Noether, Partial Noether Operators and First Integrals for Systems

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ABSTRACT

The notions of partial Lagrangians, partial Noether operators and partial Euler-Lagrange equations are used in the construction of first integrals for ordinary differential equations (ODEs) that need not be derivable from variational principles. We obtain a Noether-like theorem that provides the first integral by means of a formula which has the same structure as the Noether integral. However, the invariance condition for the determination of the partial Noether operators is different as we have a partial Lagrangian and as a result partial Euler-Lagrange equations. In order to investigate the effectiveness of the partial Lagrangian approach, some models such as the oscillator systems both linear and nonlinear, Emden and Ermakov-pinnery equations and the Hamiltonian system with two degrees of freedom are considered in this work. We study a general linear system of two second-order ODEs with variable coefficients. Note that, a Lagrangian exists for the special case only but, in general, the system under consideration does not have a standard Lagrangian. However, partial Lagrangians do exist for all such equations in the absence of Lagrangians. Firstly, we classify all the Noether and partial Noether operators for the case when the system admits a standard Lagrangian. We show that the first integrals that result due to the partial Noether approach is the same as for the Noether approach. First integrals are then constructed by the partial Noether approach for the general case when there is in general no Lagrangian for the system of two second-order ODEs with variable coefficients. We give an easy way of constructing first integrals for such systems by utilization of a partial Noether's theorem with the help of partial Noether operators associated with a partial Lagrangian.

Furthermore, we classify all the potential functions for which we construct first integrals for a system with two degrees of freedom. Moreover, the comparison of Lagrangian and partial Lagrangian approaches for the two degrees of freedom Lagrangian system is also given.

In addition, we extend the idea of a partial Lagrangian for the perturbed ordinary differential equations. Several examples are constructed to illustrate the definition of a partial

Lagrangian in the approximate situation. An approximate Noether-like theorem which gives the approximate first integrals for the perturbed ordinary differential equations without regard to a Lagrangian is deduced.

We study the approximate partial Noether operators for a system of two coupled nonlinear oscillators and the approximate first integrals are obtained for both resonant and non-resonant cases. Finally, we construct the approximate first integrals for a system of two coupled van der Pol oscillators with linear diffusive coupling. Since the system mentioned above does not satisfy a standard Lagrangian, the approximate first integrals are still constructed by invoking an approximate Noether-like theorem with the help of approximate partial Noether operators. This approach can give rise to further studies in the construction of approximate first integrals for perturbed equations without a variational principle.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any institution.

Imran Naeem

DEDICATION

To my parents

my wife Rehana, my daughter Ayesha

and my mentor Prof Mahomed

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Contents

INTRODUCTION

The relationship between symmetries and conservation laws by means of a formula is one of the great discoveries of the twentieth century. Amalie Emmy Noether, a great German mathematician discovered a systematic way of constructing conservation laws for differential equations with variational formulations. Now, we briefly mention how she came to prove Noether's theorem. David Hilbert invented the variational principle and its utilization yields the field equations of general relativity. Then he was faced with the problem of the failure of energy conservation. The correspondence between Hilbert and Klein in 1918, regarding this issue provides evidence that Hilbert requested Noether to resolve this problem. She proved two useful theorems (published in 1918) to solve the problem of failure of energy conservation in general relativity. The physicists collectively refer to these as Noether's theorem which gives the relation between symmetries and conservation laws.

The importance of Noether's theorem in the calculus of variations and physics is that the differential equations with variational structures are physical models and their variational symmetries generate conservation laws.

Now, in particle mechanics, we pay attention to systems described by the Euler-Lagrange equations of second-order. Infinitesimal transformations of variables (dependent and dependent) in which the first-order variations are taken to be functionally dependent on variables only but not on the velocity gives rise to first integrals for many systems (see, e.g. Hill 1951, Lovelock and Rand 1975, Logan 1977 and Palmieri and Vitale 1970 amongst others). Also velocity dependent transformations have been considered to account for the Kepler problem Laplace-Runge-Lenz vector (see, e.g. Mahomed and

Vawda 1996).

Noether's theorem (see Noether 1918 and Bessel-Hagen 1921) provides explicit elegant formulae for the construction of conservation laws for Euler-Lagrange differential equations once their Noether symmetries are known. In order to invoke this powerful theorem, one requires a Lagrangian of the underlying differential equation system. Thus a central problem in variational calculus is the determination of a Lagrangian so that the differential equation system of interest is cast as an Euler-Lagrange system. This is called the inverse problem in variational calculus and much work has been devoted in this direction (see, e.g. Douglas 1941 and Anderson and Duchamp 1984).

Most of the equations that arise in applications do not admit standard Lagrangians, e.g. scalar evolution equations. Likewise, for a simple system of two equations $y'' = y^2 + z^2$, $z'' = 0$, no corresponding variational problem exists which can be verified by Douglas (1941). However, one can still construct first integrals as we do here.

Notwithstanding, for ordinary differential equations of mechanics, application of the Noether theorem have yielded numerous interesting results which are scattered in the literature see, e.g. Levy-Leblond (1971), Lutzky (1978), Prince and Eliezer (1980), Sarlet and Cantrijn (1981), Leach (1985), Mahomed, Kara and Leach (1993), Wafo and Mahomed (1999) for a sample. There are other methods as well which provide first integrals without making use of a Lagrangian. The most elementary method is the direct method (Hietarinta 1986, Lewis and Leach 1982), which is oftenly used for constructing the conserved quantities. There are some other methods (see Olver 1986, Sarlet and Cantrijn 1981, Steudel 1962, Moyo and Leach 2005, Anco and Bluman 1998), in which the equations can be expressed in the characteristic form. Ibragimov in 2007 provided a method for constructing first integrals without making use of a Lagrangian. In 2006, Anco and Kara wrote an algorithm for the construction of conservation laws. Recently, a computer algebra programm has been developed for the construction of first integrals without regard to a Lagrangian (see Wolf 2002).

Now the question is how one can construct first integrals in a systematic way without the existence of Lagrangians. In 2006, Kara and Mahomed introduced the concept of partial Lagrangians and partial Euler-Lagrange equations. A Noether-like theorem, known as partial Noether theorem, was presented for partial differential equations which has the same structure as the Noether theorem. Applications included several examples including those for which Lagrangians exist. Conservation laws are constructed by utilization of a partial Noether theorem in this approach.

In this work, we introduce the notions of partial Lagrangians and partial Euler-Lagrange equations for ordinary differential equations and the idea is also extended for perturbed ordinary differential equations. Approximate partial Noether theorem is deduced for the perturbed ordinary differential equations which provides approximate first integrals without a variational principle. Several examples are considered.

Detail Outline of Work

The detail outline of this work is as follows.

In Chapter 1, the basic operators and definitions and the notion of first integrals are presented for ODEs. The concept of a partial Lagrangian is introduced for the exact and perturbed ordinary differential equations. Noether-like theorems are obtained which yield exact and approximate first integrals which need not be derivable from variational principles. The alternative way of constructing first integrals for ODEs without regard to a Lagrangian is established.

In Chapter 2, a Noether-like theorem which gives the first integrals for exact ordinary differential equations corresponding to partial Euler-Lagrange equations associated with a partial Lagrangian is invoked. The results are applied to the harmonic oscillator, modified Emden and Ermakov-Pinney equation for which standard Lagrangians exist and also to systems of two second-order equations for which we do not have standard Lagrangians. Moreover, we show that if the partial Euler-Lagrange equations are free of derivatives then the partial Noether operators and Noether symmetries are equivalent and the difference arises in the guage terms due to Lagrangians being different for the respective approaches.

Chapters 3 and 4 are concerned with a general linear system of two second order ordinary differential equations (ODEs) with variable coefficients. In Chapter 3 we invoke the canonical form for a system of two second-order ODEs and construct the Noether and partial Noether operators for the special case in which a Lagrangian exists for the system under study. It is also shown that the first integrals that result due to the partial Lagrangian approach is exactly the same as for the standard Lagrangian approach. In Chapter 4 we report on the partial Noether operators for the general linear system of two second-order ODEs for which a Lagrangian need not exist. The notion of partial Lagrangians is used and it is shown how one can construct first integrals without a variational principle for such systems.

Chapter 5 deals with first integrals for a system with two degrees of freedom. A Lagrangian exists for the Hamiltonian systems with two degrees of freedom. However, we show the effectiveness of a partial Lagrangian approach and a complete classification of the potential functions is deduced for which first integrals are constructed. Finally, in this chapter we give a comparison of the Lagrangian and partial Lagrangian approaches for a Hamiltonian system with two degrees of freedom which was considered before by Damianou and Sophocleus (2004). We give an alternative viewpoint to construct potential functions by using the concept of a partial Lagrangian. Then the first integrals are obtained by utilizing a partial Noether theorem corresponding to the partial Noether operators associated with a partial Lagrangian. This work appears in Naeem and Mahomed (2008a).

The results presented in Chapter 2, 3 and 4 have been published in Kara, Mahomed, Naeem and Wafo Soh (2007), Naeem and Mahomed (2008b) and Naeem and Mahomed

(2008c).

In Chapter 6, we further investigate first integrals of nonlinear systems which are not variational. Some examples from mechanics, nonlinear oscillations, nonlinear dynamics are constructed and it is shown how the nonlinear systems for which no standard Lagrangians exist can be reduced by finding first integrals via the partial Noether theorem.

Chapter 7 is the extension of the partial Noether theorem for perturbed ordinary differential equations. An approximate Noether-like theorem is presented to construct approximate first integrals without making use of a standard Lagrangian. Applications include a system of two coupled nonlinear oscillators. It is shown how one can construct approximate first integrals for perturbed equations without regard to a Lagrangian. We also show that these approximate partial Noether operators are not in general approximate symmetries of the system and they do not form an approximate Lie algebra.

In the last chapter (Chapter 8), we classify all the approximate partial Noether operators for a system of two coupled van der Pol oscillator with linear diffusive coupling. All the cases for coupling parameters that give partial Noether operators are discussed in detail. Then we construct approximate first integrals by utilization of an approximate partial Noether theorem corresponding to a partial Lagrangian. We show that in the absence of Lagrangians, approximate first integrals still can be constructed by using the idea of partial Lagrangians as partial Lagrangians do exist for all such type of equations in which we do not have Lagrangians and they are very important in obtaining approximate first integrals for perturbed ordinary differential equations.

In the conclusion we provide a summary of the results as well as point to further works.

Chapter 1

Preliminaries

The goal of this chapter is to present the basic operators, introduce the concepts of Noether symmetries, partial Noether operators, Euler-Lagrange equations, partial Euler-Lagrange equations and first integrals for ordinary differential equations (ODEs). To understand the notions of Noether symmetries and first integrals for Euler-Lagrange equations, there are many books on this topic but we mainly refer to the books by Olver (1986), Bluman and Kumei (1989), Stephani (1989), CRC handbook edited by Ibragimov (1994-1996) and some recent papers from the literature. The method of constructing partial Noether operators and corresponding first integrals for ODEs is derived which was considered in Kara and Mahomed (2006), for partial differential equations and Kara, Mahomed, Naeem and Wafo Soh (2007) for ordinary differential equations. Then several examples are constructed to clarify the idea of a partial Lagrangian for perturbed p-th order system of equations which is easily adaptable from Kara and Mahomed (2006), Kara, Mahomed, Naeem and Wafo Soh (2007) and Naeem and Mahomed (2008a). The formula of the partial Noether's theorem is extended to the case of perturbed ordinary differential equations and the approximate partial Noether's theorem is derived which is given at the end of this chapter.

Here we present the basic operators needed in the sequel after introducing the necessary notation.

Let x be the independent variable and $u = (u^1, \ldots, u^m)$ the dependent variable with coordinates u^α. The derivatives of u^α with respect to x are

$$u_x^\alpha \equiv u_1^\alpha = D_x(u^\alpha), \;\; u_s^\alpha = D_x^s(u^\alpha), \quad s \geq 2, \; \alpha = 1, ..., m, \tag{1.1}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x^\alpha \frac{\partial}{\partial u^\alpha} + u_{xx}^\alpha \frac{\partial}{\partial u_x^\alpha} + \cdots \;, \tag{1.2}$$

is the *total derivative operator* with respect to x. Note that the summation convention is adopted for repeated indices. The collection of first- and higher-order derivatives of u^α is denoted by $u_{(k)}$, $k \geq 1$. All variables $x, u, u_{(k)}$, $k \geq 1$ are functionally independent connected only by the differential relations (1.1). The universal space $\mathcal{A}$ (see, e.g. Ibragimov 1983, Olver 1993) is utilised here. The reader is reminded that the space $\mathcal{A}$ is the *vector space of all differential functions of all finite orders* and forms an algebra. Apart from the total derivative operator (1.2), there are other operators properly defined on $\mathcal{A}$ and the ones which we use are recalled below in Definitions 1.1 to 1.4. These are known and can be found in the books by Ovsyannikov (1982), Ibragimov (1983), Bluman and Kumei (1989), Olver (1993) (see also Kara and Mahomed 2006, Ibragimov Kara and Mahomed 1998). However, here we write them for one independent variable x.

Definition 1.1. The *Euler operator*, for each α, is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-D_x)^s \, \frac{\partial}{\partial u_s^\alpha} \;, \quad \alpha = 1, \ldots, m \;. \tag{1.3}$$

The Euler operator is also referred to as the *Euler-Lagrange operator*.

Definition 1.2. The *Lie-Bäcklund* or *generalized operator* is given by

$$X = \xi \, \frac{\partial}{\partial x} + \eta^\alpha \, \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_s^\alpha \, \frac{\partial}{\partial u_s^\alpha}, \tag{1.4}$$

where

$$\zeta_s^\alpha = D_x(\zeta_{s-1}^\alpha) - u_s^\alpha D_x(\xi), \quad s \geq 1, \, \alpha = 1, \ldots, m, \tag{1.5}$$

in which $\zeta_0^\alpha \equiv \eta^\alpha$.

One can write equation (1.5) in the characteristic form as

$$\zeta_s^\alpha = D_x^s(W^\alpha) + \xi u_{s+1}^\alpha, \tag{1.6}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi u_x^\alpha, \quad \alpha = 1, \ldots, m. \tag{1.7}$$

One may easily deduce that one can write the Lie-Bäcklund operator (1.4) in the characteristic form as

$$X = \xi D_x + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_x^s(W^\alpha) \frac{\partial}{\partial u_s^\alpha}. \tag{1.8}$$

Definition 1.3. Lie-Bäcklund operators $\tilde{X}$ and X are *equivalent* if

$$X - \tilde{X} = \lambda D_x, \quad \lambda \in \mathcal{A}. \tag{1.9}$$

In particular, a Lie-Bäcklund operator of the form

$$\tilde{X} = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots \tag{1.10}$$

is called a *canonical* or *evolutionary representation* of X.

Definition 1.4. The *Noether operator* associated with a Lie-Bäcklund operator X is

$$N = \xi + W^\alpha \frac{\delta}{\delta u_x^\alpha} + \sum_{s\geq 1} D_x^s(W^\alpha) \frac{\delta}{\delta u_{s+1}^\alpha}\ , \tag{1.11}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.3) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_x^\alpha} = \frac{\partial}{\partial u_x^\alpha} + \sum_{s\geq 1} (-D_x)^s \frac{\partial}{\partial u_{s+1}^\alpha} \quad \alpha = 1, \ldots, m. \tag{1.12}$$

Definition 1.5. The operator X in (1.4) is called a Noether symmetry generator corresponding to a Lagrangian $L \in \mathcal{A}$ if there exists a function $B \in \mathcal{A}$ such that

$$X(L) + (D_x\xi)L = D_x(B), \tag{1.13}$$

where D_x is the total derivative operator given in (1.2).

We next recall the main results of Kara, Mahomed, Naeem and Wafo Soh (2007).

1.1 Partial Noether operators and partial Lagrangians

Consider a kth-order ordinary differential equation system

$$E_\alpha\big(x,\ u,\ u_{(1)},\ u_{(2)}, \ldots, u_{(k)}\big) = 0\ , \qquad \alpha = 1, \ldots, m. \tag{1.14}$$

The following definition is well-known.

Definition 1.6. A *first integral* of system (1.14) is a differential function $I \in \mathcal{A}$, such that

$$D_x I = 0 \tag{1.15}$$

is satisfied for every solution of equations (1.14).

Note that when Definition 1.6 holds, (1.15) is called a local *conservation law* for system (1.14). Moreover, $D_x I = Q^\alpha E_\alpha$ is referred to as the *characteristic form* of the conservation law (1.15) and the function $Q = (Q^1, \ldots, Q^m)$ the associated *characteristic* of the conservation law (1.15) (see, e.g. Olver 1993).

We now assume that equations (1.14) can be written in the split form

$$E_\alpha \equiv E^0_\alpha + E^1_\alpha = 0\ , \qquad \alpha = 1, \ldots, m. \tag{1.16}$$

The definition of *partial Lagrangians* is as follows (Kara, Mahomed, Naeem and Wafo Soh 2007).

Definition 1.7. If there exists a function $L = L(x,\ u,\ u_{(1)}, \ldots,\ u_{(p)}) \in \mathcal{A}$, $p \leq k$ and nonzero functions $f_\alpha^\beta \in \mathcal{A}$ such that (1.16) can be written as $\delta L/\delta u^\alpha = f_\alpha^\beta E_\beta^1$ then, provided $E_\beta^1 \neq 0$ for some β, L is called a *partial Lagrangian* of equation (1.16). Otherwise it is called a *standard Lagrangian.* The matrix (f_α^β) is invertible.

Differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = 0, \qquad \alpha = 1, \ldots,\ m\ , \tag{1.17}$$

are the usual *Euler-Lagrange equations.* We term differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = f_\alpha^\beta E_\beta^1, \qquad \alpha = 1, \ldots,\ m\ , \tag{1.18}$$

where $E_\beta^1 \neq 0$, *partial Euler-Lagrange equations.* These were referred to as *Euler-Lagrange-type equations* in Kara and Mahomed (2006).

We present the following examples to illustrate Definition 1.7.

1. The linear harmonic oscillator equation $u'' + u = 0$ is written as the partial Euler-Lagrange equation $\delta L/\delta u - u = 0$ with respect to the partial Lagrangian $L = u'^2/2$.

2. The linear homogeneous equation $u'' + a(x)u' + b(x)u = 0$ after introducing the factor $f = \exp(\int^x a(z)dz$ can be written as the partial Euler-Lagrange equation $\delta L/\delta u - fbu = 0$ with respect to the partial Lagrangian $L = \exp(\int^x a(z)dz)(u'^2/2)$.

The definition of *partial Noether* operator now follows (Kara, Mahomed, Naeem and Wafo Soh 2007).

Definition 1.8. A Lie-Bäcklund operator X of the form (1.4) is a *partial Noether* operator corresponding to a partial Lagrangian $L \in \mathcal{A}$ if there exists a function $B \in \mathcal{A}$,

$B \neq NL + C$, C is a constant, such that

$$X(L) + LD_x(\xi) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_x(B) \ , \tag{1.19}$$

where $W = (W^1, \ldots, W^m)$, $W^\alpha \in \mathcal{A}$, is the characteristic of X.

In Kara and Mahomed (2006), the partial Noether operator was called a *Noether-type* operator.

If B is identically zero, the Lie-Bäcklund operator X is referred to as a *strict partial Noether* operator.

If L is a Lagrangian, i.e. $\delta L/\delta u^\alpha = 0$, then equation (1.19) becomes the usual determining equation for Noether symmetries.

It can be seen from (1.19) that if X and Y are partial Noether operators, then so is a linear combination of these. The partial Noether operators span a vector space.

Theorem 1.1. A Lie-Bäcklund operator X of the form (1.4) is a partial Noether operator of a partial Lagrangian $L \in \mathcal{A}$ corresponding to a partial Euler-Lagrange system of the form (1.18) if and only if the characteristic $W = (W^1, \ldots, W^m)$, $W^\alpha \in \mathcal{A}$, of X is also the characteristic of the conservation law $D_x I = 0$, where

$$I = B - N(L) \ , \tag{1.20}$$

of the partial Euler-Lagrange equations (1.18).

Theorem 1.2. Let two Lie-Bäcklund operators X and $\tilde{X}$ be equivalent, i.e. $X = \tilde{X} + \lambda D_x$, $\lambda \in \mathcal{A}$. Then X is a partial Noether operator if and only if $\tilde{X}$ is.

Under what conditions is the partial Noether operator also a symmetry generator of the partial Euler-Lagrange equations? The following theorem gives the answer. Here we let

$f_\alpha^\beta E_\beta^1 = G_\alpha$.

Theorem 1.3. If X of the form (1.4) is a partial Noether operator of a partial Lagrangian L, then X is the Lie-Bäcklund symmetry operator of the corresponding partial Euler-Lagrange equations $\delta L/\delta u^\alpha = G_\alpha$ if

$$XG_\alpha = \xi D_x G_\alpha + (D_G^* W)_\alpha, \tag{1.21}$$

where D_G^* is the adjoint of the Fréchet derivative of G with $G = (G_1, \ldots, G_m)$.

We now look at the situation when the partial Noether operators form a Lie algebra.

Theorem 1.4. If $\bar{X}$ and $\bar{Y}$ are partial Noether operators in canonical form with respective 'gauge' terms $\bar{B}$ and $\bar{C}$, then

$$[\bar{X}, \bar{Y}]L = D_x(\bar{X}\bar{C} - \bar{Y}\bar{B}) + (\bar{X}Y^\alpha - \bar{Y}X^\alpha)\frac{\delta L}{\delta u^\alpha} + (Y^\alpha \bar{X} - X^\alpha \bar{Y})\frac{\delta L}{\delta u^\alpha}\,. \tag{1.22}$$

We see from (1.22) that the commutator does not yield a partial Noether operator in general. One has the following theorem on closure of the Lie algebra.

Theorem 1.5. If $\bar{X}$ and $\bar{Y}$ are partial Noether operators of an L for a scalar partial Euler-Lagrange equation such that $\delta L/\delta u$ is independent of derivatives u_s, $s \geq 1$, then $[\bar{X}, \bar{Y}]$ is also a partial Noether operator of L.

The above theorems and definitions are also given in Kara and Mahomed (2006) and are invoked here for systems of ODEs. However, apart from the change in terminology, we restrict here to perturbed ordinary differential equations as well. We invoke the definition of a partial Lagrangian, partial Euler-Lagrange equations, partial Noether's theorem and we extend to the case of approximate ordinary differential equations. We also show when approximate partial Noether operators form an approximate Lie algebra. We proceed as follows.

Consider a pth-order system of perturbed ordinary differential equations with a small parameter ϵ

$$E_\alpha(x,\ u,\ u_{(1)},\ u_{(2)}, ..., u_{(p)}; \epsilon) = O(\epsilon^{k+1}),\ \alpha = 1, ..., m, \tag{1.23}$$

where x is the independent variable and $u = (u^1, u^2, ..., u^m)$ the dependent variable. The derivatives of u^α with respect to x are defined in (1.1), where D_x is the total derivative operator with respect to x defined in (1.2). The summation convention is utilized throughout for repeated indices. The collection of first derivative is denoted by $u_{(1)}$, second and higher-order derivatives are denoted by $u_{(2)},\ u_{(3)}, ...$ as before.

Now we recall some basic definitions from Ünal (2000), Baikov (1996), Kara, Mahomed and Ünal (1999), Johnpillai, Kara and Mahomed (2006), Johnpillai, Kara and Mahomed (2008) and Naeem and Mahomed (2008d).

Definition 1.9. The kth-order ($k \geq 1$) approximate Lie-Bäcklund symmetry operator can be expressed as

$$\chi = X_0 + \epsilon X_1 + \cdots + \epsilon^k X_k, \tag{1.24}$$

where

$$X_b = \xi_b \frac{\partial}{\partial x} + \eta_b^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta_{b,s}^\alpha \frac{\partial}{\partial u_s^\alpha} + \cdots,\quad b = 0, ..., k,\ \xi_b,\ \eta_b^\alpha \in \mathcal{A}, \tag{1.25}$$

in which $\mathcal{A}$ is the vector space of all differential functions of finite orders that form an algebra.

The additional coefficient is defined by

$$\zeta_{b,j}^\alpha = D_x^j(W_b^\alpha) + \xi_b u_{j+1}^\alpha,\quad \alpha = 1, ..., m,\quad b = 0, ..., k;\quad j \geq 1. \tag{1.26}$$

In the above equation W_b^α is the Lie characteristic function given as

$$W_b^\alpha = \eta_b^\alpha - \xi_b u_x^\alpha,\quad b = 0, ..., k;\quad \alpha = 1, ..., m. \tag{1.27}$$

One can write the approximate Lie-B$\ddot{a}$cklund operator (1.24) in the characteristic form as

$$\chi = \xi D_x + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_x^s(W^\alpha)\frac{\partial}{\partial u_s^\alpha}, \tag{1.28}$$

where

$$\xi = \xi_0 + \epsilon\xi_1 + \cdots + \epsilon^k \xi_k \tag{1.29}$$

and

$$W^\alpha = W_0^\alpha + \epsilon W_1^\alpha + \cdots + \epsilon^k W_k^\alpha, \ \ \alpha = 1, ..., m. \tag{1.30}$$

Definition 1.10. A differential function $I \in \mathcal{A}$ is an approximate first integral of system (1.23) if it satisfies

$$D_x I = O(\epsilon^{k+1}), \tag{1.31}$$

for every solution of system (1.23).

In the next section, we extend the definition of a partial Lagrangian and the partial Noether theorem obtained in Section 1.1 to the case of perturbed ordinary differential equations. In Section 1.2, the definition of a partial Lagrangian is applied to simple equations.

1.2 Approximate Partial Noether Operators and Partial Lagrangians

We adapt the definition of a partial Lagrangian from Kara and Mahomed (2006), Kara, Mahomed, Naeem and Wafo Soh (2007), Naeem and Mahomed (2008a) to the approximate situation.

Definition 1.11. Suppose that equation (1.23) can be expressed as

$$E_\alpha = E_\alpha^0 + \epsilon E_\alpha^1 + \cdots + \epsilon^k E_\alpha^k, \ \alpha = 1, ..., m. \tag{1.32}$$

Also that we have a differential function $L = L(x, \ u, \ u_{(1)}, ..., u_{(r)}) \in \mathcal{A}, \ r \leq p$ and there exists non zero functions f_α^β such that equation (1.32) can be expressed as $\delta L/\delta u^\alpha = \epsilon^k f_\alpha^\beta E_\beta^k$ or more generally $\delta L/\delta u^\alpha = F_\alpha$. Then provided E_β^k are not all zero (F_α not all zero), L is said to be a *partial Lagrangian* of the system (1.32) otherwise it is a *standard Lagrangian.* The matrix (f_α^β) is invertible.

The differential equations of the form

$$\delta L/\delta u^\alpha = \epsilon^k f_\alpha^\beta E_\beta^k, \quad \alpha = 1, ..., m; \quad k = 1, ..., p \tag{1.33}$$

or

$$\delta L/\delta u^\alpha = F_\alpha, \quad \alpha = 1, ..., m \tag{1.34}$$

are termed approximate partial Euler-Lagrange equations.

Remark. For the case of singular perturbations, one can introduce stretching transformations as in Nayfeh (1979) to transform away the singular terms and to write the system as in (1.32).

The following examples are constructed to illustrate Definition 1.11.

1. The equation

$$(y' \exp(x))' + \epsilon y \exp(x) = 0, \tag{1.35}$$

has a partial Lagrangian

$$L = \frac{1}{2} y'^2 \exp(x). \tag{1.36}$$

The partial Euler-Lagrange equation of (1.35) is

$$\frac{\delta L}{\delta y} = -(y' \exp(x))' = \epsilon y \exp(x), \tag{1.37}$$

which in comparison with equation (1.33) yields $f_1^1 = \exp(x)$ and $E_1^1 = y$.

Similarly the equation

$$y'' + y' + \epsilon y = 0, \tag{1.38}$$

satisfies a partial Lagrangian $L = 1/2y'^2$, so that

$$\frac{\delta L}{\delta y} = -y'' = y' + \epsilon y. \tag{1.39}$$

The comparison of equation (1.34) and (1.39) results in $F = y' + \epsilon y$, which is upto first order in ϵ.

2. The linear system of two second-order equations

$$y'' + \epsilon y' + z\epsilon^2 = 0,$$

$$z'' + \epsilon z' + y = 0, \tag{1.40}$$

do not admit a Lagrangian. However, it possesses a partial Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2. \tag{1.41}$$

The partial Euler-Lagrange equations with the help of equation (1.34) become

$$\frac{\delta L}{\delta y} = -y'' = \epsilon y' + \epsilon^2 z = F_1,$$

$$\frac{\delta L}{\delta z} = \epsilon z' + y = F_2. \tag{1.42}$$

3. The simple two dimensional system

$$y'' + \epsilon y' + \epsilon^2 z = 0,$$

$$z'' + \epsilon^2 z = 0, \tag{1.43}$$

or

$$(y' \exp(x))' + \epsilon^2 z \exp(x) = 0,$$

$$z'' + \epsilon^2 z = 0, \tag{1.44}$$

has no Lagrangian (see Douglas 1941).

A partial Lagrangian of (1.43) or (1.44) is

$$L = \frac{1}{2}y'^2 \exp(x) + \frac{1}{2}z'^2, \tag{1.45}$$

so we have

$$\frac{\delta L}{\delta y} = \epsilon^2 z \exp(x),$$
$$\frac{\delta L}{\delta z} = \epsilon^2 z. \tag{1.46}$$

The system (1.46) in comparison with (1.33) straightforwardly yields $f_1^1 E_1^2 = z\exp(x)$ and $f_1^2 E_2^2 = 0$, which in turn results in $f_1^1 = \exp(x)$, $E_1^2 = z$, $E_2^1 = 0$, $E_2^2 = z$ and $E_1^1 = y'$.

Definition 1.12. The approximate Lie-Bäcklund operator χ of the form (1.24) is said to be an approximate partial Noether operator corresponding to a partial Lagrangian if it can be calculated from

$$\chi(L) + LD_x(\xi) = W^\beta \frac{\delta L}{\delta u^\beta} + D_x(B), \tag{1.47}$$

with respect to a suitable function defined as

$$B = B_0 + \epsilon B_1 + \cdots + \epsilon^k B_k,$$

and $W = (W^1, \cdots W^m)$, $W^\alpha \in \mathcal{A}$, is the characteristic of χ given in (1.30).

If B is identically zero, then the approximate generalized operator χ is a strict approximate partial Noether operator associated with a partial Lagrangian L.

Definition 1.13. The approximate Noether operator associated with an approximate Lie Bäcklund operator χ is

$$\mathcal{N} = \xi + W^\alpha \frac{\delta}{\delta u_x^\alpha} + \sum_{s\geq 1} D_x^s(W^\alpha)\frac{\delta}{\delta u_{s+1}^\alpha}, \tag{1.48}$$

where ξ and W are given by (1.29) and (1.30) and the Euler-Lagrange operators with respect to derivatives of u^α are obtained from

$$\frac{\delta}{\delta u_k^\alpha} = \frac{\partial}{\partial u_k^\alpha} + \sum_{s\geq 1}(-D_x)^s \frac{\partial}{\partial u_{k+s}^\alpha}, \quad \alpha = 1, \cdots, m, \quad k \geq 1. \tag{1.49}$$

The following theorem on approximate first integrals is easily adapted from Kara, Mahomed, Naeem and Wafo (2007).

Theorem 1.6. If the approximate generalized operator χ of the form (1.24) or (1.28) is an approximate partial Noether operator associated with a partial Lagrangian L of a partial Euler-Lagrange system (1.33) or (1.34), then the approximate first integrals can be constructed from

$$I = B - \mathcal{N}(L) + O(\epsilon^{k+1}), \tag{1.50}$$

and the perturbed characteristic $W = (W^1, ..., W^m)$, $W^\alpha \in \mathcal{A}$, of the perturbed k-th order operator χ is also the characteristic of the approximate conservation law $D_x I = O(\epsilon^{k+1})$.

Theorem 1.7. Let two approximate Lie-Backlund operators χ and $\tilde{\chi}$ be equivalent i.e. $\chi = \tilde{\chi} + \lambda D_x, \lambda \in \mathcal{A}$. Then χ is an approximate partial Noether operator if and only if $\tilde{\chi}$ is.

Now the question is under what conditions is the approximate partial Noether operator also an approximate symmetry generator of the approximate partial Euler-Lagrange equations? The answer of this question is provided by the following theorem.

Theorem 1.8. Let us consider $f_\alpha^\beta E_\beta^k = H_\alpha$. If the approximate operator χ of the form (1.24) or (1.28) is an approximate partial Noether operator of a Lagrangian L, then χ is the approximate Lie-Backlund symmetry operator of the corresponding approximate partial Euler-Lagrange equations $\delta L/\delta u^\alpha = H_\alpha$ if

$$\chi H_\alpha = \xi D_x H_\alpha + (D_H^* W)_\alpha, \tag{1.51}$$

where D_H^* is the adjoint of the Frechet derivative of H with $H = (H_1, \cdots, H_m)$.

Now we come to the situation when the approximate partial Noether operators form an approximate Lie algebra.

Theorem 1.9. If $\bar{\chi_1}$ and $\bar{\chi_2}$ are approximate partial Noether operators in canonical form corresponding to 'gauge' terms $\bar{B}$ and $\bar{C}$, then

$$[\bar{\chi_1}, \bar{\chi_2}]L = D_x(\bar{\chi_1}\bar{C} - \bar{\chi_2}\bar{B}) + (\bar{\chi_1}\chi_2^\alpha - \bar{\chi_2}\chi_1^\alpha)\frac{\delta L}{\delta u^\alpha} + (\chi_2^\alpha\bar{\chi_1} - \chi_1^\alpha\bar{\chi_2})\frac{\delta L}{\delta u^\alpha}. \tag{1.52}$$

The commutator, in general, does not yield an approximate partial Noether operator from equation (1.52). In order to see the closure of the Lie algebra, we present the following theorem.

Theorem 1.10. If χ_1 and χ_2 are approximate partial Noether operators corresponding to a partial Lagrangian L for a scalar partial Euler Lagrange equation such that $\delta L/\delta y$ is independent of derivatives $u_l, l \geq 1$, then $[\bar{\chi_1}, \bar{\chi_2}]$ becomes a partial Noether operator of L.

Chapter 2

Partial Noether Operators and First Integrals via Partial Lagrangians for some Systems

A systematic way of constructing first integrals for ODEs without a variational principle is investigated for some equations. Using the idea of a partial Lagrangian, a Noether-like theorem (see Chapter 1) which gives the first integrals corresponding to partial Euler-Lagrange equations is invoked. The formula which provides the first integrals is similar to the Noether's theorem but the invariance condition to determine partial Noether operators is different as a consequence of a partial Lagrangian and partial Euler-Lagrange equations. The reader is referred to the previous chapter. Applications given here include those that admit a standard Lagrangian such as the harmonic oscillator, modified Emden and Ermakov-Pinney equations and systems of two second-order equations that do not have standard Lagrangians.

2.1 Introduction

The relationship between Noether symmetries and first integrals has been a subject of rigorous investigation for Euler-Lagrange equations (see the works of Noether 1918 and later works Olver 1986, Sarlet and Cantrijn 1981, Ibragimov et al 1998). The classical

Noether's theorem (1918) possesses the beauty in the elegant explicit formula for the construction of the first integrals once the Noether symmetries are known. It establishes a relationship between equivalence class of point symmetries and first integrals. In order to use this powerful theorem one needs a Lagrangian to obtain the Noether symmetries and to construct first integrals. There are equations that arise in applications which do not admit Lagrangians, e.g.

$$y'' = y^2 + z^2,\ z'' = 0,$$

and

$$y'' = y^2 + z^2,\ z'' = y.$$

For information on the classification of Lagrangians, the interested reader is referred to the paper Douglas (1941), in which the author has provided the complete solution to the inverse problem for the system of two second-order ODEs (3-dimensional space). The procedure relies on Riquier theory of systems of partial differential equations. The classification is made of all curve families that are extremal and nonextremal and the complete solution is obtained with respect to all the possible cases of the differential system. Now the question is how to find first integrals in the absence of a Lagrangian for such systems as mentioned above. There are other methods as well for obtaining first integrals which do not make use or even assume existence of a Lagrangian. The most elementary method is the so-called direct method. This approach, even when invoked for a Hamiltonian system, has proved fruitful in a number of applications in mechanics (see, e.g. Hietarinta 1986 and Lewis and Leach 1982). Indirect methods have also been used (see, e.g. Hietarinta 1986, Leach 1987). There are yet other methods as well. The first of these involves writing the conservation law in characteristic form and is due to Steudel (1962) (see also Olver 1993, Sarlet and Cantrijn 1981, Moyo and Leach 2005). Here the characteristics are the multipliers of the system. In order to find the first integrals one has to obtain the characteristics as well. The second of these uses the variational derivative. In this approach one first calculates the characteristics and then from these the first integrals. The work of Anco and Bluman (1998) uses the latter approach and gives an algorithm for the determination of first integrals. There have been many papers dealing with symmetries and first integrals. In Kara and Mahomed (2000), the relationship between symmetries and first integrals, irrespective of the existence of a Lagrangian of the system, is explored in a general setting. This approach though was prevalent even earlier (see, e.g. Leach 1981) albeit in a particular context.

Here we utilise the powerful approach of the partial Noether theorem which also has an explicit formula for the determination of first integrals (see Chapter 1) as the Noether theorem. We consider many examples.

2.2 Applications

Ample examples are given starting with the simple harmonic oscillator, which is considered a paradigm of mechanical systems, as well as the modified Emden and Ermakov-Pinney equations. These equations admit standard Lagrangian formulations. We show that they have partial Lagrangians too that result in first integrals. To show that our method works in more general settings, we have included examples of equations that do not have standard Lagrangians. However, these do admit partial Lagrangians and lead to first integrals using our partial Noether operators. These equations are well-known and our purpose is to provide an alternative and systematic way of constructing first integrals for such systems. As is known, first integrals are useful, e.g. in stability analysis of the system. We do not look at this aspect here.

Example 1 (Simple harmonic oscillator)
The equation

$$u'' + u = 0 \tag{2.1}$$

is familiar and its Lie point symmetry generators are well-known as well as the standard Lagrangian picture has been considered in Lutzky (1978). Here, we take an alternative view by considering its partial Lagrangian $L = u'^2/2$ so that (2.1) becomes

$$u = \frac{\delta L}{\delta u}. \tag{2.2}$$

The partial Noether operators $X = \xi\partial/\partial x + \eta\partial/\partial u$ corresponding to L satisfy (1.19), i.e.,

$$[\eta_x + \eta_u u' - u'\xi_x - u'^2\xi_u]u' + (\xi_x + \xi_u u')\frac{1}{2}u'^2 = \eta u - u'\xi u + B_x + u'B_u. \tag{2.3}$$

The usual separation by derivatives of u gives

$$\begin{aligned}
u'^3 :&\quad \xi = a(x)\\
u'^2 :&\quad \eta = \frac{1}{2}a'u + b(x)\\
u' :&\quad \eta_x = -\xi u + B_u\\
u'^0 :&\quad 0 = \eta u + B_x.
\end{aligned} \tag{2.4}$$

System (2.4) yields

$$\begin{aligned}
B &= -\frac{1}{2}a_1u^2\sin 2x - \frac{1}{2}a_2u^2\cos 2x + \frac{1}{2}a_0u^2 + b_1u\cos x - b_2u\sin x,\\
\xi &= a_1\sin 2x + a_2\cos 2x + \frac{1}{4}a_0,\\
\eta &= u(a_1\cos 2x - a_2\sin 2x) + b_1\sin x + b_2\cos x.
\end{aligned} \tag{2.5}$$

which are the same as found earlier in Lutzky (1978). The first integrals by Theorem 1.1 are

$$\begin{aligned}
&I_1 = \frac{1}{2}u'^2 + \frac{1}{2}u^2, \quad I_2 = -uu'\cos 2x + \frac{1}{2}u'^2\sin 2x - \frac{1}{2}u^2\sin 2x,\\
&I_3 = uu'\sin 2x + \frac{1}{2}u'^2\cos 2x - \frac{1}{2}u^2\cos 2x, \quad I_4 = -u'\sin x + u\cos x,\\
&I_5 = -u'\cos x - u\sin x\,.
\end{aligned} \tag{2.6}$$

So that the conservation law in each case is $D_xI = W(-u'' - u) = 0$ with respective characteristic W. Here the partial Lagrangian gave all nontrivial conservation laws as obtained in Lutzky (1978). The difference here lies in the forms of B and L which are distinct from the ones used in the Noether approach.

Example 2 (Ermakov-Pinney equation)

The equation

$$u'' = c/u^3, \quad c = \text{ constant}, \tag{2.7}$$

is an example of the well-known Ermakov-Pinney equation. It has partial Lagrangian $L = u'^2/2$ and the partial Noether operator $X = \xi\partial/\partial x + \eta\partial/\partial u$ by (1.19) satisfy (note $\delta L/\delta u = -u'' = -c/u^3$)

$$\zeta_x u' + (D_x\xi)\frac{1}{2}u'^2 = W(-c/u^3) + D_x B. \tag{2.8}$$

Equation (2.8) after usual expansion yields

$$(\eta_x + \eta_u u')u' + \frac{1}{2}u'^2(\xi_x + \xi_u u') = (\eta - u'\xi)(-c/u^3) + B_x + B_u u'. \tag{2.9}$$

Here we obtain

$$\xi = A_0 + A_1 x + A_2 x^2, \quad \eta = \frac{1}{2}A_1 u + A_2 x u,$$
$$B = \frac{1}{2}A_2 u^2 + \frac{1}{2}c u^{-2}(A_0 + A_1 x + A_2 x^2) + C_0 \tag{2.10}$$

that gives rise to the three-dimensional Noether algebra $sl(2,R)$ which is isomorphic to the Lie algebra. The first integrals are

$$I_1 = \frac{1}{2}u'^2 + \frac{1}{2}cu^{-2}, \quad I_2 = \frac{1}{2}xu'^2 - \frac{1}{2}uu' + \frac{1}{2}cu^{-2}x,$$
$$I_3 = \frac{1}{2}x^2u'^2 - xuu' + \frac{1}{2}u^2 + \frac{1}{2}cx^2u^{-2}. \tag{2.11}$$

Again the differences reside in B and L, they are not the same as for the usual Noether approach.

Example 3 (Modified Emden equation, see Leach (1985) for the Noether symmetries)
A particular example of the general modified Emden equation studied in Leach (1985) is the following one

$$u'' + \frac{2}{x}u' = 3u^5. \tag{2.12}$$

Here we take $L = \frac{1}{2}u'^2$ again (as in Example 2). So we have

$$\frac{\delta L}{\delta u} = -u'' = \frac{2}{x}u' - 3u^5.$$

The operator $X = \xi\partial/\partial x + \eta\partial/\partial u$ is known as the partial Noether operator corresponding to the partial Lagrangian as given above for the equation (2.12) if it holds

$$(\eta_x + \eta_u u')u' + (\xi_x + \xi_u u')(1/2u'^2) = (\eta - u'\xi)(\frac{2}{x}u' - 3u^5) + B_x + B_u u'. \tag{2.13}$$

The separation of equation (2.13) with respect to powers of u' gives rise to

$$
\begin{aligned}
u'^3: &\quad \xi_u = 0 \\
u'^2: &\quad \eta_u + \frac{1}{2}\xi_x = -\frac{2}{x}\xi \\
u': &\quad \eta_x = \frac{2}{x}\eta - 3u^5\xi + B_u \\
u'^0: &\quad 0 = -3u^5\eta + B_x.
\end{aligned} \tag{2.14}
$$

The determining equations (2.14) give

$$
\xi = Ax^3, \quad \eta = -\frac{1}{2}Aux^2, \quad B = -\frac{1}{2}x^3u^6. \tag{2.15}
$$

The partial Noether operator is

$$
X = 2x^3\frac{\partial}{\partial x} - ux^2\frac{\partial}{\partial u} \tag{2.16}
$$

(note that it is not a Lie symmetry of the equation!) and the first integral is

$$
I = x^3u'^2 + x^2uu' - \frac{1}{2}x^3u^6. \tag{2.17}
$$

The same integral (2.17) arises in the Noether approach of Leach (1985).

Example 4 (linear system)

The simple linear system of two second-order equations

$$
y'' + z' = 0, \quad z'' + z = 0, \tag{2.18}
$$

has no Lagrangian (see Douglas 1941 and Hojman 1981). However, it does have a partial Lagrangian

$$
L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{1}{2}z^2. \tag{2.19}
$$

The partial Euler-Lagrange equations are

$$
\frac{\delta L}{\delta y} = z', \quad \frac{\delta L}{\delta z} = 0. \tag{2.20}
$$

The partial Noether operators associated with (2.19)

$$
X = \xi\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial y} + \eta^2\frac{\partial}{\partial z} + \zeta^1_x\frac{\partial}{\partial y'} + \zeta^2_x\frac{\partial}{\partial z'} \tag{2.21}
$$

are obtained from (see Definition 1.8)

$$XL + (D_x\xi)L = W^1\frac{\delta L}{\delta y} + D_xB, \tag{2.22}$$

which straightforwardly yields

$$\zeta^1_x y' + \zeta^2_x z' - \eta^2 z + (D_x\xi)L = W^1\frac{\delta L}{\delta y} + D_xB. \tag{2.23}$$

The simple expansion of (2.23) gives

$$\Big[\eta^1_x + \eta^1_y y' + \eta^1_z z' - y'(\xi_x + \xi_y y' + \xi_z z')\Big]y' + \Big[\eta^2_x + \eta^2_y y' + \eta^2_z z' - z'(\xi_x + \xi_y y' + \xi_z z')\Big]z'$$

$$-\eta^2_z + (\xi_x + \xi_y y' + \xi_z z')(\frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{1}{2}z^2) = (\eta^1 - y'\xi)z' + B_x + B_y y' + B_z z'. \tag{2.24}$$

The separation in terms of derivatives gives the determining equations

$$\begin{aligned}
&\xi_y = 0, \quad \xi_z = 0, \quad \eta^1_y = \tfrac{1}{2}\xi_x, \quad \eta^2_z = \tfrac{1}{2}\xi_x,\\
&\eta^1_z + \eta^2_y + \xi = 0, \quad \eta^1_x = B_y, \quad \eta^2_x = \eta^1 + B_z,\\
&\eta^2 z + \tfrac{1}{2}z^2\eta_x + B_x = 0.
\end{aligned} \tag{2.25}$$

The solution of system (2.25) results in

$$\begin{aligned}
&\xi = c_0, \quad \eta^1 = -c_0 z + c_1 x + c_2,\\
&\eta^2 = c + 1 + c_3\sin x + c_4\cos x,\\
&B = yc_1 + \tfrac{1}{2}c_0 z^2 + z(c_3\cos x - c_4\sin x) - z(c_1 x + c_2)
\end{aligned} \tag{2.26}$$

which in turn gives rise to the partial Noether operators

$$X_1 = \frac{\partial}{\partial x} - z\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$X_3 = \frac{\partial}{\partial y}, \quad X_4 = \sin x\frac{\partial}{\partial z}, \quad X_5 = \cos x\frac{\partial}{\partial z}. \tag{2.27}$$

Operators (2.27) do not form a Lie algebra as can easily be verified. Nor are the operators, except for X_3, generators of symmetry of system (1). This can be checked from the Lie symmetry condition or by Theorem 1.3.

The corresponding first integrals, using Theorem 1.1, are

$$
\begin{aligned}
&I_1 = z^2 + zy' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2, \quad I_2 = y - xz - xy' - z', \\
&I_3 = z + y', \quad I_4 = z\cos x - z'\sin x, \quad I_5 = -z\sin x - z'\cos x. \qquad (2.28)
\end{aligned}
$$

These integrals (2.28) easily provides the general solution of system (2.18). The integrals I_3, I_4 and I_5 are in fact directly evident from the system (2.18) itself. The point here is that our approach leads to that which is expected as well as more in an algorithmic way. The characteristics in each case are also consequences of our method.

Example 5 (nonlinear system)

The nonlinear system

$$
y'' = y^2 + z^2, \quad z'' = 0, \qquad (2.29)
$$

admits no Lagrangian (see Douglas 1941). A partial Lagrangian of (2.29) is

$$
L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 \qquad (2.30)
$$

which has partial Euler-Lagrange equations

$$
\frac{\delta L}{\delta y} = -y^2 - z^2, \quad \frac{\delta L}{\delta z} = 0. \qquad (2.31)
$$

The partial Noether operators (2.21) associated with (2.30) are calculated from (1.19). The resulting equation is

$$
\zeta^1_x y' + \zeta^2_x z' + (\xi_x + \xi_y y' + \xi_z z')(\frac{1}{2}y'^2 + \frac{1}{2}z'^2) = (\eta^1 - y'\xi)\frac{\delta L}{\delta y} + B_x + B_y y' + B_z z'. \quad (2.32)
$$

Simple expansion shows that

$$
\begin{aligned}
&\left[\eta^1_x + \eta^1_y y' + \eta^1_z z' - y'(\xi_x + \xi_y y' + \xi_z z')\right]y' + \left[\eta^2_x + \eta^2_y y' + \eta^2_z z' - z'(\xi_x + \xi_y y' + \xi_z z')\right]z' \\
&+(\xi_x + \xi_y y' + \xi_z z')(\frac{1}{2}y'^2 + \frac{1}{2}z'^2) = -(\eta^1 - y'\xi)(y^2 + z^2) + B_x + B_y y' + B_z z'. \quad (2.33)
\end{aligned}
$$

The determining equations after separation in terms of derivatives are

$$
\begin{aligned}
&\xi_y = 0, \quad \xi_z = 0, \quad \eta^1_y = \tfrac{1}{2}\xi_x, \quad \eta^2_z = \tfrac{1}{2}\xi_x, \\
&\eta^1_z + \eta^2_y = 0, \quad \eta^1_x = \xi(y^2 + z^2) + B_y, \quad \eta^2_x = B_z, \quad \eta^1(y^2 + z^2) = B_x.
\end{aligned} \qquad (2.34)
$$

The solution of (2.34) is

$$\xi = 0, \quad \eta^1 = 0, \quad \eta^2 = c_0 + c_1 x, \quad B = c_1 z \tag{2.35}$$

and corresponding first integrals are

$$I_1 = -z', \quad I_2 = z - xz'. \tag{2.36}$$

These integrals again should be obvious from just looking at system (2.29). However, one would not be sure if there are more. Now, since we know the solutions $z'' = 0$ of (2.29), we can substitute these into $y'' = y^2 + z^2$ of (2.29). This gives

$$y'' = y^2 + (z_0 + z_1 x)^2. \tag{2.37}$$

A partial Lagrangian of (2.37) is

$$L = \frac{1}{2}y'^2 + \frac{1}{3}y^3 \tag{2.38}$$

so that $\delta L/\delta y = -(z_0 + z_1 x)^2$. The solution of condition (1.19) in this case results in

$$\xi = c_0, \quad \eta = 0, \quad B = -c_0 z_0^2 y \tag{2.39}$$

provided that $z_1 = 0$, i.e. for the autonomous equation (2.37). There is no partial Noether operator for $z_1 \neq 0$. The corresponding first integral, for $z_1 = 0$, is

$$J = \frac{1}{2}y'^2 - \frac{1}{3}y^3 - z_0^2 y. \tag{2.40}$$

Going back to the original system (2.29), we have obtained its configurational invariants (see Sarlet, Leach and Cantrijn 1985 for a detailed discussion on these) given by (2.40) and (2.36) in which we have set $z' = 0$. Note that the scalar equation (2.37) has no Lie point symmetry for $z_1 \neq 0$.

2.3 Concluding remarks

We have shown how for various systems of ordinary differential equations which admit partial Lagrangians, one can construct first integrals via partial Noether operators by means of a formula. These partial Noether operators are in general not symmetry generators of the underlying equations as we have seen. Moreover, the partial Noether operators do not form a Lie algebra in general. We have applied our results to simple examples where a Lagrangian exists as well as for differential equations which do not admit standard Lagrangians but have partial Lagrangians.

Chapter 3

Noether, Partial Noether Operators and First Integrals for a Linear System

We obtain Noether and partial Noether operators corresponding to a Lagrangian and a partial Lagrangian for a system of two linear second-order ordinary differential equations (ODEs) with variable coefficients. The canonical form for a system of two second-order ordinary differential equations is invoked and a special case of this system is studied for both Noether and partial Noether operators. Then the first integrals with respect to Noether and partial Noether operators are obtained for the linear system under consideration. We show that the first integrals for both the Noether and partial Noether operators are the same. This can give rise to further studies on systems from a partial Lagrangian viewpoint as systems in general do not admit Lagrangians. This further aspect is presented in Chapter 4 for a general linear system.

3.1 Introduction

The idea of a partial Lagrangian, partial Euler-Lagrange equations was already presented in Chapter 2 in which we have deduced the partial Noether theorem for ODEs and we showed how one can construct first integrals for equations without variational structures.

First integrals are important both to reduce the order of the equations and from the physical point of view (see, e.g. Goldstein 1950). Since a Lagrangian does not exist in all the cases; for instance in xyz space for the system $y'' = y^2 + z^2,\ z'' = y$, the curve family is non extremal (see Douglas 1941). However, there are methods, as mentioned earlier, for obtaining first integrals which do not make use or even assume existence of a Lagrangian. For such approaches the reader is referred to Hietarinta (1986), Lewis and Leach (1982), Leach (1987), Steudel (1962), Olver (1996), Anco and Bluman (1998) and Kara and Mahomed (2000), Kara and Mahomed (2006), Ibragimov (2006) amongst others.

The purpose of this work is twofold: one is simply to add to the algebraic properties obtained before in the works cited and find Noether symmetries corresponding to a Lagrangian of a system of two variable coefficient equations and secondly to obtain partial Noether operators associated with what is termed a partial Lagrangian of the same system. The reason being that a system of two equations need not have a Lagrangian formulation (see Douglas 1941). So we wish to investigate whether the first integrals arising from a Lagrangian formulation and that of a partial Lagrangian is the same for the system under study. This can then give rise to further studies on first integrals of a more general system of two second-order ODEs using the notion of partial Lagrangians as they in general do not admit standard Lagrangians (see Chapter 4).

The outline of this chapter is as follows. Section 3.2 is related to canonical form of a system of two second-order ODEs. In Sections 3.3 and 3.4 we discuss Noether and partial Noether operators, respectively. First integrals corresponding to Noether and partial Noether operators are given in Section 3.5.

3.2 Canonical form

For the algebraic classification of the general linear equation difficulties arise due to a huge number of arbitrary elements. In general for a system of n second-order nonhomogeneous linear ODEs, there are $2n^2+n$ arbitrary elements. Since the invertible transformations do not affect the number of symmetries, one can obtain a system in simpler form (see Wafo and Mahomed 2000) before performing group classification and in searching for Noether symmetries.

The system of n second-order nonhomogeneous linear ODEs

$$\mathbf{x}'' = A\mathbf{x}' + B\mathbf{x} + \mathbf{c} \tag{3.1}$$

can be mapped invertibly to one of the forms given below (see Wafo and Mahomed 2000)

$$\mathbf{y}'' = \bar{A}\mathbf{y}', \tag{3.2}$$

$$\mathbf{z}'' = \bar{B}\mathbf{z}, \tag{3.3}$$

where $A,\ B,\ \bar{A},\ \bar{B}$ are $n \times n$ matrices and $\mathbf{x},\ \mathbf{y},\ \mathbf{z}$ and $\mathbf{c}$ are vectors. Now we discuss the above theorem for the case $n = 2$. In this case the number of arbitrary elements reduces from $2 \times 2^2 + 2 = 10$ to 4.

Any system of two linear second-order ODEs maps invertibly to the linear system (Wafo and Mahomed 2000)

$$\begin{cases} y'' = a(x)y + b(x)z, \\ z'' = c(x)y - a(x)z. \end{cases} \tag{3.4}$$

Here we consider the special case $b(x) = c(x)$ for which a Lagrangian exists. Moreover, we wish to also construct partial Noether operators via a partial Lagrangian for the same special case in order to compare. Note that in general (3.4) does not have a Lagrangian (see Douglas 1941). The above system then reduces to

$$\begin{cases} y'' = a(x)y + b(x)z \\ z'' = b(x)y - a(x)z. \end{cases} \tag{3.5}$$

3.3 Noether operators of (3.5)

The operator X given in (1.4) is called a Noether symmetry operator corresponding to Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 + \frac{1}{2}ay^2 - \frac{1}{2}az^2 + byz, \tag{3.6}$$

for a system of two linear second-order ordinary differential equations (3.5) if it satisfies (1.13) with respect to some function $B(x,u)$. The operator X satisfies the following system

$$\xi_y = 0,\ \xi_z = 0, \tag{3.7}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0,\ \eta^2_z - \frac{1}{2}\xi_x = 0,\ \eta^1_z + \eta^2_y = 0, \tag{3.8}$$

$$\eta^1_x + \frac{1}{2}ay^2\xi_y - \frac{1}{2}az^2\xi_y + byz\xi_y = B_y, \tag{3.9}$$

$$\eta^2_x + \frac{1}{2}ay^2\xi_z - \frac{1}{2}az^2\xi_z + byz\xi_z = B_z, \tag{3.10}$$

$$(\frac{1}{2}a'y^2 - \frac{1}{2}a'z^2 + b'yz)\xi + \eta^1(ay+bz) + \eta^2(-az+by)$$
$$+\frac{1}{2}ay^2\xi_x - \frac{1}{2}az^2\xi_x + byz\xi_x = B_x. \tag{3.11}$$

The solution of (3.7)-(3.9) yields

$$\xi = \alpha(x), \tag{3.12}$$

$$\eta^1 = \frac{1}{2}\alpha' y + \beta(x,z), \tag{3.13}$$

$$\eta^2 = \frac{1}{2}\alpha' z + \gamma(x,y), \tag{3.14}$$

$$\beta_z + \gamma_y = 0, \tag{3.15}$$

$$B = \frac{1}{4}\alpha'' y^2 + y\beta_x + S(x,z). \tag{3.16}$$

The substitution of (3.12)-(3.16) in (3.9) and (3.10) gives

$$(\frac{1}{2}a'y^2 - \frac{1}{2}a'z^2 + b'yz)\alpha + (\frac{1}{2}\alpha'y + \beta(x,z))(ay+bz) + (\frac{1}{2}\alpha'z + \gamma(x,y))(-az+by)$$
$$+\frac{1}{2}ay^2\alpha' - \frac{1}{2}az^2\alpha' + byz\alpha' = \frac{1}{4}\alpha''' y^2 + y\beta_{xx} + S_x, \tag{3.17}$$

$$\frac{1}{2}\alpha''z + \gamma_x = S_z. \tag{3.18}$$

Equation (3.15) results in

$$\beta = -\gamma_y z + A(x). \tag{3.19}$$

The differentiation of (3.19) with respect to y and then integration gives rise to

$$\gamma = C_1(x)y + C_2(x). \tag{3.20}$$

The replacement of the value of γ in (3.19) gives

$$\beta = -C_1(x)z + A(x). \tag{3.21}$$

The substitution of β and γ from (3.20) and (3.21) in (3.16) and (3.18) yields

$$B = \frac{1}{4}\alpha''y^2 + y(-C_1'(x)z + A'(x)) + S(x,z), \tag{3.22}$$

where

$$S = \frac{1}{4}\alpha''z^2 + C_2'(x)z + C_3(x),\ C_1(x) = A_1. \tag{3.23}$$

Equation (3.17) reduces to the following system by substituting the values of β and S from (3.21) and (3.23)

$$\frac{1}{4}\alpha''' + \frac{1}{2}a'\alpha + a\alpha' + A_1 b = 0, \tag{3.24}$$

$$2b\alpha' + b'\alpha - 2aA_1 = 0, \tag{3.25}$$

$$C_2'''(x) + aC_2(x) = bA(x), \tag{3.26}$$

$$\frac{1}{4}\alpha''' - \frac{1}{2}a'\alpha - a\alpha' - A_1 b = 0, \tag{3.27}$$

$$A'' - aA(x) = bC_2(x), \tag{3.28}$$

$$C_3'(x) = 0. \tag{3.29}$$

Equation (3.29) implies that

$$C_3(x) = c_0. \tag{3.30}$$

The addition of (3.24) and (3.27) results in $\alpha''' = 0$, which in turn results in

$$\alpha = A_2 + A_3 x + A_4 x^2. \tag{3.31}$$

The insertion of α in (3.24), (3.25) and (3.27) gives

$$\frac{1}{2} a'(A_2 + A_3 x + A_4 x^2) + a(A_3 + 2A_4 x) + A_1 b = 0, \tag{3.32}$$

$$2b(A_3 + 2A_4 x) + b'(A_2 + A_3 x + A_4 x^2) - 2aA_1 = 0. \tag{3.33}$$

The following cases arise.

Case I: a, b are arbitrary.

In this case we find that

$$\alpha(x) = 0 \, , C_1(x) = 0,$$

$$A(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) + \alpha_3 u_3(x) + \alpha_4 u_4(x),$$

$$C_2(x) = \alpha_1 v_1(x) + \alpha_2 v_2(x) + \alpha_3 v_3(x) + \alpha_4 v_4(x). \tag{3.34}$$

For this case we get the symmetry generators in the general form as

$$X_i = u_i \frac{\partial}{\partial y} + v_i \frac{\partial}{\partial z}, \ i = 1, ..., 4, \tag{3.35}$$

where (u_i, v_i) are linearly independent solutions of our system (3.5) and

$$B = yA'(x) + zC_2'(x). \tag{3.36}$$

Case 2: $a = a_0, \ b = b_0, \ a_0$ and b_0 are constants.

The substitution of a and b in (3.26), (3.28), (3.32) and (3.33) gives

$$a_0(A_3 + 2A_4 x) + A_1 b_0 = 0, \tag{3.37}$$

$$2b_0(A_3 + 2A_4 x) - 2a_0 A_1 = 0, \tag{3.38}$$

$$A'' - a_0 A = b_0 C_2(x), \tag{3.39}$$

$$C_2''(x) + a_0 C_2(x) = b_0 A(x). \tag{3.40}$$

The following subcases arise for Case 2.

Case 2.1: If $a_0 \neq 0,\ \ b_0 \neq 0$, then from the above system we get

$$A_1 = 0, A_3 = 0, A_4 = 0,$$

$$C_2(x) = A_5 \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x) + A_6 \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x)$$
$$+A_7 \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x + A_8 \sin(a_0^2 + b_0^2)^{\frac{1}{4}}\ x,$$
$$A(x) = \frac{1}{b_0}[A_5(\sqrt{a_0^2 + b_0^2} + a_0) \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x)$$
$$+A_6(\sqrt{a_0^2 + b_0^2} + a_0) \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x) + A_7(-\sqrt{a_0^2 + b_0^2} + a_0) \cos(a_0^2 + b_0^2)^{\frac{1}{4}}$$
$$+A_8(-\sqrt{a_0^2 + b_0^2} + a_0) \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x]. \tag{3.41}$$

In each case the symmetries and B are given by setting one constant equal to one and the others equal to zero.

$$X_1 = \frac{\partial}{\partial x},\ B = 0,$$
$$X_2 = \frac{1}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x)\frac{\partial}{\partial y} + \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x)\frac{\partial}{\partial z},$$
$$B = \frac{y}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0)(a_0^2 + b_0^2)^{\frac{1}{4}} \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x)$$
$$+z(a_0^2 + b_0^2)^{\frac{1}{4}} \exp(a_0^2 + b_0^2)^{\frac{1}{4}} x,$$
$$X_3 = \frac{1}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x)\frac{\partial}{\partial y}$$
$$+\exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x)\frac{\partial}{\partial z},$$
$$B = \frac{-y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0^2} + a_0) \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x)$$
$$-z(a_0^2 + b_0^2)^{\frac{1}{4}} \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x),$$
$$X_4 = \frac{1}{b_0}(-\sqrt{a_0^2 + b_0^2} + a_0) \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x\frac{\partial}{\partial y} + \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x\frac{\partial}{\partial z},$$
$$B = \frac{-y}{b_0}(-\sqrt{a_0^2 + b_0^2} + a_0)(a_0^2 + b_0^2)^{\frac{1}{4}} \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x$$
$$-z(a_0^2 + b_0^2)^{\frac{1}{4}} \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x,$$

$$X_5 = \frac{1}{b_0}(-\sqrt{a_0^2+b_0^2}+a_0)\sin(a_0^2+b_0^2)^{\frac{1}{4}}x\frac{\partial}{\partial y} + \sin(a_0^2+b_0^2)^{\frac{1}{4}}x\frac{\partial}{\partial z},$$

$$B = \frac{y}{b_0}(a_0^2+b_0^2)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0^2}+a_0)\cos(a_0^2+b_0^2)^{\frac{1}{4}}x$$

$$+z(a_0^2+b_0^2)^{\frac{1}{4}}\cos(a_0^2+b_0^2)^{\frac{1}{4}}x. \quad (3.42)$$

Case 2.2: If $a_0 = 0,\ b_0 \neq 0$, then from (3.37)-(3.40), we have

$$A_1 = 0,\ A_3 = 0,\ A_4 = 0,$$

$$C_2(x) = A_9\exp(\sqrt{b_0}x) + A_{10}\exp(-\sqrt{b_0}x) + A_{11}\cos\sqrt{b_0}x$$

$$+A_{12}\sin\sqrt{b_0}x,$$

$$A(x) = A_9\exp(\sqrt{b_0}x) + A_{10}\exp(-\sqrt{b_0}x) - A_{11}\cos\sqrt{b_0}x$$

$$-A_{12}\sin\sqrt{b_0}x. \quad (3.43)$$

Straightforward manipulations yield the operators

$$X_1 = \frac{\partial}{\partial x},\ B = 0,$$

$$X_2 = \exp(\sqrt{b_0}x)\frac{\partial}{\partial y} + \exp(\sqrt{b_0}x)\frac{\partial}{\partial z},$$

$$B = y\sqrt{b_0}\exp(\sqrt{b_0}x) + z\sqrt{b_0}\exp(\sqrt{b_0}x),$$

$$X_3 = \exp(-\sqrt{b_0}x)\frac{\partial}{\partial y} + \exp(-\sqrt{b_0}x)\frac{\partial}{\partial z},$$

$$B = -y\sqrt{b_0}\exp(-\sqrt{b_0}x) - z\sqrt{b_0}\exp(-\sqrt{b_0}x),$$

$$X_4 = -\cos\sqrt{b_0}\ x\frac{\partial}{\partial y} + \cos\sqrt{b_0}\ x\frac{\partial}{\partial z},$$

$$B = y\sqrt{b_0}\sin\sqrt{b_0}x - z\sqrt{b_0}\sin\sqrt{b_0}x,$$

$$X_5 = -\sin\sqrt{b_0}\ x\frac{\partial}{\partial y} + \sin\sqrt{b_0}\ x\frac{\partial}{\partial z},$$

$$B = -y\sqrt{b_0}\cos\sqrt{b_0}\ x + z\sqrt{b_0}\cos\sqrt{b_0}\ x. \quad (3.44)$$

Case 2.3: $a_0 \neq 0,\ b_0 = 0.$

System (3.37)-(3.40) becomes

$$A_1 = 0,\ A_3 = 0,\ A_4 = 0,$$

$$A(x) = A_{13}\exp(\sqrt{a_0}x) + A_{14}\exp(-\sqrt{a_0}x),$$

$$C_2(x) = A_{15}\cos\sqrt{a_0}x + A_{16}\sin\sqrt{a_0}x. \tag{3.45}$$

The generators are

$$X_1 = \frac{\partial}{\partial x},\ B = 0,\ X_2 = \exp(\sqrt{a_0}\ x)\frac{\partial}{\partial y},\ B = y\sqrt{a_0}\exp(\sqrt{a_0}x),$$

$$X_3 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y},\ B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x),$$

$$X_4 = \cos\sqrt{a_0}x\frac{\partial}{\partial z},\ B = -z\sqrt{a_0}\sin\sqrt{a_0}x,$$

$$X_5 = \sin\sqrt{a_0}x\frac{\partial}{\partial z},\ B = z\sqrt{a_0}\cos(\sqrt{a_0}x). \tag{3.46}$$

Case 2.4: $a_0 = 0,\ b_0 = 0.$

In this case the operators and guage terms are

$$X_1 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},\ B = 0,$$

$$X_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z},\ B = \frac{1}{2}(y^2 + z^2),$$

$$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z},\ B = 0,$$

$$X_4 = \frac{\partial}{\partial x},\ B = 0,$$

$$X_5 = x\frac{\partial}{\partial y},\ B = y,\ X_6 = \frac{\partial}{\partial y},\ B = 0,$$

$$X_7 = x\frac{\partial}{\partial z},\ B = z,\ X_8 = \frac{\partial}{\partial z},\ B = 0.$$

Case 3: $a = a_0,\ b \neq$ constant.

The insertion of a and b in (3.26), (3.28), (3.32) and (3.33) yields

$$a_0(A_3 + 2A_4x) + A_1b = 0,$$

$$2a_0A_1 = 2b(A_3 + 2A_4x) + b'(A_2 + A_3x + A_4x^2),$$

$$A'' - a_0A = bC_2(x),$$

$$C_2'' + a_0C_2 = bA(x). \tag{3.47}$$

The following subcases arise for Case 3.

Case 3.1: If $a_0 \neq 0$ and $b \neq$ constant, then

$$A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = 0,$$

For this case we get the same generators as in Case 1.

Case 3.2: $a_0 = 0,\ b \neq$constant.
In this case, we get

$$A_1 = 0,\ b = d_1(A_2 + A_3x + A_4x^2)^{-2},$$

where d_1 is a constant and

$$\alpha = A_2 + A_3x + A_4x^2,$$

$$A(x) = \alpha_5u_1(x) + \alpha_6u_2(x) + \alpha_7u_3(x) + \alpha_8u_4(x), \tag{3.48}$$

$$C_2(x) = \alpha_5v_1(x) + \alpha_6v_2(x) + \alpha_7v_3(x) + \alpha_8v_4(x). \tag{3.49}$$

The generators are

$$X_i = \alpha(x)\frac{\partial}{\partial x} + u_i\frac{\partial}{\partial y} + v_i\frac{\partial}{\partial z},\ i = 1, ..., 4, \tag{3.50}$$

$$B = \frac{1}{2}(y^2 + z^2)A_4 + yA'(x) + zC_2'(x). \tag{3.51}$$

Case 4: $a \neq$ constant, $b = b_0$.

The replacement of a and b in (3.26), (3.28), (3.32) and (3.33) gives

$$\frac{1}{2}a'(A_2 + A_3x + A_4x^2) + a(A_3 + 2A_4x) + A_1b_0 = 0, \tag{3.52}$$

$$2b_0(A_3 + 2A_4x) - 2aA_1 = 0, \tag{3.53}$$

$$A'' - aA = b_0C_2(x), \tag{3.54}$$

$$C_2'' + aC_2 = b_0A(x). \tag{3.55}$$

This case is similar to Case 3 and we have

Case 4.1: $a \neq$ constant, $b_0 \neq 0$.

$$A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = 0, \tag{3.56}$$

For this case we get exactly the same generators and B as given in Case 1.

Case 4.2: $a \neq$ constant, $b = 0$.
In this case we have

$$A_1 = 0,\ a = d_2(A_2 + A_3x + A_4x^2)^{-2}, \tag{3.57}$$

where d_2 is a constant and

$$\alpha = A_2 + A_3x + A_4x^2. \tag{3.58}$$

The generators and B in this case are the same as we obtained in Case 3.2.

Case 5: $a = \lambda b,\ \lambda =$Constant.
For this case the results agree with those given in Case 3.2.

3.4 Partial Noether operators of (3.5)

The operator X in (1.4) is called a partial Noether operator corresponding to partial Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2, \tag{3.59}$$

for a system of two second-order ordinary differential equations given in (3.5) if there exists a function $B(x,u)$ such that it satisfies (1.19). The operator X satisfies the following system

$$\xi_y = 0,\ \xi_z = 0, \tag{3.60}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0,\ \eta^2_z - \frac{1}{2}\xi_x = 0,\ \eta^1_z + \eta^2_y = 0, \tag{3.61}$$

$$\eta^1_x = (ay+bz)\xi + B_y,\ \eta^2_x = (by-az)\xi + B_z, \tag{3.62}$$

$$\eta^1(ay+bz) + \eta^2(by-az) - B_x = 0. \tag{3.63}$$

The solution of (3.60)-(3.62) gives

$$\begin{aligned}
\xi &= \alpha(x),\\
\eta^1 &= \frac{1}{2}\alpha' y - C_1(x)z + A(x),\\
\eta^2 &= \frac{1}{2}\alpha' z + C_1(x)y + C_2(x),\\
B &= \frac{1}{4}\alpha'' y^2 + y(-C_1'(x)z + A'(x)) - (\frac{1}{2}ay^2 + byz)\alpha + \frac{1}{4}\alpha'' z^2\\
&+C_2' z + \frac{1}{2}a\alpha z^2 + C_3(x).
\end{aligned} \tag{3.64}$$

The replacement of η^1, η^2 and B in (3.63) reduces to the following system

$$\begin{aligned}
&\frac{1}{4}\alpha''' + \frac{1}{2}a'\alpha + a\alpha' + A_1 b = 0,\\
&2b\alpha' + b'\alpha - 2aA_1 = 0,\\
&C_2''(x) + aC_2(x) = bA(x),\\
&\frac{1}{4}\alpha''' - \frac{1}{2}a'\alpha - a\alpha' - A_1 b = 0,\\
&A''(x) - aA(x) = bC_2(x),\\
&C_3'(x) = 0.
\end{aligned} \tag{3.65}$$

The above system is the same as we obtained in the case of Noether symmetries, only B is different in some cases. For the partial Noether operators the following cases arise.

Case 1: $a,\ b$ are arbitrary.

In this case the partial Noether operators and guage terms B are the same as we obtained in the case of Noether symmetries.

Case 2: $a = a_0,\ b = b_0$.

We consider the following subcases for Case 2.

Case 2.1: $a_0 \neq 0,\ b_0 \neq 0$.

In Case 2.1 all the partial Noether operators and B are identical to the Noether symmetries except B in first equation of system (3.42) is different in this case which is given below

$$B = -(\frac{1}{2}a_0y^2 + b_0yz) + \frac{1}{2}a_0z^2. \tag{3.66}$$

Case 2.2: $a_0 = 0,\ b_0 \neq 0$.

The partial Noether operators and B for this case are also same as Noether symmetries. The difference occurs in one guage term given as

$$B = -b_0yz. \tag{3.67}$$

Case 2.3: $a_0 \neq 0,\ b_0 = 0$.

This case is similar to the Noether symmetries in Section 3 except B in the first equation of (3.46) is different which is given below

$$B = \frac{1}{2}a_0(-y^2 + z^2). \tag{3.68}$$

Case 2.4: $a_0 = 0,\ b_0 = 0$.

The results for this case agree with those found in the case of Noether symmetries.

Case 3: $a = a_0,\ b \neq$ constant.

The following are the subcases of Case 3.

Case 3.1: $a_0 \neq 0,\ b \neq$ constant.

All the operators and B in this case are similar to the case of Noether symmetries.

Case 3.2: $a_0 = 0,\ b \neq$ constant.

All the operators and B in this case are identical to the case of Noether symmetries. The difference resides in one guage term which is given below

$$B = \frac{1}{2}(y^2 + z^2)A_4 + yA'(x) - byz\alpha + C_2'(x)z. \tag{3.69}$$

Case 4: $a \neq$ constant, $b = b_0$.

In each of the subcases of Case 4 we get the same results as given for the Noether symmetries.

Case 5: $a = \lambda b,\ \lambda =$constant.

We get the same operators as constructed in Noether symmetries.

The interpretation of the results of all cases are given below.

Noether symmetries:

Case 1: In this case we have four dimensional Lie algebra.

Case 2.1: The Lie algebra is five-dimensional.

Case 2.2: In this case we also obtain a five-dimensional Lie algebra.

Case 2.3: The Lie algebra for this case is five-dimensional too.

Case 2.4: The Lie algebra is eight-dimensional.

Case 3 and *Case 4*: In Case 3 and Case 4 we obtain four dimensional Lie algebras.

Case 5: We deduce four dimensional Lie algebra.

These are subalgebras of the Lie algebras for the corresponding cases considered in Wafo and Mahomed (2000).

Partial Noether operators:

For partial Noether operators obtained herein we get the same results as for the Noether symmetry generators. In fact, the algebras for both cases are isomorphic. The reason for this is that $\delta L/\delta y$ and $\delta L/\delta z$ are independent of derivatives. However, the Bs are different for both cases due to the Lagrangian being different for the respective approaches.

3.5 First integrals

If X in (1.4) is a Noether symmetry generator corresponding to a Lagrangian given in (3.6), then I given by (1.20) is a first integral of the system of two linear second-order ODEs (3.5) associated with X. The first integrals for each case are given below.

Case 1: a, b are arbitrary.

The first integrals for this case can be expressed in the general form as

$$I_i = yA'(x) + zC_2'(x) - u_i y' - v_i z', \ i = 1, ...4, \tag{3.70}$$

where $A(x)$ and $C_2(x)$ are given in (3.34).

Case 2: $a = a_0, \ b = b_0$.

We consider the following subcases for Case 2.

Case 2.1: $a_0 \neq 0, \ b_0 \neq 0$.

The first integrals for Case 2.1 are given below

$$I_1 = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 + \frac{1}{2}a_0 z^2 - \frac{1}{2}a_0 y^2 - b_0 yz,$$

$$I_2 = \exp((a_0^2 + b_0^2)^{\frac{1}{4}}x)[\frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0^2} + a_0) + (a_0^2 + b_0^2)^{\frac{1}{4}}z$$

$$-\frac{y'}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) - z'],$$

$$I_3 = \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}}x)[-\frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0^2} + a_0)$$

$$-(a_0^2 + b_0^2)^{\frac{1}{4}}z - \frac{y'}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) - z'],$$

$$I_4 = -\frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}} \sin{(a_0^2 + b_0^2)^{\frac{1}{4}}x}(-\sqrt{a_0^2 + b_0^2} + a_0)$$

$$-z(a_0^2 + b_0^2)^{\frac{1}{4}} \sin{(a_0^2 + b_0^2)^{\frac{1}{4}}x} - [\frac{y'}{b_0} \cos{(a_0^2 + b_0^2)^{\frac{1}{4}}x}(-\sqrt{a_0^2 + b_0^2} + a_0)$$

$$+z' \cos{(a_0^2 + b_0^2)^{\frac{1}{4}} x}],$$

$$I_5 = \frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}} \cos{(a_0^2 + b_0^2)^{\frac{1}{4}} x}(-\sqrt{a_0^2 + b_0^2} + a_0)$$

$$+z(a_0^2 + b_0^2)^{\frac{1}{4}} \cos{(a_0^2 + b_0^2)^{\frac{1}{4}} x} - [\frac{y'}{b_0} \sin{(a_0^2 + b_0^2)^{\frac{1}{4}} x}(-\sqrt{a_0^2 + b_0^2} + a_0)$$

$$+z' \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x]. \tag{3.71}$$

Case 2.2: $a_0 = 0, \ b_0 \neq 0.$

For this case we get the following integrals

$$I_1 = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - b_0 yz,$$

$$I_2 = \exp(\sqrt{b_0}x)[y\sqrt{b_0} + \sqrt{b_0}z - y' - z'],$$

$$I_3 = \exp(-\sqrt{b_0}x)[-y\sqrt{b_0} - \sqrt{b_0}z - y' - z'],$$

$$I_4 = y\sqrt{b_0} \sin{\sqrt{b_0}x} - z\sqrt{b_0} \sin{\sqrt{b_0}x} + y' \cos{\sqrt{b_0}x} - z' \cos{\sqrt{b_0}x},$$

$$I_5 = -y\sqrt{b_0} \cos{\sqrt{b_0}x} + z\sqrt{b_0} \cos{\sqrt{b_0}x} + y' \sin{\sqrt{b_0}x} - z' \sin{\sqrt{b_0}x}. \tag{3.72}$$

Case 2.3: $a_0 \neq 0, \ b_0 = 0.$

First integrals for this case are given as

$$I_1 = -\frac{1}{2}a_0 y^2 + \frac{1}{2}a_0 z^2 + \frac{1}{2}y'^2 + \frac{1}{2}z'^2,$$

$$I_2 = y\sqrt{a_0} \exp(\sqrt{a_0}x) - y' \exp(\sqrt{a_0}x),$$

$$I_3 = -y\sqrt{a_0} \exp(-\sqrt{a_0}x) - y' \exp(-\sqrt{a_0}x),$$

$$I_4 = -z\sqrt{a_0} \sin(\sqrt{a_0}x) - z' \cos(\sqrt{a_0}x),$$

$$I_5 = z\sqrt{a_0} \cos(\sqrt{a_0}x) - z' \sin(\sqrt{a_0}x). \tag{3.73}$$

Case 2.4: $a_0 = 0, \ b_0 = 0.$

We find that

$$I_1 = y'z - yz',$$

$$I_2 = \frac{1}{2}(y^2+z^2) + \frac{x^2}{2}(y'^2+z'^2) - x(yy'+zz'),$$
$$I_3 = \frac{x}{2}(y'^2+z'^2) - \frac{1}{2}(yy'+zz'),$$
$$I_4 = \frac{1}{2}(y'^2+z'^2),\ I_5 = y - xy',$$
$$I_6 = -y',\ I_7 = z - xz',\ I_8 = -z'. \quad (3.74)$$

Case 3: $a = a_0,\ b \neq$constant.

We consider the following subcases.

Case 3.1: $a_0 \neq 0,\ b \neq$ constant.

In this case we can write the first integrals as in Case 1.

Case 3.2: $a_0 = 0$, $b \neq$ constant.

First integrals for this case can be written in general form

$$I_i = \frac{1}{2}(y^2+z^2)A_4 + yA'(x) + zC_2'(x) + \frac{1}{2}y'^2\alpha + \frac{1}{2}z'^2\alpha - byz\alpha$$
$$-u_i y' - v_i z',\ i = 1, ...4. \quad (3.75)$$

Case 4: $a \neq$constant, $b = b_0$.

The first integrals for this case are similar to Case 3.

Now if X in (1.4) is a partial Noether operator corresponding to the partial Lagrangian as given in (3.59) then I in (1.20) is a first integral for the system of two linear second-order ODEs (3.5) associated with X. The first integrals for partial Noether operators are the same as for the Noether symmetries in each case.

3.6 Concluding remarks

We have studied Noether and partial Noether operators corresponding to a Lagrangian and partial Lagrangian of a system of two variable coefficient second-order equations.

This contributes to the algebraic study of systems which is growing as mentioned in the Introduction. Also this study points to new ways of obtaining first integrals for a system. A Lagrangian need not exist for a given system of ODEs. However, partial Lagrangians may exist and are just as useful in finding the first integrals of the underlying system as demonstrated here via linear equations. Thus if a Lagrangian does not exist even in the simplest types of equations such as linear ones, we can still construct first integrals via partial Noether operators corresponding to a partial Lagrangian of the system. We have seen that the first integrals for the Noether and the partial Noether operators are the same for the linear system of two equations with variable coefficients considered here. These form a four five and eight dimensional subalgebras of the Lie algebra of point symmetries of the linear system as obtained in Wafo Soh and Mahomed (2000).

Chapter 4

First Integrals For a General Linear System of Two Second-Order ODEs Via a Partial Lagrangian

The partial Noether operators and first integrals of a general system of two linear second-order ordinary differential equations (ODEs) with variable coefficients are studied by means of a partial Lagrangian. The canonical form for the general system of two second-order ordinary differential equations is invoked (see also Chapter 3) and all cases of this system are discussed with respect to partial Noether operators. We also refer to the results for the special case $b(x) = c(x)$ of the system which was considered in the previous chapter using a Lagrangian and a partial Lagrangian. The first integrals are obtained explicitly by exploiting a Noether-like theorem with the help of partial Noether operators (see Chapter 1). Physical applications to conservative and oscillators mechanical systems are given.

4.1 Introduction

Systems of two second-order ODEs in general arise in relativity, classical mechanics, non-linear oscillations, quantum and fluid mechanics etc. Some important works have been done relating to a system of two second-order ODEs. The Lie point symmetries of a system of two second-order ODEs with constant coefficients were obtained by Gorringe and Leach (1988). The variable coefficient case of this system was later done by Wafo and Mahomed (2000). The point symmetry properties of a Lagrangian system with two degrees of freedom were considered by Sen (1987). The symmetries of the Hamiltonian system with two degrees of freedom were also studied by Damianou and Sophocleous (1999). Damianou and Sophocleous obtained the Noether point symmetries for a three degrees of freedom Lagrangian system and the results for one and two degrees of freedom were also reviewed in their paper. A similar approach is adapted in this work for the classification of partial Noether operators. The difference is that they have considered the nonlinear system whereas we are dealing with a general linear system of two second-order ODEs with variable coefficients but using a different approach. In the previous chapter we showed that the first integrals corresponding to Noether and partial Noether operators for a linear system of two second-order ODEs with variable coefficients are the same (see also Naeem and Mahomed 2008b). The difference arises in the guage terms only. The algebraic criteria for linearization via point transformations for a system of two second-order ODEs was considered in Wafo and Mahomed (2001). The canonical forms for a system of two second-order ODEs were attempted by Wafo and Mahomed (2001). Moreover Fels in (1995), considered the equivalence problem for a system of two second-order ODEs. The linearizability criteria for a system of two second-order quadratically semi-linear ODEs using invertible transformations were investigated by Mahomed and Qadir (2007).

The objective of this chapter is to construct the partial Noether operators and first integrals of a general linear system of two variable coefficient equations that in general do not admit a standard Lagrangian. Since partial Lagrangians do exist for the equations in the absence of a standard Lagrangian, we use an alternative way to construct the first integrals. We find the partial Noether operators and then first integrals by utilizing the partial Noether's theorem of Chapter 1.

The outline of the work is as follows. Section 4.2 is related to partial Noether operators of a general linear system of two second-order ODEs that in general has no Lagrangian. The first integrals corresponding to partial Noether operators are given in Section 4.3. Herein, even in the cases for which the linear system does not have a Lagrangian we obtain first integrals.

4.2 Partial Noether operators

In this section, we derive the partial Noether operators for a general linear system of two second-order ODEs (3.4) for which a standard Lagrangian does not exists. Several cases for arbitrary functions $a(x)$, $b(x)$ and $c(x)$ are discussed in detail.

Now, the operator X given in (1.4) is a partial Noether operator corresponding to a partial Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2, \tag{4.1}$$

with $\delta L/\delta y = -(a(x)y + b(x)z)$ and $\delta L/\delta z = a(x)z - c(x)y$ of (3.4) if it satisfies (1.19) which splits as

$$\xi_y = 0,\ \xi_z = 0, \tag{4.2}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0,\ \eta^2_z - \frac{1}{2}\xi_x = 0,\ \eta^1_z + \eta^2_y = 0, \tag{4.3}$$

$$\eta^1_x = (ay + bz)\xi + B_y, \tag{4.4}$$

$$\eta^2_x = (cy - az)\xi + B_z, \tag{4.5}$$

$$\eta^1(ay + bz) + \eta^2(cy - az) - B_x = 0. \tag{4.6}$$

From equations (4.2) and (4.3), we conclude that

$$\xi = \alpha(x), \tag{4.7}$$

$$\eta^1 = \frac{1}{2}\alpha' y - C_1(x)z + A(x), \tag{4.8}$$

$$\eta^2 = \frac{1}{2}\alpha' z + C_1(x)y + C_2(x). \tag{4.9}$$

The replacement of ξ, η^1 and η^2 from equations (4.7)-(4.9) into equations in (4.4) and (4.5) gives

$$C_1'(x) = \frac{1}{2}(c - b)\alpha, \tag{4.10}$$

$$B = \frac{1}{4}\alpha'' y^2 - C_1'(x)yz + yA'(x) - (\frac{1}{2}ay^2 + byz)\alpha + S(x, z), \tag{4.11}$$

where

$$S(x, z) = \frac{1}{4}\alpha'' z^2 + C_2'(x)z + \frac{1}{2}az^2\alpha + C_3(x). \tag{4.12}$$

Equation (4.6) with the help of (4.7 - 4.9) and (4.11 - 4.12) reduces to

$$\begin{aligned}
&(\frac{1}{2}\alpha' y - C_1(x)z + A(x))(ay + bz) + (\frac{1}{2}\alpha' z + C_1(x)y + C_2(x))(cy - az) \\
&-\frac{\alpha'''}{4}(y^2 + z^2) + C_1''(x)yz - C_2''(x)z - yA''(x) - C_3'(x) - \frac{z^2}{2}(a'\alpha + a\alpha') \\
&+(\frac{a'}{2}y^2 + b'yz)\alpha + (\frac{a}{2}y^2 + byz)\alpha' = 0.
\end{aligned} \tag{4.13}$$

The insertion of $C_1(x)$ from equation (4.10) into equation (4.13) and then separation with respect to powers of y and z yields the following system

$$\frac{1}{4}\alpha''' + a\alpha' + \frac{1}{2}a'\alpha + \frac{1}{2}b\int (c - b)\alpha dx + bA_1 = 0, \tag{4.14}$$

$$\frac{1}{2}b'\alpha + b\alpha' + \frac{1}{2}c'\alpha + c\alpha' - a\int (c - b)\alpha dx - 2aA_1 = 0, \tag{4.15}$$

$$C_2''(x) + aC_2(x) = bA(x), \tag{4.16}$$

$$\frac{1}{4}\alpha''' - a\alpha' - \frac{1}{2}a'\alpha - \frac{1}{2}c\int (c - b)\alpha dx - cA_1 = 0, \tag{4.17}$$

$$A''(x) - aA(x) = cC_2(x), \tag{4.18}$$

$$C_3'(x) = 0, \tag{4.19}$$

where A_1 is a constant.

From equation (4.19), we find that

$$C_3(x) = c_0. \tag{4.20}$$

In order to solve the system (4.14 - 4.18) and (4.10), the following cases need to be considered.

Case 1.1: $b + c \neq 0$.

In this case we easily find that

$$\alpha(x) = 0, \ C_1(x) = 0,$$

$$A(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) + \alpha_3 u_3(x) + \alpha_4 u_4(x),$$

$$C_2(x) = \alpha_1 v_1(x) + \alpha_2 v_2(x) + \alpha_3 v_3(x) + \alpha_4 v_4(x). \tag{4.21}$$

The operators for this case can be written in the general form as

$$X_i = u_i \frac{\partial}{\partial y} + v_i \frac{\partial}{\partial z}, \ i = 1, ..., 4,$$

where (u_i, v_i) are linearly independent solutions of the adjoint of the system (3.4) and

$$B = yA'(x) + zC_2'(x).$$

Case 1.2: $b + c = 0$.

The subcases of Case 1.2 are:

Case 1.2.1: $a \neq 0$.

In this case we get the similar results as in Case 1.1.

Case 1.2.2: $a = 0$.

For this case we express the solution of the system (4.14)-(4.18) and (4.10) in the general form as

$$\alpha(x) = \sum_{i=1}^{4} \beta_i w_i(x), \ \beta_i \text{ constants}$$

$$C_1(x) = \frac{1}{4c} \sum_{i=1}^{4} \beta_i w_i'''(x),$$

$$A(x) = \sum_{i=5}^{8} \alpha_i u_i(x), \ \alpha_i \text{ constants}$$

$$C_2(x) = \sum_{i=5}^{8} \alpha_i v_i(x).$$

The operators and the guage terms are

$$X_i = w_i \frac{\partial}{\partial x} + (\frac{y}{2} w_i'(x) - \frac{z}{4c} w_i'''(x)) \frac{\partial}{\partial y} + (\frac{z}{2} w_i'(x) + \frac{y}{4c} w_i'''(x)) \frac{\partial}{\partial z}, \ i = 1, ..., 4,$$

$$X_j = u_j \frac{\partial}{\partial y} + v_j \frac{\partial}{\partial z}, \ j = 5, ..., 8,$$

where w_i represent the independent solutions of the resulting system of (4.14), (4.15) and (4.17) which are solutions of the linear system

$$\alpha^{(iv)} - \alpha''' \frac{c'}{c} - c^2 \alpha = 0,$$

$$C_1(x) = \frac{1}{4c} \alpha'''$$

and (u_j, v_j) are the independent solutions of the adjoint of the system (3.4) with

$$B = \frac{1}{4}(y^2 + z^2)\alpha'' + y(-zC_1'(x) + A'(x)) - (\frac{ay^2}{2} - \frac{az^2}{2} + byz)\alpha + zC_2'(x).$$

Case 1.3: $a = x + s$, s constant.

For this case we obtain similar results as in Case 1.1.

Case 2: $a = a_0$, $b = b_0$, $c = c_0$, a_0, b_0 and c_0 are constants.

Equations (4.14 - 4.18) become

$$\frac{1}{4}\alpha''' + a_0 \alpha' + \frac{1}{2} b_0 \int (c_0 - b_0)\alpha dx + b_0 A_1 = 0,$$

$$b_0 \alpha' + c_0 \alpha' - a_0 \int (c_0 - b_0) \alpha dx - 2a_0 A_1 = 0,$$

$$C_2''(x) + a_0 C_2(x) = b_0 A(x),$$

$$\frac{1}{4}\alpha''' - a_0 \alpha' - \frac{1}{2} c_0 \int (c_0 - b_0) \alpha dx - c_0 A_1 = 0,$$

$$A''(x) - a_0 A(x) = c_0 C_2(x). \qquad (4.22)$$

After some simple calculations, the following subcases arise.

Case 2.1: $a_0 \neq 0,\ b_0 \neq 0$ and $c_0 \neq 0$.

Whence the following subcases of Case 2.1 should be looked at.

Case 2.1.1: $b_0 - c_0 \neq 0$.

The straightforward but lengthy calculations lead to

$$\alpha'''' - (b_0 - c_0)^2\alpha = 0,$$

$$c_0\alpha'' + b_0\alpha'' + (b_0 - c_0)\alpha = 0,$$

$$c_0\alpha'' + b_0\alpha'' + (b_0 - c_0)\alpha = 0,$$

$$C_2''(x) + a_0C_2(x) = b_0A(x),$$

$$A''(x) - a_0A(x) = c_0C_2(x). \tag{4.23}$$

The solution of system (4.23) gives rise to

$$\alpha(x) = 0,\ C_1(x) = 0,$$

$$C_2(x) = A_2\exp((a_0^2 + b_0c_0)^{\frac{1}{4}}x) + A_3\exp(-(a_0^2 + b_0c_0)^{\frac{1}{4}}x)$$

$$+A_4\cos((a_0^2 + b_0c_0)^{\frac{1}{4}}x) + A_5\sin((a_0^2 + b_0c_0)^{\frac{1}{4}}x),$$

$$A(x) = \frac{1}{b_0}[A_2(\sqrt{a_0^2 + b_0c_0} + a_0)\exp((a_0^2 + b_0c_0)^{\frac{1}{4}}x)$$

$$+A_3(\sqrt{a_0^2 + b_0c_0} + a_0)\exp(-(a_0^2 + b_0c_0)^{\frac{1}{4}}x)$$

$$+A_4(-\sqrt{a_0^2 + b_0c_0} + a_0)\cos((a_0^2 + b_0c_0)^{\frac{1}{4}}x)$$

$$+A_5(-\sqrt{a_0^2 + b_0c_0} + a_0)\sin((a_0^2 + b_0c_0)^{\frac{1}{4}}x). \tag{4.24}$$

The partial Noether operators and B in each case are constructed by choice of constant equal to one and the remaining constants equal to zero.

$$X_1 = \frac{1}{b_0}(\sqrt{a_0^2 + b_0c_0} + a_0)\exp((a_0^2 + b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp((a_0^2 + b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z},$$

$$B = \frac{y}{b_0}(a_0^2 + b_0c_0)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0c_0} + a_0)\exp((a_0^2 + b_0c_0)^{\frac{1}{4}}x)$$

$$+z(a_0^2+b_0c_0)^{\frac{1}{4}}\exp((a_0^2+b_0c_0)^{\frac{1}{4}}x),$$

$$X_2=\frac{1}{b_0}(\sqrt{a_0^2+b_0c_0}+a_0)\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y}+\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z},$$

$$B=\frac{-y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(\sqrt{a_0^2+b_0c_0}+a_0)\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)$$

$$-z(a_0^2+b_0c_0)^{\frac{1}{4}}\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x),$$

$$X_3=\frac{1}{b_0}(-\sqrt{a_0^2+b_0c_0}+a_0)\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial y}+\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial z},$$

$$B=\frac{-y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0c_0}+a_0)\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x$$

$$-z(a_0^2+b_0c_0)^{\frac{1}{4}}\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x,$$

$$X_4=\frac{1}{b_0}(-\sqrt{a_0^2+b_0c_0}+a_0)\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial y}+\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial z},$$

$$B=\frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0c_0}+a_0)\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x$$

$$+z(a_0^2+b_0c_0)^{\frac{1}{4}}\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x. \tag{4.25}$$

Case 2.1.2: $b_0-c_0=0$.

For this case, the results are given in the previous Chapter 3.

Case 2.2: $a_0=0,\ b_0\neq 0,\ c_0\neq 0$.

The subcases of Case 2.2 are:

Case 2.2.1: $b_0-c_0\neq 0$.

We find that

$$\alpha(x)=0,\ C_1(x)=0,\ C_2(x)=A_6\exp((b_0c_0)^{\frac{1}{4}}x)+A_7\exp(-(b_0c_0)^{\frac{1}{4}}x)$$

$$+A_8\cos(b_0c_0)^{\frac{1}{4}}x+A_9\sin(b_0c_0)^{\frac{1}{4}}x,$$

$$A(x)=\sqrt{\frac{c_0}{b_0}}[A_6\exp((b_0c_0)^{\frac{1}{4}}x)+A_7\exp(-(b_0c_0)^{\frac{1}{4}}x)-A_8\cos(b_0c_0)^{\frac{1}{4}}x$$

$$-A_9\sin(b_0c_0)^{\frac{1}{4}}x]. \tag{4.26}$$

The operators are

$$
\begin{aligned}
X_1 &= \sqrt{\frac{c_0}{b_0}} \exp((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z},\\
B &= \frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} \exp((b_0c_0)^{\frac{1}{4}}x) + z(b_0c_0)^{\frac{1}{4}} \exp((b_0c_0)^{\frac{1}{4}}x),\\
X_2 &= \sqrt{\frac{c_0}{b_0}} \exp(-(b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp(-(b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z},\\
B &= -\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} \exp(-(b_0c_0)^{\frac{1}{4}}x) - z(b_0c_0)^{\frac{1}{4}} \exp(-(b_0c_0)^{\frac{1}{4}}x),\\
X_3 &= -\sqrt{\frac{c_0}{b_0}} \cos((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \cos((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z},\\
B &= \frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} \sin{(b_0c_0)^{\frac{1}{4}}x} - z(b_0c_0)^{\frac{1}{4}} \sin{(b_0c_0)^{\frac{1}{4}}x},\\
X_4 &= -\sqrt{\frac{c_0}{b_0}} \sin((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \sin((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z},\\
B &= -\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} \cos{(b_0c_0)^{\frac{1}{4}}x} + z(b_0c_0)^{\frac{1}{4}} \cos{(b_0c_0)^{\frac{1}{4}}x}.
\end{aligned}
\tag{4.27}
$$

Case 2.2.2: $b_0 - c_0 = 0$.

In this case, the results are also given in Chapter 3.

Case 2.3: If $a_0 \neq 0,\ b_0 = 0,\ c_0 \neq 0$, the system (4.14 - 4.18) becomes

$$
\begin{aligned}
&\alpha''' - c_0^2 \int \alpha dx = 0,\\
&c_0\alpha' - a_0c_0 \int \alpha dx = 0,\\
&C_2''(x) + a_0C_2(x) = 0,\\
&A''(x) - a_0A(x) = c_0C_2(x).
\end{aligned}
\tag{4.28}
$$

The routine calculations show that

$$
\alpha(x) = 0,\ C_1(x) = 0,\ C_2(x) = A_{10}\cos\sqrt{a_0}x + A_{11}\sin\sqrt{a_0}x,
$$

$$
A(x) = A_{12}\exp(\sqrt{a_0}x) + A_{13}\exp(-\sqrt{a_0}x) - \frac{c_0}{2a_0}A_{10}\cos(\sqrt{a_0}x) - \frac{c_0}{2a_0}A_{11}\sin(\sqrt{a_0}x). \tag{4.29}
$$

The operators for this case are given below

$$
\begin{aligned}
&X_1 = -\frac{c_0}{2a_0}\cos(\sqrt{a_0}x)\frac{\partial}{\partial y} + \cos(\sqrt{a_0}x)\frac{\partial}{\partial z},\\
&B = \frac{c_0}{2\sqrt{a_0}}y\sin(\sqrt{a_0}x) - z\sqrt{a_0}\sin(\sqrt{a_0}x),\\
&X_2 = -\frac{c_0}{2a_0}\sin(\sqrt{a_0}x)\frac{\partial}{\partial y} + \sin(\sqrt{a_0}x)\frac{\partial}{\partial z},\\
&B = -\frac{c_0}{2\sqrt{a_0}}y\cos(\sqrt{a_0}x) + z\sqrt{a_0}\cos(\sqrt{a_0}x),\\
&X_3 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y},\ B = y\sqrt{a_0}\exp(\sqrt{a_0}x),\\
&X_4 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y},\ B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x).
\end{aligned}
\tag{4.30}
$$

Case 2.4: $a_0 \neq 0,\ b_0 \neq 0,\ c_0 = 0.$

For this case, the system (4.14 - 4.18) reduces to

$$
\begin{aligned}
&\alpha''' - b_0^2\int \alpha dx = 0,\\
&b_0\alpha' + a_0b_0\int \alpha dx = 0,\\
&C_2''(x) + a_0C_2(x) = b_0A(x),\\
&A''(x) - a_0A(x) = 0.
\end{aligned}
\tag{4.31}
$$

The simple but lengthy manipulations result in

$$
\begin{aligned}
&\alpha(x) = 0,\ C_1(x) = 0,\\
&C_2(x) = A_{16}\cos(\sqrt{a_0}x) + A_{17}\sin(\sqrt{a_0}x) + \frac{b_0}{2a_0}A_{14}\exp(\sqrt{a_0}x)\\
&+\frac{b_0}{2a_0}A_{15}\exp(-\sqrt{a_0}x),\\
&A(x) = A_{14}\exp(\sqrt{a_0}x) + A_{15}\exp(-\sqrt{a_0}x).
\end{aligned}
\tag{4.32}
$$

The partial Noether operators are

$$
X_1 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y} + \frac{b_0}{2a_0}\exp(\sqrt{a_0}x)\frac{\partial}{\partial z},
$$

$$B = y\sqrt{a_0}\exp(\sqrt{a_0}x) + z\frac{b_0}{2\sqrt{a_0}}\exp(\sqrt{a_0}x),$$

$$X_2 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y} + \frac{b_0}{2a_0}\exp(-\sqrt{a_0}x)\frac{\partial}{\partial z},$$

$$B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - z\frac{b_0}{2\sqrt{a_0}}\exp(-\sqrt{a_0}x),$$

$$X_3 = \cos\sqrt{a_0}x\frac{\partial}{\partial z},\ B = -z\sqrt{a_0}\sin(\sqrt{a_0}x),$$

$$X_4 = \sin\sqrt{a_0}x\frac{\partial}{\partial z},\ B = z\sqrt{a_0}\cos(\sqrt{a_0}x). \tag{4.33}$$

Note that if a_0 is negative in Cases 2.3 - 2.4, then one has trigonometric hyperbolic functions.

Case 2.5: $a_0 = 0,\ b_0 = 0,\ c_0 = 0.$

For this case the results were derived in Chapter 3.

Case 3: $a = a_0,\ b = b_0,\ c \neq \text{constant}.$

Equations (4.14 - 4.18) and (4.10) yield

$$\frac{1}{4}\alpha''' + a_0\alpha' + \frac{1}{2}b_0\int(c - b_0)\alpha dx + b_0A_1 = 0, \tag{4.34}$$

$$b_0\alpha' + \frac{1}{2}c'\alpha + c\alpha' - a_0\int(c - b_0)\alpha dx - 2a_0A_1 = 0, \tag{4.35}$$

$$C_2'''(x) + a_0C_2(x) = b_0A(x), \tag{4.36}$$

$$\frac{1}{4}\alpha''' - a_0\alpha' - \frac{1}{2}c\int(c - b_0)\alpha dx - cA_1 = 0, \tag{4.37}$$

$$A''(x) - a_0A(x) = cC_2(x), \tag{4.38}$$

$$C_1'(x) = \frac{1}{2}(c - b_0)\alpha. \tag{4.39}$$

The following subcases arise.

Case 3.1: $a_0 \neq 0,\ b_0 \neq 0,\ c \neq \text{constant}.$

Subtracting equation (4.34) from equation (4.37) and then using equation (4.39), we arrive at

$$C_1(x) = \frac{-2a_0\alpha'}{b_0 + c}, \;\; b_0 + c \neq 0. \tag{4.40}$$

The substitution of $C_1(x)$ from above equation into equation (4.35) and then the solution of resulting equation yields

$$\alpha(x) = A_{18}(c^2 + 4a_0^2 + 2b_0c + b_0^2)^{-1/4}. \tag{4.41}$$

Equations (4.34 - 4.37) with the replacement of $C_1(x)$ and $\alpha(x)$ from equations (4.40) and (4.41) give

$$\alpha(x) = 0, \; C_1(x) = 0.$$

The solution of equations (4.36) and (4.38) can be expressed as

$$\begin{aligned} A(x) &= \alpha_1 u_1(x) + \alpha_2 u_2(x) + \alpha_3 u_3(x) + \alpha_4 u_4(x), \\ C_2(x) &= \alpha_1 v_1(x) + \alpha_2 v_2(x) + \alpha_3 v_3(x) + \alpha_4 v_4(x), \end{aligned} \tag{4.42}$$

where (u_i, v_i) are linearly independent solution of the adjoint of the system (3.4).
The operators and guage terms for this case are the same as we obtained in Case 1.1.

Case 3.2: $a_0 = 0, \; b_0 \neq 0, \; c \neq \text{constant}.$
The system (4.14)-(4.18) reduces to

$$\begin{aligned} &\alpha''' + 2b_0 \int (c - b_0)\alpha dx + 4b_0 A_1 = 0, \\ &2b_0\alpha' + c'\alpha + 2c\alpha' = 0, \\ &C_2''(x) = b_0 A(x), \\ &\alpha''' - 2c \int (c - b_0)\alpha dx - 4cA_1 = 0, \\ &A''(x) = cC_2(x). \end{aligned} \tag{4.43}$$

In this case we get the same operators as given in Case 1.1.

Case 3.3: $a_0 \neq 0,\ b_0 = 0,\ c \neq \text{constant}.$

In this case the system (4.14 - 4.18) is

$$\begin{aligned}
&\alpha''' + 4a_0\alpha' = 0,\\
&c'\alpha + 2c\alpha' - 2a_0 \int c\alpha dx - 4a_0 A_1 = 0,\\
&C_2'''(x) + a_0 C_2(x) = 0,\\
&\alpha''' - 4a_0\alpha' - 2c \int c\alpha dx - 4cA_1 = 0,\\
&A''(x) - a_0 A(x) = cC_2(x).
\end{aligned} \tag{4.44}$$

From above system, we find that

$$\begin{aligned}
&\alpha(x) = 0,\ C_1(x) = 0,\\
&A(x) = \alpha_7 \exp(\sqrt{a_0}x) + \alpha_8 \exp(-\sqrt{a_0}x) + \alpha_5 u_1(x) + \alpha_6 u_2(x),\\
&C_2(x) = \alpha_5 \cos\sqrt{a_0}x + \alpha_6 \sin\sqrt{a_0}x.
\end{aligned} \tag{4.45}$$

Straightforward calculations reveal that the guage term B with the help of above values is

$$B = yA'(x) + zC_2'(x). \tag{4.46}$$

The operators are

$$\begin{aligned}
&X_1 = u_1(x)\frac{\partial}{\partial y} + \cos\sqrt{a_0}x\frac{\partial}{\partial z},\ B = yu_1'(x) - z\sqrt{a_0}\sin\sqrt{a_0}x,\\
&X_2 = u_2(x)\frac{\partial}{\partial y} + \sin\sqrt{a_0}x\frac{\partial}{\partial z},\ B = yu_2'(x) + z\sqrt{a_0}\cos\sqrt{a_0}x,\\
&X_3 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y},\ B = y\sqrt{a_0}\exp(\sqrt{a_0}x),\\
&X_4 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y},\ B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x),
\end{aligned} \tag{4.47}$$

where $u_i,\ i = 1, 2$, are linearly independent solutions of the system (4.18). If $a_0 < 0$ in Case 3.3, then one gets exponential functions of (4.16).

Case 3.4: $a_0 = 0,\ b_0 = 0,\ c \neq \text{constant}.$

In this case, we have

$$\alpha''' = 0, \quad c'\alpha + 2c\alpha' = 0,$$

$$C_2'''(x) = 0, \quad A''(x) = cC_2(x),$$
$$c \int c\alpha dx + cA_1 = 0. \tag{4.48}$$

We deduce from the system (4.48) that

$$\alpha(x) = 0,\ C_1(x) = 0,$$

$$A(x) = \alpha_{11} + \alpha_{12}x + \alpha_9 u_1(x) + \alpha_{10}u_2(x),$$

$$C_2(x) = \alpha_9 + \alpha_{10}x. \tag{4.49}$$

In this case the guage term becomes

$$B = yA'(x) + zC_2'(x). \tag{4.50}$$

Routine calculations yield the following operators

$$X_1 = u_1(x)\frac{\partial}{\partial y} + \frac{\partial}{\partial z},\ B = yu_1'(x),$$

$$X_2 = u_2(x)\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ B = yu_2'(x) + z,$$

$$X_3 = \frac{\partial}{\partial y},\ B = 0,\ X_4 = x\frac{\partial}{\partial y},\ B = y, \tag{4.51}$$

where $u_i,\ i = 1, 2$, represent linearly independent solutions of (4.18) given by

$$u_1 = \int\int c(x)dxdx, \quad u_2 = \int\int xc(x)dxdx.$$

Case 4: $a = a_0,\ b \neq \text{constant}, c = c_0.$

The replacement of a, b and c in equations (4.14 - 4.18) and (4.10) yields

$$\frac{1}{4}\alpha''' + a_0\alpha' + \frac{1}{2}b\int(c_0 - b)\alpha dx + bA_1 = 0, \tag{4.52}$$

$$\frac{1}{2}b'\alpha + b\alpha' + c_0\alpha' - a_0\int(c_0 - b)\alpha dx - 2a_0A_1 = 0, \tag{4.53}$$

$$C_2'''(x) + a_0 C_2(x) = bA(x), \tag{4.54}$$
$$\frac{1}{4}\alpha''' - a_0\alpha' - \frac{1}{2}c_0 \int (c_0 - b)\alpha dx - c_0 A_1 = 0, \tag{4.55}$$
$$A''(x) - a_0 A(x) = c_0 C_2(x), \tag{4.56}$$
$$C_1'(x) = \frac{1}{2}(c_0 - b)\alpha. \tag{4.57}$$

At this point the following subcases arise.

Case 4.1: $a_0 \neq 0,\ b \neq \text{constant}, c_0 \neq 0.$
In this case we get the similar operators as given in Case 1.1.

Case 4.2: $a_0 = 0,\ b \neq \text{constant},\ \ c_0 \neq 0.$
One may easily deduce that the system (4.52 - 4.57) with the replacement of above values takes the following form

$$\frac{1}{4}\alpha''' + \frac{1}{2}b \int (c_0 - b)\alpha dx + bA_1 = 0,$$
$$\frac{1}{2}b'\alpha + b\alpha' + c_0\alpha' = 0,$$
$$C_2'''(x) = bA(x),$$
$$\frac{1}{4}\alpha''' - \frac{1}{2}c_0 \int (c_0 - b)\alpha dx - c_0 A_1 = 0,$$
$$A''(x) = c_0 C_2(x),$$
$$C_1'(x) = \frac{1}{2}(c_0 - b)\alpha. \tag{4.58}$$

For this case we also obtain the same operators as given in Case 1.1.

Case 4.3: If $a_0 \neq 0,\ b \neq \text{constant}, c_0 = 0$, then system (4.52 - 4.57) becomes

$$\frac{1}{4}\alpha''' + a_0\alpha' - \frac{1}{2}b \int b\alpha dx + bA_1 = 0,$$
$$\frac{1}{2}b'\alpha + b\alpha' + a_0 \int b\alpha dx - 2a_0 A_1 = 0,$$
$$C_2'''(x) + a_0 C_2(x) = bA(x),$$

$$
\begin{aligned}
&\frac{1}{4}\alpha''' - a_0\alpha' = 0,\\
&A''(x) - a_0 A(x) = 0,\\
&C_1'(x) = -\frac{1}{2}b\alpha. \qquad (4.59)
\end{aligned}
$$

The solution of the system (4.59) straightforwardly yields

$$
\begin{aligned}
&\alpha(x) = 0,\ C_1(x) = 0,\\
&A(x) = \alpha_{13}\exp(\sqrt{a_0}x) + \alpha_{14}\exp(-\sqrt{a_0}x),\\
&C_2(x) = \alpha_{15}\cos\sqrt{a_0}x + \alpha_{16}\sin\sqrt{a_0}x + \alpha_{13}v_1(x) + \alpha_{14}v_2(x),\\
&B = yA'(x) + zC_2'(x), \qquad (4.60)
\end{aligned}
$$

v_i are the particular solutions of (4.16). If $a_0 < 0$, then one obtains trigonometric solutions of (4.18) and exponential solutions of (4.16).

For this case the operators are

$$
\begin{aligned}
&X_1 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y} + v_1(x)\frac{\partial}{\partial z},\ B = y\sqrt{a_0}\exp(\sqrt{a_0}x) + zv_1'(x),\\
&X_2 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y} + v_2(x)\frac{\partial}{\partial z},\ B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) + zv_2'(x),\\
&X_3 = \cos(\sqrt{a_0}x)\frac{\partial}{\partial z},\ B = -z\sqrt{a_0}\sin\sqrt{a_0}x,\\
&X_4 = \sin(\sqrt{a_0}x)\frac{\partial}{\partial z},\ B = z\sqrt{a_0}\cos\sqrt{a_0}x. \qquad (4.61)
\end{aligned}
$$

Case 4.4: $a_0 = 0,\ b \neq \text{constant}, c_0 = 0.$

In this case

$$
\begin{aligned}
&\frac{1}{2}b\int b\alpha dx - bA_1 = 0,\\
&\frac{1}{2}b'\alpha + b\alpha' = 0,\ \ C_2''(x) = bA(x),\\
&\frac{1}{4}\alpha''' = 0,\ \ A''(x) = 0,\\
&C_1'(x) = -\frac{1}{2}b\alpha. \qquad (4.62)
\end{aligned}
$$

System (4.62) returns

$$\alpha(x) = 0,\ C_1(x) = 0,$$

$$A(x) = \alpha_{17} + \alpha_{18}x,$$

$$C_2(x) = \alpha_{19} + \alpha_{20}x + \alpha_{17}v_1(x) + \alpha_{18}v_2(x), \tag{4.63}$$

v_i are the particular solutions of (4.16).

The operators for this case can be expressed as

$$X_1 = \frac{\partial}{\partial y} + v_1(x)\frac{\partial}{\partial z},\ B = zv_1'(x),$$

$$X_2 = x\frac{\partial}{\partial y} + v_2(x)\frac{\partial}{\partial z},\ B = y + zv_2'(x),$$

$$X_3 = \frac{\partial}{\partial z},\ B = 0,\ X_4 = x\frac{\partial}{\partial z},\ B = z, \tag{4.64}$$

Case 5: $a \neq$ constant, $b = b_0,\ c = c_0,\ b_0$ and c_0 are constants.
Equations (4.14 - 4.18) and (4.10) with the help of Case 5 reduce to

$$\frac{1}{4}\alpha''' + a\alpha' + \frac{1}{2}a'\alpha + \frac{1}{2}b_0\int (c_0 - b_0)\alpha dx + b_0A_1 = 0, \tag{4.65}$$

$$b_0\alpha' + c_0\alpha' - a\int (c_0 - b_0)\alpha dx - 2aA_1 = 0, \tag{4.66}$$

$$C_2''(x) + aC_2(x) = b_0A(x), \tag{4.67}$$

$$\frac{1}{4}\alpha''' - a\alpha' - \frac{1}{2}a'\alpha - \frac{1}{2}c_0\int (c_0 - b_0)\alpha dx - c_0A_1 = 0, \tag{4.68}$$

$$A''(x) - aA(x) = c_0C_2(x), \tag{4.69}$$

$$C_1'(x) = \frac{1}{2}(c_0 - b_0)\alpha. \tag{4.70}$$

At this point the following subcases are considered.

Case 5.1: $a \neq$ constant, $b_0 \neq 0,\ c_0 \neq 0$.
The subcases of Case 5.1 are:

Case 5.1.1: $b_0 + c_0 \neq 0$.

Case 5.1.2: $b_0 + c_0 = 0$.

For Cases 5.1.1 and 5.1.2, we get the same operators as given in Case 1.1.

Case 5.2: $a \neq \text{constant},\ b_0 = 0,\ c_0 \neq 0$.

In this case the system (4.65 - 4.70) results in

$$\begin{aligned}
&\frac{1}{4}\alpha''' + a\alpha' + \frac{1}{2}a'\alpha,\\
&c_0\alpha' - a\int c_0\alpha dx - 2aA_1 = 0,\\
&C_2'''(x) + aC_2(x) = 0,\\
&\frac{1}{4}\alpha''' - a\alpha' - \frac{1}{2}a'\alpha - \frac{1}{2}c_0^2\int \alpha dx - c_0A_1 = 0,\\
&A''(x) - aA(x) = c_0C_2(x),\\
&C_1'(x) = \frac{1}{2}c_0\alpha.
\end{aligned} \tag{4.71}$$

Case 5.3: $a \neq \text{constant},\ b_0 \neq 0,\ c_0 = 0$.

Using the values of a, b_0 and c_0 in the system (4.65 - 4.70), we obtain

$$\begin{aligned}
&\frac{1}{4}\alpha''' + a\alpha' + \frac{1}{2}a'\alpha - \frac{1}{2}b_0^2\int \alpha dx + b_0A_1 = 0,\\
&b_0\alpha' + ab_0\int \alpha dx - 2aA_1 = 0,\\
&C_2'''(x) + aC_2(x) = b_0A(x),\\
&\frac{1}{4}\alpha''' - a\alpha' - \frac{1}{2}a'\alpha,\\
&A''(x) - aA(x) = 0,\\
&C_1'(x) = -\frac{1}{2}b_0\alpha.
\end{aligned} \tag{4.72}$$

Case 5.4: $a \neq \text{constant},\ b_0 = 0,\ c_0 = 0.$

For this case, we have

$$\begin{aligned}
&\frac{1}{4}\alpha''' + a\alpha' + \frac{1}{2}a'\alpha = 0,\\
&-2aA_1 = 0,\\
&C_2'''(x) + aC_2(x) = 0,\\
&\frac{1}{4}\alpha''' - a\alpha' - \frac{1}{2}a'\alpha = 0,\\
&A''(x) - aA(x) = 0,\\
&C_1'(x) = 0.
\end{aligned} \tag{4.73}$$

For Cases 5.2 - 5.4, we also obtain the similar operators as given in Case 1.1.

Case 6: If a, b and c are relating to each other and $a \neq 0,\ b \neq 0,\ c \neq 0.$

The following subcases need to be considered.

Case 6.1: $a = \lambda_1 b,\ \ b = \lambda_2 c,\ \lambda_1,\ \lambda_2$ are constants.

Case 6.2: $a = \lambda b$, c is arbitrary, $\lambda =$constant.

Case 6.3: $a = \lambda c$, b is arbitrary, $\lambda =$constant.

Case 6.4: $b = \lambda c$, a is arbitrary, $\lambda =$constant.

For Cases 6.1-6.4 we obtain similar results as in Case 1.1.

The interpretation of the results of all the cases above are as follows:

Partial Noether operators:

Case 1.1: We find a four dimensional Lie algebra in this case.

Case 1.2.1: We have a four dimensional algebra.

Case 1.2.2: We deduce an eight dimensional algebra which is distinct from the algebra of Case 2.5.
Case 1.3: The Lie algebra for this case is four dimensional.
Case 2.1.1: The Lie algebra is four dimensional.
Case 2.1.2: In this case we get a five dimensional Lie algebra.
Case 2.2.1: For this case the Lie algebra is four dimensional.
Case 2.2.2: We have a five dimensional Lie algebra.
Case 2.3 and Case 2.4: In Case 2.3 and Case 2.4, we obtain a four dimensional Lie algebra.
Case 2.5: The Lie algebra is eight dimensional.
Case 3.1, Case 3.2, Case 3.2 and Case 3.4: The Lie algebra for each of the Cases 3.1-3.4 is four dimensional too.
Case 4.1, Case 4.2, Case 4.2 and Case 4.4: We also obtain a four dimensional Lie algebra in Cases 4.1-4.4.
Case 5: In all the subcases of Case 5, we get a four dimensional Lie algebra.
Case 6.1, Case 6.2, Case 6.3 and Case 6.4: The Lie algebra for each of the Cases 6.1-6.4 is four dimensional.

These form subalgebras of the Lie algebras of the linear system studied in Wafo and Mahomed (2000). Since $\delta L/\delta y = -(a(x)y + b(x)z)$ and $\delta L/\delta z = a(x)z - c(x)y$, which means these are independent of derivatives, the partial Noether operators are symmetry generators (see Chapter 1 and Kara, Mahomed, Naeem and Wafo Soh 2007) of the Euler-Lagrange equations. Note that in general if the partial Euler-Lagrange equations are free of derivatives then the partial Noether operators become symmetry generators of the equations and the Lie algebras for both are isomorphic.

As the system under consideration does not in general have a standard Lagrangian, the partial Lagrangian approach is very useful in constructing first integrals for such equations. Partial Lagrangians do exist for equations in the absence of standard Lagrangians.
In the next section we derive the first integrals for each of the Cases 1-7.

4.3 First integrals

In this section, we construct the first integrals of system (3.4) corresponding to the partial Noether operators obtained in Section 4.2. All the cases are discussed explicitly.

Now, if the operator X in (1.4) is a partial Noether operator corresponding to a partial Lagrangian L given in (4.1) of the general linear system of two second-order ODEs (3.4), then the first integrals can be found from the formula (1.20).

The first integrals for each case are given below.

Case 1: $a,\ b,\ c$ are arbitrary.
At this point following case should be investigated.

Case 1.1: $b + c \neq 0$.
In this case the first integrals can be written in the general form as

$$I_i = yA'(x) + zC_2'(x) - u_i y' - v_i z',\ i = 1, ...4,$$

where $A(x)$ and $C_2(x)$ are given in equation (4.21).

Case 1.2: $b + c = 0$.
The subcases of Case 1.2 are

Case 1.2.1: $a \neq 0$.
For this case we obtain the similar results as in Case 1.1.

Case 1.2.2: $a = 0$.
The first integrals for this case can be expressed in the general form as

$$I_i = \frac{1}{4}(y^2 + z^2)w_i'' + y[-\frac{z}{4c}w_i'''' + \frac{z}{4c^2}w_i'''] - (\frac{ay^2}{2} - \frac{az^2}{2} + byz)w_i + \frac{1}{2}(y'^2 + z'^2)w_i$$
$$-[\frac{1}{2}(yy' + zz')w_i' + \frac{1}{4c}(y'z - yz')w_i''',\ i = 1, ..., 4,$$

$$I_j = yu'_j + zv'_j - u_j y' - v_j z', \ j = 5, ...8.$$

Case 2: $a = a_0, \ b = b_0, \ c = c_0$.

The following subcases should be considered for Case 2.

Case 2.1: $a_0 \neq 0, \ b_0 \neq 0, \ c_0 \neq 0$.

At this point we consider the following subcases of Case 2.1:

Case 2.1.1: $b_0 - c_0 \neq 0$.

We find that

$$I_1 = \exp((a_0^2 + b_0 c_0)^{\frac{1}{4}} x)[\frac{y}{b_0}(a_0^2 + b_0 c_0)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0 c_0} + a_0) + z(a_0^2 + b_0 c_0)^{\frac{1}{4}}$$
$$-\frac{y'}{b_0}(\sqrt{a_0^2 + b_0 c_0} + a_0) - z'],$$
$$I_2 = -\exp(-(a_0^2 + b_0 c_0)^{\frac{1}{4}} x)[\frac{y}{b_0}(a_0^2 + b_0 c_0)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0 c_0} + a_0) + z(a_0^2 + b_0 c_0)^{\frac{1}{4}}$$
$$+\frac{y'}{b_0}(\sqrt{a_0^2 + b_0 c_0} + a_0) + z'],$$
$$I_3 = -\frac{y}{b_0}(a_0^2 + b_0 c_0)^{\frac{1}{4}}(-\sqrt{a_0^2 + b_0 c_0} + a_0) \sin{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x} - z(a_0^2 + b_0 c_0)^{\frac{1}{4}} \sin{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x}$$
$$-[\frac{y'}{b_0}(-\sqrt{a_0^2 + b_0 c_0} + a_0) \cos{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x} + z' \cos{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x}],$$
$$I_4 = \frac{y}{b_0}(a_0^2 + b_0 c_0)^{\frac{1}{4}}(-\sqrt{a_0^2 + b_0 c_0} + a_0) \cos{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x} + z(a_0^2 + b_0 c_0)^{\frac{1}{4}} \cos{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x}$$
$$-[\frac{y'}{b_0}(-\sqrt{a_0^2 + b_0 c_0} + a_0) \sin{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x} + z' \sin{(a_0^2 + b_0 c_0)^{\frac{1}{4}} x}].$$

Case 2.1.2: $b_0 - c_0 = 0$.

The first integrals for Case 2.1.2 are given in Chapter 3.

Case 2.2: $a_0 = 0, \ b_0 \neq 0, \ c_0 \neq 0$.

The following subcases arise.

Case 2.2.1: $b_0 - c_0 \neq 0$.

Simple calculations yield the following integrals

$$I_1 = \exp((b_0c_0)^{\frac{1}{4}}x)[\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} + z(b_0c_0)^{\frac{1}{4}} - y'\sqrt{\frac{c_0}{b_0}} - z'],$$

$$I_2 = -\exp(-(b_0c_0)^{\frac{1}{4}}x)[\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} + z(b_0c_0)^{\frac{1}{4}} + y'\sqrt{\frac{c_0}{b_0}} + z'],$$

$$I_3 = \frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\sin(b_0c_0)^{\frac{1}{4}}x - z(b_0c_0)^{\frac{1}{4}}\sin(b_0c_0)^{\frac{1}{4}}x$$

$$+y'\sqrt{\frac{c_0}{b_0}}\cos(b_0c_0)^{\frac{1}{4}}x - z'\cos(b_0c_0)^{\frac{1}{4}}x,$$

$$I_4 = -\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\cos(b_0c_0)^{\frac{1}{4}}x + z(b_0c_0)^{\frac{1}{4}}\cos(b_0c_0)^{\frac{1}{4}}x$$

$$+y'\sqrt{\frac{c_0}{b_0}}\sin(b_0c_0)^{\frac{1}{4}}x - z'\sin(b_0c_0)^{\frac{1}{4}}x.$$

Case 2.2.2: $b_0 - c_0 = 0$.

For this case, the results are also given in Chapter 3.

Case 2.3: $a_0 \neq 0,\ b_0 = 0,\ c_0 \neq 0$.

We find that

$$I_1 = \frac{c_0}{2\sqrt{a_0}}y\sin\sqrt{a_0}x - z\sqrt{a_0}\sin\sqrt{a_0}x + \frac{c_0}{2a_0}y'\cos\sqrt{a_0}x - z'\cos\sqrt{a_0}x,$$

$$I_2 = -\frac{c_0}{2\sqrt{a_0}}y\cos\sqrt{a_0}x + z\sqrt{a_0}\cos\sqrt{a_0}x + \frac{c_0}{2a_0}y'\sin\sqrt{a_0}x - z'\sin\sqrt{a_0}x,$$

$$I_3 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x),$$

$$I_4 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x).$$

Case 2.4: $a_0 \neq 0,\ b_0 \neq 0,\ c_0 = 0$.

The first integrals are

$$I_1 = \exp(\sqrt{a_0}x)[y\sqrt{a_0} + \frac{b_0}{2\sqrt{a_0}}z - y' - \frac{b_0}{2a_0}z'],$$

$$I_2 = -\exp(-\sqrt{a_0}x)[y\sqrt{a_0} + \frac{b_0}{2\sqrt{a_0}}z + y' + \frac{b_0}{2a_0}z'],$$

$$I_3 = -z\sqrt{a_0}\sin\sqrt{a_0}x - z'\cos\sqrt{a_0}x,$$

$$I_4 = z\sqrt{a_0}\cos\sqrt{a_0}x - z'\sin\sqrt{a_0}x.$$

Note that if a_0 is negative in Cases 2.3 - 2.4, then one replaces "sin" and "cos" by trigonometric hyperbolic functions.

Case 2.5: $a_0 = 0,\ b_0 = 0,\ \ c_0 = 0.$

We have reported the results for this case in Chapter 3.

Case 3: $a = a_0,\ b = b_0,\ c \neq \text{constant}.$

The following cases arise.

Case 3.1: $a_0 \neq 0,\ b_0 \neq 0,\ c \neq \text{constant}.$

The results of this case agree with those given in Case 1.1.

Case 3.2: $a_0 = 0,\ b_0 \neq 0,\ c \neq \text{constant}.$

The first integrals in this case are similar to Case 1.1.

Case 3.3: $a_0 \neq 0,\ b_0 = 0,\ c \neq \text{constant}.$

We find the following integrals

$$I_1 = yu_1'(x) - y'u_1(x) - z\sqrt{a_0}\sin\sqrt{a_0}x - z'\cos\sqrt{a_0}x,$$

$$I_2 = yu_2'(x) - y'u_2(x) + z\sqrt{a_0}\cos\sqrt{a_0}x - z'\sin\sqrt{a_0}x,$$

$$I_3 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x),$$

$$I_4 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x).$$

Case 3.4: $a_0 = 0,\ b_0 = 0,\ c \neq \text{constant}.$

We find that

$$I_1 = yu_1'(x) - y'u_1(x) - z',$$

$$I_2 = yu_2'(x) - y'u_2(x) + z - xz',$$

$$I_3 = -y',\ I_4 = y - xy'.$$

Case 4: $a = a_0,\ b \neq \text{constant}, c = c_0$.

The following subcases of Case 4 are considered.

Case 4.1: $a_0 \neq 0,\ b \neq \text{constant}, c_0 \neq 0$.

Case 4.2: $a_0 = 0,\ b \neq \text{constant}, c_0 \neq 0$.

For Cases 4.1 and 4.2, the results are similar as in Case 1.1.

Case 4.3: $a_0 \neq 0,\ b \neq \text{constant}, c_0 = 0$.

The simple calculations yield

$$I_1 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x) + zv_1'(x) - z'v_1(x),$$

$$I_2 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x) + zv_2'(x) - z'v_2(x),$$

$$I_3 = -z\sqrt{a_0}\sin\sqrt{a_0}x - z'\cos\sqrt{a_0}x,$$

$$I_4 = z\sqrt{a_0}\cos\sqrt{a_0}x - z'\sin\sqrt{a_0}x.$$

Case 4.4: $a_0 = 0,\ b \neq \text{constant}, c_0 = 0$.

For this case the first integrals are

$$I_1 = zv_1'(x) - z'v_1(x) - y',$$

$$I_2 = zv_2'(x) - z'v_2(x) + y - xy',$$

$$I_3 = -z',\ I_3 = z - xz'.$$

Case 5: $a \neq \text{constant},\ b = b_0,\ c = c_0,\ \ b_0$ and c_0 are constants.

The following subcases arise.

Case 5.1: $a \neq \text{constant},\ b_0 \neq 0,\ c_0 \neq 0.$

The subcases of Case 5.1:

Case 5.1.1: $b_0 + c_0 \neq 0.$

Case 5.1.2: $b_0 + c_0 = 0.$

For Cases 5.1.1 and 5.1.2, we get the same first integrals as given in Case 1.1.

Case 5.2: $a \neq \text{constant},\ b_0 = 0,\ c_0 \neq 0.$

Case 5.3: $a \neq \text{constant},\ b_0 \neq 0,\ c_0 = 0.$

Case 5.4: $a \neq \text{constant},\ b_0 = 0,\ c_0 = 0.$

For Cases (5.2 - 5.4), we also obtain the similar first integrals as given in Case 1.1.

Case 6: If $a,\ b$ and c are relating to each other and $a \neq 0,\ b \neq 0,\ c \neq 0.$

At this point the following subcases arise.

Case 6.1: $a = \lambda_1 b,\ \ b = \lambda_2 c,\ \lambda_1,\ \lambda_2$ are constants.

Case 6.2: $a = \lambda b$, c is arbitrary.

Case 6.3: $a = \lambda c$, b is arbitrary, $\lambda =$constant.

Case 6.4: $b = \lambda c$, a is arbitrary, $\lambda =$constant.

For Cases 6.1-6.4 we get the same integrals as in Case 1.1.

Applications to physical systems

We consider some examples of physical systems of two linear nonhomogeneous second-order ODEs including ones that do not admit standard Lagrangians. We utilise transformations that reduce systems of two linear second-order ODEs into canonical form (3.3). Then any system of two linear second-order ODEs can be mapped to the linear system (3.4) (see Wafo and Mahomed 2000).

We employ the linear change of variables (Wafo and Mahomed 2001)

$$y = Mz + y^*, \quad M = [m_{ij}(x)], \tag{4.74}$$

where $y^* = (y, z)^T$ is a particular solution of (3.1) and $m_j = (m_{ij})^T, \ i, j = 1, 2$ are two linearly independent solutions of $2y' - Ay = 0$. Under (4.74) the system (3.1) is transformed to the canonical form (3.3) in which $\bar{B} = M^{-1}(AM' + BM - M'')$.

1. In the first example we consider the conservative system with two degrees of freedom (see Vujanovic 1981)

$$y'' = 3y - 2z,$$

$$z'' = -y + 2z. \tag{4.75}$$

It can be easily seen that the system (4.75) is in canonical form (3.3).

Applying the change of variables (see Wafo and Mahomed 2000)

$$\bar{y} = y/\phi(x), \quad \bar{z} = z/\phi(x), \quad \bar{x} = \int^x \phi^{-2}(s)ds, \tag{4.76}$$

where ϕ satisfies

$$\phi'' - \frac{5}{2}\phi = 0, \tag{4.77}$$

which results in

$$\phi = c_1 \exp\,(\sqrt{\frac{5}{2}}x) + c_2 \exp\,(-\sqrt{\frac{5}{2}}x), \quad c_i = \text{constants}, \tag{4.78}$$

we find that the system (4.75) becomes

$$\bar{y''} = \frac{\phi^4}{2}\bar{y} - 2\phi^4\bar{z},$$

$$z'' = -\phi^4\bar{y} - \frac{\phi^4}{2}\bar{z}. \tag{4.79}$$

Note that the system (4.79) belongs to the Case 6.1 and we construct four first integrals as listed in the Table. In Vujanovic (1981) three first integrals were reported.

2. The time-dependent oscillator system (note that x is taken as the time here)

$$y'' + \omega_1^2(x)y = 0,$$

$$z'' + \omega_2^2(x)z = 0, \tag{4.80}$$

where $\omega_1(x)$ and $\omega_2(x)$ are the frequencies, is investigated. We are interested in finding the canonical form of system (4.80). Simple inspection shows that system (4.80) associated with equation (3.1) is already in canonical form (3.3). The system (4.80) will be reduced to the following form under the change of variables as in (4.76)

$$\bar{y}'' = -\nu(x)\bar{y},$$

$$\bar{z}'' = \nu(x)\bar{z}, \tag{4.81}$$

where

$$\nu(x) = (\omega_1^2(x) - \omega_2^2(x))\frac{\phi^4}{2}, \tag{4.82}$$

in which ϕ is the solution of the one dimensional time-dependent oscillator

$$\phi'' + \frac{\omega_1^2 + \omega_2^2}{2}\phi = 0. \tag{4.83}$$

At this point we consider the following assumptions.

2.1. If $\omega_1 = \omega_2$ (resonant case), then system (4.81) transforms to

$$\bar{y}'' = 0,$$

$$\bar{z}'' = 0, \tag{4.84}$$

which corresponds to the Case 2.5 and we have computed eight first integrals for this cases given in the Table.

2.2. If $\omega_1 \neq \omega_2$ (non-resonant case) then system (4.81) falls into Case 5.4 which results in four integrals as mentioned in the Table.

3. A linearly damped vibrating system with two degrees of freedom can be governed by the equations of motion (Vujanovic 1986)

$$y'' = -\omega^2 y - a_{11}y' - a_{12}z',$$
$$z'' = -\Omega^2 z - a_{21}y' - a_{22}z', \tag{4.85}$$

in which ω^2, Ω^2 and a_{ij} are constant parameters.

Perform the linear change of variables (4.74) and for sake of simplicity we choose $a_{11} = -4,\ a_{12} = -6,\ a_{21} = -4,\ a_{22} = -2$. Simple manipulations show that

$$M = \begin{pmatrix} \exp(-x) & 3\exp(4x) \\ -\exp(-x) & 2\exp(4x) \end{pmatrix}$$

is the solution of $2y' - Ay = 0$ and the system (4.85) takes the form given in (3.3). One can easily check that

$$\bar{B} = \begin{pmatrix} \frac{2}{5}\omega^2 - \frac{3}{5}\Omega^2 + 1 & \frac{6}{5}(\omega^2 + \Omega^2)\exp(5x) \\ \frac{1}{5}(\omega^2 + \Omega^2)\exp(-5x) & \frac{3}{5}\omega^2 - \frac{2}{5}\Omega^2 + 16 \end{pmatrix}.$$

The resulting system can also be expressed as

$$y'' = ay + b(x)z,$$
$$z'' = c(x)y + dz, \tag{4.86}$$

where

$$a = \frac{2}{5}\omega^2 - \frac{3}{5}\Omega^2 + 1,\ \ b(x) = \frac{6}{5}(\omega^2 + \Omega^2)\exp(5x),$$
$$c(x) = \frac{1}{5}(\omega^2 + \Omega^2)\exp(-5x),\ \ d = \frac{3}{5}\omega^2 - \frac{2}{5}\Omega^2 + 16. \tag{4.87}$$

The system (4.86) reduces to the following by use of the transformations (4.76)

$$\bar{y''} = \alpha(\bar{x})\bar{y} + \beta(\bar{x})\bar{z},$$

$$\bar{z}'' = \gamma(\bar{x})\bar{y} - \alpha(\bar{x})\bar{z}, \tag{4.88}$$

where

$$\bar{x} = \frac{\exp(2\sqrt{7}x)}{2\sqrt{7}}, \ \alpha = \frac{\phi^4(a-d)}{2}, \ \ \beta = \phi^4 b(x), \ \ \gamma = \phi^4 c(x). \tag{4.89}$$

In (4.89) ϕ satisfies

$$\phi'' - \frac{a+d}{2}\phi = 0. \tag{4.90}$$

If we choose $\omega = 1, \ \Omega = 2$, then from (4.90) $\phi = c_3 \exp(\sqrt{7}x) + c_4 \exp(-\sqrt{7}x)$ or we can select $\phi = \exp(-\sqrt{7}x)$ for $c_3 = 0$ and $c_4 = 1$. The equation (4.89) finally results in

$$\alpha = -8\left[2\sqrt{7}\bar{x}\right]^{-2}, \ \ \beta = 6\left[2\sqrt{7}\bar{x}\right]^{\frac{5-4\sqrt{7}}{2\sqrt{7}}}, \ \ \gamma = \left[2\sqrt{7}\bar{x}\right]^{\frac{-(5+4\sqrt{7})}{2\sqrt{7}}}. \tag{4.91}$$

The system (4.86) in comparison with Case 1.1 gives four first integrals. In Vujanovic (1986), the case $\omega = \Omega$ was considered but only two integrals were reported.

4.4 Conclusion

The first integrals for the general linear system of two second-order ODEs with variable coefficients are derived by using the partial Lagrangian approach. In general the underlying system of two equations does not have a Lagrangian which can be verified from Douglas (1941). In this chapter we have provided the complete classification of partial Noether operators of the general linear system of two second-order ODEs and all the first integrals are constructed with the help of partial Noether operators via a partial Lagrangian. The results for the special case $b(x) = c(x)$ of this system was considered in Chapter 3. The main system is self adjoint in this special case and a Lagrangian exists. This was also reviewed. These equations model important physical phenomena in dynamics such as oscillator and are thus important. These form four, five and eight-dimensional subalgebras of the Lie algebras of the linear system of two second-order ODEs studied in Wafo and Mahomed (2000). This study provides a new way to construct first integrals for equations for which we do not have Lagrangians as partial Lagrangians do exist for such equations in the absence of standard Lagrangians.

Chapter 5

Partial Noether Operators and First Integrals for a System with Two Degrees of Freedom

We construct all partial Noether operators corresponding to a partial Lagrangian for a system with two degrees of freedom. Then all the first integrals are obtained explicitly by utilizing the Noether-like theorem with the help of the partial Noether operators (see Chapter 1). We show how the first integrals can be constructed for the system without the need of a variational principle although the Lagrangian $L = y'^2/2 + z'^2/2 - v(y,z)$ does exist for the system. Our objective is twofold: one is to see the effectiveness of the partial Noether approach for a nonlinear system and the other to determine all the first integrals of the system under study which have not been reported before. Thus, we deduce a complete classification of the potentials $v(y,z)$ for which first integrals exist. This can give rise to further studies on systems which are not Hamiltonian via partial Noether operators and the construction of first integrals from a partial Lagrangian viewpoint.

5.1 Introduction

The notion of partial Noether operators and partial Lagrangians are important in the construction of first integrals for ordinary differential equations that in general do not admit a Lagrangian. Most of the equations that arise in applications do not have a Lagrangian. So the problem is how one can construct first integrals for equations without a variational principle. The objective of this chapter is to classify all the partial Noether operators and to construct all first integrals for a nonlinear system with two degrees of freedom which is Hamiltonian in order to see the effectiveness of using the partial Noether approach. Moreover, we obtain all the first integrals of the Hamiltonian system under consideration. These have not been obtained before in Damianou and Sophocleus (2004).

Hamiltonian systems frequently arise in classical mechanics, in non-linear oscillations and non-linear dynamics (see Barenblatt 1983, Goldstein 1950 and Nayfeh 1979). In classical mechanics Hamiltonian systems appear as physical systems. There are some important papers dealing with Hamiltonian systems. The point symmetries of a Hamiltonian system for two degrees of freedom were obtained by Damianou and Sophocleous (1999). The point symmetry properties of a Lagrangian system for two degrees of freedom were also considered by Sen (1987). Symmetry group classification of a three dimensional Hamiltonian system was investigated by Damianou and Sophocleous (2000). Classification of Noether symmetries for Lagrangian systems with three degrees of freedom were also attempted by Damianou and Sophocleous (2004). Herein, the two degrees of freedom system Noether symmetries are reported.

Noether's theorem (see Noether 1918) provides the relationship between symmetries and the conserved quantities for Euler-Lagrange differential equations once their Noether symmetries are known and is indeed a powerful method to construct conservation laws as mentioned earlier. However, we remind the reader that there are some direct methods (see e.g. Anco and Bluman 1998, Hietarinta 1986, Lewis and Leach 1982, Olver 1986, Sarlet and Cantrijn 1981 and Steudel 1962) as well. In Kara and Mahomed (2006), Kara, Mahomed, Naeem and Wafo (2007), a Noether-like theorem is invoked, which gives the first integrals for the differential equations without the use of a Lagrangian. This was reviewed in Chapter 1. We showed that the first integrals corresponding to the Noether and partial Noether operators of a variable coefficient linear system of two equations are the same (see Chapter 3 and also Naeem and Mahomed 2008b). The difference occurs in the guage terms.

In this chapter, we obtain all the partial Noether operators and first integrals for the two dimensional system that has been studied before via a standard Lagrangian in Damianou and Sophocleous (2004). The previous works Damianou and Sophocleous (2004) did not present the first integrals for the Lagrangian system of two degrees of freedom $L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - v(y,z)$. Here we take an alternative viewpoint. We first obtain the partial Noether operators and then construct the corresponding first integrals of such a system. This, hopefully, will give rise to further studies in the classification of the partial Noether operators for more general systems of two, three and four degrees of freedom and the construction of first integrals from a partial Lagrangian viewpoint.

Suppose that a particle is moving in the (y, z) plane with potential $v(y, z)$. The Hamiltonian in two dimensions is given by

$$H(p_1, p_2, x, y) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + v(y, z), \tag{5.1}$$

where $p_1 = \partial L/\partial y' = y',\ p_2 = \partial L/\partial z' = z'$.

In Newton's form one has

$$\begin{cases} y'' + v_y = 0, \\ z'' + v_z = 0, \end{cases} \tag{5.2}$$

corresponding to the Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - v(y, z). \tag{5.3}$$

The Hamiltonian (5.1) is the total energy of the system and (5.2) is in terms of the force laws. The Lagrangian (5.3) is the natural Lagrangian which is in terms of the kinetic and potential energies of the system.

5.2 Partial Noether operators of (5.2)

A partial Lagrangian for system (5.2) satisfying partial Euler-Lagrange equations $\delta L/\delta y = v_y$ and $\delta L/\delta z = v_z$, is

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2. \tag{5.4}$$

In order to construct the partial Noether operators, we invoke (1.4) upto first-order derivatives

$$X = \xi\ \frac{\partial}{\partial x} + \eta^1\ \frac{\partial}{\partial y} + \eta^2\ \frac{\partial}{\partial z} + \zeta_x^1\ \frac{\partial}{\partial y'} + \zeta_x^2\ \frac{\partial}{\partial z'}, \tag{5.5}$$

where

$$\zeta_x^1 = D_x(\eta^1) - y'D_x(\xi),\ \ \zeta_x^2 = D_x(\eta^2) - z'D_x(\xi), \tag{5.6}$$

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots\ . \tag{5.7}$$

The partial Noether operator determining equation is (see Chapter 1, Section 1.1)

$$X(L) + LD_x(\xi) = W^1\ \frac{\delta L}{\delta y} + W^2\ \frac{\delta L}{\delta z} + D_x(B)\ , \tag{5.8}$$

where $B = B(x, y, z)$ is the guage term. The partial Noether operators corresponding to the partial Lagrangian (5.4) satisfies

$$\begin{aligned}&[\eta_x^1 + \eta_y^1 y' + \eta_z^1 z' - y'(\xi_x + \xi_y y' + \xi_z z']y' + ([\eta_x^2 + \eta_y^2 y' + \eta_z^2 z'\\ &-z'(\xi_x + \xi_y y' + \xi_z z']y' + (\xi_x + \xi_y y' + \xi_z z')(\frac{1}{2}y'^2 + \frac{1}{2}z'^2) = (\eta^1 - y'\xi)v_y\\ &+(\eta^2 - z'\xi)v_z + B_x + B_y y' + B_z z'.\end{aligned} \tag{5.9}$$

After separation, the following occurs

$$\xi_y = 0,\ \xi_z = 0, \tag{5.10}$$

$$\eta_y^1 - \frac{1}{2}\xi_x = 0,\ \eta_z^2 - \frac{1}{2}\xi_x = 0,\ \eta_z^1 + \eta_y^2 = 0, \tag{5.11}$$

$$\eta_x^1 = -\xi v_y + B_y, \tag{5.12}$$

$$\eta_x^2 = -\xi v_z + B_z, \tag{5.13}$$

$$\eta^1 v_y + \eta^2 v_z + B_x = 0. \tag{5.14}$$

From equations (5.10) and (5.11), we find that

$$\xi = \alpha(x), \tag{5.15}$$

$$\eta^1 = \frac{1}{2}\alpha' y - C(x)z + S(x), \tag{5.16}$$

$$\eta^2 = \frac{1}{2}\alpha' z + C(x)y + F(x). \tag{5.17}$$

The substitution of these expressions into (5.12) and (5.13) results in

$$B = \frac{1}{4}\alpha'' y^2 - C'(x)yz + yS'(x) + \alpha(x)v + T(x, z), \tag{5.18}$$

where

$$T = \frac{1}{4}\alpha'' z^2 + F'(x)z + U(x),\ C(x) = A. \tag{5.19}$$

The replacement of η^1, η^2 and B in (5.14) leads to the following equation

$$(\frac{1}{2}\alpha' y - Az + S(x))v_y + (\frac{1}{2}\alpha' z + Ay + F(x))v_z$$
$$+\frac{1}{4}\alpha'''(y^2 + z^2) + yS''(x) + \alpha'(x)v + F''(x)z + U'(x) = 0. \tag{5.20}$$

First of all if v is arbitrary, then we get the translation in x partial Noether operator with $B = v$ and obvious integral which is the Hamiltonian itself.

The differentiation of (5.20) with respect to x yields

$$(\frac{1}{2}\alpha'' y + S'(x))v_y + (\frac{1}{2}\alpha'' z + F'(x))v_z + \frac{1}{4}\alpha''''(y^2 + z^2)$$
$$+yS'''(x) + \alpha''(x)v + F'''(x)z + U''(x) = 0. \tag{5.21}$$

In order to solve (5.21), two cases arise, viz. $\alpha'' \neq 0$ and $\alpha'' = 0$.

Case 1: $\alpha'' \neq 0$.

The division of equation (5.21) by α'' and then differentiation with respect to x gives

$$(\frac{S'}{\alpha''})'v_y + (\frac{F'}{\alpha''})'v_z + \frac{1}{4}(\frac{\alpha'''}{\alpha''})'(y^2 + z^2) + (\frac{S'''}{\alpha''})'y + (\frac{F'''}{\alpha''})'z + (\frac{U''}{\alpha''})' = 0. \qquad (5.22)$$

In equation (5.22) v satisfies the first order partial differential equation of the following form

$$\lambda_1 v_y + \lambda_2 v_z + \lambda_3(y^2 + z^2) + \lambda_4 y + \lambda_5 z + \lambda_6 = 0. \qquad (5.23)$$

For Case 1 (when $\alpha'' \neq 0$), there are three subcases:

Case 1.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0$.

Case 1.2: $\lambda_1 \neq 0,\ \lambda_2 = 0$.

Case 1.3: $\lambda_1 = 0,\ \lambda_2 \neq 0$.

For Case 2 (when $\alpha'' = 0$), we obtain

$$S'(x)v_y + F'(x)v_z + yS'''(x) + F'''(x)z + U''(x) = 0. \qquad (5.24)$$

or

$$\lambda_1 v_y + \lambda_2 v_z + \lambda_3 y + \lambda_4 z + \lambda_5 = 0. \qquad (5.25)$$

In order to solve (5.25) the following three subcases are considered.

Case 2.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0$.

Case 2.2: $\lambda_1 \neq 0,\ \lambda_2 = 0$.

Case 2.3: $\lambda_1 = 0,\ \lambda_2 \neq 0$.

We provide details of the calculations for all the subcases of Case 1 and Case 2.

Case 1.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0$.

The following subcases arise.

Case 1.1.1: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0.$

The equation (5.23) gives

$$v = -\frac{\lambda_1^2\lambda_3 z^3}{3\lambda_2^3} - \frac{\lambda_3 y^2 z}{\lambda_2} + \frac{\lambda_1\lambda_3 y z^2}{\lambda_2^2} - \frac{\lambda_3 z^3}{3\lambda_2} + \frac{\lambda_1\lambda_4 z^2}{2\lambda_2^2} - \frac{\lambda_4 y z}{\lambda_2} - \frac{\lambda_5 z^2}{2\lambda_2}$$
$$-\frac{\lambda_6}{\lambda_2} z + f(\lambda_2 y - \lambda_1 z), \tag{5.26}$$

where

$$\lambda_2 y - \lambda_1 z = \phi \text{ or } y = \frac{\lambda_1 z}{\lambda_2} + \frac{\phi}{\lambda_2}. \tag{5.27}$$

The substitution of v (in terms of y from (5.27)) in equation (5.21) and then separation with respect to powers of z gives

$$\lambda_3 = 0,\ -\frac{\alpha'\lambda_1\lambda_4}{\lambda_2^2} + \frac{A\lambda_4}{\lambda_2} - \frac{\alpha'\lambda_5}{\lambda_2} - \frac{A\lambda_1\lambda_5}{\lambda_2^2} + \frac{1}{4}\alpha''' + \frac{\alpha'''\lambda_1^2}{4\lambda_2^2} = 0, \tag{5.28}$$

$$(\frac{\alpha'\lambda_1}{2\lambda_2} - A)(\lambda_2 f'(\phi)) - \frac{\lambda_4}{\lambda_2}(\frac{\alpha'\phi}{2\lambda_2} + S(x)) + (\frac{\alpha'}{2} + \frac{A\lambda_1}{\lambda_2})(-\frac{\lambda_4\phi}{\lambda_2^2} - \frac{\lambda_6}{\lambda_2} - \lambda_1 f'(\phi))$$
$$-\frac{\lambda_5}{\lambda_2}(\frac{A\phi}{\lambda_2} + F(x)) + \frac{\alpha'''\lambda_1\phi}{2\lambda_2^2} + \frac{\lambda_1}{\lambda_2}S''(x) - \frac{\alpha'\lambda_4\phi}{\lambda_2^2} - \frac{\alpha'\lambda_6}{\lambda_2} + F''(x) = 0, \tag{5.29}$$

$$(\frac{\alpha'\phi}{2\lambda_2} + S(x))(\lambda_2 f'(\phi)) + (\frac{A\phi}{\lambda_2} + F(x))(-\frac{\lambda_4\phi}{\lambda_2^2} - \frac{\lambda_6}{\lambda_2} - \lambda_1 f'(\phi))$$
$$+\frac{\alpha'''\phi^2}{4\lambda_2^2} + \frac{\phi}{\lambda_2}S''(x) + \alpha' f(\phi) + U'(x) = 0. \tag{5.30}$$

Routine but lengthy calculations lead to

$$v = -\frac{\lambda_6}{\lambda_2} z + f(\phi), \tag{5.31}$$

where

$$f(\phi) = d_4 + d_5\phi \text{ and } \phi = \lambda_2 y - \lambda_1 z. \tag{5.32}$$

In this case we find that $\lambda_4 = 0,\ \lambda_5 = 0$ and

$$\alpha = d_1 + d_2 x + d_3 x^2, \tag{5.33}$$

$$S(x) = -\frac{3d_5\lambda_2}{2}(\frac{d_2x^2}{2} + \frac{d_3x^3}{3}) + \frac{A\lambda_6x^2}{2\lambda_2} + \frac{A\lambda_1 d_5x^2}{2} + d_6x + d_7, \tag{5.34}$$

$$F(x) = \frac{d_5}{2}(A\lambda_2 + \frac{A\lambda_1^2}{\lambda_2})x^2 + \frac{3\lambda_6}{2\lambda_2}(\frac{d_2x^2}{2} + \frac{d_3x^3}{3}) + \frac{A\lambda_1\lambda_6x^2}{2\lambda_2^2}$$

$$-\frac{\lambda_1}{\lambda_2}[-\frac{3}{2}d_5\lambda_2(\frac{d_2x^2}{2} + \frac{d_3x^3}{3}) + \frac{A\lambda_6x^2}{2\lambda_2} + \frac{A\lambda_1d_5x^2}{2}] + d_8x + d_9, \tag{5.35}$$

$$U(x) = -d_5\lambda_2[-\frac{3}{2}d_5\lambda_2(\frac{d_2x^3}{6} + \frac{d_3x^4}{12}) + \frac{A\lambda_6x^3}{6\lambda_2} + \frac{A\lambda_1d_5x^3}{6} + \frac{d_6x^2}{2}$$

$$+d_7x] + (\frac{\lambda_6}{\lambda_2} + \lambda_1d_5)[\frac{d_5}{6}(A\lambda_2 + \frac{A\lambda_1^2}{\lambda_2})x^3 + \frac{3\lambda_6}{2\lambda_2}(\frac{d_2x^3}{6} + \frac{d_3x^4}{12})$$

$$+\frac{A\lambda_1\lambda_6x^3}{6\lambda_2^2} - \frac{\lambda_1}{\lambda_2}(-\frac{3}{2}d_5\lambda_2(\frac{d_2x^3}{6} + \frac{d_3x^4}{12}) + \frac{A\lambda_6x^3}{6\lambda_2}$$

$$+\frac{A\lambda_1d_5x^3}{6}) + \frac{d_8x^2}{2} + d_9x] - d_4(d_2x + d_3x^2) + d_{10}. \tag{5.36}$$

The partial Noether operators and B in each case are given by different choice of constants equal to one and the other constants equal to zero.

$$X_1 = (-z + \frac{\lambda_6}{2\lambda_2}x^2)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},\ B = \frac{\lambda_6}{2\lambda_2}xy,$$

$$X_2 = \frac{\partial}{\partial x},\ B = -\frac{\lambda_6}{\lambda_2}z,$$

$$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + (\frac{z}{2} + \frac{3\lambda_6}{4\lambda_2}x^2)\frac{\partial}{\partial z},\ B = \frac{\lambda_6}{2\lambda_2}xz + \frac{\lambda_6^2}{4\lambda_2^2}x^3,$$

$$X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (xz + \frac{\lambda_6}{2\lambda_2}x^3)\frac{\partial}{\partial z},$$

$$B = \frac{1}{2}(y^2 + z^2) + \frac{\lambda_6}{2\lambda_2}x^2z + \frac{\lambda_6}{2\lambda_2}x^2z + \frac{\lambda_6^2}{4\lambda_2^2}x^3,$$

$$X_5 = x\frac{\partial}{\partial y},\ B = y,$$

$$X_6 = x\frac{\partial}{\partial y},\ B = 0,$$

$$X_7 = x\frac{\partial}{\partial z},\ B = z - \frac{\lambda_1\lambda_6}{2\lambda_2^2}x^2,$$

$$X_8 = x\frac{\partial}{\partial z},\ B = -\frac{\lambda_1\lambda_6}{\lambda_2^2}x. \tag{5.37}$$

Case 1.1.2: $A=0,\ S(x)=0,\ F(x)=0.$

Equations (5.21) with the substitution of $A=0,\ S(x)=0$ and $F(x)=0$ reduce to

$$\lambda_3=0,\ -\frac{\alpha'\lambda_1\lambda_4}{\lambda_2^2}-\frac{\alpha'\lambda_5}{\lambda_2}+\frac{1}{4}\alpha'''+\frac{\alpha'''\lambda_1^2}{4\lambda_2^2}=0, \tag{5.38}$$

$$\alpha'''\lambda_1-4\alpha'\lambda_4=0, \tag{5.39}$$

$$\alpha'\phi f(\phi')+\frac{\alpha'''\phi^2}{4\lambda_2^2}+\alpha' f(\phi)+U'(x)=0. \tag{5.40}$$

The solution of equations (5.38)-(5.40) and with the help of (5.15)-(5.19), results in the equations

$$v=-\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{1}{y^2}f(\frac{z}{y}), \tag{5.41}$$

$$\xi=d_{12}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_{13}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-d_{11}\frac{\lambda_1}{4\lambda_4}, \tag{5.42}$$

$$\eta^1=\frac{1}{2}\alpha' y=\left[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{12}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\sqrt{\frac{\lambda_4}{\lambda_1}}d_{13}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\right]y, \tag{5.43}$$

$$\eta^2=\frac{1}{2}\alpha' z=\left[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{12}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\sqrt{\frac{\lambda_4}{\lambda_1}}d_{13}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\right]z, \tag{5.44}$$

$$B=\left[\frac{\lambda_4}{\lambda_1}d_{12}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{\lambda_4}{\lambda_1}d_{13}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\right](y^2+z^2)$$
$$+\left[d_{12}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_{13}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-d_{11}\frac{\lambda_1}{4\lambda_4}\right]\left(-\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{1}{y^2}f(\frac{z}{y})\right). \tag{5.45}$$

It can be easily verified that the above equations yield the following partial Noether operators and

$$X_1=-\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x},\quad B=\frac{1}{8}(y^2+z^2)-\frac{\lambda_1}{4\lambda_4 y^2}f(\frac{z}{y}),$$

$$X_2=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\left(\frac{\partial}{\partial x}+\sqrt{\frac{\lambda_4}{\lambda_1}}y\frac{\partial}{\partial y}+\sqrt{\frac{\lambda_4}{\lambda_1}}z\frac{\partial}{\partial z}\right),$$

$$B=\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\left[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{1}{y^2}f(\frac{z}{y})\right],$$

$$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big(\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}y\frac{\partial}{\partial y} - \sqrt{\frac{\lambda_4}{\lambda_1}}z\frac{\partial}{\partial z}\Big),$$

$$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{1}{y^2}f(\frac{z}{y})\Big]. \tag{5.46}$$

Case 1.1.3: $A = 0,\ F(x) = \frac{\lambda_2}{\lambda_1}S(x)$.

In this case, we obtain

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2}, \tag{5.47}$$

$$\xi = \alpha(x) = d_{15}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) + d_{16}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - d_{14}\frac{\lambda_1}{4\lambda_4}, \tag{5.48}$$

$$\eta^1 = \frac{1}{2}\alpha' y + S(x) = \Big[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{15}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - \sqrt{\frac{\lambda_4}{\lambda_1}}d_{16}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big]y$$
$$+d_{17}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x) + +d_{18}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x), \tag{5.49}$$

$$\eta^2 = \frac{1}{2}\alpha' z + \frac{\lambda_2}{\lambda_1}S(x) = \Big[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{15}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - \sqrt{\frac{\lambda_4}{\lambda_1}}d_{16}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big]z$$
$$+\frac{\lambda_2}{\lambda_1}\Big(d_{17}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x) + d_{18}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big), \tag{5.50}$$

$$B = \Big[\frac{\lambda_4}{\lambda_1}d_{15}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) + \frac{\lambda_4}{\lambda_1}d_{16}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big](y^2+z^2)$$
$$+(y + \frac{\lambda_2}{\lambda_1}z)\sqrt{\frac{\lambda_4}{\lambda_1}}\Big[d_{17}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x) - d_{18}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big] + \Big[d_{18}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$
$$+d_{15}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - d_{16}\frac{\lambda_1}{4\lambda_4} - d_{14}\frac{\lambda_1}{4\lambda_4}\Big]\Big(-\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\nu}{(z - \frac{\lambda_2}{\lambda_1}y)^2}\Big). \tag{5.51}$$

Setting of each constant equal to one and the remaining one to zero yield the partial Noether operators which are summarized in the table at the end of Section 5.3.

Case 1.1.4: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) = 0$.

The replacement of the above values of $U = 0$ as well as keeping others nonzero in equation (5.21), after some simple calculations, gives

$$v = -\frac{\lambda_4}{2\lambda_1}(y^2+z^2), \tag{5.52}$$

$$\xi = \alpha(x) = d_{20} \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) + d_{21} \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - d_{19}\frac{\lambda_1}{4\lambda_4}, \quad (5.53)$$

$$\eta^1 = \frac{1}{2}\alpha' y - Az + S(x) = \Big[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{20} \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - \sqrt{\frac{\lambda_4}{\lambda_1}}d_{21} \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big]y$$

$$-Az + d_{22} \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x) + d_{23} \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x), \quad (5.54)$$

$$\eta^2 = \frac{1}{2}\alpha' z + \frac{\lambda_2}{\lambda_1}S(x) = \Big[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{20} \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - \sqrt{\frac{\lambda_4}{\lambda_1}}d_{21} \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big]z$$

$$+Ay + d_{24} \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x) + d_{25} \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x), \quad (5.55)$$

$$B = \Big[\frac{\lambda_4}{\lambda_1}d_{20} \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x) + \frac{\lambda_4}{\lambda_1}d_{21} \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big](y^2+z^2) + y\sqrt{\frac{\lambda_4}{\lambda_1}}\Big[d_{22} \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$

$$-d_{23} \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big] + z\sqrt{\frac{\lambda_4}{\lambda_1}}\Big[d_{24} \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x) - d_{25} \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\Big] + \Big[d_{20} \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$

$$+d_{21} \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x) - d_{19}\frac{\lambda_1}{4\lambda_4}\Big]\Big(-\frac{\lambda_4}{2\lambda_1}(y^2+z^2)\Big). \quad (5.56)$$

Equations (5.53)-(5.56) provide the operators and guage terms which are presented in the table.

Case 1.2: $\lambda_1 \neq 0,\ \lambda_2 = 0.$
At this point, the following cases should be considered.

Case 1.2.1: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0.$
Equation (5.23) implies that

$$v = -\frac{1}{\lambda_1}[\lambda_3(\frac{y^3}{3} + yz^2) + \frac{\lambda_4 y^2}{2} + \lambda_5 yz + \lambda_6 y] + f(z), \quad (5.57)$$

Insertion of v from (5.57) into equation (5.21) and then separation with powers of y gives rise to

$$\lambda_3 = 0,\ \frac{\alpha'''}{4} - \frac{\lambda_4\alpha'}{\lambda_1} + \frac{A\lambda_5}{\lambda_1} = 0, \quad (5.58)$$

$$S''(x) - \frac{\lambda_4}{\lambda_1}S(x) - \frac{\lambda_5}{\lambda_1}F(x) - \frac{2\lambda_5\alpha' z}{\lambda_1} - \frac{3\lambda_6\alpha'}{2\lambda_1} + \frac{A\lambda_4 z}{\lambda_1} + Af'(z) = 0, \quad (5.59)$$

$$(-Az+S(x))(-\frac{\lambda_5 z}{\lambda_1}-\frac{\lambda_6}{\lambda_1})+(\frac{\alpha' z}{2}+F(x)f'(z)+\alpha' f(z)$$
$$+\frac{1}{4}\alpha''' z^2+F''(x)z+U'(x)=0. \tag{5.60}$$

After some simple calculations the equations (5.58)-(5.60) yield

$$\lambda_5=0,\ \alpha(x)=d_3+d_4\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_5\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x), \tag{5.61}$$

$$S(x)=d_6\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_7\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{\lambda_6 d_4}{\sqrt{\lambda_1\lambda_4}}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$
$$-\frac{\lambda_6 d_5}{\sqrt{\lambda_1\lambda_4}}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{Ad_1\lambda_1}{\lambda_4}, \tag{5.62}$$
$$F(x)=d_8\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_9\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\sqrt{\frac{\lambda_1}{\lambda_4}}d_1d_4\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$
$$+\sqrt{\frac{\lambda_1}{\lambda_4}}d_1d_5\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{\lambda_6 A}{\lambda_4}, \tag{5.63}$$
$$U(x)=\frac{\lambda_6}{\lambda_1}[\sqrt{\frac{\lambda_1}{\lambda_4}}d_6\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\sqrt{\frac{\lambda_1}{\lambda_4}}d_7\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{\lambda_6 d_4}{2\lambda_4}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$
$$+\frac{\lambda_6 d_5}{2\lambda_4}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{Ad_1\lambda_1}{\lambda_4}x]-d_1[\sqrt{\frac{\lambda_1}{\lambda_4}}d_8\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\sqrt{\frac{\lambda_1}{\lambda_4}}d_9\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$$
$$-\frac{d_1d_4\lambda_1}{2\lambda_4}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\frac{d_1d_5\lambda_1}{2\lambda_4}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+\frac{A\lambda_6}{\lambda_4}x]$$
$$-d_2[d_4\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_5\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)]+d_{10}. \tag{5.64}$$

From equations (5.64) it follows that the system (5.2) admits eight operators for Case 1.2.1. The operators and guage terms for this case are listed in the table in Section 5.3.

Case 1.2.2: $A\neq 0,\ S(x)=0,\ F(x)=0.$

In this case

$$v=-\frac{\lambda_4}{2\lambda_1}(y^2+z^2)+\frac{\mu}{y^2+z^2}, \tag{5.65}$$

$$\xi=\alpha(x)=d_{15}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)+d_{16}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-d_{14}\frac{\lambda_1}{4\lambda_4}, \tag{5.66}$$

$$\eta^1=\frac{1}{2}\alpha' y=\left[\sqrt{\frac{\lambda_4}{\lambda_1}}d_{15}\exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)-\sqrt{\frac{\lambda_4}{\lambda_1}}d_{16}\exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\right]y, \tag{5.67}$$

$$\eta^2 = \frac{1}{2}\alpha' z = \Big[\sqrt{\frac{\lambda_4}{\lambda_1}} d_{12} \exp{(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)} - \sqrt{\frac{\lambda_4}{\lambda_1}} d_{13} \exp{(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)}\Big] z, \tag{5.68}$$

$$B = \Big[\frac{\lambda_4}{\lambda_1} d_{15} \exp{(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)} + \frac{\lambda_4}{\lambda_1} d_{16} \exp{(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)}\Big](y^2 + z^2)$$

$$+\Big[d_{15} \exp{(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)} + d_{16} \exp{(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)} - d_{14}\frac{\lambda_1}{4\lambda_4}\Big]\Big(-\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\mu}{y^2+z^2}\Big). \tag{5.69}$$

The partial Noether operators computed for Case 1.2.2 when $A \neq 0,\ S(x) = 0,\ F(x) = 0$ are presented in Section 5.3.

Case 1.3: $\lambda_1 = 0,\ \lambda_2 \neq 0.$

From equation (5.23), we obtain

$$v = -\frac{1}{\lambda_2}[\lambda_3(y^2 z + \frac{z^3}{3}) + \lambda_4 yz + \frac{\lambda_5 z^2}{2} + \lambda_6 z] + f(y). \tag{5.70}$$

Equation (5.21) with the help of v splits into the following

$$\lambda_3 = 0,\ \frac{\alpha'''}{4} - \frac{\lambda_5 \alpha'}{\lambda_2} + \frac{A\lambda_4}{\lambda_2} = 0, \tag{5.71}$$

$$F''(x) - \frac{\lambda_5}{\lambda_2}F(x) - \frac{\lambda_4}{\lambda_2}S(x) - \frac{2\lambda_4\alpha' y}{\lambda_2} - \frac{3\lambda_6\alpha'}{2\lambda_2} - \frac{A\lambda_5 y}{\lambda_2} - Af'(y) = 0, \tag{5.72}$$

$$(\frac{\alpha' y}{2} + S(x))f'(y) + (Ay + F(x))(-\frac{\lambda_4 y}{\lambda_2} - \frac{\lambda_6}{\lambda_2}) + \alpha' f(y)$$

$$+\frac{1}{4}\alpha''' y^2 + F''(x)y + U'(x) = 0. \tag{5.73}$$

The solution of equations (5.71)-(5.73) can be written as

$$\lambda_4 = 0,\ \alpha(x) = d_3 + d_4 \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x) + d_5 \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x), \tag{5.74}$$

$$F(x) = d_6 \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x) + d_7 \exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x) + \frac{\lambda_6 d_4}{\sqrt{\lambda_2\lambda_5}} \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)$$

$$-\frac{\lambda_6 d_5}{\sqrt{\lambda_2\lambda_5}} \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \frac{Ad_1\lambda_2}{\lambda_5}, \tag{5.75}$$

$$S(x) = d_8 \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x) + d_9 \exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \sqrt{\frac{\lambda_2}{\lambda_5}} d_1 d_4 \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)$$

$$+\sqrt{\frac{\lambda_2}{\lambda_5}}d_1 d_5 \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \frac{\lambda_6 A}{\lambda_5} \tag{5.76}$$

and

$$U(x) = \frac{\lambda_6}{\lambda_1}[\sqrt{\frac{\lambda_2}{\lambda_5}}d_6 \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \sqrt{\frac{\lambda_2}{\lambda_5}}d_7 \exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x) + \frac{\lambda_6 d_4}{2\lambda_5}\exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)$$
$$+\frac{\lambda_6 d_5}{2\lambda_5}\exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \frac{Ad_1\lambda_2}{\lambda_5}x] - d_1[\sqrt{\frac{\lambda_2}{\lambda_5}}d_8 \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \sqrt{\frac{\lambda_2}{\lambda_5}}d_9 \exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)$$
$$-\frac{d_1 d_4 \lambda_2}{2\lambda_5}\exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \frac{d_1 d_5 \lambda_2}{2\lambda_5}\exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x) - \frac{A\lambda_6}{\lambda_5}x]$$
$$-d_2(d_4 \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x) + d_5 \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)) + d_{10}. \tag{5.77}$$

Routine calculations show that system (5.2) also admits eight operators for Case 1.3 which are given in the next section.

Case 2: If $\alpha'' = 0$, then

$$\alpha(x) = d_1 + d_2 x. \tag{5.78}$$

From equation (5.21), we have

$$S'(x)v_y + F'(x)v_z + yS'''(x) + F'''(x)z + U''(x) = 0. \tag{5.79}$$

or

$$\lambda_1 v_y + \lambda_2 v_z + \lambda_3 y + \lambda_4 z + \lambda_5 = 0. \tag{5.80}$$

Whence the following cases should be considered.
Case 2.1: $\lambda_1 \neq 0,\ \lambda_2 \neq 0.$

The subcases of Case 2.1:

Case 2.1.1: $A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0.$

In this case

$$v = -\frac{1}{\lambda_2}[\lambda_3 yz - \frac{\lambda_1\lambda_3 z^2}{2\lambda_2} + \frac{\lambda_4 z^2}{2} + \lambda_5 z] + f(\lambda_2 y - \lambda_1 z), \tag{5.81}$$

where

$$\lambda_2 y - \lambda_1 z = \phi \text{ or } y = \frac{\lambda_1}{\lambda_2}z + \frac{\phi}{\lambda_2}. \tag{5.82}$$

The replacement of v in terms of y from equation (5.81) in equation (5.79) and separation with respect to z to yield

$$A\lambda_2\lambda_3 - A\lambda_1\lambda_4 - d_2\lambda_1\lambda_3 - d_2\lambda_2\lambda_4 = 0, \tag{5.83}$$

$$\begin{aligned}&(\frac{d_2\lambda_1}{2\lambda_2} - A)\lambda_2 f'(\phi) - \frac{\lambda_3}{\lambda_2}(\frac{d_2\phi}{2\lambda_2} + S(x)) - (\frac{d_2}{2} + \frac{A\lambda_1}{\lambda_2})(\frac{\lambda_3\phi}{\lambda_2^2} + \frac{\lambda_5}{\lambda_2}\\ &+\lambda_1 f'(\phi)) - \frac{\lambda_4}{\lambda_2}(\frac{A\phi}{\lambda_2} + F(x)) + \frac{\lambda_1}{\lambda_2}S''(x) - \frac{d_2\lambda_3\phi}{\lambda_2^2} - \frac{d_2\lambda_5}{\lambda_2} + F''(x) = 0,\end{aligned} \tag{5.84}$$

$$\begin{aligned}&(\frac{d_2\phi}{2\lambda_2} + S(x))\lambda_2 f'(\phi) - (\frac{A\phi}{\lambda_2} + F(x))(\frac{\lambda_3\phi}{\lambda_2^2} + \frac{\lambda_5}{\lambda_2} + \lambda_1 f'(\phi))\\ &+\frac{\phi}{\lambda_2}S''(x) + d_2 f(\phi) + U'(x) = 0.\end{aligned} \tag{5.85}$$

The equations (5.83)-(5.85) give the solution

$$v = -\frac{\lambda_5 z}{\lambda_2} + f(\phi), \tag{5.86}$$

where

$$f(\phi) = d_3\phi + d_4, \tag{5.87}$$

and

$$\lambda_3 = 0,\ \lambda_4 = 0,\ f(\phi) = d_3\phi + d_4, \tag{5.88}$$

$$S(x) = \frac{A\lambda_1 d_3 x^2}{2} + \frac{A\lambda_5 x^2}{2\lambda_2} - \frac{3d_2 d_3\lambda_2 x^2}{4} + d_5 x + d_6, \tag{5.89}$$

$$\begin{aligned}F(x) = &-\frac{\lambda_1}{2\lambda_2}(Ad_3\lambda_1 + \frac{A\lambda_5}{\lambda_2} - \frac{3d_2 d_3\lambda_2}{2})x^2 + (Ad_3\lambda_2 + \frac{3d_2\lambda_5}{2\lambda_2}\\ &+\frac{A\lambda_1\lambda_5}{\lambda_2^2} + \frac{Ad_3\lambda_1^2}{\lambda_2})\frac{x^2}{2} + d_7 x + d_8,\end{aligned} \tag{5.90}$$

$$U(x) = -d_3\lambda_2[(\frac{A\lambda_1 d_3}{6} + \frac{A\lambda_5}{6\lambda_2} - \frac{d_2 d_3 \lambda_2}{4})x^3 + \frac{d_5 x^2}{2} + d_6 x]$$
$$+(\frac{\lambda_5}{\lambda_2} + d_3\lambda_1)[(-\frac{Ad_3\lambda_1^2}{6\lambda_2} - \frac{A\lambda_1\lambda_5}{6\lambda_2^2} + \frac{d_2 d_3 \lambda_1}{4})x^3 + (Ad_3\lambda_2 + \frac{3d_2\lambda_5}{2\lambda_2}$$
$$+\frac{A\lambda_1\lambda_5}{\lambda_2^2} + \frac{Ad_3\lambda_1^2}{\lambda_2})\frac{x^3}{6} + \frac{d_7 x^2}{2} + d_8 x] - d_2 d_4 x + d_9. \tag{5.91}$$

Simple manipulations give the following operators and guage terms for Case 2.1.1.

$$X_1 = (-z + \frac{\lambda_5}{2\lambda_2}x^2)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},\ B = \frac{\lambda_5}{\lambda_2}xy,$$
$$X_2 = \frac{\partial}{\partial x},\ B = -\frac{\lambda_5}{\lambda_2}z,$$
$$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + (\frac{z}{2} + \frac{3\lambda_5}{4\lambda_2}x^2)\frac{\partial}{\partial z},\ B = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3,$$
$$X_4 = x\frac{\partial}{\partial y},\ B = y,$$
$$X_5 = \frac{\partial}{\partial y},\ B = 0,$$
$$X_6 = x\frac{\partial}{\partial z},\ B = z + \frac{\lambda_5}{2\lambda_2}x^2,$$
$$X_7 = \frac{\partial}{\partial z},\ B = \frac{\lambda_5}{\lambda_2}x. \tag{5.92}$$

Case 2.1.2: $A \neq 0,\ d_2 = 0,\ S(x) = 0,\ F(x) = 0.$

The partial Noether operators for this case are constructed by using equations (5.79)-(5.80) together with (5.15)-(5.19) and after some simple but lengthy calculations, we find that

$$v = \frac{U'(x)}{A}\arcsin\frac{y}{\sqrt{y^2+z^2}} + f(y^2+z^2), \tag{5.93}$$

$$\xi = d_1,\ \eta^1 = -Az,\ \eta^2 = Ay,\ \ B = d_1 v + U(x). \tag{5.94}$$

The choice of constants give the following operators

$$X_1 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},\ \ B = U(x), \tag{5.95}$$

$$X_2 = \frac{\partial}{\partial x}, \quad B = \frac{U'(x)}{A} \arcsin \frac{y}{\sqrt{y^2+z^2}} + f(y^2+z^2). \tag{5.96}$$

Case 2.1.3: $A \neq 0,\ d_2 \neq 0,\ S(x) = 0,\ F(x) = 0.$

We now substitute $A \neq 0,\ d_2 \neq 0,\ S(x) = 0$ and $F(x) = 0$ in equations (5.79) and (5.80) and we work out for ξ, η^1,η^2 and the guage terms B. The solution of the resulting equations computed for this case with the utilization of (5.15)-(5.19) are as follows

$$v = \frac{1}{y^2+z^2} f((y^2+z^2)^A \exp(-d_2 \arctan \frac{z}{y}), \tag{5.97}$$

$$\xi = d_1 + d_2 x, \quad \eta^1 = \frac{d_2}{2} y - Az, \quad \eta^2 = \frac{d_2}{2} z + Ay, \tag{5.98}$$

$$B = (d_1 + d_2 x)\left[\frac{1}{y^2+z^2} f((y^2+z^2)^A \exp(-d_2 \arctan \frac{z}{y}))\right] + U(x). \tag{5.99}$$

We take the constant one by one equal to one and the remaining one zero to construct the partial Noether operators and the guage terms. The table in Section 5.3 shows the results for this case.

Case 2.1.4: $A = 0,\ d_2 = 0,\ F(x) = \frac{\lambda_2}{\lambda_1} S(x).$

Equations (5.79)-(5.80) and (5.15)-(5.19) collectively yield

$$v = -\frac{\lambda_3}{2\lambda_1}(y^2+z^2) + f(\lambda_2 y - \lambda_1 z), \tag{5.100}$$

$$\xi = d_1, \quad \eta^1 = S(x) = c_1 \exp(\sqrt{\frac{\lambda_3}{\lambda_1}} x) + c_2 \exp(-\sqrt{\frac{\lambda_3}{\lambda_1}} x), \tag{5.101}$$

$$\eta^2 = F(x) = \frac{\lambda_2}{\lambda_1}\left[c_1 \exp(\sqrt{\frac{\lambda_3}{\lambda_1}} x) + c_2 \exp(-\sqrt{\frac{\lambda_3}{\lambda_1}} x)\right], \tag{5.102}$$

$$B = y\sqrt{\frac{\lambda_3}{\lambda_1}}\left[c_1 \exp \sqrt{\frac{\lambda_3}{\lambda_1}} x) - c_2 \exp - \sqrt{\frac{\lambda_3}{\lambda_1}} x)\right] + d_1\left[\frac{-\lambda_3}{2\lambda_1}(y^2+z^2) + f(\lambda_2 y - \lambda_1 z)\right]. \tag{5.103}$$

The operators and guage terms constructed in this case are given in the next section.

Case 2.2: $\lambda_1 \neq 0,\ \lambda_2 = 0.$

The following cases need to be considered for the classification of potential functions.

Case 2.2.1: If $A \neq 0,\ \lambda_3 = 0$, then

$$v = -\frac{1}{\lambda_1}[\frac{\lambda_3 y^2}{2} + \lambda_4 yz + \lambda_5 y] + f(z). \quad (5.104)$$

The insertion of v into equation (5.79) and then splitting with respect to powers of z yields

$$d_2\lambda_3 + A\lambda_4 = 0, \quad (5.105)$$

$$S''(x) - \frac{\lambda_3}{\lambda_1}S(x) - \frac{\lambda_4}{\lambda_1}F_(x) + Af'(z) - \frac{2d_2\lambda_4 z}{\lambda_1} - \frac{3d_2\lambda_5}{2\lambda_1} + \frac{A\lambda_3 z}{\lambda_1} = 0, \quad (5.106)$$

$$(Az - S(x))(\frac{\lambda_4 z}{\lambda_1} + \frac{\lambda_5}{\lambda_1}) + (\frac{d_2 z}{2} + F(x))f'(z) + d_2 f(z) + F''(x)z + U'(x) = 0. \quad (5.107)$$

The solution of system of equations (5.105)-(5.107) can be expressed as

$$\lambda_3 = 0,\ \lambda_4 = 0,\ v = -\frac{\lambda_5 y}{\lambda_1} + f(z), \quad (5.108)$$

where

$$f(z) = d_3 z + d_4, \quad (5.109)$$

and

$$S(x) = -\frac{Ad_3 x^2}{2} + \frac{3d_2\lambda_5 x^2}{4\lambda_1} + d_5 x + d_6, \quad (5.110)$$

$$F(x) = -\frac{A\lambda_5 x^2}{2\lambda_1} - \frac{3d_2 d_3 x^2}{4} + d_7 x + d_8, \quad (5.111)$$

$$U(x) = \frac{\lambda_5}{\lambda_1}[\frac{d_2\lambda_5 x^3}{4\lambda_1} - \frac{Ad_3 x^3}{6} + \frac{d_5 x^2}{2} + d_6 x] - d_2 d_4 x + d_9$$
$$-d_3[-\frac{A\lambda_5 x^3}{6\lambda_1} - \frac{d_2 d_3 x^3}{4} + \frac{d_7 x^2}{2} + d_8 x]. \quad (5.112)$$

The operators are listed in the table in Section 5.3.

Case 2.2.2: $A \neq 0, \ \ \lambda_3 \neq 0$.

We find that

$$v = -\frac{\lambda_3}{2\lambda_1}(y^2+z^2) - \frac{\lambda_5}{\lambda_1}y + h_1 z + h_2, \tag{5.113}$$

$$S(x) = \frac{Ad_3\lambda_1}{\lambda_3} + d_{10}\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x) + d_{11}\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x), \tag{5.114}$$

$$F(x) = \frac{A\lambda_5}{\lambda_3} + d_{12}\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x) + d_{13}\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x), \tag{5.115}$$

$$U(x) = \frac{\lambda_5}{\lambda_1}\left[\sqrt{\frac{\lambda_1}{\lambda_3}}d_{10}\exp(\sqrt{\frac{\lambda_1}{\lambda_3}})x - \sqrt{\frac{\lambda_1}{\lambda_3}}d_{11}\exp(-\sqrt{\frac{\lambda_1}{\lambda_3}})x + \frac{Ad_3\lambda_1 x}{\lambda_3}\right] \tag{5.116}$$

$$-d_3\left[\sqrt{\frac{\lambda_1}{\lambda_3}}d_{12}\exp(\sqrt{\frac{\lambda_1}{\lambda_3}})x - \sqrt{\frac{\lambda_1}{\lambda_3}}d_{13}\exp(-\sqrt{\frac{\lambda_1}{\lambda_3}})x + \frac{A\lambda_5 x}{\lambda_3}\right]. \tag{5.117}$$

Now, from above equations one can easily find the operators as mentioned in the table.

Case 2.3: If $\lambda_1 = 0, \ \lambda_2 \neq 0$, then from (5.80), we have

$$v = -\frac{1}{\lambda_2}[\lambda_3 yz + \frac{\lambda_4 z^2}{2} + \lambda_5 z] + f(y). \tag{5.118}$$

After replacing the value of v in equation (5.79) and splitting with respect to powers of z, we get

$$A\lambda_3 - d_2\lambda_4 = 0, \tag{5.119}$$

$$F''(x) - \frac{\lambda_4}{\lambda_2}F(x) - \frac{\lambda_3}{\lambda_2}S(x) - Af'(y) - \frac{2d_2\lambda_3 y}{\lambda_2} - \frac{3d_2\lambda_5}{2\lambda_2} - \frac{A\lambda_4 y}{\lambda_2} = 0, \tag{5.120}$$

$$-(Ay+F(x))(\frac{\lambda_3 y}{\lambda_2} + \frac{\lambda_5}{\lambda_2}) + (\frac{d_2 y}{2} + S(x))f'(y) + d_2 f(y) + S''(x)y + U'(x) = 0. \tag{5.121}$$

The solution of equations (5.119)-(5.121) is given by

$$\lambda_3 = 0, \ \lambda_4 = 0, \ f(y) = d_3 y + d_4, \tag{5.122}$$

$$S(x) = \frac{A\lambda_5 x^2}{2\lambda_2} - \frac{3d_2 d_3 x^2}{4} + d_7 x + d_8, \tag{5.123}$$

$$F(x) = \frac{Ad_3 x^2}{2} + \frac{3d_2\lambda_5 x^2}{4\lambda_2} + d_5 x + d_6, \tag{5.124}$$

$$U(x) = \frac{\lambda_5}{\lambda_2}[\frac{d_2\lambda_5 x^3}{4\lambda_2} + \frac{Ad_3x^3}{6} + \frac{d_5x^2}{2} + d_6x] - d_2d_4x + d_9$$
$$-d_3[\frac{A\lambda_5x^3}{6\lambda_2} - \frac{d_2d_3x^3}{4} + \frac{d_7x^2}{2} + d_8x]. \quad (5.125)$$

The operators and the guage terms are given at the end of next section. Note that in all the subcases of Cases 1 and 2, $\lambda_i s$, $e_i s$, $f_i s$, $g_i s$, $h_i s$ and $k_i s$ are constants.

Comparison of Lagrangian and partial Lagrangian approaches

In 2004, Damianou and Sophocleous obtained the Noether point symmetries for a two degrees of freedom Lagrangian system and the results for one degree of freedom system were also reviewed in their paper. They did not provide the complete classification for the Noether symmetries as all the higher dimensional symmetry cases are missing for the two-dimensional Lagrangian system. Moreover, the first integrals corresponding to Noether symmetries for the system under study were also not given in their paper.

In this chapter we have provided the complete classification for the partial Noether operators and the first integrals are constructed by means of a partial Lagrangian approach with the help of partial Noether operators for a system with two degrees of freedom. For partial Noether operators obtained herein we have recovered all the cases as given in the case of Noether symmetries and we have found some new results summarized in the table that have not been obtained in the earlier work (see Damianou and Sophocleous 2004). The system of determining equations obtained for partial Noether operators are similar to the case of Noether symmetries (see Damianou and Sophocleous 2004). The reason being that $\delta L/\delta y$ and $\delta L/\delta z$ are independent of derivatives and the algebras for both cases are isomorphic. We give an alternative viewpoint to construct potential functions using the notion of a partial Lagrangian. In fact, a Lagrangian exists for the system under consideration but we wanted to see the effectiveness of a partial Lagrangian approach. We used a similar classification criteria for partial Noether operators as Damianou and Sophocleus in (2004) have performed for the case of Noether symmetries. Then the first integrals are obtained by utilizing the partial Noether's theorem with the help of partial Noether operators via a partial Lagrangian.

Damianou and Sophocleous in (2004) could also have provided the complete classification for a two degrees of freedom Lagrangian system and the first integrals could have been constructed by using the classical Noether's theorem but they did not do so.

The partial Noether operators and B in each case are given in the Table (Section 5.3) by choosing each constant equal to one and the rest equal to zero.

5.3 First integrals

In this section we are concerned with the first integrals of system (5.2) via a partial Noether approach. All the cases for the potential function v are discussed briefly.

Now, if X in (1.4) is a partial Noether operator corresponding to the partial Lagrangian (5.4) for the system (5.2), then a first integral of (5.2) associated with X is constructed from the formula (1.20).

For each case, the first integrals are given in the following table. Precisely, the partial Noether operators and the first integrals of the two cases that arise are listed in the table.

Partial Operators X_i	Guage Terms B	First Integrals I_i
Case 1.1.1 $(A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0)$		
$X_1 = (-z + \frac{\lambda_6}{2\lambda_2}x^2)\frac{\partial}{\partial y}$ $+y\frac{\partial}{\partial z}$,	$B = \frac{\lambda_6}{\lambda_2}xy$,	$I_1 = y'z - yz' + \frac{\lambda_6}{\lambda_2}xy$ $-\frac{\lambda_6}{2\lambda_2}x^2y'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_6}{\lambda_2}z$,	$I_2 = -\frac{\lambda_6}{\lambda_2}z + \frac{1}{2}(y'^2 + z'^2)$,
$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}$ $+(\frac{z}{2} + \frac{3\lambda_6}{4\lambda_2}x^2)\frac{\partial}{\partial z}$,	$B = \frac{\lambda_6}{2\lambda_2}xz + \frac{\lambda_6^2}{4\lambda_2^2}x^3$,	$I_3 = \frac{\lambda_6}{2\lambda_2}xz + \frac{\lambda_6^2}{4\lambda_2^2}x^3 - \frac{3\lambda_6}{4\lambda_2}x^2z'$ $+\frac{x}{2}(y'^2 + z^2) - \frac{1}{2}(yy' + zz')$,
$X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$ $+(xz + \frac{\lambda_6}{2\lambda_2}x^3)\frac{\partial}{\partial z}$,	$B = \frac{1}{2}(y^2 + z^2)$ $+\frac{\lambda_6}{2\lambda_2}x^2z + \frac{\lambda_6^2}{8\lambda_2^2}x^4$,	$I_4 = \frac{1}{2}(y^2 + z^2) + \frac{\lambda_6}{2\lambda_2}x^2z$ $+\frac{\lambda_6^2}{8\lambda_2^2}x^4 - \frac{\lambda_6}{2\lambda_2}x^3z'$ $+\frac{x^2}{2}(y'^2 + z^2) - x(yy' + zz')$,
$X_5 = x\frac{\partial}{\partial y}$,	$B = y$,	$I_5 = y - xy'$,
$X_6 = \frac{\partial}{\partial y}$,	$B = 0$,	$I_6 = -y'$,
$X_7 = x\frac{\partial}{\partial z}$,	$B = z + \frac{\lambda_6}{2\lambda_2}x^2$,	$I_7 = z - xz' + \frac{\lambda_6}{2\lambda_2}x^2$,
$X_8 = \frac{\partial}{\partial z}$.	$B = \frac{\lambda_6}{\lambda_2}x$.	$I_8 = -z' + \frac{\lambda_6}{\lambda_2}x$.
Case 1.1.2 $(A = 0,\ S(x) = 0,\ F(x) = 0)$		
$X_1 = -\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B = \frac{1}{8}(y^2 + z^2)$ $-\frac{\lambda_1}{4\lambda_4 y^2}f(\frac{z}{y})$,	$I_1 = \frac{1}{8}(y^2 + z^2) - \frac{\lambda_1}{4\lambda_4 y^2}f(\frac{z}{y})$ $-\frac{\lambda_1}{8\lambda_4}(y'^2 + z'^2)$,
$X_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y})]$,	$I_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y})$ $+\frac{1}{2}(y'^2 + z'^2) - \sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz')]$,
$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$.	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y})]$.	$I_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{1}{y^2}f(\frac{z}{y})$ $+\frac{1}{2}(y'^2 + z'^2) + \sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz')]$.

Case 1.1.3 $(A \neq 0,\ F(x) = \frac{\lambda_2}{\lambda_1} S(x))$		
$X_1 = -\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B = \frac{1}{8}(y^2+z^2)$ $-\frac{\lambda_1 \nu}{4\lambda_4 (z-\frac{\lambda_2}{\lambda_1}y)^2}$,	$I_1 = \frac{1}{8}(y^2+z^2) - \frac{\lambda_1 \nu}{4\lambda_4 (z-\frac{\lambda_2}{\lambda_1}y)^2}$ $-\frac{\lambda_1}{8\lambda_4}(y'^2+z'^2)$,
$X_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\nu}{(z-\frac{\lambda_2}{\lambda_1}y)^2}]$,	$I_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\nu}{(z-\frac{\lambda_2}{\lambda_1}y)^2}$ $+\frac{1}{2}(y'^2+z'^2) - \sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\nu}{(z-\frac{\lambda_2}{\lambda_1}y)^2}]$,	$I_3 = \exp-(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\nu}{(z-\frac{\lambda_2}{\lambda_1}y)^2}$ $+\frac{1}{2}(y'^2+z'^2) + \sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_4 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial y} + \frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z}]$,	$B = \sqrt{\frac{\lambda_4}{\lambda_1}}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[y + \frac{\lambda_2}{\lambda_1}z]$,	$I_4 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}(y + \frac{\lambda_2}{\lambda_1}z) - y' - \frac{\lambda_2}{\lambda_1}z']$,
$X_5 = \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial y} + \frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z}]$.	$B = -\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[y + \frac{\lambda_2}{\lambda_1}z]$.	$I_5 = -\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}(y + \frac{\lambda_2}{\lambda_1}z) + y' + \frac{\lambda_2}{\lambda_1}z']$.
Case 1.1.4 $(A = 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) = 0)$		
$X_1 = -\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x}$,	$B = \frac{1}{8}(y^2+z^2)$,	$I_1 = \frac{1}{8}(y^2+z^2) - \frac{\lambda_1}{8\lambda_4}(y'^2+z'^2)$,
$X_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)]$,	$I_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{1}{2}(y'^2+z'^2)$ $-\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})]$,	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2)]$,	$I_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{1}{2}(y'^2+z'^2)$ $+\sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')]$,
$X_4 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B = y\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_4 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y - y']$,
$X_5 = \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B = -y\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_5 = -\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y + y']$,

$X_6 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B = z\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_6 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}z - z']$,
$X_7 = \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B = -z\sqrt{\frac{\lambda_4}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_7 = -\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}z + z']$,
$X_8 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$.	$B = 0$.	$I_8 = zy' - yz'$.
Case 1.2.1 $(A \neq 0,\ S(x) \neq 0,\ F(x) \neq 0,\ U(x) \neq 0)$		
$X_1 = -z\frac{\partial}{\partial y} + (y + \frac{\lambda_6}{\lambda_4})\frac{\partial}{\partial z}$,	$B = 0$,	$I_1 = y'z - yz' - \frac{\lambda_6}{\lambda_4}z'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_4}{2\lambda_1}(y^2 + z^2)$ $-\frac{\lambda_6}{\lambda_1}y$,	$I_2 = -\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) - \frac{\lambda_6}{\lambda_1}y$ $+\frac{1}{2}(y'^2 + z'^2)$,
$X_3 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} + (\sqrt{\frac{\lambda_4}{\lambda_1}}y + \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}})\frac{\partial}{\partial y}$ $+\sqrt{\frac{\lambda_4}{\lambda_1}}z\frac{\partial}{\partial z}]$,	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4}]$,	$I_3 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4} + \frac{1}{2}(y'^2 + z'^2)$ $-\sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz') - \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}y']$,
$X_4 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} - (\sqrt{\frac{\lambda_4}{\lambda_1}}y + \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}})\frac{\partial}{\partial y}$ $-\sqrt{\frac{\lambda_4}{\lambda_1}}z\frac{\partial}{\partial z}]$,	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4}]$,	$I_4 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2 + z^2) + \frac{\lambda_6}{\lambda_1}y$ $+\frac{\lambda_6^2}{2\lambda_1\lambda_4} + \frac{1}{2}(y'^2 + z'^2)$ $+\sqrt{\frac{\lambda_4}{\lambda_1}}(yy' + zz') + \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}y']$,
$X_5 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)[\sqrt{\frac{\lambda_4}{\lambda_1}}y$ $+\frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}]$,	$I_5 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y + \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}} - y']$,
$X_6 = \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B = \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[-\sqrt{\frac{\lambda_4}{\lambda_1}}y - \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}}]$,	$I_6 = -\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}y + \frac{\lambda_6}{\sqrt{\lambda_1\lambda_4}} + y']$,
$X_7 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B = \sqrt{\frac{\lambda_4}{\lambda_1}}z\exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)$,	$I_7 = \exp(\sqrt{\frac{\lambda_4}{\lambda_1}}x)[\sqrt{\frac{\lambda_4}{\lambda_1}}z - z']$,
$X_8 = \exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\frac{\partial}{\partial z}$.	$B = -\sqrt{\frac{\lambda_4}{\lambda_1}}z\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)$.	$I_8 = -\exp(-\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_4}{\lambda_1}}z + z']$.

Case 1.2.2 $(A \neq 0,\ S(x)=0,\ F(x)=0)$		
$X_1 = -\frac{\lambda_1}{4\lambda_4}\frac{\partial}{\partial x},$	$B = \frac{1}{8}(y^2+z^2)$ $-\frac{\lambda_1\mu}{4\lambda_4}(y^2+z^2),$	$I_1 = \frac{1}{8}(y^2+z^2) - \frac{\lambda_1\mu}{4\lambda_4(y^2+z^2)}$ $-\frac{\lambda_1}{8\lambda_4}(y'^2+z'^2),$
$X_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})],$	$B = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\mu}{y^2+z^2}],$	$I_2 = \exp(2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\mu}{y^2+z^2}$ $+\frac{1}{2}(y'^2+z'^2) - \sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')],$
$X_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_4}{\lambda_1}}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})],$	$B = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\mu}{y^2+z^2}],$	$I_3 = \exp(-2\sqrt{\frac{\lambda_4}{\lambda_1}}x)\times$ $[\frac{\lambda_4}{2\lambda_1}(y^2+z^2) + \frac{\mu}{y^2+z^2}$ $+\frac{1}{2}(y'^2+z'^2) + \sqrt{\frac{\lambda_4}{\lambda_1}}(yy'+zz')],$
$X_4 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$	$B = 0.$	$I_4 = zy' - yz'.$
Case 1.3 $(\lambda_1 = 0,\ \lambda_2 \neq 0)$		
$X_1 = -(z + \frac{\lambda_6}{\lambda_5})\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},$	$B = 0,$	$I_1 = y'z - yz' + \frac{\lambda_6}{\lambda_5}y',$
$X_2 = \frac{\partial}{\partial x},$	$B = -\frac{\lambda_5}{2\lambda_2}(y^2+z^2)$ $-\frac{\lambda_6}{\lambda_2}z,$	$I_2 = -\frac{\lambda_5}{2\lambda_2}(y^2+z^2)$ $-\frac{\lambda_6}{\lambda_2}z + \frac{1}{2}(y'^2+z'^2),$
$X_3 = \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\frac{\partial}{\partial x} + \sqrt{\frac{\lambda_5}{\lambda_2}}y\frac{\partial}{\partial y}$ $+(\sqrt{\frac{\lambda_5}{\lambda_2}}z + \frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}})\frac{\partial}{\partial z}],$	$B = \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\frac{\lambda_5}{2\lambda_2}(y^2+z^2) + \frac{\lambda_6}{\lambda_2}z$ $+\frac{\lambda_6^2}{2\lambda_2\lambda_5}],$	$I_3 = \exp(2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\frac{\lambda_5}{2\lambda_2}(y^2+z^2) + \frac{\lambda_6}{\lambda_2}z$ $+\frac{\lambda_6^2}{2\lambda_2\lambda_5} + \frac{1}{2}(y'^2+z'^2)$ $-\sqrt{\frac{\lambda_5}{\lambda_2}}(yy'+zz') - \frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}z'],$
$X_4 = \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\frac{\partial}{\partial x} - \sqrt{\frac{\lambda_5}{\lambda_2}}y\frac{\partial}{\partial y}$ $-(\sqrt{\frac{\lambda_5}{\lambda_2}}z + \frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}})\frac{\partial}{\partial z}],$	$B = \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\frac{\lambda_5}{2\lambda_2}(y^2+z^2) + \frac{\lambda_6}{\lambda_2}z$ $+\frac{\lambda_6^2}{2\lambda_2\lambda_5}],$	$I_4 = \exp(-2\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\frac{\lambda_5}{2\lambda_2}(y^2+z^2) + \frac{\lambda_6}{\lambda_2}z$ $+\frac{\lambda_6^2}{2\lambda_2\lambda_5} + \frac{1}{2}(y'^2+z'^2)$ $+\sqrt{\frac{\lambda_5}{\lambda_2}}(yy'+zz') + \frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}z'],$
$X_5 = \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial z},$	$B = \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\sqrt{\frac{\lambda_5}{\lambda_2}}z + \frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}],$	$I_5 = \exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\sqrt{\frac{\lambda_5}{\lambda_2}}z + \frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}} - z'],$

$X_6=\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial z}$,	$B=-\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}]$,	$I_6=-\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\sqrt{\frac{\lambda_5}{\lambda_2}}z+\frac{\lambda_6}{\sqrt{\lambda_2\lambda_5}}+z']$,
$X_7=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial y}$,	$B=\sqrt{\frac{\lambda_5}{\lambda_2}}y\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)$,	$I_7=\exp(\sqrt{\frac{\lambda_5}{\lambda_2}}x)[\sqrt{\frac{\lambda_5}{\lambda_2}}y-y']$,
$X_8=\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\frac{\partial}{\partial y}$.	$B=-\sqrt{\frac{\lambda_5}{\lambda_2}}y\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)$.	$I_8=-\exp(-\sqrt{\frac{\lambda_5}{\lambda_2}}x)\times$ $[\sqrt{\frac{\lambda_5}{\lambda_2}}y+y']$.
Case 2.1.1 $(A\neq 0,\ S(x)\neq 0,\ F(x)\neq 0,\ U(x)\neq 0)$		
$X_1=(-z+\frac{\lambda_5}{2\lambda_2}x^2)\frac{\partial}{\partial y}$ $+y\frac{\partial}{\partial z}$,	$B=\frac{\lambda_5}{\lambda_2}xy$,	$I_1=y'z-yz'+\frac{\lambda_5}{\lambda_2}xy$ $-\frac{\lambda_5}{2\lambda_2}x^2y'$,
$X_2=\frac{\partial}{\partial x}$,	$B=-\frac{\lambda_5}{\lambda_2}z$,	$I_2=-\frac{\lambda_5}{\lambda_2}z+\frac{1}{2}(y'^2+z'^2)$,
$X_3=x\frac{\partial}{\partial x}+\frac{y}{2}\frac{\partial}{\partial y}$ $+(\frac{z}{2}+\frac{3\lambda_5}{4\lambda_2}x^2)\frac{\partial}{\partial z}$,	$B=\frac{\lambda_5}{2\lambda_2}xz+\frac{\lambda_5^2}{4\lambda_2^2}x^3$,	$I_3=\frac{\lambda_5}{2\lambda_2}xz+\frac{\lambda_5^2}{4\lambda_2^2}x^3$ $-\frac{3\lambda_5}{4\lambda_2}x^2z'+\frac{x}{2}(y'^2+z^2)$ $-\frac{1}{2}(yy'+zz')$,
$X_4=x\frac{\partial}{\partial y}$,	$B=y$,	$I_4=y-xy'$,
$X_5=\frac{\partial}{\partial y}$,	$B=0$,	$I_5=-y'$,
$X_6=x\frac{\partial}{\partial z}$,	$B=z+\frac{\lambda_5}{2\lambda_2}x^2$,	$I_6=z-xz'+\frac{\lambda_5}{2\lambda_2}x^2$,
$X_7=\frac{\partial}{\partial z}$.	$B=\frac{\lambda_5}{\lambda_2}x$.	$I_7=-z'+\frac{\lambda_5}{\lambda_2}x$.
Case 2.1.2 $(A\neq 0,\ d_2=0,\ S(x)=0,\ F(x)=0)$		
$X_1=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$,	$B=U(x)$,	$I_1=U(x)+zy'-yz'$,
$X_2=\frac{\partial}{\partial x}$.	$B=v$.	$I_2=v+\frac{1}{2}(y'^2+z'^2)$.
Case 2.1.3 $(A\neq 0,\ d_2\neq 0,\ S(x)=0,\ F(x)=0)$		
$X_1=\frac{\partial}{\partial x}$,	$B=v$,	$I_1=v+\frac{1}{2}(y'^2+z'^2)$,
$X_2=x\frac{\partial}{\partial x}+\frac{y}{2}\frac{\partial}{\partial y}+\frac{z}{2}\frac{\partial}{\partial z}$,	$B=xv$,	$I_2=xv+\frac{x}{2}(y'^2+z'^2)$ $-\frac{1}{2}(yy'+zz')$,
$X_3=-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$.	$B=0$.	$I_3=zy'-yz'$.

Case 2.1.4 $(A=0,\ d_2=0,\ F(x)=\frac{\lambda_2}{\lambda_1}S(x))$		
$X_1=\frac{\partial}{\partial x}$,	$B=v$,	$I_1=v+\frac{1}{2}(y'^2+z'^2)$,
$X_2=\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $(\frac{\partial}{\partial y}+\frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z})$,	$B=\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $(y+\frac{\lambda_2}{\lambda_1}z)$,	$I_2=\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_3}{\lambda_1}}(y+\frac{\lambda_2}{\lambda_1}z)-y'-z']$,
$X_3=\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $(\frac{\partial}{\partial y}+\frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial z})$.	$B=-\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $(y+\frac{\lambda_2}{\lambda_1}z)$.	$I_3=-\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $[\sqrt{\frac{\lambda_3}{\lambda_1}}(y+\frac{\lambda_2}{\lambda_1}z)+y'+z']$.
Case 2.2.1 $(\lambda_1\neq 0,\ \lambda_2=0,\ \lambda_3=0)$		
$X_1=-z\frac{\partial}{\partial y}+(y-\frac{\lambda_5}{2\lambda_1}x^2)\frac{\partial}{\partial z}$,	$B=-\frac{\lambda_5}{\lambda_1}xz$,	$I_1=y'z-yz'-\frac{\lambda_5}{\lambda_1}xz$ $+\frac{\lambda_5}{2\lambda_1}x^2z'$,
$X_2=\frac{\partial}{\partial x}$,	$B=-\frac{\lambda_5}{\lambda_1}y$,	$I_2=-\frac{\lambda_5}{\lambda_1}y+\frac{1}{2}(y'^2+z'^2)$,
$X_3=x\frac{\partial}{\partial x}+(\frac{y}{2}+\frac{3\lambda_5}{4\lambda_1}x^2)\frac{\partial}{\partial y}$ $+\frac{z}{2}\frac{\partial}{\partial z}$,	$B=\frac{\lambda_5}{2\lambda_1}xy+\frac{\lambda_5^2}{4\lambda_1^2}x^3$,	$I_3=\frac{\lambda_5}{2\lambda_1}xy+\frac{\lambda_5^2}{4\lambda_1^2}x^3-\frac{3\lambda_5}{4\lambda_1}x^2y'$ $+\frac{x}{2}(y'^2+z^2)-\frac{1}{2}(yy'+zz')$,
$X_4=x\frac{\partial}{\partial y}$,	$B=y+\frac{\lambda_5}{2\lambda_1}x^2$,	$I_4=y-xy'+\frac{\lambda_5}{2\lambda_1}x^2$,
$X_5=\frac{\partial}{\partial y}$,	$B=\frac{\lambda_5}{\lambda_1}x$,	$I_5=\frac{\lambda_5}{\lambda_1}x-y'$,
$X_6=x\frac{\partial}{\partial z}$,	$B=z$,	$I_6=z-xz'$,
$X_7=\frac{\partial}{\partial z}$.	$B=0$.	$I_7=-z'$.
Case 2.2.2 $(\lambda_1\neq 0,\ \lambda_2=0,\ \lambda_3\neq 0)$		
$X_1=-z\frac{\partial}{\partial y}+(y+\frac{\lambda_5}{\lambda_3})\frac{\partial}{\partial z}$,	$B=0$,	$I_1=-y'z+yz'+\frac{\lambda_5}{\lambda_3}z'$,
$X_2=\frac{\partial}{\partial x}$,	$B=-\frac{\lambda_3}{2\lambda_1}(y^2+z^2)-\frac{\lambda_5}{\lambda_1}y$,	$I_2=-\frac{\lambda_3}{2\lambda_1}(y^2+z^2)-\frac{\lambda_5}{\lambda_1}y$ $+\frac{1}{2}(y'^2+z'^2)$,
$X_3=\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B=\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}})$,	$I_3=\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}}-y')$,
$X_4=\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial y}$,	$B=-\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}})$,	$I_4=-\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)(y\sqrt{\frac{\lambda_3}{\lambda_1}}$ $+\frac{\lambda_5}{\sqrt{\lambda_1\lambda_3}}+y')$,

$X_5 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial z}$,	$B = z\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)$,	$I_5 = \exp(\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $(z\sqrt{\frac{\lambda_3}{\lambda_1}} - z')$,
$X_6 = \exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\frac{\partial}{\partial z}$.	$B = -z\sqrt{\frac{\lambda_3}{\lambda_1}}\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)$.	$I_6 = -\exp(-\sqrt{\frac{\lambda_3}{\lambda_1}}x)\times$ $(z\sqrt{\frac{\lambda_3}{\lambda_1}} + z')$.
Case 2.3 $(\lambda_1 = 0,\ \lambda_2 \neq 0)$		
$X_1 = (-z + \frac{\lambda_5}{2\lambda_2}x^2)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{\lambda_2}xy$,	$I_1 = y'z - yz' + \frac{\lambda_5}{\lambda_2}xy$ $-\frac{\lambda_5}{2\lambda_2}x^2y'$,
$X_2 = \frac{\partial}{\partial x}$,	$B = -\frac{\lambda_5}{\lambda_2}z$,	$I_2 = -\frac{\lambda_5}{\lambda_2}z + \frac{1}{2}(y'^2 + z'^2)$,
$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}$ $+(\frac{z}{2} + \frac{3\lambda_5}{4\lambda_2}x^2)\frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3$,	$I_3 = \frac{\lambda_5}{2\lambda_2}xz + \frac{\lambda_5^2}{4\lambda_2^2}x^3 - \frac{3\lambda_5}{4\lambda_2}x^2z'$ $+\frac{x}{2}(y'^2 + z^2) - \frac{1}{2}(yy' + zz')$,
$X_4 = x\frac{\partial}{\partial z}$,	$B = z + \frac{\lambda_5}{2\lambda_2}x^2$,	$I_4 = z - xz' + \frac{\lambda_5}{2\lambda_2}x^2$,
$X_5 = \frac{\partial}{\partial z}$,	$B = \frac{\lambda_5}{\lambda_2}x$,	$I_5 = \frac{\lambda_5}{\lambda_2}x - z'$,
$X_6 = x\frac{\partial}{\partial y}$,	$B = y$,	$I_6 = y - xy'$,
$X_7 = \frac{\partial}{\partial y}$.	$B = 0$.	$I_7 = -y'$.

5.4 Conclusion

We have studied the partial Noether operators corresponding to a partial Lagrangian for a Hamiltonian system with two degrees of freedom. This problem was studied before via Noether symmetries in Damianou and Sophocleous (2004) wherein the authors did not provide the first integrals. In this work we have obtained both the partial Noether operators and the corresponding first integrals. We investigated the effectiveness of the partial Lagrangian approach which has provided all the first integrals. This study provides an alternative way to construct first integrals for nonlinear equations for which we do not need a Lagrangian. The previous work (Damianou and Sophocleous 2004), does not give the complete classification for the Hamiltonian system considered. In this chapter we have given the complete classification for the underlying system via a partial Lagrangian approach and we have obtained more general results that were not discussed in Damianou and Sophocleous (2004). This approach can give rise to further studies to classify nonlinear systems which are not variational and to derive first integrals from a partial Lagrangian viewpoint.

Chapter 6

First Integrals of Nonlinear Systems which are not Variational

The partial Noether operators and first integrals for systems of two second-order ODEs which are not variational are derived with the help of the partial Noether approach. We show how the nonlinear systems can be reduced via first integrals without making use of standard Lagrangians. Furthermore, it is shown that partial Lagrangians do exist for all such type of equations which do not admit standard Lagrangians and are significant in constructing first integrals.

6.1 Introduction

There has been great interest in studying the system of two second-order ODEs, especially for first integrals, which are important due to physical point of view and reduction as well. The system of two second-order ordinary differential equations (ODEs) frequently arise in nonlinear oscillations, nonlinear dynamics, relativity, fluid mechanics etc. These models, in general, describe different mechanical systems of practical importance (see e.g. Dimentberg and Bratus 2000, Dimentberg and Bratus 2002).

Our purpose is twofold: one is how first integrals can be constructed for the nonlinear ODEs which are not variational; secondly we want to investigate the effectiveness of the partial Lagrangian approach for nonlinear systems of two second-order ODEs for which standard Lagrangians do not exist. We have obtained first integrals for scalar ODEs and some systems in Chapter 2. There only Example 5 was nonlinear Here we look at further systems. In Chapter 2 the emphasis was comparative.

6.2 Partial Noether operators for nonlinear systems

In this section we find the partial Noether operators of various nonlinear systems of two second-order ODEs corresponding to partial Lagrangians via a partial Noether approach. These systems under consideration do not admit standard Lagrangians. We construct the partial Lagrangian and then partial Noether operators by invoking formula (1.19) for two dependent variables y and z and one independent variable x.

1. In the first example we consider the free oscillations of a two degrees of freedom gyroscopic system with quadratic nonlinearities (see Nayfeh 1981)

$$y'' + z' + 2y = 2yz,$$

$$z'' - y' + 2z = y^2. \tag{6.1}$$

The system (6.1) satisfies the partial Euler-Lagrange equations $\delta L/\delta y = z' + 2y - 2yz$ and $\delta L/\delta z = -y' + 2z - y^2$ has a partial Lagrangian

$$L = \frac{1}{2}(y'^2 + z'^2). \tag{6.2}$$

Now, the partial Noether operator X of system (6.1) corresponding to the partial Lagrangian (6.2) satisfies

$$\left[\eta^1_x + y'\eta^1_y + z'\eta^1_z - y'(\xi_x + y'\xi_y + z'\xi_z)\right] y' + \left[\eta^2_x + y'\eta^2_y + z'\eta^2_z - z'(\xi_x + y'\xi_y + z'\xi_z)\right] z'$$

$$+(\xi_x + y'\xi_y + z'\xi_z)\left[\frac{y'^2}{2} + \frac{z'^2}{2}\right] = (\eta^1 - y'\xi)\left[z' + 2y - 2yz\right]$$

$$+(\eta^2 - z'\xi)\left[-y' + 2z - y^2\right] + B_x + y'B_y + z'B_z. \tag{6.3}$$

The equation (6.3) splits into the following by comparing the coefficients of powers of y' and z'

$$\xi_y = 0,\ \xi_z = 0, \tag{6.4}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0,\ \eta^2_z - \frac{1}{2}\xi_x = 0,\ \eta^1_z + \eta^2_y = 0, \tag{6.5}$$

$$\eta^1_x = -2y(1-z)\xi - \eta^2 + B_y, \tag{6.6}$$

$$\eta^2_x = -(2z - y^2)\xi + \eta^1 + B_z, \tag{6.7}$$

$$\eta^1(2y - 2yz) + \eta^2(2z - y^2) + B_x = 0. \tag{6.8}$$

From equations (6.4)-(6.5), it is concluded that

$$\xi = \alpha(x),\ \eta^1 = \frac{1}{2}\alpha' y - zC_1(x) + C_3(x), \tag{6.9}$$

$$\eta^2 = \frac{1}{2}\alpha' z + yC_1(x) + C_2(x). \tag{6.10}$$

Equations (6.6) and (6.7) with the substitution of ξ, η^1 and η^2 from (6.9)-(6.10) give

$$C_1(x) = \frac{\alpha}{2} + b, \tag{6.11}$$

$$\begin{aligned} B = {} & \frac{1}{4}\alpha''(y^2+z^2) + yC_3'(x) + zC_2'(x) + \frac{1}{4}(2b + 5\alpha)(y^2+z^2) \\ & + yC_2(x) - zC_3(x) - \alpha y^2 z + C_4(x). \end{aligned} \tag{6.12}$$

Inserting ξ, η^1, η^2, $C_1(x)$ and B from (6.9)-(6.12) into (6.6) and after lengthy calculations it is found that

$$\alpha = -2b,\ C_2(x) = 0,\ C_3(x) = 0. \tag{6.13}$$

Thus equations (6.9)-(6.12) finally yield

$$\begin{aligned} & C_1(x) = 0,\ \xi = -2b,\ \eta^1 = 0,\ \eta^2 = 0, \\ & B = -2b(y^2+z^2) + 2by^2 z. \end{aligned} \tag{6.14}$$

Choosing $b = 1$, we find that the partial Noether operator and guage term of system (6.1) is

$$X = -2\frac{\partial}{\partial x},\ B = -2(y^2+z^2) + 2y^2 z. \tag{6.15}$$

2. In the second example we study the nonlinear mechanical system of two second-order ODEs

$$y'y'' = z^2 - 2xy^3,$$
$$z'' = 2(xzz' - yy') - 3x^2y^2y'. \quad (6.16)$$

A partial Lagrangian for system (6.16) satisfying the partial Euler-Lagrange equations $\delta L/\delta y = -z^2 + 2xy^3$ and $\delta L/\delta z = -2xzz' + 2yy' + 3x^2y^2y'$ is

$$L = \frac{y'^3}{6} + \frac{z'^2}{2}. \quad (6.17)$$

The partial Noether operator X corresponding to the partial Lagrangian for the system (6.16) are calculated from the formula (1.19) with respect to some function $B(x, y, z)$. The partial Noether operator determining equation is

$$\left[\eta^1_x + y'\eta^1_y + z'\eta^1_z - y'(\xi_x + y'\xi_y + z'\xi_z)\right] y' + \left[\eta^2_x + y'\eta^2_y + z'\eta^2_z - z'(\xi_x + y'\xi_y + z'\xi_z)\right] z'$$
$$+(\xi_x + y'\xi_y + z'\xi_z)\left[\frac{y'^3}{6} + \frac{z'^2}{2}\right] = (\eta^1 - y'\xi)\left[-z^2 + 2xy^3\right]$$
$$+(\eta^2 - z'\xi)\left[-2xzz' + 2yy' + 3x^2y^2y'\right] + B_x + y'B_y + z'B_z. \quad (6.18)$$

We split equation (6.18) with respect to derivatives of y and z and after simplification we obtain

$$\xi_x = 0, \ \xi_y = 0, \ \xi_z = 0, \quad (6.19)$$

$$\eta^1_y = 0, \ \eta^2_z = 2xz\xi, \quad (6.20)$$

$$\eta^1_z + \eta^2_y = -2y\xi - 3x^2y^2\xi, \quad (6.21)$$

$$\eta^1_x = (z^2 - 2xy^3)\xi + (2y + 3x^2y^2)\eta^2 + B_y, \ \eta^2_x = -2xz\eta^2 + B_z, \quad (6.22)$$

$$\eta^1(-z^2 + 2xy^3) + B_x = 0. \quad (6.23)$$

After some simple calculations, the equations (6.19)-(6.20) give rise to

$$\xi = b_1, \quad (6.24)$$

$$\eta^1 = \beta(x, z), \quad (6.25)$$

$$\eta^2 = xz^2 b_1 + \gamma(x, y). \quad (6.26)$$

The substitution of η^1 and η^2 from (6.25)-(6.26) into (6.21) results in

$$\beta(x, z) = zC_1(x) + C_2(x), \quad (6.27)$$

$$\gamma(x, y) = -y^2 b_1 - x^2 y^3 b_1 - yC_1(x) + C_3(x), \quad (6.28)$$

Equations (6.24)-(6.26) together with (6.27)-(6.28) yield

$$\xi = b_1, \quad (6.29)$$

$$\eta^1 = zC_1(x) + C_2(x), \quad (6.30)$$

$$\eta^2 = xz^2 b_1 - y^2 b_1 - x^2 y^3 b_1 - yC_1(x) + C_3(x). \quad (6.31)$$

The simple manipulation shows that the solution of (6.22) with the help of (6.29)-(6.32) can be expressed as

$$b_1 = 0, \ B = yC_2'(x) - (y^2 + x^2 y^3)C_3(x) + \frac{2}{3} y^3 b_2 + \frac{3}{4} x^2 y^4 b_2 + S(x, z), \quad (6.32)$$

where

$$S(x, z) = zC_3'(x) + xz^2(-b_2 y + C_3(x)) + T(x). \quad (6.33)$$

Equation (6.23) after replacing the values of ξ, η^1, η^2 and B from equations (6.29)-(6.33) reduces to

$$C_1(x) = b_2 = 0, \ C_2(x) = b_3, \ C_3(x) = b_3, \ T(x) = b_4. \quad (6.34)$$

Hence

$$\xi = 0, \ \eta^1 = b_3, \ \eta^2 = b_3, \quad (6.35)$$

$$B = -(y^2 + x^2 y^3) b_3 + xz^2 b_3. \quad (6.36)$$

Setting the constant $b_3 = 1$ gives the following operator and guage term for system (6.16)

$$X = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, B = xz^2 - y^2 - x^2 y^3. \quad (6.37)$$

3. We consider the nonlinear mechanical system with two degrees of freedom

$$y'' = 4x^3z + 2xy^3 + 2z,$$

$$z'' = 3x^2y^2y' + 2xz' + x^4z'. \tag{6.38}$$

A partial Lagrangian for system (6.38) is

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2, \tag{6.39}$$

so that system of partial Euler-Lagrange equations can be written as

$$\frac{\delta L}{\delta y} = -4x^3z - 2xy^3 - 2z,\ \frac{\delta L}{\delta z} = -3x^2y^2y' - 2xz' - x^4z'. \tag{6.40}$$

The partial Noether operator X corresponding to a partial Lagrangian for the system (6.38) can be determined from (1.19) with respect to some function $B(x, y, z)$. The partial Noether operator determining equation is

$$\left[\eta^1_x + y'\eta^1_y + z'\eta^1_z - y'(\xi_x + y'\xi_y + z'\xi_z)\right] y' + \left[\eta^2_x + y'\eta^2_y + z'\eta^2_z - z'(\xi_x + y'\xi_y + z'\xi_z)\right] z'$$
$$+(\xi_x + y'\xi_y + z'\xi_z)\left[\frac{y'^2}{2} + \frac{z'^2}{2}\right] = (\eta^1 - y'\xi)\left[-4x^3z - 2xy^3 - 2z\right]$$
$$+(\eta^2 - z'\xi)\left[-3x^2y^2y' - 2xz' - x^4z'\right] + B_x + y'B_y + z'B_z. \tag{6.41}$$

The usual separation of equation (6.41) with respect to derivatives of y and z results in

$$\xi_y = 0,\ \xi_z = 0, \tag{6.42}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0,\ \eta^2_z - \frac{1}{2}\xi_x = (2x + x^4)\xi, \tag{6.43}$$

$$\eta^1_z + \eta^2_y = 3x^2y^2\xi, \tag{6.44}$$

$$\eta^1_x = (4x^3z + 2xy^3 + 2z)\xi - 3x^2y^2\eta^2 + B_y, \tag{6.45}$$

$$\eta^2_x = -(2x + x^4)\eta^2 + B_z, \tag{6.46}$$

$$\eta^1(4x^3z + 2xy^3 + 2z) - B_x = 0. \tag{6.47}$$

The solution of equations (6.42)-(6.43) is of the following form

$$\xi = \alpha(x), \tag{6.48}$$

$$\eta^1 = \frac{1}{2}\alpha' y + \beta(x,z), \tag{6.49}$$

$$\eta^2 = \frac{1}{2}\alpha' z + (2xz + x^4 z)\alpha + \gamma(x,y). \tag{6.50}$$

Equation (6.43) with the substitution of η^1 and η^2 from (6.49)-(6.50) reduces to

$$\beta = zC_1(x) + C_2(x),\ \gamma = -yC_1(x) + x^2y^3\alpha + C_3(x). \tag{6.51}$$

Hence equations (6.48)-(6.50) become

$$\xi = \alpha(x), \tag{6.52}$$

$$\eta^1 = \frac{1}{2}\alpha' y + zC_1(x) + C_2(x), \tag{6.53}$$

$$\eta^2 = \frac{1}{2}\alpha' z + (2xz + x^4 z)\alpha - yC_1(x) + x^2y^3\alpha + C_3(x). \tag{6.54}$$

The insertion of (6.52)-(6.54) into (6.45)-(6.46) results in

$$B = \frac{1}{4}\alpha'' y^2 + yzC_1'(x) + yC_2'(x) - (4x^3yz + \frac{1}{2}xy^4 + 2yz)\alpha$$

$$+x^2y^3(\frac{1}{2}\alpha' z + 2xz\alpha + x^4 z\alpha + C_3(x)) - \frac{3}{4}x^2y^4C_1(x) + \frac{1}{2}x^4y^6\alpha + S(x,z), \tag{6.55}$$

where

$$S(x,z) = \frac{1}{4}\alpha'' z^2 + z^2\alpha + xz^2\alpha' + 2x^3z^2\alpha + \frac{1}{2}x^4z^2\alpha' + zC_3'(x)$$

$$+(2x + x^4)\left[\alpha' z^2 + xz^2\alpha + \frac{1}{2}x^4z^2\alpha + zC_3(x)\right] + T(x). \tag{6.56}$$

Finally equation (6.47) with the substitution of ξ, η^1, η^2 and B from (6.52)-(6.56) gives

$$\alpha = 0,\ C_1(x) = 0,\ C_2(x) = b_3,\ C_3(x) = b_3. \tag{6.57}$$

Hence

$$\xi = 0,\ \eta^1 = b_3,\ \eta^2 = b_3, \tag{6.58}$$

$$B = x^2y^3 + (2x + x^4)z. \tag{6.59}$$

6.3 First integrals

In this section we construct first integrals of the nonlinear systems of two second-order ODEs of Section 6.2 which are not variational. We use the partial Noether theorem with the help of partial Noether operators constructed in the previous section with the help of the partial Lagrangians. We restrict the formula of first integrals for two dependent variables y and z.

$$I = B - \left[\xi L + (\eta^1 - y'\xi)\frac{\delta L}{\delta y} + (\eta^2 - z'\xi)\frac{\delta L}{\delta z}\right].$$

The first integrals computed for the systems under study by using the above formula are summarized below.

The first integral for the two degrees of freedom gyroscopic system (6.1) is

$$I = -2(y^2 + z^2) + 2y^2 z - y'^2 - z'^2.$$

For the mechanical system (6.16), the first integral is

$$I = -y^2 - x^2 y^3 + xz^2 - \frac{1}{2}y'^2 - z',$$

and for the system (6.38), we have the first integral

$$I = x^2 y^3 + (2x + x^4)z - y' - z'.$$

These first integrals can be used to reduce the order of the systems. Also they can effectively be utilized for stability analysis.

6.4 Concluding remarks

The partial Noether operators corresponding to partial Lagrangians of systems of two nonlinear second-order ODEs have been constructed. We have computed the first integrals associated with the partial Noether operators by utilization of the partial Noether's theorem. Since no variational problem exists for the equations under study one can utilize a partial Lagrangian approach which we have shown here to be effective for the systems of two second-order ODEs. The first integrals still can be constructed by using a Noether-like theorem.

Chapter 7

Approximate Partial Noether Operators and First Integrals for Coupled Nonlinear Oscillators

We study the construction of approximate first integrals of approximate partial Euler-Lagrange equations via approximate partial Noether operators corresponding to a partial Lagrangian of a coupled nonlinear oscillator system. An approximate Noether-like theorem (see Chapter 1) which gives the approximate first integrals for the perturbed equations without regard to a standard Lagrangian is invoked. These approximate partial Noether operators, in general, do not form an approximate Lie algebra. The approximate first integrals are obtained for both the resonant and non-resonant cases with the help of approximate partial Noether operators associated with a partial Lagrangian. This approach can give rise to further studies in the construction of approximate first integrals for perturbed equations without a variational principle.

7.1 Introduction

The relationship between approximate symmetries and first integrals has been of great interest for differential equations with a small parameter. One can construct approximate first integrals for approximate Euler-Lagrange equations by utilizing an approximate Noether's theorem (see Ünal 2000, Baikov, Gazizov and Ibragimov 1996, Johnpillai and Kara 2001, Johnpillai, Kara and Mahomed 2006), once their approximate Noether symmetries are known. To invoke this powerful theorem one requires a Lagrangian to obtain the approximate Noether symmetries and to construct the approximate first integrals. There are perturbed differential equations that arise in applications which do not admit a Lagrangian such as for the system of two coupled van der Pol oscillators with linear diffusive coupling

$$y'' + \epsilon(y^2 - 1)y' + y = A(z - y) + B(z' - y'),$$
$$z'' + \epsilon(z^2 - 1)z' + z = A(y - z) + B(y' - z'), \tag{7.1}$$

where A and B are constants and ϵ is a small parameter, the curve family is non-extremal. We refer to the interesting paper Douglas (1941) for the classification of Lagrangians. Notice that a Lagrangian does not, in general, exist for such systems under consideration. So the question arises here is how to construct first integrals for perturbed equations without a variational principle. The approximate first integrals with the help of approximate Noether symmetries associated with a Lagrangian were studied in Baikov, Gazizov and Ibragimov (1996). In 1999, Kara, Mahomed and Ünal deduced a relationship between the approximate Lie-Bäcklund symmetries and approximate conserved vectors for the perturbed equations that do not admit a Lagrangian. The idea to construct an approximate Lagrangian for the perturbed equation by utilizing the Lie-Bäcklund symmetries and approximate conserved vectors were given by Johnpillai and Kara (2001), Johnpillai, Kara and Mahomed (2006). Recently, Kara and Mahomed (2006) and Kara, Mahomed, Naeem and Wafo (2007) found a relationship between operators and first integrals for unperturbed equations without regard to a Lagrangian. The notion of partial Lagrangian was introduced in their paper. This was reviewed in Chapter 1.

The objective of this chapter is to construct approximate first integrals of a nonlinear oscillator system via approximate partial Noether operators that are not in general the approximate symmetry generators of the given system of equations. We give an alternative viewpoint as to how the first integrals can be constructed for the perturbed equations without making use of a Lagrangian. Firstly we obtain approximate partial Noether operators corresponding to a partial Lagrangian and then obtain the approximate first integrals for such a system by utilizing the approximate Noether-like theorem of Chapter 1. Such a system is said to be a partial Euler-Lagrange system. This work follows naturally from our previous work (see Kara, Mahomed, Naeem and Wafo 2007). The reader is also referred to Chapter 1 and Kara and Mahomed (2006).

The outline of the chapter is as follows. Section 7.2 gives the approximate partial Noether operators associated with a partial Lagrangian of a system of two weakly coupled nonlinear oscillators. In Section 7.3 we obtain approximate first integrals of the system of two weakly coupled non-linear oscillators that have been studied before via a field theory approach in Kovacic (2006). The previous work Kovacic (2006) did not obtain approximate partial Noether operators. The approach of Kovacic (2006) was quite different. Moreover we construct new approximate first integrals for both resonant and non-resonant cases. The concluding remarks are summarized in Section 7.4.

7.2 Application to nonlinear oscillators

We consider the system of two weakly coupled nonlinear oscillators

$$\begin{cases} y'' = -\omega_1^2 y + \epsilon\alpha_1 z^2, \\ z'' = -\omega_2^2 z + 2\epsilon\alpha_1 yz, \end{cases} \tag{7.2}$$

where ω_1, ω_2, ϵ ($\epsilon \ll 1$) and α_1 are positive constants and prime represents the derivative with respect to x. The model in (7.2) arises in stellar structures and it describes the behavior of stellar orbits in a glaxy. The coordinate y represents the radial displacement of the orbit from a star to a reference circular orbit and the coordinate z shows the direction from the galactic plane. Note that the system (7.2) was studied before via a field theory approach in Kovacic (2006). Both cases $\omega_1 = \omega_2$ and $\omega_1 \neq \omega_2$ are considered.

The system (7.2) has the partial Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{1}{2}\omega_1^2 y^2 - \frac{1}{2}\omega_2^2 z^2. \tag{7.3}$$

Thus the approximate partial Euler-Lagrange equations are

$$\frac{\delta L}{\delta y} = -\epsilon\alpha_1 z^2, \ \frac{\delta L}{\delta z} = -2\epsilon\alpha_1 yz. \tag{7.4}$$

We derive the approximate partial Noether operators for the system of two weakly coupled nonlinear oscillators (7.2) by using the approximate Noether-like theorem obtained in Chapter 1. The approximate partial Noether operator corresponding to a partial Lagrangian (7.3) of system (7.2) satisfies

$$(X_0 + \epsilon X_1)L + D_x(\xi_0 + \epsilon\xi_1)L = [\eta_0^1 - \xi_0 y' + \epsilon(\eta_1^1 - \xi_1 y')]\frac{\delta L}{\delta y}$$

$$+[\eta_0^2 - \xi_0 z' + \epsilon(\eta_1^2 - \xi_1 z')]\frac{\delta L}{\delta z} + D_x(B_0 + \epsilon B_1). \tag{7.5}$$

The equation (7.5) results the following systems corresponding to zeroth and first order approximations of ϵ respectively.

$$X_0 L + D_x(\xi_0)L = D_x(B_0), \tag{7.6}$$

$$X_1 L + D_x(\xi_1)L = -\alpha_1 z^2(\eta_0^1 - \xi_0 y') - 2\alpha_1 yz(\eta_0^2 - \xi_0 z') + D_x(B_1). \tag{7.7}$$

The utilization of total derivative operator from (1.2) and X_b from equation (1.25) into (7.6) and (7.7) results in

$$\begin{aligned}
&\left[\eta_{0x}^1 + \eta_{0y}^1 y' + \eta_{0z}^1 z' - y'(\xi_{0x} + \xi_{0y} y' + \xi_{0z} z')\right] y' \\
&+ \left[\eta_{0x}^2 + \eta_{0y}^2 y' + \eta_{0z}^2 z' - z'(\xi_{0x} + \xi_{0y} y' + \xi_{0z} z')\right] z' \\
&-\omega_1^2 y\eta_0^1 - \omega_2^2 z\eta_0^2 + (\xi_{0x} + \xi_{0y} y' + \xi_{0z} z')\left[\frac{1}{2}(y'^2 + z'^2 - \omega_1^2 y^2 - \omega_2^2 z^2)\right] \\
&= B_{0x} + B_{0y} y' + B_{0z} z',
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
&\left[\eta_{1x}^1 + \eta_{1y}^1 y' + \eta_{1z}^1 z' - y'(\xi_{1x} + \xi_{1y} y' + \xi_{1z} z')\right] y' \\
&+ \left[\eta_{1x}^2 + \eta_{1y}^2 y' + \eta_{1z}^2 z' - z'(\xi_{1x} + \xi_{1y} y' + \xi_{1z} z')\right] z' \\
&-\omega_1^2 y\eta_1^1 - \omega_2^2 z\eta_1^2 + (\xi_{1x} + \xi_{1y} y' + \xi_{1z} z')\left[\frac{1}{2}(y'^2 + z'^2 - \omega_1^2 y^2 - \omega_2^2 z^2)\right]
\end{aligned}$$

$$= -\alpha_1 z^2(\eta_0^1 - \xi_0 y') - 2\alpha_1 yz(\eta_0^2 - \xi_0 z') + B_{1x} + B_{1y}y' + B_{1z}z'. \tag{7.9}$$

The separation of (7.8) and (7.9) with respect to the powers of derivatives of y and z yield the following system.

Zeroth-order approximation:

$$\xi_{0y} = 0,\ \xi_{0z} = 0, \tag{7.10}$$

$$\eta_{0y}^1 - \frac{1}{2}\xi_{0x} = 0,\ \eta_{0z}^2 - \frac{1}{2}\xi_{0x} = 0,\ \eta_{0z}^1 + \eta_{0y}^2 = 0, \tag{7.11}$$

$$\eta_{0x}^1 = B_{0y},\ \eta_{0x}^2 = B_{0z}, \tag{7.12}$$

$$\omega_1^2 y\eta_0^1 + \omega_2^2 z\eta_0^2 + (\frac{1}{2}\omega_1^2 y^2 + \frac{1}{2}\omega_2^2 z^2)\xi_{0x} = -B_{0x}. \tag{7.13}$$

First-order approximation:

$$\xi_{1y} = 0,\ \xi_{1z} = 0, \tag{7.14}$$

$$\eta_{1y}^1 - \frac{1}{2}\xi_{1x} = 0,\ \eta_{1z}^2 - \frac{1}{2}\xi_{1x} = 0,\ \eta_{1z}^1 + \eta_{1y}^2 = 0, \tag{7.15}$$

$$\eta_{1x}^1 = \alpha_1 z^2\xi_0 + B_{1y},\ \eta_{1x}^2 = 2\alpha_1 yz\xi_0 + B_{1z}, \tag{7.16}$$

$$\omega_1^2 y\eta_1^1 + \omega_2^2 z\eta_1^2 + (\frac{1}{2}\omega_1^2 y^2 + \frac{1}{2}\omega_2^2 z^2)\xi_{1x} = \alpha_1 z^2\eta_0^1 + 2\alpha_1 yz\eta_0^2 - B_{1x}. \tag{7.17}$$

From equations (7.10)-(7.12), we find that

$$\xi_0 = \alpha(x),$$

$$\eta_0^1 = \frac{1}{2}\alpha' y + C_1(x)z + C_2(x),$$

$$\eta_0^2 = \frac{1}{2}\alpha' z - C_1(x)y + C_3(x),$$

$$B_0 = \frac{1}{4}\alpha'' y^2 + C_2'(x)y + S(x,z), \tag{7.18}$$

where

$$S(x,z) = \frac{1}{4}\alpha'' z^2 + C_3'(x)z + T(x),\ C_1(x) = f_0. \tag{7.19}$$

The insertion of these values in equation (7.13) gives

$$\alpha''' + 4\omega_1^2\alpha' = 0, \tag{7.20}$$

$$\alpha''' + 4\omega_2^2\alpha' = 0, \tag{7.21}$$

$$f_0(\omega_1^2 - \omega_2^2) = 0, \tag{7.22}$$

$$C_2''(x) + \omega_1^2 C_2(x) = 0, \tag{7.23}$$

$$C_3''(x) + \omega_2^2 C_3(x) = 0, \tag{7.24}$$

$$T'(x) = 0. \tag{7.25}$$

The equation (7.25) yields

$$T(x) = f_1. \tag{7.26}$$

In order to solve equations (7.20)-(7.24), the following cases are considered.

Case 1: $\omega_1 \neq \omega_2$ (non-resonant).
In this case we have

$$\alpha(x) = f_6,\ C_2(x) = f_4 \cos\omega_1 x + f_5 \sin\omega_1 x,$$

$$C_3(x) = f_2 \cos\omega_2 x + f_3 \sin\omega_2 x.$$

Hence the partial Noether operators and guage terms B for the unperturbed equation in this case are

$$X_0^1 = \frac{\partial}{\partial x},\ B_0 = 0,$$

$$X_0^2 = \cos\omega_2 x\frac{\partial}{\partial z},\ B_0 = \omega_2 z \sin\omega_2 x,$$

$$X_0^3 = \sin\omega_2 x\frac{\partial}{\partial z},\ B_0 = -\omega_2 z \cos\omega_2 x,$$

$$X_0^4 = \cos\omega_1 x \frac{\partial}{\partial y},\ B_0 = \omega_1 y \sin\omega_1 x,$$

$$X_0^5 = \sin\omega_1 x \frac{\partial}{\partial y},\ B_0 = -\omega_1 y \cos\omega_1 x.$$

Case 2: $\omega_1 = \omega_2$ (resonant).

From equations (7.20)-(7.24), we obtain

$$\alpha(x) = f_6 + f_7 \cos 2\omega_1 x + f_8 \sin 2\omega_1 x,\ C_2(x) = f_4 \cos\omega_1 x + f_5 \sin\omega_1 x,$$

$$C_3(x) = f_2 \cos\omega_1 x + f_3 \sin\omega_1 x.$$

The partial Noether operators and the corresponding guage terms are

$$X_0^1 = \frac{\partial}{\partial x},\ B_0 = 0,$$

$$X_0^2 = \cos\omega_1 x \frac{\partial}{\partial z},\ B_0 = \omega_1 z \sin\omega_2 x,$$

$$X_0^3 = \sin\omega_1 x \frac{\partial}{\partial z},\ B_0 = -\omega_1 z \cos\omega_2 x,$$

$$X_0^4 = \cos\omega_1 x \frac{\partial}{\partial y},\ B_0 = \omega_1 y \sin\omega_1 x,$$

$$X_0^5 = \sin\omega_1 x \frac{\partial}{\partial y},\ B_0 = -\omega_1 y \cos\omega_1 x,$$

$$X_0^6 = \cos 2\omega_1 x \frac{\partial}{\partial x} - \omega_1 y \sin 2\omega_1 x \frac{\partial}{\partial y} - \omega_1 z \sin 2\omega_1 x \frac{\partial}{\partial z},$$

$$B_0 = \omega_1^2 (y^2 + z^2) \cos 2\omega_1 x,$$

$$X_0^7 = \sin 2\omega_1 x \frac{\partial}{\partial x} + \omega_1 y \cos 2\omega_1 x \frac{\partial}{\partial y} + \omega_1 z \cos 2\omega_1 x \frac{\partial}{\partial z},$$

$$B_0 = \omega_1^2 (y^2 + z^2) \sin 2\omega_1 x.$$

The determining equations (7.14)-(7.17) in the non-resonant case (Case 1) reduce to

$$\omega_1^2 y \eta_1^1 + \omega_2^2 z \eta_1^2 - [\eta_{1x}^1 + \eta_{1y}^1 y' + \eta_{1z}^1 z' - y'(\xi_{1x} + \xi_{1y} y') + \xi_{1z} z'] y'$$

$$-[\eta_{1x}^2 + \eta_{1y}^2 y' + \eta_{1z}^2 z' - z'(\xi_{1x} + \xi_{1y} y' + \xi_{1z} z')] z' + (\xi_{1x} + \xi_{1y} y' + \xi_{1z} z')(-\frac{1}{2} y'^2$$

$$-\frac{1}{2} z'^2 + \frac{1}{2}\omega_1^2 y^2 + \frac{1}{2}\omega_2^2 z^2) = [f_4 \cos\omega_1 x + f_5 \sin\omega_1 x - f_6 y'](\alpha_1 z^2)$$

$$+[f_2\cos\omega_2 x + f_3\sin\omega_2 x - f_6 z'](2\alpha_1 yz) + B_{1x} + B_{1y}y' + B_{1z}z'.$$

Equating the coefficients in above equation with respect to derivatives of various monomials of x and y yield the following system

$$\xi_{1y} = 0,\ \xi_{1z} = 0, \tag{7.27}$$

$$\eta^1_{1y} - \frac{1}{2}\xi_{1x} = 0,\ \eta^2_{1z} - \frac{1}{2}\xi_{1x} = 0,\ \eta^1_{1z} + \eta^2_{1y} = 0, \tag{7.28}$$

$$\eta^1_{1x} = \alpha_1 z^2 f_6 + B_{1y},\ \eta^2_{1x} = 2\alpha_1 yz f_6 + B_{1z}, \tag{7.29}$$

$$\omega_1^2 y\eta_1^1 + \omega_2^2 z\eta_1^2 + (\frac{1}{2}\omega_1^2 y^2 + \frac{1}{2}\omega_2^2 z^2)\xi_{1x} = (f_4\cos\omega_1 x + f_5\sin\omega_1 x)\alpha_1 z^2$$

$$+2\alpha_1 yz(f_2\cos\omega_2 x + f_3\sin\omega_2 x) + B_{1x}. \tag{7.30}$$

Equations (7.27)-(7.29) after some simple manipulations lead to

$$\xi_1 = \alpha(x), \tag{7.31}$$

$$\eta_1^1 = \frac{1}{2}\alpha'(x)y + C_1(x)z + C_2(x), \tag{7.32}$$

$$\eta_1^2 = \frac{1}{2}\alpha'(x)z - C_1(x)y + C_3(x), \tag{7.33}$$

$$B_1 = \frac{1}{4}\alpha'' y^2 + C_2'(x)y - \alpha_1 yz^2 f_6 + S(x,z), \tag{7.34}$$

where

$$S(x,z) = \frac{1}{4}\alpha'' z^2 + C_3'(x)z + T(x),\quad C_1(x) = g_0. \tag{7.35}$$

The equation (7.30) with the substitution of above values gives rise to

$$\alpha''' + 4\omega_1^2\alpha' = 0, \tag{7.36}$$

$$\alpha''' + 4\omega_2^2\alpha' - 4\alpha_1(f_4\cos\omega_1 x + f_5\sin\omega_1 x) = 0, \tag{7.37}$$

$$g_0(\omega_1^2 - \omega_2^2) - 2\alpha_1(f_2\cos\omega_2 x + f_3\sin\omega_2 x) = 0, \tag{7.38}$$

$$C_2''(x) + \omega_1^2 C_2(x) = 0, \tag{7.39}$$

$$C_3''(x) + \omega_2^2 C_3(x) = 0, \tag{7.40}$$

$$T'(x) = 0. \tag{7.41}$$

The solution of equations (7.36)-(7.40) with the aid of (7.31)-(7.35), lead to the system

$$f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,$$

$$\xi_1 = g_1,\ \eta_1^1 = g_4 \cos\omega_1 x + g_5 \sin\omega_1 x,$$

$$\eta_1^2 = g_6 \cos\omega_2 x + g_7 \sin\omega_2 x,$$

$$B_1 = (-\omega_1 g_4 \sin\omega_1 x + \omega_1 g_5 \cos\omega_1 x)y + (-\omega_2 g_6 \sin\omega_2 x + \omega_2 g_7 \cos\omega_2 x)z - \alpha_1 y z^2 f_6 + g_8. \tag{7.42}$$

Hence the approximate partial Noether operators in the non-resonant case for system (7.2):

$$X_1 = \frac{\partial}{\partial x},\ B_1 = 0,\ X_2 = \cos\omega_1 x \frac{\partial}{\partial y},\ B_1 = -\omega_1 y \sin\omega_1 x,$$

$$X_3 = \sin\omega_1 x \frac{\partial}{\partial y},\ B_1 = \omega_1 y \cos\omega_1 x,\ X_4 = \cos\omega_2 x \frac{\partial}{\partial z},\ B_1 = -\omega_2 z \sin\omega_2 x,$$

$$X_5 = \sin\omega_2 x \frac{\partial}{\partial z},\ B_1 = \omega_2 z \cos\omega_2 x,\ X_6 = 0,\ B_1 = -\alpha_1 y z^2. \tag{7.43}$$

The system (7.14)-(7.17) in the resonant case ($\omega_1 = \omega_2$) become

$$\xi_{1y} = 0,\ \xi_{1z} = 0, \tag{7.44}$$

$$\eta_{1y}^1 - \frac{1}{2}\xi_{1x} = 0,\ \eta_{1z}^2 - \frac{1}{2}\xi_{1x} = 0,\ \eta_{1z}^1 + \eta_{1y}^2 = 0, \tag{7.45}$$

$$\eta_{1x}^1 = \alpha_1 z^2 (f_6 + f_7 \cos 2\omega_1 x + f_8 \sin 2\omega_1 x) + B_{1y}, \tag{7.46}$$

$$\eta_{1x}^2 = 2\alpha_1 y z (f_6 + f_7 \cos 2\omega_1 x + f_8 \sin 2\omega_1 x) + B_{1z}, \tag{7.47}$$

$$\begin{aligned} -\omega_1^2 y \eta_1^1 - \omega_1^2 z \eta_1^2 + (\frac{1}{2}\omega_1^2 y^2 - \frac{1}{2}\omega_1^2 z^2)\xi_{1x} &= -\alpha_1 z^2(-\omega_1 f_7 \sin 2\omega_1 x + \omega_1 f_8 \cos 2\omega_1 x)y \\ &+ f_4 \cos\omega_1 x + f_5 \sin\omega_1 x + f_0 z - 2\alpha_1 y z((-\omega_1 f_7 \sin 2\omega_1 x + \omega_1 f_8 \cos 2\omega_1 x)z \\ &+ f_2 \cos\omega_1 x + f_3 \sin\omega_1 x - f_0 y) + B_{1x}. \end{aligned} \tag{7.48}$$

The solution of equations (7.44)-(7.47) are as follows

$$\xi_1 = \alpha(x), \tag{7.49}$$

$$\eta_1^1 = \frac{1}{2}\alpha'(x)y + C_1(x)z + C_2(x), \tag{7.50}$$

$$\eta_1^2 = \frac{1}{2}\alpha'(x)z - C_1(x)y + C_3(x), \tag{7.51}$$

$$B_1 = \frac{1}{4}\alpha'' y^2 + C_2'(x)y - \alpha_1 y z^2(f_6 + f_7\cos 2\omega_1 t + f_8 \sin 2\omega_1 t + S(x,z), \tag{7.52}$$

where

$$S(x,z) = \frac{1}{4}\alpha'' z^2 + C_3'(x)z + T(x), \quad C_1(x) = h_0. \tag{7.53}$$

Routine but lengthy calculations show that after replacement of values from equations (7.49)-(7.53) in equation (7.48) yield the following results

$$\alpha''' + 4\omega_1^2\alpha' = 0, \tag{7.54}$$

$$\alpha''' + 4\omega_2^2\alpha' - 4\alpha_1(f_4\cos\omega_1 x + f_5\sin\omega_1 x) = 0, \tag{7.55}$$

$$\alpha_1\omega_1(-f_7\sin 2\omega_1 x + f_8\cos 2\omega_1 x) = 0, \tag{7.56}$$

$$h_0(\omega_1^2 - \omega_2^2) - 2\alpha_1(f_2\cos\omega_1 x + f_3\sin\omega_1 x) = 0, \tag{7.57}$$

$$C_2''(x) + \omega_1^2 C_2(x) = 0, \tag{7.58}$$

$$C_3''(x) + \omega_2^2 C_3(x) = 0, \tag{7.59}$$

$$T'(x) = 0. \tag{7.60}$$

From equations (7.54)-(7.59) together with the help of (7.49)-(7.53), we obtain

$$f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,\ f_7 = 0,\ f_8 = 0,$$

$$\xi_1 = h_1 + h_2\cos 2\omega_1 x + h_3 \sin 2\omega_1 x,$$

$$\eta_1^1 = (-\omega_1 h_2 \sin 2\omega_1 x + \omega_1 h_3\cos 2\omega_1 x)y + h_0 z$$

$$+h_4\cos\omega_1 x + h_5\sin\omega_1 x,$$

$$\eta_1^2 = (-\omega_1 h_2 \sin 2\omega_1 x + \omega_1 h_3\cos 2\omega_1 x)z - h_0 y$$

$$+h_6\cos\omega_1 x + h_7\sin\omega_1 x,$$

$$B_1 = -(\omega_1^2 h_2\cos 2\omega_1 x + \omega_1^2 h_3 \sin 2\omega_1 x)(y^2+z^2) + (-\omega_1 h_4 \sin\omega_1 x + \omega_1 h_5\cos\omega_1 x)y$$

$$+(-\omega_1 h_6 \sin\omega_1 x + \omega_1 h_7 \cos\omega_1 x)z - \alpha_1 y z^2 f_6 + h_8. \quad (7.61)$$

In this case we find that the approximate partial Noether operators for system (7.2) are

$$X_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\ B_1 = 0,\ X_2 = \frac{\partial}{\partial x},\ B_1 = 0,$$
$$X_3 = \cos 2\omega_1 x\frac{\partial}{\partial x} - \omega_1 y \sin 2\omega_1 x\frac{\partial}{\partial y} - \omega_1 z \sin 2\omega_1 x\frac{\partial}{\partial z},$$
$$B_1 = -\omega_1^2(y^2+z^2)\cos 2\omega_1 x,$$
$$X_4 = \sin 2\omega_1 x\frac{\partial}{\partial x} + \omega_1 y \cos 2\omega_1 x\frac{\partial}{\partial y} + \omega_1 z \cos 2\omega_1 x\frac{\partial}{\partial z},$$
$$B_1 = -\omega_1^2(y^2+z^2)\sin 2\omega_1 x,$$
$$X_5 = \cos\omega_1 x\frac{\partial}{\partial y},\ B_1 = -\omega_1 y \sin\omega_1 x,\ X_6 = \sin\omega_1 x\frac{\partial}{\partial y},\ B_1 = \omega_1 y\cos\omega_1 x,$$
$$X_7 = \cos\omega_1 x\frac{\partial}{\partial z},\ B_1 = -\omega_1 z \sin\omega_1 x,\ X_8 = \sin\omega_1 x\frac{\partial}{\partial z},\ B_1 = \omega_1 z\cos\omega_1 x,$$
$$X_9 = 0,\ B_1 = -\alpha_1 y z^2. \quad (7.62)$$

7.3 First integrals

The first integrals for the approximate partial Noether operators for the system (7.2) corresponding to the partial Lagrangian (7.3) are (by (1.50))

$$I = B - \left[\xi L + (\eta^1 - \xi y')y' + (\eta^2 - \xi z')z'\right] + O(\epsilon^2), \quad (7.63)$$

where we have used $\partial L/\partial y' = y'$ and $\partial L/\partial z' = z'$ which hold for all the cases. The first integrals for both cases computed with the help of (7.63) are summarized below.

Case 1: $\omega_1 \neq \omega_2$ (non-resonant).
The first integrals are

$$I_1 = \frac{\epsilon}{2}(y'^2 + z'^2 + \omega_1^2 y^2 + \omega_2^2 z^2),\ I_2 = \epsilon(-\omega_1 y \sin\omega_1 x - y'\cos\omega_1 x),$$
$$I_3 = \epsilon(\omega_1 y\cos\omega_1 x - y'\sin\omega_1 x),\ I_4 = \epsilon(-\omega_2 z\sin\omega_2 x - z'\cos\omega_2 x),$$
$$I_5 = \epsilon(\omega_2 z\cos\omega_2 x - z'\sin\omega_2 x),\ I_6 = \frac{1}{2}(y'^2 + z'^2 + \omega_1^2 y^2 + \omega_2^2 z^2) - \epsilon\alpha_1 y z^2.$$

Case 2: $\omega_1 = \omega_2$ (resonant).

In this case we find that

$$I_1 = \epsilon(yz' - y'z),\ I_2 = \frac{\epsilon}{2}(y'^2 + z'^2 + \omega_1^2 y^2 + \omega_1^2 z^2),$$

$$I_3 = \epsilon(-\omega_1^2(y^2 + z^2)\cos 2\omega_1 x + \frac{1}{2}\cos 2\omega_1 x(y'^2 + z'^2 + \omega_1^2 y^2 + \omega_1^2 z^2)$$
$$+\omega_1(yy' + zz')\sin 2\omega_1 x),$$

$$I_4 = \epsilon(-\omega_1^2(y^2 + z^2)\sin 2\omega_1 x + \frac{1}{2}\sin 2\omega_1 x(y'^2 + z'^2 + \omega_1^2 y^2 + \omega_1^2 z^2)$$
$$-\omega_1(yy' + zz')\cos 2\omega_1 x),$$

$$I_5 = \epsilon(-\omega_1 y\sin\omega_1 x - y'\cos\omega_1 x),\ I_6 = \epsilon(\omega_1 y\cos\omega_1 x - y'\sin\omega_1 x),$$

$$I_7 = \epsilon(-\omega_1 z\sin\omega_1 x - z'\cos\omega_1 x),\ I_8 = \epsilon(\omega_1 z\cos\omega_1 x - z'\sin\omega_1 x),$$

$$I_9 = \frac{1}{2}(y'^2 + z'^2 + \omega_1^2 y^2 + \omega_1^2 z^2) - \epsilon\alpha_1 yz^2.$$

Note that the first integrals obtained herein, I_6 (non-resonant case) and I_9 (resonant case) are stable. All other first integrals for resonant and non-resonant cases of the system under study are unstable–they are just ϵ multipliers.

Comparison of the field theory method and partial Lagrangian approaches

In 2006, Kovacic obtained the conservation laws of two coupled nonlinear oscillators for both the resonant and non-resonant cases via a field theory approach. Kovacic (2006) looked at higher perturbations in ϵ which is not appropriate as the given system is first order in ϵ. Moreover, he constructed the third integral for a non-resonant case which he compared with other results obtained in Contopoulos (1960) and Hori (1967).

In this work we have constructed all the first integrals for both the resonant and non-resonant cases upto first order in ϵ for a system of two coupled nonlinear oscillators which is also perturbed upto first order in ϵ. We have obtained fifteen first integrals for both resonant and non-resonant cases in which two of them are stable as in Kovacic (2006). All the other first integrals obtained herein corresponding to the resonant and non-resonant cases are unstable (trivial). The reason to take first order approximation in ϵ is that the given system of two equations include first order approximations only. Therefore it is not useful to construct first integrals upto order ϵ^2.

We have provided an easier way to construct approximate first integrals for the perturbed equations than given in Kovacic (2006).

7.4 Concluding remarks

In this chapter we have shown how one can construct approximate first integrals for perturbed nonlinear ordinary differential equations via approximate partial Noether operators associated with a partial Lagrangian. These approximate partial Noether operators, in general, do not form an approximate Lie algebra. Notice that the system of two equations (under consideration) does not have a Lagrangian. In this work we have showed how first integrals can be constructed without making use of a Lagrangian. We have applied our results to a system of two coupled non-linear oscillators that has been studied before via a field theory approach in Kovacic (2006). These equations model important physical phenomena in nonlinear oscillations (see e.g. Nayfeh 1979) and in nonlinear dynamics etc. We have constructed approximate partial Noether operators corresponding to a partial Lagrangian and also obtained new approximate first integrals for both resonant and non-resonant cases of the system of two coupled nonlinear oscillators. This study can give rise to new ways of constructing approximate first integrals for perturbed systems of nonlinear equations for which we do not have Lagrangians. Partial Lagrangians do exist for such equations in the absence of standard Lagrangians and are useful to construct approximate first integrals.

Chapter 8

Approximate First Integrals for a System of Two Coupled van der Pol Oscillators with Linear Diffusive Coupling

The approximate partial Noether operators for a system of two coupled van der Pol oscillators with linear diffusive coupling are presented via a partial Lagrangian approach. The underlying system of two equations, in general, do not admit a standard Lagrangian. However, the approximate first integrals are constructed by utilization of the partial Noether's theorem with the help of approximate partial Noether operators associated with a partial Lagrangian (see Chapter 1). These approximate partial Noether operators are not approximate symmetries of the system and they do not form an approximate Lie algebra. Moreover, we show how approximate first integrals can be constructed for the perturbed equations without making use of standard Lagrangians. This is our second application (see also Chapter 7).

8.1 Introduction

In this chapter, we will study the system of two coupled van der Pol oscillators with linear diffusive coupling (proposed by Rand and Holmes in 1980)

$$y'' + \epsilon(y^2 - 1)y' + y = A(z - y) + B(z' - y'),$$

$$z'' + \epsilon(z^2 - 1)z' + z = A(y - z) + B(y' - z'), \tag{8.1}$$

where A and B are coupling parameters that gives the strength of the interaction and y and z are dependent variables which model the state of the oscillators and prime denotes the differentiation with respect to x. The study of coupled van der Pol oscillators have been of great interest due to their extensive range of applications in particular in biological sciences. It gives the state of two neighboring cells or group of cells each of which is able to oscillate itself. The system of two equations under study also frequently appear in nonlinear oscillations, nonlinear dynamics, engineering, mathematical physics etc. The main goal of study the system (8.1) is to find the approximate partial Noether operators and the corresponding approximate first integrals which are important to see the physical phenomena and reduction of the system as well for the solution space. Secondly we show how one can obtain approximate first integrals for perturbed equations without a variational principle.

The approximate first integrals for perturbed equations which have Lagrangians can be constructed by using the classical Noether's theorem. For various equations it is difficult to find the standard Lagrangian and most of the equations that arise in applications do not have Lagrangians e.g scalar evolution equations. Similarly, for the system of two weekly coupled nonlinear oscillators (7.1) where ω_1, ω_2, ϵ and α_1 are positive constant and prime denotes the differentiation with respect to x, no variational problem exists. Likewise, for the simple systems as illustrated in previous chapters, the curve family is also non-extremal. For an account of this theory of the inverse problem, we refer to Douglas (1941). In the absence of a Lagrangian, there are some approaches mentioned in the previous chapters that work without a variational principle.

We give an easy way to construct approximate partial Noether operators and approximate first integrals without making use of a Lagrangian. The theory of the approximate partial Noether operators and approximate conservation laws for differential equations with a small parameter has been recently introduced by the Johnpillai, Kara and Mahomed (2007) (see also Naeem and Mahomed 2008d for ODEs). The reader is also referred to Chapter 1. This approach works with the notion of a partial Lagrangian. The alternative way to construct approximate first integrals is as follows. We first compute the approximate partial Noether operators and then approximate first integrals are constructed by utilization of the approximate Noether-like theorem with the help of approximate partial Noether operators associated with a partial Lagrangian. Such a system is known as an approximate partial Euler-Lagrange system.

8.2 Partial Noether's approach

In this section we derive the approximate partial Noether operators for zeroth and first order of approximations of ϵ for the system of two coupled Van der Pol oscillators (8.1) via a partial Lagrangian.

8.2.1 Partial Noether operators for coupled van der Pol duffing oscillators

Now, the approximate generalized operator X in (1.24) is said to be approximate partial Noether operators corresponding to a partial Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{(A+1)}{2}(y^2 + z^2), \tag{8.2}$$

for system (8.1) satisfying partial Euler-Lagrange equations $\delta L/\delta y = \epsilon(y^2-1)y' - B(z'-y') - Az$ and $\delta L/\delta z = \epsilon(z^2-1)z' - B(y'-z') - Ay$, if it satisfies zeroth and first approximation equations of ϵ, respectively

$$X_0 L + (D_x\xi_0)L = (\eta_0^1 - \xi_0 y')\Big[- Az - B(z'-y')\Big]$$
$$+(\eta_0^2 - \xi_0 z')\Big[- Ay - B(y'-z')\Big] + D_x f_0, \tag{8.3}$$

$$X_1L + (D_x\xi_1)L = (\eta_0^1 - \xi_0 y')(y^2-1)y' + (\eta_0^2 - \xi_0 z')(z^2-1)z'$$

$$+(\eta_1^1 - \xi_1 y')\Big[-Az - B(z'-y')\Big] + (\eta_1^2 - \xi_1 z')\Big[-Ay - B(y'-z')\Big] + D_x f_1. \quad (8.4)$$

Expansion of total derivative operators and ζ_x^1, ζ_x^2 and then separation with respect to powers of y' and z' yield the following systems

$$\xi_{0y} = 0,\ \xi_{0z} = 0, \quad (8.5)$$

$$\eta_{0y}^1 - \frac{1}{2}\xi_{0x} = -B\xi_0,\ \eta_{0z}^2 - \frac{1}{2}\xi_{0x} = -B\xi_0,\ \eta_{0z}^1 + \eta_{0y}^2 = 2B\xi_0, \quad (8.6)$$

$$\eta_{0x}^1 = B\eta_0^1 - B\eta_0^2 + Az\xi_0 + f_{0y},\ \eta_{0x}^2 = -B\eta_0^1 + B\eta_0^2 + Ay\xi_0 + f_{0z}, \quad (8.7)$$

$$\eta_0^1(A+1)y + \eta_0^2(A+1)z + \frac{(A+1)}{2}(y^2+z^2)\xi_{0x} - Az\eta_0^1 - Ay\eta_0^2 - f_{0x} = 0. \quad (8.8)$$

$$\xi_{1y} = 0,\ \xi_{1z} = 0, \quad (8.9)$$

$$\eta_{1y}^1 - \frac{1}{2}\xi_{1x} = -(y^2-1)\xi_0 - B\xi_1,\ \eta_{1z}^2 - \frac{1}{2}\xi_{1x} = -(z^2-1)\xi_0 - B\xi_1,\ \eta_{1z}^1 + \eta_{1y}^2 = 2B\xi_1, \quad (8.10)$$

$$\eta_{1x}^1 = (y^2-1)\eta_0^1 + B\eta_1^1 - B\eta_1^2 + Az\xi_1 + f_{1y},\ \eta_{1x}^2 = (z^2-1)\eta_0^2 - B\eta_1^1 + B\eta_1^2 + Ay\xi_0 + f_{1z}, \quad (8.11)$$

$$\eta_1^1(A+1)y + \eta_1^2(A+1)z + \frac{(A+1)}{2}(y^2+z^2)\xi_{1x} - Az\eta_1^1 - Ay\eta_1^2 - f_{1x} = 0. \quad (8.12)$$

Solving equations (8.5-8.7), we obtain

$$\xi_0 = \alpha(x), \quad (8.13)$$

$$\eta_0^1 = (\frac{\alpha'}{2} - B\alpha)y + zE(x) + F(x), \quad (8.14)$$

$$\eta_0^2 = (\frac{\alpha'}{2} - B\alpha)z - yE(x) + 2By\alpha + H(x), \quad (8.15)$$

$$f_0 = (\frac{\alpha''}{2} - B\alpha')\frac{y^2}{2} + yzE' + yF' - B\frac{y^2}{2}(\frac{\alpha'}{2} - B\alpha) - ByzE - ByF - Ayz\alpha$$

$$+Byz(\frac{\alpha'}{2} - B\alpha) - B\frac{y^2}{2}E + B^2\alpha y^2 + BHy + K(x,z), \quad (8.16)$$

where E, F, H and K are arbitrary functions of x and

$$K(x,z) = \frac{z^2}{4}\alpha'' - \frac{3}{4}Bz^2\alpha' + \frac{B}{2}z^2E + H'z + BFz + \frac{B^2}{2}\alpha z^2 - BHz + M(x). \quad (8.17)$$

The replacement of (8.13)-(8.17) in (8.8) and then comparing the coefficients of powers of y and z lead to

$$B\alpha' - B^2\alpha - E' + BE = 0, \quad (8.18)$$

$$\frac{\alpha'''}{4} - \frac{3}{4}B\alpha'' + (A+1)\alpha' + \frac{3}{2}B^2\alpha' - B\alpha - 3AB\alpha - \frac{B}{2}E' + AE = 0, \quad (8.19)$$

$$\frac{\alpha'''}{4} - \frac{3}{4}B\alpha'' + (A+1)\alpha' + \frac{B^2\alpha'}{2} - AB\alpha - B\alpha + \frac{B}{2}E' - AE = 0, \quad (8.20)$$

$$\frac{B}{2}\alpha'' - 2A\alpha' - B^2\alpha' + 4AB\alpha + 2B\alpha + E'' - BE' = 0, \quad (8.21)$$

$$F'' - BF' + BH' + (A+1)F - AH = 0, \quad (8.22)$$

$$H'' + BF' - BH' + (A+1)H - AF = 0. \quad (8.23)$$

To derive the partial Noether operators we work out for the coefficients ξ, η^1 and η^2 respectively. In order to proceed with equations (8.18-8.23), two cases arise, viz. $B = 0$ and $B \neq 0$.

Case 1: $B = 0$.

The simple calculations reveal that the following subcases arise when we solve equations (8.18-8.23) for $B = 0$.

Case 1.1: $A = 0$, $B = 0$.

If $A = 0$, $B = 0$, then equations (8.18-8.23) after lengthy but simple calculations yield the following results

$$\alpha(x) = d_2 + d_3\cos 2x + d_4\sin 2x, \quad (8.24)$$

$$E(x) = d_1, \quad F(x) = d_5\cos x + d_6\sin x, \quad (8.25)$$

$$H(x) = d_7\cos x + d_8\sin x, \quad (8.26)$$

where $d_1, \cdots, d_8$ are constants.

Equations (8.13)-(8.17) together with (8.24)-(8.26) give rise to

$$\xi_0 = d_2 + d_3 \cos 2x + d_4 \sin 2x, \tag{8.27}$$

$$\eta_0^1 = y(-d_3 \sin 2x + d_4 \cos 2x) + d_1 z + d_5 \cos x + d_6 \sin x, \tag{8.28}$$

$$\eta_0^2 = z(-d_3 \sin 2x + d_4 \cos 2x) - d_1 y + d_7 \cos x + d_8 \sin x, \tag{8.29}$$

$$f_0 = (-d_3 \cos 2x - d_4 \sin 2x)(y^2 + z^2) + y(-d_5 \sin x + d_6 \cos x) + z(-d_7 \sin x + d_8 \cos x). \tag{8.30}$$

Case 1.2: $A \neq 0$, $B = 0$.

The substitution of $B = 0$ and keep $A \neq 0$ as it is in equations (8.18-8.23) and after some straightforward exploitations, two subcases arise:

Case 1.2.1: $A + 1 \neq 0, \quad B = 0.$

Case 1.2.2: $A + 1 = 0, \quad B = 0.$

Case 1.2.1: $A + 1 \neq 0, \quad B = 0.$

We obtain

$$\xi_0 = \alpha(x) = d_{10}, \tag{8.31}$$

$$\eta_0^1 = \frac{\alpha'}{2} y + zE(x) + F(x) = \frac{1}{2}(d_{11} \cos x + d_{12} \sin x)$$
$$-d_{13} \cos[(\sqrt{2A+1})x] - d_{14} \sin[(\sqrt{2A+1})x], \tag{8.32}$$

$$\eta_0^2 = \frac{\alpha'}{2} z - yE(x) + H(x) = \frac{1}{2}(d_{11} \cos x + d_{12} \sin x)$$
$$+d_{13} \cos[(\sqrt{2A+1})x] + d_{14} \sin[(\sqrt{2A+1})x], \tag{8.33}$$

$$f_0 = y\left[\frac{1}{2}(-d_{11} \sin x + d_{12} \cos x) + \sqrt{2A+1} d_{13} \sin[(\sqrt{2A+1})x]\right.$$
$$\left. - \sqrt{2A+1} d_{14} \cos[(\sqrt{2A+1})x]\right] - d_9 A y z + z\left[\frac{1}{2}(-d_{11} \sin x + d_{12} \cos x)\right.$$
$$\left. - \sqrt{2A+1} d_{13} \sin[(\sqrt{2A+1})x] + \sqrt{2A+1} d_{14} \cos[(\sqrt{2A+1})x]\right]. \tag{8.34}$$

Case 1.2.2: $A+1=0, \quad B=0$.

The utilization of the solution of equations (8.18-8.23) together with (8.13)-(8.17) yield the following results

$$\xi_0=\alpha(x)=d_{15}+d_{16}, \tag{8.35}$$

$$\eta_0^1=\frac{\alpha'}{2}y+zE(x)+F(x)=\frac{1}{2}(d_{17}\cos x+d_{18}\sin x)$$

$$-d_{19}\exp{(x)}-d_{(20)}\exp{(-x)}, \tag{8.36}$$

$$\eta_0^2=\frac{\alpha'}{2}z-yE(x)+H(x)=\frac{1}{2}(d_{17}\cos x+d_{18}\sin x)$$

$$+d_{19}\exp{(x)}+d_{(20)}\exp{(-x)}, \tag{8.37}$$

$$f_0=y\left[\frac{1}{2}(-d_{17}\sin x+d_{18}\cos x)-d_{19}\exp(x)+d_{20}\exp(-x)\right]-(d_{15}+d_{16})Ayz$$

$$+z\left[\frac{1}{2}(-d_{17}\sin x+d_{18}\cos x)+d_{19}\exp(x)-d_{20}\exp(-x)\right]. \tag{8.38}$$

Case 2: $B\neq 0$.

There are two subcases for the solution of equations (8.18-8.23) when $B\neq 0$.

Case 2.1: If $A\neq B^2/2, \quad B\neq 0$, then equations (8.18-8.23) combine with (8.13)-(8.17) to yield

$$\xi_0=\alpha(x)=0, \quad E(x)=d_{21}=0, \tag{8.39}$$

$$\eta_0^1=(\frac{\alpha'}{2}-B\alpha)y+zE(x)+F(x)=\frac{1}{2}(d_{22}\cos x+d_{23}\sin x)$$

$$-d_{24}\exp\left[(B-\sqrt{B^2-2A-1})x\right]-d_{25}\exp\left[(B+\sqrt{B^2-2A-1})x\right], \tag{8.40}$$

$$\eta_0^2=(\frac{\alpha'}{2}-B\alpha)z-yE(x)+H(x)=\frac{1}{2}(d_{22}\cos x+d_{23}\sin x)$$

$$+d_{24}\exp\left[(B-\sqrt{B^2-2A-1})x\right]+d_{25}\exp\left[(B+\sqrt{B^2-2A-1})x\right], \tag{8.41}$$

$$f_0=y\left[-(B-\sqrt{B^2-2A-1})d_{24}\exp\left[(B-\sqrt{B^2-2A-1})x\right]+\frac{1}{2}(-d_{22}\sin x\right.$$

$$+d_{23}\cos x) - (B+\sqrt{B^2-2A-1})d_{25}\exp\left[(B+\sqrt{B^2-2A-1})x\right]\Bigg]$$

$$+z\left[\frac{1}{2}(-d_{22}\sin x + d_{23}\cos x) + (B-\sqrt{B^2-2A-1})d_{24}\exp\left[(B-\sqrt{B^2-2A-1})x\right.\right.$$

$$+(B+\sqrt{B^2-2A-1})d_{25}\exp\left[(B+\sqrt{B^2-2A-1})x\right]\Bigg]+B(z-y)\left[\frac{1}{2}(d_{22}\cos x+d_{23}\sin x)\right.$$

$$-d_{24}\exp\left[(B-\sqrt{B^2-2A-1})x\right] - d_{25}\exp\left[(B+\sqrt{B^2-2A-1})x\right]\Bigg]$$

$$+B(y-z)\left[\frac{1}{2}(d_{22}\cos x + d_{23}\sin x) + d_{24}\exp\left[(B-\sqrt{B^2-2A-1})x\right.\right.$$

$$+d_{25}\exp\left[(B+\sqrt{B^2-2A-1})x\right]\Bigg]. \tag{8.42}$$

Case 2.2: $A = B^2/2, \quad B \neq 0.$

Using $A = B^2/2$ and $B \neq 0$ in equations (8.18-8.23), we deduce that

$$\xi_0 = \alpha(x) = 0, \quad E(x) = d_{26}\exp(Bx), \tag{8.43}$$

$$\eta_0^1 = (\frac{\alpha'}{2} - B\alpha)y + zE(x) + F(x) = d_{26}z\exp(Bx) + \frac{1}{2}(d_{27}\cos x + d_{28}\sin x)$$

$$-\Bigg(d_{29}\sin x + d_{30}\cos x\Bigg)\exp(Bx), \tag{8.44}$$

$$\eta_0^2 = (\frac{\alpha'}{2} - B\alpha)z - yE(x) + H(x) = -d_{26}y\exp(Bx) + \frac{1}{2}(d_{27}\cos x + d_{28}\sin x)$$

$$+\Bigg(d_{29}\sin x + d_{30}\cos x\Bigg)\exp(Bx), \tag{8.45}$$

$$f_0 = y\left[\frac{1}{2}(-d_{27}\sin x + d_{28}\cos x) - (d_{29}\cos x - d_{30}\sin x)\exp(Bx)\right.$$

$$\left.-B(d_{29}\sin x + d_{30}\cos x)\exp(Bx)\right] + z\left[\frac{1}{2}(-d_{27}\sin x + d_{28}\cos x)\right.$$

$$\left.+(d_{29}\cos x - d_{30}\sin x)\exp(Bx) + B(d_{29}\sin x + d_{30}\cos x)\exp(Bx)\right]$$

$$+B(z-y)\left[\frac{1}{2}(d_{27}\cos x + d_{28}\sin x) - (d_{29}\sin x + d_{30}\cos x)\exp(Bx)\right]$$

$$+B(y-z)\left[\frac{1}{2}(d_{27}\cos x + d_{28}\sin x) + (d_{29}\sin x + d_{30}\cos x)\exp(Bx)\right]$$

$$+\frac{B}{2}(z^2-y^2)d_{26}\exp(Bx). \tag{8.46}$$

In order to construct the partial Noether operators for the first order approximation of ϵ for system (8.1), we solve equations (8.9-8.12) with the help of the cases considered above.

One can easily construct the solution of determining equations (8.9-8.12) for the case when $A = 0$, $B = 0$ as

$$d_1 = 0, \ \ d_2 = 0, \ \ d_3 = 0, \ \ d_4 = 0, \ \ d_5 = 0, \ \ d_6 = 0, \ \ d_7 = 0, \ \ d_8 = 0, \tag{8.47}$$

or

$$\xi_0 = 0, \ \ \eta_0^1 = 0, \ \ \eta_0^2 = 0, \ \ f_0 = 0 \tag{8.48}$$

and

$$\xi_1 = g_2 + g_3 \cos 2x + g_4 \sin 2x, \tag{8.49}$$

$$\eta_1^1(-g_3 \sin 2x + g_4 \cos 2x)y + g_1 z + g_5 \cos x + g_6 \sin x, \tag{8.50}$$

$$\eta_1^2(-g_3 \sin 2x + g_4 \cos 2x)z - g_1 y + g_7 \cos x + g_8 \sin x, \tag{8.51}$$

$$f_1 = (y^2+z^2)(-g_3 \cos 2x - g_4 \sin 2x) + y(-g_5 \sin x + g_6 \cos x) + z(-g_7 \sin x + g_8 \cos x). \tag{8.52}$$

The approximate partial Noether operators are constructed by setting the constants one by one equal to one and the remaining constants to zero. In this case the approximate partial Noether operators are

$$\chi_1 = \epsilon(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}), \ \ f = 0,$$

$$\chi_2 = \epsilon\frac{\partial}{\partial x}, \ \ f = 0,$$

$$\chi_3 = \epsilon\left[\cos 2x\frac{\partial}{\partial x} - y \sin 2x\frac{\partial}{\partial y} - z \sin 2x\frac{\partial}{\partial z}\right], \ \ f = -\epsilon(y^2 + z^2)\cos 2x,$$

$$\chi_4 = \epsilon\left[\sin 2x\frac{\partial}{\partial x} + y \cos 2x\frac{\partial}{\partial y} + z \cos 2x\frac{\partial}{\partial z}\right], \ \ f = -\epsilon(y^2 + z^2)\sin 2x,$$

$$\chi_5 = \epsilon \cos x\frac{\partial}{\partial y}, \ \ f = -\epsilon y \sin x, \ \ \chi_6 = \epsilon \sin x\frac{\partial}{\partial y}, \ \ f = -\epsilon y \cos x,$$

$$\chi_7 = \epsilon \cos x \frac{\partial}{\partial z}, \quad f = -\epsilon z \sin x, \quad X_8 = \epsilon \sin x \frac{\partial}{\partial z}, \quad f = \epsilon z \cos x. \tag{8.53}$$

In order to solve equations (8.9-8.12) for first order approximation of ϵ and with the help of Case 1.2.1, two subcases arise:

Case 1.2.1.1: $A \neq -1/2, \quad B = 0.$

In this case, we have

$$d_{10} = 0, \quad d_{11} = 0, \quad d_{12} = 0, \quad d_{13} = 0, \quad d_{14} = 0, \tag{8.54}$$

or

$$\xi_0 = 0, \quad \eta_0^1 = 0, \quad \eta_0^2 = 0, \quad f_0 = 0 \tag{8.55}$$

and

$$\xi_1 = g_9, \tag{8.56}$$

$$\eta_1^1 = \frac{1}{2}(g_{10} \cos x + g_{11} \sin x) - g_{12} \cos[(\sqrt{2A+1})x] - g_{13} \sin[(\sqrt{2A+1})x], \tag{8.57}$$

$$\eta_1^2 = \frac{1}{2}(g_{10} \cos x + g_{11} \sin x) + g_{12} \cos[(\sqrt{2A+1})x] + g_{13} \sin[(\sqrt{2A+1})x], \tag{8.58}$$

$$f_1 = \left[\frac{1}{2}(-g_{10} \sin x + g_{11} \cos x) + g_{12}\sqrt{2A+1} \sin[(\sqrt{2A+1})x] - g_{13}\sqrt{2A+1} \cos[(\sqrt{2A+1})x]\right] y - g_9 A y z + \left[\frac{1}{2}(-g_{10} \sin x + g_{11} \cos x) - g_{12}\sqrt{2A+1} \sin[(\sqrt{2A+1})x] + g_{13}\sqrt{2A+1} \cos[(\sqrt{2A+1})x]\right] z. \tag{8.59}$$

For this case the approximate partial Noether operators are

$$\chi_1 = \epsilon \frac{\partial}{\partial x}, \quad f = -\epsilon A y z,$$

$$\chi_2 = \frac{\epsilon}{2} \cos x (\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = -\frac{\epsilon}{2}(y+z) \sin x,$$

$$\chi_3 = \frac{\epsilon}{2} \sin x (\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = \frac{\epsilon}{2}(y+z) \cos x,$$

$$\chi_4 = \epsilon \cos\sqrt{2A+1}x(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = \epsilon\sqrt{2A+1}\sin\sqrt{2A+1}x(y-z),$$

$$\chi_5 = \epsilon \sin\sqrt{2A+1}x(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = -\epsilon\sqrt{2A+1}\cos\sqrt{2A+1}x(y-z).$$

Case 1.2.1.2: $A = -1/2, \quad B = 0.$

To derive the approximate partial Noether operators for system (8.1) corresponding to Case 1.2.1.2, we solve (8.9-8.12) and with the aid of (8.31-8.33) result in

$$\xi_0 = 0, \quad \eta_0^1 = -d_{13}, \quad \eta_0^2 = d_{13}, \quad f_0 = 0 \tag{8.60}$$

and

$$\xi_1 = g_{14}, \tag{8.61}$$

$$\eta_1^1 = \frac{1}{2}(g_{15}\cos x + g_{16}\sin x) - g_{17} - g_{18}x, \tag{8.62}$$

$$\eta_1^2 = \frac{1}{2}(g_{15}\cos x + g_{16}\sin x) + g_{17} + g_{18}x, \tag{8.63}$$

$$f_1 = \left[\frac{1}{2}(-g_{15}\sin x + g_{16}\cos x) - g_{18}\right]y + (\frac{y^3}{3} - y)d_{13} + \frac{g_{14}}{2}yz$$
$$+\left[\frac{1}{2}(-g_{15}\sin x + g_{16}\cos x) + g_{18}\right]z + d_{13}(\frac{z^3}{3} - z).$$

Simple calculations result in

$$\chi_1 = \epsilon\frac{\partial}{\partial x}, \quad f = \frac{\epsilon}{2}yz,$$

$$\chi_2 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = -\frac{\epsilon}{2}(y+z)\sin x,$$

$$\chi_3 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = \frac{\epsilon}{2}(y+z)\cos x,$$

$$\chi_4 = -\epsilon(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = 0,$$

$$\chi_5 = -\epsilon x(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}), \quad f = -\frac{\epsilon}{2}(y-z),$$

$$\chi_6 = -\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad f = \epsilon(\frac{1}{3}(y^3 - z^3) - y + z).$$

Now, for the next case when $A + 1 = 0$, $B = 0$, the equations (8.9-8.12) jointly with equations (8.35-8.38) after some lengthy manipulations give rise to

$$d_{15} = 0, \;\; d_{16} = 0, \;\; d_{17} = 0, \;\; d_{18} = 0, \;\; d_{19} = 0, \;\; d_{20} = 0, \tag{8.64}$$

or

$$\xi_0 = 0, \;\; \eta_0^1 = 0, \;\; \eta_0^2 = 0, \;\; f_0 = 0 \tag{8.65}$$

and

$$\xi_1 = g_{19}, \tag{8.66}$$

$$\eta_1^1 = \frac{1}{2}(g_{20}\cos x + g_{21}\sin x) - g_{22}\exp(x) - g_{23}\exp(-x), \tag{8.67}$$

$$\eta_1^2 = \frac{1}{2}(g_{20}\cos x + g_{21}\sin x) + g_{22}\exp(x) + g_{23}\exp(-x), \tag{8.68}$$

$$f_1 = \left[\frac{1}{2}(-g_{20}\sin x + g_{21}\cos x) - g_{22}\exp(x) - g_{23}\exp(-x)\right]y + g_{19}yz$$
$$+\left[\frac{1}{2}(-g_{20}\sin x + g_{21}\cos x) + g_{22}\exp(x) + g_{23}\exp(-x)\right]z. \tag{8.69}$$

The operators are

$$\chi_1 = \epsilon\frac{\partial}{\partial x}, \;\; f = \epsilon yz,$$

$$\chi_2 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \;\; f = -\frac{\epsilon}{2}(y+z)\sin x,$$

$$\chi_3 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \;\; f = \frac{\epsilon}{2}(y+z)\cos x,$$

$$\chi_4 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\exp(x), \;\; f = -\epsilon(y+z)\exp(x),$$

$$\chi_5 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\exp(-x), \;\; f = -\epsilon(y+z)\exp(-x).$$

Next we construct the approximate partial Noether operators for first order approximation of ϵ when $B \neq 0$. The utilization of the solution of equations (8.9-8.12) together with (8.39-8.41) when $A \neq B^2/2$ and $B \neq 0$ result in two subcases:

Case 2.1.1: $A \neq -1/2, \;\; B \neq 0.$

In this case or

$$\xi_0 = 0, \;\; \eta_0^1 = 0, \;\; \eta_0^2 = 0, \;\; f_0 = 0 \tag{8.70}$$

and

$$\xi_1 = 0, \tag{8.71}$$

$$\eta_1^1 = \frac{g_{24}}{2}\cos x + \frac{g_{25}}{2}\sin x - g_{26}\exp\left(B - \sqrt{-1-2A+B^2}\right)$$
$$-g_{27}\exp\left(B + \sqrt{-1-2A+B^2}\right), \tag{8.72}$$

$$\eta_1^2 = \frac{g_{24}}{2}\cos x + \frac{g_{25}}{2}\sin x + g_{26}\exp\left(B - \sqrt{-1-2A+B^2}\right)$$
$$+g_{27}\exp\left(B + \sqrt{-1-2A+B^2}\right), \tag{8.73}$$

$$f_1 = \Bigg[-\left(B - \sqrt{-1-2A+B^2}\right)g_{26}\exp\left(B - \sqrt{-1-2A+B^2}\right) - \frac{g_{24}}{2}\sin x$$
$$+\frac{g_{25}}{2}\cos x - \left(B + \sqrt{-1-2A+B^2}\right)g_{27}\exp\left(B + \sqrt{-1-2A+B^2}\right)\Bigg] y +$$
$$\Bigg[\left(B - \sqrt{-1-2A+B^2}\right)g_{26}\exp\left(B - \sqrt{-1-2A+B^2}\right) - \frac{g_{24}}{2}\sin x$$
$$+\frac{g_{25}}{2}\cos x + \left(B + \sqrt{-1-2A+B^2}\right)g_{27}\exp\left(B + \sqrt{-1-2A+B^2}\right)\Bigg] z$$
$$+2B(y-z)\Bigg[g_{26}\exp\left(B - \sqrt{-1-2A+B^2}\right) + g_{27}\exp\left(B + \sqrt{-1-2A+B^2}\right)\Bigg]. \tag{8.74}$$

From the above equations we conclude that

$$\chi_1 = \frac{\epsilon}{2}\cos x\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \;\; f = -\frac{\epsilon}{2}(y+z)\sin x,$$

$$\chi_2 = \frac{\epsilon}{2}\sin x\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \;\; f = \frac{\epsilon}{2}(y+z)\cos x,$$

$$\chi_3 = \epsilon\left(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\exp\left(B - \sqrt{-1-2A+B^2}\right),$$

$$f = \epsilon\Bigg[(-y+z)\left(B - \sqrt{-1-2A+B^2}\right) + 2B(y-z)\Bigg]\exp\left(B - \sqrt{-1-2A+B^2}\right),$$

$$\chi_4 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\exp{(B+\sqrt{-1-2A+B^2})},$$

$$f = \epsilon\left[(-y+z)(B+\sqrt{-1-2A+B^2})+2B(y-z)\right]\exp{(B+\sqrt{-1-2A+B^2})}. \quad (8.75)$$

Case 2.1.2: $A = -1/2, \quad B \neq 0$.

The substitution of $A = -1/2,\ A \neq B^2/2,\ B \neq 0$ in equations (8.9-8.12) and solution together with equations (8.39-8.42) gives

$$\xi_0 = 0, \quad \eta_0^1 = -d_{24}, \quad \eta_0^2 = d_{24}, \quad f_0 = 2d_{24}B(y-z) \quad (8.76)$$

and

$$\xi_1 = 0, \quad (8.77)$$

$$\eta_1^1 = \frac{1}{2}(g_{28}\cos x + g_{29}\sin x) - g_{30} - \frac{g_{31}}{2B}\exp{(2Bx)}, \quad (8.78)$$

$$\eta_1^2 = \frac{1}{2}(g_{28}\cos x + g_{29}\sin x) + g_{30} + \frac{g_{31}}{2B}\exp{(2Bx)}, \quad (8.79)$$

$$f_1 = \left[\frac{1}{2}(-g_{28}\sin x + g_{29}\cos x) - g_{31}\exp{(2Bx)}\right]y + \left[\frac{1}{2}(-g_{28}\sin x + g_{29}\cos x)\right.$$

$$\left. -g_{31}\exp{(2Bx)}\right]z + 2B(y-z)\left[g_{30} + \frac{g_{31}}{2B}\exp{(2Bx)}\right] + (\frac{y^3}{3} - \frac{z^3}{3} - y + z)d_{24}.$$

The partial Noether operators are

$$\chi_1 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = -\frac{\epsilon}{2}(y+z)\sin x,$$

$$\chi_2 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = \frac{\epsilon}{2}(y+z)\cos x,$$

$$\chi_3 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = 2\epsilon B(y-z),$$

$$\chi_4 = \frac{\epsilon}{2B}\exp{(2Bx)}(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad f = 0,$$

$$\chi_5 = -\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad f = 2B(y-z) + \epsilon(\frac{y^3}{3} - \frac{z^3}{3} - y + z).$$

Routine but lengthy manipulations yield that the solution of the system (8.9-8.12) with (8.43-8.46) and $A = B^2/2$, $B \neq 0$ is

$$\xi_0 = 0, \quad \eta_0^1 = 0, \quad \eta_0^2 = 0, \quad f_0 = 0 \tag{8.80}$$

and

$$\xi_1 = 0,$$

$$\eta_1^1 = g_{32} z \exp(Bx) + \frac{1}{2}(g_{33}\cos x + g_{34}\sin x) - (g_{35}\cos x + g_{36}\sin x)\exp(Bx),$$

$$\eta_1^2 = -g_{32} y \exp(Bx) + \frac{1}{2}(g_{33}\cos x + g_{34}\sin x) + (g_{35}\cos x + g_{36}\sin x)\exp(Bx),$$

$$f_1 = y\left[\frac{1}{2}(-g_{33}\sin x + g_{34}\cos x) - (-g_{35}\sin x + g_{36}\cos x)\exp(Bx)\right.$$

$$\left.-B(g_{35}\cos x + g_{36}\sin x)\exp(Bx)\right] + z\left[\frac{1}{2}(-g_{33}\sin x + g_{34}\cos x)\right.$$

$$\left.+(-g_{35}\sin x + g_{36}\cos x)\exp(Bx) + B(g_{35}\cos x + g_{36}\sin x)\exp(Bx)\right]$$

$$-g_{32}\frac{B}{2}(y^2 - z^2)\exp(Bx) + 2B(y - z)(g_{35}\cos x + g_{36}\sin x)\exp(Bx).$$

In this case, we find the following operators

$$\chi_1 = \epsilon(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})\exp(Bx), \quad f = -\frac{\epsilon B}{2}(y^2 - z^2)\exp(Bx),$$

$$\chi_2 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\sin x \exp(Bx),$$

$$f = \epsilon\left[(-y + z)(\cos x + B\sin x)\exp(Bx) + 2B(y - z)\sin x \exp(Bx)\right],$$

$$\chi_3 = \epsilon(-\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\cos x \exp(Bx),$$

$$f = \epsilon\left[(y - z)(\sin x - B\cos x)\exp(Bx) + 2B(y - z)\cos x \exp(Bx)\right],$$

$$\chi_4 = \frac{\epsilon}{2}\cos x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}),$$

$$f = \frac{-\epsilon}{2}(y + z)\sin x,$$

$$\chi_5 = \frac{\epsilon}{2}\sin x(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}),$$
$$f = \frac{\epsilon}{2}(y+z)\cos x.$$

8.2.2 First integrals

The first integrals for a system of coupled Van der Pol oscillators (8.1) with the help of approximate partial Noether operators corresponding to the partial Lagrangian (8.2) are determined from the formula given in (1.50). Equation (1.50) for two dependent variables y and z can be written as

$$I = B - \left[\xi L + (\eta^1 - \xi y')y' + (\eta^2 - \xi z')z'\right] + O(\epsilon^2). \tag{8.81}$$

Here, we have used $\partial L/\partial y' = y'$ and $\partial L/\partial z' = z'$ which hold for all cases. The first integrals calculated for each case are summarized below.

Case 1.1: $A = 0, \quad B = 0.$

The partial Lagrangian in (8.2) for $A = 0$ reduces to

$$L = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - \frac{1}{2}(y^2 + z^2), \tag{8.82}$$

The invocation of the approximate partial Noether operators and guage terms for this case with the use of equation (8.81), we give the following first integrals:

$$I_1 = \epsilon(-y'z + yz'),$$
$$I_2 = \frac{\epsilon}{2}(y^2 + z^2 + y'^2 + z'^2),$$
$$I_3 = \epsilon\left(\frac{1}{2}\cos 2x(y'^2 + z'^2 - y^2 - z^2) + (yy' + zz')\sin 2x\right),$$
$$I_4 = \epsilon\left(\frac{1}{2}\sin 2x(y'^2 + z'^2 - y^2 - z^2) - (yy' + zz')\cos 2x\right),$$
$$I_5 = \epsilon(-y\sin x - y'\cos x), \quad I_6 = \epsilon(y\cos x - y'\sin x),$$

$$I_7 = \epsilon(-z\sin x - z'\cos x), \quad I_8 = \epsilon(z\cos x - z'\sin x). \tag{8.83}$$

Case 1.2.1: $A+1 \neq 0, \quad B=0.$

At this point we need to consider two subcases.

Case 1.2.1.1: $A \neq -1/2, \quad B=0.$

The first integrals for this case are

$$I_1 = \epsilon\bigg(-Ayz + \frac{1}{2}(y'^2+z'^2) + \frac{A+1}{2}(y^2+z^2)\bigg),$$
$$I_2 = \epsilon\bigg(-\frac{1}{2}(y+z)\sin x - \frac{1}{2}(y'+z')\cos x\bigg),$$
$$I_3 = \epsilon\bigg(\frac{1}{2}(y+z)\cos x - \frac{1}{2}(y'+z')\sin x\bigg),$$
$$I_4 = \epsilon\bigg(\sqrt{2A+1}(y-z)\sin(\sqrt{2A+1})x + (y'-z')\cos(\sqrt{2A+1})x\bigg),$$
$$I_5 = \epsilon\bigg(-\sqrt{2A+1}(y-z)\cos(\sqrt{2A+1})x + (y'-z')\sin(\sqrt{2A+1})x\bigg).$$

Case 1.2.1.2: $A = -1/2, \quad B=0.$

Simple manipulations lead to the following integrals

$$I_1 = \epsilon\bigg(\frac{1}{2}\bigg[yz + \frac{1}{2}(y^2+z^2) + y'^2 + z'^2\bigg]\bigg), \tag{8.84}$$
$$I_2 = \epsilon\bigg(-\frac{1}{2}(y+z)\sin x - \frac{1}{2}(y'+z')\cos x\bigg),$$
$$I_3 = \epsilon\bigg(\frac{1}{2}(y+z)\cos x - \frac{1}{2}(y'+z')\sin x\bigg),$$
$$I_4 = \epsilon\bigg(y'-z'\bigg), \quad I_5 = \epsilon\bigg(z-y+x(y'-z')\bigg),$$
$$I_6 = y'-z' + \epsilon\bigg[\frac{1}{3}(y^3-z^3) - y + z\bigg].$$

Case 1.2.2: $A+1=0, \quad B=0.$

Straightforward calculations result in

$$I_1 = \epsilon\bigg(yz + \frac{1}{2}(y'^2+z'^2)\bigg), \tag{8.85}$$

$$
\begin{aligned}
I_2 &= \epsilon\bigg(-\frac{1}{2}(y+z)\sin x - \frac{1}{2}(y'+z')\cos x \bigg), \\
I_3 &= \epsilon\bigg(\frac{1}{2}(y+z)\cos x - \frac{1}{2}(y'+z')\sin x \bigg), \\
I_4 &= \epsilon\bigg(\Big[-y+z+y'-z' \Big] \exp{(x)} \bigg), \\
I_5 &= \epsilon\bigg(\Big[y-z+y'-z' \Big] \exp{(-x)} \bigg).
\end{aligned}
$$

These are the first integrals for the case when $B = 0$. Next we use the formula given in (8.81) and construct the first integrals for the case when $B \neq 0$.

The following cases should be considered when $B \neq 0$.

Case 2: $B \neq 0$.
The following subcases arise:

Case 2.1: $A \neq B^2/2, \;\; B \neq 0$.
The subcases of Case 2.1 are:

Case 2.1.1: $A \neq -1/2, \;\; B \neq 0$.
We find that

$$
\begin{aligned}
I_1 &= \epsilon\bigg(-\frac{1}{2}(y+z)\sin x - \frac{1}{2}(y'+z')\cos x \bigg), \\
I_2 &= \epsilon\bigg(\frac{1}{2}(y+z)\cos x - \frac{1}{2}(y'+z')\sin x \bigg),
\end{aligned}
$$

$$
I_3 = \epsilon\bigg(\Big[-(y-z)(B-\sqrt{-1-2A+B^2}) + 2B(y-z) + y' - z' \Big] \exp{(B-\sqrt{-1-2A+B^2})} \bigg),
$$

$$
I_4 = \epsilon\bigg(\Big[-(y-z)(B+\sqrt{-1-2A+B^2}) + 2B(y-z) + y' - z' \Big] \exp{(B+\sqrt{-1-2A+B^2})} \bigg).
$$

Case 2.1.2: $A = -1/2, \;\; B \neq 0$.

For this case, the approximate first integrals are

$$I_1 = \epsilon\left(-\frac{1}{2}(y+z)\sin x - \frac{1}{2}(y'+z')\cos x\right),$$

$$I_2 = \epsilon\left(\frac{1}{2}(y+z)\cos x - \frac{1}{2}(y'+z')\sin x\right),$$

$$I_3 = \epsilon\left(2B(y-z)+y'-z'\right), \quad I_4 = \epsilon\left(\frac{1}{2B}\exp{(2Bx)}(y'-z')\right),$$

$$I_5 = 2B(y-z)+y'-z'+\epsilon\left[\frac{1}{3}(y^3-z^3)-y+z\right].$$

Case 2.2: $A = B^2/2, \quad B \neq 0.$

Equation (8.81) with the help of approximate partial Noether operators for Case 2.2 results in

$$I_1 = \epsilon\left(\left[yz'-y'z+\frac{B}{2}(z^2-y^2)\right]\exp{(Bx)}\right),$$

$$I_2 = \epsilon\left(\left[(-y+z)(\cos x + B\sin x) + 2B(y-z)\sin x + (y'-z')\sin x\right]\exp{(Bx)}\right),$$

$$I_3 = \epsilon\left(\left[(y-z)(\sin x - B\cos x) + 2B(y-z)\cos x + (y'-z')\cos x\right]\exp{(Bx)}\right),$$

$$I_4 = \epsilon\left(-\frac{1}{2}(y+z)\sin x - \frac{1}{2}(y'+z')\cos x\right),$$

$$I_5 = \epsilon\left(\frac{1}{2}(y+z)\cos x - \frac{1}{2}(y'+z')\sin x\right).$$

8.3 Concluding remarks

The approximate partial Noether operators corresponding to a partial Lagrangian of a system of two coupled van der Pol oscillators were constructed by using the partial Lagrangian approach. The approximate first integrals were computed by invoking the partial Noether's theorem corresponding to partial Noether operators via a partial Lagrangian for the system under consideration. These oscillators in general describe the important physical phenomena in nonlinear dynamics, nonlinear oscillations etc. Out of all the first integrals obtained here, two were stable and the rest unstable. Moreover, we have shown how one can construct approximate first integrals without a variational principle. We provided an elegant way of constructing approximate first integrals by using the notion of a partial Lagrangian which exist for such type of equations in the absence of standard Lagrangians. This is a second application of the partial Lagrangian approach to perturbed ordinary differential equations. The first approach was given in the previous chapter.

CONCLUSIONS

A systematic way of constructing first integrals for exact ordinary differential equations that need not be derivable from variational principles is presented in this work. These results were extended to perturbed ordinary differential equations and it was shown how approximate first integrals can be computed for non variational problems without regard to Lagrangians. There are approaches that provide first integrals for such type of equations without consideration of standard Lagrangians but we gave an elegant alternative way of constructing first integrals by means of a formula. The concepts of a partial Lagrangian and partial Euler-Lagrange equations were introduced for ordinary differential equations. We deduced a Noether-like theorem which gives the first integrals for ordinary differential equations with the help of partial Noether operators. The formula which provides the partial Noether operator was similar to that of the Noether operator but the invariance condition was different due to partial Lagrangians and resulting partial Euler-Lagrange equations.

We illustrated our approach for system of ordinary differential equations with or without standard Lagrangians. It was shown that partial Noether operators do not form Lie algebra in general as they are not symmetry generators of the system under consideration. We presented the theorem when the partial Euler-Lagrange equations are free of derivatives. In this situation the partial Noether operators become symmetry generators and also form a Lie algebra. The canonical form for systems of two second-order ODEs had been invoked and it was shown that the general linear second-order system converted into one of the canonical forms mentioned in Chapter 3. We constructed the partial Noether operators for a linear system of two second-order ODEs with variable coefficients which do not have Lagrangian formulation. Then the first integrals were computed by the partial Noether theorem associated with the partial Lagrangian. The case for which a Lagrangian existed for the linear system was also discussed. We derived the Noether and partial Noether operators for this special case and it was found that they were identical due to independence of derivatives in the associated partial Euler-Lagrange equations. The difference occurred in the guage terms as the Lagrangians were different for the Noether and partial Noether approaches and the algebras in this special case were isomorphic. Finally the first integrals resulting from both approaches were determined. They were the same.

We provided the complete classification for potential functions of a Hamiltonian system with two degrees of freedom. This problem was considered before via the Noether approach where the authors could not give the complete classification. The Lagrangian exists for the system under study but we wanted to know the effectiveness of the partial Noether approach which has provided all the first integrals. Then the idea was implemented to systems of nonlinear second-order ODEs which were non-variational and it was shown how the order could be reduced for such type of systems by constructing first integrals without regard to Lagrangians.

We extended the partial Noether theorem to the case of the approximate situation. An approximate partial Noether theorem was deduced for which it was shown how the approximate first integrals could be constructed for perturbed ordinary differential equations for which we have no Lagrangian formulation. Ample examples to illustrate the understanding of partial Lagrangians for the perturbed ordinary differential equations were also presented. We constructed the partial Noether operators for coupled nonlinear oscillators. Then first integrals for both resonant and non-resonant cases of coupled oscillators were computed by the approximate partial Noether theorem with the use of approximate partial Noether operators and partial Lagrangian. In the last application we considered two coupled van der pol oscillators with linear diffusive coupling. The approximate first integrals of this system were derived via the approximate partial Noether theorem and all the cases for the coupling parameters were discussed in detail.

In this work, we have shown the effectiveness of be partial Lagrangian approach for exact and perturbed ordinary differential equations for which we do not have standard Lagrangians. We showed that partial Lagrangians do exist for all such type of exact and perturbed ordinary differential equations and are very important in constructing exact and approximate first integrals.

We conclude by mentioning some further works. It would be of interest to apply the partial Noether approach to scalar higher-order ODEs as well as to second-order systems of three or more equations. Perturbed ODEs also need further investigation using this approach.

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