

$$\mathbb{H} = \{x \in \beta\mathbb{N} : q_{2^n}(x) = 0 \ (n \in \mathbb{N})\}.$$

From [8], $\mathbb{H}$ is a subsemigroup of $(\mathbb{N}^*, +)$ and a G_δ-set in $\mathbb{N}^*$ and also that $\mathbb{H} \supset E(\mathbb{N}^*)$, and so that $\mathbb{H} \cap K(\mathbb{N}^*) \neq \emptyset$.

The following result shall be needed from [8].

Let G be a group, and suppose that there is a monomorphism $\gamma : G \to C$, where C is a compact topological group. We identify G as a subset of C; we may suppose that G is dense in C. In the case where G is countable, we may also suppose that C is metrizable. Note that every commutative group can be embedded in a compact group which is a product of copies of the circle group; also, a free group of 2 generators can be algebraically embedded in a compact topological group [8, Proposition 2.24]. There is an extension of to a continuous epimorphism $\gamma : (\beta G, \Box) \to C$; we define V to be the kernel of γ, and, for $x, y \in \beta G$, we set $x \sim y$ if $\gamma(x) = \gamma(y)$.

The next result follows from [14, Proposition 7.28].

3.4.35 Theorem. *[8] Let G be a countably infinite group, and let V be as above. Then V contains $E(G^*)$, and V is topologically isomorphic to $\mathbb{H}$.*

3.4.36 Lemma. [8] Let G be a countably infinite group which is embeddable in a compact group, and let E be an equivalence class determined by the relation $\sim$. Then there is a cancellable element $u \in \beta G$ such that $uG \subset V$.

Proof. We suppose that G is embedded in a compact topological group C, as above, and take $c \in C$ such that $\gamma(x) = c$ for each $x \in E$. Let (U_n) be a sequence which is a basis for the family of open nbds of the identity of C. For each $n \in \mathbb{N}$, choose $s_n \in G$ such that $s_n c \in U_n$, and set $S = \{s_n : n \in \mathbb{N}^*\}$, so that S is a clopen subset of G^*. Clearly, for each $s \in S$, we have $sE \subset V$. By [[14], Theorem 8.34], S contains a cancellable element of βG. □

Our first critical results are modifications of Theorems 6.56 and 6.57 of [14]. We adopt the following notation, which we shall maintain throughout this section. Let S be a countable semigroup that is a subsemigroup of a group G; we may assume that G is also countable. For example, starting from a countable, cancellative, commutative semigroup S, we can take G to be the group of quotients of S. We order the group G by a total ordering, which we call '$<$'.

Let $(x_n : n \in \mathbb{N})$ be a sequence in $S^* \setminus K(S^*)$, and fix $q \in K(S^*)$. Then clearly we have $(\beta G)qx_n = \mathrm{c\ell}(Gqx_n) \subset K(S^*)$ $(n \in \mathbb{N})$, and so

$$\{x_1, \ldots, x_n\} \cap \bigcup_{i=1}^{n} (\beta G)qx_i = \emptyset \quad (n \in \mathbb{N}).$$

Therefore, for each $n \in \mathbb{N}$, we can choose a clopen subset W_n in βS such that

$$\{x_1, \ldots, x_n\} \subset W_n \quad \text{and} \quad W_n \cap \bigcup_{i=1}^{n} (\beta G)qx_i = \emptyset. \tag{3.4.1}$$

For each $n \in \mathbb{N}$ and $r \in G$, we set

$$U_{n,r} = \{U \in S^* : \; rux_i \notin W_n \; (i = 1, \ldots, n)\}. \tag{3.4.2}$$

Note that each $U_{n,r}$ is a clopen subset of S^* containing q.

Since we have that G is countable, the intersection $\bigcap\{U_{n,r} : n \in \mathbb{N}, r \in G\}$ is a G_δ-set in S^*, and so it has a non-empty interior [14, Theorem 3.36]. Thus we can find and fix a nonempty, clopen subset U of S^* such that $U \subset U_{n,r}$ for each $n \in \mathbb{N}$ and $r \in G$. By [[14], Theorem 8.34], the set of cancellable elements of S contains a dense, open subset of S^*, and so, by intersecting U with such a set, we may suppose that every element of U is cancellable in βS.

In the special case in which $S = \mathbb{N}$ and $G = \mathbb{Z}$, we can assume that we have chosen $q \in \mathbb{H}$ (because $\mathbb{H} \cap K(\mathbb{N}^*) \neq \emptyset$) and that $U \subset \mathbb{H}$. This follows from the fact that $\mathbb{H} \cap \bigcap\{U_{n,r} : n \in \mathbb{N}, r \in \mathbb{Z}\}$ is a non-empty, G_δ-set in $\mathbb{N}^*$.

The set U has the form A^* for some infinite subset A of S; we write

$$A = \{a_1, a_2, \ldots\}.$$

By passing to a subset of A, if necessary, we may suppose that

$$ba_m \neq a_n \quad \text{whenever } m < n \text{ in } \mathbb{N} \quad \text{and} \quad b < a_m \text{ in } G. \tag{3.4.3}$$

For each $r \in G$, we set $A_r = \{a \in A : r < a\}$. Of course, $A \setminus A_r$ is finite, and so $A_r^* = A^*$ $(r \in G)$.

The following results were adapted from [8].

3.4.37 Lemma. [8] For each $u \in U$ and $m, n \in \mathbb{N}$, we have $x_m \notin (\beta G)ux_n$.

Proof. Let $u \in U$, and assume towards a contradiction that there exist $m, n \in \mathbb{N}$ such that $x_m \in (\beta G)ux_n = c\ell(Gux_n)$. Let $k \in \mathbb{N}$ with $k > \max\{m, n\}$. Then W_k is an open neighbourhood of x_m, and so there exists $y \in G$ such that $yux_n \in W_k$. But this contradicts the fact that $u \in U_{k,y}$. Thus $x_m \notin (\beta G)ux_n$. □

For the proof of Lemma (3.4.39), we will need the following result.

3.4.38 Theorem. *[8] Assume that S is either $(\mathbb{N}, +)$ or a countable group. For each $p \in S^*$ the following are equivalent:*

(i) In βG, p is right cancellable.

(ii) $p \notin S^*p$.

(iii) There is no idempotent $e \in S^*$ for which $p = ep$.

Proof. See [14, Theorem 8.18]. □

3.4.39 Lemma. [8] For each $u \in U$ and $n \in \mathbb{N}$, the element ux_n is right cancellable in βG.

Proof. Suppose towards a contradiction that ux_n is not right cancellable in βG. By Theorem (3.4.38)[(3) ⇒ (1)], there is an element $x \in G^*$ such that $ux_n = xux_n$. Since $ux_n \in c\ell(Ax_n)$ and $xux_n \in c\ell(Gux_n)$, it follows by Theorem (2.3.21) that one of the following two alternatives must hold:

(i) $vx_n = rux_n$ for some $v \in c\ell\, A$ and some $r \in G$;

(ii) $axn = yux_n$ for some $a \in A$ and some $y \in \beta G$.

Suppose that (i) occurs. Assume that $v \in S$. Then $v^{-1}1rux_n = x_n \in W_n$, a contradiction of the fact that $u \in U_{n,v^{-1}r}$. Thus $v \in A^*$, and so $ux_n \in c\ell(Ax_n) = c\ell(A_r x_n)$. Also, $rux_n \in c\ell(rA_r x_n)$. By a second application of Theorem (2.3.21), one of the following two alternatives must hold:

(iii) $bx_n = ru_1x_n$ for some $u_1 \in c\ell\, A_r$ and some $b \in A_r$;

(iv) $u_2x_n = rcux_n$ for some $u_2 \in c\ell\, A_r$ and some $c \in A_r$.

Now case (iii) cannot hold when $u_1 \in A_r$ by (3.4.3), and case (iii) cannot hold when $u_1 \in A_r^*$ because, in this case, $u_1 \in U_{n,b^{-1}r}$ and so $x_n = b^{-1}ru_2x_n \notin W_n$, a contradiction of the fact that $x_n \in W_n$. Thus (iii) cannot hold. Similarly, (iv) cannot hold.
We have obtained a contradiction in the case where (i) holds.
Now suppose that (ii) occurs. Since $a^{-1}yux_n \in c\ell(Gux_n)$ and $a^{-1}yux_n = x_n \in W_n$, it follows that there is $t \in G$ such that $tux_n \in W_n$, a contradiction of the fact that $u \in U_{n,t}$. Thus we have obtained a contradiction also in the case where (ii) holds. □

3.4.40 Lemma. [8] For each $u \in U$ and $m, n \in \mathbb{N}$, either $x_m \in Gx_n$ or

$$(\beta G)ux_n \cap (\beta G)ux_m = \emptyset.$$

Proof. Let $k \in \mathbb{N}$ with $k > \max\{m, n\}$.
Suppose that $(\beta G)ux_n \cap (\beta G)ux_m \neq \emptyset$. We may suppose that $xux_m = ux_n$ for some $x \in \beta G$. Now $xux_m \in c\ell(Gux_n)$ and $ux_n \in c\ell(Ax_n)$, and so it again follows from Theorem (2.3.21) that one of the following two alternatives must hold:

(i) $sux_m = vx_n$ for some $v \in c\ell\, A$ and some $s \in G$;

(ii) $yux_m = ax_n$ for some $a \in A$ and some $y \in \beta G$.

Suppose towards a contradiction that (i) holds. Again we see that $sux_m \in c\ell(sA_rx_m)$ and $vx_n \in c\ell(Ax_n)$, and so it again follows from Theorem (2.3.21) that one of the following two alternatives must hold:

(iii) $su_1x_m = bx_n$ for some $u_1 \in c\ell\, A$ and some $b \in A$;

(iv) $scux_m = u_2x_n$ for some $u_2 \in c\ell\, A$ and some $c \in A$.

However (iii) cannot hold in the case where $u_1 \in A^*$ because this would contradict the fact that $u_1 \in U_{k,b^{-1}s}$. Hence $u_1 \in A$, and so we can conclude that $x_m \in Gx_n$. Similarly, (iii) cannot hold in the case where $u_2 \in A^*$, and so again $x_m \in Gx_n$. □

We define $x \equiv y$ for $x, y \in \beta G$ if $x \in Gy$.
It follows from the above lemmas that we have the following result.

3.4.41 Theorem. *[8] Let S be a countable semigroup that is a subsemigroup of a group G, and suppose that $(x_n : n \in \mathbb{N})$ is a sequence in $S^* \setminus K(S^*)$. Then there is an infinite subset A of S such that, for each $u \in A^*$, the following properties hold:*

(i) u is cancellable;

(ii) ux_n is right cancellable for each $n \in \mathbb{N}$;

(iii) for each $m, n \in \mathbb{N}$, either $x_m \equiv x_n$ or $(\beta G)ux_n \cap (\beta G)ux_m = \emptyset$.

4. Banach algebra theory

In this chapter, we will be mainly concerned with the spectra of a normed space and its connection with the Banach algebras. First we begin with some basic concepts. We will learn that first of all a Banach algebra is an algebra, which has a topology to make algebraic operations continuous. In particular, the topology is given by a norm.

4.1 Preliminaries

In this section, we recall some notions on Banach spaces that we will need in the next section and for the rest of this chapter.

4.1.1 Definition. [5] Let X be a *linear space*. A norm on X is a map $||\cdot|| : X \to \mathbb{R}$ such that:

(i) $||x|| \geq 0$ $(x \in X)$; $||x|| = 0$ if and only if $x = 0$;

(ii) $||\alpha x|| = |\alpha| ||x||$ $(\alpha \in \mathbb{C},\ x \in X)$;

(iii) $||x + y|| \leq ||x|| + ||y||$ $(x, y\ \in X)$.

Then $(X, ||\cdot||)$ is a normed space. If necessary, we will denote the norm on the space X by $||\cdot||_X$. We will sometimes use the term *normed space* as an abbreviation. If it ever happens that $||x|| \neq 0$ if and only if $x \neq 0$ then the function $||\cdot|| : x \mapsto ||x||$ is said to be a *seminorm* on X.

4.1.2 Definition. [1] Let X be a normed space with norm $||\cdot||$.

(i) Then there is a natural metric d on X defined by

$$d(x, y) = ||x - y||, \quad (x, y \in X).$$

The topology on X, defined by this metric, is called the *norm-topology*.

(ii) A sequence, $(x_n) \subset X$ converges in norm to a limit $x \in X$, if for all real numbers $\epsilon > 0$ there exists a natural number N such that $d(x_n, y) = ||x_n - x|| < \epsilon$ whenever $n \geq N$. In other words, we say that the sequence (x_n) converges in norm to a limit $x \in X$ if

$$\lim_{n\to\infty} ||x_n - x|| = 0.$$

(iii) A sequence, $(x_n) \subset X$ is said to be a *Cauchy sequence* if for all real numbers $\epsilon > 0$ there exists a natural number N such that $d(x_n, x_m) = ||x_m - x_n|| < \epsilon$ whenever $m, n \geq N$.

(iv) X is said to be *complete* if every Cauchy sequence converges in norm to some limit in X. A normed space is a *Banach space* if $||\cdot||$ is complete.

4.1.3 Example. [1]

(a) Let S be a non-empty set. Denote $\mathbb{C}^S$, the set of functions from S to $\mathbb{C}$. Define *pointwise* algebraic operations by

$$\begin{aligned}(\alpha f+\beta g)(s)&=\alpha f(s)+\beta g(s),\\(fg)(s)&=f(s)g(s),\\1(s)&=1,\end{aligned}$$

for each $s\in S$, each $f,g\in\mathbb{C}^S$, and each $\alpha,\beta\in\mathbb{C}$. Then $\mathbb{C}^S$, is a commutative unital algebra. We write $\ell^\infty(S)$ for the subset of bounded functions on S, and then define the *uniform norm* $|\cdot|_S$ on S by

$$|f|_S=\sup\{|f(s)|:s\in S\}\ (f\in\ell^\infty(S)).$$

It then follows that $(\ell^\infty(S),|\cdot|_S)$ is a normed space, and that $|f_n-f|_S\to 0$ if and only if $f_n\to f$ uniformly on S. It then follows that $\ell^\infty(S)$ is complete (i.e. a Banach space).

(b) Let s be the space of all complex-valued sequences $\underline{x}=(x_n)_{n\geq 1}$, with the natural 'coordinatewise' linear space operations so that $s=\mathbb{C}^{\mathbb{N}}$. Whilst s itself is not, in any way you look at, a normed space, there are several subspaces of s that are important examples of normed spaces. The basic examples we will work with are the following:

(i) The space ℓ^∞ is the space of all *bounded* sequences of s, and the norm is the uniform norm given by

$$||\underline{x}||_\infty=\sup_{n\geq 1}|x_n|\quad(\underline{x}\in\ell^\infty).$$

The space ℓ^∞ is a Banach space. Similarly, we have that $\ell^\infty(\mathbb{Z},||\cdot||_\infty$ is a Banach space.

(ii) The space $c_0=\{\underline{x}\in s:x_n\to 0$ as $n\to\infty\}$ is a Banach space as a closed subspace of $(\ell^\infty(S),||\cdot||_\infty)$.

(iii) The space $\ell^1=\{\underline{x}\in s:\sum_{n=1}^\infty|x_n|<\infty\}$. The norm on ℓ^1 is given by

$$||\underline{x}||_1=\sum_{n=1}^\infty|x_n|\quad(\underline{x}\in\ell^1).$$

Also, ℓ^1 is a Banach space.

There is a slight generalization of examples (iii) above. Indeed, Let S be a nonempty set. For $f\in\mathbb{C}^S$, we set

$$\sum\{|f(s)|:s\in S\}=\sup\sum\{|f(s)|:s\in F\},$$

where the supremum is taken over all finite subsets F of S. Then

$$\ell^1(S)=\big\{f\in\mathbb{C}^S\ :\ ||f||_1=\sum\{|f(s)|:s\in F\}<\infty\big\}.$$

The space $(\ell^1(S);||\cdot||_1)$ has similar properties to the space $\ell^1=\ell^1(\mathbb{N})$. For example, we shall at sometime consider $\ell^1(\mathbb{Z})$ and $\ell^1(\mathbb{Z}^+)$; the space $\ell^1(\mathbb{R})$ has different properties. For $s\in S$, let δ_s be the characteristic function of $\{s\}$. Then the generic element of $\ell^1(S)$ can be denoted by $\sum_{s\in S}f(s)\delta_s$. We will study more of this space and explore more of its properties.

4.1.4 Example. [1] Let X be a normed space over $\mathbb{C}$. Then when equipped with the norm topology, X becomes a topological linear space where the mappings

$$X \times X \ni (x, y) \mapsto x + y \in X$$
$$\mathbb{C} \times X \ni (\alpha, y) \mapsto \alpha x \in X$$

are continuous.

4.1.5 Definition. [1] Let X and Y be vector spaces over a scalar field $\mathbb{K}$. Then a map $T : X \to Y$ is *linear* if

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty \quad (x, y \in X, \lambda, \mu \in \mathbb{C}),$$

and, in the case where $\mathbb{K} = \mathbb{C}$, the map T is *conjugate-linear* if

$$T(\lambda x + \mu y) = \overline{\lambda} Tx + \overline{\mu} Ty \quad (x, y \in X, \lambda, \mu \in \mathbb{C}).$$

The kernel and image of a linear (or conjugate-linear) mapping $T : X \to Y$ are denoted by $\ker T$ and $\operatorname{im} T$, respectively, so that

$$\ker T = \{x \in X : Tx = 0\} \text{ and } \operatorname{im} T = \{Tx : x \in X\};$$

these are linear subspaces of X and Y, respectively. The space of linear maps from X to Y is denoted by $\mathcal{L}(X, Y)$. It's follows that, $\mathcal{L}(X, Y)$ is itself a linear space in the usual way: for $S, T \in \mathcal{L}(X, Y)$ and $\lambda, \mu \in \mathbb{C}$, define

$$(\lambda S + \mu T)(x) = \lambda Sx + \mu Tx \quad (x \in X).$$

We shall write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. The identity operator on X is $I_X : x \mapsto x$. It is clear, as we shall note in the next section, that $\mathcal{L}(X)$ is a *unital algebra* with respect to composition of operators and with I_X as the identity; it is sometimes called the algebra of *linear endomorphisms* of X.

4.1.6 Theorem. *[1] Let X and Y be normed spaces, and let $T : X \to Y$ be a linear mapping. Then the following assertions are equivalent:*

(a) T is continuous on X;

(b) T is continuous at 0_X;

(c) there is a constant $M \geq 0$ such that $||Tx|| \leq M||x||$ $(x \in X)$.

Any linear mapping between normed spaces that satisfies condition (c) of Theorem (4.1.6) is usually called a *bounded (linear) operator*, and so Theorem (4.1.6) says that any linear mapping between normed spaces is continuous if and only if it is bounded. Observe also that it is clear that a continuous linear mapping between normed spaces is *uniformly continuous*, since there exists $M \geq 0$ such that

$$||Tx - Ty|| \leq M||x - y|| \quad \text{for all} \quad x, y \in X.$$

4.1.7 Definition. [1] Let X and Y be linear spaces, and let $T : X \to Y$ be a linear map such that T is a bijection, with inverse $T^{-1} : Y \to X$. Then for sure T^{-1} is a linear map. In the case where X and Y are normed spaces and both T and T^{-1} are continuous, we say that T is a *linear homeomorphism*, and that the space X and Y are *linearly homeomorphic*; sometimes we say they are *isomorphic*.
The linear operator $T : X \to Y$ is an *isometric* mapping if

$$||Tx|| = ||x|| \ (x \in X);$$

the map T is an *isometry* if, further, T is a bijection, and then T surely is a linear homeomorphism; and in this case, X and Y are *isometrically isomorphic*.

Two norms $||\cdot||$ and $|||\cdot|||$ on a linear space X are said to be equivalent, written $||\cdot|| \sim |||\cdot|||$, if and only if they define the same topology, that is, if and only if the identity mapping $\iota : (X; ||\cdot||) \to (X; |||\cdot|||)$ is a homeomorphism. Since ι is surely linear, it follows immediately from Theorem (4.1.6) the following characterization of the equivalence of norms.

4.1.8 Corollary. [1] Two norms $||\cdot||$ and $|||\cdot|||$ on a linear space X are equivalent if and only if there are constants $C, D > 0$ such that

$$C||x|| \leq |||x||| \leq D||x|| \quad (x \in X).$$

For normed spaces X and Y, we denote by $\mathcal{B}(X, Y)$ the space of all bounded continuous linear operators $T : X \to Y$. For $T \in \mathcal{B}(X, Y)$, we define

$$||T|| = \sup_{x \neq 0} \frac{||T||}{||x||} = \sup_{||x||=1} ||Tx||.$$

Observe that $||T||$ is well defined and that, in particular, $||T||$ is the smallest possible value of the constant M in Theorem (4.1.6)(c). In particular, we have the central inequality

$$||Tx|| \leq ||T||||x|| \ (T \in \mathcal{B}(X, Y), x \in X).$$

The mapping $||\cdot|| : T \mapsto ||T||$ is said to be the *operator norm* on $\mathcal{B}(X, Y)$ (the next proposition shows it's a norm).

4.1.9 Proposition. [1] Let X, Y and Z be normed spaces. Then:

(i) $\mathcal{B}(X, Y)$ is a normed space in the operator norm;

(ii) if $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$, then $ST \in \mathcal{B}(X, Z)$ and

$$||ST|| \leq ||S||||T||;$$

(iii) if Y is Banach space, then so is $\mathcal{B}(X, Y)$.

The map $ST \in \mathcal{B}(X, Z)$ in (ii), above, is the composition of S and T, defined by $(ST)(x) = S(Tx)$ $(x \in X)$; this composition is sometimes written as $S \circ T$. Yes indeed, ST is linear whenever $S : Y \to Z$ and $T : X \to Y$ are linear mappings.

4.1.10 Example. [1] The following are two important special cases of the spaces $\mathcal{B}(X, Y)$.

(i) Let$X$ be a normed space. Then we write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. We have shown that $ST = S \circ T \in \mathcal{B}(X, Y)$ whenever $S, T \in \mathcal{B}(X)$. Now suppose that X is complete. Then $\mathcal{B}(X)$ is a Banach space; later we will show that $\mathcal{B}(X)$ is a Banach algebra with respect to the product $(S, T) \mapsto ST$.

(ii) Let X be any normed space over the scalar field $\mathbb{C}$. Then the *dual space* $X^* := \mathcal{B}(X, \mathbb{C})$ is a Banach space; it consists of the *continuous linear functionals* on X. The operator norm on X^* is called the *dual norm*. Thus, the dual norm on x^* is defined by

$$||f|| = \sup\{|f(x)| : x \in X, ||x|| \leq 1\} \quad (f \in X^*).$$

The preceding 'central inequality' becomes

$$|f(x)| \leq ||f||||x|| \quad (x \in X, f \in X^*).$$

A Banach space Y that is isometrically isomorphic to X^* for some normed space X is a *dual Banach space*.

(iii) Since X^* is itself a Banach space, it also has a dual space, namely $(X^*)^*$ this latter is called the *second dual* space of X, and it is written as X^{**}. For $x \in X$, we define a functional $\widehat{x}$ on X^* by:

$$\widehat{x}(f) = f(x) \quad (f \in X^*).$$

It follows that $\widehat{x}$ is a linear functional on X^*, and that $|\widehat{x}(f)| \leq ||f||||x||$, so that $||\widehat{x}|| \leq ||x||$. We have that, in fact, $||\widehat{x}|| = ||x||$ $(x \in X)$. To see this, consider an arbitrary $x \in X$, and let $Y = \mathbb{C}x = \{\lambda x : \lambda \in \mathbb{C}\}$ be the linear subspace of X. Define $g : Y \to \mathbb{C}$ by $g(\lambda x) = \lambda||x||$, then it follows that $g(x) = ||x||$ and $|g(y)| \leq ||y||$ $(y \in Y)$. By the Hahn-Banach theorem [[1], Theorem 3.1], we can choose $f : X \to \mathbb{C}$ such that $f|Y = g$ and $|f(z)| \leq ||z||$ $(z \in X)$. This implies that $||f|| \leq 1$ while the first condition implies that $f(x) = g(x) = ||x||$. In particular, we have that $||x|| = |f(x)| = |\widehat{x}(f)| \leq ||\widehat{x}||||f|| \leq ||\widehat{x}||$. Therefore,

$$||\widehat{x}|| = ||x||.$$

With this we have the linear isometric map

$$J : x \mapsto \widehat{x}, X \to X^{**}.$$

Since X^{**} is a Banach space, we observe that $J : X \to c\ell\ J(X)$ is an isometric isomorphism. In particular, X is a Banach space, if and only if, $J(X)$ is closed in X^{**}. The linear map J is the *canonical embedding*.

Let X be a linear space and let Y be a closed linear subspace of X. look at the quotient space X/Y. We recall that X/Y is defined as the space of equivalence classes corresponding to the relation

$$x_1 \sim x_2 \text{ if and only if } x_1 - x_2 \in Y$$

The equivalence class of an element $x \in X$ is the set

$$[x] = x + Y = \{x + y : y \in Y\}.$$

Addition and scalar multiplication are defined on the equivalence classes by

$$[x] + [y] = [x + y] \text{ and}$$
$$\alpha[x] = [\alpha x] \ (\alpha \in \mathbb{K}).$$

The linear structure on X/Y is then the unique one which makes the quotient map

$$X \ni x \mapsto [x] \in X/Y$$

linear. Define for every equivalence class $\underline{v} \in X/Y$, the quantity

$$||\underline{v}||_{X/Y} = \inf\{||x|| : x \in \underline{v}\}.$$

4.1.11 Proposition. [1] Let X and Y be as above. Then

(i) the map $||\cdot||_{X/Y} : X/Y \to [0, \infty)$ is a norm, called the *quotient norm*.

((ii) the linear map $Q : X \ni x \mapsto [x] \in X/Y$ is continuous and has $||Q|| \leq 1$.

The following is a Factorization Theorem for linear continuous mappings.

4.1.12 Proposition. [1] Let X and Y be normed spaces, let $T : X \to Y$ be a linear continuous map.

(i) The linear space $N = \ker T$ is closed.

(ii) There exists a unique linear continuous map $\widehat{T} : X/N \to Y$ such that $T = \widehat{T} \circ Q$, where $Q : X \to X/N$ is the quotient map.

(iii) We have $||T|| = ||\widehat{T}||$.

Note that completeness is a topological property. That is, if $||\cdot||$ is a norm on X, such that $(X, ||\cdot||)$ is a Banach space, then so is $(X, |||\cdot|||)$, where $||\cdot||$ is equivalent to $|||\cdot|||$.

4.1.13 Remark. [1] From the notion of metric spaces the following are equivalent

(i) X is a Banach space.

(ii) given any sequence $(x)_{n\geq 1} \subset X$ with $\sum_{n=1}^{\infty} ||x_n|| < \infty$, the sequence $(y_n)_{n\geq 1}$ of partial sums, defined by $y_n = \sum_{k=1}^{n} x_k$ is convergent;

(iii) every Cauchy sequence in X has a convergent subsequence.

There are several ways to construct new Banach spaces from old ones. What follows is one example.

4.1.14 Proposition. [1] Let X be a Banach space, let Y be a normed space, and let $T : X \to Y$ be a surjective linear continuous map. Suppose that there exists some constant $C > 0$ such that

$$\text{for every } y \in Y, \text{ there exists } x \in X \text{ with } Tx = y, \text{ and } ||x|| \leq C||y||.$$

then Y is a Banach space.

4.1.15 Corollary. [1] Let X be a Banach spaces, and let Y be a closed linear subspace of X. When equipped with the quotient norm, the quotient space X/Y is a Banach space.

The following are examples of Banach spaces arising from topology.

4.1.16 Example. [1]

(i) Let X be a nonempty topological space, and let $C^b(X)$ be the space of all bounded, continuous, complex-valued functions, with the uniform norm $|\cdot|_X$. This space is a Banach space. If X is a compact Hausdorff space, then every function is automatically bounded. In this case the Banach space is simply denoted by $C(X)$ and it is a subspace of $\ell^\infty(X)$.

(ii) Let $C_0(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, with the uniform norm $|\cdot|_{\mathbb{R}}$ (which is sometimes written as $||\cdot||_\infty$). This is also a Banach space.
In general, let X be a nonempty, locally-compact Hausdorff space. A function f on X *vanishes at infinity* if, for each $\epsilon > 0$, there is a compact subset K of X such that $|f(x)| < \epsilon$ $(x \in X \setminus K)$. Let $C_0(X)$ be the space of all functions on X which vanish at infinity, with the uniform norm $|\cdot|_X$ on X. This is also a Banach space. To see this, we simply have to show that $C_0(X)$is closed in $C^b(X)$.

Indeed, let $(f_n)_{n\geq 1} \subset C_0(X)$ be a sequence which converges in norm to some $f \in C^b(X)$, and we now show that $f \in C_0(X)$. Let $\epsilon > 0$, we construct the compact set K such that $|f(x)| < \epsilon$ $(x \in X \setminus K)$. We begin by choosing $n \geq 1$ such that $||f_n - f|| < \epsilon/2$, then we can choose the compact set $K \subset X$, such that

$$|f_n(x)| < \epsilon/2 \ (x \in X \setminus K).$$

Observe that now, if $x \in X \setminus K$, then

$$|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq ||f_n(x) - f(x)|| + |f_n(x)| < \epsilon.$$

Thus we get the required results.
The space $C_0(X)$ contains as a dense subspace the space $C_{00}(X)$ of all continuous functions on X with *compact support* (that is, those $f \in C(X)$ such that $\mathrm{supp} f = \mathrm{c}\ell \ \{x \in X : f(x) \neq 0\}$ is compact).

Another way of constructing Banach spaces is the completion.

4.1.17 Definition. [1] Let X be a normed space. Then the Banach space $\tilde{X}$ is called a completion of X if there is a linear isometry $i : X \to X$ such that the image $i(X)$ is dense in $\tilde{X}$.

4.1.18 Proposition. [1] Let X and Y be normed spaces, with Y complete. Let X_0 be a subspace of X, and let $T : X_0 \to Y$ be a continuous linear mapping. Then T extends uniquely to a continuous linear mapping $\tilde{T} : \mathrm{c}\ell X_0 \to Y$, and $||\tilde{T}|| = ||T||$.

Let X be a normed space. Then a completion of X is a complete normed space $\tilde{X}$ together with an isometric linear mapping $J : X \to \tilde{X}$ such that $J(X)$ is dense in X. We usually 'identify' in an informal way the space X with $J(X)$ and regard $X \subset \tilde{X}$; then the mapping J is identified with the inclusion mapping $X \hookrightarrow \tilde{X}$.

Completions are unique and every normed space has a completion [1].

4.1.19 Theorem. *[1] Let X be a normed space, with a completion $\tilde{X}$. Then there is a unique Banach space structure on $\tilde{X}$ such that X is isometrically embedded as a dense subspace of $\tilde{X}$. Thus $\tilde{X}$ is a Banach space completion of X.*

4.1.20 Corollary. [1] Let X be a normed space, let Y be a Banach space, and let $T : X \to Y$ be an isometric linear map.

(i) Let $\tilde{T} : \tilde{X} \to Y$ be the linear continuous map such that

$$\widehat{T}\langle x\rangle = Tx.$$

Then $\tilde{T}$ is linear, isometric, and $\tilde{T}(\tilde{X}) = \mathrm{c}\ell \ T(X)$.

(ii) X is complete, if and only if, $T(X)$ is closed in Y.

4.1.21 Integration of vector-valued functions. [1] Let E be a (real or complex) Banach space. For a compact interval $[a, b]$ of $\mathbb{R}$, we write $C_E[a, b]$ for the Banach space of all continuous functions from $[a, b]$ to E, normed by the uniform norm $|\cdot|_{[a,b]}$, which is defined by

$$|f|_{[a,b]} = \sup\{||f(x)|| : a \leq x \leq b\} \quad (f \in C_E[a, b]).$$

Let $C_E^0[a, b]$ be the subset of $C_E[a, b]$ consisting of all functions with the special form:

$$f(t) = g_1(t)e_1 + \ldots + g_n(t)e_n \quad (a \leq t \leq b),$$

where $n \in \mathbb{N}$ and where $e_k \in E$ and $g_k \in C_E[a, b]$ for $k = 1, \ldots, n$. The space $C_E^0[a, b]$ is a vector subspace of $C_E[a, b]$. In particular, $C_E^0[a, b]$ is dense in $(C_E[a, b], |\cdot|_{[a,b]})$ see [[1], Lemma 3.9].

4.1.22 Theorem. *[1] Let $[a, b]$ be a compact interval in $\mathbb{R}$, and let $(E, ||\cdot||)$be a complex Banach space. Then, for each $f \in C_E[a, b]$, there is a unique element $\int_a^b f$ of E such that*

$$\psi\left(\int_a^b f\right) = \int_a^b \psi \circ f$$

for every $\psi \in E^*$. Moreover, the mapping $f \mapsto \int_a^b f$ is a continuous linear mapping from $C_E[a, b]$ into E such that:

(i) if T is any bounded linear operator from E to some Banach space, then

$$T\left(\int_a^b f\right) = \int_a^b T \circ f;$$

(ii) for every $f \in C_E[a, b]$, we have

$$\left|\left|\int_a^b f\right|\right| \leq \int_a^b ||f(x)||\mathrm{d}x \leq (b - a)|f|_{[a,b]}.$$

We employ Theorem (4.1.22) to define some E-valued path integrals.
Let $\gamma : [a, b] \to \mathbb{C}$ be a contour, and suppose that $F : [\gamma] \to E$ is continuous, E is a Banach space. Define

$$\int_\gamma F(z)\mathrm{d}z = \int_a^b F(\gamma(t))\gamma'(t)\mathrm{d}t.$$

This definition has all the properties by analogy with the case of complex-valued functions.

Let U be a nonempty, open subset of $\mathbb{C}$, and let E be a Banach space. A function $f : U \to E$ is an *analytic* (or *holomorphic*) E-valued function if

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists in E (with convergence in the norm topology) for each $z \in U$; f is *weakly analytic* (or *weakly holomorphic*) if $\lambda \circ f : U \to \mathbb{C}$ is analytic in the usual sense for each λ in E^*.

The following results will be referred later on. We begin with the vector-valued forms of Cauchy's integral theorem (formula).

4.1.23 Theorem. *[1] Let U be a nonempty, open subset of $\mathbb{C}$, and let E be a complex Banach space, and let f be an* analytic *E-valued function on U. Let γ be a contour in U such that $n(\gamma; z) = 0$ for every $z \in \mathbb{C} \setminus U$. Then:*

(i) $\int_\gamma f(z)\mathrm{d}z = 0$;

(ii) for every $w \in U \setminus [\gamma]$, we have

$$n(\gamma; z)f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\mathrm{d}z.$$

Proof. See [1]. □

4.1.24 Proposition. [1] (Morera's theorem) Let U be a nonempty, open subset of $\mathbb{C}$, and let $f \in C(U)$. Suppose that $\int_{\partial T} f(z)\mathrm{d}z = 0$ for every triangle T contained in U. Then $f \in O(U)$ the collection of all analytic function on U.

4.1.25 Theorem. *[1] (Liouville's theorem for vector-valued functions) Let E be a complex normed space, and let $f : \mathbb{C} \to E$ be a function which is weakly analytic and bounded. Then f is bounded.*

Proof. For every continuous linear functional $\lambda \in E^*$, the function $\lambda \circ f$ is an entire [1] function on $\mathbb{C}$. Since f is bounded, each function $\lambda \circ f$ is also bounded, and so is constant by the classical Liouville's theorem, [1, Proposition 1.39].
And so for $z_1, z_2 \in \mathbb{C}$ and each $\lambda \in E^*$, we have $\lambda(f(z_1)) = \lambda(f(z_2))$. By the Hahn-Banach theorem, $f(z_1) = f(z_2)$ and so f is constant. □

4.1.26 The weak topology. [1] In this section we examine the topology of dual spaces, we do this by describing the weak and weak-$*$ topologies associated with the Banach spaces in a suitable context.

We begin by defining something more general than a norm, which is the following.

4.1.27 Definition. [1] Let X be a real linear space. A *sublinear functional* on X is a mapping $p : X \to \mathbb{R}$ such that:

(i) $p(x + y) \leq p(x) + p(y) \quad (x, y \in X)$;

(ii) $p(\alpha x) = \alpha p(x) \quad (x \in X, \alpha > 0)$.

Observe that a sublinear functional is allowed to assume negative real values and that clause (ii) is only required to hold for $\alpha > 0$. It follows easily, that a seminorm on X is a sublinear functional.

4.1.28 Definition. [1] A (Hausdorff) *locally convex space* is a real or complex valued linear space X equipped with a given collection $\mathcal{P}$ of seminorms on X that together separate the points of X, in the sense that, for every $x \neq 0$ in X , there is some $p \in \mathcal{P}$ such that $p(x) \neq 0$.

In terms of $\mathcal{P}$, we define the standard topology on X as follows.

[1] an analytic function on the whole of $\mathbb{C}$ is an entire function

4.1.29 Definition. [1] A subset U of X is *open* if and only if for each $x \in U$, there are finitely many $p_1, \ldots, p_n$ in $\mathcal{P}$ and $\epsilon > 0$ such that

$$U \supset \{y \in X : p_k(y - x) < \epsilon \quad (1 \leq k \leq n)\}.$$

This is just a statement that the sets of the form $p(U)$ with $p \in \mathcal{P}$ and U open form a subbase. Equivalently, we can say that a set is open if and only if it's a union of sets of the above form. We sometimes write $(X, \mathcal{P})$ for the space X with the topology defined by $\mathcal{P}$.

$\underline{Notation}$: If given $p_1, \ldots, p_n \in \mathcal{P}$ and $\epsilon > 0$, then we write

$$W(x, p_1, \ldots, p_n, \epsilon) = \{y \in X : p_k(y - x) < \epsilon \quad (1 \leq k \leq n)\}.$$

$W(x, p_1, \ldots, p_n, \epsilon)$ then forms an open neighbourhood base at x.

If $\mathcal{P}, \mathcal{Q}$ are two sets of seminorms on X that defines the same topology, then they are considered to be equivalent. Actually, we shall always assume that such a family $\mathcal{P}$ has the property that $\max\{p_k; 1 \leq k \leq n\}$ belongs to $\mathcal{P}$ whenever $p_1, \ldots, p_n \in \mathcal{P}$; imposing this extra condition does not change the topology. A seminorm q on X is now continuous if there exists $M > 0$ and $p \in \mathcal{P}$ such that $q(x) \leq Mp(x)$ $(x \in X)$.

This next result is analogous to Theorem (4.1.6) where X and Y are normed spaces.

4.1.30 Lemma. [1] Let $(X; \mathcal{P})$ and $(Y; \mathcal{Q})$ be normed spaces, and let $T : X \to Y$ be a linear map. Then the following assertions are equivalent:

(a) T is continuous on X;

(b) T is continuous at 0_X;

(c) for each $q \in \mathcal{Q}$, there are seminorms $p_1, \ldots, p_n \in \mathcal{P}$ on X and $M > 0$ such that $q(Tx) \leq M \max\{p_1(x), \ldots, p_n(x)\}$ for all $x \in X$.

4.1.31 Corollary. [1] Let X be a locally convex space (in particular, a norm) space. Then a linear functional λ on X is continuous if and only if $\ker \lambda$ is closed in X.

We are now ready to discuss the weak topology on a Banach space and the weak-$*$ topology on its dual. We set this up in a useful context by pairing the linear spaces.

Let X and Y be linear spaces over $\mathbb{K}$. A *pairing* of (X, Y) is a specified non-degenerate, bilinear form $(x, y) \mapsto \langle x, y \rangle$ on $X \times Y$. [By 'non-degenerate' we mean: for each $0 \neq x \in X$ there is some $y \in Y$ such that $\langle x, y \rangle \neq 0$, and, for each $0 \neq y \in Y$, there is some $x \in X$ such that $\langle x, y \rangle \neq 0$. The two parts of this conditions aren't equivalent except for the finite-dimensional case.]

4.1.32 Example. [1]

(i) Let X be a $\mathbb{K}$-linear space, and let Y be its algebraic dual space, with $\langle x, f \rangle = f(x)$ for $x \in X$ and $f \in Y$. Then $\langle \cdot, \cdot \rangle$ is a pairing of (X, Y).

(ii) Let X be a locally convex space (we can take a normed space), and let $Y = X^*$, the 'continuous dual'; the pairing is the same as in (i). By the Hahn-Banach theorem this pairing is non-degenerate.[1]

(iii) Let Y be a normed space, and $X = Y^*$ to be its dual space. Then (Y^*, Y) is a non- degenerate pairing; this pairing is not, usually, a particular case of (ii). However, the pairing (Y^*, Y^{**}) is a particular case of (ii).

(iv) If (X, Y) is any pairing, then there is always the 'reverse pairing', in which the bilinear form remains the same, but the rôles of X and Y are interchanged.

We employ the above notation $(x, f) \mapsto \langle x, f\rangle$ for the pairing between a normed space and its dual space X', where $x \in X$ and $f \in X'$ and also the notation $(f, \Lambda) \mapsto \langle f, \Lambda\rangle$ for the pairing between X' and X''. Then the canonical embedding $K : x \mapsto \widehat{x}, X \to X''$, takes the form

$$\langle f, Kx\rangle = \langle x, f\rangle \quad (x \in X, f \in X').$$

It is significant to know that, associated with any pairing, there is a *weak topology*.

4.1.33 Definition. [1] Let (X, Y) be a pairing. Then the mappings

$$p_f : x \mapsto |\langle x, f\rangle|, \quad X \to \mathbb{R},$$

as f goes through Y, form a point-separating family of seminorms on X. The associated locally convex topology on X is called the *weak topology* associated with the pairing; it is denoted by $\sigma(X, Y)$.

It easily follows that the topology $\sigma(X, Y)$ is the weakest topology that makes all of the linear functionals p_f continuous. For if $\mathcal{T}$ is a topology to which all the p_f are continuous, then $\mathcal{T}$ must contain the family $W(x, p_f, \epsilon)$. In particular, $\mathcal{T}$ must contain arbitrary union of families of subsets of U open in $\sigma(X, Y)$, that is τ must contain $\sigma(X, Y)$. Therefore $\sigma(X, Y)$ is the weakest topology topology to which all the p_f are continuous.

Actually, the following is the exact result.

4.1.34 Theorem. *[1] Let (X, Y) be a pairing. A linear functional φ on X is continuous for the weak topology $\sigma(X, Y)$ if and only if there is some $f \in Y$ such that $\varphi(x, y) = \langle x, f\rangle$ $(x \in X)$, and so $(X; \sigma(X, Y))' = Y$.*

Proof. Let $f \in Y$ be such that for each $x \in X$, $\varphi(x, y) = \langle x, f\rangle$. Then $|\varphi(x, y)| = p_f(x)$ which then shows that φ is continuous.
Conversely, φ be a $\sigma(X, Y)$–continuous linear functional on X. By Lemma (4.1.30), we can choose $f_1, \ldots, f_n$ in Y and $M > 0$ such that

$$|\varphi(x)| \leq M \max\{|\langle x, f_1\rangle|, \ldots, |\langle x, f_n\rangle|\} \quad (x \in X).$$

In particular, therefore, if we define $\theta : X \to \mathbb{K}^n$ by

$$\theta(x) = \big(\langle x, f_1\rangle, \ldots, \langle x, f_n\rangle\big) \quad (x \in X),$$

we have $\ker\theta \subset \ker\varphi$. Indeed, if $x \in \ker\theta$ then $\langle x, f_1\rangle = \cdots = \langle x, f_n\rangle = 0$, and so $\varphi(x) = 0$; hence $x \in \ker\varphi$. It follows then that there is a linear functional, $\phi : \mathbb{K}^n \to \mathbb{K}$ such that$\varphi = \phi \circ \theta$. By the general form of a linear functional on $\mathbb{K}^n$, we can choose elements $a_1, \ldots, a_n \in \mathbb{K}$ such that

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i z_i \quad (a_1, \ldots, a_n \in \mathbb{K}).$$

It follows that $\varphi(x) = \langle x, f\rangle$ $(x \in X)$, where $f = \sum_{i=1}^{n} a_i f_i \in X$. □

4.1.35 Example. [1]

(i) Let X be any linear space over $\mathbb{K}$, and take X' to be its algebraic dual. Then every linear functional on X is $\sigma(X, X')$–continuous.

(ii) Let X be a normed space, and let X' be dual space. Then we have the topology $\sigma(X, X')$.

4.1.36 Definition. [1] Let X be a normed space. Then the *weak topology* on X is the topology $\sigma(X, X')$ from the pairing (X, X'), and the *weak-$*$ topology* on X' is the topology $\sigma(X', X)$ from the pairing (X', X).

Observe that, the weak topology on a normed space X always means $\sigma(X, X')$. Thus the weak topology on X^* is $\sigma(X', X'')$, on the other hand the weak-$*$ topology on X^* is $\sigma(X', X)$. We then have

$$\sigma(X', X) \leq \sigma(X', X'') \leq ||\cdot||,$$

where $||\cdot||$ denotes the norm topology on X' and '$\leq$' implies that the topology on the left is weaker. Actually, Theorem (4.1.34) shows that the $\sigma(X', X)$, $\sigma(X', X'')$ topologies coincide if and only if the canonical isometric embedding $x \mapsto \widehat{x}$ of X into X'' is surjective.
We will use the term 'weakly' for topological properties of a space with respect to the weak topology. For example, let X be a normed space. Then a subset S of X is *weakly compact* if it is compact with respect to the $\sigma(X, X')$–topology on X. Yet, if we say 'S is closed', we mean that it is closed in the norm topology of X. By Theorem (4.1.34), a linear functional on X is weakly continuous if and only if it is continuous. On a similar way, the term 'weak-$*$' is used.

4.1.37 Theorem. *[1] Let X be a normed space. Then a subspace Y of X is closed if and only if it is weakly closed.*

The next result is important and fundamental [1].

4.1.38 Theorem. *[1] (Banach-Alaoglu theorem) Let X be a normed space, and let B be the closed unit ball of X'. Then B is weak-$*$-compact.*

4.2 Definitions and Basic properties

Naturally in the literature the symbol $\mathbb{F}$ is used to denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. However, our interest lies in the complex field.

4.2.1 Definition. [1] Let A be a linear space over $\mathbb{C}$. Then A is an *algebra* if it has the operation

$$m_A : (a, b) \mapsto ab = a \cdot b, \; A \times A \to A,$$

known as *multiplication* or *product*, which satisfies the following axioms for all $x, y, z \in A$ and every $\alpha \in \mathbb{F}$:

(i) $(ab)c = a(bc)$;

(ii) $\alpha(ab) = (\alpha a)b = a(\alpha b)$;

(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

An algebra A is said to be *commutative* (or *abelian*) if the multiplication is commutative, so that for all $a, b \in A$,

(iv) $ab = ba$.

An algebra A has an *identity* element, e or e_A, if $e_A a = a e_A = a$ for every $a \in A$. The identity element is unique, because if another identity f exists, then $e = fe = ef = f$. An algebra with an identity is said to be a *unital* algebra.

4.2.2 Remark. [1]

(1) The field $\mathbb{F}$ is said to be the *scalar field* of A. If $\mathbb{F} = \mathbb{R}$, A is called the *real algebra*, and if $\mathbb{F} = \mathbb{C}$, a *complex algebra*.

(2) The mapping $(a, b) \mapsto ab$ is called the *product* in A, and the vector ab the product of a and b.

(3) Axiom (i) affirm that the set A with its product is a semigroup. Axiom (ii) is equivalent to

(ii') $(\alpha\beta)(ab) = (\alpha a)(\beta b) \quad (a, b \in A \ \alpha, \beta \in \mathbb{C})$.

We now begin to discuss the notion of general Banach algebras. A Banach algebra is first of all an algebra [5]. We begin with an algebra A and put a topology on A given by a norm to make the algebraic operations continuous.

4.2.3 Definition. [5] Let A be an algebra. An algebra norm on A is a map $||\cdot|| : A \to \mathbb{R}$ such that $(A, ||\cdot||)$ is a normed space, and further:

$$||ab|| \leq ||a|| ||b|| \quad (a, b \in A)$$

The normed algebra $(A, ||\cdot||)$ is a *Banach algebra* if $||\cdot||$ is a complete norm.

If the product is commutative (abelian), then we say that A is a commutative Banach algebra, and A is said to be *unital* if A has an identity of norm 1; that is, e is an identity element in A and $||e|| = 1$. We call e the *unit* of A. We sometimes write $e = e_A$ if a need arises to justify that e_A belongs to$A$ [1].

If a linear subspace $B \subset A$ is a closed subalgebra of A then it is complete, and so a Banach algebra (under the same operation and norm of A). The space B is said to be a *Banach subalgebra* of A. In other words, a subalgebra of A is a linear subspace of A that is also a subsemigroup of A. B is *unital Banach subalgebra* of A if B also contains the unit of A [1].

Observe that, for each $x, y, x_0, y_0 \in A$

$$||xy - x_0 y_0|| = ||x(y - y_0) + (x - x_0)y_0|| \leq ||x|| ||y - y_0|| + ||x - x_0|| ||y_0||$$

and so (i) relates multiplication and norm and this makes the product a jointly continuous function of its factors [1].

If A is a Banach algebra with identity, then by moving to an equivalent norm, we may suppose that A is unital under the new norm. This is demonstrated as (that is) if e is the identity of A, then $e = e^2$ and so we have $||e|| \leq ||e|| ||e||$, which implies that $||e|| \geq 1$. To obtain equality we have:

4.2.4 Proposition. [15] Let A be a unital Banach algebra. Then there is a norm $|||\cdot|||$ on A, equivalent to the norm $||\cdot||$ on A, such that $(A, |||\cdot|||)$ is a unital Banach algebra with $|||e||| = 1$.

Proof. For all $a \in A$, let L_a be the linear operator $L_a : A \to A,\ b \mapsto ab \in A,\ b \in A$. Then L_a is well defined, because if $L_a = L_{a'}$, we have that $L_a e = L_{a'} e$ and so $a = a'$. Thus the mapping $L : a \mapsto L_a$ is an injective map from A into the set of all linear operators on A, that is, $\mathcal{L}(A)$. In particular, $L_a \in \mathcal{B}(A)$ since multiplication is distributive and continuous.

$$L_a(\alpha x + \beta y) = a(\alpha x + \beta y) = \alpha ax + \beta by = \alpha L_a(x) + \beta L_a(y),$$
$$||L_a(b)|| = ||ab|| \leq ||a||||b||, \text{ for } b \in A,$$

and so $||L_a||$ is bounded, and $||L_a|| \leq ||a||$. Next we have that, for each $x \in A$

$$L_{\alpha a+\beta b}(x) = (\alpha a + \beta b)x = \alpha ax + \beta bx = \alpha L_a(x) + \beta L_b(x),$$
$$L_{ab}(x) = (ab)x = a(bx) = L_a L_b(x),$$
$$L_e(x) = ex = I(x),$$
$$||L_a|| = ||a||$$

since $a = ae = L_a e$, and so $||a|| = ||L_a e|| \leq ||L_a||$. So the space of linear operators $\{L_a\}$ is a subalgebra of $\mathcal{B}(A)$, and the mapping $L : A \to \mathcal{B}(A)$ is an isometric isomorphism.
Furthermore it is closed, because if $L_{a_n} \to T$ in $\mathcal{B}(A)$; then $L_{a_n} x = a_n x = (L_{a_n} e)x$ and so as $n \to \infty$, $Tx = Tex$ by continuity, that is, $T = L_{Te}$. Therefore the subalgebra is complete with norm $||\cdot||$.

Define $|||a||| = ||L_a||$. Then we have just seen that $|||a||| \leq ||a||$, for any $a \in A$.
On the opposites,

$$\begin{aligned} |||a||| = ||L_a|| &= \sup\{||L_a b|| : ||b|| \leq 1\} \\ &= \sup\{||ab|| : ||b|| \leq 1\} \\ &\geq \left|\left| a \cdot \frac{e}{||e||} \right|\right| \\ &= \frac{||a||}{||e||}. \end{aligned}$$

Therefore, $\frac{||a||}{||e||} \leq |||a||| \leq ||a||$, for all $a \in A$, which shows that the two norms $|||\cdot|||$ and $||\cdot||$ are equivalent. Furthermore, for any $a, b \in A$,

$$\begin{aligned} |||ab||| &= ||L_{ab}|| \\ &= ||L_a L_b|| \\ &\leq ||L_a||||L_b|| \\ &= |||a||||||b||| \end{aligned}$$

and so $(A, |||\cdot|||)$ is a Banach algebra. Lastly we have $|||e||| = ||L_e|| = 1$. □

4.2.5 Remark. [15] Let A be an algebra, and given $a \in A$, let λ_a, ρ_a be the mappings of A into A given by

$$\lambda_a(x) = ax, \quad \rho_a(x) = xa \quad (x \in A).$$

The axioms (ii) and (iii) of Definition (4.2.1) are equivalent to the statement $\lambda_a, \rho_a \in L(A)$. Axiom (i) is equivalent to each of the following identities

$$\lambda_a \lambda_b = \lambda_{ab}, \quad \rho_a \rho_b = \rho_{ab}, \quad \lambda_a \rho_b = \rho_b \lambda_a.$$

According to the above proposition we can always assume that the unit of a unital Banach algebra has unit norm 1. That is why this is often taken as part of the definition of a unital Banach algebra. The operator L_a is actually a topological isomorphism of A which allows us to restrict the resultant exposition to studying Banach algebras with norm-one unity. For such an algebra A the operator L_a implements an isometric embedding of A into the algebra $\mathcal{B}(A)$. In this case, L_a implements an isometric isomorphism between the algebras A and $L(A)$. The usual terminology that is used for studying representations of arbitrary Banach algebras.

If A does not have a unit, then one can always be adjoined to it. That is, if A is an algebra over the scalar field $\mathbb{C}$ such that A does not have an identity. Then the *unitization* of A is the unital algebra

$$A_+ := A \oplus \mathbb{C} \cdot 1$$

with the product that makes the symbol 1 into an identity. Thus,

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu) \quad (a, b \in A, \lambda, \mu \in \mathbb{C}).$$

We take $A_+ = A$ if A already has an identity.
If A is a non-unital normed algebra, A_+ is normed as an ℓ^1-sum, that is,

$$||(a, \lambda)|| = ||a|| + |\lambda| \quad (a \in A, \lambda \in \mathbb{C}).$$

Then it follows.

4.2.6 Proposition. [1] A non-unital Banach algebra can be embedded into a unital Banach algebra A_+.

Proof. The algebra A_+ is associative and distributive. Furthermore, the element $(0, 1)$ is a unit of A_+:

$$(a, \lambda)(0, 1) = (a0 + 1a + 0\lambda, \lambda 1) = (\lambda, a) = (0, 1)(a, \lambda).$$

A_+ is a Banach space when equipped with the ℓ^1-sum. Moreover,

$$\begin{aligned} ||(a, \lambda)(b, \mu)|| &= ||(ab + \mu a + \lambda b, \lambda\mu)|| \\ &= ||ab + \mu a + \lambda b|| + |\lambda\mu| \\ &\leq ||a||||b|| + |\mu|||a|| + |\lambda|||b|| + |\lambda||\mu| \\ &= (||a|| + |\lambda|)(||b|| + |\mu|) \\ &= ||(a, \lambda)||||(b, \mu)||. \end{aligned}$$

Hence, A_+ is a Banach algebra with unit. The space A can be identified with the ideal $\{(x, 0) : \; x \in A\}$ in A_+ via the isometric isomorphism $x \mapsto (x, 0)$. □

The following examples shows that many important spaces are Banach algebras.

4.2.7 Example. [1], [15]

(i) Let S be a non-empty set. Recall from (4.1.3)(a) that $\mathbb{C}^S$ is a set of functions from S to $\mathbb{C}$, then $\mathbb{C}^S$ is a commutative unital algebra. We have that the space $\ell^\infty(S)$ of the subset of bounded functions on S, and with the uniform norm $|\cdot|_S$ on S defined by

$$|f|_S = \sup\{|f(s)| : s \in S\} \quad (f \in \ell^\infty(S)).$$

is a unital Banach algebra.

(ii) For X be a topological space. The space $C(X)$ of the algebra of all continuous functions on X, and $C^b(X)$ for the algebra of bounded, continuous functions on X are unital Banach algebras with the respect to the uniform norm $|\cdot|_X$. If X is a compact space, then as already stated $C^b(X) = C(X)$, and so $(C(X), |\cdot|_X)$ is a unital Banach algebra. This example is very important.

(iii) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disc. The *disc algebra* is the unital Banach subalgebra of $C(\overline{\mathbb{D}})$:

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}.$$

Indeed, we show that $A(\overline{\mathbb{D}})$ is a closed subspace of $(C(\overline{\mathbb{D}}), |\cdot|_{\mathbb{D}})$. Let $(f_n)_{n\geq 1}$ be a sequence in $A(\overline{\mathbb{D}})$ with $\lim_{n\to\infty} f_n = f$ in $C(\overline{\mathbb{D}})$. So $(f_n)_{n\geq 1}$ is analytic on $\mathbb{D}$, then for each triangle T contained in $\mathbb{D}$, we have

$$\int_{\partial T} f_n(z)\mathrm{d}z = 0 \quad (n \in \mathbb{N})$$

by Cauchy's theorem, Theorem (4.1.23), and so $\int_{\partial T} f(z)\mathrm{d}z = 0$. Hence f is analytic on $\mathbb{D}$ by Morera's theorem, Proposition (4.1.24), and so $f \in A(\overline{\mathbb{D}})$.

(iv) Let $C_0(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, with the uniform norm $|\cdot|_{\mathbb{R}}$ is a non-unital Banach algebra.

(v) Let $\ell^1(\mathbb{Z})$ denote the Banach space of complex sequences $(a_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ such that $||a|| = \sum_{n\in\mathbb{Z}} |a_n| < \infty$. Then $\ell^1(\mathbb{Z})$ is a commutative, unital Banach algebra when equipped with the multiplication

$$a * b : \mathbb{Z} \to \mathbb{C};\ n \mapsto \sum_{m\in\mathbb{Z}} a_m b_{n-m} \quad (a, b \in \ell^1(\mathbb{Z})).$$

Observe that $\sum_{n\in\mathbb{Z}} |a_n| < \infty$ actually implies that $(a_n)_{n\in\mathbb{Z}}$ is bounded, and so, if $a, b \in \ell^1(\mathbb{Z})$ then the series $\sum_{m\in\mathbb{Z}} a_m b_{n-m}$ is absolutely convergent for all $n \in \mathbb{Z}$. Furthermore,

$$\begin{aligned} ||a * b|| = \sum_{m\in\mathbb{Z}} |(a * b)_m| &= \sum_{n\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} a_m b_{n-m} \right| \\ &\leq \sum_{m,n\in\mathbb{Z}} |a_m||b_{n-m}| = \sum_{m\in\mathbb{Z}} |a_m| \sum_{n\in\mathbb{Z}} |b_{m-n}| = ||a|| ||b|| < \infty \end{aligned}$$

which shows that $a * b \in \ell^1(\mathbb{Z})$, and that $||\cdot||$ is submultiplicative in $\ell^1(\mathbb{Z})$. Furthermore,

$$(a * b)_m = \sum_{m\in\mathbb{Z}} a_m b_{n-m} = \sum_{p\in\mathbb{Z}} b_p a_{n-p} = (a * b)_p$$

and so $*$ is commutative, linear in each variable follows easily, and to see associativity, if $a, b, c \in \ell^1(\mathbb{Z})$ and $p \in \mathbb{Z}$ then

$$\begin{aligned} ((a * b) * c)_m = \sum_{n\in\mathbb{Z}} \left(\sum_{p\in\mathbb{Z}} a_p b_{n-p} \right) c_{m-n} &= \sum_{p\in\mathbb{Z}} a_p \sum_{p\in\mathbb{Z}} b_{n-p} c_{m-n} \\ = \sum_{p\in\mathbb{Z}} a_p \sum_{r\in\mathbb{Z}} b_r c_{m-p-r} &= \sum_{p\in\mathbb{Z}} a_p (b * c)_{m-p} = (a * (b * c))_m. \end{aligned}$$

If $e \in \ell^1(\mathbb{Z})$ is defined by setting $e_0 = 1$ and $e_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ then $e \in \ell^1(\mathbb{Z})$, and $||e|| = 1$ and

$$(e * a)_m = \sum_{m \in \mathbb{Z}} e_m a_{n-m} = a_n \quad \forall\, n \in \mathbb{Z},\; a \in \ell^1(\mathbb{Z}).$$

(vi) If X is a Banach space, let $\mathcal{B}(X)$ denote the set of all bounded linear operators $T : X \to X$ with the operator norm

$$||T|| = \sup_{x \in X, ||x|| \leq 1} ||Tx||.$$

$\mathcal{B}(X)$ is a Banach space. Define a product on $\mathcal{B}(X)$ by $ST = S \circ T$, called the composition product. This is clearly associative and bilinear, and if $x \in X$ with $||x|| \leq 1$ then $||ST|| \leq ||S||||T||$. Hence $\mathcal{B}(X)$ is a Banach algebra. If dim$X > 1$ then $\mathcal{B}(X)$ is not commutative.

(vii) Let $n \in \mathbb{N}$. Denote by $\mathbb{M}_n$ the set of all $n \times n$ matrices over $\mathbb{C}$, identified with $\mathcal{L}(\mathbb{C}^n)$. Then $\mathbb{M}_n$ is an algebra with an identity. It's a Banach algebra with respect to any of the equivalent norms already described.

(viii) Let G be a group, and let

$$\ell^1(G) = \left\{ f \in \mathbb{C}^G : ||f||_1 = \sum_{s \in G} |f(s)| < \infty \right\}.$$

Then $(\ell^1(G), ||\cdot||_1)$ is a Banach space. We can think of an element of $\ell^1(G)$ as

$$\sum_{s \in G} \alpha_s \delta_s,$$

where $\sum_{s \in G} |\alpha_s| < \infty$, here $\delta_s(s) = 1$ and $\delta_s(t) = 0\ (t \neq s)$. We define a product on $\ell^1(G)$ that is not the pointwise product, it is denoted by $\star$ and is sometimes called *convolution multiplication*. In this multiplication,

$$\delta_s \star \delta_t = \delta_{st}\ (s, t\ \in G),$$

where st is the product in G. Thus

$$(f \star g)((t) = \sum \{f(r)g(s) : rs = t\}\ (t \in G).$$

With this, $\ell^1(G)$ is a unital Banach algebra for this product and the norm $||\cdot||_1$. It is commutative if and only if G is an abelian group. A special case is given when $G = \mathbb{Z}$, a group with respect to addition.

More examples can be found in [7].

We now begin our study of general Banach algebras by considering invertible elements in such algebras.

4.2.8 Definition. [1, 15] Let A be a unital Banach algebra.

(i) An element $a \in A$ is *invertible* if there exists an element $b \in A$ with $ab = ba = e_A$. The element b is unique; it is called the *inverse* of a and written as a^{-1}. The set of invertible elements of A is denoted by $\mathrm{Inv}\, A$.

(ii) Let B be a unital Banach subalgebra of A. If an element b is invertible in B, then it is invertible in A; so $\mathrm{Inv\ B} \subset B \cap \mathrm{Inv}\, A$.

(iii) Let B be a subalgebra of A with $e \in B$, then B is *inverse closed* if $\mathrm{Inv}\, A = B \cap \mathrm{Inv}\, A$. That is, if every $b \in B$ which is invertible in A also has an inverse $b^{-1} \in B$.

4.2.9 Remark. [15] The following are properties of $\mathrm{Inv}\, A$.

(i) $\mathrm{Inv}\, A$ forms a group under multiplication.

Indeed, we have that if e is the identity for A then $e \in \mathrm{Inv}\, A$ and that $ae = ea = a$ for all $a \in \mathrm{Inv}\, A$. So e is an identity for $\mathrm{Inv}\, A$ and has the property that for each $a \in \mathrm{Inv}\, A$, there is a $b \in A$ such that $ab = ba = e$ and so b is an inverse of A thus $b \in \mathrm{Inv}\, A$.
Lastly, if $a, b \in \mathrm{Inv\ A}$ then $ac = ca = e$ and $bd = db = e$ for some $c, d \in \mathrm{Inv}\, A$. Then $dc \in A$ and so $abdc = dcab = e$. Thus $ab \in \mathrm{Inv}\, A$ and since a, b are invertible we have that $b^{-1}a^{-1}ab = b^{-1}b = e$ and $abb^{-1}a^{-1} = aa^{-1} = e$. Thus ab is invertible and so lies in $\mathrm{Inv}\, A$.

(ii) If $a \in A$ is left invertible and right invertible such that $ba = e$ and $ac = e$ for some $b, c \in A$, then a is invertible.

Suppose that $ba = e$ and $ac = e$ for some $b, c \in A$. Then multiply the first equation by c to get $(ba)c = c$ and the second by b to get $b(ac) = b$. It follows by associativity from both equation that, $b = c$. Hence, $ab = ba = e$ and so a is invertible with inverse b.

(iii) if $a = bc = cb$ then a is invertible if and only if b and c are invertible.

Let $a = bc = cb$. If b and c are invertible, $b, c \in \mathrm{Inv\ A}$ and so $a = bc \in \mathrm{Inv}\, A$ since $\mathrm{Inv}\, A$ is a group, that is, a is invertible. Conversely, suppose a is invertible. Then $ad = da = e$ for some $d \in A$. So $(cb)d = c(bd) = e$ and $d(bc) = (db)c = e$ so c is left and right invertible, whence invertible from (ii). Similarly, $(bc)d = b(cd) = e$ and $d(cb) = (dc)b = e$ so b is left and right invertible hence invertible.

(iv) If $b_1, \dots, b_n$ are commutative elements of A (that is, $b_ib_j = b_jb_i$ for each $i, j = 1, \dots, n$) then $b_1b_2 \cdots b_n$ is invertible if and only if $b_1, b_2, \dots, b_n$ all are invertible. [The commutativity is essential].

Suppose that $b_1, \dots, b_n$ are all invertible, then the product $b_1 \cdots b_n$. Conversely, we do an inductive proof. If b_1b_2 is invertible then by (iii) b_1 and b_2 are invertible. Suppose that $b_1 \cdots b_k$ is invertible implies that $b_1, \dots, b_k$ are all invertible. Then if $b_1 \cdots b_{k+1}$ is invertible we have that $(b_1 \cdots b_k)b_{k+1}$ is also and so $b_1, \dots, b_k$ and b_{k+1} are invertible by (iii). Hence by principle of mathematical induction $b_1, \dots, b_{k+1}$ are all invertible.

4.2.10 Example. [5, 15]

(i) If X is a compact topological space then

$$\mathrm{Inv}\, C(X) = \{f \in C(X) : f(x) \neq 0 \text{ for all } x \in X\}.$$

(ii) Let X be a nonempty topological space. then

$$\mathrm{Inv}\, C^b(X) = \{f \in C^b(X) : 0 \notin c\ell\, f(X)\}.$$

Where $c\ell\, f(X)$ denotes the closure of the range of f. In particular, if X is a locally compact Hausdorff space and $f \in C(X)$, we have $f \in \operatorname{Inv} C(X)$ if and only $\mathbf{Z}(f) = \emptyset$. The set $\mathbf{Z}(f)$ is the *zero set* of f given by

$$\mathbf{Z}(f) = \{x \in X : f(x) = 0\} = f^{-1}(\{0\}).$$

Thus $\mathbf{Z}(f)$ is always a closed subset of X.

(iii) If If X is a Banach space then

$$\operatorname{Inv} \mathcal{B}(X) \subset \{T \in \mathcal{B}(X) : \ker T = \{0\}\}.$$

Equality holds if dim $X < \infty$ otherwise we have proper inclusion.

(iv) $\operatorname{Inv} \mathbb{M}_n = \{T \in \mathbb{M}_n : \det T \neq 0\}$. Where $T = (\alpha_{ij})$ is an element of $\mathbb{M}_n$ and the function $\det$ is continuous on $\mathbb{M}_n$, and the set of all invertible matrices is a dense, open subset of $\mathbb{M}_n$.

(v) It is possible that the inclusion in Definition (4.2.8)(ii) to be strict. This is seen by recalling that $A(\overline{\mathbb{D}})$ is the disc algebra of continuous functions $\overline{\mathbb{D}} \to \mathbb{C}$ which are holomorphic on $\mathbb{D}$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then by the maximum modulus principle $||f|| = ||f|_{\mathbb{T}}||$, the restriction of f to $\mathbb{T}$, and the map $A(\overline{\mathbb{D}}) \to C(\mathbb{T}), f \mapsto f|_{\mathbb{T}}$ is a unital isometric isomorphism. Then we can identify $A(\overline{\mathbb{D}})$ with $A(\mathbb{T}) = \{f|_{\mathbb{T}} : f \in A(\overline{\mathbb{D}})\}$, which is a closed unital subalgebra of $C(\mathbb{T})$.

Now then consider the following function, $f(z) = z$ for $z \in \overline{\mathbb{D}}$ which is not invertible in $A(\overline{\mathbb{D}})$ because $f(0) = 0$. Hence $f|_{\mathbb{T}}$ is not invertible either in $A(\mathbb{T})$. However, f is invertible in $C(\mathbb{T})$ with an inverse given by $g(e^{i\theta}) = e^{-i\theta}$, and so $\operatorname{Inv} A(\mathbb{T}) \subsetneq A(\mathbb{T}) \cap \operatorname{Inv} C(\mathbb{T})$.

4.2.11 Theorem. *[5, 15] Let $(A, ||\cdot||)$ be a unital Banach algebra.*

(i) If $a \in A$ with $||a|| < 1$ then $e_A - a \in \operatorname{Inv} A$ and

$$(e_A - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

(ii) $\operatorname{Inv} A \supset \{b \in A : ||e_A - b|| < 1\}$.

(iii) $\operatorname{Inv} A$ is an open subset of A.

(iv) The map $\theta : \operatorname{Inv} A \to \operatorname{Inv} A, \quad a \mapsto a^{-1}$ is a homeomorphism.

Proof. (i) Since $||a^n|| \leq ||a||^n$ and $||a|| < 1$, the series $\sum_{n=0}^{\infty} a^n$ is absolutely convergent and so convergent by completeness of A, (see Remark (4.1.13)) say to $b \in A$. Let b_n be the nth partial sum of this series, that is

$$b_n = \sum_{k=0}^{n} a^k \quad \text{and} \quad b = \sum_{n=0}^{\infty} a^n,$$

then we have that $||b(e_A - a) - 1|| = \lim_{n\to\infty} ||b_n(e_A - a) - 1||$. Nonetheless,

$$||b_n(e_A - a) - 1|| = || - a^{n+1}|| \leq ||a||^{n+1} \to 0 \quad \text{as} \quad n \to \infty,$$

so that $b(e_A - a) = 1$. Also, for each $n \in \mathbb{N}$, we have $b_n(e_A - a) = (e_A - a)b_n$ and so $b(e_A - a) = (e_A - a)b = 1$. Therefore, $e_A - a \in \operatorname{Inv} A$ and $b = (e_A - a)^{-1}$.

(ii) This follows immediately from (i). Note that we could have written (i) in the following: for each $b \in A$ with $||e_A - a|| < 1$ then we have $b \in \text{Inv } A$ with

$$b^{-1} = \sum_{n=0}^{\infty} (e_A - a)^n.$$

(iii) Let $a \in \text{Inv } A$ and let $r_a = ||a^{-1}||^{-1}$. We claim that the open ball $B(a, r_a) = \{b \in A : ||a - b|| < r_a\}$ is contained in $\text{Inv } A$; for if $b \in B(a, r_a)$ then $||a - b|| < r_a$ and

$$b = (a - (a - b))a^{-1}a = (1 - (a - b)a^{-1})a.$$

Since $||(a - b)a^{-1}|| < r_a||a^{-1}|| < 1$, the element $1 - (a - b)a^{-1}$ is invertible by part (i). Hence b is the product of two invertible elements, so is invertible. This shows that every element of $\text{Inv } A$ may be surrounded by an open ball which is contained in $\text{Inv } A$, hence $\text{Inv } A$ is open.

(iv) Since $(a^{-1})^{-1} = a$, the map θ is a bijection with $\theta = \theta^{-1}$. So we only need to show that θ is continuous.

If $a \in \text{Inv } A$ and $b \in \text{Inv } A$ with $||a - b|| < \frac{1}{2}||a^{-1}||^{-1}$ then using the triangle inequality and the identity

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1} \tag{4.2.1}$$

we have $||b^{-1}|| \leq ||a^{-1} - b^{-1}|| + ||a^{-1}|| \leq ||a^{-1}||||a - b||||b^{-1}|| + ||a^{-1}|| \leq \frac{1}{2}||b^{-1}||||a^{-1}||$, so $||b^{-1}|| \leq 2||a^{-1}||$. Using (4.2.1) again, we have $||\theta(a) - \theta(b)|| = ||a^{-1} - b^{-1}|| \leq ||a^{-1}||||a - b||||b^{-1}|| < 2||a^{-1}||^2||a - b||$, which shows that θ is continuous at a.

□

4.3 Ideals and the Spectrum

We now demonstrate some basic results about Banach algebras. The *spectrum* of an element is a key idea here. Throughout A is a unital algebra with identity e_A, unless otherwise stated.

4.3.1 The spectrum. We first look at the spectrum of an element in a Banach algebra. The notion generalizes that of the eigenvalues of a matrix.

4.3.2 Definition. [5] Let A be a unital algebra, and let $a \in A$. The *spectrum* of a is

$$\sigma_A(a) = \{z \in \mathbb{C} : ze_A - a \notin \text{Inv } A\};$$

the *resolvent set* of a is $\rho_A(a)$, the complement of $\sigma_A(a)$ in $\mathbb{C}$, so that

$$\rho_A(a) = \mathbb{C} \setminus \sigma_A(a);$$

the *resolvent function* of a is the function

$$R_a : z \mapsto (ze_A - a)^{-1},\ \rho_A(a) \to \text{Inv } A.$$

Usually we write $\rho(a)$ for $\rho_A(a)$, etc.

Note that if A has no identity element and $a \in A$, then we define $\sigma_A(a) = \sigma_{\tilde{A}}(a)$ where $\tilde{A} = A \cup \{e\}$. In this case we have $0 \in \sigma(A)$ for all $a \in A$.

We shall use the following identity which follows easily from (4.2.1) for each $z, w \in \rho(a)$, we have

$$R_a(w) - R_a(z) = (z - w)R_a(z)R_a(w). \tag{4.3.1}$$

To see this, let $z, w \in \rho(a)$ then

$$\begin{aligned} R_a(w) - R_a(z) = (we - a)^{-1} - (ze - a)^{-1} &= (we - a)^{-1}(e - (we - a)(ze - a)^{-1}) \\ &= (we - a)^{-1}\big((ze - a) - (we - a)\big)(ze - a)^{-1} \\ &= (z - w)(we - a)^{-1}(ze - a)^{-1} \\ &= (z - w)R_a(z)R_a(w). \end{aligned}$$

Now if A is a Banach algebra, with $a \in A$. It follows from Theorem 4.2.11(i) that

$$\sigma(a) \subset \{z \in \mathbb{C} : |z| \leq ||a||\}.$$

This means that $\sigma(a)$ is bounded. Indeed, if $|z| > ||a||$ then $ze - a = z(e - z^{-1}a)$ and $||z^{-1}a|| = |z|^{-1}||a|| < 1$ and so $ze - a$ is invertible by Theorem 4.2.11(i), and so $z \notin \sigma(a)$.

4.3.3 Definition. [5] Let A be a unital algebra, and let $a \in A$. the *spectral radius* of a is

$$\nu_A(a) = \nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

The element a is *quasi-nilpotent* if $\nu(a) = 0$ (i.e., $\sigma(a) = \{0\}$ or $\sigma(a) = \emptyset$); the set of *quasi-nilpotents* is denoted $\mathcal{Q}(A)$.
An element $a \in A$ (A doesn't need to be unital) is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$; the minimum such n is called the *index* of a; the set of nilpotent elements of A is denoted by $\mathcal{N}(A)$. Obviously $\mathcal{N}(A) \subset \mathcal{Q}(A)$.

4.3.4 Example. [5, 15]

(i) For any $\lambda \in \mathbb{C}$ $\sigma(\lambda e) = \{\lambda\}$.

(ii) Let X be a compact space. For $f \in C(X)$, $\sigma(f) = f(X)$, the range of f, and $\nu(f) = |f|_X$, so that the only quasi-nilpotent in $C(X)$ is 0.

(iii) For $T \in \mathcal{B}(\mathbb{C}^n) = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{M}_n$, the spectrum of the matrix T is the finite set of eigenvalues of T.

Recall that if A is an algebra, then an idempotent in A is an element p such that $p^2 = p$. If A has an identity e and that p is an idempotent in A. Then $1 - p$ is an idempotent.

4.3.5 Proposition. [1, 15] Let be an A be an algebra and let $a, b \in A$.

(i) Let p be an idempotent in A. Then $\sigma(p) \subset \{0, 1\}$.

(ii) If $e - ab \in \mathrm{Inv}A$ then $e - ba \in \mathrm{Inv}\,A$, and

$$(e - ab)^{-1} = e + b(e - ab)^{-1}a.$$

(iii) $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.

Proof. (i) Let $z \in \mathbb{C}$, and let $b = (1-z)e - p$. Then $(ze-p)b = b(ze-p) = (z-z^2)e$. Hence, whenever $z \notin \{0,1\}$, necessarily $z \notin \sigma(a)$.

(ii) Suppose that $c(e-ab) = (e-ab)c = e$. Then

$$(e+bca)(e-ba) = e - ba + bca - (bca)ba = e - ba + bc(e-ab)a = e - ba + ba = e,$$

and also $(e-ba)(e+bca) = e$

(iii) Let $z \in \sigma(ab) \setminus \{0\}$. Then $z - ab = z(e - z^{-1}ab) \notin \operatorname{Inv} A$, and so $e - z^{-1}ab \notin \operatorname{Inv} A$. By (a), $e - z^{-1}ba \notin \operatorname{Inv} A$, and so $z - ba = z(e - z^{-1}ba) \notin \operatorname{Inv} A$. Then it follows that $z \in \sigma(ba) \setminus \{0\}$. The other inclusion follows symmetrically.

□

4.3.6 Definition. [1, 15] If a is an element of a unital Banach algebra and $p \in \mathbb{C}[w]$ is a complex polynomial, say $p = z_0 1 + z_1 w + \cdots + z_n w^n$ where $z_0, z_1, \ldots, z_n$ are complex numbers, then we have that

$$p(a) = z_0 1 + z_1 a + \cdots + z_n a^n.$$

4.3.7 Theorem. *[1, 15] (The spectral mapping theorem for polynomials) If p is a complex polynomial and a is an element of a unital Banach algebra then*

$$\sigma(p(a)) = p(\sigma a) = \{p(z) : z \in \sigma(a)\}.$$

Proof. If p is a constant then this follows easily since $\sigma(ze_A) = \{e_A\}$. Now suppose that $\deg p = n \geq 1$ and let $y \in \mathbb{C}$. Since $\mathbb{C}$ is closed algebraically, we can always write

$$y - p = C(z_1 - w) \cdots (z_n - w)$$

for some $C, z_1, \ldots, z_n \in \mathbb{C}$. Then

$$y - p(a) = C(z_1 - a) \cdots (z_n - a)$$

and the factors $z_i - a$ all commute. So we have that

$$\begin{aligned} y \in \sigma(p(a)) &\Leftrightarrow y - p(a) \quad \text{is not invertible} \\ &\Leftrightarrow \text{some } z_i - a \quad \text{is not invertible by Remark (4.2.9)(iv)} \\ &\Leftrightarrow \text{some } z_i \in \sigma(a) \\ &\Leftrightarrow \sigma(a) \quad \text{contains a root of } y - p \\ &\Leftrightarrow y = p(z) \quad \text{for some } z \in \sigma(a). \end{aligned}$$

□

The following is a key basic results of our thesis.

4.3.8 Proposition. [5, 15] Let A be a unital Banach algebra, and let $a \in A$.

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$.

(ii) For each $\lambda \in A^*$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$.

(iii) The spectrum $\sigma(a)$ is compact and non-empty.

(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have

$$a^n = \frac{1}{2\pi i} \int_{|\zeta|=r} \zeta^n (\zeta e_A - a)^{-1} \mathrm{d}\zeta.$$

(v) $\nu(a) = \lim_{n\to\infty} ||a^n||^{1/n}$.

Proof. (i) The map

$$\theta : z \mapsto z e_A - a, \quad \mathbb{C} \to A,$$

is continuous, and $\mathrm{Inv}\, A$ is open by Theorem (4.2.11)(iii). So the set $\rho(a) = \theta^{-1}(\mathrm{Inv}\, A)$ is open in $\mathbb{C}$.

(ii) Let $z \in \rho(a)$, and let $w \in \rho(a) \setminus \{z\}$. Set $f = \lambda \circ R_a$. Then

$$\frac{f(w) - f(z)}{w - z} = \frac{\lambda(R_a(w)) - \lambda(R_a(z))}{w - z} = \lambda\left(\frac{R_a(w) - R_a(z)}{w - z}\right) = \lambda(-R_a(w)R_a(z))$$
$$\to -\lambda(R_a(z)^2) \quad \text{as } w \to z,$$

using Theorem (4.2.11)(iv). Thus f is analytic on $\rho(a)$.

(iii) By (i), the spectrum $\sigma(a)$ is closed. We already know that $\sigma(a)$ is bounded. Thus by the Heine-Borel theorem $\sigma(a)$ is compact.

Suppose towards a contradiction that $\sigma(a) = \emptyset$. Let $\lambda \in A^*$. By (ii), the function $\lambda \circ R_a$ is entire. But

$$R_a(z) = (z e_A - a)^{-1} = z^{-1}(e_A - z^{-1} a) = \sum_{n=0}^{\infty} z^{-(n+1)} a^n$$

and $||R_a(z)|| = \frac{|z|^{-1}}{1 - ||z^{-1} a||}$ and so $R_a(z) \to 0$ as $|z| \to \infty$. By Liouville's theorem Theorem (4.1.25), $\lambda \circ R_a = 0$. Hence $\lambda(R_a(0) = 0$. This is true for each $\lambda \in A^*$, and so $R_a(0) = 0$ by the Hahn-Banach theorem. But this is a contradiction since $R_a(z)$ is invertible for all $z \in \mathbb{C}$.

(iv) For $r > ||a||$, the series $\sum_{n=0}^{\infty} z^{-(n+1)} a^n$ is convergent to $(z e_A - a)^{-1}$ on $\{z \in \mathbb{C} : \ |z| = r\}$, and so the equation holds for this value of r. Since R_a is analytic on $\rho(a)$, the equation holds true for all $r > \nu(a)$ by Cauchy's theorem (4.1.23).

(v) Let $z \in \sigma(a)$ and $n \in \mathbb{N}$. Then $z^n \in \sigma(a^n)$ by Theorem (4.3.7), and so $|z|^n \leq ||a^n||$, hence $\nu(a) \leq \inf ||a^n||^n$.
Let $r > \nu(a)$, and set $M_r = \sup\{||R_a(z)|| : \ |z| = r\}$. Then, by (iv), we have

$$\lambda(a^n) = \left| \frac{1}{2\pi i} \int_{|z|=r} (\lambda \circ R_a)(z) z^n \mathrm{d}z \right| \leq r^{n+1} ||\lambda|| M_r$$

and so $||a^n|| \leq r^{n+1} M_r$ by the Hahn-Banach theorem. This shows that $\limsup ||a^n||^n \leq r$ and the result follows.

□

Part (iii) of the above result is the *fundamental theorem of Banach algebras*; part (v) is the *spectral radius formula*. The consequences are:

4.3.9 Corollary. [5] Let A be a unital Banach algebra, and let $a \in A$. Then $a \in \mathfrak{Q}(A)$ if and only if $||a^n||^{1/n} \to 0$;

$$a^n \to 0 \text{ as } n \to \infty \text{ if and only if } \nu(a) < 1.$$

4.3.10 Corollary. [5] If A is a unital Banach algebra and B is a closed unital Banach subalgebra of A then $\nu_A(b) = \nu_B(b)$ for all $b \in B$.

Proof. By the spectral radius formula, $\nu_A(b) = \lim ||b^n||^n = \nu_B(b)$ since B is closed in A and the norm is the same through out B as well as in A. □

The following result says that $\mathbb{C}$ is essentially the only unital Banach algebra which is a *division algebra* (*field*). Recall that a unital algebra A is a division algebra if $\operatorname{Inv} A = A \setminus \{0\}$.
It is a surprise that a theorem with a short proof as the following has such important consequences [5].

4.3.11 Theorem. *[1, 5](Gel'fand-Mazur) Let A is a unital Banach algebra which is a division algebra. Then $A = \mathbb{C}e_A$.*

Proof. Define $\theta : z \mapsto ze_A$, $\mathbb{C} \to A$. Then θ is a monomorphism. Let $a \in A$. By Proposition (4.3.8)(iii), $\sigma(a) \neq \emptyset$ and by hypothesis, $\operatorname{Inv} A = A \setminus \{0\}$ and so we can choose $z \in \mathbb{C}$ such that

$$z \in \sigma(a) \Leftrightarrow ze_A - a \notin \operatorname{Inv} A \Leftrightarrow ze_A - a = 0 \Leftrightarrow a = ze_A.$$

Hence $\theta(z) = a$, and θ is a surjection. □

If K is a non-empty compact subset of $\mathbb{C}$, then exactly one of the connected components of $\mathbb{C} \setminus K$ is unbounded. The bounded components of $\mathbb{C} \setminus K$ are called the *holes* of K. Note that the bounded connected components of $\rho_A(a)$ are precisely the holes of $\sigma_A(a)$ [1], [15].

4.3.12 Theorem. *[1, 15] Let B be a closed subalgebra of a unital Banach algebra A with $e \in B$. If $b \in B$ then $\sigma_B(a)$ is the union of $\sigma_A(a)$ with zero or more of the holes of $\sigma_A(b)$. In particular, if $\sigma_A(b)$ has no holes then $\sigma_B(a) = \sigma_A(b)$.*

Proof. See [1]. □

4.3.13 Definition. [1, 15] Let A be a Banach algebra. If $S \subseteq A$ then the *commutant* of A in A is $S' = \{a \in A : ab = ba \quad (b \in S)\}$. The *bicommutant* of S in A is $S'' = (S')'$. A set $S \subseteq A$ is commutative if $ab = ba$ for all $a, b \in S$. Hence S is commutative if and only if $S \subseteq S'$.

4.3.14 Lemma. [1, 15] Let A be a Banach algebra. If $T \subseteq S \subseteq A$, then $T' \supseteq S'$ and $S \subseteq S''$. furthermore, $S' = S'''$.

Proof. Let $a \in S'$. Let $b \in T$, $b \in S$ and so $ab = ba$ for all $b \in S$. Hence $ab = ba$ for all $b \in T$, and so $a \in T'$.
It follows easily that $S \subseteq S''$, since for any $a \in S$, $ab = ba$ for all $b \in S'$. Hence $a \in S''$.
Since $S \subseteq S''$, then $S' \supseteq (S'')' = S'''$ and $S' \subseteq (S')'' = S'''$. Therefore, $S' = S'''$. □

4.3.15 Proposition. [1] Let A be a unital Banach algebra and let $S \subseteq A$.

(i) S' is a closed, inverse-closed unital subalgebra of A.

(ii) If S is commutative then so is $B = S''$, and $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

Proof. (i) S' is closed, since multiplication is continuous on A. It follows that $e_A \in S$, and by linearity of multiplication S' is a linear subspace of A, and it is a subalgebra by associativity. Lastly we show that S' is inverse-closed. Let $b \in S' \cap \operatorname{Inv} A$ then $bc = cb$ for all $c \in S$, and so $cb^{-1} = b^{-1}c$ for all $c \in S$, and so $b^{-1} \in S'$.

(ii) Since $S \subseteq S'$, $S' \supseteq S''$ and $S'' \subseteq S'''$ by Lemma (4.3.14). Hence $B = S''$ is commutative. Furthermore, B is an inverse-closed subalgebra of A by (i), and so $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

□

4.3.16 Ideals and the radical. For the following definition see [5]. Let A be an algebra. For subsets S and T of A, we write

$$S \cdot T = \{ab : a \in S, b \in T\} \text{ and}$$
$$ST = \left\{ \sum_{j=1}^{n} \alpha_j a_j b_j : \alpha_j \in \mathbb{C}, a_j \in S, b_j \in T \right\}$$

so that $ST = \operatorname{lin} S \cdot T$, where 'lin' denotes the linear span. We write $S^{[2]}$ for $S \cdot S$ and S^2 for $\operatorname{lin} S^{[2]}$. Recall that a linear subspace I of A is a *left ideal* if $AI \subset A$, *right ideal* if $IA \subset A$ and an *ideal* if $AI \cup IA \subset I$. A *proper* left ideal of a Banach algebra A is a left ideal of A which is not equal to A, i.e. $I \subsetneq A$. A left ideal M is *maximal* if $M \neq A$ and if there are no left ideals I with $M \subsetneq I \subsetneq A$. Every left ideal is contained in a maximal left ideal(in the case where A is unital).
Let A be a unital algebra. We shall write $\mathcal{M}_A$ for the set of all maximal ideals of A.

For an ideal I in A, A/I is the quotient algebra with the product defined as

$$(a + I)(b + I) = ab + I \ (a, b \in A).$$

4.3.17 Theorem. *[5, 15] If I is a closed ideal of a Banach algebra A then A/I is a Banach algebra. If A is commutative then so is A/I. If A is unital then so is A/I, and $e_{A/I} = e_A + I$.*

Proof. We know that A/I is a Banach space. Also the product is well-defined in A/I, since if $a_1 + I = a_2 + I$ and $b_1 + I = b_2 + I$ then $a_1 - a_2 \in I$ and $b_1 - b_2 \in I$, and so $a_1(b_1 - b_2) + (a_1 - a_2)b_2 \in I$ since I is an ideal, and so

$$(a_1b_1 + I) - (a_2b_2 + I) = a_1b_1 - a_2b_2 + I = a_1(b_1 - b_2) + (a_1 - a_2)b_2 + I = I$$

thus $a_1b_1 + I = a_2b_2 + I$. This multiplication is linear in each variable.
To see that A/I is a Banach algebra, let $a, b \in A$.Then we have

$$\begin{aligned}
||a + I||||b + I|| &= \inf_{y,z \in I} ||a + y||||b + z|| \\
&\geq \inf_{y,z \in I} ||(a + y)(b + z)|| \\
&= \inf_{y,z \in I} ||ab + (az + yb + yz)|| \\
&\geq \inf_{x \in I} ||ab + x|| \quad (\text{since } az + yb + yz \in I \ (y, z \in I)) \\
&= ||ab + I|| = ||(a + I)(b + I)||.
\end{aligned}$$

If A is commutative then $(a+I)(b+I) = ab+I = ba+I = (b+I)(a+I)$ for all $a, b \in A$ and so A/I is commutative.
Lastly we show that A/I is unital. Since $(e_A+I)(a+I) = a+I = (a+I)(e_A+I)$, it follows that e_A+I is the identity of A/I. Also, since I is a proper ideal of A, $e_A \notin I$, and so $e_A+I \neq 0$, hence $||e_A+I|| \neq 0$. Furthermore,

$$||e_A+I|| = ||(e_A+I)^2|| \leq ||e_A+I||^2$$

and cancelling leads to $||e_A+I|| \geq 1$. For the other inequality observe that

$$||e_A+I|| = \inf_{b \in I} ||e_A - b|| \leq ||e_A - 0|| = 1$$

. □

4.3.18 Lemma. [5, 15] Let A be a unital Banach algebra. If I is an ideal of A, then I is a proper ideal if and only if $I \cap \operatorname{Inv} A = \emptyset$.

Proof. Since $e_A \in \operatorname{Inv} A$, then if $I \cap \operatorname{Inv} A = \emptyset$, $e_A \notin I$ and so $I \neq A$, hence I is proper. Conversely, if $I \cap \operatorname{Inv} A \neq \emptyset$, then let $b \in I \cap \operatorname{Inv} A$. For each $a \in A$, $a = b(b^{-1}a) \in I$, since I is an ideal and $b \in I$. So $I = A$. □

4.3.19 Theorem. *[1, 15] Let A be a unital Banach algebra.*

(i) Let I be a proper ideal of A. Then $c\ell\, I$ is also a proper ideal ideal of A.

(ii) Any maximal ideal of A is closed.

Proof. (i) If $a \in A$ and $\{x_n\}$ is a sequence in I converging to $x \in c\ell\, I$ then $ax_n \to ax$ and $x_n a \to xa$ as $n \to \infty$. Since each ax_n and $x_n a$ are in I, this shows that ax and xa are in $c\ell\, I$, which is therefore an ideal of A.
Since I is a proper ideal, we have $I \cap \operatorname{Inv} A = \emptyset$ by Lemma (4.3.18). Since $\operatorname{Inv} A$ is open by Theorem (4.2.11)(iii), it follows that $c\ell\, I \cap \operatorname{Inv} A = \emptyset$ and so $c\ell\, I \neq A$.

(ii) Let M be a maximal ideal. Since $M \subset c\ell\, M$ and $c\ell\, M$ is a proper ideal by (i), we have $M = c\ell\, M$, and so M is closed.

□

The following result shall oft-times be referred to.

4.3.20 Proposition. [1] Let L be a maximal left ideal in a unital algebra A, and take $a \in A \setminus L$. Then there exists $b \in A$ with $e + ba \in L$

Proof. Let $J = \{ba + L : b \in A\} = Aa + L$. Then J is a left ideal in A with $J \supsetneq L$, and so $J = A$ by maximality of J. Hence we can choose some $b \in A$ and $x \in L$ with $-e = ba + x$. Then $e + ba = -x \in L$. □

4.3.21 Lemma. [1] Let A be an algebra with an identity e, and let I be a left ideal in A. Let $a \in I$ with $e + a \in \operatorname{Inv} A$. Then $e + a \in \operatorname{Inv}(I + \mathbb{C} \cdot e)$.

Proof. There exist $b \in A$ with $(e+b)(e+a) = (e+a)(e+b) = e$. But now $b = -a - ba \in I$, and $e + b$ is the inverse of $e + a$ in $I + \mathbb{C} \cdot e$. □

Recall that the *kernel* of a homomorphism $\theta : A \to B$ is the set

$$\ker \theta = \{a \in A : \theta(a) = 0\}.$$

With this we would like to note that if θ is a nonzero homomorphism of Banach algebras then $\ker \theta$ is a closed ideal of A.

4.3.22 Proposition. [1, 15] Let A and B be a unital Banach algebra and let $\theta : A \to B$ be a unital homomorphism.

(i) $\theta(\mathrm{Inv}\, A) \subset \mathrm{Inv}\, B$ and $\theta^{-1}(a) \supset \theta^{-1}(A)$ for $a \in \mathrm{Inv}\, A$.

(ii) For all $a \in A$ we have $\sigma_A(a) \supset \sigma_B(\theta(a)$.

(iii) If θ is an isomorphism then $\sigma_A(a) = \sigma_B(\theta(a)$ for all $a \in A$.

Proof. (i) If $a \in \mathrm{Inv}\, A$, then $\theta(a)\theta(a^{-1}) = \theta(aa^{-1}) = \theta(e) = e$ and $\theta(a^{-1})\theta(a) = \theta(a^{-1}a) = \theta(e) = e$, and so $\theta(a)$ is invertible in B, with inverse $\theta(a^{-1})$.

(ii) if $\lambda \in \sigma_B(\theta(a))$, then $\lambda - \theta(a) = \theta(\lambda - a) \notin \mathrm{Inv}\, A$ by (i). Hence $\lambda \in \sigma_A(a)$.

(iii) Since θ^{-1} is a homomorphism, by (ii) we have

$$\sigma_A(a) = \sigma_A(\theta^{-1}(\theta(a))) \subset \sigma_B(\theta(a)) \subset \sigma_A(a),$$

and we have equality.

□

4.3.23 Definition. [1, 5] Let A be a Banach algebra. The ideal defined by

$$J(A) = \bigcap\{M : \; M \in \mathcal{M}_{A_+}\},$$

where $\mathcal{M}_{A_+}$ is the set of all maximal ideals for A_+, is called the *(Jacobson) radical* of A. Clearly, $J(A) = \mathcal{Q}(A)$ is a closed ideal in A. A (necessarily non-unital) algebra A is *radical* if $J(A) = A$; necessarily a radical algebra doesn't have an identity. The algebra A is *semisimple* if $J(A) = \{0\}$.

In the unital case,

$$J(A) = \{a \in A : e_A - ba \in \mathrm{Inv}\ A\ (b \in A)\}.$$

In particular, $J(A)$ is an ideal in A, and $A/J(A)$ is a semisimple algebra.

There is a variety of the different characterisation of the Jacobson radical, we give some in the next result.

4.3.24 Theorem. *[1] Let A be an algebra with an identity, and let $J = J(A)$ be the Jacobson radical of A. Then:*

(i) J is the intersection of the maximal left ideals of A;

(ii) $J(A) = \{a \in A : e_A + Aa \subset \mathrm{Inv}\, A\}$;

(iii) $J(A) = \{a \in A : e_A + aA \subset \mathrm{Inv}\, A\}$;

(i) J *is the intersection of the maximal right ideals of* A.

Proof. (i) See [1, Theorem 5.9(i)].

(ii) Let $a \in J$ and $b \in A$. Suppose that $e_A + ba$ is not invertible in A. Then there is a maximal left ideal L of A with $e_A + ba \in L$. Nonetheless, $ba \in L$ (according to (i)), and so $e_A \in L$, contrary to the fact that L is a proper ideal. Therefore $e_A + ba$ is left invertible in A.
Choose $c \in A$ so that $(e_A + c)(e_A + ba) = e_A$, so that $c = -ba - cba \in J$. By the same argument as above, $e_A + c$ is left invertible in A. But $e_A + c$ is right invertible, and so $e_A + c$ is invertible, and thus so is $e_A + ba$. Therefore $e_A + Aa \subset \mathrm{Inv}\, A$.
Conversely, suppose that $a \in A$ has the property that $e_A + ba$ is left invertible for every $b \in A$. Let L be the maximal left ideal of A, so that $L + Aa$ is a left ideal in A containing L. Suppose that $a \notin L$. Then $L + Aa \neq L$, and so $L + Aa = A$ by maximality of L. Thus there is some $b \in A$ with $e_A + ba \in L$, contrary to the left invertibility that $e_A + ba$. Hence a is in each maximal left ideal, so that $a \in J$ according to (i).

(iii) This is immediate from (ii), since by Proposition (4.3.5)(ii), $e_A + ba$ is invertible if and only if $e_A + ab$ is invertible.

(iv) This follows from the equivalences of (i) and (ii).

□

4.3.25 Corollary. [8] Let A be an algebra. Then $J(A)$ is an ideal in A and $A/J(A)$ is semisimple.

Proof. We may suppose that A has an identity. Let $J = J(A)$ be an ideal in A. Let $\pi : A \to A/J$ be the quotient map. Choose $x \in J(A/J)$ and $y \in A/J$, say $x = \pi(a)$ and $y = \pi(b)$, where $a, b \in A$. Since $e_A + xy$ is left invertible in A/J, there is some $c \in A$ such that the element $u := c(e_A + ab) - e_A \in J$. But then $c(e_A + ab) = e_A + u$ is left invertible in A. Thus $a \in J$ and $x = \pi(a) = 0$. Hence $J(A/J) = \{0\}$, and so A/J is semisimple. □

4.3.26 Corollary. [1] Let A be an algebra. Then $J(A)$ is a quasi-nilpotent ideal that is a union of all the quasi-nilpotent left ideal in A.

Proof. We may suppose that A has an identity. Choose $a \in J(A)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $ze_A - a = z(e_A - z^{-1}a) \in \mathrm{Inv}\, A$ by Theorem (4.3.24)(ii), and so $a \in \mathcal{Q}(A)$. Thus $J(A)$ is a quasi-nilpotent ideal.
Conversely, suppose that L is a quasi-nilpotent left ideal, and choose $a \in L$. Then $Aa \subset \mathrm{Inv}\, A$, and hence $e_A + Aa \subset \mathrm{Inv}\, A$. By Theorem (4.3.24)(ii), $a \in J(A)$. Hence $L \subset J(A)$. □

4.3.27 Corollary. [1, 8] Let I be an ideal in an algebra A. Then $J(I) = I \cap J(A)$.

Proof. We may suppose that A has an identity. Let $a \in J(A)$ and $b \in A$. Then $ba \in I$ and $e_A + Iba \subset e_A + Ia \subset \mathrm{Inv}\, A_+$ according to Theorem (4.3.24)(ii). By the same results, $ba \in J(I)$. Thus $J(I)$ is a left ideal in A. It follows that, $J(I)$ is quasi-nilpotent in A, and so $J(I) \subset J(A)$ according to Corollary (4.3.26).

Immediately suppose that $a \in I \cap J(A)$ and $b \in I$. Then $e_A + ba \in \operatorname{Inv} A$, and so $e_A + ba \in \operatorname{Inv}(I + \mathbb{C} \cdot e_A)$ according to Lemma (4.3.21), and so once again, by Theorem (4.3.24)(ii), $a \in J(I)$. Hence $I \cap J(A) \subset J(I)$. □

In general, $J(A) \subsetneq \mathcal{Q}(A)$, and neither $\mathcal{N}(A)$ nor $\mathcal{Q}(A)$ is closed under either sums or products; this is shown by easy examples of 2×2 matrices.

For example, the commutative Banach algebra $C(X)$, X compact, is semisimple.

4.3.28 Proposition. [4, 8] Let A be an Banach algebra, and let I be an ideal in A.

(i) Suppose that $I \subset J(A)$. Then $J(A/I) = J(A)/I$.

(ii) Suppose that A/I is semisimple. Then $J(A) \subset I$.

Proof. See Proposition 1.5.4 [4]. □

4.3.29 Proposition. [4, 8] Let A be a Banach algebra. Then $J(A)$ is a closed ideal and $A/J(A)$ is a semisimple Banach algebra.

Proof. We may suppose that A has an identity. We know that $J(A)$ is an ideal in A. By definition, $J(A)$ is the intersection of the maximal ideals of A. However, each maximal ideal of A is closed, and so $J(A)$ is closed. According to Corollary (4.3.25), $A/J(A)$ is semisimple. □

4.3.30 Proposition. [4, 8] Let A be a Banach algebra, and let B be a closed unital subalgebra of A. Then $J(A) \cap B \subset J(B)$.

Proof. Let $a \in J(A) \cap B$, and let $b \in B$. Then $\nu(ba) = 0$ and so $\nu_B(ba) = 0$. Thus $ba \in \mathcal{Q}(A)$, and so $a \in J(B)$. □

4.3.31 Proposition. [8] Let A be a Banach algebra, and let I be a closed, left ideal in A. Suppose that there is an element $u \in I$ with $au \neq 0$ and $ua \neq 0$ whenever $a \in A \setminus \{0\}$. Then I is semisimple if and only if A is semisimple.

Proof. Suppose that I is not semisimple, and let $a \in J(I) \setminus \{0\}$. Then $ua \in J(I)$. Let $b \in A$. Then $bu \in I$, and so $bua \in \mathcal{Q}(I) \subset \mathcal{Q}(A)$. Thus $ua \in J(A)$. Since $ua \neq 0$, A is not semisimple.
Suppose that A is not semisimple, and let $a \in J(A) \setminus \{0\}$. Then $au \in J(A) \cap I$. Let $b \in I$. Then $bau \in \mathcal{Q}(A) \cap I = \mathcal{Q}(I)$, and so $au \in J(I)$. Since $au \neq 0$, I is not semisimple. □

In general, a closed subalgebra of a unital, semisimple Banach algebra is not necessarily semisimple. In [8] it was given as an example that, $A = \mathbb{M}_n$, the algebra of $n \times n$ matrices over $\mathbb{C}$, so that A is a unital, semisimple, finite-dimensional Banach algebra, and let B be the closed, unital subalgebra of upper-triangular matrices. Then $J(B)$ consists of the matrices that are zero on the diagonal, and so $J(B) \neq 0$ whenever $n \geq 2$. There are also easy examples of commutative, radical Banach algebras with a dense, semisimple subalgebra.

4.4 Unital Commutative Banach Algebras

Throughout this section, we assume that A is a unital commutative Banach algebra.

4.4.1 Definition. [1, 15] A *character* on A is a non-zero homomorphism $\tau : A \to \mathbb{C}$, which satisfies $\tau(ab) = \tau(a)\tau(b)$ for $a, b \in A$. We write $\Omega(A)$ for the set of characters on A.

4.4.2 Example. [15] Let $A = C(X)$ where X is a compact topological space and $\varepsilon_{x_0}(f) = f(x_0)$ $(f \in A)$ for some $x_0 \in X$. Then $\varepsilon_{x_0} \in \Omega(A)$.
Indeed, since $\varepsilon_{x_0}(\alpha f + \beta g) = (\alpha f + \beta g)(x_0) = \alpha f(x_0) + \beta g(x_0) = \alpha\varepsilon_{x_0}(f) + \beta\varepsilon_{x_0}(g)$, and $\varepsilon_{x_0}(fg) = (fg)(x_0) = f(x_0)g(x_0) = \varepsilon_{x_0}(f)\varepsilon_{x_0}(g)$ and lastly $\varepsilon_{x_0}(e_{C(X)}) = 1 \neq 0$ the results follows.

Note that this definition holds even if A is not commutative. However, $\Omega(A)$ is often not very interesting in that case. As an example, if $A = \mathbb{M}_n$, and $n > 1$ then $\Omega(A) = \emptyset$. Surely this is true, as it follows that A is spanned by $\{ab - ba : a, b \in A\}$. If $\varphi : A \to \mathbb{C}$ is a homomorphism, then $\varphi(ab - ba) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0$, so $\varphi = 0$ by linearity; hence $\Omega(A) = \emptyset$.

4.4.3 Lemma. [1, 15] Let $\tau \in \Omega(A)$. Then τ is continuous and

$$||\tau|| = \tau(e_A) = 1.$$

In particular, $\Omega(A)$ is a subset of the closed unit ball of A^*.

Proof. Note that $\tau(e_A) = \tau(e_A^2) = \tau(e_A)^2$, and so $\tau(e_A) \in \{0, 1\}$. If $\tau(e_A) = 0$, then $\tau(a) = \tau(ae_A) = \tau(a)\tau(e_A) = 0$ for any $a \in A$, and so $\tau = 0$. Which is a contradiction since $\tau \in \Omega(A)$. And so $\tau(e_A) = 1$.
We have $\tau(\mathrm{Inv}\, A) \subset \mathrm{Inv}\, \mathbb{C} = \mathbb{C} \setminus \{0\}$ by Proposition (4.3.22)(i). For any $a \in A$, it follows that $\tau(\tau(a)e_A - a) = \tau(a)\tau(e_A) - \tau(a) = 0$, and so $\tau(a)e_A - a \notin \mathrm{Inv}\, A$. Hence $\tau(a) \in \sigma(a)$, and so $|\tau(a)| \leq ||a||$ by Proposition (4.3.8)(iii) and so $||\tau|| \leq 1$. Since $|\tau(e_A)| = 1$, and so $||\tau|| \geq 1$ and hence $||\tau|| = 1$.
Any $\tau \in \Omega(A)$ is a linear functional $A \to \mathbb{C}$ by definition, and we just saw that it is continuous with norm 1. Hence $\Omega(A)$ is contained in the closed unit ball of A'. □

We have just seen that $\Omega(A)$ is a subset of the set

$$\{\lambda \in A^* : \ \lambda(e_A) = 1 = ||\lambda||\},$$

which is called the *state space* of A.

4.4.4 Example. [1, 15] Let A be a Banach algebra, and set $B = (A'', \cdot)$, a Banach algebra with the usual product. Let $\tau \in \Omega(B)$, and let

$$V_\tau = \{\Phi \in A'' : \ ||\Phi|| = \tau(\Phi) = 1\}.$$

Then $(V_\tau, \cdot)$ is a compact, right topological semigroup (with respect to the $\sigma(A'', A')$-topology), and so the minimum ideal $K(V_\tau)$ exists.

4.4.5 Definition. [5] The space $\Omega(A)$ is taken to have the subspace topology obtained from the weak-$*$ topology on A'. It is said to be the *Gel'fand topology*.
This topology is sometimes called the weak-$*$ topology on $\Omega(A)$, and written $\sigma(\Omega(A), A)$. It is the

weakest topology on $\Omega(A)$ that makes each of the maps $\tau \mapsto \tau(a),\ \Omega(A) \to \mathbb{C}$, for $a \in A$, continuous. Let $a \in A$. We define $\widehat{a}$ on A' by

$$\widehat{a}(f) = f(a) \quad (f \in A').$$

The definition of the weak-$*$ topology shows that $\widehat{a}$ is continuous on (A', σ).

Let A be a Banach algebra. Then we shall call the topological space

$$(\Omega(A); \sigma(\Omega(A), A))$$

the *character space* of A; it is sometimes called the *spectrum* of A or the *carrier space* of A. In the case where A is commutative and has an identity, this space is called the *maximal ideal space* of A; the latter term arises because, as we shall see in Theorem (4.4.8), there is a bijection from $\Omega(A)$ onto $\mathcal{M}_A$, the set of all maximal ideals.

Let A be a Banach algebra with an identity. Then it follows that a base for the Gel'fand topology of $\Omega(A)$ is formed by sets of the form

$$\{\tau \in \Omega(A) : \ |\tau(a_j)| < 1 \ (j = 1, \dots, n)\},$$

where $a_1, \dots, a_n \in A$ [5].

4.4.6 Theorem. *[5, 15] $\Omega(A)$ is a compact Hausdorff space (in the weak-$*$ topology).*

Proof. The space $\Omega(A)$ is Hausdorff since the weak-$*$ topology is Hausdorff. By Lemma (4.4.3) $\Omega(A)$ is contained in the unit ball of A', which is compact in the weak-$*$ topology according to Theorem (4.1.38). It suffices to show that $\Omega(A)$ is weak-$*$ closed in A'. But

$$\begin{aligned}\Omega(A) &= \{\tau \in A^* : \ \tau(e_A) = 1, \ \tau(ab) = \tau(a)\tau(b) \text{ for } a, b \in A\} \\ &= \{\tau \in A^* : \ \tau(e_A) = 1\} \bigcap_{a,b \in A} \{\tau \in A^* : \ \tau(ab) - \tau(a)\tau(b) = 0\} \\ &= K_{e_A}^{-1}(\{e_A\}) \bigcap_{a,b \in A} (K_{ab} - K_a \cdot K_b)^{-1}(\{0\}).\end{aligned}$$

So each evaluation functional $K_{a,b}$ and K_{e_A} are weak-$*$ continuous. Thus the sets in the intersection are weak-$*$ closed, and so $\Omega(A)$ is weak-$*$ closed. □

We now regard $\widehat{a}$ as being defined on $\Omega(A)$; for each $a \in A$, it follows that $\widehat{a} \in C(\Omega(A))$ and that $|\widehat{a}|_{\Omega(A)} \leq ||a||$.

Let A be an algebra, and let $\tau \in \Omega(A)$. We write

$$M_\tau = \ker \tau = \{a \in A : \ \tau(a) = 0\}.$$

4.4.7 Lemma. [5, 15] Let A be a unital commutative Banach algebra.

(i) If $\tau \in \Omega(A)$ then M_τ is a maximal ideal of A.

(ii) If M is a maximal ideal of A, then the map $\mathbb{C} \to A/M, \ \lambda \mapsto \lambda e_A + M$ is an isometric isomorphism.

Proof. (i) Let $\tau \in \Omega(A)$. Since τ is a non-zero homomorphism, its kernel M_τ is a proper ideal of A. Suppose that L is an ideal of A with $M_\tau \subsetneq L$ and let $a \in L \setminus M_\tau$. Then $\tau(a) \neq 0$, and so $b = \tau(a)^{-1}a \in L$ and $\tau(b) = 1$. Since $\tau(e_A) = 1$ by Lemma (4.4.3), it follows that $e_A - b \in M_\tau$ and so $e_A = b + e_A - b \in L$. By Lemma (4.3.18), $L = A$. This shows that M_τ is contained in any strictly larger proper ideal of A, and so M_τ is a maximal ideal of A.

(ii) Let M be a maximal ideal of A. According to Theorems (4.3.19)(ii) and (4.3.17), A/M is a unital Banach algebra with the unit given by $e_A + M$. If $a + M$ is a non-zero element of A/M then $a \in A \setminus M$. Let $I = \{ab + m : \ m \in M, \ b \in A\}$. Since A is commutative and M is an ideal, it follows that I is an ideal of A, and $M \subsetneq I$. By maximality of M, $I = A$. And so $e_A \in I$, and $ab + m = e_A$ for some $b \in A$ and $m \in M$. Immediately

$$(a + M)(b + M) = ab + M = ab + m + M = e_A + M,$$

which is the unit of A/M. By the Gel'fand-Mazur theorem (4.3.11) and Lemma (4.3.21), $A/M = \mathbb{C}e_{A/M} = \mathbb{C}(e_A + M)$. Then the quotient homomorphism $\lambda \mapsto \lambda e_A + M$ 'is' a character, with $\ker \lambda = M$. Therefore the specified map is a surjection.

□

4.4.8 Theorem. *[5, 15] Let A be a commutative, unital Banach algebra. The mapping $\tau \mapsto M_\tau$ is a bijection from $\Omega(A)$ onto $\mathcal{M}_A$.*

Proof. If $\tau \in \Omega(A)$, then M_τ is a maximal ideal of A, by Lemma (4.4.7)(i). Hence the mapping is well defined.
Suppose τ_1 and τ_2 are in $\Omega(A)$ with $\ker \tau_1 = \ker \tau_2$, then for any $a \in A$ we have $a - \tau(a)e_A \in \ker \tau_2 = \ker \tau_1$, and so $\tau_1(a - \tau_2(a)e_A) = 0$, hence $\tau_1(a) = \tau_2(a)$. Thus $\tau_1 = \tau_2$, and so the map $\tau \mapsto M_\tau$ is injective.
Next we show that the mapping is surjective. Let M be a maximal ideal of A, and let $q : A \to A/M, a \mapsto a + M$ be the representing quotient map. Note that q is a homomorphism and $\ker q = M$. According to Lemma (4.4.7)(ii), the map $\lambda : \ \mathbb{C} \to A/M, b \mapsto be_A + M$ is an isomorphism. Let $\tau = \lambda^{-1} \circ q : \ A \to \mathbb{C}$. Since τ is a composition of two homomorphism, it is a homomorphism, and $\tau(e_A) = \lambda^{-1}(q(e_A)) = \lambda^{-1}(e_A + M) = 1$, and so $\tau \neq 0$. Thus $\tau \in \Omega(A)$. Since λ is an isomorphism, we have $\ker \tau = \ker q = M$. The results follows. □

4.4.9 Example. This example was adapted from [15] and [1].

(i) Let $A = C(X)$, with X a nonempty compact Hausdorff space. Then A is a commutative, unital Banach algebra. For each $x \in X$, the map

$$\varepsilon_x(f) = f(x) \quad (f \in C(X)),$$

is a character, i.e. ε_x is 'evaluation at x', with the corresponding maximal ideal

$$M_x = \{f \in C(X) : \ f(x) = 0\}.$$

These are the only characters of this form. See [1].

(ii) Let $A(\overline{\mathbb{D}})$ be the disc algebra. For each $w \in \mathbb{D}$, the map $\varepsilon_w(f) = f(w)$ is a character on $A(\overline{\mathbb{D}})$. Same as in (i), every character arises in this way.
Indeed, let $g \in A(\overline{\mathbb{D}})$ be the function defined as $g(w) = w, \ w \in \mathbb{D}$. If $\tau \in \Omega(A(\overline{\mathbb{D}}))$ then

$\tau(e_A) = 1$ and $|\tau(g)| \leq ||g|| = 1$, and so $\tau(g) \in \overline{\mathbb{D}}$. For any polynomial $p : \mathbb{D} \to \mathbb{C}$, given by $p = \lambda_0 e_A + \lambda_1 g + \cdots + \lambda_n g^n$ for constants λ_i, and so $\tau(p) = \lambda_0 + \lambda_1 \tau(g) + \cdots + \lambda_n \tau(g)^n$ $= p(\tau(g))$. As a result of Fejér's theorem ([1]) that the set of polynomials form a dense unital subalgebra of $A(\overline{\mathbb{D}})$, this shows that $\tau(f) = f(\tau(g))$ for all $f \in A(\overline{\mathbb{D}})$, and so $\tau = \varepsilon_{\tau(g)}$. Thus $\Omega(A(\overline{\mathbb{D}})) = \{\varepsilon_w : w \in \overline{\mathbb{D}}\}$ and the map $w \mapsto \varepsilon_w$ is a homeomorphism $\overline{\mathbb{D}} \to \Omega(A(\overline{\mathbb{D}}))$.

(iii) Let $\ell^1(\mathbb{Z})$ be the commutative, unital Banach algebra of Example (4.2.7)(v). If $\tau \in \Omega(A(\ell^1(\mathbb{Z}))$ then $\tau(e_0) = 1$. Furthermore, $e_n * e_{-n} = 1$ and so $e_n = (e_{-n})^{-1})$ and $|\tau(e_0)| \leq ||e_n|| = 1$ for each $n \in \mathbb{Z}$, and so

$$1 \leq |\tau(e_{-n})|^{-1} = |(\tau(e_{-n})^{-1})| = |\tau(e_n)| \leq 1.$$

Whence equality. In particular, $|\tau(e_1)| = 1$, and so $\tau(e_1) \in \mathbb{T}$. Since $e_n = e_1^n$, it follows that $\tau(e_n) = \tau(e_1)^n$. Thus if $x \in \ell^1(\mathbb{Z})$, then

$$\tau(x) = \tau\left(\sum_{n\in\mathbb{Z}} x_n e_n\right) = \sum_{n\in\mathbb{Z}} x_n \tau(e_1)^n.$$

Whence τ is determined by the complex number $\tau(e_1) \in \mathbb{T}$. Conversely, for any $z \in \mathbb{T}$, there is a character $\tau_z \in \Omega(A(\ell^1(\mathbb{Z}))$ with $\tau_z(e_1) = z$. Surely, let $A_0 = \operatorname{span}\{e_n : n \in \mathbb{Z}\}$, which is dense in $\ell^1(\mathbb{Z})$. Let $\tau_0 : A_0 \to \mathbb{C}$ be the unique linear functional such that $\tau_0(e_n) = z^n$ for all $n \in \mathbb{Z}$. If $x, y \in A_0$, then

$$\begin{aligned}\tau_0(x * y) = \tau_0\left(\sum_{n\in\mathbb{Z}} x_n y_{n-m} e_n\right) &= \sum_{n\in\mathbb{Z}} x_n y_{n-m} z^n \\ &= \sum_{n\in\mathbb{Z}} x_n z^m y_{n-m} z^{n-m} = \tau_0(x)\tau_0(y),\end{aligned}$$

and so τ_0 is continuous. Thus τ_0 can be extended to a continuous linear homomorphism $\tau_z : \ell^1(\mathbb{Z}) \to \mathbb{C}$, which is a character on $\ell^1(\mathbb{Z})$. It is clear that, $\tau_z(e_1) = z$. This proves that the map $\gamma : \mathbb{T} \to \Omega(A(\ell^1(\mathbb{Z}))$, $z \mapsto \tau_z$ is a bijection. The map γ is a homeomorphism. To see this, γ is continuous since for $x \in \ell^1(\mathbb{Z})$ and $z \in \mathbb{T}$ it follows that

$$K_x(\gamma(z)) = K_x(\tau_z) = \tau_z(x) = \sum_{n\in\mathbb{Z}} x_n z^n.$$

This series converges absolutely since $x \in \ell^1(\mathbb{Z})$ for any $z \in \mathbb{T}$. Hence $K_x(\gamma(z))$ is continuous and so γ is continuous. Since $\mathbb{T}$ is compact and $\Omega(A(\ell^1(\mathbb{Z}))$ is Hausdorff it follows that γ is a homeomorphism.

The following result is purely algebraic.

4.4.10 Lemma. [15] Let A be a commutative, unital Banach algebra and let $a \in A$. The following statements are equivalent:

(i) $a \notin \operatorname{Inv} A$;

(ii) $a \in I$ for some proper ideal I of A;

(iii) $a \in M$ for some maximal ideal M of A.

Proof. (i) implies (ii). If $a \notin \operatorname{Inv} A$, then let $I = \{ab : \ b \in A\}$. Since A is commutative, this is an ideal of A, and $a = ae_A \in I$ since A is unital. If $e_A \in I$, then $ab = e_A$ for some $b \in A$, and so $a \in \operatorname{Inv} A$, a contradiction. And so $e_A \notin I$, and I is a proper ideal of A.

(ii) implies (iii). Suppose that $a \in I$ for some proper ideal I of A. It follows that $I \subset M$ for some maximal ideal M of A, and so $a \in M$.

(iii) implies (i). If M is a maximal ideal of A then M is a proper ideal, and so $M \cap \operatorname{Inv} A = \emptyset$ by Lemma (4.3.18). Hence $a \notin \operatorname{Inv} A$ for every $a \in M$. □

4.4.11 Theorem. *[15] (Gel'fand representation theorem) Let A be a commutative, unital Banach algebra and let $a \in A$.*

(i) the map $\gamma : A \to C(\Omega(A)), \ a \mapsto \widehat{a}$ is a norm-decreasing (and hence continuous) homomorphism;

(ii) $\sigma(a) = \widehat{a}(\Omega(A)) = \{\widehat{a}(\tau) : \tau \in \Omega(A)\}$;

(iii) $\nu(a) = |\widehat{a}|_{\Omega(A)} = \sup_{\tau \in \Omega(A)} |\tau(a)|$;

(iv) $a \in \operatorname{Inv} A$ if and only if $\widehat{a} \in \operatorname{Inv} C(\Omega(A))$;

(v) $J(A) = \mathcal{Q}(A) = \ker \gamma$.

Proof. (i) The map γ is a homomorphism; to see this, we have

$$\widehat{ab}(\gamma) = \gamma(ab) = \gamma(a)\gamma(b) = (\widehat{a}\widehat{b})(\gamma) \quad (\gamma \in \Omega(A), \ a, \ b \in A).$$

Immediately we have

$$||\widehat{a}||_\infty = \sup\{|\tau(a)| : \ \tau \in \Omega(A)\} = \nu(a) \leq ||a|| \quad (a \in A),$$

and so $a \mapsto \widehat{a}$ is norm-decreasing.

(ii) Indeed,

$$\begin{aligned} z \in \sigma(a) &\Leftrightarrow ze_A - a \notin \operatorname{Inv} A \\ &\Leftrightarrow ze_A - a \quad \text{belongs to some maximal ideal} \\ &\Leftrightarrow ze_A - a \in M_\tau \quad \text{for some } \tau \in \Omega(A) \\ &\Leftrightarrow z = \tau(a) \quad \text{for some } \tau \in \Omega(A), \text{ since } \tau(z) = z \text{ by Lemma(4.4.3)} \\ &\Leftrightarrow z \in \widehat{a}(\Omega(A)). \end{aligned}$$

(iii) This is immediate from (ii) and by definition of $\nu(a)$.

(iv) Indeed,

$$\begin{aligned} a \in \operatorname{Inv} A &\Leftrightarrow a \notin M \quad \text{for all maximal ideals } M \text{ of } A, \text{ by Lemma(4.4.10)} \\ &\Leftrightarrow a \notin \ker \tau \quad \text{for all } \tau \in \Omega(A), \text{ by Theorem(4.4.8)} \\ &\Leftrightarrow \tau(a) \neq 0 \quad \text{for all } \tau \in \Omega(A) \\ &\Leftrightarrow \widehat{a} \in \operatorname{Inv} C(\Omega(A)) \quad \text{by Example(4.2.10)(i).} \end{aligned}$$

(v) Since $\nu(a) = \sup\{|\tau(a)| : \tau \in \Omega(A)\}$, it follows that $\nu(a) = 0$ if and only if $a \in \ker \tau$ for all $\tau \in \Omega(A)$.

□

4.4.12 Corollary. [5] The following are equivalent:

(i) A is semisimple;

(ii) γ is a monomorphism;

(iii) ν is a norm on A.

Hence a semisimple, commutative, unital Banach algebra is identified as a subalgebra of $C(\Theta)$ for the compact space $\Theta = \Omega(A)$. This kinds of algebras are called *Banach function algebras*. Recall that there are commutative, unital Banach algebras which are not semisimple. [5].

4.4.13 Definition. Let A be a unital commutative Banach algebra. For each $a \in A$, the *Gel'fand transform* of a is the mapping

$$\widehat{a} : \Omega(A) \to \mathbb{C},\ \tau \mapsto \tau(a).$$

In other words, $\widehat{a}$ is the restriction to $\Omega(A)$ of the map $\tau \mapsto \tau(a)$ on A'.

4.4.14 Example. [5, 15]

(i) Let X be a compact Hausdorff space. We know that the map $X \to \Omega(C(X)),\ x \mapsto \epsilon_x$ is a homeomorphism. If $f \in C(X)$ then

$$f :\ \Omega(C(X)) \to \mathbb{C}, \quad \epsilon_x \mapsto \epsilon_x(f).$$

This implies that if we identify $\Omega(C(X))$ with X by assuming that $x = \epsilon_x$, then $\widehat{f} = f$.

(ii) We know that $\Omega(\ell^1(\mathbb{Z}))$ can be identified with $\mathbb{T}$, by assuming that $z \in \mathbb{T}$ is the character $\tau_z :\ \ell^1(\mathbb{Z}) \to \mathbb{C},\ x \mapsto \sum_{n\in\mathbb{Z}} x_n z^n$. Hence for $x \in \ell^1(\mathbb{Z})$, it follows that

$$\widehat{x} :\ \Omega(\ell^1(\mathbb{Z})) \to \mathbb{C}, \quad \tau_z \mapsto \tau_z(x) = \sum_{n\in\mathbb{Z}} x_n z^n.$$

By writing $z = e^{i\theta}$ and $x_n = x(n)$, then this form happens

$$\widehat{x}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} x(n) e^{in\theta},$$

so $\widehat{x}$ can possibly be viewed as the inverse of the Fourier transform of x.

5. Semigroup algebras

In this chapter, we determine the Jacobson radical of certain algebras based on semigroups, and in particular on the semigroups $(\beta S, \Box)$, where S is a cancellative, countable, commutative semigroup and βS is its Stone-Čech semigroup compactification. In particular, we determine the radical of $\ell^1(\beta S)$.

In the first section it is discussed that the fundamental analytic object associated with a semigroup S is its semigroup algebra $\ell^1(S)$. It is shown that the maximal ideal space of $\ell^1(S)$ is the space Φ_S of semi-characters of S, and that $\ell^1(S)$ is semisimple if and only if Φ_S separates points on S.

5.1 Semigroup algebra $\ell^1(G)$

Let S be a non-empty set. Then $\ell^1(S)$ is the usual Banach space consisting of the functions $f \in \mathbb{C}^S$ such that

$$||f|| = \sum_{s \in S} |f(s)| < \infty.$$

For an element $f \in \ell^1(S)$, the *support* of f is $\operatorname{supp} f = \{s \in S : f(s) \neq 0\}$. Of course, supp f is always countable. The characteristic function of $\{s\}$ for an element $s \in S$ is denoted by δ_s, and a generic element of $\ell^1(S)$ is written as $\sum_{s \in S} f(s)\delta_s$. The linear space spanned by the functions δ_s is $\mathbb{C}S$; these are the elements of *finite support*. Thus $\mathbb{C}S$ is a dense subspace of $(\ell^1(S), \cdot)$ [8].

In this chapter, we consider algebras $\mathbb{C}S$ and $\ell^1(S)$ based on certain semigroups S, of which basic examples has been provided thus far. See Chapter (2).

Let S be a semigroup. Then there is a unique product $\star$ on $\ell^1(S)$ such that

$$\delta_s \star \delta_t = \delta_{st} \quad (s, t \in S)$$

and such that $(\ell^1(S), \star, \cdot)$ is a Banach algebra; this is the *semigroup algebra* of S. Thus, given $f, g \in \ell^1(S)$, we have

$$(f \star g)(t) = \sum \{f(r)g(s) : r, s \in S, rs = t\} \quad (t \in S),$$

where the sum is zero when there are no elements $r, s \in S$ with $rs = t$. The space $\mathbb{C}S$, the 'algebraist's semigroup algebra', is a dense subalgebra of our Banach algebra $\mathbb{C}S$. For $n \in \mathbb{N}$, the nth power of $f \in \ell^1(S)$ is denoted by f^{*n}.
An extensive study of this Banach algebra is done at [5].

5.1.1 Definition. [8] Let S be a semigroup. The radical of the semigroup algebra $(\ell^1(S), \star)$ is denoted by $J(S)$, and the sets of nilpotents and quasi-nilpotents in $\ell^1(S)$ are denoted by $\mathcal{N}(S)$ and $\mathcal{Q}(S)$, respectively. The radical of the algebra $(\mathbb{C}S, \star)$ is denoted by $J_0(S)$.

Let S be a semigroup. Then it follows from Theorem (4.3.24) that

$$J(S) = \{f \in \ell^1(S) : g \star f \in \mathcal{Q}(S) \ (g \in \ell^1(S))\}.$$

Easy examples show that there are finite, commutative semigroups S such that $\ell^1(S)$ is not semisimple. For example, set $S = \{o, s\}$ where $o^2 = os = so = s^2 = o$, so that S is an commutative semigroup (and S is a zero semigroup). Then set $f = \delta_o - \delta_s$. Clearly f is nilpotent of index 2 and $\delta_o \star f$ and $\delta_s \star f$ are zero, and so $J(S) = \mathbb{C}f \neq \{0\}$.

It is not known whether it is a general truth that the semisimplicity of one of the algebras $\mathbb{C}S$ and $\ell^1(S)$ follows from the semisimplicity of the other [8].

In the case where S is a commutative semigroup, the condition for $\ell^1(S)$ to be semisimple is given by the following result.

5.1.2 Proposition. [8] Let S be a commutative semigroup. Then $\ell^1(S)$ is semisimple if and only if $\Omega(S)$ separates points of S, in the sense that, for each $s, t \in S$ with $s \neq t$, there exists $\tau \in \Omega(S)$ such that $\tau(s) \neq \tau(t)$.

In the case where G is a group, $J(G) = \{0\}$, and so $\ell^1(G)$ is semisimple [4, Corollary 3.3.35]. It is also true that $J_0(G) = \{0\}$; this is a theorem of Rickart, see [8], for example. Further, $\mathcal{Q}(G) = \{0\}$ for each commutative group G.

For S a cancellative semigroup, it is not known whether $\ell^1(S)$ or $\mathbb{C}S$ is necessarily semisimple; in particular, it seems surprising that algebraists have not determined whether $\mathbb{C}S$ is always semisimple for a cancellative semigroup S. Both algebras are semisimple if S is either finite or commutative [8]. It is not true that every cancellative semigroup is a subsemigroup of a group [8]; it is not known whether $\ell^1(S)$ or $\mathbb{C}S$ is necessarily semisimple whenever this is the case.

Let $\mathbb{S}_n$ be the free semigroup on n generators. Then it is true that $\ell^1(\mathbb{S}_n)$ is semisimple: indeed, $J(\mathbb{S}_n) = \mathcal{Q}(\mathbb{S}_n) = \{0\}$ [4, theorem 2.3.14]. For some partial results on when $\mathbb{C}S$ is semisimple for particular cancellative semigroups, see [8]. For example, each ordered semigroup S is cancellative and such that $\mathbb{C}S$ is semisimple.

The following is a corollary of Proposition (4.3.30).

5.1.3 Proposition. [8] Let S be a semigroup with a subgroup G. Suppose that $f \in J(S)$ with $\mathrm{supp}\ f \subset G$. Then $f = 0$.

5.1.4 Proposition. [8] Let S be a compact, right topological semigroup, and suppose that $p, q \in K(S)$. Take $f \in J_0(S)$ with $\mathrm{supp}\, f \subset pK(S)q$. Then $f = 0$.

Proof. Set $G = pK(S)q$, so that G is a group. By Theorem (3.3.8), there exists $r \in E(S) \cap G$ such that $rp = p$ and $qr = q$ and such that $rK(S)r = rSr$ is a group.
Take $g \in \mathbb{C}G \subset \mathbb{C}S$. Since $f \in J_0(S)$, there exists $h \in \mathbb{C}S$ with $g \star f + h = h \star g \star f$. We have

$$g \star f + \delta_r \star h \star \delta_r = \delta_r \star h \star \delta_r \star g \star f$$

because $\delta_r \star g = g$ and $f \star \delta_r = f$, and also supp $(\delta_r \star h \star \delta_r) \subset rSr \subset G$, so that $\delta_r \star h \star \delta_r \in \mathbb{C}G$. Thus $f \in J_0(G) = \{0\}$. □

Let $\mathbb{F}_2$ be the free group on two generators. It is shown in [6, Lemma 7.23] that there are nilpotent elements of every index in $\ell^1(\mathbb{F}_2)$ and that there are quasi-nilpotent elements that are not nilpotent. Thus $\{0\} = J(\mathbb{F}_2) \subsetneq \mathcal{N}(\mathbb{F}_2) \subsetneq \mathcal{Q}(\mathbb{F}_2)$ [8].

5.2 The semigroup $(S^*, \Box)$

In this section, we focus on the set $S^* = \beta S \setminus S$, where βS is the Stone-Čech compactification of a discrete topological space S. In general, we set $A^* = c\ell\, A \cap S^*$ for a subset A of S, where $c\ell\, A$ is the closure of A in βS.

We will take S to be a semigroup through out, with the particular example being $S = (\mathbb{N}, +)$. As shown in Chapter (3), that, in the case where S is a semigroup, there exists a unique binary operation $\Box$ on βS such that $(\beta S, \Box)$ is a semigroup containing S as a subsemigroup and such that $(\beta S, \Box)$ is a compact, right topological semigroup. Therefore, for a semigroup S, the semigroup $(\beta S, \Box)$ is a *Stone-Čech compactification* of S.

In the general case, where the product in S is denoted by juxtaposition, we shall usually denote the operation $\Box$ in βS by juxtaposition and write just βS for $(\beta S, \Box)$; the corresponding product in $\ell^1(\beta S)$ is denoted by $\star$. In the special case where S is commutative (and especially where $S = (\mathbb{N}, +)$), we shall sometimes write $(\beta S, +)$ for the semigroup $(\beta S, \Box)$, as in Chapter (3), where we recall that, in general, $x + y \neq y + x$ for all $x, y \in \beta S$.

There is also a unique binary operation $\Diamond$ on βS such that $(\beta S, \Diamond)$ is a semigroup containing S as a subsemigroup and satisfying $(\beta S, \Diamond)$ is compact, left topological semigroup. In the case where the semigroup S is commutative, the two semigroups $(S^*, \Box)$ and $(S^*, \Diamond)$ have the same minimal ideal and $\ell^1(\beta S, \Diamond)$ is just the opposite algebra to $\ell^1(\beta S, \Box)$, and so these two algebras have the same Jacobson radical. In the case where G is a group, the map $s \mapsto s^{-1}$ on G can be extended to a continuous homeomorphism $\eta : \beta G \to \beta G$ such that $\eta(x \Box y) = \eta(y) \Diamond \eta(x)$ $(x, y \in \beta G)$. It then follows that $\ell^1(\beta G, \Box)$ is semisimple if and only if $\ell^1(\beta S, \Diamond)$ is semisimple; it is not known whether this is always true when G is replaced by a (cancellative) semigroup [8].

Let S be a semigroup, and let $u \in \beta S$. We recall that the left ideal $(\beta S)u$ is closed in βS and that $(\beta S)u = c\ell(Su)$; we shall use this fact several times. The set S^* is an ideal in βS if and only if S is weakly cancellative [2, Theorem 6.16(ii)], and then $S^* = (S^*, \Box)$ is also a compact, right topological semigroup; furthermore, $\ell^1(S^*)$ is a closed ideal in $\ell^1(\beta S, \star)$, and hence $\ell^1(\beta S) = \ell^1(S) \ltimes \ell^1(S^*)$ is a semi-direct product. The structure theorem applies to both βS and S^*; in particular, βS and S^* each have a (unique) minimum ideal. In the case where S is weakly cancellative, $K(S^*) = K(\beta S)$ [8].

5.2.1 Proposition. [8] Let S be a weakly cancellative semigroup such that $\ell^1(S)$ is semisimple. Then $J(\beta S) = J(S^*)$.

Proof. By Corollary (4.3.27), $J(\beta S) \cap \ell^1(S^*) = J(S^*)$. Since $\ell^1(S)$ is semisimple, $J(\beta S) \subset \ell^1(S^*)$ by Proposition (4.3.28)(ii). □

5.2.2 Proposition. [8] The algebra $\ell^1(\mathbb{N}^*)$ is semisimple if and only if $\ell^1(\mathbb{Z}^*)$ is semisimple.

Proof. Since $\mathbb{N}^*$ is a left ideal in $\mathbb{Z}^*$, and so $\ell^1(\mathbb{N}^*)$ is a closed left ideal in $\ell^1(\mathbb{Z}^*)$. By [14, Theorem 8.3.4], there is an element $x \in \mathbb{N}^*$ such that x is cancellable in $\mathbb{Z}^*$, and so $u := \delta_x \in \ell^1(\mathbb{N}^*)$ has the property that $au = 0$ and $ua = 0$ whenever $a \in \ell^1(\mathbb{Z}^*) \setminus \{0\}$. Thus the result follows from Proposition (4.3.31). □

5.2.3 Example. [8] For $m, n \in \mathbb{N}$, define $m \vee n = \max\{m, n\}$, and set $S = (\mathbb{N}, \vee)$. Then S is a countable, weakly cancellative, commutative semigroup, and $\ell^1(S)$ is semisimple because S is separating (see also [6, Example 4.9]), and so $J(\beta S) = J(S^*)$. Take $u, v \in S^*$, then $u\Box v = v$, and so $(S^*, \Box)$ is a right zero semigroup. It is easy to see [6, Example 7.32] that

$$J(\beta S, \Box) = \left\{ f \in \ell^1(S^*) : \sum_{u \in S^*} f(u) = 0 \right\}$$

and so $\ell^1(S^*)$ is not semisimple.

The following result implies immediately that $\{0\} \subsetneq \mathcal{N}(\mathbb{N}^*, +) \subsetneq \mathcal{Q}(\mathbb{N}^*, +)$. The theorem is due to Hindman and Pym see [8].

From now on throughout this chapter, we shall seek to determine the ideal $J(S^*)$ for a semigroup S, centering on the example where S is cancellative, countable, and commutative, and more generally for a countable semigroup S that can be embedded in a group. We shall point out that it seems to be hard to determine even whether $J(\mathbb{N}^*)$ is equal to $\{0\}$, and thence that $\ell^1(\mathbb{N}^*, +)$ is a semisimple Banach algebra. In [8], it is shown that this question is closely related to a well-known open question in the theory of $\beta\mathbb{N}$ which we would like to express here. For a general cancellative, countable, commutative semigroup S, we should like to determine $J(S^*)$ if it should occur that $\ell^1(S^*)$ is not semisimple [8].

5.3 The Radical of some Semigroup algebras

Here, we begin the study of $J(S^*)$, the radical of $\ell^1(S^*)$, for suitable commutative semigroups S. Thus, for the remainder of the thesis, S will denote a commutative semigroup.

5.3.1 Theorem. *[8] Let S be a cancellative, countable, commutative semigroup, and suppose that $f \in J(S^*)$ or $f \in J_0(S^*)$. Then* $\operatorname{supp} f \subset K(S^*)$.

Proof. We take G to be the group of quotients of S, so that S is a subsemigroup of G and Theorem (3.4.41) applies. We now denote the semigroup operation in G^* by '+'; for $x \in \beta G$ and $n \in \mathbb{N}$, we write $n * x$ for $x + \cdots + x$, where there are n copies of x. Assume towards a contradiction that $\operatorname{supp} f \not\subset K(S^*)$, and set

$$X = \text{supp } f \setminus K(S^*),$$

so that X is a countable, non-empty set.
By Theorem (3.4.41), there exists $u \in \beta S$ such that u is cancellable, and $u + x$ is right cancellable for each $x \in X$, and, furthermore, for each $x, y \in X$, either $x \equiv y$ or $(\beta G + u + x) \cap (\beta G + u + y) = \emptyset$. By replacing each $x \in X$ by $u + x$ and replacing f by $\delta_u \star f$, we may suppose that x is right cancellable for each $x \in X$ and that, for each $x, y \in X$, either $x \equiv y$ or $(\beta G + u + x) \cap (\beta G + u + y) = \emptyset$. Note that it remains true that $f \in J(S^*)$ or $f \in J_0(S^*)$ because $J(S^*)$ and $J_0(S^*)$ are ideals in $\ell^1(S^*)$ and $\mathbb{C}S^*$, respectively, and so, in either case, $\lim_{n\to\infty} ||f^{*n}||^{1/n} = 0$. Further, $||f|| = ||\delta_u \,\Box\, f||$ because u is cancellable, and so we have not changed the value of $||f||$.
Suppose that

$$x_{i_1} + \cdots + x_{i_k} \equiv x_{j_1} + \cdots + x_{j_m}, \tag{5.3.1}$$

where $x_{i_1} + \cdots + x_{i_k}, x_{j_1} + \cdots + x_{j_m} \in X$. Then $(\beta G + x_{i_k}) \cap (\beta G + x_{j_m}) \neq \emptyset$, and so $x_{i_k} \equiv x_{j_m}$. Since x_{i_k} and x_{j_m} are right cancellable, it follows that

$$x_{i_1} + \cdots + x_{i_{k-1}} \equiv x_{j_1} + \cdots + x_{j_{m-1}}.$$

By repeating this argument, we see that it follows from (5.3.1) that $k = m$ and $x_{i_r} \equiv x_{j_r}$ for all $r \in \{1, \ldots, k\}$.
Choose $x \in X$, and set $T_n = G + n * x$ for $n \in \mathbb{N}$. Set $h = f \mid T_1$, so that $h \in \ell^1(S^*)$. Since $f(x) \neq 0$, we have $h(x) \neq 0$. By the remark of the previous paragraph, it follows that, for each $n \in \mathbb{N}$, we have $h^{*n} = f^{*n} \mid T_n$, and so $||h^{*n}|| \leq ||f^{*n}||$. Consequently, $\lim_{n\to\infty} ||h^{*n}||^{1/n} = 0$ and $h \in \mathcal{Q}(S^*)$. Now define $\varphi \in \ell^1(G)$ by

$$\varphi(y) = h(y + x) \quad (y \in G).$$

Then $||h^{*n}|| \leq ||f^{*n}||$ $(n \in \mathbb{N})$, and so $\varphi \in \mathcal{Q}(G)$. However $\mathcal{Q}(G) = \{0\}$ because G is an commutative group, and so $\varphi = 0$. Hence $h(x) = 0$, a contradiction.
We conclude that $\operatorname{supp} f \subset K(S^*)$. □

5.3.2 Corollary. [8] Let S be a cancellative, countable, commutative semigroup. Then $\ell^1(S^*)$ is semisimple if and only if $\ell^1(K(S^*))$ is semisimple.

Proof. Assume that $\ell^1(K(S^*))$ is semisimple, and take $f \in J(S^*)$. Then, by theorem (5.3.1), $\operatorname{supp}\, f \subset K(S^*)$, and so $f \in J(S^*) \cap \ell^1(K(S^*)) \subset J(K(S^*)) = \{0\}$. Thus $f = 0$, and so $\ell^1(S^*)$ is semisimple.
Assume that $\ell^1(S^*)$ is semisimple. By Corollary (4.3.27), $J(K(S^*)) = \{0\}$, and so $\ell^1(K(S^*))$ is semisimple. □

5.3.3 Proposition. [8] Let $f \in J(\mathbb{H})$. Then $\operatorname{supp}\, f \subseteq K(\mathbb{N}^*) \cap \mathbb{H}$.

Proof. We observed in the course of the above discussion that, in the case where $S = \mathbb{N}$ and $G = \mathbb{Z}$, we could have chosen our non-empty subset U to be a subset of $\mathbb{H}$. Then the given proof leads to the stated result. □

Recall that a *rectangular semigroup* is a semigroup R that, as a set, has the form $A \times B$, where A and B are non-empty sets, and the product is given by $(a, b)(c, d) = (a, d)$ for $a, c \in A$ and $b, d \in B$, so that all elements of R are idempotents. Let $R = A \times B$ be such a semigroup. In the following, we denote the semigroup multiplication by juxtaposition, and we write π_A and π_B for the projections onto A and B, respectively. Let b_1 and b_2 be two distinct elements in B, and take the set U of pairs $\{u, v\}$ of elements R such that $\pi_B(u) = b_1$ and $\pi_B(v) = b_2$. Observe that, for $\{u_1, v_1\}$ and $\{u_2, v_2\}$ in U, we have

$$u_1u_2 = u_1, \quad u_1v_2 = u_2, \quad v_1u_2 = u_1, \quad v_1v_2 = v_1. \tag{5.3.2}$$

Also observe that the set U is closed under left-translation by elements of R.
Take the set N of elements $f \in \ell^1(R)$ of the form $f = \delta_u - \delta_v$, where $\{u, v\} \in U$, so that $N \subset \mathbb{C}R$. Then it follows from (5.3.2) that $f_1 \star f_2 = 0$ whenever $f_1, f_2 \in N$. Further, N is closed under left-translations by elements of R. Take $f \in N$ and $g \in \ell^1(R)_+$. Then $g \star f$ has the form $h = \sum_{i=1}^{\infty} \alpha_i f_i$, say, where $\alpha_i \in \mathbb{C}$ $(i \in \mathbb{N})$, $\sum_{i=1}^{\infty} |\alpha_i| < \infty$, and $f_i \in N$ $(i \in \mathbb{N})$. Thus $h \star h = 0$. We close with the fact that each such element $g \star f$ is nilpotent of index at most 2, and so $f \in J(R) \cap J_0(R)$. Thus $N \subset J(R) \cap J_0(R)$.
The next result follows from the above paragraph.

5.3.4 Proposition. [8] Let $R = A \times B$ be a rectangular semigroup with $|B| \geq 2$. Then $\dim\, J(R) \geq |A|$ and $\dim\, J_0(R) \geq |A|$. In particular, the algebras $\ell^1(R)$ and $\mathbb{C}R$ are not semisimple.

This result has relevance to the main problem because it is a result of Y. Zelenyuk [14, Theorem 9.41] that $\mathbb{N}^*$ contains a copy of such a rectangular semigroup $R = A \times B$ with $|A| = |B| = 2^{\mathfrak{c}}$. Hence there are many 'very large' semigroups R in $\mathbb{N}^*$ such that $\ell^1(R)$ and $\mathbb{C}R$ are not as close as we might think from being semisimple [8].

5.4 A condition for Semisimplicity

We now give our main description of $J(\beta S)$ and $J(S^*)$ for a cancellative, countable, commutative semigroup S.

5.4.1 Theorem. *[8] Let W be a compact, right topological semigroup, and suppose that $\ell^1(K(W))$. Then the following conditions are mutually equivalent:*

(a) $\delta_p \star f \star \delta_q = 0$ for each $p,\ q\ \in\ K(W)$;

(b) $g_1 \star f \star g_2 \star f \star g_3 \star f = 0$ for each $g_1,\ g_2,\ g_3\ \in \ell^1(W)$;

(c) $f \in J(W)$.

Proof. We suppose that $f = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ where $\alpha_i \in \mathbb{C}$ and $x_i \in K(W)$ for $i \in \mathbb{N}$ and where $\sum_{i=1}^{\infty} |\alpha_i| < \infty$.

(a) $\implies$ (b) It suffices to prove (b) in the special case in which $g_1 = \delta_{y_1}$, $g_2 = \delta_{y_2}$, $g_3 = \delta_{y_3}$ for some $y_1, y_2, y_3 \in W$. But in this case

$$(g_1 \star f) \star (g_2 \star f) \star (g_3 \star f) = \delta_{y_1} \sum_{\alpha_i} \alpha_j (\delta_{x_i y_2} \star f \star \delta_{y_3 x_j}).$$

Since $x_i y_2,\ y_3 x_j\ \in K(W)$ for each $i,\ j\ \in \mathbb{N}$, it follows from (a) that each term in the bracket is 0, and so $g_1 \star f \star g_2 \star f \star g_3 \star f = 0$.

(b) $\implies$ (c) By (b), $g \star f$ is nilpotent of index at most 3 for each $g \in \ell^1(W)$. More directly, f itself is nilpotent of index at most 3. Thus (c) follows from Theorem (4.3.24).

(c) $\implies$ (a) Take $p,\ q \in K(W)$, and set $G = pK(W)q$, so that G is a subgroup of W by Theorem (3.3.8). Since $\operatorname{supp}(\delta_p \star f \star \delta_q) \subset G$, it follows from Proposition (5.1.3) that $\delta_p \star f \star \delta_q = 0$, giving (a). □

Suppose that $f \in \mathbb{C}K(W)$, in the above notation. Then the theorem still holds, with clause (c) replaced by '$f \in J_0(K(W))$'; in the proof of the implication (c) $\implies$ (a), we use Proposition (5.1.4), rather than Proposition (5.1.3). Thus $J(W) \cap \mathbb{C}W = J_0(W)$.

5.4.2 Theorem. *[8] Let S be a cancellative, countable, commutative semigroup, and suppose that $f \in \ell^1(S^*)$. Then $f \in J(S^*)$ if and only if $\operatorname{supp} f \subset K(S^*)$ and $\delta_p \star f \star \delta_q = 0$ for each $p,q\ \in\ K(S^*)$.*
Further, in this case, $g \star f$ is nilpotent of index at most 3 for each $g \in \ell^1(S^*)$.

Proof. Suppose that $f \in J(S^*)$. Then $\operatorname{supp} f \subset K(S^*)$ by Theorem (5.3.1), and hence $f \in \ell^1(K(S^*))$. Now take $p,\ q \in K(S^*)$. Then we have $\delta_p \star f \star \delta_q = 0$ by the implication (c) $\implies$ (a) of Theorem (5.4.1) (applied with $W = \beta S$).

Conversely, suppose that f satisfies the two stated conditions. Then $f \in \ell^1(K(S^*))$, and so $f \in J(S^*)$ by the implication (c) $\Longrightarrow$ (a) of Theorem (5.4.1).
Now suppose that $f \in J(S^*)$. Then $(g \star f)^{*3} = 0$ for each $g \in \ell^1(S^*)$ by the implication (c) $\Longrightarrow$ (b) of Theorem (5.4.1). $\square$

Similarly, for $f \in \mathbb{C}S^*$, we have $f \in J_0(S^*)$ if and only if $\operatorname{supp} f \subset K(S^*)$ and $\delta_p \star f \star \delta_q = 0$ for each $p,\ q \ \in \ K(S^*)$.
The above theorem concerns the algebra $(\ell^1(S^*), \Box)$. However, our earlier remarks show that the same characterization applies to the radical of $\ell^1(S^*, \Diamond)$.
We further remark that, for each $f \in J(S^*)$, there exists $p, \ \in \ K(S^*)$ such that $g \star f \star \delta_p$ is nilpotent of index at most 2 for each $g \in \ell^1(S^*)$. Indeed, suppose that $f \star \delta_p = 0$ for each $p \ \in \ K(S^*)$. Then this is immediate. Otherwise, $f \star \delta_p \neq 0$ for some $p \ \in \ K(S^*)$, and then $\delta_p \star f \star \delta_q = 0$ for each $q \ \in \ K(S^*)$, again giving the result.

5.4.3 Theorem. *[8] The following statements are mutually equivalent:*

(a) $\ell^1(G^*)$ is semisimple for some infinite, countable, commutative group G;

(b) $\ell^1(G^*)$ is semisimple for each infinite, countable, commutative group G;

(c) $\ell^1(\mathbb{H})$ is semisimple;

(d) $\ell^1(\mathbb{N}^*)$ is semisimple.

Proof. We consider the subset V of βG that was defined before Theorem (3.4.35). We note that $K(\mathbb{N}^*) \cap \mathbb{H} = K(\mathbb{H})$, which is topologically isomorphic to $K(V) = V \cap K(\beta G)$ (see Theorem (3.2.29)). Thus, the equivalence of (a), (b), and (c) will follow once we have shown that, for a fixed infinite, countable, commutative group G, the algebra $\ell^1(G^*)$ is semisimple if and only if $\ell^1(V)$ is semisimple.
First, assume that there exists $f \in J(G^*)$ with $f = 0$. Then $\operatorname{supp} f \subset K(S^*)$ by Theorem (5.3.1), and so we may suppose that

$$f = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i},$$

where $\{x_i : i \in \mathbb{N}\} \subset K(G^*)$. By Theorem (5.4.2), $\delta_p \star f \star \delta_q = 0$ for each $p,\ q \ \in \ K(G^*)$. We partition the set $\{x_i : i \in \mathbb{N}\}$ into equivalence classes with respect to $\sim$, say into the disjoint subsets $\{E_m : m \in \mathbb{N}\}$, and set $f_m = f \mid E_m$ for $m \in \mathbb{N}$. Since $f \neq 0$, there exists $m_0 \in \mathbb{N}$ with $f_{m_0} \neq 0$.
Now suppose that $m_1, m_2 \in \mathbb{N}$ with $m_1 \neq m_2$. For each $x, y \in \beta G$ with $p + x + q = p + y + q$ for some $p,\ q \ \in \ K(S^*)$, necessarily $x \sim y$, and so the elements $\delta_p \star f_{m_1} \star \delta_q$ and $\delta_p \star f_{m_2} \star \delta_q$ have disjoint support for each $p,\ q \ \in \ K(G^*)$. Hence $\delta_p \star f_m \star \delta_q = 0$ for each $m \in \mathbb{N}$ and each $p,\ q \ \in \ K(G^*)$. Since $\operatorname{supp} f_m \subset K(G^*)$, Theorem (5.4.1) applies to show that $f_m \in J(G^*)$ for each $m \in \mathbb{N}$; in particular, $f_{m_0} \in J(G^*)$.
By Lemma (3.4.36), there is a cancellable element $u \in \beta G$ such that $u + x \in V$ for each $x \in E_{m_0}$. Thus $\delta_u \star f_{m_0} \neq 0$ and $\delta_u \star f_{m_0} \in J(V)$. This shows that $J(V) \neq \{0\}$.
Secondly, assume that there exists $f \in J(V)$ with $f \neq 0$. Then $\operatorname{supp} f \subset K(V)$ by Proposition (5.3.3); in particular, $f \in \ell^1(K(V))$, and so, again, Theorem (5.4.1) applies.
Take $r \in E(K(\beta G)) \subset V$. Since $r + K(V) + r$ is a subgroup of V,we have $\delta_r \star f \star \delta_r = 0$. Now take $p,\ q \ \in \ K(\beta G)$. By Theorem (3.3.8), there exists $r \in E(K)$ with $p + r = p$ and $r + q = q$, and so

$$\delta_p \star f \star \delta_q = \delta_p \star \delta_r \star f \star \delta_r \star \delta_q = 0.$$

By Theorem (5.4.1), (a) $\implies$ (c), $f \in J(G^*)$. This shows that $J(G^*) \neq \{0\}$. By Proposition (5.2.2), $\ell^1(\mathbb{N}^*)$ is semisimple if and only if $\ell^1(\mathbb{Z}^*)$ is semisimple, and so (d) is also equivalent to the other statements. □

5.4.4 Theorem. *[8] Let S be a cancellative, countable, commutative semigroup. Consider the following conditions on $(S^*, +)$:*

(a) there exist $n \in \mathbb{N}$ and two disjoint sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of $K(S^*)$ such that, for each $p,\ q \in K(S^*)$, the set $\{p + x_1 + q, \ldots, p + x_n + q\}$ is a permutation of the set $\{p + y_1 + q, \ldots, p + y_n + q\}$;

(b) $J(S^*, +) \neq \{0\}$ nor $J_0(S^*, +) \neq \{0\}$;

(c) there is a non-empty, finite subset F of distinct elements of $K(S^*)$ and $x \in F$ such that, for each $p,\ q \in K(S^*)$, there exist $y \in F$ such that $y \neq x$ and $p + y + q = p + x + q$.

Then (a) $\implies$ (b) $\implies$ (c).

Proof. (a) $\implies$ (b) Let $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be as specified in (a), and set

$$f = \sum_{i=1}^{n} \delta_{x_i} - \sum_{i=1}^{n} \delta_{y_i},$$

so that $f \in \mathbb{C}S$ with $\operatorname{supp} f \subset K(S^*)$ and $f \neq 0$. Take $p,\ q \in K(S^*)$. Then clearly $\delta_p \star f \star \delta_q = 0$, and so, by Theorem (5.4.2), $f \in J_0(S^*) \cap J(S^*)$. Hence $J(S^*, +) \neq \{0\}$ and $J_0(S^*, +) \neq \{0\}$.
(b) $\implies$ (c) Take $f \in J(S^*)$ with $f \neq 0$; we may suppose that $||f|| = 1$. By Theorem (5.3.1), $\operatorname{supp} f \subset K(S^*)$, and so $f = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ where $\alpha_i \in \mathbb{C} \setminus \{0\}$ $(i \in \mathbb{N})$, $\sum_{i=1}^{n} |\alpha_i| = 1$, and $\{x_i : i \in \mathbb{N}\}$ is a set of distinct points in $K(S^*)$. Choose $k \in \mathbb{N}$ such that $\sum_{i=k+1}^{\infty} |\alpha_i| < |\alpha_1|$ and set $F = \{x_1, \ldots, x_k\}$, so that F is a non-empty, finite subset of distinct elements of $K(S^*)$. Set $g = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ and $h = \sum_{i=k+1}^{\infty} \alpha_i \delta_{x_i}$ so that $g, h \in \ell^1(\mathbb{N}^*)$, $f = g + h$, $||g|| \geq |\alpha_1|$, and $||h|| < |\alpha_1|$. Set $F = \{x_1, \ldots, x_k\}$ and $x = x_1$.
By Theorem (5.4.2), $\delta_p \star f \star \delta_q = 0$ for each $p,\ q \in K(S^*)$. Take $p,\ q \in K(S^*)$, and assume towards a contradiction that, for each $y \in F$ with $y = x$, we have

$$p + x + q \neq p + y + q$$

. Then $||\delta_p \star g \star \delta_p|| = g \geq |\alpha_1|$ and $||\delta_p \star h \star \delta_p|| < |\alpha_1|$, and so $||\delta_p \star f \star \delta_p|| > 0$, a contradiction of the fact that $\delta_p \star f \star \delta_p = 0$. Thus there exist $y \in F$ with $y \neq x$ such that $p + y + q = p + x + q$. □

The question whether or not clause (a) of the above theorem holds is a well-known open question in the theory of Stone-Čech semigroup compactification; in particular, it is open for the case where $S = (\mathbb{N}, +)$. Indeed, it may be that there exist $x, y \in K(S^*)$ with $x \neq y$ and such that $p + x + y = p + y + q$ for each $p,\ q \in K(S^*)$, a condition that implies (a). Unfortunately, it is not known whether the conditions in clauses (a) and (c) are equivalent see [8].

6. Conclusion

The operation defined on a discrete semigroup S can be extended to the Stone-Čech compactication βS of S so that for all $a \in S$, the left translation $\beta S \ni x \mapsto ax \in \beta S$ is continuous, and for all $q \in \beta S$, the right translation $\beta S \ni x \mapsto xq \in \beta S$ is continuous. We let the points of βS to be the ultrafilters on S and we identified the principal ultrafilters with the points of S. We saw that being a compact right topological semigroup, βS contains a smallest two-sided ideal $K(\beta S)$ which is a completely simple semigroup. In this dissertation we gave an exposition of some basic results related to the semigroup βS and to the semigroup algebra $\ell^1(\beta S)$. Particularly we proved that $\ell^1(\beta\mathbb{N})$ is semisimple if and only if $\ell^1(K(\beta\mathbb{N}))$ is semisimple and established how this problem was reduced to whether $\ell^1(K(\beta\mathbb{N}))$ is semisimple to a problem about $K(\beta\mathbb{N})$.

The results we exposed which are known in the literature were

(i) Let S be a cancellative, countable, commutative semigroup, and suppose that $f \in J(S^*)$ or $f \in J_0(S^*)$. Then $\mathrm{supp}\ f \subset K(S^*)$.

(ii) Let S be a cancellative, countable, commutative semigroup. Then $\ell^1(sawS^*)$ is semisimple if and only if $\ell^1(K(S^*))$ is semisimple.

These results allowed us to get the main description or characterisation of $J(\beta S)$ and $J(S^*)$ for a cancellative, countable, commutative semigroup S. Particularly, they gave us an exposition of some basic results related to the semigroup βS and to the semigroup algebra $\ell^1(\beta S)$.

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