INVERSE OPERATIONS ON TENSOR PRODUCTS OF MATRICES

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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(Signature of candidate)

23rd day of September 20 22 at Windhoek, Namibia

Abstract

A variety of products of matrices arise by considering different algebraic structures – for example, linear transformations (matrix multiplication), product vector spaces which leads to entrywise products (known as the Hadamard or Schur product) and bilinear transformations (tensor products).

Inverse operations of linear transformations have been extensively studied in the literature, but inverse tensor products are less well known.

This dissertation considers these inverse operations from different perspectives, by focusing on characterising such operations and examining certain desirable properties. Primarily, by abiding to an algebraic perspective, quotients of vector spaces of $(ms) \times (nt)$ matrices are considered by characterising linear quotient functions. Requirements for such functions to satisfy desirable properties, in addition to linear properties, are considered. Additional quotients, which do not appear in the literature, are derived. Multiplicative (monoidal) quotients are also considered. These quotients only exist on restricted structures, and their limitations are briefly examined.

Lastly, by relaxing the requirement for a purely algebraic quotient and finite-dimensional spaces, an analytic approach is considered by assessing a least squares minimisation of objects on reproducing kernel Hilbert spaces. In this method, Tikhonov regularisation is employed to ensure boundedness in obtaining inverse operations.

Contents

List of Symbols

$\mathbb{F}$	Arbitrary field
$\mathbb{R}$	The field of real numbers
$\mathbb{R}_{\geq 0}$	The semi-ring of non-negative real numbers
$\mathbb{R}_{>0}$	The set of positive real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{F}^n$	Vector space of n-dimensional vectors over some field $\mathbb{F}$
$e_{i,n}$	The i^{th} element in the standard basis for $\mathbb{F}^n$
$\mathbb{F}^{m,n}$	Vector space of $m \times n$ matrices over some field $\mathbb{F}$
$\mathbb{F}^{m,n}_{nz}$	Vector space of $m \times n$ matrices over some field $\mathbb{F}$, excluding $\mathbf{0}^{m,n}$
$\mathbf{0}^{m,n}$	The $m \times n$ zero matrix
I_m	The $m \times m$ identity matrix
$E^{m,n}_{k,l}$	Standard basis vector of the vector space $\mathbb{F}^{m,n}$, with a non-zero entry of 1 in the $(k,l)^{\text{th}}$ position and zero entries elsewhere
$[\,\cdot\,]_{i,j}$	Entry of a matrix in the i^{th} row and j^{th} column
$[\,\cdot\,]_i$	i^{th} entry of a vector in $\mathbb{F}^n$
$\lVert \cdot \rVert_F$	Frobenius norm
$\langle \cdot, \cdot \rangle$	Inner product
M^{-1}	Inverse of a matrix M
$\overline{M}$	Complex conjugate of a matrix M
M^*	Complex conjugate transpose of a matrix M, or adjoint of an operator M
M^T	Transpose of a matrix M
$\underline{R}_{\,k,l}$	The $(k,l)^{\text{th}}$ block of a $m \times n$ block matrix $R \in \mathbb{F}^{m,n} \otimes \mathbb{F}^{s,t}$
$\otimes$	Tensor product or Kronecker product
$\oslash$	Right Kronecker quotient
$\circleddash$	Left Kronecker quotient
$\oslash^{\times}$	Uniform multiplicative (right) Kronecker quotient
$\odot$	Partial Frobenius product
$\mathrm{nz}(M)$	Set of indices (i,j) of a matrix M for which $[M]_{i,j} \neq 0$
$\delta_{i,j}$	Kronecker delta of variables i and j, defined by $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise
$\forall$	Universal quantifier “for all”
$\exists$	Existential quantifier “there exists”
$\implies$	Shorthand notation denoting “implies”
$\iff$	Shorthand notation denoting “if and only if”

Chapter 1

Introduction

A tensor product $\otimes : V \times W \to V \otimes W$ of vector spaces V and W is a universal bilinear map of vector spaces to another vector space. An extensive development of tensor products in multilinear algebra is presented by Merris in [11], which is considered as a notable resource and reference for this research. This tensor product mentioned above is characterised by the Universal Factorisation Property, which is defined formally in [11]. The following definitions are due to Merris. For simplicity, we consider bilinearity to define the tensor product below. The definitions and results are easily extended to the multilinear (m-linear) case, either by appropriate definitions or by composing tensor products. The definition of a bilinear function is provided in Chapter 2. In this dissertation, with the exception of Chapter 4, we will be considering finite-dimensional vector spaces.

Definition 1. Let V and W be vector spaces. A vector space T and a bilinear function

$$\Psi : V \times W \to T$$

are said to satisfy the **Universal Factorisation Property** if, for every vector space U, and every bilinear function

$$f : V \times W \to U$$

there exists a unique linear function $h : T \to U$ such that

$$f = h \circ \Psi.$$

Lemma 1. *If (T_1, Ψ_1) and (T_2, Ψ_2) satisfy the Universal Factorisation Property, then T_1 and T_2 are isomorphic.*

Definition 2. Let $\Psi : V \times W \to T$ be a bilinear map. If (T, Ψ) satisfies the Universal Factorisation Property for V and W then the vector space T is called the **tensor product** of V and W, written

$$T = V \otimes W.$$

Definition 3. Let $\Psi : V \times W \to T$ be a bilinear map with $T = V \otimes W$ the tensor product of V and W. Then each element of T is called a tensor. In particular, the elements

$$v \otimes w := \Psi(v, w) \ , \quad v \in V, w \in W$$

are called **decomposable tensors.** The decomposable tensors in T generate T.

So, given a vector space U, for every bilinear function $f : V \times W \to U$ there exists a unique linear function $h : V \otimes W \to U$, as demonstrated by the following commutative diagram:

$$\begin{array}{ccc} V \times W & \xrightarrow{\Psi} & V \otimes W \\ \Big\downarrow f & \swarrow_{!h} & \\ U & & \end{array} \tag{1.1}$$

The notation $!h$ signifies that h is a unique function. This research project aims to understand how inverse operations in terms of decomposable tensors may be approached and established for tensor products; and to investigate the mathematical properties of such operations.

An example of a tensor product on matrices is the Kronecker product [14], which has also been extensively researched in various publications, including [13]. The Kronecker product realises the tensor product for finite-dimensional vector spaces (column vector spaces) and induces a Kronecker product $A \otimes B$ of two matrices A and B (i.e. linear transforms on finite-dimensional vector spaces).

1.1 The Kronecker Product as Realisation of the Tensor Product

Since matrices represent linear transformations under some choice of basis, a Kronecker product of matrices arises naturally from the Kronecker product of vectors. To illustrate this assertion, consider the following definition for the Kronecker product of vectors $a \in \mathbb{F}^m$ and $b \in \mathbb{F}^n$, where $\mathbb{F}^k$ denotes the space of k-dimensional vectors over some field $\mathbb{F}$.

Definition 4. The **Kronecker product of vectors** $a \in \mathbb{F}^m$ and $b \in \mathbb{F}^n$ is defined in terms of decomposable tensors $a \otimes b \in \mathbb{F}^m \otimes \mathbb{F}^n$ by:

$$a \otimes b = \begin{bmatrix} a_1 b \\ a_2 b \\ \vdots \\ a_m b \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ \vdots \\ a_1 b_n \\ a_2 b_1 \\ a_2 b_2 \\ \vdots \\ \vdots \\ a_m b_n \end{bmatrix}.$$

Here, $\mathbb{F}^m \otimes \mathbb{F}^n$ is identified with $\mathbb{F}^{mn}$. The Kronecker product is bilinear and it can be shown that $(\mathbb{F}^{mn}, \otimes)$ satisfies the Universal Factorisation Property.

Now, consider the linear function $g : \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}^p \times \mathbb{F}^q$ defined as:

$$g(a, b) = (Aa, Bb),$$

for all $a \in \mathbb{F}^m$ and $b \in \mathbb{F}^n$, where A is a $p \times m$ matrix and B a $q \times n$ matrix. By their actions on the standard bases, A and B may be considered as the linear transformations $A : \mathbb{F}^m \to \mathbb{F}^p$ and $B : \mathbb{F}^n \to \mathbb{F}^q$. Furthermore, g can be considered as the transformation given by $A \times B$ (the direct sum of matrices) – and will be denoted by $A \times B$ in diagram (1.3) below.

Furthermore, consider the function $k : \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}^p \otimes \mathbb{F}^q$ defined as:

$$k(a, b) = (Aa) \otimes (Bb).$$

It can be shown that k is a bilinear function. It is apt to introduce the definition of the Kronecker product of matrices, as a simple extension of the Kronecker product of vectors defined above. Let $\mathbb{F}^{r,s}$ denote the space of $r \times s$ matrices over the field $\mathbb{F}$, where $r, s \in \mathbb{N}$.

Definition 5. The **Kronecker product** of matrix $A \in \mathbb{F}^{p,m}$ with the matrix $B \in \mathbb{F}^{q,n}$ is defined as:

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pm}B \end{bmatrix}.$$

Given the definition of the Kronecker product of A and B above, we can explicitly show that:

$$(A \otimes B)(a \otimes b) = (Aa) \otimes (Bb). \tag{1.2}$$

Furthermore, we will define the linear function $A \otimes B : \mathbb{F}^m \otimes \mathbb{F}^n \to \mathbb{F}^p \otimes \mathbb{F}^q$ as the matrix transformation given by the matrix $A \otimes B$. Now, consider the following diagram, expanded from diagram (1.1). The function Ψ induces, as per Definition 1 above, the tensor product of two vector spaces $V = \mathbb{F}^m$ and $W = \mathbb{F}^n$. Similarly, we introduce a tensor product Ψ', where the underlying vector spaces will be clear from the context.

$$\begin{array}{ccc} \mathbb{F}^m \times \mathbb{F}^n & \xrightarrow{\Psi} & \mathbb{F}^m \otimes \mathbb{F}^n \\ {\scriptstyle A\times B}\downarrow & \searrow^{k} & \downarrow{\scriptstyle A\otimes B} \\ \mathbb{F}^p \times \mathbb{F}^q & \xrightarrow[\Psi']{} & \mathbb{F}^p \otimes \mathbb{F}^q \end{array} \tag{1.3}$$

By considering this diagram together with properties of tensor products, we can also show that equation (1.2) holds, where the function "$A \otimes B$" is a unique linear transformation from $\mathbb{F}^m \otimes \mathbb{F}^n$ to $\mathbb{F}^p \otimes \mathbb{F}^q$. Furthermore, by this uniqueness, and our choice of the Kronecker product to realise the tensor product, we must have that this linear map is the Kronecker product of matrices A and B. To further illustrate this reasoning, we consider the operator space analogue of diagram (1.3). For all (a, b) in the bilinear dual space of $\mathbb{F}^{p,m} \times \mathbb{F}^{q,n}$, there exists a unique h' such that

$$\begin{array}{ccc} \mathbb{F}^{p,m}\times\mathbb{F}^{q,n} & \xrightarrow{\Psi'} & \mathbb{F}^{p,m}\otimes\mathbb{F}^{q,n} \\ {\scriptstyle (a,b)}\big\downarrow & \swarrow {\scriptstyle !h' : A\otimes B\mapsto(Aa,Bb)} & \\ \mathbb{F}^{p}\times\mathbb{F}^{q} & & \end{array} \tag{1.4}$$

commutes. So, given that the tensor product of vectors is realised via the Kronecker product, the Kronecker product of matrices becomes "the" product of interest, and will therefore be considered in detail in the study.

1.2 Quotients of Tensor Products

To understand how inverse operations may be approached for tensor products and established for such products from an algebraic perspective, a study on quotients and how these may be defined for tensor products is of relevance. For the purpose of introducing this study, it is therefore apt to introduce inverse operations by discussing the notion of an equivalence relation and a pre-image.

Definition 6. Let Z be a set. An **equivalence relation** on Z is a subset $E \subseteq Z \times Z$, which satisfies the following properties. For all $x, y \in Z$, we use the symbol "xEy" to denote that $(x, y) \in E$.

1. $xEx \quad \forall x \in Z$
2. $xEy \iff yEx \quad \forall x, y \in Z$
3. xEy and $yEz \implies xEz \quad \forall x, y, z \in Z$

Definition 7. Let $f : X \to Y$ be a map between sets X and Y. The **pre-image** of f is denoted f^{-1} and defined as:

$$f^{-1}(y) = \{x \in X \ : \ f(x) = y\}, \quad y \in Y.$$

Should f be invertible, the image of every subset $Z \subseteq Y$ under the inverse map is identified with $\bigcup_{z\in Z} f^{-1}(z)$ in the obvious way, and indeed $f^{-1}(f(z)) = \{z\}$ so that $f^{-1} \circ f$ may be taken as the identity in the usual way.

Definition 8. Let E be an equivalence relation on a set Z. An **equivalence class** of E for an $x \in Z$, denoted by $[x]_E$, is defined as:

$$[x]_E = \{y \in Z : xEy\}.$$

It can be shown that every function, by its pre-images, induces a partition

$\{Z_i, i \in I\}$ of Z; and that every partition induces an equivalence relation.

Definition 9. A **quotient** of a vector space V, is a partition $\{V_i, i \in I\}$ of V which is a vector space, with the same (set) operations as V, i.e.

$$V_i + V_j = \{a + b : a \in V_i, b \in V_j\},$$

and:

$$\alpha V_i = \{\alpha\, a : a \in V_i\}.$$

Therefore, each partition defined as a quotient defines an equivalence class, given by the pre-images of the function $f : V \to I$, given by:

$$f(a) = i \quad \text{if and only if} \quad a \in V_i.$$

However, consider $f^{-1}(-1)$ in the case where $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$. Here, $f^{-1}(-1)$ is an empty set. We may approach the problem of assessing the inverse of f at this point by considering the "closest" pre-image. So, we consider the function $f^{\sim}(y) = \{x \in \mathbb{R} : |y - f(x)| \text{ is a minimum}\}$. For example,

$$\begin{aligned} f^{\sim}(-1) &= \{x \in \mathbb{R} : |-1 - f(x)| \text{ is a minimum}\} \\ &= \{0\} \cong 0, \end{aligned}$$

i.e. $\{0\}$ is identified with 0. On the other hand, we note that $f^{\sim}(1) = \{-1, 1\}$. This is a high-level analogue to the approximation methods of defining the Kronecker quotient by minimizing the Frobenius norm

$$\|M - C \otimes D\|_F$$

for $M \in \mathbb{F}^{ac,bd}, C \in \mathbb{F}^{a,b}$ and $D \in \mathbb{F}^{c,d},\ a, b, c, d \in \mathbb{N}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ [16]. The Frobenius norm is given by:

$$\|M\|_F = \sqrt{\operatorname{tr}(M^* M)},$$

where M^* is the complex conjugate transpose of M.

This optimisation approach may be compared to a more algebraic consideration, which emphasises quotients. The Kronecker quotient has first been conceptualised by Leopardi in [9]. Hardy has expanded on this notion extensively in [3]. For the purpose of introduction, we will focus on the right Kronecker quotient as defined by [3].

Definition 10. A **right Kronecker quotient**, denoted $\oslash$, is an operation $\oslash : (\mathbb{F}^{p,m} \otimes \mathbb{F}^{q,n}) \times \mathbb{F}^{q,n}_{nz} \to \mathbb{F}^{p,m}$ satisfying:

$$(A \otimes B) \oslash B = A,$$

for $A \in \mathbb{F}^{p,m}$ and $B \in \mathbb{F}^{q,n}_{nz}$, where $\mathbb{F}^{q,n}_{nz} := \mathbb{F}^{q,n} \backslash \{\mathbf{0}^{q,n}\}$, with $\mathbf{0}^{q,n}$ the $q \times n$ zero matrix.

A Kronecker quotient $\oslash : V \otimes W \times W \to V$ is an algebraic analogue of a projection from $V \otimes W$ to W. These projections are not unique, and have a variety of algebraic qualities, which is the topic of the proposed study. A review of this work will involve studying how properties of Kronecker products reflect in Kronecker quotients; as well as understanding the various ways in which such Kronecker quotients may also be defined, for example by approximation – through optimisation problems [16] or by Tikhonov regularisation [12].

1.3 Properties of the Kronecker Product

As part of understanding inverse operations on tensor products, it is of relevance to understand the key properties of Kronecker products; and in which conditions these reflect naturally in their analogue quotients (if at all). The basic properties of Kronecker products are summarised here, in a similar manner to how they have been set out in [13].

Associativity:
$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad \forall A \in \mathbb{F}^{m,n}, B \in \mathbb{F}^{p,q}, C \in \mathbb{F}^{r,s}.$

Right-distributivity:
$(A + B) \otimes C = A \otimes C + B \otimes C \quad \forall A, B \in \mathbb{F}^{p,q}, C \in \mathbb{F}^{r,s}.$

Left-distributivity:
$A \otimes (B + C) = A \otimes B + A \otimes C \quad \forall A \in \mathbb{F}^{p,q}, B, C \in \mathbb{F}^{r,s}.$

Scalar factorisation:
$(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B) \quad \forall \alpha \in \mathbb{F}, A \in \mathbb{F}^{p,q}, B \in \mathbb{F}^{r,s}.$

Transpose:
$(A \otimes B)^T = A^T \otimes B^T \quad \forall A \in \mathbb{F}^{p,q}, B \in \mathbb{F}^{r,s}.$

Mixed product:

$(A \otimes B)(C \otimes D) = AC \otimes BD.$

Complex conjugate:

$\overline{A \otimes B} = \overline{A} \otimes \overline{B} \quad \forall A \in \mathbb{C}^{p,q}, B \in \mathbb{C}^{r,s},$

where $\overline{M}$ denotes the complex conjugate of the matrix M.

Determinant:

$\det(A \otimes B) = \det(B \otimes A) = [\det(A)]^n[\det(B)]^m \quad \forall A \in \mathbb{F}^{m,m}, B \in \mathbb{F}^{n,n}.$

Inverse:

$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \quad \forall$ nonsingular matrices $A \in \mathbb{F}^{m,m}, B \in \mathbb{F}^{n,n}.$

Trace:

$\operatorname{tr}(A \otimes B) = \operatorname{tr}(B \otimes A) = \operatorname{tr}(A)\operatorname{tr}(B) \quad \forall A \in \mathbb{F}^{m,m}, B \in \mathbb{F}^{n,n}.$

A Kronecker quotient may be defined such that analogues of certain properties follow naturally – while expressions for other properties might prove more problematic. For example, the associativity property may lead naturally to the following expression:

$$(M \oslash C) \oslash B = M \oslash (B \otimes C) \quad \forall A \in \mathbb{F}^{m,n}, B \in \mathbb{F}^{p,q}, C \in \mathbb{F}^{r,s},$$

where $M = A \otimes B \otimes C$.

The right-distributivity property may then follow in a similar manner:

$$[(A \otimes C) + (B \otimes C)] \oslash C = (A \otimes C) \oslash C + (B \otimes C) \oslash C \quad \forall A, B \in \mathbb{F}^{p,q}, C \in \mathbb{F}^{r,s}.$$

In Chapter 2, we will consider some of the fundamental properties in more detail. In Chapter 3, we will consider the quotient analogues of these properties, some already proposed in [3]; and consider how possible definitions and characterisations of Kronecker quotients vary in order to satisfy certain properties. For example, [3] provides a demonstration of how a Kronecker quotient satisfying an associativity property is generally irreconcilable with the trace property (as defined and proposed in [3]). Certain properties, like the trace property, may also impose restrictions on the underlying field. We will examine these restrictions, and potential methods to circumvent them.

1.4 Beyond Vector Spaces

These considerations of algebraic properties of the Kronecker quotient are not to be constrained to the (graded) algebra of vector spaces. Section 3.5 and 3.6 consider other properties and associated Kronecker quotients which do not appear in the literature. Appendix A provides a novel investigation into quotients that provide tensor decompositions. Square matrices form a multiplicative monoid so, it is natural to consider monoid quotients as well. Monoid quotients will be described and discussed in Section 3.6.

Finally, matrices over the real and complex fields form a Banach algebra. Thus, quotients in Banach algebras and spaces will also be investigated in Chapter 4. In addition to the algebraic perspective on- and considerations of algebraic properties of the Kronecker quotient, the analytic approach to these quotients are also of concern. In addition to the optimisation approach of [16] mentioned above, Saitoh utilises the theories of reproducing kernels and Tikhonov regularisation to obtain an inversion for A in the Kronecker product $A \otimes B$ [12]. This approach is analytical of nature; and generalised for both the Hadamard- and tensor product of matrices. Saitoh's approach is within the context of reproducing kernel Hilbert spaces, as is defined in [12]. Details on these methods will be provided in Chapter 4.

Chapter 2

On Tensor Products and the Kronecker Product

2.1 Tensor Products

In the introduction, we provided a definition of a tensor product, and described the Universal Factorisation Property. Owing to this property, it is possible to "factorise" – and thus study – any multilinear function of vector spaces in terms of linear functions (which are thoroughly studied in literature) and a more well-studied multilinear function with its resulting tensor product – as also explained in [11], in which the latter pair is called a "Universal Pair". This will be of particular importance in Section 2.2, where we will motivate the universality of the Kronecker product and we will use this product to realise the tensor product of vectors and matrices. We will confine our discussion to bilinear functions, as defined below.

Definition 11. Let V and W be vector spaces over some field $\mathbb{F}$. A function $f : V \times W \to U$ is **bilinear** when:

$$\begin{aligned} f(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2) = \alpha_1\beta_1 f(v_1, w_1) + \alpha_1\beta_2 f(v_1, w_2) \\ + \alpha_2\beta_1 f(v_2, w_1) + \alpha_2\beta_2 f(v_2, w_2), \end{aligned}$$

for all $v_1, v_2 \in V, w_1, w_2 \in W$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$. Thus, f is linear in both V and W.

It is important to note that, as mentioned in Chapter 1, the decomposable tensors $\{v \otimes w : v \in V, w \in W\}$ span the vector space $V \otimes W$. So, despite the fact that $v \otimes w \in V \otimes W$, in general:

$$V \otimes W \neq \{v \otimes w : v \in V, w \in W\}.$$

This statement is, in part, motivated by the following theorem presented in [11].

Theorem 1. *Let* $\{e_{V,1}, e_{V,2}, \ldots, e_{V,n}\}$ *and* $\{e_{W,1}, e_{W,2}, \ldots, e_{W,m}\}$ *denote the standard bases of vector spaces* V *and* W*, respectively. Then the set:*

$$\{e_{V,i} \otimes e_{W,j} : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}$$

is the standard basis of $V \otimes W$*. Therefore, the dimension of* $V \otimes W$ *is* nm*.*

As explained by Merris in [11], it is practical to define a linear (or bilinear) transformation through its action on a basis, given that any linear- or bilinear transformation is completely and uniquely described by its action on a basis of the domain [11]. This concept is known as linear extension (or bilinear extension) and is defined below.

Theorem 2. *Let* $\{e_i : i = 1, 2, \ldots, n\}$ *be a basis of the vector space* V*. Then, for arbitrary scalars* $\alpha_i,\ i = 1, 2, \ldots, n$*, we have the (arbitrary) vector:*

$$v = \sum_{p=1}^{n} \alpha_p e_p.$$

As a result, there exists precisely one linear function $f : V \to U$ *such that:*

$$f(v) = \sum_{p=1}^{n} \alpha_p f(e_p).$$

Similarly, we can have extensions for bilinear functions and multilinear functions, as illustrated in the following theorem.

Theorem 3. *Let* $\{e_{V,i} : i = 1, 2, \ldots, n\}$ *and* $\{e_{W,j} : j = 1, 2, \ldots, m\}$ *be bases of vector spaces* V *and* W*, respectively, then for arbitrary scalars* $\alpha_i,\ i = 1, 2, \ldots, n$ *and* $\beta_j,\ j = 1, 2, \ldots, m$ *there exists precisely one bilinear function* $g : V \times W \to U$ *such that:*

$$g(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j g(e_{V,i}, e_{W,j}),$$

where $v = \sum_{i=1}^{n} \alpha_i e_{V,i}$ *and* $w = \sum_{j=1}^{m} \beta_j e_{W,j}$.

As mentioned previously, the bilinear case can be extended to the multilinear case (i.e. for vector spaces $V_1, V_2, \ldots, V_p$ for any $p \in \mathbb{N}$) through composing tensor products and an associativity argument (see Section 2.2). Relevant definitions for this can be found in [11]. As an example, the dimension of $V_1 \otimes V_2 \otimes \cdots \otimes V_p$ becomes $\prod_{i=1}^{p} \dim(V_i)$.

Since decomposable tensors provide a clear algebraic model for inverse operations, we will focus on providing more depth and development around them. A convenient interpretation may be achieved within the dual space. Imposing additional structure on non-decomposable tensors will yield different inverse operations.

In [11], Merris provides a fundamental understanding of decomposable tensors through the viewpoint of dual spaces. An adapted and condensed demonstration is provided below.

Denote the set of bilinear functionals on $V \times W$ by

$$M(V, W) := \{f : V \times W \to \mathbb{F} \quad : f \text{ is a bilinear function}\}$$

and linear functionals on V by

$$L(V) := \{g : V \to \mathbb{F} \quad : g \text{ is a linear function}\}.$$

Furthermore, $M(V, W)$ is a vector space under componentwise addition and scalar multiplication, defined by:

$$(f_1 + f_2)(v, w) := f_1(v, w) + f_2(v, w)$$

and

$$(\alpha f)(v, w) := \alpha f(v, w),$$

where $\alpha \in \mathbb{F}$. Now, consider the dual space of $M(V, W)$, denoted by:

$$D_M := \{\Theta_{(v,w)} : M(V, W) \to \mathbb{F} \, : \, (v, w) \in V \times W\},$$

and, for each $(v, w) \in V \times W$, define $\Theta_{(v,w)} \in D_M$ by:

$$\Theta_{(v,w)}(f) := f(v, w),$$

where $f \in M(V, W)$.

To show that $\Theta_{(v,w)}$ is linear, consider the following, for $f, g \in M(V, W)$ and $\alpha, \beta \in \mathbb{F}$, then by definition:

$$\begin{aligned}\Theta_{(v,w)}(\alpha f + \beta g) &= (\alpha f + \beta g)(v, w) \\ &= \alpha f(v, w) + \beta g(v, w) \\ &= \alpha \Theta_{(v,w)}(f) + \beta \Theta_{(v,w)}(g).\end{aligned}$$

As an analogue to Theorem 1, we have the following:

Theorem 4. *The set:*

$$E := \{\Theta_{(e_{V,i}, e_{W,j})} : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}$$

is a basis for D_M, the dual space of the bilinear functionals on $V \times W$.

Proof. Let $E_V := \{e_{V,i} : i = 1, 2, \ldots, n\}$ and $E_W := \{e_{W,j} : j = 1, 2, \ldots, m\}$ be bases of vector spaces V and W respectively. Furthermore, let the sets $B_V := \{b_{V,i} : i = 1, 2, \ldots, n\}$ and $B_W := \{b_{W,j} : j = 1, 2, \ldots, m\}$ be bases of $L(V)$ and $L(W)$ respectively, dual to E_V and E_W.

We may, therefore, define the elements of B_V and B_W by:

$$b_{V,i}(e_{V,t}) := \delta_{i,t} \quad \text{and} \quad b_{W,j}(e_{W,t}) := \delta_{j,t}, \tag{2.1}$$

and linear extension. The concept of linear extension will be explained briefly forthwith. For every $v \in V$ and $w \in W$, we may write, where $\alpha_i, \beta_j \in \mathbb{F}, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$:

$$v = \alpha_1 e_{V,1} + \alpha_2 e_{V,2} + \cdots + \alpha_n e_{V,n},$$

and:

$$w = \beta_1 e_{W,1} + \beta_2 e_{W,2} + \cdots + \beta_m e_{W,m}.$$

So, given linear extension, equation (2.1) defines $b_{V,i} \in B_V$ as follows:

$$\begin{aligned}b_{V,i}(v) &= b_{V,i}(\alpha_1 e_{V,1} + \alpha_2 e_{V,2} + \cdots + \alpha_n e_{V,n}) \\ &= \alpha_1 b_{V,i}(e_{V,1}) + \alpha_2 b_{V,i}(e_{V,2}) + \cdots + \alpha_n b_{V,i}(e_{V,n}) \\ &= \alpha_i,\end{aligned}$$

and similarly, $b_{W,j} \in B_W$ by $b_{W,j}(w) = \beta_j$.

Now, consider the set of functionals:

$$B_M := \{b_{V,i}b_{W,j} : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\},$$

where we define, for $v \in V$ and $w \in W$:

$$b_{V,i}b_{W,j}(v, w) := b_{V,i}(v)b_{W,j}(w). \tag{2.2}$$

By this definition it follows that $b_{V,i}b_{W,j}$ is bilinear, so if $v_1, v_2 \in V, w_1, w_2 \in W$, then:

$$\begin{aligned}
&b_{V,i}b_{W,j}(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2)\\
&= b_{V,i}(\alpha_1 v_1 + \alpha_2 v_2)b_{W,j}(\beta_1 w_1 + \beta_2 w_2)\\
&= [\alpha_1 b_{V,i}(v_1) + \alpha_2 b_{V,i}(v_2)][\beta_1 b_{W,j}(w_1) + \beta_2 b_{W,j}(w_2)]\\
&= \alpha_1\beta_1 b_{V,i}b_{W,j}(v_1, w_1) + \alpha_1\beta_2 b_{V,i}b_{W,j}(v_1, w_2) + \alpha_2\beta_1 b_{V,i}b_{W,j}(v_2, w_1)\\
&\qquad + \alpha_2\beta_2 b_{V,i}b_{W,j}(v_2, w_2).
\end{aligned}$$

Hence, we have that $b_{V,i}(v)b_{W,j}(w) \in M(V, W),\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. Following from the results above, we have, for arbitrary $v \in V, w \in W$ and $f \in M(V, W)$:

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} f(e_{V,i}, e_{W,j})b_{V,i}b_{W,j}(v, w)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} f(e_{V,i}, e_{W,j})[b_{V,i}(\alpha_1 e_{V,1} + \alpha_2 e_{V,2} + \cdots + \alpha_n e_{V,n})\\
&\qquad \times b_{W,j}(\beta_1 e_{W,1} + \beta_2 e_{W,2} + \cdots + \beta_m e_{W,m})]\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j f(e_{V,i}, e_{W,j})\\
&= f(\alpha_1 e_{V,1} + \alpha_2 e_{V,2} + \cdots + \alpha_n e_{V,n}, \beta_1 e_{W,1} + \beta_2 e_{W,2} + \cdots + \beta_m e_{W,m})\\
&= f(v, w).
\end{aligned}$$

From this equation, we may deduce that B_M spans $M(V, W)$. Furthermore, consider the set B_M and let $\gamma_{i,j} \in \mathbb{F}, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$ be the scalars such that:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\gamma_{i,j}b_{V,i}b_{W,j} = \mathfrak{o}, \tag{2.3}$$

where $\mathfrak{o}$ is the zero bilinear functional, defined by $\mathfrak{o}(v, w) = 0$ for all $v \in V$ and $w \in W$. Now, consider the sum of functionals in equation (2.3), evaluated at $(e_{V,1}, e_{W,1}) \in V \times W$:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\gamma_{i,j}b_{V,i}b_{W,j}(e_{V,1}, e_{W,1}) = \mathfrak{o}(e_{V,1}, e_{W,1})$$
$$\implies \gamma_{1,1} = 0,$$

since, by equation (2.2) and (2.1), $b_{V,i}b_{W,j}(e_{V,p}, e_{W,t}) = \delta_{i,p}\delta_{j,t}$. Evaluating equation (2.3) in a similar fashion for every $(e_{V,i}, e_{W,j})$, we have that $\gamma_{i,j} = 0$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Hence, the elements $b_{V,i}b_{W,j}, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$ are linearly independent.

Therefore, B_M is a basis for $M(V, W)$ and given that the set E is the dual basis of B_M, it may be concluded that E is a basis for D_M. □

Now, define $\theta : V \times W \to D_M$ by:

$$\theta(v, w) := \Theta_{(v,w)}.$$

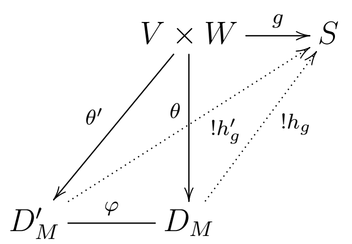
Furthermore, let the set E in Theorem 4 be the basis of D_M. The function θ is bilinear and acts as an embedding of the pair $(v, w) \in V \times W$ into a function space. To show that θ is bilinear, let $f \in M(V, W)$ and consider the following expression:

$$\begin{aligned}
&\theta(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2)(f)\\
&= \Theta_{(\alpha_1 v_1+\alpha_2 v_2,\beta_1 w_1+\beta_2 w_2)}(f)\\
&= f(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2)\\
&= \alpha_1\beta_1 f(v_1, w_1) + \alpha_1\beta_2 f(v_1, w_2) + \alpha_2\beta_1 f(v_2, w_1) + \alpha_2\beta_2 f(v_2, w_2)\\
&= \alpha_1\beta_1\Theta_{(v_1,w_1)}(f) + \alpha_1\beta_2\Theta_{(v_1,w_2)}(f) + \alpha_2\beta_1\Theta_{(v_2,w_1)}(f) + \alpha_2\beta_2\Theta_{(v_2,w_2)}(f)\\
&= \alpha_1\beta_1\theta(v_1, w_1)(f) + \alpha_1\beta_2\theta(v_1, w_2)(f) + \alpha_2\beta_1\theta(v_2, w_1)(f) + \alpha_2\beta_2\theta(v_2, w_2)(f).
\end{aligned}$$

Thus, bilinearity of θ follows from the bilinearity of $f \in M(V, W)$. Now, let S be any vector space and define a bilinear function $g : V \times W \to S$. Furthermore, define a linear function $h_g : D_M \to S$ by its action on the basis of D_M (and linear extension):

$$h_g(\Theta_{(e_{V,i},e_{W,j})}) := g(e_{V,i}, e_{W,j}),$$

for $i = 1, 2, \dots, n,\ j = 1, 2, \dots, m$. So, since g and $h_g \circ \theta$ both take prescribed values for each basis element $(e_{V,i}, e_{W,j})$; we have that h_g is unique and that $g = h_g \circ \theta$ for all $(v, w) \in V \times W$. Additionally, from Theorem 4, we may infer that the image of θ spans all of D_M. The pair (D_M, θ), therefore, satisfies the Universal Factorisation Property – visualised through the following commutative diagram:



(2.4)

The Universal Factorisation Property follows since, by construction, θ is surjective. In Chapter 1, this universality is suggested by Lemma 1. If we suppose that D'_M is another such vector space and $\theta' : V \times W \to D'_M$; then there exists unique linear transformations $k : D_M \to D'_M$ and $k' : D'_M \to D_M$. Here, θ' is surjective (otherwise h'_g is not unique). In this case, we have:

$$\theta = k' \circ \theta',\ \theta' = k \circ \theta$$
$$\implies \theta = k' \circ k \circ \theta$$
$$\implies k' = k^{-1}.$$

The last implication follows by surjectivity of θ. Therefore, given that there exists an invertible transformation between D_M and D'_M, we have that there exists some isomorphism $k : D_M \to D'_M$.

We may thus (in accordance with definitions and notations introduced in Chapter 1) relabel our conventions above to the conventional notations for tensor products and decomposable tensors:

$$D_M = V \otimes W$$

and

$$\Theta_{(v,w)} = v \otimes w.$$

2.2 The Kronecker Product

The theory of Kronecker products has been developed in line with solving matrix equations with respect to an unknown matrix X [13, 7], which include some classical equations such as the Lyapunov equation (for given matrices A and H):

$$XA + A^*X = H,$$

and the Sylvester equation [8] (for given matrices A, B and C):

$$AX + XB = C.$$

For example, the matrix expression $AX = B$ is equivalent to the expression $(I \otimes A)\,\mathrm{vec}\,(X) = \mathrm{vec}\,(B)$, where $\mathrm{vec}\,(\cdot)$ is the vec operator [13], which is discussed briefly in Section 2.3.

Given a fundamental understanding of the tensor product, we may further our discussion of Kronecker products, as they pertain to the focus of this work. Notations introduced throughout Chapter 1 – in particular the products denoted by $A \otimes B$ and $a \otimes b$ – will continue to be relevant, unless explicitly stated otherwise.

The Kronecker product $A \otimes B$ is considered a realisation of the tensor product of vector spaces A and B. So, akin to how A and B are defined as matrices and identified as linear transformations due to their actions on the standard basis; $A \otimes B$ is a realisation of a tensor product of linear transformations by the choice of basis. In this case, $\mathbb{F}^m \otimes \mathbb{F}^n$ becomes isomorphic to $\mathbb{F}^{mn}$.

This characteristic – which is briefly introduced in Chapter 1 and illustrated in diagram (1.4) – is established formally in [11], through a theorem equivalent to the following.

Theorem 5. *Let $L(X, Y) := \{\, f : X \to Y \ : f \text{ is linear} \}$ denote the space of linear transformations on vector spaces X and Y. Then the space*

$$L(V \otimes W, S \otimes T)$$

is isomorphic to the tensor product

$$L(V, S) \otimes L(W, T).$$

Furthermore, $\{T_A \otimes T_B : T_A \in L(V,S), T_B \in L(W,T)\}$ *is the set of decomposable tensors.*

Proof. Let $T_1 \in L(V,S)$ and $T_2 \in L(W,T)$. Now, consider the transformation $\Phi : V \times W \to S \otimes T$ defined by $\Phi(v,w) := T_1(v) \otimes T_2(w)$. Since Φ is bilinear, there exists a unique linear transformation $k_{T_1,T_2} \in L(V \otimes W, S \otimes T)$ such that $\Phi(v,w) = k_{T_1,T_2}(v \otimes w)$, which may be defined by $k_{T_1,T_2}(e_{V,i} \otimes e_{W,j}) := T_1(e_{V,i}) \otimes T_2(e_{W,j})$ and bilinear extension. Here, $e_{V,i}$ and $e_{W,j}$ denote basis elements of V and W as in Section 2.1.

Now, let $T_1 \otimes T_2 \in L(V,S) \otimes L(W,T)$ denote decomposable tensors and consider the following diagram.

$$\begin{array}{ccc} L(V,S) \times L(W,T) & \longrightarrow & L(V,S) \otimes L(W,T) \\ & \searrow^{\Psi} & \downarrow {!h} \\ & & L(V \otimes W, S \otimes T) \end{array} \tag{2.5}$$

Define Ψ, as indicated in the diagram above, by $\Psi(T_1, T_2) := k_{T_1,T_2}$. To show that Ψ is bilinear, consider the following, for arbitrary $v \in V, w \in W$ and $T_1, T_1' \in L(V,S),\ T_2, T_2' \in L(W,T),\ \lambda, \lambda', \mu, \mu' \in \mathbb{F}$:

$$\begin{aligned}
&\Psi(\lambda T_1 + \lambda' T_1', \mu T_2 + \mu' T_2')(v \otimes w) \\
&= k_{\lambda T_1 + \lambda' T_1', \mu T_2 + \mu' T_2'}(v \otimes w) \\
&= [\lambda T_1(v) + \lambda' T_1'(v)] \otimes [\mu T_2(w) + \mu' T_2'(w)] \\
&= \lambda\mu[T_1(v) \otimes T_2(w)] + \lambda\mu'[T_1(v) \otimes T_2'(w)] + \lambda'\mu[T_1'(v) \otimes T_2(w)] \\
&\qquad + \lambda'\mu'[T_1'(v) \otimes T_2'(w)] \\
&= [\lambda\mu\Psi(T_1,T_2) + \lambda\mu'\Psi(T_1,T_2') + \lambda'\mu\Psi(T_1',T_2) + \lambda'\mu'\Psi(T_1',T_2')](v \otimes w).
\end{aligned}$$

As a result, there exists a unique linear transformation:

$$h : L(V,S) \otimes L(W,T) \to L(V \otimes W, S \otimes T),$$

such that $h(T_1 \otimes T_2) = k_{T_1,T_2}$. In order to show that h is a bijection, let $\{s_j : j = 1, 2, \ldots, p\}$ and $\{v_k : k = 1, 2, \ldots, n\}$ denote bases of S and V respectively, and consider the following set:

$$\{[s_j, v_k] : j = 1, 2, \ldots, p,\ k = 1, 2, \ldots, n\}. \tag{2.6}$$

If we define each $[s_j, v_k]$, for scalars $\alpha_i, i = 1, 2, \ldots, n$, by:

$$[s_j, v_k](\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) := \alpha_k s_j,$$

we have that $[s_j, v_k] \in L(V, S)$ and, moreover, that the set (2.6) is a basis of $L(V, S)$. Similarly, the set $\{[t_i, w_l] : i = 1, 2, \ldots, q,\ l = 1, 2, \ldots, m\}$ is a basis of $L(W, T)$.

Now, consider the map h, as defined above:

$$h : [s_j, v_k] \otimes [t_i, w_l] \mapsto k_{[s_j,v_k],[t_i,w_l]},$$

where:

$$\begin{aligned} k_{[s_j,v_k],[t_i,w_l]}(v_k \otimes w_l) &= [s_j, v_k](v_k) \otimes [t_i, w_l](w_l) \\ &= s_j \otimes t_i. \end{aligned}$$

Hence, we have that $k_{[s_j,v_k],[t_i,w_l]} = [s_j \otimes t_i, v_k \otimes w_l]$. Note that, given Theorem 1, the set:

$$\{[s_j \otimes t_i, v_k \otimes w_l] : j = 1, 2, \ldots, p,\ k = 1, 2, \ldots, n,\ i = 1, 2, \ldots, q,\ l = 1, 2, \ldots, m\}$$

is a basis of $L(V \otimes W, S \otimes T)$.

Therefore, h is clearly bijective and we may conclude that:

$$L(V, S) \otimes L(W, T) \cong L(V \otimes W, S \otimes T). \qquad \square$$

The Kronecker product is not a unique realisation of the tensor product of vector spaces. However, it is generally the method of choice, partly due to extensive literature available and its simple description – as will be investigated in the next section.

Of more importance, however, is the universality principle demonstrated in the previous section. There will always be an isomorphism between an alternative choice of product (perhaps denoted by "$A \otimes_* B$") and $A \otimes B$; and, hence, properties in $A \otimes B$ carry over, by isomorphism, to properties in $A \otimes_* B$.

2.3 Properties of the Kronecker Product

The Kronecker product has many practical properties. Following the list of properties provided in Chapter 1, a more detailed breakdown and consideration of selected properties is of key importance in selecting properties desired for an inverse operation, or quotient. These properties are well documented, with concise summaries presented in (but not limited to) [13] and [7].

As stated by numerous sources, including van Loan in [15] and Laub throughout [8], the Kronecker product $A \otimes B$ manifests a number of structural properties common to both A and B; whether or not, for example, the components and their product are nonsingular, orthogonal, positive definite or (upper or lower) triangular.

Bilinearity (over vector spaces) is the defining property of the Kronecker product. This fact necessitates the consideration of a vector space quotient, as will be discussed in Chapter 3. Naturally, associativity of Kronecker products is such an example. We reiterate this property, along with the other important and useful properties of Kronecker products for convenience. Within each property listing below, the matrices A, B, C, D and scalar α are defined assuming that the various operations involved (such as matrix multiplication and addition) are defined.

KP Property 1. $(A \otimes B)^T = A^T \otimes B^T$.

KP Property 2. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

In [11], a proof is presented to show that the tensor products of vector spaces V_1, V_2, V_3 respects the isomorphism $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$. Furthermore, the Kronecker product is left and right distributive, and commutes with scalar multiplication in the left (separately, right) arguments.

KP Property 3. $A \otimes (B + C) = A \otimes B + A \otimes C$, $(A + B) \otimes C = A \otimes C + B \otimes C$.

KP Property 4. $(\alpha A) \otimes B = \alpha(A \otimes B)$.

KP Property 5. $A \otimes (\alpha B) = \alpha(A \otimes B)$.

An interpretation of the matrices A and B as linear transformations emphasises the product's multiplicative property and the matrix inverse property (KP Property 7) follows as a consequence (as demonstrated in [7]).

KP Property 6. $(A \otimes B)(C \otimes D) = AC \otimes BD$.

KP Property 7. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

KP Property 8. Let λ be the eigenvalue of A corresponding to the eigenvector x_A. Similarly, let μ be the eigenvalue of B corresponding to the eigenvector x_B. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$, corresponding to the eigenvector $x_A \otimes x_B$.

Additionally, several algebraic properties are also of interest, which provide the relation between tensor products and ordinary products.

KP Property 9. $\operatorname{tr}(A \otimes B) = \operatorname{tr}(B \otimes A) = \operatorname{tr}(A)\operatorname{tr}(B)$.

KP Property 10. $\det(A \otimes B) = \det(B \otimes A) = [\det(A)]^n[\det(B)]^m$.

The singular value decomposition of $A \otimes B$, as substantiated in [13], leads to another useful property concerning the rank of this product.

KP Property 11. $\operatorname{rank}(A \otimes B) = \operatorname{rank}(B \otimes A) = \operatorname{rank}(A)\operatorname{rank}(B)$.

The trace and determinant properties above may also be deduced from- and understood in terms of the eigenvalue- property above. Let $\sigma(A)$ and $\sigma(B)$ denote the set of all eigenvalues of matrices $A \in \mathbb{C}^{m,m}$ and $B \in \mathbb{C}^{n,n}$ respectively. Then, for KP Property 9, we have:

$$\operatorname{tr}(A \otimes B) = \sum_{\lambda \in \sigma(A)} \sum_{\mu \in \sigma(B)} (\lambda\mu) = \sum_{\lambda \in \sigma(A)} \lambda \sum_{\mu \in \sigma(B)} \mu = \operatorname{tr}(A)\operatorname{tr}(B),$$

and for KP Property 10:

$$\begin{aligned}\det(A \otimes B) &= \prod_{\lambda \in \sigma(A)} \left(\prod_{\mu \in \sigma(B)} \lambda\mu \right) \\ &= \prod_{\lambda \in \sigma(A)} \left(\lambda^n \prod_{\mu \in \sigma(B)} \mu \right) \\ &= \left(\prod_{\lambda \in \sigma(A)} \lambda \right)^n \left(\prod_{\mu \in \sigma(B)} \mu \right)^m \\ &= [\det(A)]^n[\det(B)]^m.\end{aligned}$$

As explained in [7], it is clear that $A \otimes B$ is in some relation with $B \otimes A$, since these products commute by trace, determinant or rank; and both contain all permutations of the product of entries in A and B. It can be shown that:

$$A \otimes B = P_{s,m}(B \otimes A)P_{n,t}, \tag{2.7}$$

for $A \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{s,t}$. The permutation matrices $P_{s,m} \in \mathbb{F}^{sm,sm}$ and $P_{n,t} \in \mathbb{F}^{nt,nt}$ may be defined as follows:

$$P_{r,k} := \sum_{i=1}^{r}\sum_{j=1}^{k}(e_{j,k} \otimes e_{i,r})(e_{i,r} \otimes e_{j,k})^T, \tag{2.8}$$

where $e_{i,n}$ denotes the i^{th} element in the standard basis for $\mathbb{F}^n$. These matrices are known as the "vec-permutation" matrices in [5]. In some sources, such as [15], these are also known as "perfect shuffle" matrices.

Although not of specific interest to this dissertation, a brief outline of the vec operator is appropriate as part of an introductory understanding of the Kronecker product. The vec operator, commonly denoted vec(·), stacks consecutive columns of an $m \times n$ matrix $A = [\underline{A}_1\ \underline{A}_2 \dots \underline{A}_n]$ vertically to form an $mn \times 1$ column vector:

$$\operatorname{vec}(A) = \begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \\ \vdots \\ \underline{A}_n \end{bmatrix}.$$

The vec operator has several uses, which include solving linear matrix equations, deriving Jacobian matrices and various applications within matrix differentiation [4]. An important result in the study of vec operators and their relation to the Kronecker product is the following, for $A \in \mathbb{F}^{m,n}, B \in \mathbb{F}^{n,p}$ and $C \in \mathbb{F}^{p,q}$:

$$\operatorname{vec}(ABC) = (C^T \otimes A)\operatorname{vec}(B). \tag{2.9}$$

This result follows from the simpler relation, $\operatorname{vec}(ab^T) = b \otimes a$, for column vectors $a \in \mathbb{F}^n$ and $b \in \mathbb{F}^m$. In addition, Henderson and Searle extensively characterize the "vec-permutation" matrix, denoted in [5] as $P_{u,v}$ and expressed by the following equation:

$$\operatorname{vec}(A) = P_{m,n}\operatorname{vec}(A^T), \tag{2.10}$$

where $A \in \mathbb{F}^{m,n}$. Various formulations for $P_{m,n}$ equivalent to (2.8), most of which derive from (2.9) and (2.10), are discussed in [5].

Chapter 3

On The Kronecker Quotient

3.1 Properties for the Kronecker Quotient

Considering the list of key properties of Kronecker products in Section 2.2 above, it is useful to understand how these reflect in Kronecker quotients. In [3], properties for Kronecker quotients, as they are defined in Section 1.2, are proposed and studied. These properties are listed below as they will be considered throughout this chapter. Firstly, we recall the definition of Kronecker quotient.

Definition 10. A **right Kronecker quotient**, denoted $\oslash$, is an operation satisfying:

$$(A \otimes B) \oslash B = A,$$

for $A \in \mathbb{F}^{p,m}$ and $B \in \mathbb{F}^{q,n}_{nz}$, where $\mathbb{F}^{q,n}_{nz} := \mathbb{F}^{q,n} \backslash \{\mathbf{0}^{q,n}\}$, with $\mathbf{0}^{q,n}$ the $q \times n$ zero matrix.

Although properties are listed and discussed for the right Kronecker quotient – the left Kronecker quotient analogues follow suit in a natural way. Within each property listing, the matrices A, B, M, M_1, M_2 and scalar α, where A and B are non-zero, are defined assuming that the operations involved are defined. For example, in KQ Property 1 below, we must have $M \in \mathbb{F}^{ms,nt}$ and $B \in \mathbb{F}^{s,t}_{nz}$, for some $m, n, s, t \in \mathbb{N}$. Additionally, we note that if the left hand side of a Kronecker quotient is the correct form of Kronecker product, these properties are actually identities. However, they do not hold in general, but under certain restrictions. These restrictions will be described when the properties are considered in detail.

KQ Property 1. $(M \oslash B)^T = M^T \oslash B^T$.

KQ Property 2. $(M \oslash A) \oslash B = M \oslash (B \otimes A)$.

KQ Property 3. $(M_1 + M_2) \oslash B = (M_1 \oslash B) + (M_2 \oslash B)$.

KQ Property 4. $(\alpha M) \oslash B = \alpha(M \oslash B)$.

KQ Property 5. $M \oslash (\alpha B) = \frac{1}{\alpha}(M \oslash B)$.

KQ Property 6. $\mathrm{tr}(M) = \mathrm{tr}(B)\,\mathrm{tr}(M \oslash B)$.

KQ Property 7. $\mathrm{rank}(M \oslash B) = \dfrac{\mathrm{rank}(M)}{\mathrm{rank}(B)}$.

KQ Property 8. $(M_1 \oslash A)(M_2 \oslash B) = (M_1 M_2) \oslash (AB)$.

KQ Property 9. $\det(M) = [\det(B)]^m [\det(M \oslash B)]^s$.

3.2 Algebraic Perspectives

In Chapter 2, we summarised certain algebraic properties of the Kronecker product. These properties are restricted to those which we may naturally deem of most importance or interest. We are interested in defining an operation $\oslash : (\mathbb{F}^{p,m} \otimes \mathbb{F}^{q,n}) \times \mathbb{F}^{q,n}_{nz} \to \mathbb{F}^{p,m}$ which satisfies:

$$(A \otimes B) \oslash B = A.$$

Due to the reduction in dimension of the domain, such an operation is necessarily a quotient. The choice of algebraic properties which may be required (or desirable) impacts the way in which such a quotient could be defined and characterised.

To illustrate this point, let V and W be vector spaces. Owing to its definition in Chapter 1, the tensor product $V \otimes W$ is also a vector space. Now, suppose $L(V, X)$ and $L(W, Y)$ are the vector spaces of linear operators on $V \to X$ and $W \to Y$ (vector space homomorphisms). So, for example, if $V = X = \mathbb{R}^n$ and $W = Y = \mathbb{R}^m$, then $L(V, X) \cong \mathbb{R}^{n,n}$ and $L(W, Y) \cong \mathbb{R}^{m,m}$. (In this example $L(V, X)$ and $L(W, Y)$ are endomorphisms – though they need not have this restriction.)

Therefore, we may similarly form the tensor product $L(V, X) \otimes L(W, Y)$. Investigating the quotient of this space allows for two main points of view.

1. **As a vector space**

   We may define a quotient that reflects the distributive and scalar factorisation properties KP Property 3 and 4 in Section 2.2, such as KQ Property 3 and 4:

$$\begin{aligned}(M_1 + M_2) \oslash B &= (M_1 \oslash B) + (M_2 \oslash B),\\ (\alpha M) \oslash B &= \alpha(M \oslash B).\end{aligned}$$

   These are described as "linear Kronecker quotients" in [3] and will be discussed in Section 3.3 below.

2. **As a monoid**

   Alternatively, we may consider $L(V, V)$ and $L(W, W)$ as operations and hence as monoids (under composition). So, we may define a multiplicative quotient $\oslash$ such that KQ Property 8 may be satisfied:

$$(M_1 \oslash A)(M_2 \oslash B) = (M_1 M_2) \oslash (AB).$$

   Monoids do not lend themselves to quotients with "neat" algebraic properties. Little attention is afforded to characterising this type of multiplicative quotient in [3], and will be considered briefly in Section 3.6.

3.3 The Kronecker Quotient as a Linear Operator

As introduced above in Section 3.2, Kronecker quotients satisfying KQ Property 3 and KQ Property 4, may be expressed as vector spaces (of linear operators). In [3], Hardy provides such a description for the left Kronecker quotient in terms of linear operators. In this section, a similar operation - albeit with focus on a right quotient will be defined and characterised. Left Kronecker quotient analogues of the definitions below follow naturally.

Definition 12. A **linear (right) Kronecker quotient** is a linear operation defined, for a fixed but arbitrary $B \in \mathbb{F}_{nz}^{s,t}$ and $m, n \in \mathbb{N}$, by a linear map:

$$q(B, m, n) : M \mapsto M \oslash B,$$

for all $M \in \mathbb{F}^{ms,nt}$.

For given matrices $M \in \mathbb{F}^{ms,nt}$ and $B \in \mathbb{F}^{s,t}$, it is clear that a matrix $A \in \mathbb{F}^{m,n}$ does not necessarily exist such that $M = A \otimes B$. Continuing in the spirit of an algebraic view, we may express any matrix $M \in \mathbb{F}^{ms,nt}$ in terms of the (left) coset of $\ker(q(B, m, n))$, as expressed in [3]:

$$M = A \otimes B + R_M, \tag{3.1}$$

for some $R_M \in \ker(q(B, m, n))$. Furthermore, in addition to the definition for a quotient in Chapter 1, a quotient of the vector space $\mathbb{F}^{ms,nt}$ may further be described as the set of cosets (quotient space):

$$\{A \otimes B + \ker(q(B, m, n)) : A \otimes B \in (\mathbb{F}^{m,n} \otimes \operatorname{span}\{B\})\}.$$

So, dealing with the quotient of $\mathbb{F}^{ms,nt}$ involves dealing with a set of cosets as described above. Let $K := \ker(q(B, m, n))$ and let $f : \mathbb{F}^{ms,nt} \to \mathbb{F}^{ms,nt}/K$ denote the canonical vector space homomorphism given by:

$$f(M) := A \otimes B + K,$$

for $M \in \mathbb{F}^{ms,nt}$ as defined in (3.1). Consider the following diagram, which illustrates the role of the operations $q(B, m, n)$, for a $B \in \mathbb{F}_{nz}^{s,t}$, $m, n \in \mathbb{N}$:

$$\begin{array}{ccc} M = A \otimes B + R_M & \xmapsto{\;f\;} & A \otimes B + K \\ & \searrow^{q(B,m,n)} & \downarrow \\ & & A \otimes B \\ & & \downarrow \\ & & A = M \oslash B \in \mathbb{F}^{m,n} \end{array} \tag{3.2}$$

In the diagram, $A \otimes B \in (\mathbb{F}^{m,n} \otimes \operatorname{span}\{B\})$ depicts the "representative matrix" for the relevant coset of matrices $M \in \mathbb{F}^{ms,nt}$ such that $M = A \otimes B + R_M$, for some $R_M \in K$.

More generally, the description of the kernel of a vector space homomorphism lends itself to a description of the quotient in question. In [3], conditions for such a linear Kronecker quotient to satisfy KQ Property 1, 2 and 5 (in addition to KQ Property 3 and 4) has been derived in virtue of expression (3.1).

Theorem 6. *Let $\oslash$ be a right Kronecker quotient satisfying* KQ Property 3 *and* KQ Property 4, *with the corresponding linear operator $q(B, m, n)$ as defined above. The following holds true, for all $B \in \mathbb{F}^{s,t}_{nz}$, $A \in \mathbb{F}^{m,n}$, $C \in \mathbb{F}^{m,n}_{nz}$, $M \in \mathbb{F}^{ms,nt}$, $N \in \mathbb{F}^{msp,ntr}$ and $\alpha \in \mathbb{F}\backslash\{0\}$, where $M = A \otimes B + R_M$ and $R_M \in \ker(q(B, m, n))$:* *

1. $(M \oslash B)^T = M^T \oslash B^T$ *if and only if*
$$R_M \in \ker(q(B, m, n)) \implies R_M^T \in \ker(q(B^T, n, m)).$$
2. $M \oslash (\alpha B) = \frac{1}{\alpha}(M \oslash B)$ *if and only if*
$$q(\alpha B, m, n)(M) = \frac{1}{\alpha} q(B, m, n)(M) \text{ if and only if}$$
$$R_M \in \ker(q(B, m, n)) \implies R_M \in \ker(q(\alpha B, m, n)).$$
3. $(N \oslash B) \oslash C = N \oslash (C \otimes B)$ *if and only if*
$$q(C \otimes B, p, r)(N) = q(C, p, r) \circ q(B, mp, nr)(N).$$

Proof.

1. Firstly, suppose that $(M \oslash B)^T = M^T \oslash B^T$. Since $M = A \otimes B + R_M$, and by the linearity of $q(B, m, n)$ (for all $B \in \mathbb{F}^{s,t}$, $m, n \in \mathbb{N}$) we have the following:
$$\begin{aligned}(M \oslash B)^T &= [q(B, m, n)(A \otimes B + R_M)]^T \\ &= [q(B, m, n)(A \otimes B)]^T \\ &= A^T.\end{aligned}$$
Furthermore:
$$\begin{aligned}M^T \oslash B^T &= q(B^T, n, m)(A^T \otimes B^T + R_M^T) \\ &= A^T + q(B^T, n, m)(R_M^T).\end{aligned}$$
Therefore, we have that $R_M^T \in \ker(q(B^T, n, m))$. Now, conversely, suppose that $R_M \in \ker(q(B, m, n))$ implies $R_M^T \in \ker(q(B^T, n, m))$ and consider the following:
$$\begin{aligned}M^T \oslash B^T &= q(B^T, n, m)(A^T \otimes B^T + R_M^T) \\ &= A^T\end{aligned}$$

*The properties implied by the statements above are not only implied, but also if and only if statements.

$$= [q(B,m,n)(A \otimes B + R_M)]^T$$
$$= (M \oslash B)^T.$$

2. Consider the following:

$$\begin{aligned} M \oslash (\alpha B) &= q(\alpha B,m,n)(A \otimes B + R_M) \\ &= q(\alpha B,m,n)\left(\left(\frac{1}{\alpha}A\right) \otimes (\alpha B)\right) + q(\alpha B,m,n)(R_M) \\ &= \frac{1}{\alpha}A + q(\alpha B,m,n)(R_M). \end{aligned}$$

Given that $\frac{1}{\alpha}(M \oslash B) = \frac{1}{\alpha}A$, we may conclude that $M \oslash (\alpha B) = \frac{1}{\alpha}(M \oslash B)$ if and only if

$$q(\alpha B,m,n)(M) = \frac{1}{\alpha}q(B,m,n)(M),$$

or, equivalently, if and only if

$$R_M \in \ker(q(\alpha B,m,n)).$$

3. The proof for this property follows directly using Definition 12. □

For a $B \in \mathbb{F}^{s,t}_{nz}$ we have the following set of linear operators:

$$\{q(B,m,n) : \mathbb{F}^{ms,nt} \to \mathbb{F}^{m,n} : m,n \in \mathbb{N}\}, \tag{3.3}$$

where each of these operators are dependent on the order of the matrix they operate on, without any clear relation between them. Little to no universal statements can be made applicable to the entire scheme of operators – apart from separate statements on individual operators.

It is desirable to impose (in addition to linearity properties) a property of uniformity and consider a scheme of linear operators $\bar{q}(B,m,n)$, such that, for $A \in \mathbb{F}^{m,n}$ and $D \in \mathbb{F}^{sp,tr}$:

$$\bar{q}(B,mp,nr)(A \otimes D) = A \otimes \bar{q}(B,p,r)(D). \tag{3.4}$$

Some examples of uniform quotients given by $\bar{q}(B,m,n)$ are given in Section 3.4. It is clear that such linear operators are independent of the order of D – as well as completely described by equation (3.4). Based on (3.3), such a scheme may simply be defined as:

$$\{\bar{q}(B,m,n) : \mathbb{F}^{ms,nt} \to \mathbb{F}^{m,n} : m,n \in \mathbb{N},$$
$$\bar{q}(B,m,n)(A \otimes C) := \bar{q}(B)(C)A \quad \forall C \in \mathbb{F}^{s,t}, A \in \mathbb{F}^{m,n}\}, \tag{3.5}$$

where $\bar{q}(B) := q(B,1,1)$ for $B \in \mathbb{F}^{s,t}_{nz}$.

The property specification for the set above relies on the fact that any $M \in \mathbb{F}^{ms,nt}$, $m,n,s,t \in \mathbb{N}$ can be written as the sum of simple tensors. Other schemes may also be defined – for example, a scheme that incorporates qualities from operators across all $m,n \in \mathbb{N}$.

Definition 13. Let $B \in \mathbb{F}^{s,t}_{nz}$ and $m,n \in \mathbb{N}$ be fixed but arbitrary. A **uniform linear (right) Kronecker quotient** is a map $\bar{q}(B,m,n) : \mathbb{F}^{ms,nt} \to \mathbb{F}^{m,n}$ defined by, for all $A \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{s,t}$:

1. $\bar{q}(B,m,n)(A \otimes C) = A\,\bar{q}(B)(C)$,
2. $\bar{q}(B)(B) = 1$,

and linear extension, where $\bar{q}(B) := \bar{q}(B,1,1)$.

The linearity and uniformity of these operators enables independence from their left linear Kronecker quotient analogues, which may be defined, for $A \in \mathbb{F}^{m,n}_{nz}$ and $s,t \in \mathbb{N}$, as:

$$\bar{q}_L(A,s,t) : \mathbb{F}^{ms,nt} \to \mathbb{F}^{s,t},$$

such that $\bar{q}_L(A,s,t)(C \otimes B) = \bar{q}_L(A)(C)B$ for all $C \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{s,t}$.

For all $M = C \otimes R \otimes D$ where $C \in \mathbb{F}^{m,n}, R \in \mathbb{F}^{p,r}$, and $D \in \mathbb{F}^{s,t}$, this (lack of) interaction between the right and left operators may, therefore, be expressed as the following composition:

$$\begin{aligned}\bar{q}(B,p,r) \circ \bar{q}_L(A,ps,rt)(M) &= \bar{q}(B,p,r)\ [\bar{q}_L(A)(C)(R \otimes D)] \\ &= \bar{q}_L(A)(C)\ \bar{q}(B)(D)\ R \\ &= \bar{q}_L(A,p,r) \circ \bar{q}(B,mp,nr)(M).\end{aligned}$$

In [3], by defining a partial Frobenius product:

$$\odot : \mathbb{F}^{ms,nt} \times \mathbb{F}^{s,t} \to \mathbb{F}^{m,n},\ m,n,s,t \in \mathbb{N}$$

as an extension of the Frobenius inner product, Hardy characterizes uniform Kronecker quotients, a class of the linear Kronecker quotients described above, through a series of derivations. Definitions throughout [3] are based on a left Kronecker quotient operation. This dissertation will focus on the right Kronecker quotient – therefore, definitions and equations from [3] have been adapted to reflect the right Kronecker quotients and operations in this paper.

Definition 14. The **partial Frobenius product** of $M \in \mathbb{F}^{ms,nt}$ and $B \in \mathbb{F}^{s,t}$, the $m \times n$ matrix $M \odot B$, is the bilinear, commutative operation defined entrywise by:

$$[M \odot B]_{j,k} = \sum_{u=1}^{s} \sum_{v=1}^{t} [B]_{u,v} [M]_{(j-1)s+u,(k-1)t+v},$$

for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots n$. Furthermore, we define $B \odot M = M \odot B$ (equality is immediate if $m = n = 1$).

Lemma 2. *The following identity holds for all* $C \in \mathbb{F}^{s,s},\ s \in \mathbb{N}$*:*

$$C \odot I_s = \mathrm{tr}(C).$$

Proof. Given Definition 14 with $m = n = 1$ and $s = t$, we have that $C \odot I_s \in \mathbb{F}^{1,1} \cong \mathbb{F}$. Applying Definition 14 yields:

$$\begin{aligned} C \odot I_s &= \sum_{u,v=1}^{s} [I]_{u,v} [C]_{u,v} \\ &= \sum_{u=1}^{s} [C]_{u,u} \\ &= \mathrm{tr}(C). \end{aligned}$$

□

Given the scheme defined in (3.5), let $E_{i,j}^{m,n}$ denote the standard basis vector of $\mathbb{F}^{m,n}$ defined by $[E_{i,j}^{m,n}]_{k,l} = \delta_{i,k}\delta_{j,l}$. To further the discussions below, we will introduce a different rendering of linear Kronecker quotients in the form of matrices denoted $R_q(B)$ (originally termed the realisation of $\oslash$ in [3]).

Definition 15. Let $B \in \mathbb{F}_{nz}^{s,t},\ s, t \in \mathbb{N}$ be fixed but arbitrary. The **quotient realisation matrix** for B is defined entrywise by:

$$[R_q(B)]_{u,v} := \bar{q}(B)(E_{u,v}^{s,t}).$$

Lemma 3. *For $B \in \mathbb{F}^{s,t}_{nz}$ and $M \in \mathbb{F}^{ms,nt}$, and a linear Kronecker quotient $\bar{q}(B,m,n)$ conforming to the uniformity property given by* (3.4)*, we have the following identity:*

$$\bar{q}(B,m,n)(M) = M \odot R_q(B).$$

Proof. Consider the following:

$$\begin{aligned}
\bar{q}(B,m,n)(M) &= \bar{q}(B,m,n)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{u=1}^{s}\sum_{v=1}^{t}[M]_{(i-1)s+u,(j-1)t+v}\,(E^{m,n}_{i,j}\otimes E^{s,t}_{u,v})\right)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{u=1}^{s}\sum_{v=1}^{t}[M]_{(i-1)s+u,(j-1)t+v}\;\bar{q}(B,m,n)(E^{m,n}_{i,j}\otimes E^{s,t}_{u,v})\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{u=1}^{s}\sum_{v=1}^{t}[M]_{(i-1)s+u,(j-1)t+v}\;\bar{q}(B)(E^{s,t}_{u,v})\;E^{m,n}_{i,j}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{u=1}^{s}\sum_{v=1}^{t}[M]_{(i-1)s+u,(j-1)t+v}\;[R_q(B)]_{u,v}\;E^{m,n}_{i,j}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{u=1}^{s}\sum_{v=1}^{t}[R_q(B)]_{u,v}\;[M]_{(i-1)s+u,(j-1)t+v}\;E^{m,n}_{i,j}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}[M\odot R_q(B)]_{i,j}\;E^{m,n}_{i,j}\\
&= M\odot R_q(B).
\end{aligned}$$

□

As discovered in [3], by characterising the matrices $\{R_q(B),\ B \in \mathbb{F}^{s,t}_{nz}\}$, conditions for satisfying KQ Property 1, 2 and 5 and reveal some appealing properties.

Lemma 4. *For all $N \in \mathbb{F}^{msp,ntr}, A \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{s,t}$:*

$$N \odot (A \otimes B) = (N \odot B) \odot A.$$

Proof. Let $u = (u_1-1)s+u_2$ and $v = (v_1-1)t+v_2$, where $u_1 \in \{1,2,\ldots,m\}, u_2 \in \{1,2,\ldots,s\}$ and $v_1 \in \{1,2,\ldots,n\}, v_2 \in \{1,2,\ldots,t\}$.

For all $i \in \{1,2,\ldots,p\}, j \in \{1,2,\ldots,r\}$, consider the following:

$$[N \odot (A \otimes B)]_{i,j} = \sum_{u=1}^{ms}\sum_{v=1}^{nt}[A\otimes B]_{u,v}[N]_{(i-1)ms+u,(j-1)nt+v}$$

$$= \sum_{u_1=1}^{m} \sum_{u_2=1}^{s} \sum_{v_1=1}^{n} \sum_{v_2=1}^{t} [A \otimes B]_{(u_1-1)s+u_2,(v_1-1)t+v_2}$$
$$\times [N]_{(i-1)ms+(u_1-1)s+u_2,(j-1)nt+(v_1-1)t+v_2}$$
$$= \sum_{u_1=1}^{m} \sum_{u_2=1}^{s} \sum_{v_1=1}^{n} \sum_{v_2=1}^{t} [A]_{u_1,v_1} [B]_{u_2,v_2}$$
$$\times [N]_{[(i-1)m+u_1-1]s+u_2,[(j-1)n+v_1-1]t+v_2}$$
$$= \sum_{u_1=1}^{m} \sum_{v_1=1}^{n} [A]_{u_1,v_1} [N \odot B]_{(i-1)m+u_1,(j-1)n+v_1}$$
$$= [(N \odot B) \odot A]_{i,j}. \qquad \square$$

Theorem 7. *Let* $\oslash$ *be a uniform Kronecker quotient (satisfying* KQ Property 3*,* KQ Property 4 *and* (3.4)*). For* $B \in \mathbb{F}_{nz}^{s,t}$*, let* $R_q(B)$ *be the matrix as defined in* Definition 15*. The following holds true, for all* $A \in \mathbb{F}^{m,n}, C \in \mathbb{F}_{nz}^{m,n}, M \in \mathbb{F}^{ms,nt}, N \in \mathbb{F}^{msp,ntr}$*, and* $\alpha \in \mathbb{F}\backslash\{0\}$*:*

1. $(M \oslash B)^T = M^T \oslash B^T$ *if and only if* $R_q(B^T) = [R_q(B)]^T$*.*
2. $M \oslash (\alpha B) = \frac{1}{\alpha}(M \oslash B)$ *if and only if* $R_q(\alpha B) = \frac{1}{\alpha} R_q(B)$*.*
3. $(N \oslash B) \oslash C = N \oslash (C \otimes B)$ *if and only if* $R_q(C \otimes B) = R_q(C) \otimes R_q(B)$*.*

Proof.

1. From Lemma 3, we have:

$$[M \oslash B]^T = [M \odot R_q(B)]^T$$
$$= \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{u=1}^{s} \sum_{v=1}^{t} [R_q(B)]_{u,v} \, [M]_{(i-1)s+u,(j-1)t+v} \, E_{i,j}^{m,n} \right]^T$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{u=1}^{s} \sum_{v=1}^{t} [(R_q(B))^T]_{v,u} \, [M^T]_{(j-1)t+v,(i-1)s+u} \, E_{j,i}^{n,m}$$
$$= M^T \odot [R_q(B)]^T,$$

and:

$$M^T \oslash B^T = M^T \odot R_q(B^T).$$

So, if we suppose that $[M \oslash B]^T = M^T \oslash B^T$, since M is arbitrary we must have that $R_q(B^T) = [R_q(B)]^T$, and vice versa.

2. From the bilinearity of $\odot$, we have:

$$\begin{aligned}\frac{1}{\alpha}(M \oslash B) &= \frac{1}{\alpha}(M \odot R_q(B)) \\ &= M \odot \left(\frac{1}{\alpha} R_q(B)\right).\end{aligned}$$

So, if we suppose that $\frac{1}{\alpha}(M \oslash B) = M \oslash (\alpha B)$, we must have that:

$$M \odot \frac{1}{\alpha}(R_q(B)) = M \odot R_q(\alpha B),$$

if and only if

$$\frac{1}{\alpha}(R_q(B)) = R_q(\alpha B).$$

3. Given Lemma 4, the proof for this identity follows straightforwardly:

$$\begin{aligned}(N \oslash B) \oslash C &= (N \odot R_q(B)) \odot R_q(C) \\ &= N \odot (R_q(C) \otimes R_q(B)).\end{aligned}$$ □

Another result of interest provided in [3] involves basis expansion, and is a generalisation of [9, Lemma 7.5].

Theorem 8. *Let* $\{B_1, B_2, \ldots, B_{st}\}$ *be a basis for* $\mathbb{F}^{s,t}$*. For all* $M \in \mathbb{F}^{ms,nt}$*,* $m, n \in \mathbb{N}$*, the following holds:*

$$M = \sum_{i=1}^{st} \bar{q}(B_i, m, n)(M) \otimes B_i = \sum_{i=1}^{st} (M \oslash B_i) \otimes B_i$$

if and only if:

$$\bar{q}(B_k)(B_p) = \delta_{k,p}.$$

The proof for the theorem above is provided in Appendix A. In [3], Hardy has considered the case of $\bar{q}(\cdot)$ and $\{B_1, B_2, \ldots, B_{st}\}$ appropriate to this theory. It could prove useful to consider whether, given a quotient operation $\bar{q}(\cdot)$, such a basis exists and if so, to describe the necessary conditions. An initial investigation is provided in Appendix A.2. On the other hand, given that $(A \otimes B_i) \oslash B_i = A$, we have that $\sum_{i=1}^{st} (A \otimes B_i) \oslash B_i = st\, A$, and hence:

$$A = \sum_{i=1}^{st} (A \otimes B_i) \oslash \left(\frac{1}{st}\, B_i\right).$$

Theorem 9. *Let $\bar{q}(X)$ be a uniform linear Kronecker quotient given $X \in \mathbb{F}_{nz}^{s,t}$. If there exists a basis $\{B_1, B_2, \ldots, B_{st}\}$ for $\mathbb{F}^{s,t}$ such that $\bar{q}(B_p)(B_k) = \delta_{p,k},\ p,k \in \{1,2,\ldots,st\}$ then the matrix:*

$$\Omega := \begin{bmatrix} \bar{q}(B_1)(C_1) & \bar{q}(B_2)(C_1) & \cdots & \bar{q}(B_{st-1})(C_1) & \bar{q}(B_{st})(C_1) \\ \bar{q}(B_1)(C_2) & \bar{q}(B_2)(C_2) & \cdots & \bar{q}(B_{st-1})(C_2) & \bar{q}(B_{st})(C_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{q}(B_1)(C_{st}) & \bar{q}(B_2)(C_{st}) & \cdots & \bar{q}(B_{st-1})(C_{st}) & \bar{q}(B_{st})(C_{st}) \end{bmatrix},$$

is invertible for every other basis $\{C_1, C_2, \ldots, C_{st}\}$.

The proof and derivation of the matrix Ω of Theorem 9 above is provided in Appendix A.

Uniform linear Kronecker quotients lay down the basis for deriving a generalisation of Leopardi's approach to describing a Kronecker quotient in [9].

3.4 Examples of Uniform Linear Kronecker Quotients

In Section 3.3, a definition for the uniform linear Kronecker quotient is provided, by requiring:

$$\bar{q}(B)(B) = 1. \tag{3.6}$$

So, given the scalar-valued function $\bar{q}(B) : \mathbb{F}^{s,t} \to \mathbb{F}$ and equation (3.6), along with the axiom of uniformity in Definition 13, we are able to generate $\bar{q}(B,m,n),\ m,n \in \mathbb{N}$ such that $\bar{q}(B,m,n)$ is well-defined and describes a quotient (for arbitrary $B \in \mathbb{F}_{nz}^{s,t}$).

Lemma 5. *Let $\bar{q}(B,m,n)$ be a uniform linear Kronecker quotient with $B \in \mathbb{F}_{nz}^{s,t}$ and $m,n \in \mathbb{N}$. The following properties hold for all $m > 1,\ n > 1,\ M,N \in \mathbb{F}^{ms,nt}$ and $\alpha \in \mathbb{F}$:*

1. $\bar{q}(B,m,n)(M+N) = \bar{q}(B,m,n)(M) + \bar{q}(B,m,n)(N)$,
2. $\bar{q}(B,m,n)(\alpha M) = \alpha\, \bar{q}(B,m,n)(M)$,
3. $\bar{q}(B^T,n,m)(M^T) = [\bar{q}(B,m,n)(M)]^T$,
4. $\bar{q}(\alpha B,m,n)(M) = \frac{1}{\alpha}\, \bar{q}(B,m,n)(M)$,

if and only if these properties hold for $\bar{q}(B)$ (thus for $\bar{q}(B,m,n)$, where $m =$

$n = 1$).

Proof. Since we have that $M \in \mathbb{F}^{ms,nt} \cong \mathbb{F}^{m,n} \otimes \mathbb{F}^{s,t}$ we may express this general tensor as:

$$M = \sum_{i=1}^{mn} A_i \otimes C_i, \tag{3.7}$$

where $C_i \in \mathbb{F}^{s,t},\ i \in \{1, 2, \ldots, mn\}$. The vectors $A_i \in \mathbb{F}^{m,n}_{nz},\ i \in \{1, 2, \ldots, mn\}$ are linearly independent. So, given the linearity and uniformity of $\bar{q}(B, m, n)$ we have that:

$$\bar{q}(B, m, n)(M) = \sum_{i=1}^{mn} A_i\, \bar{q}(B)(C_i). \tag{3.8}$$

1. We note that, by definition, $\bar{q}(B, m, n)$ is linear, so for 1 and 2 we need only prove that $\bar{q}(B)$ is indeed linear. Let the general tensors $M, N \in \mathbb{F}^{ms,nt}$ be expressed in similar fashion to identity (3.8). Given the bilinearity of the tensor product, we have:

$$\begin{aligned}
\bar{q}(B, m, n)(M + N) &= \bar{q}(B, m, n) \left(\sum_{i=1}^{mn} (A_i \otimes C_i) + \sum_{i=1}^{mn} (A_i \otimes D_i) \right) \\
&= \bar{q}(B, m, n) \left(\sum_{i=1}^{mn} A_i \otimes (C_i + D_i) \right) \\
&= \sum_{i=1}^{mn} A_i\, \bar{q}(B)(C_i + D_i),
\end{aligned}$$

   and:

$$\begin{aligned}
\bar{q}(B, m, n)(M + N) &= \bar{q}(B, m, n) \left(\sum_{i=1}^{mn} (A_i \otimes C_i) + \sum_{i=1}^{mn} (A_i \otimes D_i) \right) \\
&= \sum_{i=1}^{mn} \bar{q}(B, m, n)\ (A_i \otimes C_i) + \sum_{i=1}^{mn} \bar{q}(B, m, n)\ (A_i \otimes D_i) \\
&= \sum_{i=1}^{mn} A_i\, \bar{q}(B, m, n)(C_i) + \sum_{i=1}^{mn} A_i\, \bar{q}(B, m, n)(D_i) \\
&= \sum_{i=1}^{mn} A_i\ \left(\bar{q}(B, m, n)(C_i) + \bar{q}(B, m, n)(D_i) \right).
\end{aligned}$$

   It follows straightforwardly that $\bar{q}(B)$ is distributive.

2. The proof of this scalar multiplication property follows similarly from identity (3.8) and reasoning similar to the proof of 1.

Given that $\bar{q}(B)$ satisfies 1 and 2 by construction, we implicitly use the linearity of $\bar{q}(B, m, n)$ in the proof of 3 and 4.

3. Given KP Property 1, we have that:

$$M^T = \sum_{i=1}^{mn} [A_i \otimes C_i]^T = \sum_{i=1}^{mn} A_i^T \otimes C_i^T.$$

Since $\bar{q}(B)$ maps to a scalar value, we may assert the following:

$$[\bar{q}(B, m, n)(M)]^T = \bar{q}(B^T, n, m)(M^T),$$

if and only if:

$$\sum_{i=1}^{mn} A_i^T \, \bar{q}(B)(C_i) = \sum_{i=1}^{mn} A_i^T \, \bar{q}(B^T)(C_i^T),$$

if and only if:

$$\bar{q}(B)(C_i) = \bar{q}(B^T)(C_i^T),$$

on account of the linear independence of the vectors A_i^T , $i \in \{1, 2, \ldots, mn\}$. The statement follows since C_i is arbitrary.

4. Owing to identity (3.8) it follows that:

$$\frac{1}{\alpha} \, \bar{q}(B, m, n)(M) = \bar{q}(\alpha B, m, n)(M),$$

if and only if:

$$\sum_{i=1}^{mn} A_i \left[\frac{1}{\alpha} \bar{q}(B)(C_i) \right] = \sum_{i=1}^{mn} A_i \, \bar{q}(\alpha B)(C_i),$$

if and only if:

$$\frac{1}{\alpha} \, \bar{q}(B)(C_i) = \bar{q}(\alpha B)(C_i).$$

The statement follows since C_i is arbitrary. □

Define:

$$\mathrm{nz}(B) := \{(i, j) \in \{1, 2, \ldots, s\} \times \{1, 2, \ldots, t\} : [B]_{i,j} \neq 0\},$$

where $B \in \mathbb{F}_{nz}^{s,t}$. It is straightforward to show that such an operator $\bar{q}(\cdot)$ exists.

For example, the operator $\bar{q}_{(i,j)}(B),\ (i,j) \in \mathrm{nz}(B)$ defined, for all $C \in \mathbb{F}^{s,t}$ as:

$$\bar{q}_{(i,j)}(B)(C) := ([B]_{i,j})^{-1}\,[C]_{i,j}, \tag{3.9}$$

satisfies linearity and (3.6). However, this quotient operation – with its reliance on a specific (i,j), in particular – might impose restrictions on desirable properties set out in Section 3.1. For example:

$$\bar{q}_{(i,j)}(B^T)(C^T) = [\bar{q}_{(i,j)}(B)(C)]^T \text{ if and only if } [C]_{i,j} = [C]_{j,i},$$

for all $C \in \mathbb{F}^{s,t}$ and $(i,j) \in \mathrm{nz}(B)$. In this case, an averaging method may be more suitable. A weighted average uniform Kronecker quotient is developed and introduced in [3] by defining the transformation $W_{s,t} : \mathbb{F}^{s,t}_{nz} \to \mathbb{F}^{s,t}_{nz}$ for $s,t \in \mathbb{N}$ such that, for all $B \in \mathbb{F}^{s,t}_{nz}$:

$$\sum_{(i,j)\in\mathrm{nz}(B)} [W_{s,t}(B)]_{i,j} = 1.$$

So, we define, for all $C \in \mathbb{F}^{s,t}$:

$$\bar{q}_\mu(B)(C) := \sum_{(u,v)\in\mathrm{nz}(B)} [W_{s,t}(B)]_{u,v}([B]_{u,v})^{-1}[C]_{u,v},$$

and extended for all $M \in \mathbb{F}^{ms,nt}$ and $m,n \in \mathbb{N}$:

$$\bar{q}_\mu(B,m,n)(M) := \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{(u,v)\in\mathrm{nz}(B)} [W_{s,t}(B)]_{u,v}([B]_{u,v})^{-1}[\underline{M}_{\,i,j}]_{u,v}\right] E^{m,n}_{i,j},$$

where $\underline{R}_{\,k,l} \in \mathbb{F}^{s,t}$ denotes the $(k,l)^{\text{th}}$ block of a $m \times n$ block matrix $R \in \mathbb{F}^{m,n} \otimes \mathbb{F}^{s,t}$.

Showing that $\bar{q}_\mu(B,m,n)$ is a quotient operation, by way of satisfying the axioms in Definition 13, is straightforward. For all $A \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{s,t}$ we have:

$$\begin{aligned}
[\bar{q}_\mu(B,m,n)(A\otimes C)]_{i,j} &= \sum_{(u,v)\in\mathrm{nz}(B)} [W_{s,t}(B)]_{u,v}([B]_{u,v})^{-1}[\underline{A\otimes C}_{\,i,j}]_{u,v} \\
&= \sum_{(u,v)\in\mathrm{nz}(B)} [W_{s,t}(B)]_{u,v}([B]_{u,v})^{-1}[A]_{i,j}[C]_{u,v} \\
&= [A]_{i,j}\,\bar{q}_\mu(B)(C),
\end{aligned}$$

therefore $\bar{q}_\mu(B,m,n)(A\otimes C) = A\,\bar{q}_\mu(B)(C)$. Furthermore, given the definitions for $\bar{q}_\mu(B)$ and $W_{s,t}$ above, it is clear that $\bar{q}_\mu(B)(B) = 1$.

It is worth noting that this generic weighted average quotient does not impose a restriction on the underlying field. However, certain quotients may be restricted to certain fields. For example, [3] defines the transformation:

$$W^L_{s,t} := \frac{1-\delta_{[B]_{u,v},0}}{|\mathrm{nz}(B)|},$$

where $\mathbb{F}$ is a field with characteristic zero, such that:

$$\bar{q}_{\mu,L}(B)(C) := \frac{1}{|\mathrm{nz}(B)|}\sum_{(u,v)\in\mathrm{nz}(B)}([B]_{u,v})^{-1}[C]_{u,v}.$$

The operation above, $\bar{q}_{\mu,L}$, is equivalent to the Kronecker quotient defined by Leopardi in [9].

The restriction to fields with characteristic zero in the example above limits the calculations to the most general of fields. Should the weighted average uniform Kronecker quotient exist for certain finite fields, fields with a small number of elements will have fewer weight transformation $W_{s,t}$ possibilities (as opposed to larger fields). An extreme example of this is to consider the Galois field of order two, GF(2).

In [16], as introduced in Chapter 1, van Loan and Pitsianis defines a Kronecker quotient by way of determining the "nearest Kronecker product". This approach determines a right quotient, given (where $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$) $M\in\mathbb{F}^{ms,nt}$ and $C\in\mathbb{F}^{s,t}$, by seeking the $A\in\mathbb{F}^{m,n}$ which minimises:

$$\|M - A\otimes C\|_F.$$

A weighted average Kronecker quotient counterpart of this optimisation-based approach may be constructed by the operation defined by, for $B\in\mathbb{F}^{s,t}_{nz}$ and all $C\in\mathbb{F}^{s,t}$:

$$\bar{q}_F(B)(C) := \frac{\|C\|_F}{\|B\|_F}.$$

An alternative representation of this operation is given by [3]:

$$\bar{q}_F(B,m,n)(M) = \frac{\|M\|^B_{m,n,F}}{\|B\|_F},$$

where $\|\cdot\|_{m,n,F}^{D} : \mathbb{F}^{ms,nt} \to \mathbb{F}^{m,n}$ is the linear (in A) map which obeys:

$$\|A \otimes D\|_{m,n,F}^{D} = A\|D\|_F,$$

for all $A \in \mathbb{F}^{m,n}$ and $D \in \mathbb{F}^{s,t}$. Alternatively, another example is given by defining:

$$W_{s,t}^{F} := \frac{[B]_{u,v}}{\|B\|_F^2},$$

where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ [3].

3.5 Restricted Properties for the Kronecker Quotient

Requiring a Kronecker quotient to satisfy certain properties, such as KQ Property 6 and 7, may result in constructing a trivial and restrictive quotient. Furthermore, the domain considered for these properties is necessarily restricted. In [3], Hardy examines the limitations of a trace property for the uniform linear Kronecker quotient – resulting in the following theorem. In this case, the concept of realisation matrices and the partial Frobenius product prove useful.

Theorem 10. *Let $B \in \mathbb{F}_{nz}^{s,s}$ be an arbitrary matrix with $\operatorname{tr}(B) \neq 0$. The following property (KQ Property 6) holds for all $M \in \mathbb{F}^{ms,ms}$:*

$$\operatorname{tr}(M) = \operatorname{tr}[\bar{q}(B,m,m)(M)] \operatorname{tr}(B),$$

if and only if:

$$R_q(B) = \frac{1}{\operatorname{tr}(B)} I_s.$$

Proof. Suppose that:

$$\operatorname{tr}(M) = \operatorname{tr}[\bar{q}(B,m,m)(M)] \operatorname{tr}(B),$$

and † $M = C \otimes D$ for some $C \in \mathbb{F}^{m,m}$, $\operatorname{tr}(C) \neq 0$ and $D \in \mathbb{F}^{s,s}$. The equality above holds if either $\operatorname{tr}(B) = \operatorname{tr}(M) = 0$ or, given KP Property 9, Lemma 4 and the commutativity of $\odot$:

$$\operatorname{tr}(C)\operatorname{tr}(D) = \operatorname{tr}[\bar{q}(B,m,m)(M)] \operatorname{tr}(B)$$

†In the case where M is a general tensor, this proof extends naturally, given that trace and the partial Frobenius product are linear and bilinear operations respectively.

$$= \operatorname{tr}[M \odot R_q(B)] \operatorname{tr}(B)$$
$$= \operatorname{tr}[(C \otimes D) \odot R_q(B)] \operatorname{tr}(B)$$
$$= \operatorname{tr}[C \odot (D \odot R_q(B))] \operatorname{tr}(B)$$
$$= (D \odot R_q(B)) \operatorname{tr}(C) \operatorname{tr}(B),$$

where the last equality follows since $D \odot R_q(B) \in \mathbb{F}$. This result implies that, for all $D \in \mathbb{F}^{s,s}$:

$$\operatorname{tr}(D) = (D \odot R_q(B)) \operatorname{tr}(B).$$

Since, by Lemma 2, we have that $D \odot I_s = \operatorname{tr}(D)$, we may conclude (for arbitrary $D \in \mathbb{F}^{s,s}$):

$$D \odot \frac{1}{\operatorname{tr}(B)} I_s = D \odot R_q(B)$$
$$\implies R_q(B) = \frac{1}{\operatorname{tr}(B)} I_s.$$

Now, suppose that $R_q(B) = \frac{1}{\operatorname{tr}(B)} I_s$. Let:

$$M = \sum_{i=1}^{st} C_i \otimes D_i.$$

Then we have:

$$\operatorname{tr}(M) = \sum_{i=1}^{st} \operatorname{tr}(C_i \otimes D_i)$$
$$= \sum_{i=1}^{st} \operatorname{tr}(C_i) \operatorname{tr}(D_i),$$

and:

$$\operatorname{tr}[\bar{q}(B, m, n)(M)] = \operatorname{tr}(M \odot R_q(B))$$
$$= \frac{1}{\operatorname{tr}(B)} \operatorname{tr}(M \odot I_s)$$
$$= \frac{1}{\operatorname{tr}(B)} \operatorname{tr}\left[\left(\sum_{i=1}^{st} C_i \otimes D_i\right) \odot I_s\right]$$
$$= \frac{1}{\operatorname{tr}(B)} \operatorname{tr}\left[\sum_{i=1}^{st} C_i \odot (D_i \odot I_s)\right]$$

$$= \frac{1}{\mathrm{tr}(B)} \mathrm{tr}\left[\sum_{i=1}^{st} \mathrm{tr}(D_i) C_i\right]$$
$$= \frac{1}{\mathrm{tr}(B)} \sum_{i=1}^{st} \mathrm{tr}(D_i)\,\mathrm{tr}(C_i).$$

Consequently, $\mathrm{tr}(M) = \mathrm{tr}[\bar{q}(B, m, n)(M)]\,\mathrm{tr}(B)$. □

Given Theorem 10 above, we may define a Kronecker quotient $q^{tr}(B, m, n)(M) : \mathbb{F}^{ms,ms} \to \mathbb{F}^{m,m}$, given $B \in \mathbb{F}^{s,s}_{nz}$:

$$\begin{aligned} q^{tr}(B, m, n)(M) &:= \frac{1}{\mathrm{tr}(B)}(M \odot I_s) \\ &= \frac{1}{\mathrm{tr}(B)} I_s \odot \left(\sum_{i,j=1}^{m} E_{i,j} \otimes \underline{M}_{i,j}\right) \\ &= \frac{1}{\mathrm{tr}(B)} \sum_{i,j=1}^{m} (I_s \odot \underline{M}_{i,j}) \odot E_{i,j} \\ &= \frac{1}{\mathrm{tr}(B)} \sum_{i,j=1}^{m} \mathrm{tr}(\underline{M}_{i,j}) \odot E_{i,j} \\ &= \frac{1}{\mathrm{tr}(B)} \sum_{i,j=1}^{m} \mathrm{tr}(\underline{M}_{\ i,j})\, E_{i,j}, \end{aligned}$$

given Lemma 4 and the commutativity and bilinearity of $\odot$. The quotient operation $q^{tr}(B, m, n)(M)$ satisfies KQ Property 3, 4 (linearity) and 6, together with the uniformity identity in (3.4). Demanding that KQ Property 6 holds, restricts $R_q(B)$ (and $\bar{q}(B, m, n)$) to a relatively simplistic form, unlikely to yield additional significant properties. For example, such a quotient cannot satisfy this property in conjunction with associativity (KQ Property 2) [3].

Corollary. *Let* $B \in \mathbb{F}^{s,s}_{nz}$, $\mathrm{tr}(B) \neq 0$ *and* $C \in \mathbb{F}^{m,m}_{nz}$, $\mathrm{tr}(C) \neq 0$ *be arbitrary matrices and* $\bar{q}(X, m, m) : \mathbb{F}^{mp,mp} \to \mathbb{F}^{m,m}$ *a uniform linear Kronecker quotient associated with a matrix* $X \in \mathbb{F}^{p,p}_{nz}$. *If KQ Property 2 holds for* $\bar{q}(X, m, m)$, *that is:*

$$\bar{q}(C, p, p) \circ \bar{q}(B, mp, mp)(N) = \bar{q}(C \otimes B, p, p)(N), \tag{3.10}$$

for all $N \in \mathbb{F}^{msp,msp}$, *then KQ Property 6:*

$$\mathrm{tr}(N) = \mathrm{tr}[\bar{q}(C \otimes B, p, p)(N)]\,\mathrm{tr}(C \otimes B),$$

does not hold for all $N \in \mathbb{F}^{msp,msp}$.

Proof. As a counter-example, consider the matrix $E_{1,1}^{4,4}$ expressed by two different vector combinations (where $e_{i,n}$ denotes the i^{th} element in the standard basis for $\mathbb{F}^n$):

$$E_{1,1}^{4,4} = E_{1,1}^{2,2} \otimes E_{1,1}^{2,2} = e_{1,4} \otimes e_{1,4}^T.$$

Suppose equation (3.10) holds. Theorem 7 yields:

$$\begin{aligned}
&E_{1,1}^{2,2} \otimes E_{1,1}^{2,2} = e_{1,4} \otimes e_{1,4}^T \\
&\implies R_q(E_{1,1}^{2,2} \otimes E_{1,1}^{2,2}) = R_q(e_{1,4} \otimes e_{1,4}^T) \\
&\implies R_q(E_{1,1}^{2,2}) \otimes R_q(E_{1,1}^{2,2}) = R_q(e_{1,4}) \otimes R_q(e_{1,4}^T).
\end{aligned}$$

Additionally, suppose the trace property as expressed above holds. Then:

$$R_q(E_{1,1}^{2,2} \otimes E_{1,1}^{2,2}) = \frac{1}{\left[\operatorname{tr}(E_{1,1}^{2,2})\right]^2} I_4.$$

However, since $R_q(A) \in \mathbb{F}^{m,n}$ for $A \in \mathbb{F}^{m,n}$, $R_q(e_{1,4}) \otimes R_q(e_{1,4}^T)$ has a rank of 1, whereas $\operatorname{rank}(R_q(E_{1,1}^{2,2} \otimes E_{1,1}^{2,2})) = \operatorname{rank}(I_4) = 4$. So we can conclude that:

$$R_q(e_{1,4}) \otimes R_q(e_{1,4}^T) \neq \frac{1}{\left[\operatorname{tr}(E_{1,1}^{2,2})\right]^2} I_4.$$

Therefore, the trace and associativity properties cannot hold simultaneously for all $N \in \mathbb{F}^{msp,msp}$, $m, s, p \in \mathbb{N}$. □

Inasmuch as the rank of a matrix does not interact with many algebraic properties and has no sum-respecting structure, KQ Property 7 is another restrictive property. We may, nevertheless, construct a generic Kronecker quotient operation, satisfying KQ Property 7.

Proposition. *Let* $B \in \mathbb{F}_{nz}^{s,t}$ *be fixed but arbitrary and* $P \in \mathbb{F}^{s,s}$ *and* $Q \in \mathbb{F}^{t,t}$ *any two invertible matrices. The quotient operation* $q_{r,P,Q}(B, m, n) : F_{B,m,n} \to \mathbb{F}^{m,n}$, *where* r *in* q_r *stands for rank, defined for all* $M \in F_{B,m,n} := \{X \in \mathbb{F}_{nz}^{ms,nt} : \operatorname{rank}(B) \mid \operatorname{rank}(X)\}$*:*

$$q_{r,P,Q}(B,m,n)(M) := \begin{cases} A & \textit{if } M = A \otimes B \\ P \begin{bmatrix} I_\rho & \mathbf{0}^{\rho,t-\rho} \\ \mathbf{0}^{s-\rho,\rho} & \mathbf{0}^{s-\rho,t-\rho} \end{bmatrix} Q & \textit{if } M \neq A \otimes B \ \ \forall A \in \mathbb{F}^{m,n}, \end{cases}$$

where $\rho = \dfrac{\operatorname{rank}(M)}{\operatorname{rank}(B)}$, *satisfies:*

$$q_{r,P,Q}(B, m, n)(A \otimes B) = A,$$

for all $A \in \mathbb{F}^{m,n}$, $A \otimes B \in F_{B,m,n}$, *and:*

$$\operatorname{rank}(M) = \operatorname{rank}(q_{r,P,Q}(B, m, n)(M)) \operatorname{rank}(B).$$

Naturally, $q_{r,P,Q}(B, m, n)$ as defined above do not satisfy linearity or uniformity properties as focused on in this chapter.

Similarly, matrix eigenvalues do not behave algebraically in a number of ways, and an alternative approach to defining a Kronecker quotient may be more apt. Given $M \in \mathbb{F}^{ms,ms}$ and $B \in \mathbb{F}^{s,s}_{nz}$ with eigenvalues β_i, $i \in \{1, 2, \ldots, p\}$ and μ_j, $j \in \{1, 2, \ldots, r\}$ respectively, such a quotient operation might, intuitively, be expected to yield a matrix with eigenvalues $\frac{\beta_i}{\mu_j}$. Consequently, a restriction on the number of eigenvalues for both M and B arises, as detailed in the proposition below. In particular, since $M \oslash B$ cannot have more than m eigenvalues, a bound exists on the number of unique ratios $\frac{\beta_i}{\mu_j}$.

Proposition. *For arbitrary diagonalisable matrices* $B \in \mathbb{F}^{s,s}_{nz}$ *and* $M \in \mathbb{F}^{ms,ms}$, *let* β_i, $i \in \{1, 2, \ldots, p\}$ *and* μ_j, $j \in \{1, 2, \ldots, r\}$ *be non-zero eigenvalues for* M *and* B *(respectively), where* $p, r \in \mathbb{N}$ *such that:*

$$\left| \left\{ \frac{\beta_i}{\mu_j} \,:\, i \in \{1, 2, \ldots, p\}, j \in \{1, 2, \ldots, r\} \right\} \right| \le m,$$

with associated eigenspaces $\varepsilon_M(\beta_i)$ *and* $\varepsilon_B(\mu_j)$. *If there exists a quotient* $\oslash_{ms} : \mathbb{F}^{ms} \to \mathbb{F}^m$ *such that the set:*

$$\varepsilon_{M,B} := \left\{ z_i \oslash_{ms} x_j : z_i \in \varepsilon_M(\beta_i), x_j \in \varepsilon_B(\mu_j),\ i \in \{1, 2, \ldots, p\}, j \in \{1, 2, \ldots, r\} \right\},$$

forms a basis for $\mathbb{F}^m$ *and* $\frac{\beta_i}{\mu_j} \neq \frac{\beta_k}{\mu_l}$ *implies that* $z_i \oslash_{ms} x_j$ *and* $z_k \oslash_{ms} x_l$ *is linearly independent. Then we may define the vector* $M \oslash_I B : \mathbb{F}^m \to \mathbb{F}^m$ *by its action on the basis* $\varepsilon_{M,B}$*:*

$$(M \oslash_I B)(z_i \oslash_{ms} x_j) := \frac{\beta_i}{\mu_j}(z_i \oslash_{ms} x_j).$$

The definition of $\oslash_I$ imposes a significant restriction on the dimension of $\varepsilon_{M,B}$.

A second constraint that follows is that $\frac{\beta_i}{\mu_i} = \frac{\beta_j}{\mu_j}$ for linearly dependent pairs $\{z_i \oslash_{ms} x_i, z_j \oslash_{ms} x_j\}$.

3.6 On Multiplicative Quotients

As discussed in Section 3.2, as opposed to the vector space perspective, the Kronecker quotient of a tensor product space may also be considered through a multiplicative lens by defining a quotient $\oslash$ that satisfies KQ Property 8:

$$(M \oslash B)(N \oslash D) = (MN) \oslash (BD),$$

for all square matrices $M, N \in \mathbb{F}^{mn,mn}$ and $B, D \in \mathbb{F}^{m,m}$, $m, n \in \mathbb{N}$. Such a requirement leads to considering the matrices above as elements of a monoid.

Definition 16. A **monoid** $(S, *)$ is a set S that is closed under a binary operation $* : S \times S \to S$ such that the following axioms are satisfied:

1. The binary operation is associative, i.e. :

$$(a * b) * c = a * (b * c) \quad \forall a, b, c \in S.$$

2. There exists an identity element $I \in S$ such that:

$$I * a = a * I = a \quad \forall a \in S.$$

Clearly, monoids afford much less structure than vector spaces. In the sense of describing a quotient of $L(V, V) \otimes L(W, W)$, the best attempt to a quotient might be in finding a monoid homomorphism. As a starting point, we will confine the investigation of a multiplicative Kronecker quotient to the following subgroup (and a submonoid of the monoid of $(mn) \times (mn)$ matrices):

$$\mathrm{GL}^{\otimes}_{m,n}(\mathbb{F}) := \{X \otimes Y : X \in \mathrm{GL}_m(\mathbb{F}), Y \in \mathrm{GL}_n(\mathbb{F})\}, \tag{3.11}$$

of the general linear group $\mathrm{GL}_{mn}(\mathbb{F})$, which consists of $(mn) \times (mn)$ invertible matrices over $\mathbb{F}$. As discussed in Section 3.3, defining a quotient function for every $m, n \in \mathbb{N}$ requires the investigation of an infinite set of quotients, similar to the set described in equation (3.3). Therefore, keeping to a "uniform" quotient, similar to the property in equation (3.4), in this section is vital for simplicity.

Definition 17. A **uniform multiplicative (right) Kronecker quotient** is an operation:

$$\oslash^{\times} : \mathrm{GL}^{\otimes}_{m,n}(\mathbb{F}) \times \mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}_m(\mathbb{F}),$$

defined such that:

1. $B \oslash^{\times} B = 1$,
2. $(M \oslash^{\times} B)(N \oslash^{\times} C) = (MN) \oslash^{\times} (BC)$,
3. $(\alpha M) \oslash^{\times} B = \alpha\,(M \oslash^{\times} B)$,
4. $(A \otimes C) \oslash^{\times} B = A\,(C \oslash^{\times} B)$,

holds for all $M, N \in \mathrm{GL}^{\otimes}_{m,n}(\mathbb{F})$, $B, C \in \mathrm{GL}_n(\mathbb{F})$, $A \in \mathrm{GL}_m(\mathbb{F})$ and $\alpha \in \mathbb{F}$.

Now, let $A_1, A_2 \in \mathrm{GL}_m(\mathbb{F})$ and $C_1, C_2, B_1, B_2 \in \mathrm{GL}_n(\mathbb{F})$, and consider the following expressions:

$$[(A_1 \otimes C_1) \oslash^{\times} B_1]\,[(A_2 \otimes C_2) \oslash^{\times} B_2] = A_1 A_2\,[(C_1 \oslash^{\times} B_1)]\,[(C_2 \oslash^{\times} B_2)],$$

and:

$$\begin{aligned}[(A_1 \otimes C_1) \oslash^{\times} B_1]\,[(A_2 \otimes C_2) \oslash^{\times} B_2] &= [(A_1 \otimes C_1)(A_2 \otimes C_2)] \oslash^{\times} (B_1 B_2) \\ &= [(A_1 A_2) \otimes (C_1 C_2)] \oslash^{\times} (B_1 B_2)] \\ &= A_1 A_2[(C_1 C_2) \oslash^{\times} (B_1 B_2)].\end{aligned}$$

Hence, uniformity demands the resulting multiplicative identity:

$$(C_1 \oslash^{\times} B_1)\,(C_2 \oslash^{\times} B_2) = (C_1 C_2) \oslash^{\times} (B_1 B_2).$$

For the remainder of this chapter, we consider uniform multiplicative quotients as described in Definition 17. Generally, an examination of multiplicative functions of matrices, necessitates examination of the determinant. In [1], Djoković demonstrates that the characterisation of any multiplicative map on matrices is that it factors through the determinant function. ‡

Theorem 11. *Let* $\det(\cdot)$ *denote the Dieudonné determinant and* K *be a division ring, not of characteristic* 2*, with* K^* *its multiplicative group and* $[K^*, K^*]$ *its commutator subgroup. Furthermore, let* $\phi : \mathrm{GL}_{mn}(K) \to \mathrm{GL}_m(K)$*,* $m, n \in$

‡Djokovic's result in Theorem 11 also holds for fields of characteristic 2. See the closing comments of [1] for details.

$\mathbb{N}$ *be a homomorphism satisfying:*

$$\phi(AB) = \phi(A)\phi(B),$$

for all $A, B \in \mathrm{GL}_{mn}(K)$*. Then a homomorphism* $p_\phi : K^*/[K^*, K^*] \to \mathrm{GL}_m(K)$ *exists such that:*

$$\phi(A) = p_\phi \circ \det(A),$$

for all $A \in \mathrm{GL}_{mn}(K)$*.*

The proof of this theorem may be found in [1]. Let $B \in \mathrm{GL}_n(\mathbb{F})$, $C \in \mathrm{GL}_m(\mathbb{F})$ and $M \in \mathrm{GL}_{mn}(\mathbb{F})$. Suppose that $M = C \otimes I_n$ and consider the following:

$$\begin{aligned} M \oslash^{\times} B &= (I_{mn} \oslash^{\times} B)(M \oslash^{\times} I_n) \\ &= (I_{mn} \oslash^{\times} B)C. \end{aligned}$$

Furthermore:

$$M \oslash^{\times} B = C(I_{mn} \oslash^{\times} B).$$

So, $(I_{mn} \oslash^{\times} B)$ is in the center of $\mathrm{GL}_m(\mathbb{F})$ and can therefore be expressed as a non-zero scalar multiple of I_m [2, pp. 150].

Proposition. *Let* $B \in \mathrm{GL}_n(\mathbb{F})$ *and* $\oslash^{\times}$ *be a uniform multiplicative Kronecker quotient. The following holds:*

$$(I_{mn} \oslash^{\times} B) = \frac{1}{\beta} I_m,$$

where $\beta \in \mathbb{F}$ *is some non-zero scalar.*

With this taken into consideration, $\oslash^{\times}$ satisfies the definition for a Kronecker quotient together with Property 8, provided that the quotient $M \oslash^{\times} I_n$ exists for all $M \in \mathrm{GL}^{\otimes}_{m,n}(\mathbb{F})$.

Given Theorem 11 and the proposition above, we have the following (in our case, $\mathbb{F}^*/[\mathbb{F}^*, \mathbb{F}^*] \equiv \mathbb{F}^*$):

$$\begin{aligned} M \oslash^{\times} B &= (M \oslash^{\times} I_n)(I_{mn} \oslash^{\times} B) \\ &= p_{I_n}[\det(M)] \left(\frac{1}{\beta} I_m \right), \end{aligned}$$

where det (in the above equation) denotes the Dieudonné determinant (the usual determinant since we are working over the field $\mathbb{F}$) and β some non-zero scalar. Here, $p_{I_n} : \mathrm{GL}_m(\mathbb{F})/\mathrm{SL}_m(\mathbb{F}) \to \mathrm{GL}_m(\mathbb{F})$ is a group homomorphism, where $\mathrm{SL}_m(\mathbb{F})$ is the special linear group consisting of all $m \times m$ matrices over $\mathbb{F}$ with a determinant equal to one. We note that $\mathrm{GL}_m(\mathbb{F})/\mathrm{SL}_m(\mathbb{F}) \cong \mathbb{F}^*$.

So, since we require $(A \otimes B) \oslash^{\times} B = A$ for all $A \in \mathrm{GL}_m(\mathbb{F})$ and $B \in \mathrm{GL}_n(\mathbb{F})$ we may deduce the following:

$$\begin{aligned}(A \otimes B) \oslash^{\times} B = \left(\frac{1}{\beta} I_m\right)[A(B \oslash^{\times} I_n)] = A \\ \implies B \oslash^{\times} I_n = \beta,\end{aligned}$$

and, for all $D \otimes C \in \mathrm{GL}^{\otimes}_{m,n}(\mathbb{F})$:

$$(D \otimes C) \oslash^{\times} B = \frac{\gamma}{\beta} D,$$

where the scalars $\beta, \gamma \in \mathbb{F}^*$ are defined by $\frac{1}{\beta} I_m = (I_{mn} \oslash^{\times} B)$ and $\frac{1}{\gamma} I_m = (I_{mn} \oslash^{\times} C)$.

However, consider the matrix $X \otimes I_n,\ X \in \mathrm{SL}_m(\mathbb{F}) \setminus \{I_m\}$. Since p_{I_n} is a homomorphism, we have:

$$\begin{aligned}(X \otimes I_n) \oslash^{\times} I_n &= p_{I_n}[\det(X \otimes I_n)] \\ &= p_{I_n}[(\det(X))^n (\det(I_n))^m] \\ &= p_{I_n}(1) \\ &= I_m \\ &\neq X,\end{aligned}$$

which contradicts Definition 10. Consequently, defining such a multiplicative quotient such that it is well-defined on the entire group $\mathrm{GL}^{\otimes}_{m,n}(\mathbb{F})$ is not viable and will require identifying an appropriate structure.

Chapter 4

On Quotients in Banach Spaces

The previous chapter focused on quotients of finite-dimensional vector spaces. Both finite and infinite-dimensional vector spaces are considered in this chapter. Infinite-dimensional vector spaces are usually studied in the setting of functional analysis and Banach spaces.

Definition 18. A vector space F (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) equipped with a norm $\|\cdot\| : F \to \mathbb{R}_{\geq 0}$, with respect to which it is complete (i.e. every Cauchy sequence in F converges to a value in F) is called a **Banach space**, i.e. it obeys the axioms:

1. $\|x\| \geq 0 \quad \forall x \in F$
2. $\|x\| = 0 \iff x = 0 \quad \forall x \in F$
3. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in F$
4. $\|\alpha x\| = |\alpha|\|x\| \quad \forall x \in F$ and $\alpha \in \mathbb{F}$.
5. If $f : \mathbb{N} \to F$ is a Cauchy-convergent sequence, i.e. for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that:

   $$\forall m, n \geq N, \ \|f(m) - f(n)\| < \epsilon,$$

   then there exists $l \in F$ such that, for all ϵ, there exists $N \in \mathbb{N}$ such that:

   $$\forall n \geq N, \ \|f(n) - l\| < \epsilon.$$

Definition 19. A vector space H (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), equipped with a scalar-valued inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$, is a called a **Hilbert space** if it is complete with respect to the norm defined by (for all $f \in H$):

$$\|f\|_H := \sqrt{\langle f, f \rangle}.$$

Hilbert spaces are examples of Banach spaces.

Algebraically, quotients of Banach spaces are induced by kernels of bounded linear maps. An analytical approach may be considered by relaxing the requirement that a quotient is algebraic. In [12] (which serves as the main source for this chapter's research), Saitoh considers such an approach within Hilbert spaces (with discrete bases, i.e. separable Hilbert spaces).

Let H be a Hilbert space and consider any $h \in H$. Given a reproducing kernel Hilbert space H_K (with reproducing kernel K) and a bounded linear operator $L : H_K \to H$, Saitoh aims to find $f \in H_K$ which attains the infimum:

$$\inf_{f \in H_K} \|Lf - h\|_H. \tag{4.1}$$

Generally speaking, the desired quotient is described by the object closest to h in the equation above. A least squares minimisation is utilised – which, as shown below, is connected to obtaining a (Moore-Penrose) pseudoinverse matrix [6].

Definition 20. Let $A \in \mathbb{F}^{m,n}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The **Moore-Penrose matrix inverse**, $A^\dagger \in \mathbb{F}^{n,m}$, is a matrix unique to A which satisfies:

1. $AA^\dagger A = A$,
2. $A^\dagger A A^\dagger = A^\dagger$,
3. $(AA^\dagger)^* = AA^\dagger$,
4. $(A^\dagger A)^* = A^\dagger A$.

The criteria above are called the Moore-Penrose conditions.

Now, if given an $x \in \mathbb{C}^{mm}$ and $M \in \mathbb{C}^{mm,m}$, we define $a' \in \mathbb{C}^m$ such that:

$$\min_{a \in \mathbb{C}^m} \|Ma - x\| = \|Ma' - x\|, \tag{4.2}$$

then we have that [6, pp. 5-19]:

$$a' = M^\dagger x + r, \tag{4.3}$$

where $r \in \ker(M)$, and:

$$M^*Ma' = M^*x. \tag{4.4}$$

To serve as an example within the context of this research, a finite-dimensional analogue of Saitoh's method within $\mathbb{C}^m$, will be explored. Let M_m denote a linear transformation defined by, for all $a \in \mathbb{C}^m$ and (fixed but arbitrary) $b \in \mathbb{C}^m$:

$$M_m : a \mapsto a \otimes b.$$

We will use a matrix representation by defining the matrix M such that $Mz = M_m(z)$ for all $z \in \mathbb{C}^m$. Similarly, the notation $f(e_i) := e_i^* f$ will apply to all $f \in \mathbb{C}^m,\ m \in \mathbb{N}$.

In order to determine a suitable pseudoinverse, Saitoh applies the concept of reproducing kernels and positive definite Hermitian matrices. Moreover, Tikhonov regularisation is incorporated in this method to ensure that the infimum in (4.1) remains bounded.

Definition 21. Let $F(E) = \{f : E \to \mathbb{C}\}$ denote the set of functions with domain E and codomain $\mathbb{C}$, with E a non-empty set. A **reproducing kernel Hilbert space** on E is a Hilbert space $H_K \subset F(E)$ endowed with a function $K : E \times E \to H_K$, called the reproducing kernel, that satisfies the reproducing property:

$$f(p) = \langle f, K_p \rangle,$$

for all $p \in E$ and $f \in H_K$, where $K_p(x) := K(x, p)$. Here, $\langle f, K_p \rangle$ is the inner product of f and $K_p : E \to H_K$ in H_K.

An illustrative example of such a reproducing kernel Hilbert space is the complex-valued vector space, $\mathbb{C}^m$. We may consider every $f \in \mathbb{C}^m$ as acting like functions on the standard basis $E := \{e_i : i = 1, \ldots, m\}$ and defined by the values $[f]_i = f(e_i)$. So, in this setting, every f may be described as an inner product operation, and is related to the dual space of E.

We consider an example which shows that every non-singular Hermitian matrix may be associated with a reproducing kernel Hilbert space. Let A denote a positive definite Hermitian matrix, and define the following inner product, for all $f, g \in \mathbb{C}^m$:

$$\langle f, g \rangle_A := g^* A^{-1} f. \tag{4.5}$$

Since A is positive definite, it has a unique positive definite square root $A^{\frac{1}{2}}$.

Furthermore, since $\left\langle A^{\frac{1}{2}}e_i, A^{\frac{1}{2}}e_j \right\rangle_A = \delta_{j,i}$ for $i, j = 1, 2, \ldots, m$, the set:

$$\left\{E_i := A^{\frac{1}{2}}e_i, i = 1, 2, \ldots, m\right\}$$

is an orthonormal basis for the inner product space $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_A)$. Now, let K denote the reproducing kernel function as defined in Definition 21. By the reproducing kernel property, the j^{th} coordinate of E_i can be described as:

$$[E_i]_j = \langle E_i, K_{e_j}\rangle_A = e_j^* E_i.$$

Since $\{E_i : i = 1, 2, \ldots, m\}$ forms a basis for $\mathbb{C}^m$, we may write K_{e_q} as $K_{e_q} = \sum_{i=1}^{m} \alpha_{i,q} E_i$. Consider the following:

$$\begin{aligned} K(e_p, e_q) &= \langle K_{e_q}, K_{e_p}\rangle_A \\ &= \left\langle \sum_{i=1}^{m} \alpha_{i,q} E_i, \sum_{j=1}^{m} \alpha_{j,p} E_j \right\rangle_A \\ &= \sum_{i,j=1}^{m} \alpha_{i,q}\overline{\alpha_{j,p}}\langle E_i, E_j\rangle_A \\ &= \sum_{i=1}^{m} \alpha_{i,q}\overline{\alpha_{i,p}}. \end{aligned}$$

Moreover, since:

$$\begin{aligned} \langle E_i, K_{e_j}\rangle_A &= \sum_{k=1}^{m} \overline{\alpha_{k,j}}\langle E_i, E_k\rangle_A \\ &= \overline{\alpha_{i,j}}, \end{aligned}$$

and:

$$\begin{aligned} \langle E_i, K_{e_j}\rangle_A &= e_j^* E_i \\ &= e_j^* A^{\frac{1}{2}} e_i \\ &= [A^{\frac{1}{2}}]_{j,i}, \end{aligned}$$

we have that $\overline{\alpha_{i,j}} = [A^{\frac{1}{2}}]_{j,i}$. But $(A^{\frac{1}{2}})^* = A^{\frac{1}{2}}$, so $\alpha_{i,j} = [A^{\frac{1}{2}}]_{i,j}$.

The reproducing kernel function, K, therefore, evaluates to the following:

$$\begin{aligned} K(e_p, e_q) &= \sum_{i=1}^{m} \alpha_{i,q}\overline{\alpha_{i,p}} \\ &= \sum_{i=1}^{m} [A^{\frac{1}{2}}]_{i,q}[A^{\frac{1}{2}}]_{p,i} \\ &= [A]_{p,q}. \end{aligned}$$

Given the expression for K above, we may verify that $(\mathbb{C}^m, \langle\cdot,\cdot,\rangle_A)$ is a reproducing kernel Hilbert space:

$$\begin{aligned} \langle f, K_{e_k}\rangle_A &= [K_{e_k}]^* A^{-1} f \\ &= \sum_{i,j=1}^{m} [f]_i [A^{-1}]_{j,i}\overline{K(e_j, e_k)} \\ &= \sum_{i,j=1}^{m} [f]_i [A^{-1}]_{j,i}\overline{[A]_{j,k}} \\ &= \sum_{i=1}^{m} [f]_i [I_m]_{i,k} \\ &= [f]_k. \end{aligned}$$

Now, let K_1 and K_2 denote the reproducing kernel functions for $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_{A_1})$ and $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_{A_2})$ respectively. The reproducing kernel functions K_1 and K_2 induce a tensor product Hilbert space $H_{K_1} \otimes H_{K_2}$ with reproducing kernel $K : \mathbb{C}^m \times \mathbb{C}^m \times \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$, given by:

$$K(e_i, e_j, e_k, e_l) := K_1(e_i, e_j)\, K_2(e_k, e_l).$$

Saitoh [12] considers a "diagonal restriction" on the tensor product space $\mathbb{C}^m \otimes \mathbb{C}^m$, which is also a reproducing kernel space on $\{(e_i, e_j),\ i, j = 1, 2, \ldots m\}$, with the reproducing kernel:

$$\begin{aligned} K(e_i, e_j, e_i, e_j) &:= K_1(e_i, e_j)\, K_2(e_i, e_j) \\ &= [A_1]_{i,j}\, [A_2]_{i,j} \\ &= (e_i^* A_1 e_j)(e_i^* A_2 e_j) \\ &= (e_i \otimes e_i)^*(A_1 \otimes A_2)(e_j \otimes e_j), \end{aligned}$$

equipped with the inner product $\langle\cdot,\cdot\rangle_{\otimes} := \langle\cdot,\cdot\rangle_{A_1\otimes A_2}$, defined analogously to the inner product in (4.5). Furthermore, [12] provides the following expression, for any $z \in \mathbb{C}^m \otimes \mathbb{C}^m \cong \mathbb{C}^{mm}$:

$$[z]_k = \sum_{i,j=1}^{m} \beta_{i,j}[E_{1,i} \otimes E_{2,j}]_k, \tag{4.6}$$

where $\beta_{i,j} \in \mathbb{C},\ i,j = 1,2,\ldots,m$ are some scalars, and:

$$\left\{E_{p,i} := [A_p^{\frac{1}{2}}]e_i,\ i = 1,2,\ldots,m\right\}, \quad p = 1,2$$

the orthonormal bases for $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_{A_1})$ and $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_{A_2})$ respectively. For the general case (such as infinite-dimensional spaces), convergence criteria is given as $\sum_{i,j=1}^{m}|\beta_{i,j}|^2 < \infty$. The diagonal restriction produces the space of Hadamard products, whereas the tensor product reproducing kernel produces the space generated by Kronecker products. In the following, $M^* : H_B \to H_A$ denotes the adjoint of $M : H_A \to H_B$ i.e. the operator satisfying $\langle Mx, y\rangle_{H_B} = \langle x, M^*y\rangle_{H_A}$ for all x, y in the relevant Hilbert space.

Keeping in mind the fundamental results in equations (4.3) and (4.4) above, Saitoh derives the following requirement on any potential solution.

Theorem 12. *Given any $x \in \mathbb{C}^{mm}$ and bounded linear operator $M : \mathbb{C}^m \to \mathbb{C}^{mm}$, there exists $a' \in \mathbb{C}^m$ in the reproducing kernel Hilbert space $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_A)$ such that equation (4.2) is satisfied, if and only if:*

$$M^*x \in \mathbb{C}^m(K_M),$$

where $\mathbb{C}^m(K_M)$ is the reproducing kernel Hilbert space endowed with the kernel defined by:

$$K_M(e_p, e_q) := \left\langle M^*MK_{e_q}, M^*MK_{e_p}\right\rangle_A.$$

Furthermore, the unique $a'_x \in \mathbb{C}^m$ that minimises the norm in (4.2) can be expressed as:

$$[a'_x]_p = \left\langle M^*x, M^*MK_{e_p}\right\rangle_A.$$

Saitoh gives a more general setting for Theorem 12 where boundedness is required, but in settings where boundedness may not be assumed, an additional term is required to ensure that solutions are bounded. Thus, Saitoh applies Tikhonov regularisation, an analytical method which, instead, seeks to minimise:

$$\lambda\|a\|^2 + \|Ma - x\|^2. \tag{4.7}$$

Through introducing some factor $\lambda \in \mathbb{R}_{>0}$, Saitoh constructs the reproducing kernel Hilbert space $(\mathbb{C}^m, \langle\cdot,\cdot\rangle_{A,\lambda})$, with inner product:

$$\langle f, g\rangle_{A,\lambda} := \lambda\langle f, g\rangle_A + \langle Mf, Mg\rangle_{A_1\otimes A_2},$$

and the accompanying reproducing kernel function :

$$K_\lambda(e_p, e_q) = \left[(\lambda I_m + M^*M)^{-1}K(e_p, e_q)\right].$$

Given the construction above, a statement on minimising the adjusted norm equation in (4.7) follows.

Lemma 6. *The unique vector $a'_{x,\lambda} \in \mathbb{C}^m$ that minimises the equation in* (4.7) *satisfies the following expressions:*

$$[a'_{x,\lambda}]_k = \langle x, MK_\lambda(\cdot, e_k)\rangle_{A_1\otimes A_2},$$

and:

$$|[a'_{x,\lambda}]_k| \leq \sqrt{\frac{K(e_k, e_k)}{2\lambda}}\|x\|.$$

Finally, the solutions for the minimisation of equation (4.7) are attained in [12] by introducing scalars $d_{u,u',v},\ u, u', v = 1, 2, \ldots, m$, dependent on λ, which satisfy the following basis expansion:

$$MK_\lambda(\cdot, e_p) = \sum_{u,u'=1}^{m}\sum_{v=1}^{m} d_{u,u',v}E_{1,u}\otimes E_{1,u'}\otimes(E_{2,v}e_p),$$

where the orthonormal basis vectors are defined in equation (4.6). By introducing additional schemes of constants $\{D_{v,u,u'} : v, u, u' = 1, 2, \ldots, m\}$ and $\{C_{(u,u'),(u_*,u'_*)} : u, u', u_*, u'_* = 1, 2, \ldots, m\}$ determined by:

$$Me_v = \sum_{u,u'=1}^{m} D_{v,u,u'}(e_u \otimes e_{u'}),$$

and defined by:

$$C_{(u,u'),(u_*,u'_*)} := \sum_{v=1}^{m} D_{v,u,u'}D_{v,u_*,u'_*},$$

Saitoh [12] presents the set of equations:

$$\lambda d_{u,u',v} + \sum_{u,u'=1}^{m} d_{u_*,u'_*,v} C_{(u,u'),(u_*,u'_*)} = D_{v,u,u'}. \tag{4.8}$$

In the finite case, the scalars $d_{u,u',v}$ may be derived. In the general case, a convergence criteria is provided and required to hold. The following result, regarding an inversion $a \otimes b \mapsto a$ in terms of seeking a vector that minimises (4.7), is derived.

Theorem 13. *Let $M_m : \mathbb{C}^m \to \mathbb{C}^{mm}$ be a linear transformation, defined by $M_m : a \mapsto a \otimes b$ for all $a \in \mathbb{C}^m$, given a fixed $b \in \mathbb{C}^m$. Given a scalar $\lambda \in \mathbb{C}$, the element which minimises the expression* (4.7)*, is given by:*

$$a'_{x,\lambda} = \sum_{u,u',v}^{m} \overline{d_{u,u',v}} \, [x]_v \, E_v,$$

where $\{E_i,\ i = 1, 2, \ldots, m\}$ is an orthonormal basis for the space $\mathbb{C}^m$, with inner product $\langle \cdot, \cdot \rangle_A$ defined in (4.5)*. The constants, $d_{u,u',v},\ u, u', v = 1, 2, \ldots, m$, are determined by the set of equations in* (4.8)*.*

By the deductions and theorem above, a generalised inverse may be obtained in both the finite- and infinite-dimensional cases, albeit with the requirement of setting a Lagrange multiplier, and solving the series of equations given by (4.8) (subject to convergence criteria).

Chapter 5

Conclusion

This research initialised its aim of characterising inverse operations of Kronecker products with an introduction to the fundamentals and construction of tensor products (of vector spaces), and consequently Kronecker products. The Kronecker quotient operation, $\oslash_{m,n,s,t} : \mathbb{F}^{ms,nt} \times \mathbb{F}^{s,t} \to \mathbb{F}^{m,n}$, is defined generally; together with proposed properties.

With foremost focus on finite-dimensional vector spaces and KQ Property 3 and 4, a linear (right) Kronecker Quotient operation $q(B, s, t)(M) : M \mapsto M \oslash B$ (given an arbitrary $B \in \mathbb{F}^{s,t}$) is defined and presented in terms of quotient space cosets, the canonical vector space homomorphism kernel and representative matrices. In this algebraic spirit, conditions on $q(B, s, t)$ to (additionally) satisfy KQ Properties 1, 2 and 5 is investigated in Theorem 6.

Describing quotients in terms of $\oslash_{m,n,s,t}$ (for every $m, n, s, t \in \mathbb{N}$) suffers excessive generality and lacks a coherent theory. In order to unite the scheme of linear Kronecker quotients across multiple dimensions, the uniform linear Kronecker quotient $\bar{q}(B) : \mathbb{F}^{s,t} \to \mathbb{F}$ is adopted and defined such that $\bar{q}(B, s, t)(A \otimes C) = A\, \bar{q}(B)(C)$ for all $A \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{s,t}$.

As a dimension-reducing transformation, the Frobenius inner product, $\odot$, is described. This leads to characterising a matrix $R_q(B)$, whereby $\bar{q}(B, m, n)(M) = M \odot R_q(B)$. Conditions on KQ Properties 2, 1, and 5 are revisited, leading to a significant result – associativity holds for $\oslash$ if and only if $R_q(\cdot)$ preserves the tensor product structurally, which may warrant further research.

Additionally, a potential "basis expansion" for a matrix $M \in \mathbb{F}^{ms,nt}$ is described in Theorem 8 and a consequent investigation into the existence of an "orthogonal" basis $\{B_1, B_2, \ldots, B_n : \bar{q}(B_i)(B_j) = \delta_{i,j}\}$. Following investigation, such a basis may not always exist, and requires the invertibility of a specific matrix, defined in Theorem 9. Further investigation, perhaps a procedure to obtain such a basis, could be of interest.

As expected, linear properties – which are natural in the vector space setting – allow for a flexible linear operator. In addition, the uniform linear Kronecker quotient also facilitates useful examples of Kronecker quotients. Examples explored in this research either follow weighted average transformation methods (some as generalisation of existing methods in [9] and [3]); or utilise the partial Frobenius norm, defined in [3]. These examples, however, impose restrictions on the fields underlying to $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, or fields of characteristic 0. Further investigation into defining such examples applicable to finite fields could prove practical.

Nonetheless, characterising a Kronecker quotient with multiplicative properties, for example KQ Property 8, has been investigated. A multiplicative quotient, $\oslash^{\times}$, limited to operating on the submonoid $\mathrm{GL}_{m,n}^{\otimes}(\mathbb{F}) \subseteq \mathrm{GL}_{mn}(\mathbb{F})$, is defined in (3.11). Given Theorem 11, such a quotient operation must be factored through the determinant. Consequently, it is shown that a multiplicative and well-defined quotient on the monoid of $(mn) \times (mn)$ matrices is not viable. Identifying an appropriate structure for such a quotient might be requisite for further research.

Further properties for a Kronecker quotient are also considered. A uniform linear Kronecker quotient which satisfies KQ Property 6 is constructable. However, such a quotient is restricted to a relatively simplistic form and does not satisfy KQ Property 2 simultaneously. Similarly, a Kronecker quotient $\oslash$ satisfying KQ Property 7, as defined in Section 3.5, does not hold up with linearity or uniformity. Considering a quotient satisfying matrix spectra properties may be defined by way of a vector "$M \oslash_I B$" by its action on a particular basis. As might be expected, restrictions are imposed on the basis of $\mathbb{F}^m$ and eigenvalues of the matrices involved.

In Chapter 4, in order to characterise a quotient suitable to both finite- and infinite-dimensional vector spaces, the study of such a quotient is taken to the setting of Banach spaces. To this end, the approach in [12] is examined, which seeks an $a \in H_1$ to minimise $\|Ma - x\|_{H_2}$, given a bounded linear operator M and $x \in H_2$, where H_1 and H_2 are considered as reproducing kernel Hilbert spaces. This approach is analytical, in the sense that the expression above is not required to be zero, i.e. the quotient is not required to be algebraic. Moreover, to ensure that this infimum remains bounded, Tikhonov regularisation is applied. The outcome of this method, however, relies upon setting a factor $\lambda \in \mathbb{R}_{>0}$ – and certain constants, dependent on λ, which are determined by an infinite (or finite) set of equations in (4.8). In addition, this approach applies to infinite-dimensional spaces with discrete bases. An approach applicable to spaces with continuous bases may be considered for potential research.

Further potential directions of research, not mentioned above, include the following:

1. Characterising a Kronecker quotient for a "symmetric" Kronecker product, as defined and described in [13].
2. Investigating a Kronecker quotient, uniform or non-uniform, which satisfies KQ Property 6 together with 2 – or combinations of other properties as explored in Section 3.5.
3. Quotients of alternative matrix products – for example Tracy-Singh products and Khatri-Rao products [10].
4. Kronecker quotients for matrices over rings. Many results in the research above generalise immediately to unital commutative rings.
5. Possible inversions of Kronecker sums. For $A \in \mathbb{F}^{m,m}$ and $B \in \mathbb{F}^{n,n}$, the Kronecker sum may be defined as in [13]:

$$A \oplus B := (A \otimes I_n) + (I_m \otimes B).$$

   Future research might, for example, benefit from characterising an operation $\ominus : \mathbb{F}^{mn,mn} \to \mathbb{F}^{m,m}$ defined such that $(A \oplus B) \ominus B = A$.

Appendix A

Proof and Bases for Theorem 8

Theorem 8. *Let* $\{B_1, B_2, \ldots, B_{st}\}$ *be a basis for* $\mathbb{F}^{s,t}$*. For all* $M \in \mathbb{F}^{ms,nt}$, $m, n \in \mathbb{N}$, *the following holds:*

$$M = \sum_{i=1}^{st} \bar{q}(B_i, m, n)(M) \otimes B_i = \sum_{i=1}^{st} (M \oslash B_i) \otimes B_i$$

if and only if:

$$\bar{q}(B_k)(B_p) = \delta_{k,p}.$$

A.1 Theorem Proof

Proof. Since the general tensor M is in $\mathbb{F}^{ms,nt} \cong \mathbb{F}^{m,n} \otimes \mathbb{F}^{s,t}$, we have the following expression, by the bilinearity of the tensor product and given the basis $B := \{B_1, B_2, \ldots, B_{st}\}$:

$$M = \sum_{k=1}^{st} A_k \otimes B_k, \tag{A.1}$$

where $A_k \in \mathbb{F}^{m,n}$, $k \in \{1, 2, \ldots, st\}$.

Now, suppose that $\bar{q}(B_i)(B_l) = \delta_{i,l}$, for all $B_i, B_l \in B$. Given the expression for M above, and the linearity of $\bar{q}(B_i, m, n)$, we have that:

$$\bar{q}(B_i, m, n)(M) = \bar{q}(B_i, m, n) \left(\sum_{j=1}^{st} A_j \otimes B_j \right)$$

$$= \sum_{j=1}^{st} A_j \ \bar{q}(B_i)(B_j) \tag{A.2}$$
$$= A_i,$$

which implies that $M = \sum_{k=1}^{st} \bar{q}(B_k, m, n)(M) \otimes B_k$.

To prove the converse case, suppose that $M = \sum_{k=1}^{st} \bar{q}(B_k, m, n)(M) \otimes B_k$, for all $M \in \mathbb{F}^{ms,nt}$. Now, suppose we choose an M such that:

$$M = A_j \otimes B_j,$$

where $A_j \neq \mathbf{0}^{m,n}$. Consider the following:

$$\begin{aligned} M &= \sum_{i=1}^{st} \bar{q}(B_i, m, n)(M) \otimes B_i \\ &= \sum_{i=1}^{st} \bar{q}(B_i, m, n)(A_j \otimes B_j) \otimes B_i \\ &= \sum_{i=1}^{st} (\bar{q}(B_i)(B_j)A_j) \otimes B_i. \end{aligned}$$

Since the set $\{B_1, B_2, \ldots, B_{st}\}$ is linearly independent, we may assert the following:

$$\begin{aligned} A_j \otimes B_j = &\sum_{i=1}^{st} (\bar{q}(B_i)(B_j)A_j) \otimes B_i \\ &\implies A_j = \bar{q}(B_i)(B_j)A_j \quad \text{and} \quad \bar{q}(B_i)(B_j)A_j = \mathbf{0}^{m,n} \ \text{ when } \ i \neq j \\ &\implies \bar{q}(B_k)(B_j) = \delta_{k,j}. \end{aligned}$$ □

A.2 Necessary Conditions for a Basis with Property $\bar{q}(B_i)(B_j) = \delta_{i,j}$ to Exist

Note that it may be shown that, by construction, there exists a unique quotient $\bar{q}(\cdot)$ defined by $\bar{q}(B_j)(B_k) = \delta_{j,k}$, $j, k \in \{1, 2, \ldots, st\}$ for every basis $\{B_1, \ldots, B_{st}\}$.

Lemma 7. *There exists a basis $\{B_1, B_2, \ldots, B_{st}\} \subseteq \mathbb{F}^{s,t}$ such that $\bar{q}(B_j)(B_k) = 0$ for all $j < k$.*

Proof. Let $B_1 \in \mathbb{F}^{s,t}$ and define the subset $F \subseteq \mathbb{F}^{s,t}$:

$$F := \{M \in \mathbb{F}^{s,t} : M \neq \mathbf{0}^{s,t}, M \notin \text{span}\{B_1\}\} = \mathbb{F}^{s,t} \setminus \text{span}\{B_1\}.$$

Now, suppose that $\bar{q}(B_1)(B) \neq 0$ for all $B \in F$. There exists a $C \in \mathbb{F}^{s,t}$ such that $C := \dfrac{1}{\bar{q}(B_1)(C_1)} C_1$ where $C_1 \in F$. So, we have that, since $C, B_1 - C \in F$:

$$\begin{aligned}\bar{q}(B_1)(B_1) - \bar{q}(B_1)(C) &= 0 \\ &\neq \bar{q}(B_1)(B_1 - C),\end{aligned}$$

which contradicts the linearity of the operator $\bar{q}(B_1)$. So, for any $C_1 \in F$, there exists a $B_1 - \dfrac{1}{\bar{q}(B_1)(C_1)} C_1 \in F$ such that:

$$\bar{q}(B_1)(B_1 - \frac{1}{\bar{q}(B_1)(C_1)} C_1) = 0.$$

We may, therefore, define $B_2 \in F$ as:

$$B_2 := \bar{q}(B_1)(C_1) B_1 - C_1,$$

such that $\bar{q}(B_1)(B_2) = 0$. Similarly, if we define B_3 by:

$$B_3 := \bar{q}(B_1)(C_2) B_1 + \bar{q}(B_2)(C_2) B_2 - C_2,$$

where $C_2 \neq \mathbf{0}^{s,t}, C_2 \notin \text{span}\{B_1, B_2\}$, we have that $\bar{q}(B_1)(B_3) = \bar{q}(B_2)(B_3) = 0$.

Continuing in this fashion, given $C_{k-1} \neq \mathbf{0}^{s,t}, C_{k-1} \notin \text{span}\{B_1, B_2, \ldots, B_{k-1}\}$, there exists $B_k, k \in \{2, \ldots, st\}$ defined as:

$$B_k := \sum_{i=1}^{k-1} (\bar{q}(B_i)(C_{k-1}) B_i) - C_{k-1}, \tag{A.1}$$

such that $\bar{q}(B_n)(B_k) = 0, n < k$.

Lastly, since every $C_{k-1} \notin \text{span}\{B_1, B_2, \ldots, B_{k-1}\}$, we have that every $B_k \notin \text{span}\{B_1, B_2, \ldots, B_{k-1}\}$. Therefore, the set $\{B_i : i \in \{1, 2, \ldots, st\}\}$ is linearly independent – and subsequently, given that there are st elements in this set, a basis of $\mathbb{F}^{s,t}$. □

Let $\bar{q} : \mathbb{F}^{s,t} \to \{\bar{q}(\cdot) : \mathbb{F}^{s,t} \to \mathbb{F} : \bar{q}(\cdot) \text{ is linear}\}$ denote a uniform Kronecker quotient (as defined in Section 3.3). For all $A \in \mathbb{F}^{s,t}, A \neq \mathbf{0}^{s,t}$, we have that $\bar{q}(A)(A) = 1$. For $s = t = 1$, the basis $\{B_1\}$ trivially satisfies the requirement with no further proof required.

Since the method above only proves that $\bar{q}(B_n)(B_k) = 0$ for $n < k$, another perspective to solving the question is necessary. Suppose $st = 2$. Let $B_1 \in \mathbb{F}^{s,t}_{nz}$ and $F \subseteq \mathbb{F}^{s,t}$ define the subset:

$$F := \{M \in \mathbb{F}^{s,t} : M \neq \mathbf{0}^{s,t}, M \notin \text{span}\{B_1\}\}.$$

Now, suppose that $\bar{q}(B_1)(B) \neq 0$ for all $B \in F$. There exists a $D \in \mathbb{F}^{s,t}$ such that $D := \dfrac{1}{\bar{q}(B_1)(D_1)} D_1$ where $D_1 \in F$. So, since $B_1 - D \in F$, we have that:

$$\begin{aligned} \bar{q}(B_1)(B_1) - \bar{q}(B_1)(D) &= 0 \\ &\neq \bar{q}(B_1)(B_1 - D), \end{aligned}$$

which contradicts the linearity of the operator $\bar{q}(B_1)$. So, for any $D_1 \in F$, there exists a $B_1 - \dfrac{1}{\bar{q}(B_1)(D_1)} D_1 \in F$ such that

$$\bar{q}(B_1)(B_1 - \frac{1}{\bar{q}(B_1)(D_1)} D_1) = 0.$$

Hence, we can define $B_2 \in F$ as:

$$B_2 := \bar{q}(B_1)(D_1)B_1 - D_1,$$

such that $\bar{q}(B_1)(B_2) = 0$. Similarly, we may define $B_1 := \bar{q}(B_2)(D_2)B_2 - D_2,\ D_2 \notin \text{span}\{B_2\}$ such that $\bar{q}(B_2)(B_1) = 0$.

So, for $st = 2$, we wish to solve the following system of equations:

$$\begin{aligned} B_2 &= \bar{q}(B_1)(C_1')B_1 - C_1' \\ B_1 &= \bar{q}(B_2)(C_2')B_2 - C_2', \end{aligned} \tag{A.2}$$

The requirement (axiom in Definition 13) $\bar{q}(B_i)(B_i) = 1,\ i = 1, 2$ and $\bar{q}(B_i)(B_j) = 0,\ i \neq j$, results in the constraint:

$$\bar{q}(B_1)(C_2') = \bar{q}(B_2)(C_1') = -1. \tag{A.3}$$

In addition to defining $C_1' \notin \text{span}\{B_1\}$ and $C_2' \notin \text{span}\{B_2\}$ above, demanding that $B_1 \neq \mathbf{0}^{s,t}$ and $B_2 \neq \mathbf{0}^{s,t}$ requires $C_1' \notin \text{span}\{B_2\}$ and $C_2' \notin \text{span}\{B_1\}$.

Hence, $\{C_1', C_2'\}$ is a basis where $C_1' \notin \text{span}\{B_1\}, C_1' \notin \text{span}\{B_2\}$ and $C_2' \notin \text{span}\{B_1\}, C_2' \notin \text{span}\{B_2\}$, which may be written in the form, given any linearly independent vectors $C_1, C_2 \in \mathbb{F}_{nz}^{s,t}$:

$$\begin{aligned} C_1' &= aC_1 + bC_2, \\ C_2' &= cC_1 + dC_2. \end{aligned}$$

where $a, b, c, d \in \mathbb{F}$. Now, define the following:

$$\begin{aligned} \alpha_1 &:= \bar{q}(B_1)(C_1) \text{ and } \alpha_2 := \bar{q}(B_1)(C_2), \\ \beta_1 &:= \bar{q}(B_2)(C_1) \text{ and } \beta_2 := \bar{q}(B_2)(C_2). \end{aligned} \tag{A.4}$$

From the system of equations in (A.2), we have the corresponding matrix representation:

$$\begin{bmatrix} a\alpha_1 + b\alpha_2 & -1 \\ -1 & c\beta_1 + d\beta_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{A.5}$$

In order for solutions to B_1 and B_2 to exist, we have two constraints. We require the matrix on the left-hand side to be invertible, thus:

$$(a\alpha_1 + b\alpha_2)(c\beta_1 + d\beta_2) \neq 1,$$

and for $\bar{q}(B_k)(B_k) = 1,\ k \in \{1, 2\}$ to hold within (A.2):

$$a\beta_1 + b\beta_2 = c\alpha_1 + d\alpha_2 = -1.$$

Expressed in matrix form, we require:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} = \begin{bmatrix} x & -1 \\ -1 & y \end{bmatrix}, \tag{A.6}$$

where the scalars x, y satisfy $xy \neq 1$. Thus, the rightmost matrix is invertible, and since $\{C_1, C_2\}$ and $\{C_1', C_2'\}$ are linearly independent sets, the leftmost matrix is also invertible.

To satisfy the restraints above and, as a result, to obtain a solution for (A.2) we require the matrix:

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \tag{A.7}$$

to be invertible. Given (A.5) and (A.6), we obtain a general system:

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Therefore, given that the system of equations above is independent of the choice of C_1 and C_2, we may assert that we can choose arbitrary C_1 and C_2 to satisfy (A.2) if and only if a B_1 and B_2 exist such that the matrix in (A.7) is invertible (that is, $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$).

Higher-dimensional case

Now, suppose $st = n$, $n \geq 1$. The following derivation will follow suit from the two-dimensional case.

Theorem 9. *Let $\bar{q}(X)$ be a uniform linear Kronecker quotient given $X \in \mathbb{F}^{s,t}_{nz}$. If there exists a basis $\{B_1, B_2, \dots, B_{st}\}$ for $\mathbb{F}^{s,t}$ such that $\bar{q}(B_p)(B_k) = \delta_{p,k}$, $p, k \in \{1, 2, \dots, st\}$ then the matrix:*

$$\Omega := \begin{bmatrix} \bar{q}(B_1)(C_1) & \bar{q}(B_2)(C_1) & \cdots & \bar{q}(B_{st-1})(C_1) & \bar{q}(B_{st})(C_1) \\ \bar{q}(B_1)(C_2) & \bar{q}(B_2)(C_2) & \cdots & \bar{q}(B_{st-1})(C_2) & \bar{q}(B_{st})(C_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{q}(B_1)(C_{st}) & \bar{q}(B_2)(C_{st}) & \cdots & \bar{q}(B_{st-1})(C_{st}) & \bar{q}(B_{st})(C_{st}) \end{bmatrix},$$

is invertible for every other basis $\{C_1, C_2, \dots, C_{st}\}$.

Proof. Let $n = st$ and assume that $\{B_1, B_2, \dots, B_n\}$ obeys $\bar{q}(B_i)(B_j) = \delta_{i,j}$. Consider the system of equations for $B_1, B_2, \dots, B_n$, where:

$$B_i = \sum_{j=1, j\neq i}^{n} \bar{q}(B_j)(C'_{n+1-i})B_j - C'_{n+1-i},$$

and $\bar{q}(B_i)(C'_{n+1-i}) = -1$. We note that such $C'_1, C'_2, \dots, C'_n$ exist, for example by taking $C'_{n+1-i} = -B_i + B_j$, $j \neq i$.

Given any linearly independent set of vectors (so, any basis) in $\mathbb{F}^{s,t}$,

$\{C_1, C_2, \ldots, C_n\}$ where $C_1, \ldots, C_n \notin \text{span}\{B_j\},\ j = 1, 2, \ldots, n$, the appropriate basis $\{C_1', C_2', \ldots, C_n'\}$ may be expressed as:

$$C_i' := \sum_{j=1}^{n} a_{i,j} C_j,$$

where $a_{i,j} \in \mathbb{F}$.

Now, define the scalars $\gamma_{i,j} := \bar{q}(B_i)(C_j),\ i, j \in \{1, 2, \ldots, n\}$. Additionally, define the matrices $\Lambda := [a_{i,j}]$ and $\Omega := [\gamma_{i,j}]^T$.

Since $\sum_{j=1}^{n} a_{l,j}\gamma_{k,j} = [\Lambda\Omega]_{l,k}$, we may now express the system of equations above by:

$$B_i = \sum_{j=1, j\neq i}^{n} \left[\sum_{k=1}^{n} a_{n+1-i,k}\gamma_{j,k} \right] B_j - C_{n+1-i}' = \sum_{j=1, j\neq i}^{n} [\Lambda\Omega]_{n+1-i,j}\ B_j - C_{n+1-i}',$$

and in matrix form:

$$\Lambda \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \Delta \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix},$$

where

$$\Delta := \begin{bmatrix} [\Lambda\Omega]_{1,1} & [\Lambda\Omega]_{1,2} & \cdots & [\Lambda\Omega]_{1,n-1} & -1 \\ [\Lambda\Omega]_{2,1} & [\Lambda\Omega]_{2,2} & \cdots & -1 & [\Lambda\Omega]_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & [\Lambda\Omega]_{n,2} & \cdots & \cdots & [\Lambda\Omega]_{n,n} \end{bmatrix}.$$

For solutions to exist, we require that Δ be invertible, i.e. that $\det(\Delta) \neq 0$. A second constraint is that $\bar{q}(B_i)(B_i) = 1,\ i \in \{1, 2, \ldots, n\}$, which translates to:

$$\bar{q}(B_i)(C_{n+1-i}') = [\Lambda\Omega]_{n+1-i,i} = -1.$$

Given that:

$$\Lambda\Omega = \Delta,$$

to satisfy system (A.2), we ultimately require Ω to be invertible (since Δ is invertible). □

If Ω is invertible, we obtain the expression:

$$\Omega \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}.$$

Here, $\Omega = \Omega(C_1, C_2, \ldots, C_n)$ depends on $C_1, C_2, \ldots, C_n$. The construction of the system of equations above is independent of the choice of C_i, $i \in \{1, 2, \ldots, n\}$. Therefore, if there exists a basis $\{B_1, B_2, \ldots, B_n\}$ as described in Theorem 9, this system of equations is satisfied by any choice of basis $\{C_1, C_2, \ldots, C_n\}$.

Bibliography

[1] Dragomir Ž. Djoković, *On homomorphisms of the general linear group*, aequationes mathematicae **4** (1970), no. 1-2, 99–102.

[2] John B. Fraleigh, *A first course in abstract algebra*, Pearson Education, 2003.

[3] Yorick Hardy, *On Kronecker quotients*, Electronic Journal of Linear Algebra **27** (2014), 172–189.

[4] Harold V. Henderson and Shayle R. Searle, *Vec and vech operators for matrices, with some uses in Jacobians and multivariate statistics*, The Canadian Journal of Statistics / La Revue Canadienne de Statistique **7** (1979), no. 1, 65–81.

[5] Harold V. Henderson and Shayle R. Searle, *The vec-permutation matrix, the vec operator and Kronecker products: a review*, Linear and Multilinear Algebra **9** (1981), no. 4, 271–288.

[6] Leslie Hogben (ed.), *Handbook of linear algebra*, second ed., Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2014.

[7] Roger A. Horn and Charles R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1991.

[8] Alan J. Laub, *Matrix analysis for scientists & engineers*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.

[9] Paul Leopardi, *A generalized FFT for Clifford algebras*, Bulletin of the Belgian Mathematical Society **11** (2005), 663–688.

[10] Shuangzhe Liu and Göetz Trenkler, *Hadamard, Khatri-Rao, Kronecker and other matrix products*, International Journal of Information and Systems Sciences **4** (2008).

[11] Russel Merris, *Multilinear algebra*, Algebra, Logic and Applications, vol. 8, Gordon and Breach Science Publishers, Amsterdam, 1997.

[12] Saburou Saitoh, *Generalized inversions of Hadamard and tensor products for matrices*, Advances in Linear Algebra & Matrix Theory **4** (2014), 87–95.

[13] Kathrin Schäcke, *On the Kronecker product*, `http://orion.math.uwaterloo.ca/~hwolkowi/henry/reports/kronessaykathrin.ps`, 2004, Last accessed: 10 June 2020.

[14] Willi-Hans Steeb and Yorick Hardy, *Matrix calculus and Kronecker product: A practical approach to linear and multilinear algebra*, World Scientific, 2011.

[15] Charles F. van Loan, *The ubiquitous Kronecker product*, Journal of Computational and Applied Mathematics **123** (2000), 85–100.

[16] Charles F. van Loan and Nikos Pitsianis, *Approximation with Kronecker products*, Linear Algebra for Large Scale and Real-Time Applications (Marc S. Moonen, Gene H. Golub, and Bart L. R. De Moor, eds.), Kluwer Publications, 1993, pp. 293–314.