On The Expressivity of the Many-Valued Interval-Based Temporal Logics

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Declaration

I hereby declare that the thesis titled "On the Expressivity of the Many-Valued Interval-Based Temporal Logic" belongs to Lesibana Ledwaba under the supervision of Professor Willem Conradie. I have referenced all the material used throughout the thesis.

Signature: ______________________

Date: 11 October 2022

Abstract

Interval-Based Temporal Logics take intervals over linear orders as the primary objects of temporal analysis. The are 13 relations between the intervals known as Allen's Relations on a linear order. We use Allen's relation as the accessibility relation between intervals and interpret the interval structures as Kripke frames.
One can think of Interval-Based Temporal Logics in a Many-Valued Interval setting where propositional variables are not just true or false but they are true or false to some extent and this extent we take as members of an algebra of truth values. Moreover, intervals can be taken to arise from many-valued linear orders.

In this thesis we consider the interdefinability of modalities in the many-valued interval setting. We define truth preserving morphisms that allow us to characterize the expressivity of many-valued interval-based temporal logic (MVIBTL). We use bismimulation as our primary truth preserving morphism and characterize which of the MVIBTL modalities are expressible in terms of the other modalities.

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Introduction

Interval reasoning can be found in various fields including philosophy, linguistics, computer science and artificial intelligence [17]. Interval temporal logic take intervals over linear orders as the primitive ontological entities. The binary relations between intervals are known as Allen's relations. There are 13 such relations namely: *equals, after, later, begin, end, during, overlap* and their inverses. Together they builds on *Allen's interval algebra*, which was introduced by James. F Allen in the the 1980s [2]. Allen was mainly interested in artificial intelligence and representation of knowledge [2].
Fuzzy modal logic was introduced by Fitting [14] and has been studied further ever since. In modal logic, two points are either related or not and propositional variables are either true or false at states. Fitting generalized modal logic where propositional variables can take a range of truth values and not just true or false as states and for two states in a frame, it is not the case that they are either related by the accessibility relation or not but they can be related to a certain extent and this extent comes from the same algebra of truth values.

The thesis studies the many-valued interval-based temporal logic (also called fuzzy interval-based temporal logic, or also Fuzzy Halpern-Shoham logic (Fuzzy HS)), which follows a similar approach to Fittings. Many-valued interval temporal logic was introduced by Willem Conradie, Dario Della Monica, Emilio Monoz-Velasco and Guido Sciavicco [8]. Many-valued interval-based temporal logic is a generalization of fuzzy interval-based temporal logic where the propositional variables are not just true or false but they are true or false to some extent and this extent we take as a member of algebra of truth values with Allen's relation as the accessibility relations. The Allen's relations are also many-valued as they are based on a many-valued linear order. The logic is built on a Heyting algebra $\mathcal{A}$ which acts as an algebra of truth values.

We study the expressive power of fuzzy HS fragments in term of their ability to define fuzzy different fuzzy HS modalities. We define two notions of truth-preserving morphism in the fuzzy setting and use them to show undefinability of FHS modalities.

The structure of the thesis is as follows. In Chapter 1 we provide the algebraic and order-theoretic definitions that are used throughout the thesis. In Chapter 2 we discuss modal logic, point based temporal logic and interval-based temporal logic and give an introduction to many-valued interval-based temporal logic. Chapter 3, 4 and 5 contains most of the original work done in the thesis. In Chapter 3 we construct the truth-preserving morphisms in the many-valued interval setting with Section 3.1 focusing on bisimulation and Section 3.2 focusing on generated submodels. In Chapter 4, we prove that the modalities $\langle D \rangle, \langle O \rangle$ and $\langle L \rangle$ are undefinable in fragments of fuzzy HS corresponding to fagment of HS where they are definable in the crisp case. In chapter 5 we develop the theory of interdefinability of modalities in many-valued interval-based temporal logic in more depth.

Chapter 1

Algebraic Preliminaries

This chapter covers some of the most important background that will be needed for the new work in the thesis. We give an introduction to lattices, Heyting algebras and linear orders. The three sections plays a major role in the later chapters of the thesis.

1.1 Lattices

The section introduces the definitions from universal algebra that will be needed in the theory of the thesis. We refer the reader to [22].

Definition 1.1. We say that the algebra $\mathbb{L} = (A, \wedge, \vee)$, having two operations called *'meet'* and *'join'* ($\vee$ and $\wedge$) on A is called a *lattice* if it satisfies the following identities:

L1: Commutative laws
(a) $x \vee y = y \vee x$;
(b) $x \wedge y = y \wedge x$;
L2: Associative laws
(a) $x \vee (y \vee z) = (x \vee y) \vee z$;
(b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
L3: Idempotent laws
(a) $x \vee x = x$;
(b) $x \wedge x = x$;
L4: Absorption laws
(a) $x = x \vee (x \wedge y)$;
(b) $x = x \wedge (x \vee y)$

Additionally, we say that an algebra $\mathbb{L} = (A, \wedge, \vee, 0, 1)$ having two binary operations and two nullary operations, denoted 0 and 1 (called "bottom" and "top", respectively) is a bounded lattice if

- $(A; \wedge; \vee)$ is a lattice
- $a \wedge 0 = 0$
- $a \vee 1 = 1$

Definition 1.2. A *distributive lattice* is a lattice which satisfies either of the distributive laws:
(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

Definition 1.3. Let $\preceq$ be a relation on set A. Then we say $\preceq$ is a partial order on A if the three conditions hold:

1. $a \preceq a$, *reflexivity*;
2. $a \preceq b$ and $b \preceq a$ implies $a = b$, *antisymmetry*;
3. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$, *transitivity*.

Let P be a non-empty set and define a partial order $\preceq$ on it, then $(P; \preceq)$ is called a partially ordered set, or a poset.

Let $(P, \preceq)$ be a poset, then the expression $a \prec b$ means that $a \preceq b$ and $a \neq b$. A partial order can be defined on any lattice L as follows: For all $a, b \in L$, $a \preceq b$ iff $a = a \wedge b$.

Definition 1.4. Let $(P, \preceq)$ be a poset and S with $S \subseteq P$, then S has an upper bound $u \in P$ if for every $a \in S$ then $a \preceq u$. S has a supremum $u \in P$ if the following two conditions hold:

1. S has an upper bound u,
2. $a \preceq b$ for every $a \in S$ implies $u \preceq b$.

S has a lower bound $l \in P$ if for every $a \in S$ then $l \preceq a$. We say that S has an infimum $l \in P$ if the following two conditions are hold:

1. S has a lower bound l,

2. $b \preceq a$ for every $a \in S$ implies $b \preceq l$.

For any lattice $\mathbb{L}$ and $a, b \in T$, we difine the operations $\vee$ and $\wedge$ as follows:

$$a \vee b = sup\{a, b\}$$

$$a \wedge b = inf\{a, b\}$$

We say

$$\bigvee S = sup(S)$$

is an infinite join and

$$\bigwedge S = inf(S)$$

infinite meet

Definition 1.5. A lattice $\mathbb{L}$ is *complete* if both $\bigvee S$ and $\bigwedge S$ exists for every $S \subseteq T$.

1.2 Heyting Algebra

The section introduces the definition of Heyting algebra and the types of Heyting algebra that we will focus on. We refer the reader to [22, 8] for further information.

Definition 1.6. A *Heyting algebra* [22] $\mathcal{A}$ is a tuple $(A, \wedge, \vee, \rightarrow, 0, 1)$ where A is non-empty domain (whose members we denote by $\alpha, \beta, \dots$), $\wedge, \vee$ and $\rightarrow$ are binary operations on A (called *meet*, *join* and *Heyting implication*) and 0 and 1 are designated elements of A such that the following conditions hold:

- $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
- $(\alpha \rightarrow \alpha) = 1$ for all $\alpha \in A$;
- $(\alpha \rightarrow \beta) \wedge \beta = \beta$ and $\alpha \wedge (\alpha \rightarrow \beta) = \alpha \wedge \beta$ for all $\alpha, \beta \in A$;
- $\alpha \rightarrow (\beta \wedge \gamma) = (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)$ and $(\alpha \vee \beta) \rightarrow \gamma = (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$ for all $\alpha, \beta, \gamma \in A$.

We restrict our attention to complete Heyting algebras, which is just a Heyting algebra in which $\bigvee S$ and $\bigwedge S$ exist for every subset of elements S.

In a complete Heyting algebra, it is well known that the Heyting implication may be expressed as follows:

$$a \to b = \bigvee \{c \mid a \wedge c \leq b, c \in \mathcal{A}\}$$

Remark 1.7. For all a in A,

1. $1 \to 0 = 0 \leq a$;
2. $0 \to a = 1$.

Definition 1.8. A *Heyting chain* is a linearly ordered Heyting algebra

Example 1.9. We now give an example each of a finite and infinite Heyting algebra.

1. $\mathcal{A} = (\{0 \prec \frac{1}{4} \prec \frac{1}{2} \prec 1\}, \vee, \wedge, \to, 0, 1)$ is a finite Heyting chain.

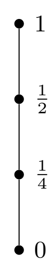


Figure 1.1: The operation $\vee$, $\wedge$ and $\to$ are defined as $a \vee b = max\{a, b\}$, $a \wedge b = min\{a, b\}$ and $a \to b = \begin{cases} 1, & a \leq b \\ b, & \text{b < a} \end{cases}$

2. $\mathcal{A} = (\{-\infty, \ldots, 0 \prec \frac{1}{4} \prec \frac{1}{2} \prec 1, \ldots, \infty\}, \vee, \wedge, \to, 0, 1)$ is an infinite Heyting chain.

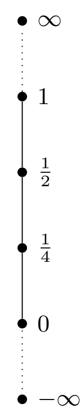


Figure 1.2: The operation $\vee$, $\wedge$ and $\rightarrow$ are defined as $a \vee b = max\{a, b\}$, $a \wedge b = min\{a, b\}$ and $a \rightarrow b = \begin{cases} \infty, & a \leq b \\ b, & \text{b < a} \end{cases}$

The following lemma collects some properties of Heyting algebras that will be used in the thesis and was taken from [22] and [9].

Lemma 1.10. Let $a, b, c \in \mathcal{A}$, where $\mathcal{A}$ is a Heyting algebra. Let $D \subseteq A$ ($D \neq \emptyset$) Then the following holds:
(1) $a \wedge (b \rightarrow c) = a \wedge ((a \wedge b) \rightarrow c)$;
(2) $a \rightarrow b = a \rightarrow (a \wedge b)$;
(3) $\bigwedge_{c \in D}(c \rightarrow a) = (\bigvee_{c \in D}) \rightarrow a$;
(4) $c \leq a \rightarrow b$ iff $c \wedge a \leq b$;
(5) $c \wedge (a \rightarrow b) = c \wedge ((c \wedge a) \rightarrow (c \wedge b))$;

1.3 Linear Orders and Fuzzy Linear Orders

The section introduces the definition of a linear order and the different types of linear orders that we will be needed in the thesis. We refer the reader to [9] for further information.

Definition 1.11. A *strict linear order* is a pair $(X, <)$ in which X is a non-empty set and $<$ is a binary relation satisfying the following properties:

1. Transitivity: if $a < b$ and $b < c$ then $a < c$ for $a, b, c \in X$.

2. Irreflexivity: $a < a$ does not hold for any $a \in X$.

3. Trichotomy: For $a, b \in X$, exactly one of the following holds
   i. $a < b$
   ii. $b > a$
   iii. $a = b$.

Definition 1.12. A relation $<$ is a *strict order* on a set X if it is

1. Irreflexive

2. Transitive

The following definitions are based on the fuzzy case.

Lemma 1.13. If $\mathcal{A}$ is a finite Heyting chain and $\bigvee_{i=0}^{n} a_i = c$, then there is $0 \leq i_0 \leq n$ such that $c = a_{i_0}$.

Proof. Suppose that $\bigvee_{i=0}^{n} a_i = c$. We procced by induction on n.
For $n = 0$, we have that $\bigvee a_0 = sup\{a_0\} = a_0$. Therefore $c = a_0$.

Assume that for $n = k$, $(\bigvee_{i=0}^{k} a_k) = a_{i_0}$ for some $0 \leq i_0 \leq k$.

We now consider the case $n = k + 1$.

$$\begin{aligned}
\bigvee_{i=0}^{k+1} a_{k+1} &= sup\{a_0, a_1, a_2, \ldots, a_{k+1}\} \\
&= (a_0 \vee a_1 \vee a_2 \vee \ldots \vee a_k) \vee a_{k+1} \\
&= a_{i_0} \vee a_{k+1} \\
&= sup\{a_{i_0}, a_{k+1}\} \\
&= max\{a_{i_0}, a_{k+1}\} && \mathcal{A} \text{ is linearly ordered}
\end{aligned}$$

□

The following definition introduces a notion of a fuzzy linear order which will be of fundamental importance in this thesis. The definition was first introduced on [8].

Definition 1.14. (Fuzzy Linear Orders)
We say that the structure $\tilde{\mathbb{D}} = \langle D, \tilde{<}, \tilde{=} \rangle$, with D a non-empty domain and $\tilde{<}, \tilde{=}$ given by

$$\tilde{<}, \cong : D \times D \to \mathcal{A}$$

and $\mathcal{A}$ a complete Heyting algebra, is an *$\mathcal{A}$-valued fuzzy linear order* if the following holds:

1. $\forall x(\tilde{=}(x,y) = 1 \Leftrightarrow x = y)$ (*reflexivity of* $\tilde{=}$) - we say that x and y are equal to the extent 1 if and only if they are the same point;
2. $\forall x, y(\tilde{=}(x,y) = \tilde{=}(y,x))$ (*symmetry of* $\tilde{=}$) - the extent to which x is equal to y is the same as the extent to which y is equal to x ;
3. $\forall x(\tilde{<}(x,x) = 0)$ (*irreflexivity of* $\tilde{<}$) - the extent to which x is less than itself is 0 ;
4. $\forall x, y, z(\tilde{<}(x,z) \succeq \tilde{<}(x,y) \wedge \tilde{<}(y,z))$ (*transivity of* $\tilde{<}$) - the extent to which x is less than z is greater than or equal to the extent to which x is less than y and the extent to which y is less than z ;
5. $\forall x, y, z(\tilde{<}(x,y) \succ 0 \;\&\; \tilde{<}(y,z) \succ 0 \Rightarrow \tilde{<}(x,z) \succ 0)$ (*transfer of positivity of* $\tilde{<}$) - if x is less than y to some positive extent and y is less than z to some positive extent then x is less than z to some positive extent;
6. $\forall x, y(\tilde{<}(x,y) = 0 \;\&\; \tilde{<}(y,x) = 0 \Rightarrow \tilde{=}(y,x)) = 1$ ((*weak*) *totality*) - if the extent to which x is less than y is 0 and the extent to which y is less than x is 0 then y and x are equal to the extent 1 ;
7. $\forall x, y(\tilde{=}(x,y) \succ 0 \Rightarrow \tilde{<}(x,y) \prec 1)$ (*non-contradiction of* $\tilde{<}$ *over* $\tilde{=}$) - if the extent to which x is equal to y is non-zero then the extent to which x is less then y is less than 1.

Definition 1.15. A fuzzy linear order $\mathbb{D} = \langle D, \tilde{<}, \tilde{=}\rangle$ is dense if for all $p, q \in \mathbb{D}$ such that $\tilde{<}(p,q) \succ 0$, there exist a $t \in \mathbb{D}$ such that $\tilde{<}(p,t) \succ 0$ and $\tilde{<}(t,q) \succ 0$.

Chapter 2

Modal Logic and Temporal Logic

In this chapter we introduce modal logic, point-based temporal logic and interval-based temporal logic which lays the foundation needed in the introduction of many-valued interval-based temporal logic.

2.1 Modal Logic, Frames and Models

This section aims to give a brief overview of modal logic and the mathematical structures that we use to interpret these modal languages, namely *models* and *frames*. Models are important as they help us define the notion of 'truth' and frames help us to establish the validity of these modal formulas. We refer the reader to [4] for further details.

Definition 2.1. The *basic modal logic language* is defined using a set of *proposition letters* $\mathcal{AP}$ whose elements are usually denoted p, q, r, and so on, and a unary operator $\Diamond$ and Boolean connectives. The formulas ϕ of the modal language are given by the rule

$$\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \theta) \mid \Diamond\phi$$

where p ranges over elements of $\mathcal{AP}$.

The definition means that a formula can either be a propositional letter, the propositional constant falsum, a negation of a formula, a disjunction of formulas, or a formula prefixed by a diamond. The dual of $\Diamond$ is defined as follows $\Box\phi := \neg\Diamond\neg\phi$ and called $\Box$ (box). We also include the classical connectives namely conjunction ($\wedge$), implication ($\rightarrow$) and constant true ($\top$) defined as follows: $\phi \wedge \theta = \neg(\neg\phi \vee \neg\theta)$, $\phi \rightarrow \theta = \neg\phi \vee \theta$ and $\top := \neg\bot$.

Definition 2.2. A *Kripke frame* for the basic modal language is a pair $\mathfrak{F} = (W, R)$ such that

1. W is a non-empty set
2. R is a binary relation on W.

Elements of W will sometimes be referred to as states.

Definition 2.3. A *model* for the basic modal language is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F}$ is Kripke frame and V is a function assigning to each propositional letter p in $\mathcal{AP}$ a subset $V(p)$ of W.

In short, a model is a frame with an assignment.

Definition 2.4. Suppose w is a state in a model $\mathcal{M} = (\mathcal{F}, V)$. Then we define the notion of ϕ being *satisfied* (or *true*) in M at state w as follows:
$\mathcal{M}, w \models p$ iff $w \in V(p), p \in \Phi$,
$\mathcal{M}, w \models \bot$ never,
$\mathcal{M}, w \models \neg\phi$ iff not $M, w \models \phi$,
$\mathcal{M}, w \models \phi \vee \theta$ iff $M, w \models \phi$ or $\mathcal{M}, w \models \theta$,
$\mathcal{M}, w \models \Diamond\phi$ iff for some $v \in W$, with Rwv we have $M, v \models \phi$.

It follows from the definition that $M, w \models \Box\phi$ iff for all $v \in W$ such that Rwv, we have $M, w \models \phi$,
$\mathcal{M}, w \models \phi \wedge \theta$ iff $M, w \models \phi$ and $\mathcal{M}, w \models \theta$,
$\mathcal{M}, w \models \top$ always.

Example 2.5. Consider the frame $\mathcal{F} = \langle\{1, 2, 4, 6, 12\}, R\rangle$, where Rxy iff y can be divided by x. We are omitting the arrows from the diagram that are implied by transitivity. We are working with proper divisibility and hence the irreflexive frame.

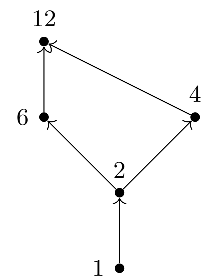


Figure 2.1

Let $V(p) = \{1, 2, 12\}, V(q) = \{2, 4, 6, 12\}$. Then from $\mathcal{M} = \langle \mathcal{F}, V \rangle$, we evaluate the following formula:

$\mathcal{M}, 1 \models p \wedge \Diamond(p \wedge \Box q)$

Proof. The successors of 2 are 4,6,12 and q is true in all of them, moreover 2 satisfies p and so $\mathcal{M}, 2 \models p \wedge \Box q$. So at 1, we have $\Diamond(p \wedge \Box q)$ true and 1 also satisfies p so we have $\mathcal{M}, 1 \models p \wedge \Diamond(p \wedge \Box q)$ true. $\Box$

Definition 2.6. A formula ϕ is *valid* at a state w in a frame $\mathcal{F}$ if ϕ is true at w in every model $(\mathcal{F}, \mathcal{V})$ based on $\mathcal{F}$.

Example 2.7. The formula given by $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$ satisfies validity in all frames.
Choose any frame $\mathcal{F}$ and a state $w \in \mathcal{F}$ and any valuation V on $\mathcal{F}$. We want to show that if $(\mathcal{F}, V), w \models \Box(p \wedge q)$ then $(\mathcal{F}, V), w \models \Box p \wedge \Box q$. Assume that the following formula is true, i.e, $(\mathcal{F}, V), w \models \Box(p \wedge q)$. Using the definition, it follows that for all v and Rwv we have that $(\mathcal{F}, V), v \models p \wedge q$. But, if $v \models p \wedge q$ then we have that $v \models p$ and $v \models q$. Hence we have $w \models \Box p$ and $w \models \Box q$. Finally, $w \models \Box p \wedge \Box q$.

2.2 Point Based Temporal Logics

With the point-based temporal logic, we take points as our primary objects when thinking about time. We capture this mathematically with Kripke frames where the states represents time instants and the relation the flow of time. Every point is related by the relation to the points that come after it in the flow of time. In general, the languages discussed in the thesis can be interpreted on any Kripke frame, but since we want to reason about time, it is reasonable to assume that the frames are trees or linear orders. We refer the reader to [17] for more details.

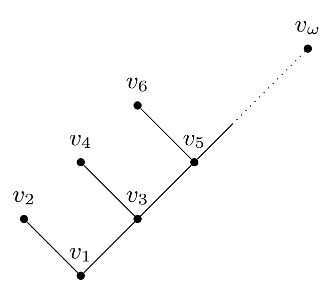
An example of a temporal logic with a linear order is the Linear Time Logic (LTL). LTL was first introduced by Amir Pnueli in 1977 [13]. It is an infinite sequence of states with each point having a unique successor [15].

Examples of temporal logics where frames are assumed to be trees are Priorean, Peircean, Ockhamist, Computation Tree Logic (CTL) and Full Computation Tree Logic (CTL*). The Peircean temporal language can be regarded as a proper fragment of the Ockhamist temporal language [17, 15]. CTL was introduced by E. A. Emerson and E. M. Clarke in 1981 [13]. CTL can be regarded as the Peircean temporal language on the class of computational trees excluding the past operator [17, 15]. CTL* was introduced by E. A. Emerson and Joseph Y. Halpern in 1983 [13] and is just the Ockhamist extention of CTL over a class of trees [17, 15]. LTL and CTL are just the fragment of the CTL*.

Definition 2.8. A tree is a strict order $T = (T, <)$ with the *left-linearity* property,i.e, if $x < y$ and $z < y$ then either $x < z$, $z < y$ or $x = z$ and is connected, i.e, $\forall x, y \in T$ there is a $z \in T$ such that $z \leq x$ and $z \leq y$.

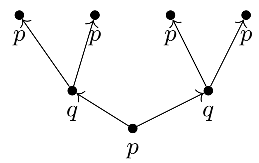
Definition 2.9. A *path* in tree is the subset T linearly ordered by $<$ and is maximal for inclusion. i.e, all the elements of T that succeeds one another form a branch.

Example 2.10. The following figure gives an example of a tree with an infinite path.



(a)

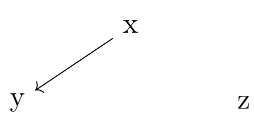
Figure 2.2: Example of a tree with an infinite path



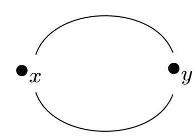
(a)

Figure 2.3: Example of a tree

The structure in figure 2.3 is a tree since it satisfies all the conditions, i.e, left-linearity and connectedness.



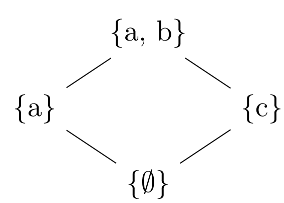
(a)



(b)

Figure 2.4: Examlple of structures which fail to be trees

The structure in figure 2.4(a) fail to be a tree because it is not connected and the structure in figure 2.4(b) fail to be a tree because by the transitivity property, it implies that x is reflexity, thus failing the irreflexive property.



(a)

Figure 2.5: Example of a structure which is not a tree

the structure in figure 2.5 fail to be a tree since it does not satisfy the left-linearity property.

Definition 2.11. A temporal model is a triple $M = (T, <, V)$ where $(T, <)$ is a tree and V is a valuation assigning to each propositional letter a set of time instants $V(p) \subseteq T$ at which p is declared true.

Definition 2.12. Given any model $M = (T, <, V)$ and any moment $t \in T$, the truth condition for F and P are
$\mathcal{M}, w \models p$ iff $w \in V(p), p \in \Phi$,
$\mathcal{M}, w \models \perp$ never,
$\mathcal{M}, w \models \neg\phi$ iff not $M, w \models \phi$,
$\mathcal{M}, w \models \phi \vee \theta$ iff $M, w \models \phi$ or $\mathcal{M}, w \models \theta$,
$M, t \models F\phi$ iff $M, t' \models \phi$ for some $t' > t$,
$M, t \models P\phi$ iff $M, t' \models \phi$ for some $t' < t$.

The F in the formula means that it will at some time be the case that ϕ is true and P means the it has at some time been the case that ϕ was true.

Example 2.13. Consider the tree $\mathcal{T} = \langle\{1, 2, 4, 6, 12\}, <\rangle$, where $< xy$ iff y can be divided by x. Let $\mathcal{M} = \langle\mathcal{F}, V\rangle$. We are omitting the arrows from the diagram that are implied by transitivity.

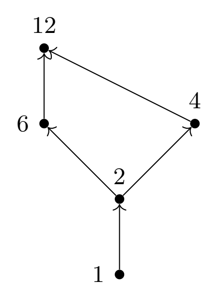


Figure 2.6

Let $V(p) = \{1, 2, 12\}, V(q) = \{2, 4, 6, 12\}$. Then

Proof. 2 is a successor of 1 where both p or q are true. So the formula $F(p \vee q)$ is true and thus $\mathcal{M}, 1 \models F(p \vee q)$ is also true. □

2.3 Interval Based Temporal Logics

With Interval-Based Temporal Logics, which is our main interest of study we take intervals as our primary objects. We take a linear order and look at the relation between the intervals. These relations are known as Allen's Relations [2]. Depending on our conviction about time, we can either include point-intervals or exclude them. We use Allen's relation as the accessibility relation between intervals and interpret the interval structures as Kripke frames.We refer the reader to [16] for more details.

Definition 2.14. Given a strict linear order $\mathbb{D} = \langle D, < \rangle$, an *interval* in $\mathbb{D}$ is a pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 \leq d_1$. $[d_0, d_1]$ is a strict interval if $d_0 < d_1$. An interval is said to be a point interval if $d_0 = d_1$.

The set of all strict intervals on $\mathbb{D}$ is dentoted by $\mathbb{I}(\mathbb{D})^-$ and the set of all non-strict intervals is denoted by $\mathbb{I}(\mathbb{D})^+$. We will denote either of these by $\mathbb{I}(\mathbb{D})$.
The set of *strict* intervals does not include the point intervals and the set of *non-strict* intervals includes the point intervals.

A *non-strict interval model* is a pair $\mathcal{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle$, where $\mathbb{D}$ is a strict linear order and V is a valuation assigning to each propositional letter a set of intervals $V(p) \subseteq \mathbb{I}(\mathbb{D})^+$ at which p is declared true. Respectively, a *strict interval model* is a pair $\mathcal{M}^- = \langle \mathbb{I}(\mathbb{D})^-, V \rangle$, where V is defined likewise. So V is a map $V : \mathcal{AP} \to P(\mathbb{I}(\mathbb{D}))$, where $P(\mathbb{I}(\mathbb{D}))$ is a powerset of $\mathbb{I}(\mathbb{D})$.

There are 13 different binary relations between intervals on a linear ordering [2] namely *equals, ends, during, begins, meets, overlaps, before* , together with their inverses, called the Allen's relations as shown in figure 2.7.

HS	**Allen's relations**
$\langle A \rangle$	$[x, y]R_A[x', y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x, y]R_L[x', y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x, y]R_B[x', y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x, y]R_E[x', y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x, y]R_D[x', y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x, y]R_O[x', y'] \Leftrightarrow x < x' < y < y'$

Graphical representation

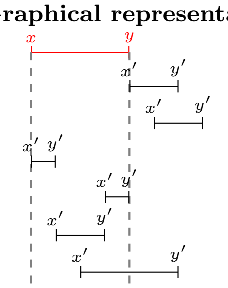


Figure 2.7: Allen's interval relations.

Definition 2.15. The formulas of Interval-Based temporal logic can be generated by the following syntax

$$\phi ::= \perp \mid p \mid \neg\phi \mid \phi \vee \theta \mid \phi \wedge \theta \mid \langle X \rangle \phi \mid \langle \bar{X} \rangle \phi$$

where $\langle X \rangle$ is existential modality and $X \in \{A, L, B, E, D, O\}$ and $\langle \bar{X} \rangle$ is the inverse of $\langle X \rangle$.

Likewise, we can define the dual of $\langle X \rangle$ which is the universal modality $[X]$ by $[X]\phi := \neg\langle X\rangle\neg\phi$, which is the converse of $[X]$.
We associate the universal and the existential modalities with each Allen's relations R_X.
The formal semantics of these modal operators is defined as follows in the non-strict semantic:

1. $\mathcal{M}^+, [d_0, d_1] \Vdash \langle X \rangle \phi$ if there is an interval $[d_2, d_3]$ such that $[d_0, d_1]R_X[d_2, d_3]$ and $\mathcal{M}^+, [d_2, d_3] \Vdash \phi$;

2. $\mathcal{M}^+, [d_0, d_1] \Vdash \left\langle \bar{X} \right\rangle \phi$ if there is an interval $[d_2, d_3]$ such that $[d_2, d_3]R_X[d_0, d_1]$ and $\mathcal{M}^+, [d_2, d_3] \Vdash \phi$;

Consequently, the formal semantics of the box is as follows:

1. $\mathcal{M}^+, [d_0, d_1] \Vdash [X]\phi$ if for all intervals $[d_2, d_3]$ such that $[d_0, d_1]R_X[d_2, d_3]$, it is the case that $\mathcal{M}^+, [d_2, d_3] \Vdash \phi$;

2. $\mathcal{M}^+, [d_0, d_1] \Vdash [\bar{X}]\phi$ if for all intervals $[d_2, d_3]$ such that $[d_2, d_3]R_X[d_0, d_1]$, it is the case that $\mathcal{M}^+, [d_2, d_3] \Vdash \phi$.

For the strict semantics we replace the $+$ with $-$. Some of these modalities can be expressed in terms of the other. For example, one can choose the modalities corresponding to *begin*, *end* and their inverses given by the syntax

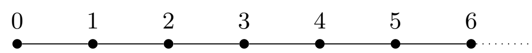
$$\phi ::= p \mid \neg\phi \mid \phi \wedge \theta \mid \langle B \rangle \phi \mid \langle E \rangle \phi \mid \left\langle \bar{B} \right\rangle \phi \mid \left\langle \bar{E} \right\rangle \phi$$

Their non-strict semantics is defined as follows:

1. $\mathcal{M}^+, [d_0, d_1] \Vdash \langle B \rangle \phi$ if $\mathcal{M}^+, [d_0, d_2] \Vdash \phi$ for some d_2 such that $d_0 \leq d_2 < d_1$;

2. $\mathcal{M}^+, [d_0, d_1] \Vdash \langle E \rangle \phi$ if $\mathcal{M}^+, [d_2, d_1] \Vdash \phi$ for some d_2 such that $d_0 < d_2 \leq d_1$;

Example 2.16. Consider the following model $\tilde{\mathcal{M}} = \langle \mathbb{I}(\mathbb{N}), \tilde{V} \rangle$ with valuation $V(p) = \{[1,2],[3,4],[5,6]\}$



Then it follows that

1. $\mathcal{M}^+, [1,3] \models \langle B \rangle p$
   Indeed, $\mathcal{M}^+, [1,2] \models p$ and $[1,3]R_B[1,2]$.
2. $\mathcal{M}^+, [1,5] \models \langle E \rangle \langle B \rangle p$
   Indeed, $\mathcal{M}^+, [3,5] \models \langle B \rangle p$ and $[1,5]R_E[3,5]$. Also, $\mathcal{M}^+, [3,4] \models p$ and $[3,5]R_B[3,4]$.
3. $\mathcal{M}^+, [2,5] \models \langle D \rangle p$
   Indeed, $\mathcal{M}^+, [3,4] \models p$ and $[2,5]R_B[3,4]$.

It should be clear that Interval-Based temporal logic is a special type of modal logic in which the Kripke frames consist of the intervals over a given linear order and the relation is given by a selection from among Allen's relations.

For a given language, we define a fragment ($\mathcal{F}$) as picking a set of modalities and index the modalities that occur in the language and keeping the propositional connectives fixed. If we given a class of linear orders, we say that two formulae ϕ and θ are equivalent relative to the class of linear orders, denoted by $\phi \equiv \theta$ if $\mathcal{M}, [a,b] \models \phi \leftrightarrow \theta$ for all $[a,b] \in \mathbb{I}(\tilde{\mathbb{D}})$.

Let $Var(\phi)$ be the set of all propositional variables occurring in ϕ and $\mathcal{A}$ be a Heyting algebra. Then we say that modality $\langle X \rangle$ is *definable* in a fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of $\mathcal{A}$-fuzzy linear orders, denoted $\langle X \rangle \lhd^{\mathcal{A}}_{\mathcal{C}} \mathcal{F}$, if $\langle X \rangle p \equiv_{\mathcal{C}} \phi$ for some $\mathcal{F}$-formula ϕ and for all p such that $Var(\phi) = \{p\}$.

Given a class of linear orders, we say that

- $\mathcal{F}_1 \preceq \mathcal{F}_2$ ($\mathcal{F}_2$ is at least as expressive as $\mathcal{F}_1$) if each modality $\langle X \rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$.

- $\mathcal{F}_1 \prec \mathcal{F}_2$ ($\mathcal{F}_1$ is strictly less expressive than $\mathcal{F}_2$) if $\mathcal{F}_1 \preceq \mathcal{F}_2$ holds but $\mathcal{F}_2 \preceq \mathcal{F}_1$ does not hold.
- $\mathcal{F}_1 \equiv \mathcal{F}_2$ ($\mathcal{F}_1$ is equally expressive as $\mathcal{F}_2$) if both $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$ hold.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are expressivle incomparable if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$ hold.

One of the focus of the thesis is the study of the interdefinability of HS modalities in a many-valued setting, over the class of all fuzzy linear orders and over the class of all dense fuzzy linear orders, which will be discussed further in Chapter 5.

2.4 Many Valued Interval Temporal Logics

So far, we have looked at the point-based temporal logic and the interval-based temporal logic. One can think of these two logics in a many-valued interval setting where propositional letters are not just true or false but they are true or false to some extent and this extent we take as a member of some algebra of truth values. Moreover, intervals can be taken to arise from many-valued linear orders. We refer the reader to [8] for further details.

Definition 2.17. Given a set of propositional letters $\mathcal{AP}$ and a complete Heyting algebra $\mathcal{A}$, we define the language of the many-valued interval-based temporal logic (MVIBTL) $\mathcal{L}_S^{\mathcal{A}}(\mathcal{AP})$ by the following syntax

$$\phi ::= \alpha \mid p \mid \phi \vee \theta \mid \phi \wedge \theta \mid \phi \to \theta \mid \langle X \rangle \, \phi \mid [X]\phi$$

where $\alpha \in \mathcal{A}, p \in \mathcal{AP}, S \subseteq \{A, L, B, E, D, O, \bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\}$ and $X \in S$.

Given a many-valued strictly linearly ordered set $\mathbb{D}$, we define the set of many-valued strict intervals in $\tilde{\mathbb{D}}$ as

$$\mathbb{I}(\tilde{\mathbb{D}}) = \{[x, y] \mid \tilde{<}(x, y) \succ 0\}$$

The following definition defines Allen's fuzzy relations, which is obtained by generalizing the original, crisp definition and substituting every $=$ with $\tilde{=}$ and $<$ with $\tilde{<}$.

Definition 2.18. Let $[x, y], [z, t] \in \mathbb{I}(\tilde{\mathbb{D}})$. The Allen's fuzzy relations are defined as follows:

- $\tilde{R}_A([x,y],[z,t]) = \tilde{=}(y,z)$;
- $\tilde{R}_L([x,y],[z,t]) = \tilde{<}(y,z)$;
- $\tilde{R}_B([x,y],[z,t]) = \tilde{=}(x,z) \wedge \tilde{<}(t,y)$;
- $\tilde{R}_E([x,y],[z,t]) = \tilde{<}(x,z) \wedge \tilde{=}(y,t)$;
- $\tilde{R}_D([x,y],[z,t]) = \tilde{<}(x,z) \wedge \tilde{<}(t,y)$;
- $\tilde{R}_O([x,y],[z,t]) = \tilde{<}(x,z) \wedge \tilde{<}(z,y) \wedge \tilde{<}(y,t)$.

Definition 2.19. An $\mathcal{A}$-*valued interval model* (fuzzy interval model) is a tuple of the type

$$\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$$

where $\tilde{\mathbb{D}}$ is a many-valued linearly ordered set and $\tilde{V}$ is a valuation defined by

$$\tilde{V} := \mathcal{AP} \times \mathbb{I}(\tilde{\mathbb{D}}) \to \mathcal{A}$$

evaluating propositional letters in $\mathcal{A}$. We also use $\mathcal{A}$-model to refer to the fuzzy interval model.

Please note that the same Heyting algebra is used in the definition of $\tilde{\mathbb{D}}$ and $\tilde{\mathbb{V}}$.

We interpret the formulas of the MVBTL in a many-valued interval temporal model $\tilde{\mathbb{M}}$ and an interval $[a,b]$ by extending the valuation $\tilde{V}$ of propositional letters as follows, where $X \in \{A, L, B, E, D, O\}$:

1. $\tilde{V}(\alpha, [a,b]) = \alpha$;
2. $\tilde{V}(\phi \wedge \theta, [a,b]) = \tilde{V}(\phi, [a,b]) \wedge \tilde{V}(\theta, [a,b])$;
3. $\tilde{V}(\phi \vee \theta, [a,b]) = \tilde{V}(\phi, [a,b]) \vee \tilde{V}(\theta, [a,b])$;
4. $\tilde{V}(\phi \to \theta, [a,b]) = \tilde{V}(\phi, [a,b]) \to \tilde{V}(\theta, [a,b])$;
5. $\tilde{V}(\langle X \rangle \phi, [a,b]) = \bigvee_{[x,y] \in \mathbb{I}(\tilde{\mathbb{D}})} \{\tilde{R}_X([a,b],[x,y]) \wedge \tilde{V}(\phi, [x,y])\}$;
6. $\tilde{V}([X]\phi, [a,b]) = \bigwedge_{[x,y] \in \mathbb{I}(\tilde{\mathbb{D}})} \{\tilde{R}_X([a,b],[x,y]) \to \tilde{V}(\phi, [x,y])\}$.

The following example shows how to evaluate the formulas of the MVIBTL.

Example 2.20. Consider the following model $\tilde{\mathcal{M}} = \langle \mathbb{I}(\tilde{\mathbb{Z}}), \tilde{V} \rangle$ with valuation $\tilde{V}(p, [1,2]) = 1$ and $\tilde{V}(p, [x,y]) = 0$ for all other $[x,y] \in \mathbb{I}(\tilde{\mathbb{Z}})$ given by the following figure.

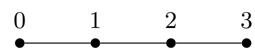


Figure 2.8

Let $\mathcal{A}$ be a three element Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$.

Define $\tilde{=}$ and $\tilde{<}$ as follows:
$\tilde{<}(x,y) = min\{1, max\{\frac{1}{2}(y-x), 0\}\}$;
$\tilde{=}(x,y) = max\{0, 1 - \frac{1}{2} \mid x - y \mid\}$.

1.

$$\begin{aligned}
\tilde{V}(\langle B \rangle p, [0,2]) &= \bigvee_{[x,y] \in \mathbb{I}(\tilde{\mathbb{Z}})} \{\tilde{R}_B([0,2],[x,y]) \wedge \tilde{V}(p,[x,y])\} \\
&\succeq \tilde{R}_B([0,2],[1,2]) \wedge \tilde{V}(p,[1,2]) \\
&= \tilde{=}(0,1) \wedge \tilde{<}(2,2) \wedge \tilde{V}(p,[1,2]) \\
&= \frac{1}{2} \wedge 0 \wedge 1 \\
&= 0
\end{aligned}$$

2.

$$\begin{aligned}
\tilde{V}(\langle D \rangle p, [0,3]) &= \bigvee_{[x,y] \in \mathbb{I}(\tilde{\mathbb{Z}})} \{\tilde{R}_D([0,3],[x,y]) \wedge \tilde{V}(p,[x,y])\} \\
&\succeq \tilde{R}_D([0,3],[1,2]) \wedge \tilde{V}(p,[1,2]) \\
&= \tilde{=}(0,1) \wedge \tilde{<}(2,3) \wedge \tilde{V}(p,[1,2]) \\
&= \frac{1}{2} \wedge \frac{1}{2} \wedge 1 \\
&= \frac{1}{2}
\end{aligned}$$

3.

$$\begin{aligned}
\tilde{V}(\langle B\rangle\langle E\rangle p,[0,2]) &= \bigvee_{[x,y]\in\mathbb{I}(\tilde{\mathbb{Z}})} \{\tilde{R}_B([0,2],[x,y]) \wedge \bigvee_{[z,t]\in\mathbb{I}(\tilde{\mathbb{Z}})} \tilde{R}_E([x,y],[z,t]) \wedge \tilde{V}(p,[z,t])\} \\
&\succeq \tilde{R}_B([0,2],[0,1]) \wedge \tilde{R}_E([0,1],[1,2]) \wedge \tilde{V}(p,[1,2]) \\
&= \tilde{=}(0,0) \wedge \tilde{<}(1,2) \wedge \tilde{<}(0,1) \wedge \tilde{=}(1,2) \wedge \tilde{V}(p,[1,2]) \\
&= 1 \wedge \frac{1}{2} \wedge \frac{1}{2} \wedge \frac{1}{2} \wedge 1 \\
&= \frac{1}{2}
\end{aligned}$$

Chapter 3

Truth-Preserving Model-Theoretic Constructions

The section introduces the truth-preserving morphisms that will be needed in the theory of the thesis. We construct two generalized truth-preserving morphisms, namely bisumulation and generated-submodels from modal logic and temporal logic. Truth-preserving morphisms allow us to study the expressity of the languages by showing that which properties they are not define. [12] introduces truth-preserving model theoretic construction in the context of many-valued model logic and the notion of such a construction parameterised by a truth value.

3.1 Bisimulation

The concept of bisimulation allows us to move back and forth in $\mathcal{A}$-valued interval mode by matching steps in both directions.
The following definition has been modified from [12] for MVIBTL. Note that we use $\perp$ to denote 0.

Definition 3.1. (α-invariance)

Let $\tilde{\mathcal{M}} = \left\langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \right\rangle$ and $\tilde{\mathcal{M}}' = \left\langle \mathbb{I}(\tilde{\mathbb{D}}'), \tilde{V}' \right\rangle$ be two $\mathcal{A}$-models, $[a, b] \in \mathbb{I}(\tilde{\mathbb{D}})$ and $[a', b'] \in \mathbb{I}(\tilde{\mathbb{D}})$ be two intervals and $\alpha \in \mathcal{A}$ ($\alpha \neq \perp$). We say that modal truth is *α-invariant for the transition between $[a, b]$ and $[a', b']$* if for every formula ϕ

$$\alpha \wedge \tilde{V}(\phi, [a, b]) = \alpha \wedge \tilde{V}'(\phi, [a', b'])$$

Definition 3.2. . Let $\mathcal{A}$ be a complete Heyting algebra and $S \subseteq \langle A, L, B, E, D, O, \bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\rangle$. Given two $\mathcal{A}$-models $\tilde{\mathcal{M}} = \left\langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \right\rangle$ and $\tilde{\mathcal{M}}' = \left\langle \mathbb{I}(\tilde{\mathbb{D}}'), \tilde{V}' \right\rangle$ and a truth value $\alpha \in \mathcal{A}$ ($\alpha \neq \bot$). A non-empty relation $\tilde{\mathcal{Z}} \subseteq \mathbb{I}(\tilde{\mathbb{D}}) \times \mathbb{I}(\tilde{\mathbb{D}}')$ is α_S-bisimuation between $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}'$ if for any pair $\langle [a,b], [a',b'] \rangle \in \tilde{\mathcal{Z}}$

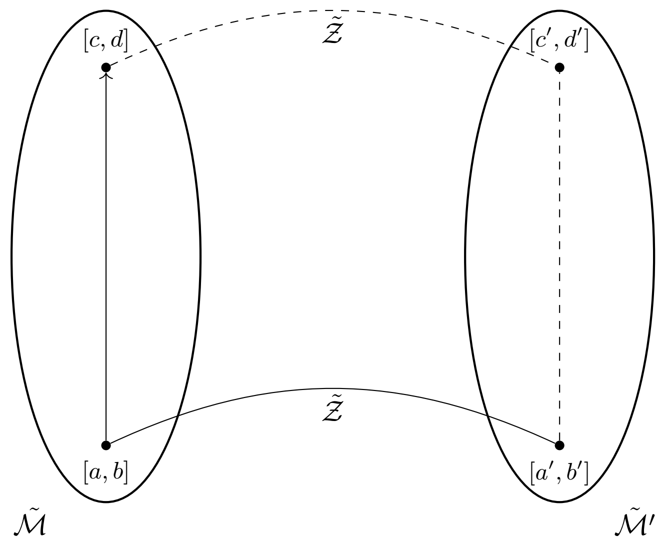
1. *base:* $\alpha \wedge \tilde{V}(p, [a,b]) = \alpha \wedge \tilde{V}'(p, [a',b'])$, for all $p \in \mathcal{AP}$;

2. *forward condition:* For every $([c,d]) \in \mathbb{I}(\tilde{\mathbb{D}})$ and $X \in S$ such that $\alpha \wedge \tilde{R}_X([a,b],[c,d]) \neq \bot$, there exist $[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\alpha \wedge \tilde{R}_X([a,b],[c,d]) = \alpha \wedge \tilde{R}'_X([a',b'],[c',d'])$ and $\langle [c,d],[c',d'] \rangle \in \tilde{\mathcal{Z}}$;

3. *backward conditions:* For every $([c',d']) \in \mathbb{I}(\tilde{\mathbb{D}}')$ and $X \in S$ such that $\alpha \wedge \tilde{R}'{}_X([a',b'],[c',d']) \neq \bot$, there exist $[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})$ such that $\alpha \wedge \tilde{R}'{}_X([a',b'],[c',d']) = \alpha \wedge \tilde{R}_X([a,b],[c,d])$ and $\langle [c,d],[c',d'] \rangle \in \tilde{\mathcal{Z}}$;

We say that the two intervals $[a,b]$ and $[a',b']$ are α-bisimilar, denoted $\tilde{\mathcal{M}}, [a,b] \underline{\leftrightarrow}_{\alpha_S} \tilde{\mathcal{M}}', [a',b']$ if there is an α-bimismulation $\tilde{\mathcal{Z}}$ between $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}'$ such that $\langle [a,b],[a',b'] \rangle \in \tilde{\mathcal{Z}}$.

One can think of definition 3.2 pictorially. The following figure shows a representation of the forward condition. Suppose that $\langle [a,b],[a',b'] \rangle \in \tilde{\mathcal{Z}}$ and $\alpha \wedge \tilde{R}_{X \in S}([a,b],[c,d]) \neq \bot$. Then the forward say there exist $[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ that completes the square.



Theorem 3.3. Let $\tilde{\mathcal{M}} = \left\langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \right\rangle$ and $\tilde{\mathcal{M}}' = \left\langle \mathbb{I}(\tilde{\mathbb{D}}'), \tilde{V}' \right\rangle$ be two $\mathcal{A}$-models, $[a,b] \in \mathbb{I}(\tilde{\mathbb{D}})$ and $[a',b'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ be two intervals and truth value $\alpha \in \mathcal{A}$ $(\alpha \neq \bot)$. If $\tilde{\mathcal{M}}, [a,b] \underline{\leftrightarrow}_{\alpha S} \tilde{\mathcal{M}}', [a',b']$, then for any fuzzy interval formula $\mathcal{L}_S^{\mathcal{A}}(\mathcal{AP})$

$$\alpha \wedge \tilde{V}(\phi, [a,b]) = \alpha \wedge \tilde{V}'(\phi, [a',b'])$$

Proof. We proceed by induction on ϕ.
If $\phi \in \mathcal{AP}$ then the result follows by base condition and if $\phi \in \mathcal{A}$ then it is trivial.

Assume that $\alpha \wedge \tilde{V}(\psi, [a,b]) = \alpha \wedge \tilde{V}'(\psi, [a',b'])$ and $\alpha \wedge \tilde{V}(\theta, [a,b]) = \alpha \wedge \tilde{V}'(\theta, [a',b'])$ holds.

If $\phi := \psi \wedge \theta$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\psi \wedge \theta, [a,b]) &= \alpha \wedge (\tilde{V}(\psi, [a,b]) \wedge \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V}(\psi, [a,b])) \wedge (\alpha \wedge \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V}'(\phi, [a',b'])) \wedge (\alpha \wedge \tilde{V}'(\theta, [a',b'])) \quad \text{By Inductive Hypothesis} \\
&= \alpha \wedge (\tilde{V}'(\psi, [a',b']) \wedge \tilde{V}'(\theta, [a',b'])) \\
&= \alpha \wedge \tilde{V}'(\psi \wedge \theta, [a',b'])
\end{aligned}$$

If $\phi := \psi \vee \theta$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\psi \vee \theta, [a,b]) &= \alpha \wedge (\tilde{V}(\psi, [a,b]) \vee \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V}(\psi, [a,b])) \vee (\alpha \wedge \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V}'(\psi, [a',b'])) \vee (\alpha \wedge \tilde{V}'(\theta, [a',b'])) \quad \text{By IH} \\
&= \alpha \wedge (\tilde{V}'(\psi, [a',b']) \vee \tilde{V}'(\theta, [a',b'])) \\
&= \alpha \wedge \tilde{V}'(\psi \vee \theta, [a',b'])
\end{aligned}$$

If $\phi := \psi \to \theta$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\psi \to \theta, [a,b]) &= \alpha \wedge (\tilde{V}(\psi, [a,b]) \to \tilde{V}(\theta, [a,b])) \\
&= \alpha \wedge (\alpha \wedge \tilde{V}(\psi, [a,b]) \to \alpha \wedge \tilde{V}(\theta, [a,b])) \qquad \text{Lemma (1.10)} \\
&= \alpha \wedge (\alpha \wedge \tilde{V}'(\psi, [a',b']) \to \alpha \wedge \tilde{V}'(\theta, [a',b'])) \quad \text{IH} \\
&= \alpha \wedge (\tilde{V}'(\psi, [a',b']) \to \tilde{V}'(\theta, [a',b'])) \\
&= \alpha \wedge \tilde{V}'(\psi \to \theta, [a',b'])
\end{aligned}$$

If $\phi := \langle X \rangle \psi$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\langle X \rangle \psi, [a,b]) &= \alpha \wedge \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{R}_X([a,b],[c,d]) \wedge \tilde{V}(\psi, [c,d]) \right] \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\alpha \wedge (\tilde{R}_X([a,b],[c,d]) \wedge \tilde{V}(\psi, [c,d])) \right] \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[(\alpha \wedge \tilde{R}_X([a,b],[c,d])) \wedge (\alpha \wedge \tilde{V}(\psi, [c,d])) \right] \\
&= \bigvee_{[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')} \left[(\alpha \wedge \tilde{R}_X([a',b'],[c',d'])) \wedge (\alpha \wedge \tilde{V}'(\psi, [c',d'])) \right] \\
&= \bigvee_{[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')} \left[\alpha \wedge (\tilde{R}'_X([a',b'],[c',d']) \wedge \tilde{V}'(\psi, [c',d'])) \right] \\
&= \alpha \wedge \bigvee_{[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')} \left[\tilde{R}'_X([a',b'],[c',d']) \wedge \tilde{V}'(\psi, [c',d']) \right] \\
&= \alpha \wedge \tilde{V}'(\langle X \rangle \psi, [a',b']).
\end{aligned}$$

The 4th step of the proof follows because $\langle [a,b],[a',b'] \rangle \in \tilde{\mathcal{Z}}$, so by Definition 3.2 (2), for each $[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})$ such that $\alpha \wedge \tilde{R}_X([a,b],[c,d]) \neq \bot$, there is $[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\alpha \wedge \tilde{R}_X([a,b],[c,d]) = \alpha \wedge \tilde{R}'_X([a',b'],[c',d'])$ and

$\langle [c,d],[c',d'] \in \tilde{\mathcal{Z}}\rangle$ and by IH $\alpha \wedge \tilde{V}(\psi,[c,d]) = \alpha \wedge \tilde{V}'(\psi,[c',d'])$ and by Definition 3.2 (3), for each $[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\alpha \wedge \tilde{R}_X([a',b'],[c',d']) \neq \bot$, there is a $[c,d]$ such that $\alpha \wedge \tilde{R}_X([a',b'],[c',d']) = \alpha \wedge \tilde{R}'_X([a,b],[c,d])$, and $\langle [c,d],[c',d'] \in \tilde{\mathcal{Z}}\rangle$ and by IH $\alpha \wedge \tilde{V}(\psi,[c,d]) = \alpha \wedge \tilde{V}'(\psi,[c',d'])$.

If $\phi := [X]\psi$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}([X]\psi,[a,b]) &= \alpha \wedge \bigwedge_{[c,d]\in\mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{R}_X([a,b],[c,d]) \to \tilde{V}(\psi,[c,d])\right] \\
&= \alpha \bigwedge_{[c,d]\in\mathbb{I}(\tilde{\mathbb{D}})} \left[\alpha \wedge (\tilde{R}_X([a,b],[c,d]) \to \tilde{V}(\psi,[c,d]))\right] \\
&= \alpha \wedge \bigwedge_{[c,d]\in\mathbb{I}(\tilde{\mathbb{D}})} \left[(\alpha \wedge \tilde{R}_X([a,b],[c,d])) \to (\alpha \wedge \tilde{V}(\psi,[c,d]))\right] \\
&= \alpha \wedge \bigwedge_{[c',d']\in\mathbb{I}(\tilde{\mathbb{D}}')} \left[(\alpha \wedge \tilde{R}'_X([a',b'],[c',d'])) \to (\alpha \wedge \tilde{V}'(\psi,[c',d']))\right] \\
&= \alpha \wedge \bigwedge_{[c',d']\in\mathbb{I}(\tilde{\mathbb{D}}')} \left[\alpha \wedge (\tilde{R}'_X([a',b'],[c',d']) \to \tilde{V}'(\psi,[c',d']))\right] \\
&= \alpha \wedge \bigwedge_{[c',d']\in\mathbb{I}(\tilde{\mathbb{D}}')} \left[\tilde{R}'_X([a',b'],[c',d']) \to \tilde{V}'(\psi,[c',d'])\right] \\
&= \alpha \wedge \tilde{V}'([X]\psi,[a',b'])
\end{aligned}$$

The 4th step follows from Lemma (1.10).

The 5th step of the proof was because $\langle [a,b],[a',b']\rangle \in \tilde{\mathcal{Z}}$, so by Definition 3.2 (2) for each $[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})$ such that $\alpha \wedge \tilde{R}_X([a,b],[c,d]) \neq \bot$, there is $[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\alpha \wedge \tilde{R}_X([a,b],[c,d]) = \alpha \wedge \tilde{R}'_X([a',b'],[c',d'])$ and $\langle [c,d],[c',d'] \in \tilde{\mathcal{Z}}\rangle$ and by IH $\alpha \wedge \tilde{V}(\psi,[c,d]) = \alpha \wedge \tilde{V}'(\psi,[c',d'])$ and by Definition 3.2 (3), for each $[c',d'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\alpha \wedge \tilde{R}_X([a',b'],[c',d']) \neq \bot$, there is a $[c,d]$ such that $\alpha \wedge \tilde{R}_X([a',b'],[c',d']) = \alpha \wedge \tilde{R}'_X([a,b],[c,d])$, so $\alpha \wedge \tilde{R}_X([a,b],[c,d]) = \alpha \wedge \tilde{R}'_X([a',b'],[c',d'])$ and $\langle [c,d],[c',d'] \in \tilde{\mathcal{Z}}\rangle$ and by IH $\alpha \wedge \tilde{V}(\psi,[c,d]) = \alpha \wedge \tilde{V}'(\psi,[c',d'])$.

□

3.2 Generated Submodels

Taking *generated submodels* is a way of generating smaller models from bigger models without affecting the information contained in the original model. These models are obtained by removing points from the original models and making a restriction on the original valuation to the remaining points and keeping the successors of the remaining points. In the definition below, taking $\mathcal{A}$ to be the two element Boolean algebra and α to be 1, we simply get the normal definition of a generated submodel for crisp interval models.

Definition 3.4. Let $\tilde{\mathcal{M}} = \left\langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \right\rangle$ and $\tilde{\mathcal{M}}' = \left\langle \mathbb{I}(\tilde{\mathbb{D}}'), \tilde{V}' \right\rangle$ be two $\mathcal{A}$-models and truth value $\alpha \in \mathcal{A}$ ($\alpha \neq \bot$). Then $\tilde{\mathcal{M}}'$ is an α-generated submodel of $\tilde{\mathcal{M}}$, denoted by $\tilde{\mathcal{M}}' \rightarrowtail_\alpha \tilde{\mathcal{M}}$ if :

1. $\mathbb{I}(\tilde{\mathbb{D}}') \subseteq \mathbb{I}(\tilde{\mathbb{D}})$;
2. For every $[a', b'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ and $p \in \mathcal{AP}$, $\alpha \wedge \tilde{V}'(p, [a', b']) = \alpha \wedge \tilde{V}(p, [a', b'])$;
3. For intervals $[a, b], [c, d] \in \mathbb{I}(\tilde{\mathbb{D}}')$, $\alpha \wedge \tilde{R}'_X([a, b], [c, d]) = \alpha \wedge \tilde{R}_X([a, b], [c, d])$;
4. For intervals $[a, b], [c, d] \in \mathbb{I}(\tilde{\mathbb{D}}')$, $\alpha \wedge \tilde{R}'_{\bar{X}}([a, b], [c, d]) = \alpha \wedge \tilde{R}_{\bar{X}}([a, b], [c, d])$;
5. If $[a, b] \in \mathbb{I}(\tilde{\mathbb{D}}')$ and $\alpha \wedge \tilde{R}_X([a, b], [c, d]) \neq \bot$, then $[c, d] \in \mathbb{I}(\tilde{\mathbb{D}}')$;
6. If $[a, b] \in \mathbb{I}(\tilde{\mathbb{D}}')$ and $\alpha \wedge \tilde{R}_{\bar{X}}([a, b], [c, d]) \neq \bot$, then $[c, d] \in \mathbb{I}(\tilde{\mathbb{D}}')$.

Theorem 3.5. For $\mathcal{A}$-models $\tilde{\mathcal{M}} = \left\langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \right\rangle$ and $\tilde{\mathcal{M}}' = \left\langle \mathbb{I}(\tilde{\mathbb{D}}'), \tilde{V}' \right\rangle$. If $\tilde{\mathcal{M}}' \rightarrowtail_\alpha \tilde{\mathcal{M}}$, then for each formula ϕ and interval $[a, b] \in \mathbb{I}(\tilde{\mathbb{D}}')$,

$$\alpha \wedge \tilde{V}(\phi, [a, b]) = \alpha \wedge \tilde{V}'(\alpha, [a, b])$$

Proof. We prove by induction on ϕ.
If $\phi \in \mathcal{AP}$ then the result follows by property 2 and if $\phi \in \mathcal{A}$ then it is trivial.

Assume that for every interval $[a, b] \in \mathbb{I}(\tilde{\mathbb{D}}')$ and formulas ψ and θ we have $\alpha \wedge \tilde{V}(\psi, [a, b]) = \alpha \wedge \tilde{V}'(\psi, [a, b])$ and $\alpha \wedge \tilde{V}(\theta, [a, b]) = \alpha \wedge \tilde{V}'(\theta, [a, b])$.

If $\phi := \psi \wedge \theta$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\psi \wedge \theta, [a,b]) &= \alpha \wedge (\tilde{V}(\psi, [a,b]) \wedge \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V}(\psi, [a,b])) \wedge (\alpha \wedge \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V'}(\phi, [a,b])) \wedge (\alpha \wedge \tilde{V'}(\theta, [a,b])) \quad \text{By Inductive Hypothesis} \\
&= \alpha \wedge (\tilde{V'}(\psi, [a,b]) \wedge \tilde{V'}(\theta, [a,b])) \\
&= \alpha \wedge \tilde{V'}(\psi \wedge \theta, [a,b])
\end{aligned}$$

If $\phi := \psi \vee \theta$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\psi \vee \theta, [a,b]) &= \alpha \wedge (\tilde{V}(\psi, [a,b]) \vee \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V}(\psi, [a,b])) \vee (\alpha \wedge \tilde{V}(\theta, [a,b])) \\
&= (\alpha \wedge \tilde{V'}(\psi, [a,b])) \vee (\alpha \wedge \tilde{V'}(\theta, [a,b])) \quad \text{By Inductive Hypothesis} \\
&= \alpha \wedge (\tilde{V'}(\psi, [a,b]) \vee \tilde{V'}(\theta, [a,b])) \\
&= \alpha \wedge \tilde{V'}(\psi \vee \theta, [a,b])
\end{aligned}$$

If $\phi := \psi \rightarrow \theta$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\psi \rightarrow \theta, [a,b]) &= \alpha \wedge (\tilde{V}(\psi, [a,b]) \rightarrow \tilde{V}(\theta, [a,b])) \\
&= \alpha \wedge (\alpha \wedge \tilde{V}(\psi, [a,b]) \rightarrow \alpha \wedge \tilde{V}(\theta, [a,b])) \quad \text{By Lemma 1.10} \\
&= \alpha \wedge (\alpha \wedge \tilde{V'}(\psi, [a,b]) \rightarrow \alpha \wedge \tilde{V'}(\theta, [a,b])) \quad \text{By IH} \\
&= \alpha \wedge (\tilde{V'}(\psi, [a,b]) \rightarrow \tilde{V'}(\theta, [a,b])) \\
&= \alpha \wedge \tilde{V'}(\psi \rightarrow \theta, [a,b])
\end{aligned}$$

If $\phi := \langle X \rangle \psi$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}(\langle X \rangle \psi, [a,b]) &= \alpha \wedge \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})} \Big[\tilde{R}_X([a,b],[c,d]) \wedge \tilde{V}(\psi,[c,d]) \Big] \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}})} \Big[\alpha \wedge (\tilde{R}_X([a,b],[c,d]) \wedge \tilde{V}(\psi,[c,d])) \Big] \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}'})} \Big[(\alpha \wedge \tilde{R}_X([a,b],[c,d])) \wedge (\alpha \wedge \tilde{V}(\psi,[c,d])) \Big] \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}'})} \Big[(\alpha \wedge \tilde{R}'_X([a,b],[c,d])) \wedge (\alpha \wedge \tilde{V}'(\psi,[c,d])) \Big] \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}'})} \Big[\alpha \wedge (\tilde{R}'_X([a,b],[c,d]) \wedge \tilde{V}'(\psi,[c,d])) \Big] \\
&= \alpha \wedge \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{D}'})} \Big[\tilde{R}'_X([a,b],[c,d]) \wedge \tilde{V}'(\psi,[c,d]) \Big] \\
&= \alpha \wedge \tilde{V}'(\langle X \rangle, [a,b]).
\end{aligned}$$

The 4th step of the proof was as a result of Inductive Hypothesis and Lemma 1.10

The proof for $\phi := \langle \bar{X} \rangle \psi$ is symmetric.

If $\phi := [X]\psi$ then

$$\begin{aligned}
\alpha \wedge \tilde{V}([X]\psi, [a,b]) &= \alpha \wedge \bigwedge_{[c,d]\in \mathbb{I}(\tilde{\mathbb{D}})} \Big[\tilde{R}_X([a,b],[c,d]) \to \tilde{V}(\psi,[c,d])\Big] \\
&= \alpha \bigwedge_{[c,d]\in \mathbb{I}(\tilde{\mathbb{D}})} \Big[\alpha \wedge (\tilde{R}_X([a,b],[c,d]) \to \tilde{V}(\psi,[c,d]))\Big] \\
&= \alpha \wedge \bigwedge_{[c,d]\in \mathbb{I}(\tilde{\mathbb{D}})} \Big[(\alpha \wedge \tilde{R}_X([a,b],[c,d])) \to (\alpha \wedge \tilde{V}(\psi,[c,d]))\Big] \\
&= \alpha \wedge \bigwedge_{[c,d]\in \mathbb{I}(\tilde{\mathbb{D}}')} \Big[(\alpha \wedge \tilde{R}'_X([a,b],[c,d])) \to (\alpha \wedge \tilde{V}'(\psi,[c,d]))\Big] \\
&= \alpha \wedge \bigwedge_{[c,d]\in \mathbb{I}(\tilde{\mathbb{D}}')} \Big[\alpha \wedge (\tilde{R}'_X([a,b],[c,d]) \to \tilde{V}'(\psi,[c,d]))\Big] \\
&= \alpha \wedge \bigwedge_{[c,d]\in \mathbb{I}(\tilde{\mathbb{D}}')} \Big[\tilde{R}'_X([a,b],[c,d]) \to \tilde{V}'(\psi,[c,d])\Big] \\
&= \alpha \wedge \tilde{V}'([X]\psi,[a,b])
\end{aligned}$$

The 3rd step of the proof was as a result of Inductive Hypothesis and Lemma 1.10.
The proof for $\phi := [\bar{X}]\psi$ is symmetric. □

Truth invariance should hold for submodels which are closed under accessibility relation of the original model

Chapter 4

Expressivity of the MVIBTL

The following chapter studies the interdefinability of the many-valued interval-based modalities over the class of all fuzzy linear orders. We show that the modalities $\langle D\rangle, \langle O\rangle$ and $\langle L\rangle$ are not expressible in terms of the respective fragments in which they are expressible in the crisp case by constructing α-bisimulations that respects the fragment but the valuation of the modality in the related models is not the same. [8] shows that the definitions in the crisp settings no longer hold in the fuzzy setting by giving counterexamples of the definitions. We are strengthening the result by showing that not only the crisp definitions fail, but it is impossible to find other definitions because they are just not expressible. We refer the reader to Chapter 5, Definition 5.2 on the definition of expressible.

Proposition 4.1. $\langle D\rangle$ is not expressible in the language $\mathcal{L}^{\mathcal{A}}_{\{B,E\}}(\mathcal{AP})$.

Proof. To show that we use $\frac{1}{2}$-bisimulation. Let $S = \{B, E\}$. Let $\mathcal{A}$ be a three element Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$. Let $\tilde{<} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathcal{A}$ and $\tilde{=} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathcal{A}$ and $\tilde{<}' : \mathbb{Z}' \times \mathbb{Z}' \rightarrow \mathcal{A}$ and $\tilde{=}' : \mathbb{Z}' \times \mathbb{Z}' \rightarrow \mathcal{A}$.

Define $\tilde{<}(x, y) = min\{1, max\{\frac{1}{2}(y - x), 0\}\}, \tilde{=}(x, y) = max\{0, 1 - \frac{1}{2} \mid x - y \mid \}, \tilde{<}(x', y') = min\{1, max\{\frac{1}{2}(y' - x'), 0\}\}, \tilde{=}(x', y') = max\{0, 1 - \frac{1}{2} \mid x' - y' \mid\}$, where x' and y' are the image of x and y under the mapping.

Consider the following models $\tilde{\mathcal{M}} = \langle \mathbb{I}(\tilde{\mathbb{Z}}), \tilde{V}\rangle$ with valuation $\tilde{V}(p, [x, y]) = 1$ for all $[x, y] \in \mathbb{I}(\tilde{\mathbb{Z}})$ and $\tilde{\mathcal{M}}' = \langle \mathbb{I}(\tilde{\mathbb{Z}}'), \tilde{V}'\rangle$ with valuation $\tilde{V}'(p, [x', y']) = 1$ for all $[x', y'] \in \mathbb{I}(\tilde{\mathbb{Z}}')$.

Define $\tilde{\mathcal{Z}}$ by:

$\tilde{\mathcal{Z}} = \{([x,y],[x',y']) \mid x,y \in \mathbb{Z}, x<y\}$
$\cup\, \{([x,y],[x',(y+1)']) \mid x,y \in \mathbb{Z}, x<y\}$
$\cup\, \{([x,y],[(x-1)',y']) \mid x,y \in \mathbb{Z}, x<y\}$
$\cup\, \{([x,y],[(x-1)',(y-1)']) \mid x,y \in \mathbb{Z}, x<y\}$
$\cup\, \{([x,y],[(x+1)',(y+1)']) \mid x,y \in \mathbb{Z}, x<y\}$
$\cup\, \{([(x-1),y],[x',y']) \mid x,y \in \mathbb{Z}, x<y\}$
$\cup\, \{([x,y+1],[x',y']) \mid x,y \in \mathbb{Z}, x<y\}$

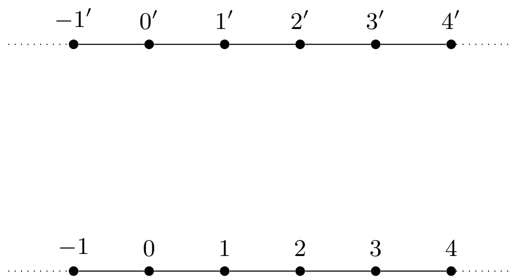


Figure 4.1: $\frac{1}{2}$-bisimulation between the two models

We now show that $\tilde{\mathcal{Z}}$ is a $\frac{1}{2}$-bisimulation.
1. It is easy to verify the local condition since all $\tilde{\mathcal{Z}}$-related satisfy the same propositional variable. So

$$\frac{1}{2} \wedge \tilde{V}(p,[x,y]) = \frac{1}{2} \wedge \tilde{V}'(p,[z',t'])$$

for all $[x,y] \in \mathbb{I}(\tilde{\mathbb{D}})$, $[z',t'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\langle [x,y],[z',t'] \rangle \in \tilde{\mathcal{Z}}$ and $p \in \mathcal{AP}$.

2. We now show the back and forth condition.
Case 1: $\langle [x,y],[x',y'] \rangle \in \tilde{\mathcal{Z}}$,

If $\tilde{R}_E([x,y],[z,t]) > 0$ then,
$\frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t]) = \frac{1}{2}$ and $\frac{1}{2} \wedge \tilde{R}'_E([x',y'],[z',t']) = \frac{1}{2}$, so $\frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R}'_E([x',y'],[z',t'])$ and $\langle [z,t],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Similar for the back condition.

Case 2: $\langle [x,y],[x',(y+1)'] \rangle \in \tilde{\mathcal{Z}}$, Forth clause for $\tilde{R}_E$: Suppose $\tilde{R}_E([x,y],[z,t]) \succeq \frac{1}{2}$, then $\frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R}'_E([x',(y+1)'],[z',(t+1)'])$ and $\langle [z,t],[z',(t+1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,t)\\ &= \tilde{<}'(x',z') \wedge \tilde{=}'((y+1)',(t+1)')\\ &= \tilde{R}'_E([x',(y+1)'],[z',(t+1)'])\end{aligned}$$

Back clause for $\tilde{R}_E$: Suppose $\tilde{R}'_E([x',(y+1)'],[z',t']) \succeq \frac{1}{2}$. Then by definition of $\tilde{<}(a,b)$ either $z-x=1$ or $z-x \geq 2$.

(1) if $z-x \geq 2$ then,

$$\frac{1}{2} \wedge \tilde{R}'_E([x',(y+1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[(z-1),(t-1)])$$

and $\langle [z-1,t-1],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\tilde{R}_E([x,y],[(z-1),(t-1)]) = \tilde{<}(x,(z-1)) \wedge \tilde{=}(y,(t-1))$$

Since $\tilde{R}'_E([x',(y+1)'],[z',t']) \succeq \frac{1}{2}$ we have that $\tilde{=}'((y+1)',t') \succeq \frac{1}{2}$ and $\tilde{=}'((y+1)',t') = \tilde{=}(y,(t-1))$ because the distance between $y+1$ and t is the same as the distance between y and $t-1$. Also, since $z-x \geq 2, (z-1)-x \geq 1$ so $\tilde{<}(x,(z-1)) \succeq \frac{1}{2}$.So $\tilde{R}_E([x,y],[(z-1),(t-1)]) \succeq \frac{1}{2}$

(2) if $z-x=1$ then in order to have $\tilde{=}'(a',b') \succeq 0$, the distance between t' and $(y+1)'$ must be at most 1. So either $t \leq y+1$ or $t > y+1$

(i) if $y \leq t \leq y+1$ then

$$\frac{1}{2} \wedge \tilde{R}'_E([x',(y+1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t])$$

and $\langle [z,t],[z',t'] \rangle \in \tilde{\mathcal{Z}}$

Since $\tilde{R}'_E([x',(y+1)'],[z',t']) \succeq \frac{1}{2}$, we have $\tilde{=}'((y+1)',t') \succeq \frac{1}{2}$. Then we have either $y=t-1$ or $t=y$. So $\tilde{=}(y,t) \succeq \frac{1}{2}$. Also since $z-x=1$, we have that $\tilde{<}(x,z) = \frac{1}{2}$. So $\tilde{R}_E([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{=}(y,t) \succeq \frac{1}{2}$.

(ii) $t = y + 2$ then

$$\frac{1}{2} \wedge \tilde{R}'_E([x',(y+1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t-1])$$

and $\langle [z,t-1],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Since

$$\begin{aligned}\tilde{R}_E([x,y],[z,(t-1)]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,(t-1)) \\ &= \tilde{<}'(x',z') \wedge \tilde{=}'((y+1)',t') \\ &= \tilde{R}'_E([x',(y+1)'],[z',t'])\end{aligned}$$

We know $z = x + 1$ and $x < y$. So $z \leq y$ and $y < t - 1$, so indeed $z < t - 1$ and $[z, t-1] \in \mathbb{I}(\tilde{\mathbb{D}})$.

Case 3: $\langle [x,y],[(x-1)',y'] \rangle \in \tilde{\mathcal{Z}}$,

Forth clause for $\tilde{R}_E$: Suppose $\tilde{R}_E([x,y],[z,t]) \succeq \frac{1}{2}$, then $\frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R}'_E([(x-1)',y'],[(z-1)',t'])$ and $\langle [z,t],[(z-1)',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,t) \\ &= \tilde{<}'((x-1)',(z-1)') \wedge \tilde{=}'(y',t') \\ &= \tilde{R}'_E([(x-1)',y'],[(z-1)',t'])\end{aligned}$$

Back clause for $\tilde{R}_E$:
Suppose $\tilde{R}'_E([(x-1)',y'],[z',t']) \succeq \frac{1}{2}$. Then by definition of $\tilde{<}(a,b)$ either $z - (x-1) \geq 2$ or $z - (x-1) = 1$.

(1) if $z - (x-1) \geq 2$ then

$$\frac{1}{2} \wedge \tilde{R}'_E([(x-1)',y'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t])$$

and $\langle [z,t],[z',t']\rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\tilde{R}_E([x,y],[z,t]) = \tilde{<}(x,z) \wedge \tilde{=}(y,t)$$

Since $\tilde{R'}_E([(x-1)',y'],[z',t']) \succeq \frac{1}{2}$ we have that $\tilde{=}'(y',t') \succeq \frac{1}{2}$.So $\tilde{=}(y,t) \succeq \frac{1}{2}$. Since $z-(x-1) \geq 2$, we have that $z-x \geq 1$ and $\tilde{<}(x,z) \succeq \frac{1}{2}$. So $\tilde{R}_E([x,y],[z,t]) \succeq \frac{1}{2}$

(2) if $z-(x-1)=1$ then in order for $\tilde{=}(z,b) \succ 0$, the distance between y and t must be 1. Sow we have either $t-1 \leq y \leq t$ or $y = t+1$
(i) if $t-1 \leq y \leq t$ then

$$\frac{1}{2} \wedge \tilde{R'}_E([(x-1)',y'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[z+1,t])$$

and $\langle [z+1,t],[z',t']\rangle \in \tilde{\mathcal{Z}}$

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[z+1,t]) &= \tilde{<}(x,z+1) \wedge \tilde{=}(y,t)\\ &= \tilde{<}'((x-1)',z') \wedge \tilde{=}'(y',t')\\ &= \tilde{R}_E([(x-1)',y'],[z',t'])\end{aligned}$$

(ii) $t=y+1$ then

$$\frac{1}{2} \wedge \tilde{R'}_E([(x-1)',y'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[z+1,t+1])$$

and $\langle [z+1,t+1],[z',t']\rangle \in \tilde{\mathcal{Z}}$.

Since $\tilde{R'}_E([(x-1)',y'],[z',t']) \succeq \frac{1}{2}$, we have that $\tilde{<}'(y',t') \succeq \frac{1}{2}$. So it must be that $y=t+1$. So $\tilde{<}(y,(t+1)) \succeq \frac{1}{2}$. Also since $z-(x-1)=1$, we have that $(z+1)-x=1$. So $\tilde{<}(x,(z+1)) \succeq \frac{1}{2}$ and $\tilde{R}_E([x,y],[z+1,t+1]) \succeq \frac{1}{2}$.

Case 4: $\langle [x,y],[(x-1)',(y-1)']\rangle \in \tilde{\mathcal{Z}}$,

Forth clause for $\tilde{R}_E$:
Suppose $\tilde{R}_E([x,y],[z,t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R'}_E([(x-1)',(y-1)'],[(z-1)',(t-1)'])$ and

$\langle [z,t], [(z-1)', (t-1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,t) \\ &= \tilde{<}'((x-1)',(z-1)') \wedge \tilde{=}'((y-1)',(t-1)'). \\ &= \tilde{R}'_E([(x-1)',(y-1)'],[(z-1)',(t-1)'])\end{aligned}$$

Back clause for $\tilde{R}_E$:
Suppose $\tilde{R}'_E([(x-1)',(y-1)'],[z',t']) \succeq \frac{1}{2}$.
$\frac{1}{2} \wedge \tilde{R}'_E([(x-1)',(y-1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[(z+1),(t+1)])$ and $\langle [z+1,t+1],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[(z+1),(t+1)]) &= \tilde{<}(x,(z+1)) \wedge \tilde{=}(y,(t+1)) \\ &= \tilde{<}'((x-1)',z') \wedge \tilde{=}'((y-1)',t') \\ &= \tilde{R}'_E([(x-1)',(y-1)'],[z',t'])\end{aligned}$$

Case 5: $\langle [x,y],[(x+1)',(y+1)'] \rangle \in \tilde{\mathcal{Z}}$

Forth clause for $\tilde{R}_E$:
Suppose $\tilde{R}_E([x,y],[z,t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R}'_E([(x+1)',(y+1)'],[(z+1)',(t+1)'])$ and $\langle [z,t],[(z+1)',(t+1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,t) \\ &= \tilde{<}'((x+1)',(z+1)') \wedge \tilde{=}'((y+1)',(t+1)') \\ &= \tilde{R}'_E([(x+1)',(y+1)'],[(z+1)',(t+1)'])\end{aligned}$$

Back clause for $\tilde{R}_E$:
Suppose $\tilde{R}'_E([(x+1)',(y+1)'],[z',t']) \succeq \frac{1}{2}$. Then

$$\frac{1}{2} \wedge \tilde{R}'_E([(x+1)',(y+1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[(z-1),(t-1)])$$

and $\langle [z-1, t-1], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[(z-1),(t-1)]) &= \tilde{<}(x,(z-1)) \wedge \tilde{=}(y,(t-1)) \\ &= \tilde{<}'((x+1)', z') \wedge \tilde{=}'((y+1)', t') \\ &= \tilde{R}'_E([(x+1)', (y+1)'], [z', t'])\end{aligned}$$

Case 6: $\langle [(x-1), y)], [x', y'] \rangle \in \tilde{\mathcal{Z}}$
Forth clause for $\tilde{R}_E$:
Suppose $\tilde{R}_E([(x-1), y], [z, t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_E([(x-1), y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_E([x', y'], [z', t'])$ and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([(x-1), y], [z, t]) &= \tilde{<}((x-1), z) \wedge \tilde{=}(y, t) \\ &= \tilde{<}'(x', z') \wedge \tilde{=}'(y', t') \\ &= \tilde{R}'_E([(x-1)', y'], [z', t'])\end{aligned}$$

Back clause for $\tilde{R}_E$: Suppose $\tilde{R}'_E([x', y'], [z', t']) \succeq \frac{1}{2}$. Then by definition of $\tilde{<}(a, b)$ either$z - x = 1$ or $z - x \geq 2$.

(1) if $z - x \geq 2$ then,

$$\frac{1}{2} \wedge \tilde{R}'_E([x', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_E([x, y], [z, t])$$

and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x, y], [z, t]) &= \tilde{<}(x, z) \wedge \tilde{=}(y, t) \\ &= \tilde{<}'(x', z') \wedge \tilde{=}'(y', t') \\ &= \tilde{R}'_E([x', y'], [z', t'])\end{aligned}$$

(2) if $z - x = 1$ then by definition of $\tilde{=}'(a, b)$ either $t \leq y$ or $t > y$

(i) if $t \leq y$ then

$$\frac{1}{2} \wedge \tilde{R}'_E([x', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_E([x, y+1], [z, t+1])$$

and $\langle [z, t+1], [z', t'] \rangle \in \tilde{\mathcal{Z}}$

Indeed,

$$\begin{aligned}\tilde{R}_E([x, y+1], [z, t+1]) &= \tilde{<}(x, z) \wedge \tilde{=}(y+1, t+1)\\ &= \tilde{<}'(x', z') \wedge \tilde{=}'(y', t')\\ &= \tilde{R'}_E([x', y'], [z', t'])\end{aligned}$$

(ii) $t > y$ then

$$\frac{1}{2} \wedge \tilde{R'}_E([x', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_E([x, y], [z, t])$$

and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Since

$$\begin{aligned}\tilde{R}_E([x, y], [z, t]) &= \tilde{<}(x, z) \wedge \tilde{=}(y, (t-1))\\ &= \tilde{<}'(x', z') \wedge \tilde{=}'((y+1)', t')\\ &= \tilde{R'}_E([x', (y+1)'], [z', t'])\end{aligned}$$

We know $z = x + 1$ and $x < y$. So $z \leq y$ and $y < t$, so indeed $z < t$ and $[z, t] \in \mathbb{I}(\tilde{\mathbb{D}})$.

Case 7: $\langle [x, (y+1)], [x', y'] \rangle \in \tilde{\mathcal{Z}}$
Forth clause for $\tilde{R}_E$: Suppose $\tilde{R}_E([x, (y+1)], [z, t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_E([x, (y+1)], [z, t]) = \frac{1}{2} \wedge \tilde{R'}_E([x', y'], [z', t'])$ and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Since $\tilde{R}_E([x, (y+1)], [z, t]) \succeq \frac{1}{2}$, we have $\tilde{=}((y+1), t) \succeq \frac{1}{2}$. Then either $y' = t'$ or $y' = t' - 1$. So $\tilde{=}'(y', t') \succeq \frac{1}{2}$. Also $\tilde{<}'(x', z') \succeq \frac{1}{2}$, so $\tilde{R'}_E([x', y'], [z', t']) \succeq \frac{1}{2}$.

Back clause for $\tilde{R}_E$: Suppose $\tilde{R'}_E([x', y'], [z', t']) \succeq \frac{1}{2}$. Then by definition of $\tilde{<}(a, b)$ either$z - x = 1$ or $z - x \geq 2$.

(1) if $z - x \geq 2$ then,

$$\frac{1}{2} \wedge \tilde{R'}_E([x', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_E([x, y], [z, t])$$

and $\langle [z,t],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,t)\\ &= \tilde{<}'(x',z') \wedge \tilde{=}'(y',t')\\ &= \tilde{R}'_E([x',y'],[z',t'])\end{aligned}$$

(2) if $z - x = 1$ then by definition of $\tilde{=}'(a,b)$ either $t \leq y$ or $t > y$

(i) if $t \leq y$ then

$$\frac{1}{2} \wedge \tilde{R}'_E([x',y'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y+1],[z,t+1])$$

and $\langle [z,t+1],[z',t'] \rangle \in \tilde{\mathcal{Z}}$

Indeed,

$$\begin{aligned}\tilde{R}_E([x,y+1],[z,t+1]) &= \tilde{<}(x,z) \wedge \tilde{=}(y+1,t+1)\\ &= \tilde{<}'(x',z') \wedge \tilde{=}'(y',t')\\ &= \tilde{R}'_E([x',y'],[z',t'])\end{aligned}$$

(ii) $t > y$ then

$$\frac{1}{2} \wedge \tilde{R}'_E([x',y'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_E([x,y],[z,t])$$

and $\langle [z,t],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Since

$$\begin{aligned}\tilde{R}_E([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{=}(y,(t-1))\\ &= \tilde{<}'(x',z') \wedge \tilde{=}'((y+1)',t')\\ &= \tilde{R}'_E([x',(y+1)'],[z',t'])\end{aligned}$$

We know $z = x + 1$ and $x < y$. So $z \leq y$ and $y < t$, so indeed $z < t$ and $[z,t] \in \mathbb{I}(\tilde{\mathbb{D}})$.

Similarly we show for B since it is symmetric to E.

But

$$\begin{aligned}\frac{1}{2} \wedge \tilde{V}(\langle D\rangle p, [-1,1]) &= \frac{1}{2} \wedge \tilde{R}_D([-1,1],[0,1]) \wedge \tilde{V}(p,[0,1]) \\ &= \frac{1}{2} \wedge 0 \wedge 1 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2} \wedge \tilde{V}'(\langle D\rangle p, [-1',2']) &= \frac{1}{2} \wedge \tilde{R}_D([-1',2'],[0',1']) \wedge \tilde{V}'(p,[0',1']) \\ &= \frac{1}{2} \wedge \frac{1}{2} \wedge 1 \\ &= \frac{1}{2}\end{aligned}$$

Since $\frac{1}{2} \wedge \tilde{V}(\langle D\rangle p, [-1,1]) = 0$ and $\frac{1}{2} \wedge \tilde{V}'(\langle D\rangle p, [-1',2']) = \frac{1}{2}$, so $\frac{1}{2} \wedge \tilde{V}(\langle D\rangle p, [-1,1]) \neq \frac{1}{2} \wedge \tilde{V}'(\langle D\rangle p, [-1',2'])$, therefore by Theorem 3.3, $\langle D\rangle$ is not expressible in the language $\mathcal{L}^{\mathcal{A}}_{\{B,E\}}(\mathcal{AP})$.

$\square$

The following proposition shows that the modality O cannot be expressed in terms of the modalities E and $\bar{B}$

Proposition 4.2. $\langle O \rangle$ is not expressible in the language $\mathcal{L}^{\mathcal{A}}_{\{E,\bar{B}\}}(\mathcal{AP})$.

Proof. To show that we use $\frac{1}{2}$-bisimulation. Let $\mathcal{A}$ be a three element Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$. Let $\tilde{<} : \mathbb{Z} \times \mathbb{Z} \to \mathcal{A}$ and $\tilde{=} : \mathbb{Z} \times \mathbb{Z} \to \mathcal{A}$ and $\tilde{<}' : \mathbb{Z}' \times \mathbb{Z}' \to \mathcal{A}$ and $\tilde{=}' : \mathbb{Z}' \times \mathbb{Z}' \to \mathcal{A}$.

Define $\tilde{<}(x,y) = min\{1, max\{\frac{1}{2}(y-x), 0\}\}, \tilde{=}(x,y) = max\{0, 1-\frac{1}{2} \mid x-y \mid \}, \tilde{<}(x',y') = min\{1, max\{\frac{1}{2}(y'-x'), 0\}\}, \tilde{=}(x',y') = max\{0, 1-\frac{1}{2} \mid x'-y' \mid\}$.

Consider the following models $\tilde{\mathcal{M}} = \langle \mathbb{I}(\tilde{\mathbb{Z}}), \tilde{V} \rangle$ with valuation $\tilde{V}(p, [x,y]) = 1$ for all $[x,y] \in \mathbb{I}(\tilde{\mathbb{Z}})$ and $\tilde{\mathcal{M}}' = \langle \mathbb{I}(\tilde{\mathbb{Z}}'), \tilde{V}' \rangle$ with valuation $\tilde{V}'(p, [x',y']) = 1$ for all $[x',y'] \in \mathbb{I}(\tilde{\mathbb{Z}}')$.

Define $\tilde{\mathcal{Z}}$ by:

$$\tilde{\mathcal{Z}} = \{([x,y],[x',y']) \mid x,y \in \mathbb{Z}, x < y\}$$
$$\cup \{([x,y],[x',(y+1)']) \mid x,y \in \mathbb{Z}, x < y\}$$
$$\cup \{([x,y],[(x-1)',y']) \mid x,y \in \mathbb{Z}, x < y\}$$
$$\cup \{([x,y],[(x-1)',(y-1)']) \mid x,y \in \mathbb{Z}, x < y\}$$
$$\cup \{([x,y],[(x+1)',(y+1)']) \mid x,y \in \mathbb{Z}, x < y\}$$
$$\cup \{([(x-1),y],[x',y']) \mid x,y \in \mathbb{Z}, x < y\}$$
$$\cup \{([x,y+1],[x',y']) \mid x,y \in \mathbb{Z}, x < y\}$$

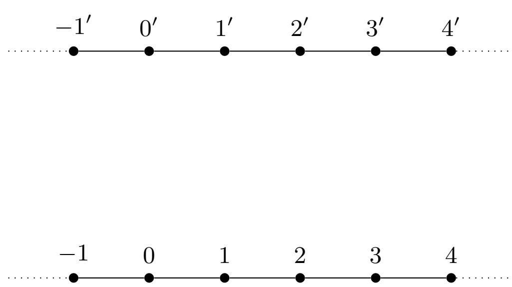


Figure 4.2: $\frac{1}{2}$-bisimulation between the two models

We now show that $\tilde{\mathcal{Z}}$ is a $\frac{1}{2}$-bisimulation.
1. It is easy to verify the local condition since all $\tilde{\mathcal{Z}}$-related satisfy the same

propositional variable. So

$$\frac{1}{2} \wedge \tilde{V}(p, [x, y]) = \frac{1}{2} \wedge \tilde{V}'(p, [z', t'])$$

for all $[x, y] \in \mathbb{I}(\tilde{\mathbb{D}})$, $[z', t'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\langle [x, y], [z', t'] \rangle \in \tilde{\mathcal{Z}}$ and $p \in \mathcal{AP}$. From Lemma 3.1, we have that the bisimulation satisfies E, so we only show for $\bar{B}$.

2. We now show the back and forward condition.
Case 1: $\langle [x, y], [x', y'] \rangle \in \tilde{\mathcal{Z}}$,

If $\tilde{R}_{\bar{B}}([x, y], [z, t]) \succeq \frac{1}{2}$ then,
$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z, t]) = \frac{1}{2}$ and $\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', y'], [z', t']) = \frac{1}{2}$, so $\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', y'], [z', t'])$ and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Similar for the back condition.

Case 2: $\langle [x, y], [x', (y + 1)'] \rangle \in \tilde{\mathcal{Z}}$,

Forth clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}_{\bar{B}}([x, y], [z, t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', (y + 1)'], [z', (t + 1)'])$ and $\langle [z, t], [z', (t + 1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}
\tilde{R}_{\bar{B}}([x, y], [z, t]) &= \tilde{=}(x, z) \wedge \tilde{<}(y, t) \\
&= \tilde{=}'(x', z') \wedge \tilde{<}'((y + 1)', (t + 1)') \\
&= \tilde{R}'_{\bar{B}}([x', (y + 1)'], [z', (t + 1)'])
\end{aligned}$$

Back clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}'_{\bar{B}}([x', (y+1)'], [z', t']) \succeq \frac{1}{2}$, then by definition of $\tilde{=}(a, b)$, the distance between x and z has to be at most 1. So we either have $t = y + 1$ or $t > y$.

(i) If $t > y$ then,

$$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', (y + 1)'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z, t - 1])$$

and $\langle [z, t - 1], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_{\bar{B}}([x,y],[z,t-1]) &= \tilde{\doteq}(x,z) \wedge \tilde{<}(y,t-1)\\ &= \tilde{\doteq}'(x',z') \wedge \tilde{<}'((y+1)',t')\\ &= \tilde{R}'_{\bar{B}}([x',(y+1)'],[z',t'])\end{aligned}$$

Since $\tilde{\doteq}'(x,z) \succeq \frac{1}{2}, z \leq x+1$ and $x < y$. So $z \leq y$ and $y < t-1$. So indeed $z < t-1$ and $[z,t-1] \in \mathbb{I}(\tilde{\mathbb{D}})$

(ii) If $t = y+1$, then

$$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x',(y+1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x,y],[z,t])$$

and $\langle [z,t],[z',t'] \rangle \in \mathbb{I}(\tilde{\mathbb{D}})$

Since $\tilde{R}_{\bar{B}}([x',(y+1)'],[z',t']) \succeq \frac{1}{2}$ we have that $\tilde{\doteq}'(x',z') \succeq \frac{1}{2}$. So $\tilde{\doteq}(x,z) \succeq \frac{1}{2}$. Again, Since $t = y+1y$ we have that $y < t$ and $\tilde{<}(y,t) \succeq \frac{1}{2}$.

Case 3: $\langle [x,y],[(x-1)',y'] \rangle \in \tilde{\mathcal{Z}}$,

Forth clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}_{\bar{B}}([x,y],[z,t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([(x-1)',y'],[(z-1)',t'])$ and $\langle [z,t],[(z-1)',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_{\bar{B}}([x,y],[z,t]) &= \tilde{\doteq}(x,z) \wedge \tilde{<}(y,t)\\ &= \tilde{\doteq}'((x-1)',(z-1)') \wedge \tilde{<}'(y',t')\\ &= \tilde{R}'_{\bar{B}}([(x-1)',y'],[(z-1)',t'])\end{aligned}$$

Back clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}'_{\bar{B}}([(x-1)',y'],[z',t']) \succeq \frac{1}{2}$, then by definition of $\tilde{\doteq}(a,b)$ and $\tilde{R}_{\bar{B}}$, the distance between $x-1$ and z has to be at most 1. So we must have $t > y$.

For $t > y$ then,

$$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([(x-1)', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z+1, t+1])$$

and $\langle [z', t'], [z+1, t+1] \rangle \in \tilde{\mathcal{Z}}$.

Since $\tilde{R}'_{\bar{B}}([(x-1)', y'], [z', t']) \succeq \frac{1}{2}$ we have that $\tilde{=}'((x-1)', z') = \tilde{=}(x, z+1)$ because the distance between $(x-1)'$ and z is the same as the distance between x and $z+1$. Again, $\tilde{<}'(y', t') \succeq \frac{1}{2}$, by definition, also $\tilde{<}(y, t+1) \succeq \frac{1}{2}$.

Case 4: $\langle [x, y], [(x-1)', (y-1)'] \rangle \in \tilde{\mathcal{Z}}$,

Forth clause for $\tilde{R}_{\bar{B}}$:
Suppose $\tilde{R}_{\bar{B}}([x, y], [z, t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([(x-1)', (y-1)'], [(z-1)', (t-1)'])$ and $\langle [z, t], [(z-1)', (t-1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned} \tilde{R}_{\bar{B}}([x, y], [z, t]) &= \tilde{=}(x, z) \wedge \tilde{<}(y, t) \\ &= \tilde{=}'((x-1)', (z-1)') \wedge \tilde{<}'((y-1)', (t-1)'). \\ &= \tilde{R}'_{\bar{B}}([(x-1)', (y-1)'], [(z-1)', (t-1)']) \end{aligned}$$

Back clause for $\tilde{R}_{\bar{B}}$:
Suppose $\tilde{R}'_{\bar{B}}([(x-1)', (y-1)'], [z', t']) \succeq \frac{1}{2}$.
$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([(x-1)', (y-1)'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [(z+1), (t+1)])$ and $\langle [z+1, t+1], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned} \tilde{R}_{\bar{B}}([x, y], [(z+1), (t+1)]) &= \tilde{=}(x, (z+1)) \wedge \tilde{<}(y, (t+1)) \\ &= \tilde{=}'((x-1)', z') \wedge \tilde{<}'((y-1)', t') \\ &= \tilde{R}'_{\bar{B}}([(x-1)', (y-1)'], [z', t']) \end{aligned}$$

Case 5: $\langle [x, y], [(x+1)', (y+1)'] \rangle \in \tilde{\mathcal{Z}}$

Forth clause for $\tilde{R}_{\bar{B}}$:
Suppose $\tilde{R}_{\bar{B}}([x, y], [z, t]) \succeq \frac{1}{2}$, then
$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([(x+1)', (y+1)'], [(z+1)', (t+1)'])$ and

$\langle [z,t], [(z+1)', (t+1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_{\bar{B}}([x,y],[z,t]) &= \tilde{=}(x,z) \wedge \tilde{<}(y,t)\\ &= \tilde{=}'((x+1)',(z+1)') \wedge \tilde{<}'((y+1)',(t+1)')\\ &= \tilde{R}'_{\bar{B}}([(x+1)',(y+1)'],[(z+1)',(t+1)'])\end{aligned}$$

Back clause for $\tilde{R}_{\bar{B}}$:
Suppose $\tilde{R}'_{\bar{B}}([(x+1)',(y+1)'],[z',t']) \succeq \frac{1}{2}$. Then

$$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([(x+1)',(y+1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x,y],[(z-1),(t-1)])$$

and $\langle [z-1,t-1],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_{\bar{B}}([x,y],[(z-1),(t-1)]) &= \tilde{=}(x,(z-1)) \wedge \tilde{<}(y,(t-1))\\ &= \tilde{=}'((x+1)',z') \wedge \tilde{<}'((y+1)',t')\\ &= \tilde{R}'_{\bar{B}}([(x+1)',(y+1)'],[z',t'])\end{aligned}$$

Case 6: $\langle [(x-1),y)],[x',y'] \rangle \in \tilde{\mathcal{Z}}$

Forth clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}_{\bar{B}}([(x-1),y],[z,t]) \succeq \frac{1}{2}$. The distance between z and $x-1$ has to be at most 1, so we have that either $z \leq x-1$ or $z = x$.

if $z \leq x-1$ then,

$$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([(x-1),y],[z,t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x',y'],[(z+1)',t'])$$

and $\langle [z,t],[(z+1)',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}\tilde{R}_{\bar{B}}([(x-1),y],[z,t]) &= \tilde{=}((x-1),z) \wedge \tilde{<}(y,t)\\ &= \tilde{=}'(x',(z+1)') \wedge \tilde{<}'(y',t')\\ &= \tilde{R}'_{\bar{B}}([x',y'],[(z+1)',t'])\end{aligned}$$

Since $z \leq x - 1$, i.e, $z + 1 \leq x$ and $x < y$, so $z + 1 < y$ and $y < t$, so indeed $z + 1 < t$ and $[(z+1)', t'] \in \mathbb{I}(\tilde{\mathbb{Z}})$.

If $z = x$ then

$$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([(x-1), y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', y'], [z', t'])$$

Indeed, since $x = z$, we have that $\tilde{=}(x, z) = 1$, $\tilde{=}(x-1, z) = \frac{1}{2}$ and $\tilde{<}(y, t) \succeq \frac{1}{2}$. So $\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([(x-1), y], [z, t]) = \frac{1}{2}$ and $\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', y'], [z', t']) = \frac{1}{2}$, hence the above equality holds.

Back clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}'_{\bar{B}}([x', y'], [z', t']) \succeq \frac{1}{2}$. By definition of $\tilde{=}(a, b)$ and $\tilde{R}_{\bar{B}}$, the distance between x and z is at most 1. By definition of $\tilde{<}(a, b)$ we have that $y < t$.

Then,

$$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, y+1], [z, t+1])$$

and $\langle [z, t+1], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}
\tilde{R}_{\bar{B}}([x, y+1], [z, t+1]) &= \tilde{=}(x, z) \wedge \tilde{<}((y+1), (t+1)) \\
&= \tilde{=}'(x', z') \wedge \tilde{<}'(y', t') \\
&= \tilde{R}'_{\bar{B}}([x', y'], [z', t'])
\end{aligned}$$

Case 7: $\langle [x, (y+1)], [x', y'] \rangle \in \tilde{\mathcal{Z}}$

Forth clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}_{\bar{B}}([x, (y+1)], [z, t]) \succeq \frac{1}{2}$. Then

$\frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x, (y+1)], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x', y'], [z', (t-1)'])$ and $\langle [z, t], [z', (t-1)'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}
\tilde{R}_{\bar{B}}([x, (y+1)], [z, t]) &= \tilde{=}(x, z) \wedge \tilde{<}((y+1), t) \\
&= \tilde{=}'(x', z') \wedge \tilde{<}'(y', (t-1)') \\
&= \tilde{R}'_{\bar{B}}([x', y'], [z', (t-1)'])
\end{aligned}$$

The distance between x and z has to be at most 1, so either $z \leq x$ or $z = x+1$. For $z \leq x$, we have $x < y$, so $z < y$ and $y+1 < t$,i.e, $y < t-1$, so indeed $z < t-1$ and $[z,'(t-1)'] \in \mathbb{I}(\tilde{\mathbb{Z}}')$.

For $z = x+1$, we have $x < y$, and so $z < y+1$ and $y+1 < t$, i.e, $y < t-1$, so $z < t-1$ and $[z',(t-1)'] \in \mathbb{I}(\tilde{\mathbb{Z}}')$.

Back clause for $\tilde{R}_{\bar{B}}$: Suppose $\tilde{R}'_{\bar{B}}([x',y'],[z',t']) \succeq \frac{1}{2}$.By definition of $\tilde{\simeq}(a,b)$ and $\tilde{R}_{\bar{B}}$, the distance between x and z is at most 1. By definition of $\tilde{<}(a,b)$ we have that $y < t$.

Then,

$$\frac{1}{2} \wedge \tilde{R}'_{\bar{B}}([x',y'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_{\bar{B}}([x,y+1],[z,t+1])$$

and $\langle [z,t+1],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned}
\tilde{R}_{\bar{B}}([x,y+1],[z,t+1]) &= \tilde{\simeq}(x,z) \wedge \tilde{<}(y+1,t+1) \\
&= \tilde{\simeq}'(x',z') \wedge \tilde{<}'(y',t') \\
&= \tilde{R}'_{\bar{B}}([x',y'],[z',t'])
\end{aligned}$$

But

$$\begin{aligned}
\frac{1}{2} \wedge \tilde{V}(\langle O \rangle p,[0,1]) &= \frac{1}{2} \wedge \bigvee_{[z,t]\in\mathbb{I}(\tilde{\mathbb{Z}})} \tilde{R}_O([0,1],[z,t]) \wedge \tilde{V}(p,[z,t]) \\
&= \frac{1}{2} \wedge 0 \wedge 1 \\
&= 0
\end{aligned}$$

$\tilde{V}(p,[z,t]) = 1$, since p is true at all the intervals. $\tilde{R}_O([0,1],[z,t]) = \tilde{<}(0,z) \wedge \tilde{<}(z,1) \wedge \tilde{<}(1,t) = 0$ because there is no $z \in \mathbb{Z}$ such that $0 < z < 1$.

and

$$\begin{aligned}\frac{1}{2} \wedge \tilde{V}'(\langle O\rangle p, [0', 2']) &\succeq \frac{1}{2} \wedge \tilde{R}_O([0', 2'], [1', 3']) \wedge \tilde{V}'(p, [1', 3']) \\ &= \frac{1}{2} \wedge \frac{1}{2} \wedge 1 \\ &= \frac{1}{2}\end{aligned}$$

Since $\frac{1}{2} \wedge \tilde{V}(\langle O\rangle p, [0,1]) = 0$ and $\frac{1}{2} \wedge \tilde{V}'(\langle O\rangle p, [0', 2']) = \frac{1}{2}$, so $\frac{1}{2} \wedge \tilde{V}(\langle O\rangle p, [0,1]) \neq \frac{1}{2} \wedge \tilde{V}'(\langle O\rangle p, [0', 2'])$, therefore by Theorem 3.3,$\langle O\rangle$ is not expressible in the language $\mathcal{L}^{\mathcal{A}}_{\{\bar{B},E\}}(\mathcal{AP})$.

$\square$

The following propositions shows that the modality L cannot be expressed in terms of modality A.

Proposition 4.3. $\langle L \rangle$ is not expressible in the language $\mathcal{L}^{\mathcal{A}}_{\{A\}}(\mathcal{AP})$.

Proof. To show that we use $\frac{1}{2}$-bi-simulation. Let $\mathcal{A}$ be a three element Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$. Let $D_1 = \{1, 2, 3\}$ and $D_2 = \{0, 1, 2, 3\}$
Let $\tilde{<} : \mathbb{D}_1 \times \mathbb{D}_1 \to \mathcal{A}$ and $\tilde{=} : \mathbb{D}_1 \times \mathbb{D}_1 \to \mathcal{A}$ and $\tilde{<}' : \mathbb{D}_2 \times \mathbb{D}_2 \to \mathcal{A}$ and $\tilde{=}' : \mathbb{D}_2 \times \mathbb{D}_2 \to \mathcal{A}$.

Define $\tilde{<}(x, y) = min\{1, max\{\frac{1}{2}(y - x), 0\}\}, \tilde{=}(x, y) = max\{0, 1 - \frac{1}{2} \mid x - y \mid \}, \tilde{<}(x', y') = min\{1, max\{\frac{1}{2}(y' - x'), 0\}\}, \tilde{=}(x', y') = max\{0, 1 - \frac{1}{2} \mid x' - y' \mid\}$.

Consider the following models $\tilde{\mathcal{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}_1), \tilde{V} \rangle$ with valuation $\tilde{V}(p, [x, y]) = 1$ for all $[x, y] \in \mathbb{I}(\tilde{\mathbb{D}}_1)$ and $\tilde{\mathcal{M}}' = \langle \mathbb{I}(\tilde{\mathbb{D}}_2), \tilde{V}' \rangle$ with valuation $\tilde{V}'(p, [x', y']) = 1$ for all $[x', y'] \in \mathbb{I}(\tilde{\mathbb{D}}_2)$.

Define $\tilde{\mathcal{Z}}$ by:

$\tilde{\mathcal{Z}} = \{([x, y], [x', y']) \mid x, y \in \mathbb{Z}, x < y\}$
$\cup \{([x, y], [(x - 1)', y']) \mid x, y \in \mathbb{Z}, x < y\}$
$\cup \{([x, y], [(x - 1)', (y - 1)']) \mid x, y \in \mathbb{Z}, x < y\}$

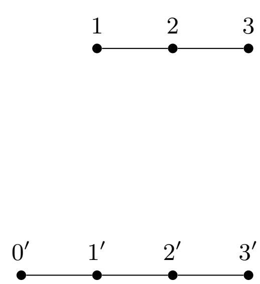


Figure 4.3: $\frac{1}{2}$-bisimulation between the two models

We now show that $\tilde{\mathcal{Z}}$ is a $\frac{1}{2}$-bisimulation.
1. It is easy to verify the local condition since all $\tilde{\mathcal{Z}}$-related satisfy the same propositional variable. So

$$\frac{1}{2} \wedge \tilde{V}(p, [x, y]) = \frac{1}{2} \wedge \tilde{V}'(p, [z', t'])$$

for all $[x, y] \in \mathbb{I}(\tilde{\mathbb{D}})$, $[z', t'] \in \mathbb{I}(\tilde{\mathbb{D}}')$ such that $\langle [x, y], [z', t'] \rangle \in \tilde{\mathcal{Z}}$ and $p \in \mathcal{AP}$.

2. We now show the back and fourth condition.
Case 1: $\langle [x, y], [x', y'] \rangle \in \tilde{\mathcal{Z}}$,

If $\tilde{R}_A([x, y], [z, t]) \succ 0$ then,
$\frac{1}{2} \wedge \tilde{R}_A([x, y], [z, t]) = \frac{1}{2}$ and $\frac{1}{2} \wedge \tilde{R}'_A([x', y'], [z', t']) = \frac{1}{2}$, so $\frac{1}{2} \wedge \tilde{R}_A([x, y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_A([x', y'], [z', t'])$ and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Similar for the back condition.

Case 2: $\langle [x, y], [(x-1)', y'] \rangle \in \tilde{\mathcal{Z}}$

Forth clause for $\tilde{R}_A$: Suppose $\tilde{R}_A([x, y], [z, t]) \succeq \frac{1}{2}$. Then

$$\frac{1}{2} \wedge \tilde{R}_A([x, y], [z, t]) = \frac{1}{2} \wedge \tilde{R}'_A([(x-1)', y'], [z', t'])$$

and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned} \tilde{R}'_A([(x-1)', y'], [z', t']) &= \tilde{=}'(y', z') \\ &= \tilde{=}(y, z) \\ &= \tilde{R}'_A([x, y], [z, t]) \end{aligned}$$

Back clause for $\tilde{R}_A$: Suppose $\tilde{R}'_A([(x-1)', y'], [z', t']) \succeq \frac{1}{2}$. Then

$$\frac{1}{2} \wedge \tilde{R}'_A([(x-1)', y'], [z', t']) = \frac{1}{2} \wedge \tilde{R}_A([x, y], [z, t])$$

and $\langle [z, t], [z', t'] \rangle \in \tilde{\mathcal{Z}}$.

Indeed,

$$\begin{aligned} \tilde{R}_A([x, y], [z, t]) &= \tilde{=}(y, z) \\ &= \tilde{=}'(y', z') \\ &= \tilde{R}'_A([(x-1)', y'], [z', t']) \end{aligned}$$

Case 3: $\langle [x, y], [(x-1)', (y-1)'] \rangle \in \tilde{\mathcal{Z}}$,

Forth clause for $\tilde{R}_A$: Suppose $\tilde{R}_A([x, y], [z, t]) \succeq \frac{1}{2}$. Then

$$\frac{1}{2} \wedge \tilde{R}_A([x,y],[z,t]) = \frac{1}{2} \wedge \tilde{R'}_A([(x-1)',(y-1)'],[(z-1)',(t-1)'])$$

and $\langle [z,t],[(z-1)',(t-1)'] \rangle \in \tilde{\mathcal{Z}}$

Indeed,

$$\begin{aligned}\tilde{R}_A([x,y],[z,t]) &= \tilde{=}(y,z)\\ &= \tilde{=}'((y-1)',(z-1)')\\ &= \tilde{R'}_A([(x-1)',(y-1)'],[(z-1)',(t-1)'])\end{aligned}$$

Back clause for $\tilde{R}_A$: Suppose $\tilde{R'}_A([(x-1)',(y-1)'],[z',t']) \succeq \frac{1}{2}$. The distance between $y-1$ and z has to be at most 1, we have that either $y-1=z$, $y-1=z-1$ or $y-1=z+1$. We consider the three cases separately:

(i) If $y-1=z$ then

$$\frac{1}{2} \wedge \tilde{R'}_A([(x-1)',(y-1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_A([x,y],[z,t])$$

and $\langle [z,t],[z',t'] \rangle \in \tilde{\mathcal{Z}}$
Indeed, $|(y-1)-z| = 0$, therefore we have that $|y-z| = 1$ and $\tilde{=}(y,z) \succeq \frac{1}{2}$ and therefore $\tilde{R}_A([x,y],[z,t]) = \frac{1}{2}$.

(ii) If $y-1=z-1$ then

$$\frac{1}{2} \wedge \tilde{R'}_A([(x-1)',(y-1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_A([x,y],[z,t])$$

and $\langle [z,t],[z',t'] \rangle \in \tilde{\mathcal{Z}}$

Indeed,$y=z$ and $\tilde{=}(y,z) = 1$ and therefore $\tilde{R}_A([x,y],[z,t]) = 1$.

(iii) If $y-1=z+1$ then

$$\frac{1}{2} \wedge \tilde{R'}_A([(x-1)',(y-1)'],[z',t']) = \frac{1}{2} \wedge \tilde{R}_A([x,y],[z+1,u])$$

where $u = min\{3, t+1\}$.

Then $|(z+1)-(y-1)| \leq 0$, therefore we have that $|(z+1)-y| \leq 1$, so $\tilde{=}(z+1, y) \succeq \frac{1}{2}$. Since $y-1 = z+1$, we have that $z = y-2$ and $z \in \{0,1,2,3\}$, the smallest y can be is 2 and the greatest it can be is 3, and so $z \leq 1$. So we have the following cases:

(a) If $\boldsymbol{y' = 2'}$ and $\boldsymbol{z' = 0'}$ then $z+1 = 1$ and $\tilde{=}(z+1, y) = \frac{1}{2}$ and therefore $\tilde{R}_A([x,y],[z+1,u]) = \frac{1}{2}$. Then t' can either be $1', 2'$ or $3'$. For $t < 3$, pick $u = t+1$ and therefore $z+1 < u$ and so $[z+1,u] \in \mathbb{I}(\tilde{\mathbb{Z}})$ and indeed $\langle [z+1,u],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.
For $t = 3$, pick $u = t$ and therefore $z+1 < u$ and so $[z+1,u] \in \mathbb{I}(\tilde{\mathbb{Z}})$ and indeed $\langle [z+1,u],[z',t'] \rangle \in \tilde{\mathcal{Z}}$.

(b) If $\boldsymbol{y' = 3'}$ and $\boldsymbol{z' = 1'}$ then we have that $z+1 = 2$ and $\tilde{=}(z+1, y) = \frac{1}{2}$ and therefore $\tilde{R}_A([x,y],[z+1,u]) = \frac{1}{2}$. Then t' can either be $2'$ or $3'$. For both cases, pick $u = 3$ and therefore $z+1 < u$ and so $[z+1,u] \in \mathbb{I}(\tilde{\mathbb{Z}})$ and indeed $\langle [z',t'],[z+1,u] \rangle \in \tilde{\mathcal{Z}}$.

But

$$\begin{aligned}\frac{1}{2} \wedge \tilde{V}(\langle L \rangle p, [1,2]) &= \frac{1}{2} \wedge \bigvee_{[z,t] \in \mathbb{I}(\tilde{\mathbb{Z}})} \tilde{R}_L([1,2],[z,t]) \wedge \tilde{V}(p,[z,t]) \\ &= \frac{1}{2} \wedge 0 \wedge 1 \\ &= 0\end{aligned}$$

$\tilde{V}(p,[z,t]) = 1$, since p is true at all the intervals. $\tilde{R}_L([1,2],[z,t]) = \tilde{<}(2,z) = 0$ because there is no $z \in \mathbb{Z}$ such that $2 < z < 3$.

and

$$\begin{aligned}\frac{1}{2} \wedge \tilde{V'}(\langle L \rangle p, [0',1']) &\succeq \frac{1}{2} \wedge \tilde{R'}_L([0',1'],[2',3']) \wedge \tilde{V'}(p,[2',3']) \\ &= \frac{1}{2} \wedge \frac{1}{2} \wedge 1 \\ &= \frac{1}{2}\end{aligned}$$

Since $\frac{1}{2} \wedge \tilde{V}(\langle L \rangle p, [1,2]) = 0$ and $\frac{1}{2} \wedge \tilde{V'}(\langle L \rangle p, [0',1']) = \frac{1}{2}$, so $\frac{1}{2} \wedge \tilde{V}(\langle L \rangle p, [1,2]) \neq \frac{1}{2} \wedge \tilde{V'}(\langle L \rangle p, [0',1'])$, therefore by Theorem 3.3,$\langle L \rangle$ is not expressible in the language $\mathcal{L}^{\mathcal{A}}_{\{A\}}(\mathcal{AP})$.

$\square$

Chapter 5

Definability and Expressiveness of MVIBTL Modalities

In this chapter we provide the theoretical background of definability, expressivity and duality and prove some of the important results that arise from the concepts. We further explore the results from Chapter 4 using theoretical background provided in the chapter and derive some of the important results that arise from the theoretical background. We extend the interdefinabilities by making an assumption on the Heyting algebra and show that under such assumption, certain interdefinability equations still hold as inequality in one direction.

5.1 Definability and Duality

The definition below was extended from [1].
We say that two modalities $\langle X\rangle$ and $\langle Y\rangle$ are dual if and only if $(\langle X\rangle, \langle Y\rangle) \in S$, where S is the relation defined as
$S = \{(\langle A\rangle, \langle \bar{A}\rangle), (\langle \bar{A}\rangle, \langle A\rangle), (\langle L\rangle, \langle \bar{L}\rangle), (\langle \bar{L}\rangle, \langle L\rangle), (\langle B\rangle, \langle E\rangle), (\langle \bar{B}\rangle, \langle \bar{E}\rangle),$
$(\langle E\rangle, \langle B\rangle), (\langle \bar{E}\rangle, \langle \bar{B}\rangle), (\langle D\rangle, \langle D\rangle), (\langle \bar{D}\rangle, \langle \bar{D}\rangle), (\langle O\rangle, \langle \bar{O}\rangle), (\langle \bar{O}\rangle, \langle O\rangle)\}$.

We lift S to a relation between fragments, denoted $\bar{S}$ and defined by
$\bar{S} = \{(\mathcal{F}_1, \mathcal{F}_2) \mid ((\forall\langle X\rangle \in \mathcal{F}_1)(\exists\langle Y\rangle \in \mathcal{F}_2)(\langle X\rangle, \langle Y\rangle) \in S)$ and $((\forall\langle Y\rangle \in \mathcal{F}_2)(\exists\langle X\rangle \in \mathcal{F}_1)(\langle Y\rangle, \langle X\rangle)) \in S\}$.

We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are dual if and only if $(\mathcal{F}_1, \mathcal{F}_2) \in \bar{S}$ and $\tilde{R}^D$ is the dual of $\tilde{R}$.

Note that in fact S a function.

Let $\phi \in \{A, L, B, E, D, O\}$. We define ϕ^D to be the dual of ϕ, where $\phi^D \in \{\bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\}$ and $(\phi, \phi^D) \in S$.

The concept of duality between the fuzzy modalities allows us to study the interdefibality for only one of the modality in each pair of the dual modalities.

Definition 5.1. We define $\langle X \rangle p \equiv_{\mathcal{C}} \phi$ to hold if

$$\tilde{V}(\langle X \rangle p, [a, b]) = \tilde{V}(\phi, [a, b])$$

for all $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$ based on a fuzzy linear order $\tilde{\mathbb{D}}$ from class $\mathcal{C}$ and all $[a, b] \in \mathbb{I}(\tilde{\mathbb{D}})$

Definition 5.2. (Definability)
Let $Var(\phi)$ be the set of all propositional variables occurring in ϕ and $\mathcal{A}$ be a Heyting algebra. Then we say that modality $\langle X \rangle$ is *definable* in a fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of $\mathcal{A}$-fuzzy linear orders, denoted $\langle X \rangle \lhd_{\mathcal{C}}^{\mathcal{A}} \mathcal{F}$, if $\langle X \rangle p \equiv_{\mathcal{C}} \phi$ for some $\mathcal{F}$-formula ϕ and for all p such that $Var(\phi) = \{p\}$. We will say that ϕ defines $\langle X \rangle$ over $\mathcal{C}$

Definition 5.3. (Dual of $\tilde{\mathbb{D}}$)
We define the dual of a fuzzy linear order $\tilde{\mathbb{D}} = \langle D, \tilde{<}, \tilde{=} \rangle$, denoted by, $\tilde{\mathbb{D}}^D = \langle D, \tilde{<}^D, \tilde{=} \rangle$, where $\tilde{<}^D$ is the inverse of $\tilde{<}$ given by $\tilde{<}^D(y, x) = \tilde{<}(x, y)$.

Proposition 5.4. $\tilde{\mathbb{D}}^D$ is an $\mathcal{A}$-valued fuzzy linear order.

Proof. Since $\tilde{\mathbb{D}}$ is an $\mathcal{A}$-valued fuzzy linear order, properties (1) and (2) of Definition 1.14 follow.
Now we show the remaining properties.

3. $\forall x(\tilde{<}^D(x, x) = 0)$(*irreflexivity of* $\tilde{<}^D$):
Since $\tilde{\mathbb{D}}$ is an $\mathcal{A}$-valued fuzzy linear order, we have that $\tilde{<}(x, x) = 0$. By definition of the dual, we have $\tilde{<}^D(x, y) = \tilde{<}(x, y)$, so using $x = y$ we get that $\tilde{<}^D(x, x) = \tilde{<}(x, x) = 0$.

4. $\forall x, y, z(\tilde{<}^D(x, z) \succeq \tilde{<}^D(x, y) \wedge \tilde{<}^D(y, z))$(*transivity of* $\tilde{<}^D$):
Since $\tilde{\mathbb{D}}$ is an $\mathcal{A}$-valued fuzzy linear order, we have $\tilde{<}(z, x) \succeq \tilde{<}(z, y) \wedge \tilde{<}(y, x)$ and by definition of the dual we have that $\tilde{<}^D(x, z) = \tilde{<}(z, x), \tilde{<}^D(x, y) = \tilde{<}(y, x), \tilde{<}^D(y, z) = \tilde{<}(z, y)$, so we have that $\tilde{<}^D(x, z) \succeq \tilde{<}^D(x, y) \wedge \tilde{<}^D(y, z)$.

5. $\forall x, y, z(\tilde{<}^D(x, y) \succ 0 \,\&\, \tilde{<}^D(y, z) \succ 0 \Rightarrow \tilde{<}^D(x, z) \succ 0)$(*transfer of positivity of* $\tilde{<}^D$):
Suppose that $\tilde{<}^D(x, y) \succ 0$ and $\tilde{<}^D(y, z) \succ 0$. Then by definition of the dual we have we have that $\tilde{<}(y, x) \succ 0$ and $\tilde{<}(z, y) \succ 0$. Since $\tilde{<}$ satisfies property (5), we have that $\tilde{<}(z, x) \succ 0$, so by definition of the dual, we have that $\tilde{<}^D(x, z) \succ 0$.

6. $\forall x, y(\tilde{<}^D(x, y) = 0 \;\&\; \tilde{<}^D(y, x) = 0 \Rightarrow \tilde{=}(y, x)) = 1$((*weak*)*totality*);
Suppose that $\tilde{<}^D(x, y) = 0$ and $\tilde{<}^D(y, x) = 0$. Then by definition of the dual we have that $\tilde{<}(y, x) = 0$ and $\tilde{<}(x, y) = 0$. Since $\tilde{<}$ satisfies property (6), we have that $\tilde{=}(y, x) = 1$. So by definition of the dual and property we have that $\tilde{=}(x, y) = 1$.

7. $\forall x, y(\tilde{=}(x, y) \succ 0 \Rightarrow \tilde{<}^D(x, y) \prec 1)$(*non-contradiction of* $\tilde{<}$ *over* $\tilde{=}$).
Suppose that $\tilde{=}(y, x) \succ 0$. Since $\tilde{=}$ is symmetric by property (2) and it satisfies property (7), we have that $\tilde{<}(y, x) \prec 1$. By definition of the dual, we have that $\tilde{<}^D(x, y) \prec 1$.

□

Definition 5.5. (Symmetric Class Of Linear Orders)
A class $\mathcal{C}$ of fuzzy linear orders is said to be symmetric if whenever $\tilde{\mathbb{D}} \in \mathcal{C}$, then $\tilde{\mathbb{D}}^D \in \mathcal{C}$.

Definition 5.6. (Dual $\mathcal{A}$-models)
Let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$ and $\mathbb{M}^D = \langle \mathbb{I}(\tilde{\mathbb{D}})^D, \tilde{V}^D \rangle$ be two $\mathcal{A}$-models. Then we say that $\mathbb{M}^D$ is the dual $\mathcal{A}$-model of $\tilde{\mathbb{M}}$ if $\tilde{V}^D : \mathcal{AP} \times \mathbb{I}(\tilde{\mathbb{D}}^D) \rightarrow \mathcal{A}$ is a valuation given by

$$\tilde{V}^D(p, [y, x]) = \tilde{V}(p, [x, y])$$

for each $p \in \mathcal{AP}$.

Remark 5.7. From Definition (5.3), we have the following

1. $(\tilde{\mathbb{D}}^D)^D = \tilde{\mathbb{D}}$;

2. $(\tilde{V}^D)^D = \tilde{V}$;

3. $(\tilde{\mathbb{M}}^D)^D = \tilde{\mathbb{M}}$;

Proposition 5.8. Let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$ and $\mathbb{M}^D = \langle \mathbb{I}(\tilde{\mathbb{D}})^D, \tilde{V}^D \rangle$ be two dual $\mathcal{A}$-models and $X \in \{A, L, B, E, D, O, \bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\}$. Then

For intervals $[x, y], [z, t] \in \mathbb{I}(\tilde{\mathbb{D}})$,

$$\tilde{R}_X([x, y], [z, t]) = \tilde{R}_Y^D([y, x], [t, z])$$

where $(\langle X \rangle, \langle Y \rangle) \in S$.

Proof. Let $[x, y], [z, t] \in \mathbb{I}(\tilde{\mathbb{D}})$ then we consider cases for $X \in \{A, L, B, E, D, O\}$. Cases for $X \in \{\bar{A}, \bar{L}, \bar{B}, \bar{E}, \bar{D}, \bar{O}\}$ are symmetric.

If $X = A$ then by definition we have that

$$\begin{aligned} \tilde{R}_A([x, y], [z, t]) &= \tilde{=}(y, z) \\ &= \tilde{=}(z, y) && \tilde{\mathbb{D}} \text{ is a fuzzy linear order, property (2)} \\ &= \tilde{R}_{\bar{A}}^D([y, x], [t, z]) \end{aligned}$$

and $(\langle A \rangle, \langle \bar{A} \rangle) \in S$

If $X = L$ then by definition we have that

$$\begin{aligned} \tilde{R}_L([x, y], [z, t]) &= \tilde{<}(y, z) \\ &= \tilde{<}^D(z, y) && \tilde{\mathbb{D}} \text{ is dual to } \tilde{\mathbb{D}}^D \\ &= \tilde{R}_{\bar{L}}^D([y, x], [t, z]) \end{aligned}$$

and $(\langle L \rangle, \langle \bar{L} \rangle) \in S$

If $X = B$ then by definition we have that

$$\begin{aligned} \tilde{R}_B([x, y], [z, t]) &= \tilde{=}(x, z) \wedge \tilde{<}(t, y) \\ &= \tilde{=}(z, x) \wedge \tilde{<}^D(y, t) && \tilde{\mathbb{D}} \text{ is dual to } \tilde{\mathbb{D}}^D, \text{ property (2)} \\ &= \tilde{R}_E^D([y, x], [t, z]) \end{aligned}$$

and $(\langle B \rangle, \langle E \rangle) \in S$

If $X = E$ then by definition we have that

$$\begin{aligned} \tilde{R}_E([x, y], [z, t]) &= \tilde{<}(x, z) \wedge \tilde{=}(y, t) \\ &= \tilde{<}^D(z, x) \wedge \tilde{=}(t, y) && \tilde{\mathbb{D}} \text{ is dual to } \tilde{\mathbb{D}}^D, \text{ property (2)} \\ &= \tilde{R}_B^D([y, x], [t, z]) \end{aligned}$$

and $(\langle E\rangle, \langle B\rangle) \in S$

If $X = D$ then by definition we have that

$$\begin{aligned}
\tilde{R}_D([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{<}(t,y) \\
&= \tilde{<}^D(z,x) \wedge \tilde{<}^D(y,t) && \tilde{\mathbb{D}} \text{ is dual to } \tilde{\mathbb{D}}^D \\
&= \tilde{R}^D_D([y,x],[t,z])
\end{aligned}$$

and $(\langle D\rangle, \langle D\rangle) \in S$

If $X = O$ then by definition we have that

$$\begin{aligned}
\tilde{R}_O([x,y],[z,t]) &= \tilde{<}(x,z) \wedge \tilde{<}(z,y) \wedge \tilde{<}(y,t) \\
&= \tilde{<}^D(z,x) \wedge \tilde{<}^D(y,z) \wedge \tilde{<}^D(t,y) && \tilde{\mathbb{D}} \text{ is dual to } \tilde{\mathbb{D}}^D \\
&= \tilde{R}^D_{\bar{O}}([y,x],[t,z])
\end{aligned}$$

and $(\langle O\rangle, \langle \bar{O}\rangle) \in S$ □

Lemma 5.9. Let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ and $\tilde{\mathbb{M}}^D = \langle \mathbb{I}(\tilde{\mathbb{D}})^D, \tilde{V}^D\rangle$ be two dual $\mathcal{A}$-models, $[x,y] \in \mathbb{I}(\tilde{\mathbb{D}})$, $[y,x] \in \mathbb{I}(\tilde{\mathbb{D}}^D)$. Then for all formulas ϕ

$$\tilde{V}(\phi,[x,y]) = \tilde{V}^D(\phi^D,[y,x])$$

where ϕ^D is the dual of ϕ

Proof. We proceed by induction on ϕ.
For $\phi \in \mathcal{AP}$ the result follows by Definition 5.6.

Assume that $\tilde{V}(\psi,[x,y]) = \tilde{V}^D(\psi^D,[y,x])$ and $\tilde{V}(\theta,[x,y]) = \tilde{V}^D(\theta^D,[y,x])$ holds for all $[x,y] \in \mathbb{I}(\mathbb{D})$.

If $\phi := \psi \wedge \theta$ then

$$\begin{aligned}
\tilde{V}(\psi \wedge \theta,[x,y]) &= \tilde{V}(\psi,[x,y]) \wedge \tilde{V}(\theta,[x,y]) \\
&= \tilde{V}^D(\phi^D,[y,x]) \wedge \tilde{V}^D(\theta^D,[y,x]) && \text{By Inductive Hypothesis} \\
&= \tilde{V}^D(\psi^D \wedge \theta^D,[y,x]) \\
&= \tilde{V}^D((\psi \wedge \theta)^D,[y,x])
\end{aligned}$$

If $\phi := \psi \vee \theta$ then

$$\begin{aligned}
\tilde{V}(\psi \vee \theta, [x,y]) &= \tilde{V}(\psi, [x,y]) \vee \tilde{V}(\theta, [x,y]) \\
&= (\tilde{V}^D(\phi^D, [y,x])) \vee (\tilde{V}^D(\theta^D, [y,x])) \quad \text{By Inductive Hypothesis} \\
&= \tilde{D}^D(\psi^D \vee \theta^D, [y,x]) \\
&= \tilde{V}^D((\psi \vee \theta)^D, [y,x])
\end{aligned}$$

If $\phi := \psi \rightarrow \theta$ then

$$\begin{aligned}
\tilde{V}(\psi \rightarrow \theta, [x,y]) &= \tilde{V}(\psi, [x,y]) \rightarrow \tilde{V}(\theta, [x,y]) \\
&= \tilde{V}^D(\psi^D, [y,x]) \rightarrow \tilde{V}^D(\theta^D, [y,x]) \quad \text{By Inductive Hypothesis} \\
&= \tilde{V}^D(\psi^D \rightarrow \theta^D, [y,x]) \\
&= \tilde{V}^D((\psi \rightarrow \theta)^D, [y,x])
\end{aligned}$$

Suppose $\phi = \langle X \rangle \psi$ and $(\langle X \rangle, \langle Y \rangle)$, then

$$\begin{aligned}
\tilde{V}(\langle X \rangle \psi, [x,y]) &= \bigvee_{[z,t] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{R}_X([x,y],[z,t]) \wedge \tilde{V}(\psi, [z,t]) \right] \\
&= \bigvee_{[t,z] \in \mathbb{I}(\tilde{\mathbb{D}})^D} \left[(\tilde{R}^D_Y([y,x],[t,z])) \wedge (\tilde{V}^D(\psi^D, [t,z])) \right] \quad \text{Proposition 5.8, IH} \\
&= \tilde{V}^D(\langle Y \rangle \psi^D, [y,x]) \\
&= \tilde{V}^D((\langle X \rangle \psi)^D, [y,x])
\end{aligned}$$

Suppose $\phi = [X]\psi$ and $(\langle X \rangle, \langle Y \rangle)$, then

$$\begin{aligned}
\tilde{V}([X]\psi, [x,y]) &= \bigwedge_{[z,t] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{R}_X([x,y],[z,t]) \rightarrow \tilde{V}(\psi, [z,t]) \right] \\
&= \bigwedge_{[t,z] \in \mathbb{I}(\tilde{\mathbb{D}})^D} \left[(\tilde{R}^D_Y([y,x],[t,z])) \rightarrow (\tilde{V^D}(\psi^D, [t,z])) \right] \quad \text{Proposition 5.8, IH} \\
&= \tilde{V^D}([Y]\psi^D, [y,x]) \\
&= \tilde{V^D}(([X]\psi)^D, [y,x])
\end{aligned}$$

□

Proposition 5.10. Let $\langle X \rangle$ and $\langle Y \rangle$ be two dual modalities and $\mathcal{C}$ be a symmetric class of $\mathcal{A}$-valued fuzzy linear orders with $\mathcal{A}$ a complete Heyting algebra. Then $\langle X \rangle$ is definable in a fragment $\mathcal{F}$ relative to the class $\mathcal{C}$ if and only if $\langle Y \rangle$ is definable in $\tilde{S}(\mathcal{F})$ relative to the class $\mathcal{C}$.

Proof. Let ϕ be an $\mathcal{F}$-formula which defines $\langle X\rangle$ relative to $\mathcal{C}$ with ϕ^D its dual formula and let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ be an arbitrary model with $\tilde{\mathbb{D}} \in \mathcal{C}$ and let $\tilde{\mathbb{M}}^D = \langle \mathbb{I}(\tilde{\mathbb{D}})^D, \tilde{V}^D\rangle$ be its dual model. It is clear that $\phi^D \in \tilde{S}(\mathcal{F})$. Since $\mathcal{C}$ is symmetric, $\tilde{\mathbb{D}}^D \in \mathcal{C}$. It will suffice to show that for all $[y, x] \in \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}(\langle Y\rangle p, [y, x]) = \tilde{V}(\phi^D, [y, x])$.

By the definability equation $\langle X\rangle p \equiv \phi$, we have that

$$\begin{aligned}
\tilde{V}(\langle Y\rangle p, [y, x]) &= (\tilde{V}^D)^D(\langle Y\rangle p, [y, x]) && \text{Remark 5.7}\\
&= \tilde{V}^D(\langle X\rangle p, [x, y]) && \text{Lemma 5.9}\\
&= \tilde{V}^D(\phi, [x, y]) && \text{By assumption and Definition 5.2}\\
&= (\tilde{V}^D)^D(\phi^D, [y, x]) && \text{Lemma 5.9}\\
&= \tilde{V}(\phi^D, [y, x])
\end{aligned}$$

□

The following proposition is useful to transfer inequalities between modalities to their duals.

Proposition 5.11. Let $\langle Y\rangle$ be the dual of $\langle X\rangle$ and $\langle Z\rangle$ be the dual of $\langle T\rangle$. Let $\mathcal{C}$ be a symmetric class of $\mathcal{A}$-valued fuzzy linear orders with $\mathcal{A}$ a complete Heyting algebra. If

$$\tilde{V}(\langle X\rangle p, [x, y]) \leq \tilde{V}(\langle T\rangle p, [x, y])$$

for all models $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ with $\tilde{\mathbb{D}} \in \mathcal{C}$ and $[x, y] \in \mathbb{I}(\tilde{\mathbb{D}})$, then

$$\tilde{V}(\langle Y\rangle p, [x, y]) \leq \tilde{V}(\langle Z\rangle p, [x, y])$$

for all models $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ with $\tilde{\mathbb{D}} \in \mathcal{C}$ and $[x, y] \in \mathbb{I}(\tilde{\mathbb{D}})$.

Proof. Suppose

$$\tilde{V}(\langle X\rangle p, [x, y]) \leq \tilde{V}(\langle T\rangle p, [x, y])$$

for all models $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ with $\tilde{\mathbb{D}} \in \mathcal{C}$ and $[x, y] \in \mathcal{C}$.

Now let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ be an arbitrary model with $\tilde{\mathbb{D}} \in \mathcal{C}$ and $[x, y] \in \mathcal{C}$. Then

$$\begin{aligned}
(\tilde{V})(\langle Y\rangle p,[x,y]) &= (\tilde{V}^D)(\langle X\rangle p,[y,x]) && \text{Lemma 5.9}\\
&\leq (\tilde{V}^D)(\langle T\rangle p,[y,x]) && \text{Assumptions}\\
&\leq (\tilde{V}^D)^D(\langle Z\rangle p,[x,y]) && \text{Lemma 5.9}\\
&= \tilde{V}(\langle Z\rangle p,[x,y])
\end{aligned}$$

□

Proposition 5.12. Suppose $\langle X\rangle$ is not definable in terms of the fragment $\mathcal{F}$ over all linear orders in the crisp setting. Then for any Heyting algebra $\mathcal{A}$, $\langle X\rangle$ is not definable in terms of $\mathcal{F}$ over all $\mathcal{A}$-fuzzy linear orders.

Proof. Let $\mathcal{C}$ be the class of all $\mathcal{A}$-fuzzy linear orders and let $\mathcal{C}^{crisp}$ be all crisp members of $\mathcal{C}$. We proceed by contrapositively.

Suppose $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$. Then there is a $\phi \in \mathcal{F}$ such that

$$\langle X\rangle p \equiv_{\mathcal{C}} \phi$$

for all models $\mathcal{M}$ based on any $\mathbb{D} \in \mathcal{C}$.

But then

$$\mathcal{M},[x,y] \models \langle X\rangle p \leftrightarrow \phi$$

for all crisp models $\mathcal{M}$ based on any $\mathbb{D} \in \mathcal{C}^{crisp}$.

□

5.2 Interdefinability of HS modalities

In this section we show the interdefinability of the HS modalities. From Chapter 4, we have the following results.

Formula	Dual Formula	Definability
$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle \bar{L}\rangle p \equiv \langle \bar{A}\rangle\langle \bar{A}\rangle p$	$L \ntriangleleft_{\mathcal{C}}^{\mathcal{A}} A$
$\langle O\rangle p \equiv \langle E\rangle\langle \bar{B}\rangle p$	$\langle \bar{O}\rangle p \equiv \langle B\rangle\langle \bar{E}\rangle p$	$O \ntriangleleft_{\mathcal{C}}^{\mathcal{A}} E\bar{B}$
$\langle D\rangle p \equiv \langle B\rangle\langle E\rangle p$	$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$D \ntriangleleft_{\mathcal{C}}^{\mathcal{A}} BE$

Table 5.1: Interdefibilities of the dual formulas

Then by Proposition 5.10, we have that $\langle \bar{L} \rangle \not\trianglelefteq^{\mathcal{A}}_{\mathcal{C}} \bar{A}$ and $\langle \bar{O} \rangle \not\trianglelefteq^{\mathcal{A}}_{\mathcal{C}} B\bar{E}$. The formula for modality $\langle D \rangle$ is symmetric to its dual formula.

Proposition 5.13. Let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$ be $\mathcal{A}$-fuzzy model. Then for all $[x,y] \in \mathbb{I}(\tilde{\mathbb{D}})$,

$$\tilde{V}(\langle \bar{D} \rangle p, [x,y]) \leq \tilde{V}(\langle \bar{B} \rangle \langle \bar{E} \rangle p, [x,y])$$

Proof.

$$\begin{aligned}\tilde{V}(\langle \bar{D} \rangle p, [x,y]) &= \bigvee_{[z,t] \in \mathbb{I}(\tilde{\mathbb{D}})} (\tilde{R}_{\bar{D}}([x,y],[z,t]) \wedge \tilde{V}(p,[z,t]) \\ &= \bigvee_{[z,t] \in \mathbb{I}(\tilde{\mathbb{D}})} (\tilde{<}(z,x) \wedge \tilde{<}(y,t) \wedge \tilde{V}(p,[z,t]))\end{aligned}$$

and

$$\begin{aligned}\tilde{V}(\langle \bar{B} \rangle \langle \bar{E} \rangle p, [x,y]) &= \bigvee_{[w,s] \in \mathbb{I}(\tilde{\mathbb{D}})} (\tilde{R}_{\bar{B}}([x,y],[w,s]) \wedge \tilde{V}(\langle E \rangle p, [w,s]) \\ &= \bigvee_{[w,s] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[(\tilde{R}_{\bar{B}}([x,y],[w,s]) \wedge \bigvee_{[u,v] \in \mathbb{I}(\tilde{\mathbb{D}})} (\tilde{R}_{\bar{E}}([w,s],[u,v]) \wedge \tilde{V}(p,[u,v]) \right] \\ &= \bigvee_{[w,s],[u,v] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{R}_{\bar{B}}([x,y],[w,s]) \wedge \tilde{R}_{\bar{E}}([w,s],[u,v]) \wedge \tilde{V}(p,[u,v]) \right] \\ &= \bigvee_{[w,s],[u,v] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{=}(w,x) \wedge \tilde{<}(y,s) \wedge \tilde{<}(u,w) \wedge \tilde{=}(v,s) \wedge \tilde{V}(p,[u,v]) \right] \\ &= \bigvee_{[w,s],[u,v] \in \mathbb{I}(\tilde{\mathbb{D}})} \left[\tilde{=}(w,x) \wedge \tilde{=}(v,s) \wedge \tilde{<}(u,w) \wedge \tilde{<}(y,s) \wedge \tilde{V}(p,[u,v]) \right]\end{aligned}$$

Let $[z,t] \in \mathbb{I}(\tilde{\mathbb{Z}})$. We want to find $[w,s],[u,v] \in \mathbb{I}(\tilde{\mathbb{D}})$ such that

$\tilde{<}(z,x) \wedge \tilde{<}(y,t) \wedge \tilde{V}(p,[z,t]) \leq \tilde{=}(w,x) \wedge \tilde{=}(v,s) \wedge \tilde{<}(u,w) \wedge \tilde{<}(y,s) \wedge \tilde{V}(p,[u,v])$

If $\tilde{<}(z,x) \wedge \tilde{<}(y,t) \wedge \tilde{V}(p,[z,t]) = 0$, then any $[w,s],[u,v]$ will do.

Suppose $\tilde{<}(z,x) \wedge \tilde{<}(y,t) \wedge \tilde{V}(p,[z,t]) > 0$. Take $u = z, w = x$ and $s = v = t$. Then we have that

$$\begin{aligned}\tilde{<}(x,y) &\succ 0 \\ \tilde{<}(y,t) &\succ 0 \\ \tilde{<}(z,x) &\succ 0\end{aligned}$$

By weak transitivity(property (4)), it follows that

$$\tilde{<}(w,s) \succ 0$$
$$\tilde{<}(u,v) \succ 0$$

So, $[w,s],[u,v] \in \mathbb{I}(\tilde{\mathbb{D}})$.

Then $\tilde{=}(w,x) = \tilde{=}(x,x) = 1, \tilde{=}(v,s) = \tilde{=}(t,t) = 1, \tilde{<}(u,w) = \tilde{<}(z,x), \tilde{<}(y,s) = \tilde{<}(y,t)$ and $[u,v] = [z,t]$

Then,
$\tilde{=}(w,x) \wedge \tilde{=}(v,s) \wedge \tilde{<}(u,w) \wedge \tilde{<}(y,s) \wedge \tilde{V}(p,[u,v]) = 1 \wedge \tilde{<}(z,x) \wedge \tilde{<}(y,t) \wedge \tilde{V}(p,[z,t])$
and so
$\tilde{<}(z,x) \wedge \tilde{<}(y,t) \wedge \tilde{V}(p,[z,t]) \leq \tilde{=}(w,x) \wedge \tilde{=}(v,s) \wedge \tilde{<}(u,w) \wedge \tilde{<}(y,s) \wedge \tilde{V}(p,[u,v])$.

□

We will now show that the other inequality does not hold in general. We do this by finding the counterexample.

Let $\mathcal{A}$ be a three element Heyting chain $\{0 \prec \frac{1}{2} \prec 1\}$. Define $\tilde{<}(x,y) = min\{1, max\{\frac{1}{2}(y-x), 0\}\}, \tilde{=}(x,y) = max\{0, 1 - \frac{1}{2} \mid x - y \mid\}$, where $x, y \in \{0,1,2,3\}$

Consider the following model $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$ with $\tilde{\mathbb{D}} = \{0,1,2,3\}$, valuation $\tilde{V}(p,[0,3]) = 1$ and $\tilde{V}(p,[x,y]) = 0$ for all $[x,y] \neq [0,3]$ given by the following figure.

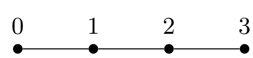


Figure 5.1

Then

$$\tilde{V}(\langle \bar{D} \rangle p, [0,2]) = 0$$

and

$$\begin{aligned}\tilde{V}(\langle\bar{B}\rangle\langle\bar{E}\rangle p,[0,2]) &\succeq \tilde{R}_{\bar{B}}([0,2],[1,3]) \wedge \tilde{V}(\langle\bar{E}\rangle p,[1,3])\\ &\succeq \frac{1}{2} \wedge (\tilde{R}_{\bar{E}}([1,3],[0,3]) \wedge \tilde{V}(p,[0,3]))\\ &= \frac{1}{2} \wedge \frac{1}{2} \wedge 1\\ &= \frac{1}{2}\end{aligned}$$

Proposition 5.14. Let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V}\rangle$ be a dense $\mathcal{A}$-fuzzy model and $\mathcal{A}$ a finite Heyting chain. Then for all $[x,y] \in \mathbb{I}(\tilde{\mathbb{D}})$,

$$\tilde{V}(\langle L\rangle p,[x,y]) \geq \tilde{V}(\langle O\rangle[E]\langle O\rangle\langle O\rangle p,[x,y])$$

Proof. Suppose that $\mathcal{A}$ is a finite Heyting chain.

($\geq$): Suppose that $\tilde{V}(\langle O\langle[E]\langle O\rangle\langle O\langle p,[a,b]) = \gamma \succ 0$ for an interval $[a,b] \in \tilde{\mathbb{M}}$. Consider the picture below

Figure 5.2

Then

$$\begin{aligned}\tilde{V}(\langle O\rangle[E]\langle O\rangle\langle O\rangle p,[a,b]) &= \bigvee_{[c,d]\in\mathbb{I}(\tilde{\mathbb{D}}} (\tilde{R}_O([a,b],[c,d]) \wedge \tilde{V}([E]\langle O\rangle\langle O\rangle p,[c,d]))\\ &= \tilde{R}_O([a,b],[c_0,d_0]) \wedge \tilde{V}([E]\langle O\rangle\langle O\rangle p,[c_0,d_0]) && \text{Lemma 1.13}\\ &= \gamma\end{aligned}$$

Let $\tilde{R}_O([a,b],[c_0,d_0]) = \sigma \succ 0$ and $\tilde{V}([E]\langle O\rangle\langle O\rangle p,[c_0,d_0]) = \beta \succ 0$.

Then

$$\tilde{R}_O([a,b],[c_0,d_0]) = \tilde{<}(a,c_0) \wedge \tilde{<}(c_0,b) \wedge \tilde{<}(b,d_0) = \sigma$$

So

$$\begin{aligned}\tilde{R}_E([c_0,d_0],[b,d_0]) &= \tilde{<}(c_0,b) \wedge \tilde{=}(d_0,d_0)\\ &= \tilde{<}(c_0,b) \wedge 1\\ &\succeq \sigma\end{aligned}$$

Therefore,

$$\beta = \tilde{V}([E]\langle O\rangle\langle O\rangle p, [c_0, d_0]) \\ \preceq \tilde{R}_E([[c_0, d], [b, d_0]) \rightarrow \tilde{V}(\langle O\rangle\langle O\rangle p, [b, d_0])$$

Then by the antitonicity of $\rightarrow$ in the first coordinate

$$\beta \preceq \sigma \rightarrow \tilde{V}(\langle O\rangle\langle O\rangle p, [b, d_0])$$

By residuation property of $\wedge$ and $\rightarrow$, we have

$$\gamma = \sigma \wedge \beta \preceq \tilde{V}(\langle O\rangle\langle O\rangle p, [b, d])$$

$$= \bigvee_{[u,v]\in\mathbb{I}(\tilde{\mathbb{D}}} \left[\tilde{R}_O([b, d], [u, v]) \wedge \bigvee_{[s,w]\in\mathbb{I}(\tilde{\mathbb{D}}} \tilde{R}_O([u, v], [s, w]) \wedge \tilde{V}(p, [s, w])\right]$$

$$= \tilde{R}_O([b, d], [u_0, v_0]) \wedge \tilde{R}_O([u_0, v_0], [s_0, w_0]) \wedge \tilde{V}(p, [s_0, w_0]) \qquad \text{Lemma 1.13}$$

where $\tilde{<}(b, u_0) \succeq \gamma$, $\tilde{<}(u_0, d) \succeq \gamma$ and $\tilde{<}(d, v_0) \succeq \gamma$, $\tilde{<}(u_0, s_0) \succeq \gamma$, $\tilde{<}(s_0, v_0) \succeq \gamma$ and $\tilde{<}(v_0, w_0) \succeq \gamma$.

Therefore, $\tilde{<}(b, u_0) \succeq \gamma$, $\tilde{<}(u_0, s_0) \succeq \gamma$, $\tilde{<}(b, s_0) \succeq \gamma$ by Property 4 of Proposition 5.4. Therefore,
$\tilde{R}_L([a, b], [s_0, w_0]) \succeq \gamma$ and $\tilde{V}(p, [s_0, w_0]) = 1$ and thus $\tilde{V}(\langle L\rangle p, [a, b]) \succeq \gamma$.
□

We now show that the other inequality does not hold in general. We do this by finding a counterexample.

Let $\mathcal{A}$ be the Heyting algebra $\{0 \prec \frac{1}{4} \prec \frac{1}{2} \prec 1\}$. Define $\tilde{<}$ and $\tilde{=}$ as follows:

$$\tilde{<}(a, b) = \begin{cases} 0 & \text{, for } b \leq a \\ 1 & \text{, for } b - a \geq 1 \\ \frac{1}{2} & \text{, for } \frac{1}{2} \leq b - a < 1 \\ \frac{1}{4} & \text{, otherwise} \end{cases}$$

and

$$\tilde{=}(a, b) = \begin{cases} 1 & \text{, for } b = a \\ \frac{1}{2} & \text{, for } \frac{1}{2} \leq |b - a| < 1 \\ \frac{1}{4} & \text{, for } 0 < |b - a| < \frac{1}{2} \\ 0 & \text{, otherwise} \end{cases}$$

where $a, b \in \mathbb{Q}$.

We now show that the above definition satisfies the properties of a fuzzy linear order.

1. $\forall x(\tilde{=}(x,y) = 1 \Leftrightarrow x = y)$ (*reflexivity of* $\tilde{=}$);
If $\tilde{=}(x,y) = 1$ then $|x-y| = 0$ and therefore $x = y$. If $x = y$ then $\tilde{=}(x,y) = 1$.

2. $\forall x, y(\tilde{=}(x,y) = \tilde{=}(y,x))$ (*symmetry of* $\tilde{=}$) ;
Since $|y - x| > 0$, we have that $|y - x| = |x - y|$ and $x = y$ is the same as $y = x$, it follows that $\tilde{=}(x,y) = \tilde{=}(y,x)$.

3. $\forall x(\tilde{<}(x,x) = 0)$ (*irreflexivity of* $\tilde{<}$);
Since $x \leq x$, it follows that $\tilde{<}(x,x) = 0$ if $x \leq x$.

4. $\forall x, y, z(\tilde{<}(x,z) \succeq \tilde{<}(x,y) \wedge \tilde{<}(y,z))$ (*transivity of* $\tilde{<}$);
Let $\tilde{<}(x,y) \wedge \tilde{<}(y,z) = min\{\tilde{<}(x,y), \tilde{<}(y,z)\} = \alpha$. Then we proceed by cases:
Case 1: $\alpha = 0$. Then we are done.

Case 2: $\alpha = \frac{1}{4}$.
Then $y - x > 0$ and $y - z > 0$ and therefore $z - x > 0$ and $\tilde{<}(x,z) \succeq \frac{1}{2}$.

Case 3: $\alpha = \frac{1}{2}$.
Then $y - x \geq \frac{1}{2}$ and $y - z \geq \frac{1}{2}$ and therefore $z - x \geq \frac{1}{2}$ and $\tilde{<}(x,z) \succeq \frac{1}{2}$.

Case 4: $\alpha = 1$.
Then $y - x \geq 1$ and $y - z \geq 1$ and therefore $z - x \geq 1$ and $\tilde{<}(x,z) \succeq 1$.

5. $\forall x, y, z(\tilde{<}(x,y) \succ 0 \;\&\; \tilde{<}(y,z) \succ 0 \Rightarrow \tilde{<}(x,z) \succ 0)$ (*transfer of positivity of* $\tilde{<}$);
Suppose that $\tilde{<}(x,y) \succ 0$ and $\tilde{<}(y,z) \succ 0$. Then we have that $x < y$ and $y < z$. Therefore $x < z$ and $\tilde{=}(x,z) \succ 0$.

6. $\forall x, y(\tilde{<}(x,y) = 0$ and $\tilde{<}(y,x) = 0 \Rightarrow \tilde{=}(y,x)) = 1((weak) totality)$;
Suppose that $\tilde{<}(x,y) = 0 \;\&\; \tilde{<}(y,x) = 0$. Then we have that $y \leq x$ and $x \leq y$. Therefore $x = y$ and by definition $\tilde{=}(x,y) = 1$.

7. $\forall x, y(\tilde{=}(x,y) \succ 0 \Rightarrow \tilde{<}(x,y) \prec 1)$(*non-contradiction of* $\tilde{<}$ *over* $\cong$).
If $\tilde{=}(x,y) \succ 0$ then $|y - x| < 1$ and therefore $y - x < 1$ and therefore

$\tilde{<}(x,y) \preceq \frac{1}{2} \prec 1$.

Therefore the definition satisfies all the properties of a fuzzy linear.

Consider the following model $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{Q}}), \tilde{V} \rangle$ with valuation $\tilde{V}(p, [\frac{1}{4}, \frac{1}{2}]) = 1$ and $\tilde{V}(p, [x,y]) = 0$ for all $[x,y] \neq [\frac{1}{4}, \frac{1}{2}]$, illustrated in the following figure.

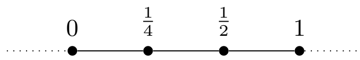


Figure 5.3

Then

$$\begin{aligned}
\tilde{V}(\langle L \rangle p, [0, \frac{1}{8}]) &= \bigvee_{[u,v] \in \mathbb{I}(\tilde{\mathbb{Q}})} (\tilde{R}_L([0, \frac{1}{8}], [u,v]) \wedge \tilde{V}(p, [u,v])) \\
&\succeq \tilde{R}_L([0, \frac{1}{8}], [\frac{1}{4}, \frac{1}{2}]) \wedge \tilde{V}(p, [\frac{1}{4}, \frac{1}{2}]) \\
&= \tilde{<}(\frac{1}{8}, \frac{1}{4}) \wedge \tilde{V}(p, [\frac{1}{4}, \frac{1}{2}]) \\
&= \frac{1}{4} \wedge 1 \\
&= \frac{1}{4}
\end{aligned}$$

But

$$\begin{aligned}
\tilde{V}(\langle O \rangle [E] \langle O \rangle \langle O \rangle p, [0, \frac{1}{8}]) &= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{Q}})} (\tilde{R}_O([0, \frac{1}{8}], [c,d]) \wedge \tilde{V}([E] \langle O \rangle \langle O \rangle p, [c,d])) \\
&\preceq \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{Q}})} (\frac{1}{4} \wedge \tilde{V}([E] \langle O \rangle \langle O \rangle p, [c,d])) \\
&= \bigvee_{[c,d] \in \mathbb{I}(\tilde{\mathbb{Q}})} (\frac{1}{4} \wedge (\bigwedge_{[e,f] \in \mathbb{I}(\tilde{\mathbb{Q}})} (\tilde{R}_E([c,d],[e,f]) \rightarrow \tilde{V}(\langle O \rangle \langle O \rangle p, [e,f]))))
\end{aligned}$$

Where the inequality follows from $\tilde{R}_O([0, \frac{1}{8}], [c,d]) = \tilde{<}(0,c) \wedge \tilde{<}(c, \frac{1}{8}) \wedge \tilde{<}(\frac{1}{8}, d)$ and c is between 0 and $\frac{1}{8}$, then $\tilde{<}(0,c) = \frac{1}{4}$ and therefore $\tilde{R}_O([0, \frac{1}{8}], [c,d]) = \frac{1}{4}$.

We now show that for each $[c,d]$ with $\tilde{R}_O([0,\frac{1}{8}],[c,d]) = \frac{1}{4}$, we can find an interval $[e,f]$ such that

$$\tilde{R}_E([c,d],[e,f]) \rightarrow \tilde{V}(\langle O\rangle\langle O\rangle p,[e,f]) = 0$$

Then we proceed by cases:

(i) $d \leq \frac{1}{4}$
Therefore we have that $0 < c < d \leq \frac{1}{4}$. Let $f := \frac{1}{2}$ and $e := c + \frac{1}{2} \mid d - c \mid$. So we have that

$$\begin{aligned}\tilde{R}_E([c,d],[e,f]) &= \tilde{<}(c,e) \wedge \tilde{=}(d,\frac{1}{2})\\ &\succeq \frac{1}{4} \wedge \frac{1}{2}\\ &= \frac{1}{4}\end{aligned}$$

Inddeed, we have that $c < d$ and by definition of e, we have have that $c < e$ and so $0 < e - c < \frac{1}{2}$, so $\tilde{<}(c,e) \succeq \frac{1}{4}$ and $0 <\mid d - \frac{1}{2} \mid< \frac{1}{2}$, so $\tilde{=}(d,\frac{1}{2}) \succeq \frac{1}{2}$.

But, $\tilde{V}(\langle O\rangle\langle O\rangle p,[\frac{1}{4},d]) = 0$ because if $\tilde{R}_O([\frac{1}{4},d],[u,v]) \wedge \tilde{R}_O([u,v],[x,y]) > 0$, then $\tilde{V}(p,[x,y]) = 0$. Therefore, $\tilde{R}_E([c,d],[\frac{1}{4},d]) \rightarrow \tilde{V}(\langle O\rangle\langle O\rangle p,[\frac{1}{4},d]) = 0$.

(ii) $d > \frac{1}{4}$
Therefore we have that $0 < c < \frac{1}{4} < d$. Let $f := d$ and $e := \frac{1}{4}$. So we have that

$$\begin{aligned}\tilde{R}_E([c,d],[\frac{1}{4},d]) &= \tilde{<}(c,\frac{1}{4}) \wedge \tilde{=}(d,d)\\ &\succeq \frac{1}{4} \wedge 1\\ &= \frac{1}{4}\end{aligned}$$

Indeed, $[\frac{1}{4},d]$ is still an interval since $d > \frac{1}{4}$. Since we have that $c < \frac{1}{4}$ and so $0 < \frac{1}{4} - c < \frac{1}{2}$, so $\tilde{<}(c,\frac{1}{4}) \succeq \frac{1}{4}$ and $\tilde{=}(d,d) = 1$.

But, $\tilde{V}(\langle O\rangle\langle O\rangle p,[\frac{1}{4},d]) = 0$ because if $\tilde{R}_O([\frac{1}{4},d],[u,v]) \wedge \tilde{R}_O([u,v],[x,y]) > 0$, then $\tilde{V}(p,[x,y]) = 0$. Therefore, $\tilde{R}_E([c,d],[\frac{1}{4},d]) \rightarrow \tilde{V}(\langle O\rangle\langle O\rangle p,[\frac{1}{4},d]) = 0$.

Proposition 5.15. Let $\tilde{\mathbb{M}} = \langle \mathbb{I}(\tilde{\mathbb{D}}), \tilde{V} \rangle$ be a dense $\mathcal{A}$-fuzzy model and $\mathcal{A}$ a finite Heyting chain. Then for all $[x, y] \in \mathbb{I}(\tilde{\mathbb{D}})$, the following hold.

$$\tilde{V}(\langle \bar{L} \rangle p, [x, y]) \geq \tilde{V}(\langle \bar{O} \rangle [B] \langle \bar{O} \rangle \langle \bar{O} \rangle p, [x, y])$$

Proof. We have that $\langle \bar{L} \rangle$ is the dual of $\langle L \rangle$ and $\langle \bar{O} \rangle [B] \langle \bar{O} \rangle \langle \bar{O} \rangle$ is the dual of $\langle O \rangle [E] \langle O \rangle \langle O \rangle$. By Proposition 5.11 and Proposition 5.14, the result follows. □

Chapter 6

Conclusion

Many-valued Interval-Based Temporal Logic (MVIBTL) was recently introduced by Willem Conradie, Dario Della Monica, Emilio Monoz-Velasco and Guido Sciavicco [8]. MVIBTL generalizes the interval-based temporal logic in such a way that propositional letters are not just true or false but they are true or false to some extent and this extent we take as a member of algebra of truth values. The Allen's relation that are used to interpret the modalities are also many-valued as they are based on many-valued linear order. Among one of the most fundamental questions one can ask about a logical language is a characterization of its expressivity, which was the main question around the thesis revolved. We studied modalities corresponding to Allen's relations that are (or not) expressible in terms of sets of other modalities corresponding to Allen's relations.

In this work we generalized the notion of the truth preserving morphisms of bisimulation and generated submodels from the crisp interval temporal logic setting to the many-valued interval-based temporal logic setting. We further showed that these generalizations preserve truth as shown in Theorem 3.3 and Theorem 3.5.

We use bisimulation to prove that many modalities which were definable in terms fragments in the crisp case are no longer definable in terms of those fragments in the fuzzy setting. In the crisp case, certain modalities were definable in terms of fragments. In [8], they only show that the specific equivalences that defined these modalities in those fragments do not hold in the fuzzy setting anymore. That brings the question of whether the isn't another formula that can define these modalities. What we have shown in this thesis is that there isn't another formula that can define these modalities. Some of these inequalities holds by assuming the Heyting algebra to be a finite Heyting chain when the class of fuzzy linear order is changed to the class of dense fuzzy linear order.

We generalized the study of definability in [1] to the fuzzy setting
The table below summarizes the interdefinabilities that were studied in the thesis on a class of all fuzzy linear order $\mathcal{C}$.

Defining equation in the crisp setting	crisp setting	fuzzy setting
$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$L \lhd_{\mathcal{C}}^{\mathcal{A}} A$	$L \not\lhd_{\mathcal{C}}^{\mathcal{A}} A$
$\langle O\rangle p \equiv \langle B\rangle\langle \bar{E}\rangle p$	$O \lhd_{\mathcal{C}}^{\mathcal{A}} B\bar{E}$	$O \not\lhd_{\mathcal{C}}^{\mathcal{A}} B\bar{E}$
$\langle \bar{L}\rangle p \equiv \langle \bar{A}\rangle\langle \bar{A}\rangle p$	$\bar{L} \lhd_{\mathcal{C}}^{\mathcal{A}} \bar{A}$	$\bar{L} \not\lhd_{\mathcal{C}}^{\mathcal{A}} \bar{A}$
$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$D \lhd_{\mathcal{C}}^{\mathcal{A}} EB$	$D \not\lhd_{\mathcal{C}}^{\mathcal{A}} EB$
$\langle \bar{D}\rangle p \equiv \langle \bar{E}\rangle\langle \bar{B}\rangle p$	$\bar{D} \lhd_{\mathcal{C}}^{\mathcal{A}} \bar{E}\bar{B}$	$\bar{D} \not\lhd_{\mathcal{C}}^{\mathcal{A}} \bar{E}\bar{B}$
$\langle \bar{O}\rangle p \equiv \langle B\rangle\langle \bar{E}\rangle p$	$\bar{O} \lhd_{\mathcal{C}}^{\mathcal{A}} B\bar{E}$	$\bar{O} \not\lhd_{\mathcal{C}}^{\mathcal{A}} B\bar{E}$

Table 6.1: Interdefinabilities of modalities on class of fuzzy linear orders

The table below summarizes the inequalities that were studied in the thesis on a class of all fuzzy dense linear order $\mathcal{C}$.

Defining equation in the crisp setting	Crisp Setting	Fuzzy Setting
$\langle L\rangle p \equiv \langle O\rangle[E]\langle O\rangle\langle O\rangle p$	$L \lhd_{\mathcal{C}}^{\mathcal{A}} EO$	$L \geq EO$
$\langle \bar{L}\rangle p \equiv \langle \bar{O}\rangle[B]\langle \bar{O}\rangle\langle \bar{O}\rangle p$	$\bar{L} \lhd_{\mathcal{C}}^{\mathcal{A}} B\bar{O}$	$\bar{L} \geq E\bar{O}$

Table 6.2: Inequalities of modalities on class of fuzzy dense linear order

There is still much to be done in the area of MVIBTL. Enverthing that is shown not be interdefinable in the crisp setting carries over to the fuzzy setting because the crisp setting is a special type of fuzzy setting. In order to arrive at a complete classification of the expressive power of MVIBTL, it remains to work on the remaining interdefinabilities that were not included in the tables above.

Another question to work on is if it is possible to study interdefinabilty over other classes of fuzzy linear orders without making additional assumption on the Heyting algebra. Looking at the interdefinabilities in Table 6.2, the assumption of the finite Heyting chain instead of a general Heyting algebra allowed us to show that its just that one inequality corresponding to the defining inequality in crisp case that still holds.

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