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Partial isospectrality of a matrix pencil and circularity of the c-numerical range ☆

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We study when functions of the eigenvalues of the pencil

$$\mathrm{Re}(e^{-it}A) = \cos(t)\mathrm{Re}A + \sin(t)\mathrm{Im}A \tag{1}$$

are constant functions of t. The results are then applied to questions regarding the numerical range, the higher rank numerical range and the c-numerical range, and we derive trace type conditions for when these numerical ranges are disks centered at 0. The theory of symmetric polynomials plays an important part in the proofs.

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1. Introduction

In this paper we study the eigenvalues of the pencil

$$\operatorname{Re}(e^{-it}A) = \cos(t)\operatorname{Re}A + \sin(t)\operatorname{Im}A. \tag{2}$$

Here $A \in \mathbb{C}^{n\times n}$ is a fixed matrix, $\operatorname{Re}(M) = \frac{1}{2}(M+M^*)$, $\operatorname{Im}(M) = \frac{1}{2i}(M-M^*)$, and M^* denotes the complex conjugate transpose of the matrix M. The study of this pencil is in large part motivated by questions regarding the numerical range of the matrix A, as well as generalizations of the numerical range, such as the higher rank numerical range and the c-numerical range.

Questions regarding rotational symmetry of the classical numerical range, the higher rank numerical range, as well as the c-numerical range have been studied in [3,6,10–12,14,15]. Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. Recall that the *c-numerical range* (see [9]) is defined as

$$W_c(A) = \left\{ \sum_{i=1}^{n} c_i u_i^* A u_i : \{u_1, \dots, u_n\} \text{ is an orthonormal basis of } \mathbb{C}^n \right\},$$

where v^* denotes the conjugate transpose of a vector v. When $c = (1, 0, \dots, 0)$ the c-numerical range reduces to the classical numerical range. The paper [9] by Chi-Kwong Li discusses many properties of the c-numerical range.

The paper is organized as follows. In Section 2 we characterize when eigenvalue quantities related to the pencil (2) that can be expressed in terms of symmetric polynomials, are constant in t. In Section 3, we give a characterization when the c-numerical range of a matrix A is a disk with center 0. This characterization is in terms of a symmetric polynomial, which in turn can be expressed as a trace polynomial involving the matrices A and A^*. In Section 4, we revisit the case of the classical numerical range and also briefly discuss when the first rank-k numerical ranges are disks.

2. Symmetric functions of the eigenvalues

Let us fix a matrix $A \in \mathbb{C}^{n\times n}$ and denote the eigenvalues of $\operatorname{Re}(e^{-it}A)$ by $\lambda_k(t) = \lambda_k(\operatorname{Re}(e^{-it}A))$, $k = 1, \dots, n$, where $\lambda_1(t) \ge \dots \ge \lambda_n(t)$. In this section we consider quantities $p(\lambda_1(t), \dots, \lambda_n(t))$, where $p(x_1, \dots, x_n)$ is a symmetric polynomial in the commuting variables $x_1, \dots, x_n$. Recall that the polynomial $p(x_1, \dots, x_n)$ is *symmetric* if for all permutations σ on $\{1, \dots, n\}$ we have $p(x_1, \dots, x_n) = p(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. For instance, $x_1 + \cdots + x_n$ and $x_1 \cdots x_n$ are examples of symmetric polynomials. Note that these two polynomials when applied to the tuple $(\lambda_1(t), \dots, \lambda_n(t))$, correspond to $\sum_k \lambda_k(t) = \operatorname{Tr}(\operatorname{Re}(e^{-it}A))$ and $\prod_k \lambda_k(t) = \det(\operatorname{Re}(e^{-it}A))$, respectively.

The *power sum symmetric polynomials* are defined as

$$p_k(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k, \quad k = 1, 2, \dots .$$

It is well-known (see, e.g., [13]) that any symmetric polynomial in n variables can be expressed as a polynomial with rational coefficients in the first n power sum polynomials. For instance, the two variable symmetric polynomial $r(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$ can be written as $r = \frac{1}{2}p_1^3 - \frac{1}{2}p_1 p_2$.

In order to state our main result, we consider words w in two letters. For instance, PPQ, $PQPQPP$ are words in the letters P and Q of lengths 3 and 6, respectively. The length of a word w is denoted by $|w|$. The empty word $\emptyset$ has length 0. When we write $\mathrm{na}(w, P) = l$ we mean that P appears l times in the word w (na=number of appearances). We also need the notion of *trace polynomials* (introduced in [16]; see also [7,8]), which in the scalar valued case are linear combinations of products of traces of matrices. An example of such a scalar valued polynomial is

$$q(P, Q) = \mathrm{Tr}(PQPQ)\mathrm{Tr}\left(P^2Q^3\right) - \frac{5}{3}\left(\mathrm{Tr}(PQ^5PQ)\right)^2,$$

where the trace of a square matrix M is denoted by $\mathrm{Tr}\, M$. We will use trace polynomials with matrices A and A^*.

Theorem 2.1. *Let $r_1, \ldots r_k$ be symmetric polynomials in n variables and let $A \in \mathbb{C}^{n\times n}$. Then there exists an integer K and scalar valued trace polynomials $q_1(A, A^*), \ldots, q_K(A, A^*)$, so that*

$$r_j(\lambda_1(t), \ldots, \lambda_n(t)), \quad j = 1, \ldots, k,$$

are constant functions of t if and only if

$$q_1(A, A^*) = \cdots = q_K(A, A^*) = 0. \tag{3}$$

Moreover, the trace polynomials q_j, $j = 1, \ldots, K$, can be derived in an algorithmic way. Finally, if $r_1, \ldots r_k$ have rational coefficients, then so will q_j, $j = 1, \ldots, K$.

Proof. We may express the symmetric polynomials $r_1, \ldots r_k$ as polynomial expressions with rational coefficients in power sum polynomials $p_1, \ldots, p_n$; say,

$$r_j = h_j(p_1, \ldots, p_n), \quad j = 1, \ldots, k,$$

where $h_1, \ldots, h_k$ are some polynomials in n variables with rational coefficients. Next notice that

$$p_k(\lambda_1(t), \ldots, \lambda_n(t)) = \sum_{j=1}^{n} \lambda_j(t)^k = \mathrm{Tr}\left(\mathrm{Re}\left(e^{-it}A\right)\right)^k = \mathrm{Tr}\left(\frac{e^{-it}A + e^{it}A^*}{2}\right)^k,$$

which is a trigonometric polynomial (in e^{it}) where the coefficients are trace polynomials (with matrices A and A^*). Therefore, $r_j(\lambda_1(t), \ldots, \lambda_n(t)), j = 1, \ldots, k$, are trigonometric polynomials (in e^{it}) where the coefficients are trace polynomials (with matrices A and A^*). If we now take the coefficients appearing in $r_j(\lambda_1(t),\dots,\lambda_n(t)), j=1,\dots,k$, that correspond to non-zero powers of e^{it}, and call these q_j, $j=1,\dots,K$, then these give the desired trace polynomials. Moreover, if r_j, $j=1,\dots,k$, have rational coefficients, then so will q_j, $j=1,\dots,K$. $\square$

Remark 2.2. In case each of the first n power sum polynomials can be expressed as a polynomial in the polynomials $r_1,\dots,r_k$, the trace conditions (3) in Theorem 2.1 will reduce to the condition [15, Theorem 1.1(ii)].

We let

$$e_k(x_1,\dots,x_n)=\sum_{1\le l_1<\cdots<l_k\le n} x_{l_1}\cdots x_{l_k},$$

denote the *elementary symmetric polynomials*. The well-known Newton identities relating the elementary symmetric polynomials with the power sum polynomials are

$$ke_k(x_1,\dots,x_n)=\sum_{i=1}^{k}(-1)^{i-1}e_{k-i}(x_1,\dots,x_n)p_i(x_1,\dots,x_n),\ 1\le k\le n,$$

where $e_0\equiv 1$. For instance, $e_1=p_1, 2e_2=e_1p_1-p_2=p_1^2-p_2$.

As an example, let us compute what the trace polynomials q_j are when we take a 3×3 matrix A and let $k=1$ and r_1 be the polynomial $e_3(x_1,x_2,x_3)=x_1x_2x_3$. In other words, we will describe when $\det\left(\mathrm{Re}\left(e^{-it}A\right)\right)$ is constant. It follows from the Newton identities (in three variables) that

$$e_3=\frac{p_1^3}{6}-\frac{p_1p_2}{2}+\frac{p_3}{3}.$$

Thus

$$\lambda_1(t)\lambda_2(t)\lambda_3(t)=\frac{1}{2^3}\left[\frac{1}{6}\left(\mathrm{Tr}\left(e^{-it}A+e^{it}A^*\right)\right)^3\right.$$
$$\left.-\frac{1}{2}\left(\mathrm{Tr}\left(e^{-it}A+e^{it}A^*\right)\right)\left(\mathrm{Tr}\left(e^{-it}A+e^{it}A^*\right)^2\right)+\frac{1}{3}\left(\mathrm{Tr}\left(e^{-it}A+e^{it}A^*\right)^3\right)\right].$$

Extracting the coefficient of e^{-3it}, we get $\left(\text{up to a factor } \frac{1}{2^3}\right)$

$$\frac{1}{6}\mathrm{Tr}(A)^3-\frac{1}{2}\mathrm{Tr}(A)\mathrm{Tr}(A^2)+\frac{1}{3}\mathrm{Tr}(A^3). \tag{4}$$

Extracting the coefficient of e^{-2it} we get

$$0. \tag{5}$$

Extracting the coefficient of e^{-it}, we get $\left(\text{up to a factor } \frac{1}{2^3}\right)$

$$\frac{1}{2}\mathrm{Tr}(A)^2\mathrm{Tr}(A^*) - \mathrm{Tr}(A)\mathrm{Tr}\left(AA^*\right) - \frac{1}{2}\mathrm{Tr}(A^*)\mathrm{Tr}\left(A^2\right) + \mathrm{Tr}\left(A^2A^*\right), \tag{6}$$

where we used the rule $\mathrm{Tr}(CD) = \mathrm{Tr}(DC)$. The coefficient of e^{ikt} is just the complex conjugate of the coefficient of e^{-ikt}. Thus we arrive at the following corollary.

Corollary 2.3. *For a 3×3 matrix A, the expression $\det(\mathrm{Re}(e^{-it}A))$ is constant if and only if the quantities in (4) and (6) are equal to zero.*

In n variables we have for $e_n(x_1, \ldots, x_n) = x_1 \cdots x_n$ that

$$e_n = (-1)^n \sum_{\substack{m_1+2m_2+\cdots+nm_n=n \\ m_1\geq 0,\ldots,m_n\geq 0}} \prod_{i=1}^{n} \frac{(-p_i)^{m_i}}{m_i!\, i^{m_i}}.$$

In a similar way as above, one may derive for an $n \times n$ matrix A necessary and sufficient conditions for $\det\left(\mathrm{Re}\left(e^{-it}A\right)\right)$ to be constant in terms of trace conditions.

3. Circularity of the c-numerical range

In this section we consider the problem of characterizing when the c-numerical range of a matrix is a disk. Note that in the special case when $c_1 = \cdots = c_k = 1$ and $c_{k+1} = \cdots = c_n = 0$, it is also called the k-numerical range.

We recall from [9, Result 5.1] the following proposition.

Proposition 3.1. *[9] Let $c = (c_1, \ldots, c_n)$, where $c_1 \geq c_2 \geq \ldots \geq c_n$. For $A \in \mathbb{C}^{n\times n}$ we have that $\mathrm{Re}(W_c(A)) = [\alpha, \beta]$, where*

$$\alpha = c_1\lambda_n + \cdots + c_n\lambda_1, \quad \beta = c_1\lambda_1 + \cdots + c_n\lambda_n$$

and $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of $\mathrm{Re}\, A$.

Corollary 3.2. *Let $c = (c_1, \ldots, c_n)$, where $c_1 \geq c_2 \geq \ldots \geq c_n$. For $A \in \mathbb{C}^{n\times n}$ we have that $W_c(A)$ is a disk with center 0 if and only if*

$$c_1\lambda_1(t) + \cdots + c_n\lambda_n(t)$$

is a constant function of t. Here $\lambda_k(t) = \lambda_k(\mathrm{Re}(e^{-it}A))$, $k = 1, \ldots, n$, which are ordered non-increasingly.

Remark 3.3. Note that $\mathrm{Re}\left(e^{-i(t+\pi)}A\right) = -\mathrm{Re}\left(e^{-it}A\right)$, so that $\lambda_k(t) = -\lambda_{n-k+1}(t+\pi)$. This implies, for instance, that if $\lambda_1(t) + \lambda_2(t)$ is constant, then so is $\lambda_n(t) + \lambda_{n-1}(t)$.

Proof. Without loss of generalization, let us assume that the disk in question is the unit disk. Clearly, $W_c(A)$ is the unit disk if and only if for each $t \in \mathbb{R}$ we have that $e^{-it}W_c(A) = W_c\left(e^{-it}A\right)$ is the unit disk. The latter happens if and only if $\mathrm{Re}(W_c(e^{-it}A)) = [-1,1]$ for all $t \in \mathbb{R}$. Notice that $W_c\left(e^{-i(t+\pi)}A\right) = -W_c\left(e^{-it}A\right)$, from which it follows that $W_c(e^{-it}A)$ is the unit disk for all $t \in \mathbb{R}$ if and only if

$$\max_{x \in \mathrm{Re}(W_c(e^{-it}A))} x = 1$$

for all $t \in \mathbb{R}$. Using Proposition 3.1 the latter happens if and only if $c_1\lambda_1(t) + \cdots + c_n\lambda_n(t) = 1$ for all $t \in \mathbb{R}$.

Clearly, when the disk with center 0 has radius r, we obtain that $c_1\lambda_1(t) + \cdots + c_n\lambda_n(t) = r$ for all $t \in \mathbb{R}$. This finishes the proof. □

Note that when $c = (1, 0, \ldots, 0)$, the circularity of $W_c(A) = W(A) = \{x^*Ax : x^*x = 1\}$, the classical numerical range of A, is characterized in [14, Theorem 4.5].

We need the following notation that was introduced in the proof of [2, Theorem 2]. Let $m_1, \ldots, m_k$ be positive integers with $m_1 + \cdots + m_k = n$. We let $P(m_1, \ldots, m_k)$ denote the collection of tuples $(P_1, \ldots, P_k)$ such that $P_1, \ldots, P_k$ form a partition of $\{1, 2, \ldots, n\}$ where P_j has m_j elements, $j = 1, \ldots, k$.

Theorem 3.4. *Let*

$$c = (c_1, \ldots, c_1, c_2, \ldots, c_2, \;\ldots\ldots\; , c_k, \ldots, c_k),$$

where $c_1 > c_2 > \ldots > c_k$, c_j *appears* m_j *times,* $m_j \geq 1$, $\sum_{j=1}^{k} m_j = n$, *and let* $A \in \mathbb{C}^{n\times n}$. *Introduce the polynomial*

$$p_t(x) = \prod_{(P_1,\ldots,P_k)\in P(m_1,\ldots,m_k)} \left(x - \sum_{j=1}^{k} c_j \sum_{s\in P_j} \lambda_s(t) \right),$$

where $\lambda_k(t) = \lambda_k(\mathrm{Re}(e^{-it}A))$, $k = 1, \ldots, n$, *which are ordered non-increasingly. Then* $p_t(x)$ *is a trigonometric polynomial in* t *of degree at most* n *(i.e., of the form* $p_t(x) = \sum_{k=-n}^{n} e^{ikt}p_k(x)$*), which can be constructed using the coefficients of the characteristic polynomial of* $\mathrm{Re}\left(e^{-it}A\right)$. *Moreover,* $W_c(A)$ *equals the unit disk if and only if* $p_t(1) = 0$ *for all* $t \in \mathbb{R}$, *and* $p_t(x) > 0$ *for all* $x > 1$ *and all* $t \in \mathbb{R}$.

Proof. It is clear from the definition of $p_t(x)$ that the coefficients are symmetric polynomials in $\lambda_1(t), \ldots, \lambda_n(t)$. As is well known (see, e.g., [13]) every symmetric polynomial can be expressed as a polynomial in the power functions $p_k(\lambda_1(t), \ldots, \lambda_n(t)) = \sum_{i=1}^{n} \lambda_i(t)^k = \mathrm{Tr}\left(\mathrm{Re}\left(e^{-it}A\right)\right)^k$, $k = 1, \ldots, n$. But then it follows that the coefficients are trigonometric polynomials in t of degree at most n. This proves the first part.

For the second part, notice that the roots of $p_t(x)$ are $\sum_{j=1}^{k} c_j \sum_{s\in P_j} \lambda_s(t)$, which are indexed by $(P_1,\ldots,P_k) \in P(m_1,\ldots,m_k)$. Among all the roots, the largest is when the tuple $(P_1,\ldots,P_k)$ is the partition where $P_1 = \{1,\ldots,m_1\}$, $P_2 = \{m_1+1,\ldots,m_1+m_2\},\ldots,P_k = \{\sum_{l=1}^{k-1} m_l + 1,\ldots,\sum_{l=1}^{k} m_l\}$. This largest root being equal to 1 for all t corresponds exactly to $p_t(1) = 0$ for all $t \in \mathbb{R}$, and $p_t(x) > 0$ for all $x > 1$ and all $t \in \mathbb{R}$. Here we used that $p_t(x) \to \infty$ when $x \to \infty$. $\square$

Let us show in an example how the polynomial $p_t(x)$ can be constructed.

Example 3.5. Let

$$c = (1,1,0,0). \tag{7}$$

Then the characteristic polynomial of $\mathrm{Re}\left(e^{-it}A\right)$ equals

$$\prod_{i=1}^{4}(x-\lambda_i(t)) = \sum_{i=0}^{4}(-1)^{4-i}x^i e_{4-i}\left(\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t)\right) = \sum_{i=0}^{4}(-1)^{4-i}x^i e_{4-i}.$$

Using the calculations from [2, Equation (16)] (or, alternatively, [1, Theorem 4.2]), we have that

$$p_t(x) = \prod_{1\le r<s\le 4}(x-\lambda_r(t)-\lambda_s(t)) =: \sum_{i=0}^{6} k_i x^{6-i},$$

where $k_0 = 1$,

$$k_1 = -3e_1, k_2 = 2e_2 + 3e_1^2, k_3 = -4e_1e_2 - e_1^3, k_4 = -4e_4 + e_1e_3 + e_2^2 + 2e_1^2e_2,$$
$$k_5 = 4e_1e_4 - e_1^2e_3 - e_1e_2^2,\ k_6 = -e_1^2e_4 - e_3^2 + e_1e_2e_3.$$

Using the Newton identities, we may express the coefficients of $p_t(x)$ in terms of the power sum polynomials $p_j = p_j\left(\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t)\right)$ as follows

$$k_1 = -3p_1,\ k_2 = 4p_1^2 - p_2,\ k_3 = -3p_1^3 + 2p_1p_2,$$
$$k_4 = \frac{5}{4}p_1^4 - p_1^2p_2 - p_1p_3 - \frac{1}{4}p_2^2 + p_4,\ k_5 = -\frac{1}{4}p_1(p_1^4 - 4p_3p_1 - p_2^2 + 4p_4),$$
$$k_6 = \frac{1}{4}p_1^2p_4 - \frac{5}{18}p_1^3p_3 + \frac{1}{12}p_1^4p_2 - \frac{1}{9}p_3^2 + \frac{1}{72}p_1^6 - \frac{1}{8}p_1^2p_2^2 + \frac{1}{6}p_1p_2p_3.$$

Next, we use that

$$p_j = \mathrm{Tr}\left(\left(\mathrm{Re}\left(e^{-it}A\right)\right)^j\right) = \frac{1}{2^j}\mathrm{Tr}\left(\left(\frac{1}{z}A + zA^*\right)^j\right),$$

where we put $z = e^{it}$. This yields

$$k_1 = -\frac{3}{2}\left(\frac{1}{z}\mathrm{Tr}(A) + z\mathrm{Tr}(A^*)\right),$$

$$k_2 = \left(\frac{1}{z}\mathrm{Tr}(A) + z\mathrm{Tr}(A^*)\right)^2 - \frac{1}{4z^2}\mathrm{Tr}\left(A^2\right) - \frac{1}{2}\mathrm{Tr}(AA^*) - \frac{z^2}{4}\mathrm{Tr}\left(A^{*2}\right),$$

$$k_3 = \left(z\mathrm{Tr}(A^*) + \frac{1}{z}\mathrm{Tr}(A)\right)\left(\frac{1}{2}\mathrm{Tr}(AA^*) + \frac{1}{4z^2}\mathrm{Tr}\left(A^2\right) + \frac{z^2}{4}\mathrm{Tr}\left(A^{*2}\right)\right) - 3\left(\frac{z}{2}\mathrm{Tr}(A^*) + \frac{1}{2z}\mathrm{Tr}(A)\right)^3.$$

We omit the expressions for k_4, k_5 and k_6 in terms of the traces as these are very long, but can be determined in a similar way. Furthermore, from $p_t(1) = 0$ for all $t \in \mathbb{R}$ follows

$$0 = 1 + k_1 + k_2 + k_3 + k_4 + k_5 + k_6 =: \sum_{i=-6}^{6} a_i z^i,$$

where $a_{-i} = \overline{a}_i = 0$ for $0 \leq i \leq 6$. This yields

$$a_{-6} = \frac{1}{4608}\big(\mathrm{Tr}(A)^6 + 6\mathrm{Tr}(A)^4\mathrm{Tr}(A^2) - 20\mathrm{Tr}(A)^3\mathrm{Tr}(A^3) - 9\mathrm{Tr}(A)^2\mathrm{Tr}(A^2)^2 + 18\mathrm{Tr}(A^4)\mathrm{Tr}(A)^2 + 12\mathrm{Tr}(A)\mathrm{Tr}(A^2)\mathrm{Tr}(A^3) - 8\mathrm{Tr}(A^3)^2\big) = 0,$$

$$a_{-5} = \frac{1}{4608}\big(144\mathrm{Tr}(A^3)\mathrm{Tr}(A)^2 - 36\mathrm{Tr}(A)^5 + 36\mathrm{Tr}(A)\mathrm{Tr}(A^2)^2 - 144\mathrm{Tr}(A^4)\mathrm{Tr}(A)\big) = 0,$$

$$a_{-3} = \frac{1}{4608}\big(36\mathrm{Tr}(A^*)\mathrm{Tr}(A^2)^2 - 180\mathrm{Tr}(A^*)\mathrm{Tr}(A)^4 + 432\mathrm{Tr}(A)^2\mathrm{Tr}(A^*A^2) - 1728\mathrm{Tr}(A)^3 + 1152\mathrm{Tr}(A)\mathrm{Tr}(A^2) - 576\mathrm{Tr}(A)\mathrm{Tr}(A^3A^*) - 144\mathrm{Tr}(A^*)\mathrm{Tr}(A^4) + 144\mathrm{Tr}(A)\mathrm{Tr}(AA^*)\mathrm{Tr}(A^2) + 288\mathrm{Tr}(A^*)\mathrm{Tr}(A)\mathrm{Tr}(A^3)\big) = 0,$$

$$a_{-1} = \frac{1}{4608}\big(144\mathrm{Tr}(A)\mathrm{Tr}(AA^*)^2 - 5184\mathrm{Tr}(A^*)\mathrm{Tr}(A)^2 - 6912\mathrm{Tr}(A) + 432\mathrm{Tr}(A)^2\mathrm{Tr}(AA^{*2}) + 144\mathrm{Tr}(A^*)^2\mathrm{Tr}(A^3) - 360\mathrm{Tr}(A^*)^2\mathrm{Tr}(A)^3 + 1152\mathrm{Tr}(A^*)\mathrm{Tr}(A^2) + 2304\mathrm{Tr}(A)\mathrm{Tr}(AA^*) - 567\mathrm{Tr}(A^*)\mathrm{Tr}(A^3A^*) - 576\mathrm{Tr}(A)\mathrm{Tr}(A^{*2}A^2) - 288\mathrm{Tr}(A)\mathrm{Tr}(A^*AA^*A) + 72\mathrm{Tr}(A)\mathrm{Tr}(A^{*2})\mathrm{Tr}(A^2) + 144\mathrm{Tr}(A^*)\mathrm{Tr}(AA^*)\mathrm{Tr}(A^2) + 864\mathrm{Tr}(A^*)\mathrm{Tr}(A)\mathrm{Tr}(A^2A^*)\big) = 0.$$

We will not write down full expressions for a_{-4}, a_{-2} and a_0, as these are very long. However, in order to follow our computations for the remainder of this example, it is important to note that in every term of a_{-4} there is a factor $\mathrm{Tr}\left(A^k\right)$ or $\mathrm{Tr}\left(A^{*k}\right)$, and that

$$a_{-2} = \frac{1}{4}\mathrm{Tr}\left(A^3A^*\right) - \frac{1}{64}\left(\mathrm{Tr}\left(A^2A^*\right)\right)^2 + \text{ terms with a factor } \mathrm{Tr}\left(A^k\right) \text{ or } \mathrm{Tr}\left(A^{*k}\right),$$
$$a_0 = 1 - \frac{1}{32}\mathrm{Tr}\left(A^2A^*\right)\mathrm{Tr}\left(AA^{*2}\right) - \frac{1}{16}\left(\mathrm{Tr}\left(AA^*\right)\right)^2 + \frac{1}{8}\mathrm{Tr}\left(AA^*AA^*\right) - \frac{1}{2}\mathrm{Tr}\left(AA^*\right) + \frac{1}{4}\mathrm{Tr}\left(A^2A^{*2}\right) + \text{terms with a factor } \mathrm{Tr}\left(A^k\right) \text{ or } \mathrm{Tr}\left(A^{*k}\right).$$

Let us apply the above techniques to

$$A = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & \beta & \gamma \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

For this particular A we have that $a_{-6} = \alpha^6$. Setting this equal to 0, gives $\alpha = 0$. Now we find

$$a_{-2} = \frac{1}{4}\beta^3\bar{\delta} - \frac{1}{64}(2\bar{\gamma}\beta^2 + 2\gamma\beta\bar{\delta})^2,$$
$$a_0 = -\frac{1}{8}|\beta|^4|\gamma|^2 - \frac{1}{8}\beta^2\bar{\beta}\,\bar{\gamma}^2\delta - \frac{1}{8}\beta\,\bar{\beta}^2\gamma^2\bar{\delta} - \frac{1}{8}|\beta|^2|\gamma|^2|\delta|^2 + \frac{5}{16}|\beta|^4 + \frac{5}{4}|\beta|^2|\gamma|^2 + \frac{1}{8}|\beta|^2|\delta|^2 + \frac{1}{4}\beta\,\bar{\gamma}^2\delta + \frac{1}{4}\bar{\beta}\,\gamma^2\bar{\delta} + \frac{1}{4}|\delta|^2|\gamma|^2 + \frac{1}{16}|\delta|^4 - \frac{3}{2}|\beta|^2 - |\gamma|^2 - \frac{1}{2}|\delta|^2 + 1,$$

and $a_{-6} = a_{-5} = a_{-4} = a_{-3} = a_{-1} = 0$. Letting β, γ, δ be real, and solving for $a_{-2} = a_0 = 0$, we find for example the solution $\beta = \delta, |\gamma| = 1$. Letting $\beta = \delta$ and $\gamma = 1$, we find that

$$p_t(x) = x^6 + x^4\left(-2\beta^2 - 1\right) + 2x^2\beta^2\left(\frac{1}{4}\beta^2\cos(2t) + \frac{1}{4}\beta^2 + 1\right) - \frac{1}{2}\beta^4\left(\cos(2t) + 1\right)$$
$$= \left(x^2 - 1\right)\left(\frac{1}{2}\beta^4\cos(2t) + \frac{1}{2}\beta^4 - 2\beta^2x^2 + x^4\right).$$

Clearly, $p_t(1) \equiv 0$. Next, if we let $x > 1$, then using that $\cos(2t) \geq -1$, we obtain that

$$p_t(x) \geq \left(x^2 - 1\right)x^2\left(x^2 - 2\beta^2\right).$$

Thus, the requirement that $p_t(x) > 0$ for $x > 1$ gives that $|\beta| \leq \frac{1}{\sqrt{2}}$. In conclusion, for c as in (7) we have that

$$A = \begin{pmatrix} 0 & \beta & 1 & \beta \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $|\beta| \leq \frac{1}{\sqrt{2}}$, has the property that $W_c(A)$ is the unit disk. □

Making use of [2, Theorem 4], we obtain the following case where we characterize when $W_c(A)$ equals the unit disk.

Proposition 3.6. *Let*

$$c = (c_1, c_2, 0), \quad A = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & -\alpha & 0 \\ 0 & 0 & \delta \end{pmatrix},$$

with $c_1 \neq 0$, and put $f = \sqrt{2|\alpha|^2 + |\beta|^2 + |\gamma|^2}$. Then $W_c(A)$ equals the unit disk if and only if one of the following holds:

- $\delta = 0$, $\alpha^2 = -\beta\gamma$ *and* $\max\{|c_1 - c_2|, |c_1|, |c_2|\} f = 2$.
- $\delta \neq 0$, $\alpha^2 = -\beta\gamma$, $|c_1 - c_2| f = 2$, $|c_1\delta| + \frac{1}{2} f|c_2| \leq 1$, *and* $|c_2\delta| + \frac{1}{2} f|c_1| \leq 1$.

Proof. We use [2, Theorem 4] to rephrase the problem to the direct sum of three matrices having numerical range equal to the unit disk. Analyzing the latter we get the above result. Note that $B := \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ can only have a circular numerical range if and only if it is of rank 1 (i.e., $\alpha^2 = -\beta\gamma$), and in that case the radius is $f/2$ (as B is unitarily similar to $\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$; note that the Frobenius norm needs to be the same). □

4. Revisiting the circularity of the numerical range

It was derived in [14, Theorem 4.5] that for a complex matrix $B \in \mathbb{C}^{n\times n}$, its numerical range is the closed unit disk if and only if there exist $P_0, P_1 \in \mathbb{C}^{(n-1)\times n}$ so that $B = 2P_0^* P_1$ and $P_0^* P_0 + P_1^* P_1 = I_n$. Of course, we can also apply the results in the previous section to the standard numerical range by choosing

$$c = (1, 0, \dots, 0). \tag{8}$$

If we do this for the case $n = 3$, we get the following result.

Proposition 4.1. *Let $A \in \mathbb{C}^{3\times 3}$. If $W(A)$ equals the closed unit disk, then*

$$1 + \frac{1}{4}\mathrm{Tr}(A)\mathrm{Tr}(A^*) - \frac{1}{4}\mathrm{Tr}(AA^*) = 0,$$
$$-\frac{1}{2}\mathrm{Tr}(A) - \frac{1}{16}\mathrm{Tr}(A)^2\ \mathrm{Tr}(A^*) + \frac{1}{8}\mathrm{Tr}(A)\ \mathrm{Tr}(AA^*) + \frac{1}{16}\mathrm{Tr}(A^*)\mathrm{Tr}(A^2) - \frac{1}{8}\mathrm{Tr}(A^2A^*) = 0,$$
$$\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2) = 0,$$

and

$$-\frac{1}{6}\mathrm{Tr}(A)^3 + \frac{1}{2}\mathrm{Tr}(A)\mathrm{Tr}(A^2) - \frac{1}{3}\mathrm{Tr}(A^3) = 0.$$

Proof. If we apply Theorem 3.4 with a $c = (1, 0, 0)$, then we have that $p_t(x)$ is just the characteristic polynomial of $\operatorname{Re}\left(e^{-it}A\right)$, which equals

$$p_t(x) = x^3 - e_1x^2 + e_2x - e_3 = x^3 - p_1x^2 + \frac{1}{2}\left(p_1^2 - p_2\right)x - \left(\frac{p_1^3}{6} - \frac{p_1p_2}{2} + \frac{p_3}{3}\right).$$

Next we use that

$$p_1 = \frac{1}{2}\operatorname{Tr}\left(e^{-it}A + e^{it}A^*\right), \quad p_2 = \frac{1}{4}\operatorname{Tr}\left(e^{-it}A + e^{it}A^*\right)^2,$$
$$p_3 = \frac{1}{8}\operatorname{Tr}\left(e^{-it}A + e^{it}A^*\right)^3.$$

Extracting the coefficient in $p_t(1)$ that is constant in t, we get

$$1 + \frac{1}{4}\operatorname{Tr}(A)\operatorname{Tr}(A^*) - \frac{1}{4}\operatorname{Tr}(AA^*). \tag{9}$$

Extracting the coefficient of e^{-it} in $p_t(1)$ we obtain

$$-\frac{1}{2}\operatorname{Tr}(A) - \frac{1}{16}\operatorname{Tr}(A)^2\operatorname{Tr}(A^*) + \frac{1}{8}\operatorname{Tr}(A)\operatorname{Tr}(AA^*) + \frac{1}{16}\operatorname{Tr}(A^*)\operatorname{Tr}(A^2) - \frac{1}{8}\operatorname{Tr}(A^2A^*). \tag{10}$$

Extracting the coefficient of e^{-2it}, we get (up to a factor $\frac{1}{8}$)

$$\operatorname{Tr}(A)^2 - \operatorname{Tr}(A^2). \tag{11}$$

Extracting the coefficient of e^{-3it}, we get (up to a factor $\frac{1}{8}$)

$$-\frac{1}{6}\operatorname{Tr}(A)^3 + \frac{1}{2}\operatorname{Tr}(A)\operatorname{Tr}(A^2) - \frac{1}{3}\operatorname{Tr}(A^3). \tag{12}$$

Thus it follows that $p_t(1) \equiv 0$ if and only if (9)-(12) are all 0. □

It should be noticed that by using the techniques developed in this paper one can also find necessary conditions in terms of traces for when the first k higher rank numerical ranges are disks with center 0. Recall that the *rank-k numerical range* of a square matrix B is defined by

$$\Lambda_k(B) = \{\lambda \in \mathbb{C} : PBP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P\}.$$

This notion, which generalizes the classical numerical range when $k = 1$ and is motivated by the study of quantum error correction, was introduced in [4]. In [5,17] it was shown that $\Lambda_k(B)$ is convex. Subsequently, in [10] a different proof of convexity was given. In [14] the situation was analyzed when one of the eigenvalues $\lambda_k(\operatorname{Re}(e^{-it}A))$ is a constant function of t, which corresponds to the rank-k numerical range of A being a circle with center 0.

Now, if for instance one wants the numerical range and the 2-rank numerical range to be disks with center 0 then one has that $\lambda_1(t)$ and $\lambda_2(t)$ are constant, which is equivalent to $\lambda_1(t)$ and $\lambda_1(t)+\lambda_2(t)$ being constant. For the latter, one applies Theorem 3.4 to both

$$c = (1, 0, \dots, 0) \text{ and } c = (1, 1, 0, \dots, 0),$$

and combine both sets of necessary trace conditions.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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