Appendix 1: Complete records of the coded transcripts of five video recorded episodes

This appendix contains the complete coded transcripts of five video recorded episodes. Each episode comprises the proof of one theorem or proposition attempted by one of the participants of my study. As discussed in Section 5.3, the proof transcripts have been divided into sub-episodes according to the following criteria:

- A sub-episode contains a complete proof or component of proof (where applicable) that is attempted by the student at the board.
- Any discussion or digression from the actual proof construction (while the proof construction is in progress) is allocated a separate sub-episodes, as although these discussions are vital and were analyzed to gain more insight into students' thought processes, I wanted to be able to focus on the actual proof construction taking place as a separate entity. Once the discussion is ended and proof construction is resumed, a new sub-episode is begun.
- When the discussion focusses on different themes such as a particular misconception or a more in-depth look at a different notion, these different notions were also isolated so that each notion or misconception can be looked at and discussed in its own sub-episode before going on to the next notion and the next sub-episode.

Session 1: Episode 1: A proof involving the implication sign

The first session of the weekly group discussion meetings was attended by ten of the participants who had agreed to take part. The first proposition tackled by the small group generates much discussion about the implication and bi-implication signs as well as the notion of subset. Included below is the first proof that the students attempted which involved the implication sign as well as all the relevant discussion that took place about the implication sign. The proposition that the first session started with was: *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$. The overall approach or main idea behind the proof is to assume that set *A* is a subset of set *B* and that set *B* is a subset of set *C* and then prove that set *A* is a subset of set *C*. The proof technique or method that is appropriate for this proof is to start off with an element, *x* in the set *A*, and show that this element is also an element of the set *C*, using the two given assumptions that *A* is a subset of *B* and *B* is a subset of *C*. Thus successful proof construction requires knowledge of the proof methodology of an implication proof and the precise definition of subset and the ability to use this definition in the logical reasoning and justification of each step in the proof. Frank came up to the board and attempted this proof.

Sub-episode 1.1: Frank's first attempt at the proof construction

	Speech and actions	**Student and teacher interactions**	**Proof comprehension**	**Interpretation according to Theoretical Framework (T.F.) and general comments on the proof construction process**
1.	*[Frank comes to the board]* **Frank**: I can show you the proof, the steps we can take to solve this proposition- So the first step is to let *x* be an element of *A*. *[writes: let* $x \in A$ *]* The first step that we must take it to let *x* be an element of *A*.	*Proposal of a new plan.* *Talking out loud while writing- saying what is being written and reasoning*	***L2a**: selection of correct opening statement* ***L1a**: selection of correct math symbols/signs* ***L2c:** selection of correct proof framework for implication and subset*	*Frank uses the word 'approximate' when referring to the double implication sign. This may be due to a lack of familiarity with the terms 'imply' and 'approximate'.* *Frank appears to have a good understanding of the proof method and the overall reasoning and justification necessary for each step.*
	So we approximate since *x* is an element of *A*, then *x* is an element of *B*. Then since here *A* is a subset of *B*.		***L1ax**: selection of inappropriate math language* ***L3a**: correct deduction from previous statement (spoken and written)*	

	[writes: $\Leftrightarrow x \in B$ *(since* $A \subseteq B$*)]*		***L1ax****: selection of inappropriate math symbols/signs (using* $\Leftrightarrow$ *instead of* $\Longrightarrow$*)* ***L1a****:besides the double implication, selection of correct math symbols/signs*	
	Since *x* is an element of *B* then we can approximate that *x* is an element of *C* since *B* is a subset of *C*.		***L3a****: correct deduction from previous statement (spoken and written)* ***L1ax****: selection of inappropriate math language (spoken)*	
	[*writes*: $\Leftrightarrow x \in C$ *(since* $B \subseteq C$*)*]		***L1ax****: selection of inappropriate math symbols/signs (using* $\Leftrightarrow$ *instead of* $\Longrightarrow$*)* ***L1a****: besides the double implication, selection of correct math symbols/signs (written)*	
	Then from here we see that *B* is a subset of *C* and the elements in *B* are contained in *C*. So even the element in *B* are contained in *A* so we said that then, let's see… *A* is a subset of *C*. Conclude [*writes*: *then* $A \subseteq C$]		***L3ax****: incorrect deduction* ***L3c****: correct conclusion* ***L1a****: correct mathematical symbols/signs in conclusion* ***L2c****: proof method of subset and overall reasoning apparently correct*	

Sub-episode 1.2: Edgar's suggestion when implemented reveals incomplete knowledge about the proof method, the implication symbol and the term 'suppose'

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to Theoretical Framework (T.F.) and general comments on the proof construction process
2.	**T**: Ok. Does everybody agree with that? Any problems there, any comments?	*Transactive prompt- request for reflection and critique*		
3.	**Edgar**: Ja, I just want to like, I don't know whether we need to start our proof… We said suppose that *A* is a subset of *B* and also *B* is a subset of *C* and then we specify what we need to do, what is it that we need to do in order to actually come up with something that completes the equation. I don't know, do we, don't we start by saying, 'Suppose is a subset of *A* and also that's a subset of *A*?	*Transactive question- request for clarification*	*H1a: trying to get the main ideas or approach used* *L2a: suggests selection of phrases that add to the logic of the proof construction such as 'suppose'*	*Edgar's description about the proof method or proof framework as an equation may be indicative of complex level thinking.*
4.	**Frank:** Okay you want me to write suppose *A* is a subset of *B* and *B* is a subset of *C* implies that *A* is a subset of *C* *[writes as he is speaking directly above his proof attempt: Suppose* $A \subseteq B$ *and* $B \subseteq C \Rightarrow A \subseteq C$*]*	*Transactive question- request for clarification*	***L2ax**: questioning purpose of opening statement involving "suppose" and writing statement which does not make sense*	*When inserting a statement that would add to the logic of proof construction, Frank shows that the word 'suppose' and the implication symbol might not be really understood.*

				This might also indicate pseudoconceptual thinking about the proof framework.
5.	**Gary**: The first thing when you said 'Suppose that *A* is a subset of *B*, right? And *B*'s a subset of *C*, so we don't have to say 'it implies' that. OK, we're thinking that if we're saying *A* is a subset of *B* and *B* is a subset of *C* it implies that we are supposed to prove that *A* is a subset of *C* so we don't have to say we suppose that it implies that.	*Contribution to proof*	*L1a: clarifying the use of the term "suppose"* *L2c: clarifying logical framework*	*Gary's description of the method of proof of an implication seems to be at concept level. Gary and Helen are developing Frank's understanding of the proof method and the terms 'suppose' and 'implies' in the EZPD.*
6.	**Frank**: OK, we don't have to say that, we rub that one out. But you show, you show some people who don't understand that that you're going to prove that you are? *[erases* $\Rightarrow A \subseteq C$ *and writes We need to show that* $A \subseteq C$*]*	*Transactive question- request for clarification*	***L2c**: clarifying logical framework*	
7.	**Helen**: But also be like, no, for the fact that we're saying that we need to show that *A* is a subset of *C* we don't, you don't have to say 'it implies, implies…'	*Contribution to an idea – point out error*	*L2c: clarifying logical framework*	*Helen's description of the proof framework seems to be at concept level.*
8.	**Frank**: OK. *[points to the board]* So when you say that all this statement means implies?	*Transactive question- clarification*	***L2a**: beginning to realize the meaning of implies*	
9.	**Helen**: Yes	*Transactive response- to agree*		
10.	**Frank**: OK, I understand.	*Moment of realization of the word 'implies'*		

Sub-episode 1.3: Discussion and clarification of the notions of implication and double implication and the method of proof of an implication

In sub-episode 1.3 Frank's attention is directed to the main flaw in the proof, that of using a double implication symbol instead of an implication symbol throughout the proof. In this sub-episode we see that the notions of implication and double implication do not seem to be well understood by the majority of the participants.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to Theoretical Framework (T.F.) and general comments on the proof construction process
11.	**Gary**: Can you explain about you're, you're saying the double implication signs?	*Transactive question- request for clarification*	*L1a: clarifying meaning of the double implication sign*	*Gary seems to have realized the incorrect use of the double implication sign and now turns his attention to address this error.*
12.	**Frank**: OK, this one?			
13.	**Gary**: Ja			
14.	**Frank**: This is for approximately. If you, we approximate that x an element of B *[points to* $\Leftrightarrow x \in B$ *(since* $A \subseteq B$*) on the board]* it can be implied, like you are implying that this x is an element of B since x is an element of A, this element can be in B *[points to the board]* since A is a subset of B, you see? Ja, you know what I'm saying? I suppose. You agree with me the way I	*Transactive response- to elaborate*	***L1ax****: incorrect language use giving way to* ***L1a:*** *correct language use for term implication* ***L1b****: correctly describes the concept of subset in own words*	*The description of the notion of subset seems to indicate concept level thinking about this mathematical object. Frank refers to the word 'approximate' when referring to the double implication indicating lack of familiarity with the term*

	did it?			*approximate and the double implication symbol.* *We see a shift from the word 'approximate' to the word 'implying'.*
15.	**Maria**: So if it was a similar proof, it would mean that *A* is not an element of *B*?	*Transactive question- request for clarification*	*L1ax: questioning meaning and consequence of the implication.* *L1ax: uses "element of" instead of "subset of" when talking about sets.*	*Maria seems to be associating the double implication sign with the notion of subset which might indicate complex thinking.*
16.	**Frank**: If it was what?			
17.	**Maria**: If like if the arrow wasn't pointing?	*Transactive question- request for clarification*	*L1ax:unclear about meaning of term, implication*	*The implication sign is likened to an arrow perhaps indicating complex thinking.*
17-31	*Unfruitful and confusing discussion taking place about arrows pointing in one way and two. The discussion is brought back to the notions of implication and double implication in line 31.*			
31.	**Maria**: Ja, what's the difference between…?	*Transactive question- request for clarification*		
32.	**Frank**: Oh there's no difference.	*Transactive response- to elaborate*	***L1ax***:*unclear about meaning of terms, implication and double implication*	*Frank seems to think that there is no difference between the implication sign and the double implication sign. This may be indicative*

				of complex thinking
33.	**Maria**: There's no difference?	*Transactive question- request for justification*		
34.	**Edgar**: OK, let me actually now try to explain *(pointing to the board)* Actually you see this one which shows an arrow going to that forward one, that one, if you use that one you are going to make sure that you prove this side, you prove that one there	*Contribution to an idea*	*L1b: describing the terms implication and double implication in own words*	*The double implication sign is described as arrows going forward and back but the method of proof seems correct perhaps indicating pseudoconcept level thinking.*
35.	**Helen**: Yes and then the other…	*Contribution to an idea*	*L1b: describing the terms implication and double implication in own words*	*Description of double implication sign as arrows going forward and back but method of proof seems correct perhaps indicating pseudoconcept level thinking.*
36.	**Edgar**: And then you are going to prove again on the other side.	*Contribution to an idea*	*L1b: describing the terms implication and double implication in own words*	*Description of double implication sign as arrows going forward and back but method of proof seems correct perhaps indicating pseudoconcept level thinking.*
37.	**Helen**: Yes	*Confirmation*		
38.	**Edgar**: So if you are using the double one	*Contribution to an idea*	*L1b: describing the terms implication and double implication in own words*	

39.	**Helen**: That means you have already shown…	*Contribution to an idea*	*L1b: describing the terms implication and double implication in own words*	*Description of double implication sign as arrows going forward and back but method of proof seems correct perhaps indicating pseudoconcept level thinking.*
40.	**Edgar**: Yes, if you are using the double one with arrows you know, that one is like what applies on one side will also apply on the other side.	*Contribution to an idea*	*L1b: describing the terms implication and double implication in own words*	*Description of double implication sign as what applies to one side will also apply to the other side but method of proof seems correct perhaps indicating pseudoconcept level thinking.*
41.	**Frank**: Ok			
42.	**Edgar**: So this one is a shortcut but as our lecturer has said, actually the best way is to use the longest method, because the other one you can explain more to, make you to understand.	*Contribution to an idea*		*Description of double implication sign as a shortcut.*
43.	**Edgar**: Ja, this one is also fine.			
44.	**Maria**: So are we supposed to write *A* is equals to *B* since *A* is a subset of *B*?	*Transactive question- request for clarification*	*L1bx: incorrect interpretation of subset, equality and implication*	*Maria is caught up in the confusion of seeing the subset relation and the double implication sign and wants to make the conclusion that sets A and B are equal perhaps indicating complex level*

				thinking.
45.	**Frank**: OK. *A* equals to *B* (*writes A = B*) Let me see *(steps back to look at the board)* Yes, you can say *A* equals to *B* because those elements in *B* are also in *A*. You can say that.	*Transactive response- to justify*	***L1bx**: incorrect interpretation of subset, equality and implication*	*Frank also seems to have the same viewpoint as Maria above. This might indicate complex level thinking of the double implication sign.*
46.	**Edgar**: Actually here, no,	*Contribution to an idea*		
47.	**S**: They are not…	*Contribution to an idea*		
48.	**Edgar**: Here we are not? We are not proving this one *(pointing to the board)* This is another one, so?	*Contribution to an idea*		
49.	**Frank**: No! If they are in *A*, they are in *B* there's no matter whether you say *A* is equal to *B*.	*Transactive response- to justify*	***L1bx**: incorrect description of subset, equality and implication*	*The subset relation seems to be associated with equality indicating complex level thinking about the notions of subset and equality of sets.*
50.	**Laura**: So, so, so okay, if you can just rub the imply sign and write equals…	*Transactive question- request for clarification*	*L1bx: incorrect description of subset, equality and implication*	*The double implication is associated with equality. This might indicate complex level thinking.*
51.	**Frank**: No, no, no you don't have to write the equals sign, *(pointing to the board)* you must write the imply sign.	*Transactive response- to clarify*		
52.	**Maria**: Yes, so what, you've just said…I didn't say we should write equals sign there, I mean at the	*Transactive question- request for clarification*	*L1bx: incorrect interpretation of subset, equality and implication*	*Maria on connecting $A \subseteq B$ with the double implication, makes the conclusion that*

	beginning of the proof aren't they supposed to say that *A* is equals to *B* and so imply that the elements which are in *A* are in *B*?			*sets A and B are equal, perhaps indicating that she is gaining an understanding of the double implication.*
53.	**Frank**: OK, you must, you're wanting me to rub here (*erases* $A = B$) and where must I write it?	*Transactive question- request for clarification*		
54.	**Maria**: Before 'suppose'	*Transactive response- to give an answer*		
55.	**Frank**: Before 'suppose'?			
56.	**Helen**: Can I say something? Before you write can I, can I, can I, can I say something? We have been given that *A* is a subset of *B*. And then we can't say *A* is equal, is not equal, is equals to *B* because we are not given that *B* is a subset of *A*.	*Contribution to an idea –point out misconceptions*	*L1b: explaining the meaning of subset and equality in own words*	*Correct description of subset and equality. This may indicate concept level thinking.*
57.	**Frank**: OK			
58.	**Helen**: Yes. We're only given that *A* is a subset of *B*. That is why we can't say that *A* is equal to *B* because we don't have *B* as a subset of *A*.	*Transactive response- to justify*	*L1b: explaining the meaning of subset and equality in own words*	*Correct description of subset and equality which may indicate concept level thinking.*
59.	**Frank**: OK			
60.	**Joseph**: Ja, it seems to say *A* is a subset of *B*, it doesn't necessarily mean that in every element that is in *A* are the same element that are in *B*. There may	*Contribution to an idea- giving examples- narrative*	*L1c: illustrating notions of subset and equality with examples*	*Correct description and explanation of notions of subset and equality which may indicate concept level*

	be…let's say *B* consists of elements of natural numbers and then *A* consists of elements that are even numbers.			*thinking.*
61.	**Frank**: The subset you are talking about is improper subsets.	*Transactive response- point of confusion*	***L1ax**: incorrect use of terms*	*The notions of subset and improper subset are associated together seemingly indicating complex level thinking.*
62.	**Joseph**: Now what I'm saying is that *A* doesn't necessarily mean it contains all the elements that are in *B*. It contains some elements.	*Transactive response- to elaborate*	*L1c: illustrating notions of subset and equality with examples*	*The description of subset and equality seems to indicate concept level thinking.*
63.	**T**: Sorry, just repeat what you were saying?	*Facilitative- attempt to re-structure discussion and focus attention on the example given by Joseph*		
64.	**Joseph**: The elements that are in *A*- they may not be all the elements in *B*. It means that those elements that are in *B*- some of those elements are in *A*.	*Transactive response- to elaborate*	*L1c: illustrating notions of subset and equality with examples*	*The description of the term subset seems to indicate concept level thinking about the notions of subset and equality of sets.*
65.	**T**: Can you tell us what that double implication means?	*Transactive prompts- request for clarification*		
66.	**Maria**: It means that… like if you are proving something which is, like you've got an equal sign like this side is equal to this..so if you put that double implication it means that what you are	*Proposal of new plan or strategy*	*L1b: explaining the meaning of the term double implication in own words*	*The double implication is associated with the more familiar equality sign which seems to indicate complex level thinking.*

	proving on the left you are sure that is equal to what you are proving on the right. Ja.			
67.	**T**: So…	*Transactive prompts- request for elaboration*		
68.	**Maria**: So you won't go back again proving that other side.	*Transactive response- to elaborate*	*L1b: explaining the meaning of the term double implication in own words*	*The description of the double implication seems to indicate that Maria's thinking is evolving from complex level towards pseudoconcept level.*
69.	**T**: OK, any other ideas about the double implication? I want to hear what everybody thinks… What is, what is the difference between the double implication and the implication? What is the… What do we mean? *(Points to a student)* Yes?	*Transactive prompts- request for elaboration*		
70.	**Gary**: Uh a double implication sign it simply means let's say if on the left hand side you have an equation, it means you can use the right hand side to go, to go back to the right hand side, to the left hand side and the other way round, you must leave the right hand side. That's how we do it.	*Proposal of new plan or strategy*	*L1b: explaining the meaning of the term double implication in own words*	*The double implication is described as having a left and right hand side. Method of proof is described as using the one side to prove the other and vice versa which may indicate pseudoconcept level thinking.*
71.	**T**: Ok. In this case what would you say about it?	*Transactive prompts- request for elaboration*		

72.	**Gary**: In this case I'd argue with that because if we say x is an element in A it does not necessarily mean like if you let x be an element in B it does not mean that we will find that x is an element in A because we are not given that A equals B. So that proof is not right.	*Transactive response- to explain*	*L1b: explaining the meaning of the term double implication in own words.*	*The error of using the double implication is pointed out showing good understanding (seemingly concept level) of the notions of implication, double implication, subset and equality of sets.*
73	**T:** And so what would you do?	*Transactive prompts- request for strategy*		
74	**Gary**: I'd remove the double implication sign	*Proposal of new plan*	*L1a: correctly replacing the symbol of the double implication with the symbol of implication.*	
75	**T**: And instead?	*Transactive response-request for strategy*		
76	**Gary**: The single implication sign.		*L1a: correctly replacing the symbol of the double implication with the symbol of implication.*	
77	**T**: Ok. Can you do that for us? Just explain. Go up and just explain what you said one more time to make it clear to everybody why you would do that.	*Transactive prompts- request for elaboration*		
78	**Gary**: *[erases the $\Leftrightarrow$]* I'll start by removing the double implication sign because if let's say we say let x be an element in B *[writes: let $x \in B$]*	*Transactive response- to explain Talking out loud while writing-reasoning*	*L1c: illustrating the notions of subset, implication and double implication with examples*	*Using a narrative example to illustrate the difference between the implication and double implication, Gary seems to exhibit concept*

	we are talking about if *A* is a subset of *B* *[points to* $A \subseteq B$ *]* and B is a subset of A, *[writes* $B \subseteq A$ *]* then we'll say if *x* is in *B* it means that we will have *x* in *A*, right?	*Giving examples-narrative*		*level thinking.*
	S: Yes			
79	**Gary**: If we are given that *B* is a subset of *A*. But in this case we are not given that *B* is a subset of *A* *[points to:* $B \subseteq A$ *]* so we cannot use the double implication sign here *[writes* $\Leftrightarrow$ *next to* $x \in A$ *]* because in this case we'll say *x* is an element of *B*. And if we use this sign *[points to* $\Leftrightarrow$ *]* it means that we will find that *x* is an element of *A* *[points to* $x \in A$ *]* But we are not given this statement that *B* is a subset of *A* *[points to* $B \subseteq A$ *]* That's why I undo that double implication sign *[points to the* $\Leftrightarrow$ *]* I simply use the single *[makes the* $\Leftrightarrow$ *into a* $\Rightarrow$ *]*	*Giving examples-narrative* *Talking out loud while writing-reasoning*	*L1c: illustrating the notions of subset, implication and double implication with examples*	*Using a narrative example to illustrate the difference between the implication and double implication, Gary seems to exhibit concept level thinking.*
	T goes on to highlight the learning of the difference between an implication and double implication. Edgar then comes up to try another version of the same proof which is very incorrect and then the lecturer brings back the discussion to the implication sign to	*Facilitative: highlighting learning*		

	ensure that everyone has clarity			
80	**T**: The discussion about the implication sign, that was very good, I think that was very helpful. Perhaps we could just clarify that a little bit more because I noticed that some of you are thinking of equality when you are thinking of the implication. *[T goes to the board and cleans some of it]* So maybe, maybe somebody can tell us what does, what really does an implication sign…, what does it mean if I say for example *P* implies *Q*. *[writes $P \Rightarrow Q$]* What does that mean? What do we mean when we say *P* and we know that this means *[points to the ⇒]* implies, right? *[under ⇒ writes implies]* This means 'implies'. What are we saying here? *[Points to a student]* Yes?	*Directive- providing immediate corrective feedback* *Transactive prompts- request for clarification* *Transactive prompts- request for clarification*		
81	**Edgar**: I think that in this case if we say that if *P* implies *Q* that means… after proving that it's true that *P* implies *Q*, we need to also prove the opposite side and the opposite way of *Q* being, implying to *P*.	*Contribution to an idea*	*L1bx: explaining the meaning of the terms implication and double implication in own words*	*The terms implication and double implication are associated together seemingly indicating complex level thinking.*
82.	**T**: To *P*?	*Transactive prompts- request for clarification*		

83	**Edgar**: Yes			
84	**T**: Then that would be a double implication. This is a double implication, what you are saying is a double implication. *[writes on board P $\Leftrightarrow$ Q]* So what does it mean first of all that P implies Q? What are we saying? What do we actually mean by this?	*Directive- providing immediate corrective feedback* *Transactive prompts- request for clarification*		
85	**Helen**: I think it actually means that *P* is not *Q* but it may be *Q*. I think	*Contribution to an idea*	*L1bx: explaining the meaning of the terms implication and double implication in own words*	*The description of the term implies seems to indicate complex level thinking.*
86	**T**: Maybe *Q*?	*Transactive prompts- request for clarification*		
87	**Helen**: Yes.			
88	**T**: Are there any other ideas about that? Any other ideas? What do we mean by *P* implies *Q*?	*Transactive prompts- request for clarification*		
89	**Edgar**: OK			
90	**T**: Yes?			
91	**Edgar**: I think that, I think if there are certain elements in *P* that means all of them they can be found in *Q*. But not all elements that are in *Q* can be found in *P*.	*Contribution to an idea*	*L1bx: explaining the meaning of the terms implication and double implication in own words*	*The term implies seems to be associated with the notion of subset which seems to indicate complex level thinking. Edgar talks about P and Q as if they are sets and not statements also showing incomplete*

				understanding and thus seeming to indicate complex level thinking for the terms implication/ implies.
92	**T**: But these are not necessarily sets.	*Directive- providing immediate corrective feedback*		
93	**Edgar**: Yes			
94	**T**: This is a statement – a statement *P* implies *Q*. *[points to a student]* Yes?	*Directive- providing immediate corrective feedback*		
95	**Joseph**: If *P* is true then *Q* will be true but you can't say if *Q* is true then *P* is true.	*Contribution to an idea*	*L1b: explaining the meaning of the terms implication and double implication in own words*	*The notion of implication is correctly defined. However when the same definition is used and interpreted to find the method of the actual proof, it is found that the understanding is not complete which seems to indicate pseudoconceptual thinking about the definition and proof method.*
96	**T**: Yes, I like that. If *P* is true then	*Facilitative- re-voicing*		
97	**Joseph**: *Q* is also	*Transactive response- to elaborate*		
98	**T**: *Q* is true *[writes If P is true, then Q is true]* That is a good definition. So that's all that this means. *P* implies *Q* means that	*Facilitative- confirming student ideas*		

	if P, if P is right, if P is true, then Q is true. *[Points to $P \Rightarrow Q$]* So when you want to prove this kind of thing that is why we start off with assuming that P is true. And then we move towards proving that Q is true. So in this case actually we could have written it as an implication. *[points to: <u>Proposition</u>: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$]* What could we have written here? You see we've got If … then, which is what is here *[points to If P is true, then Q is true]* you see? The if … then. Basically that's all an implication is. So how could we have written this statement?	*Didactive – ideas on nature of mathematics* *Transactive prompts- request for elaboration and strategy*		
99	**S**: If A implies B	*Transactive response- to give an answer*		
100	**T**: We could have had A is a subset of B and	*Transactive prompts- request for reflection and clarification*		
101	**T and some students**: B is a subset of C	*Transactive response- to give an answer*		*Students seem to be gaining understanding about the notion of implication.*
102	**T**: implies *[writes: $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$]*	*Transactive prompts- request for reflection and clarification*		

103	**Some students**: *A* is a subset of *C*	*Transactive response- to give an answer*	*L1a: correct use of implication*	*Students seem to be gaining understanding about the notion of implication.*
104	**T**: Exactly because that is what 'implies' would mean. OK, and then the double implication is basically going two ways. So it basically has this and as you said, it implies the other way as well. *[Next to $P \Leftrightarrow Q$ writes: ($P \Rightarrow Q$ and $Q \Rightarrow P$]* Yes, so then you would have to prove that if *P* is true, then *Q* is true and then the other proof would be if *Q* is true, then	*Facilitative- confirming student ideas* *Didactive – ideas on nature of mathematics* *Transactive prompts- request for reflection and clarification*		
105	**Some students**: *P* is true.	*Transactive response- to give an answer*	*L1a: correct use of implication*	*Students seem to be gaining understanding about the notion of implication.*

Session 1: Episode 2

The next theorem that I chose to focus on also took place in the first session. Theorem: *If A, B and C are sets, the following are equivalent:*

$$a) A \subseteq B$$

$$b) A \cap B = A$$

$$c) A \cup B = B$$

The proof method involves the following: a) $\Leftrightarrow$ b), b) $\Leftrightarrow$ c) and a) $\Leftrightarrow$ c). The proof of each double implication for example a) $\Leftrightarrow$ b), entails proving both implications: a) $\Rightarrow$ b) and b) $\Rightarrow$ a). I'll be highlighting Maria's attempt to prove a) $\Leftrightarrow$ b). The proof of a) $\Leftrightarrow$ b) encompasses the method of proof of an implication, the method of proof of equality of sets and the method of proof of showing that one set is a subset of another. A successful proof construction also requires knowledge of the precise definitions of set equality, subset and intersection and the ability to use these definitions in the logical reasoning and justification of each step in the proof. In this proof construction exercise Maria like many other students is seen to really be battling with most aspects of proof construction especially the proof methodologies and logical processes involved as well as the need for justification of deductions.

Sub-episode 2.1: Maria's first attempt at proof of a) $\Rightarrow$ b) or $A \subseteq B \Rightarrow A \cap B = A$

As can be seen in the following transcript Maria rushes into this proof, without having a clear idea of what showing the equivalence of statements a), b) and c) actually entails. Maria wants to start with showing that a) implies b) and then continue by showing b) implies c) and this seems to be a commonly held view since none of the other students say anything to the contrary. The lecturer does not seem to pick up the fact that the notion of equivalence is not clear and Maria starts off with attempting the proof of a) $\Rightarrow$ b) in sub-episode 2.1.

	Speech and actions	**Student and teacher utterances**	**Proof comprehension**	**Interpretation according to Theoretical Framework (T.F.) and general comments on the proof construction process**
	This theorem is put up on the board to be proved: *Theorem If A, B and C are sets, the following are equivalent.*			

	a) $A \subseteq B$ *b)* $A \cap B = A$ c) $A \cup B = B$			
1	**Maria**: *[goes to the front and attempts the proof starting with (a)* $\Rightarrow$ *(b)]* I think you have to show that (a) is equal, implies (b) and (b) implies (c). And this would mean that (a) implies (c) *[writes: (a)* $\Rightarrow$ *(b), (b)* $\Rightarrow$ *(c) (a)* $\Rightarrow$ *(c)]*	*Proposal of new idea* *Transactive argument- reasoning and explaining while writing*	***H1ax**: the main approach to be used is explained with some flaws* ***H1bx**: breaking down the proof into components, implication is used instead of double implication* ***L1a**: is able to use the term implies with ease and correctly uses the mathematical terms/ symbols/ signs*	*Recognizes that (a)* $\Rightarrow$ *(b) translates to* $A \subseteq B \Rightarrow A \cap B = A$ *She interchanges the term "equal" with the term "implies" showing that the two are associated together. This may indicate complex thinking.*
	So you start by saying to show that, OK, *A* is a subset of *B* implies that *A* is a… I forgot that name, what is it? *[writes:* $A \subseteq B \Rightarrow A \cap B = A$ *]*		***L2a**: select correct opening statement for starting the proof showing apparent knowledge of what needs to be done* ***L1a**: correct use of mathematical symbols/signs (written)*	
2	**S**: intersection	*Contribution to an idea*		
3	**Maria**: *A* intersection *B* which is equal to *A*. So from this if *A* is a subset of *B*	*Transactive argument- reasoning and explaining while writing*	***L2ax**: selecting incorrect statement to start the proof showing* ***L2cx:** lack of logical approach in the method of the proof*	*Adopting a method of proof which involves showing that the two sides of the implication are equivalent. This may indicate complex thinking of the proof method for*
	this means that, mmm, *x* is an element of *A*, which implies that *x* is also an		***L2bx**: selects non-useful or trivial deductions from*	

	element of *B*. And… *[writes: If* $A \subseteq B \quad x \in A$ $\Rightarrow x \in B$ *]*		*previous statements (spoken)* ***L1a**: the statement and non-useful deduction correctly written using mathematical symbols/signs*	*proving an implication since Maria associates this method with the more familiar method of proving an identity or equality.*
	Then we come to this side. That if *A* is an intersection of *B* which is equals to *A*		***L2ax**:select incorrect statement to continue the proof showing* ***L2cx:** lack of logical approach in the method of the proof;*	*The need for justification of each statement is not well grasped.*
	it will mean that *A* is a subset of *B*. And this would mean that *x* is an element of *A*.		***L3ax**: makes an incorrect deduction from previous statement (spoken)*	
	If and if *x* is an element of *A* it implies that it is also an element of *B*. *[writes on the other side of the board so it looks like this:* *If* $A \subseteq B$ *if* $A \cap B = A$ $x \in A$ $A \subseteq B$ $\Rightarrow x \in B$ $\Rightarrow x \in A$ $\Rightarrow x \in B$ *]*		***L2bx**: selects non-useful or trivial deductions from previous statements (spoken)* ***L1a**: the statement and non-useful deduction correctly written using mathematical symbols/signs*	
	Then I've proven this one and I come to the (b). Again let *A* and the intersection of *B* which equals to *A* and would imply that *A* is a union of *B* which is equals to *B* *[writes: let* $A \cap B = A \Rightarrow A \cup B = B$ *]* Then again if…		***L3cx**: Proof is concluded without any basis* ***L2cx**: Incorrect proof method and logic process*	

Sub-episode 2.2: Maria's explanation of first proof attempt

As the following transcript shows the lecturer and her peers try to develop Maria's understanding.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
4	**T**: Can you just explain that first part that you did one more time? I didn't get…	*Transactive prompt-request for clarification*		
5	**Maria**: Which one? This one? *[points to the first part and then to the second part she did on the board]* If, ok, here it says *A* is a subset of *B* and on this side it says *A* is an intersection of *B* which is equal to *A*.	*Transactive response- to elaborate and explain*	***L2ax**: selection of incorrect statement to begin the proof showing* ***L2cx:** lack of logical approach in the method of the proof*	*Maria's explanation seems to show that she views the implication proof as having 'two sides' and she seems to be working on arriving at the result of equality of 'both sides'.*
	And if *A* is an intersection of *B* which is equal to *A* it means that *A* is a subset of *B*.		***L3ax**: deduction made from previous statements without any justification*	*Statements are made without justification. The need for justification is not grasped.*
6	**Christine**: How? Isn't it that *A* intersection *B* is equal to *A* on the other side of the equals sign?	*Transactive question- request for clarification and justification*	*L1ax: implication referred to as "equals"* *L2c: questions reasoning used and the methodology of the proof*	*Christine refers to the implication as an equals sign and seems to look at an implication as two sides of an equation.*
7	**Maria**: Mmm?			

8	**Christine**: Aren't you supposed to say that *A* intersection *B* is a subset of *A* and the other way round?	*Proposal of new plan or strategy*	*L2c: questions reasoning used and the methodology of the proof*	*Christine tries to show Maria what she should be proving, that is the equality of the two sets A and A ∩ B indicating that her understanding of the proof method seems to be correct thus perhaps exhibiting pseudoconceptual thinking as her description or explanation of the implication as an equality is not correct.*
9	**Maria**: Ja, but we've got an equals sign here, meaning that *A* is a subset of *A* intersection *B*. At the same time *A* intersection *B* is a subset of *A*.	*Transactive response- to explain and justify*	***L1b**: recognizes implications of equality of two sets and provides the definition in her own words*	*Maria shows understanding of the notion of equality of two sets and is able to explain the meaning in her own words. She seems to think that the equality of the sets A∩B and A is a given and does not need to be proved- confirming uncertainty about the proving method.*
10	**Christine**: Would you say *A* is a subset of *B*?	*Transactive question- request for reflection and justification*	*L3b: questioning how an assertion is made from the previous statement without any basis*	*Christine's question shows that she has some understanding of the logical reasoning processes involved in the*

				proof.
11	**Maria**: Ja. Ok *[writes: $A \cap B = A$ a. $A \cap B \subseteq A$ and b. $A \subseteq A \cap B$]* *A* intersection *B* is equals *A* which means that *A* intersection *B* subset of *A*. Again *A* intersection, OK, again *A* is a subset of *A* intersection *B*.	*Transactive response- to justify and elaborate*	***L1b**: recognizes implications of equality of two sets and writes down the definition* ***L2bx**: correctly identifies the definition of equality of two sets but seems to think that this is a given rather than what is to be shown*	*Maria shows understanding of the notion of equality of two sets and is able to explain the meaning in her own words and write this in mathematical terms perhaps indicating concept level understanding of the equality of sets. However since her understanding of the proof method of an implication is lacking, this information is not used appropriately.*
	Ok, from this *[points to $A \cap B \subseteq A$]* would I be wrong if I say *A* is a subset of *B*? *[adds $\Rightarrow A \subseteq B$]*	*Transactive question- request for confirmation*	***L3ax**: new assertion made without any justification from previous ones*	*Maria is beginning to realize that she has made the assertion "A is a subset of B" without any justification. This seems to clearly indicate cognitive growth in the EZPD. Whereas before she seemed very certain of the wrong assertion, as a result of Christine's probing, she is now questioning that assertion.*
12	**T**: What does, what does everyone think? Can you make such a	*Transactive prompt- request for reflection*		

	conclusion?	*and justification*		
13	**Christine**: *[shakes her head]* I don't think so.	*Transactive response – to answer*	*L3a: recognizes that the assertion has been made incorrectly and does not follow from previous statements*	*The need for justification is being reinforced in the EZPD.*
14	**T**: You haven't shown us where it's coming from. Has she? Has she shown us where…?	*Facilitative-confirming students' ideas* *Transactive prompt-request for reflection*		
15	**Bonnie**: No	*Transactive response – to answer*	*L3a: recognizes that the assertion has been made incorrectly*	
16	**Maria**: Ok			

Sub-episode 2.3: Edgar's introduction of non-useful and trivial aspects

The lecturer tries to bring the discussion back to bear on the strategy used so that the correct proof method can be identified.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to Theoretical Framework (T.F.) and general comments on the proof construction process
17	**T**: Maybe if we just go back to the	*Transactive prompt-*		

	beginning. What are you trying to show, first of all?	*request for clarification and strategy*		
18	**Maria**: Here?			
19	**T**: Mmm			
20	**Maria**: I was trying to show that this [*underlines* $A \subseteq B$*]* implies this *[underlines* $A \cap B = A$ *in statement:* $A \subseteq B \Rightarrow A \cap B = A$*]*	*Transactive response- to clarify*	***H1a**: identifies the main idea behind the proof*	
21	**T**: So that's the first thing you want to show that *A* subset of *B* implies *A* intersection *B* equals *A*. So what do we start off with?	*Facilitative- confirming student ideas,* *Transactive prompt- request for strategy*		
22	**Edgar**: Isn't it that we know that *A* will always be a subset of *A*	*Transactive response- offer point of confusion*	*L2bx: introduces trivial implications of the notion of subset which are not useful in the proving process*	*Students are misled and taken out of ZPD as a result of trivial implications introduced. Thus knowing what aspects of mathematical objects and their definitions will be useful in the proving process is essential.*
23	**Maria**: Hmm?			
24	**Edgar**: *A* will always be a subset of *A*. Always. In other words always start with *A* being a subset of *A*.	*Transactive response- offer point of confusion*	*L2bx: introduces trivial aspects of the subset which are not useful in the proving*	*Edgar's introduction of non-useful aspects of mathematical objects or*

			process	*definitions may be indicative of his lack of strategic knowledge.*
25	**Maria**: Oh, here? Or there? *[points to the board]*			
26	**Edgar**: Ja, the first one.			
27	**Maria**: OK			
28	**Edgar**: And also *B* is a subset of *B*	*Transactive response- offer point of confusion*	*L2bx: introduces trivial aspects of the subset which are not useful in the proving process*	*Students are misled and taken out of ZPD as a result of trivial implications introduced.*
29	**Maria**: [*erases* If $A\subseteq B$ $x\in A$ $\Rightarrow x\in B$] *A* is a subset of *A*. And if *A* is a subset of *A* it means that *x* is an element of *A*. It implies that *x* is an element of *A* *[writes:* $A \subseteq A$ $\Rightarrow x \in A$ *]*	*Transactive argument- to explain while writing*	***L2bx**: tries to use aspects of the notion of subset which are not useful in the proving process* ***L2cx**: using incorrect proving process/proof framework*	*Maria is misled and taken out of ZPD as a result of the trivial aspects introduced. She desperately tries to use these aspects but doesn't really get anywhere.*
	And since *x* is an element of *A* and *A* is a subset of *B* it means that *A* is an element of *B*.		***L2bx**: trivial aspects used* ***L2ax**: incorrect choice of statement to begin proof*	
30	**S**: *x*			
31	**Maria**: Mmm? x?			
32	**S**: *x* is an element of *B*	*Contribution to the proof*	*L3a: deduction suggested from previous statement*	
33	**Maria**: Ja *[writes:* $\Rightarrow x\in B$ *(since* $A\subseteq B$*)* *erases what was written before]* *OK, let us come to this one.*		***L3a**: correct deduction from previous statement using the assumption* ***L2ax**: incorrect statement to*	*Maria is looking at the two sides of the implication as two entities on which work must be done in order to*

	A is an intersection of B is equals to A writes: $A \cap B = A$]		*begin the next part of proof*	*show equality perhaps indicating complex level thinking.*
34	**T**: Do, do you think…? Everybody, you must stop her *(laughs)* if you don't… Do you agree with what she has done so far?	*Transactive prompt- request for critique*		
35	*Other students ask Maria to stand in a way that will allow them to see the board as she tends to stand directly in front.*			
36	**Maria**: OK. *[writes: Thus $A \subseteq B$. Thus so far on the board:* $A \subseteq B$ $A \subseteq A$ $\Rightarrow x \in A$ $\Rightarrow x \in B$ *(since* $A \subseteq B$*)* *Thus* $A \subseteq B$ *]* I'll conclude that	*Transactive response- convinced of incorrect proof process and conclusion*	***L2cx**: not able to logically follow thought process of implication proof* ***L3cx**: The assumption made at beginning of proof is the same as the conclusion arrived at- no recognition of basis for a conclusion*	*Maria thinks that she has been successful in proving the implication proof although she has not done so yet- method of proof of implication has been associated with the method of proving equality perhaps indicating complex level thinking about the proof method of an implication.*

Sub-episode 2.4: Maria and other students their reveal complex thinking about proof method

The discussion continues with the lecturer drawing everyone's attention back to the proof method.

	Speech and actions	Student and	Proof comprehension	Interpretation according to

		teacher utterances		T.F. and general comments on the proof construction process
37	**T**: Ok. This is what I see a lot of, a lot of this kind of thing. I'm so glad you're, you have gone up and you have put it up because you know when I get the tests and the exams and exercises back and this is what I get and I want to understand why, what's going on here. So please explain to me what have you done, what have you started with, where are you going?	*Facilitative- highlighting misconceptions, providing encouragement Transactive prompt- request for reflection and strategy*		
38	**Maria**: Oh. I started with the first here – that one? *[points to a) $A \subseteq B$ b) $A \cap B = A$ c) $A \cup B = B$]* Here I want to prove that a) implies b)	*Transactive response- to clarify and elaborate*	***H1a**: explaining the over-arching approach used*	
39	**T**: Ok			
40	**Maria**: And b) implies c)	*Transactive response- to clarify and elaborate*	***H1a**: explaining the over-arching approach used*	
41	**T**: Ok			
42	**Maria**: And if a) implies b) and b) implies c) we know that a) implies c). So I came here, I wanted to prove that a) is true.	*Transactive response- to clarify and elaborate*	***L2cx**: has incorrect idea about how to prove an implication*	*Maria is under the impression that she should first prove that the statement to the left of the implication sign is true. This is probably as a result of associating the method of proof of an implication with an*

				implication that is given or true which may indicate complex level thinking.
43	**T**: You want to prove that a) is true?	*Transactive prompt- request for reflection and clarification*		
44	**Maria**: Ja			*Confirming that when proving the implication, Maria thinks that she must prove the statement before the implication seeming to indicate complex thinking.*
45	**T**: But write down for me, you remember we discussed what implication means? What does a) implies b) mean?	*Transactive prompt- request for reflection and clarification*		
46	**Maria**: It means that if a) is true, then we know b) is true.	*Transactive response- give an explanation of the term implies*	***L1b**: Maria defines what a) implies b) means correctly* ***L2cx**: the method of proving an implication however is not correctly understood and is at the heart of her misconception*	*Maria is associating the method of proof of an implication with the definition of the implication which may indicate complex thinking about the proof method. This is revealed as one of the biggest misunderstandings when students attempt implication proofs. They seem to think that it follows from the definition of an implication that they must first prove a) to be true and from that it will*

				automatically follow that b) is true, not realizing that they need to prove that b) is true in order to prove that the implication is true!
47	**Edgar**: But if b) is true it doesn't mean that a) can be true.	*Transactive response – offer a point of confusion*	*L2bx: brings in an aspect of the implication that is not useful to the proving process*	*This may show Edgar's lack of strategic knowledge as Edgar introduces irrelevant points in the discussion.*
48	**T**: So if a) is true then b) is true. That's what you're trying to prove, right? If a) is true, then b) is true. So you start off with assuming something. What is what you start off with? What do you assume?	*Facilitative- attempting to structure discussion and proof writing Transactive prompt- request for strategy and reflection*		
49	**Maria**: I assume that…			
50	**T**: Don't rub everything out. Let's leave it. What do you assume?			
51	**Maria**: I assume that (a) implies (b) and (b) implies (c) and I want to show that (a) implies (c).	*Transactive response- trying to clarify*	***L2cx**: showing no understanding of proof process and the method of proof of the implication that is being attempted*	*Maria brings in irrelevant information- showing incomplete understanding of proof methodology and technique. She is also using the word 'assume' incorrectly showing that she probably does not know the correct meaning of the word.*
52	**T**: Ok, somebody help her. What do you assume?	*Transactive prompt- request for strategy and reflection.*		

		T is frustrated as she doesn't understand the source of difficulty		
53	**Frank**: Assume *A* is a subset of *B* – you'll assume that. Then you'll be fine.	*Transactive response- to clarify*	*L2a: identifies the statement that needs to be assumed in the proof*	*Frank is showing good understanding of the implication proof method and what needs to be assumed which may indicate concept level thinking about the implication proof method.*
54	**Joseph**: You are saying if (a)'s true then (b) will be true. Now let's prove a) and why it's true, né? Then let *x* to be an element of *A* and see if it leads us to say *x* will be an element of *B*. Then if that is true it means that b) is true.	*Contribution to the idea- bringing in a point of confusion*	*L2cx: incorrect description of the proving process involved in an implication*	*Joseph shows similar incorrect methodology (as Maria) of proving an implication- he also wants to starts by showing that a) is true and thinks that it will automatically lead to b) being true, which may indicate complex level thinking.*
55	**S**: also true	*Transactive response- to agree*		*Other students have similar ideas.*
56	**T**: Do you agree with that?	*Transactive prompt- request for reflection*		
57	**Some students**: No			
58	**T**: Do you all agree with that?	*Transactive prompt- request for reflection*		
59	**Some students**: No			
60	**T**: No? OK, tell me some other ideas. *(laughs)* What are we trying to do here?	*Transactive prompt- request for reflection and strategy*		

61	**Gary**: First of all we are trying to show if *A* is a subset of *B* it will mean that it might take, it might lead us to *A* being an intersection *B* being equal to *A*. So what we must do now is that our assumption will be that *A* is a subset of *B*. After that we use our assumption to prove that *A* is an intersection of *B* which will be equal to *A*.	*Transactive response- to explain and elaborate*	*H1a: main idea behind the proof correct L2a: clarifies the reasons behind making a particular assumption and what needs to be proved, L2c: has correct idea of how to prove the implication*	*Gary correctly describes the method of proof to be used in this implication proof using all the correct mathematical language and terms which may indicate concept level thinking about the proof method and related notions such as implication and assumptions.*
	Between lines 61 and 62, there are quite a few transactive prompts and facilitative utterances from the lecturer, where Gary is prompted to give further elaboration and write on the board. In order to save space I have not included these.			
62	**Gary**: *[writes: Assume $A \subseteq B$] [writes: We show that $A \cap B = A$] [above Assume $A \subseteq B$ writes: $A \subseteq B \Rightarrow A \cap B = A$ Thus written on the board (a) $\Rightarrow$ (b) $A \subseteq B \Rightarrow A \cap B$ $=A$ Assume $A \subseteq B$ We show that $A \cap B = A$]* *A* is a subset of *B*	*Transactive response- giving the answer*	*L2a,L2c: identifies the assumption statement and what is needed to be shown and uses phrases that add to the logic of the proof construction. He also describes and uses the correct proof methodology L1a: correctly uses mathematical language/symbols/signs*	*Gary correctly translating his spoken description of the proof method for this implication proof into written form of mathematical terms, signs and symbols. Gary and other students are creating the EZPD in which Maria and others are helped to have a better understanding of the method of how to prove an implication.*
63	**T**: implies	*Transactive prompt-request for clarification*		
64	**Maria**: *A* intersection *B* equals *A*	*Transactive*	***L2a**: identifies statement to be*	*Maria participating and*

		response- to give an answer	*proved in an implication*	*developing understanding in the EZPD.*
65	**T**: So do you see that, just explain now why, what we are…	*Transactive prompt- request for clarification*		
66	*(some laughter)* **T**: There seems to be a lot of confusion there.			
67	**Gary**: OK. First of all we say we must show that (a) implies (b) *[points to (a) ⇒ (b)]* meaning that *A* is a subset of *B* implies that the *A* intersection of *B* will give us *A*. Right? So first of all we must show that, we must assume that a) is true. That's why we say assume that *A* is a subset of *B*. From this assumption we must show that it will lead us to *A* being a subset of *B* which will give us *A*…	*Transactive argument- explaining what has been written on the board*	*H1a: main idea behind the proof explained* *L2a: identifies the assumption statement and what is needed to be shown and identifies the purpose of sentences in the proof,* *L2c: explains the method of the proof*	*Gary may be exhibiting concept level understanding of the proof method of an implication. He and other students are creating the EZPD where the method of proof of an implication is clarified.*
68	**S**: *A* intersection *B*	*Transactive response- to give an answer*	*L2a: identifies what is needed to be shown*	*Other students participating in EZPD.*
69	**Gary**: Ja			
70	**T**: Ok. Are you happy now? Do you get where we are going?	*Transactive prompt- request for reflection*		
71	**Maria**: Ja	*Confirmation*		
72	**T**: Because you said that we were trying to prove (a) implies (b), right? But you didn't seem to know what that means. This is what it means. You've	*Transactive prompt- request for clarification*		

	got it now?			
73	**Maria**: Ja			
74	**T**: a) implies b). This is what it means, that *A* subset of *B* implies that *A* intersection *B* equals *A*. So our starting point is that *A* is a subset of *B*. We don't have to prove that. We don't have to prove that *A* is a subset of *B*, we start off with that assumption – that *A* is a subset of *B*. OK? Are you alright?	*Facilitative-confirming student's ideas and highlighting learning*		
75	**Maria**: Ja			

Sub-episode 2.5: Maria's second attempt at proof of $A \subseteq B \Rightarrow A \cap B = A$ revealing pseudoconceptual thinking of proof method of equality

Maria then continues (with Gary's and other students' help) to complete the proof of the first component. This is shown in the following excerpt of dialogue.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
76	**T**: Ok. Go ahead			
77	**Maria**: So what I don't get here is that am I supposed to prove this or this *[points to: Assume $A \subseteq B$ We show that $A \cap B = A$]*	*Transactive question-request for clarification*	***L2cx**: still not certain of what is assumed and what is to be proved in an implication*	*Although the lecturer has just explained the method of proof and the reasoning behind it, and in addition these notions have been*

				explained by Gary and other students, Maria has still not quite grasped the method but she shows improvement in that she has now realized that she must prove one of the statements $A \subseteq B$ or $A \cap B = A$. She still does not seem to have an understanding of the word 'assume', or what is written on the board- as what she needs to show is clearly written there.
78	**Helen**: The second intersection b) part	*Transactive response- to clarify*	*L2a: identifies the correct statement to be proved*	
79	**Maria**: The second one?	*Transactive question- request for clarification*	***L2cx**: still seems to be surprised at statement to be proved- still not sure of proof method*	*Maria is helped by Helen in the EZPD created to realize the assumption and what needs to be proved in an implication.*
80	**Helen**: Yes	*Transactive response- to clarify*	*L2a: identifies the correct statement to be proved*	
81	**Maria**: Ok. From this *A* intersection *B* equals to *A* *[writes: $A \cap B = A$]*	*Transactive argument- saying what's being written*	***L2ax**: incorrect statement for beginning the proof- writes down what is to be proved as if it is already true*	*It is not clear from this first step, whether Maria recognizes that this is what she needs to prove, and the proof method to be*

				used.
	It's like this, it means that Ok, let *x* be an element of *A* *[writes: let* $x \in A$		***L2a****: correctly identifies how to go about proving the statement*	*Taking the first step in the proof correctly, it seems that Maria has correct understanding of the proof method for equality.*
	and if $x \in A$ *then* $x \in (A \cap B)$*]*		***L3ax****: makes an incorrect deduction without any justification seemingly using the statement to be proved as a given fact.*	*Maria still has not grasped the need for justification for each statement and seems to want to take the easy way out reaching the conclusion immediately by making an unjustified deduction.*
82	**T:** Tell us, what are you trying to do now? It's good. You're on the right track but just explain why you do that?	*Transactive prompt- request for explanation*		
83	**Maria**: I want to prove that like if like *A* and *A* intersection *B*, these things have something in common *(pointing to* $A \cap B$ *and A)*	*Transactive response- to clarify and explain*	***L2c****: looking at the proof of equality of two sets as showing that the two sets have something in common*	*This may indicate complex or pseudoconceptual thinking of the methodology of the proof of equality of two sets (which incorporates proof method of showing that one set is a subset of another) as Maria's method seems to be correct but her description of that method is incomplete: "these things*

				have something in common"
84	**T**: Mmm			
85	**Maria**: And that thing is x. So I want to show that if x is contained in A it will also be contained in B where they intersect.	*Transactive response- to clarify and explain*	***L2c**: proving equality of two sets by showing that the two sets have element x in common*	*This may indicate pseudoconceptual thinking of the methodology of the proof of equality of two sets.*
86	**Edgar**: If they're contained in A, if x is contained in A and it's also contained in B then they will give us, the answer will be the same. You said what you want to assume. And then if A, if x is in A and x also is in B, that means we'll get A intersecting B because…	*Transactive response-offer point of confusion*	*L2cx:confusion about the logic and reasoning behind implication proof*	*Edgar going off at a tangent with the notion of intersection.*
87	**Maria**: Ja. If x is in A and also in B?			
88	**Edgar**: Yes			

Sub-episode 2.6: Discussion and clarification of the notions of intersection and union

The discussion then leads Laura (line 89) to ask about the difference between the union and intersection of sets and this leads to a very interesting discussion where examples are used to develop understanding:

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
89	**Laura**: Can I ask something? What is the	*Transactive question-*	*L1b: questioning the terms*	*Laura feels comfortable in*

	difference between intersect and union?	*request for clarification*	*intersection and union*	*the EZPD created to ask about ideas that are bothering her which she wouldn't have the opportunity to do otherwise*
90	**Edgar**: Intersection and union?			
91	**Laura**: Ja			
92	**Edgar**: OK, um, union you actually write everything. If there are, if there are elements in *A* you write them all, you include those that are in *B*.	*Transactive response-explanation*	*L1b: explaining the meaning of the term union in own words*	*Seemingly correct thinking of the term union: "writing everything", "write them all", "include elements in both sets" indicating concept level thinking of the notion of union.*
93	**Laura**: Ok			
94	**Edgar**: Ja, you write them all. But the other one of			
95	**Laura**: intersection			
96	**Edgar**: intersection, it means that element, you find that elements that are in *A* they are also in *B*. They are there in *B*. But elements that are in *B* you might not find them all in *A*	*Transactive response-explanation*	*L1b: explaining the meaning of the term intersection in own words*	*This may indicate complex level description of the intersection– seemingly associated with the notion of subset.*
97	**Laura**: Oh			
98	**Edgar**: That's what, that's what I think.			*Edgar seems to realize that he might not have the full vision of the notions of union and intersection*

99	**Gary**: OK, ah, can I say something about the union and intersection. At the union it's either let's say for example *x*, let's say *x* is in *A*, right? If we say union it's either in *A* or in *B*, it cannot be in both *A* and *B*. But then if you say intersection it means *A* and *B*, all of them, they contain *x*.	*Transactive response- clarification and explanation*	*L1bx: incorrectly describing the meaning of the terms intersection and union in own words*	*Gary's thinking of union and intersection seems to be at complex level. He seems to have been confused by the definitions of union and intersection.*
100	**T**: Would you go up and draw a Venn diagram and just show us as a picture what does union and intersection mean? I think that will clarify it.	*Transactive prompt- request for clarification through use of pictorial examples*		*The lecturer encourages the use of examples (in this case venn diagrams) to clarify confusion about mathematical objects.*
101	**Gary**: *[goes to the board and draws 2 intersecting circles A and B. The part that intersects he colours in black. Writes: Intersection]* This one will be *A*. Right. This one is for intersection. Right? Then for union. *[writes:* Union *and draws the same 2 intersecting circles. The part that intersects he leaves white, the part on the left that does not intersect he fills with dots, and the part on the right that does not intersect he fills with lines and he points to the intersection diagram]* Here *A* and *B* they have a common number, let's say for *x*, that's why we say it's an intersection – it means *x* is in both *A* and *B*. This is where they intersect *[points to the black part]* *[Points to the union diagram]* Then here	*Contribution to proof-writing and clarification of mathematical objects; giving pictorial examples*	*L1cx: incorrectly illustrating the notions of intersection and union with examples* *L1bx: incorrect description and example for union. The union is shown not to contain the intersection*	*This may indicate complex thinking about the union. Using Venn diagrams and shading Gary depicts the intersection and union of two sets. The depiction of the union of two sets clearly shows what seems to be a commonly held misconception; that the union does not contain elements in the intersection of the two sets. This could perhaps have been brought about because of the definition of the union of two sets;* $A \cup B = \{x: x \in A \text{ or } x \in B\}$*. Thus students may be*

	it's… Let's say this *A [left-hand circle]* and this *B [right-hand circle]* The value that we find in *A* is not in *B* but			*getting confused and thinking that x may be in A or in B but not in both (exclusive 'or' versus inclusive 'or'). Thus here the definition seems to bring confusion.*
102	**S**: Those, no, like what he is trying to say, he is trying to say that x is either in *A* or in *B*. Intersection is both in *A* and in *B*.	*Contribution to proof-writing and clarification*	*L1b: clarifying the meanings of intersection and union in own words.*	*Other students confirm that Gary's view is a commonly held perception that the elements of the union are:"either in A or in B", indicating complex level thinking.*
103	**T**: But in the union… Isn't the intersection in the union?	*Transactive pompt- request for clarification*		
104	*[Gary returns to his seat]* **Kenny**: It's there at the union. You can find that *x* is in *A* and *x* is in *B*, but when you write *A* union *B* you can't repeat that *x*.	*Transactive response-clarification and explanation*	*L1b: clarifying the meanings of intersection and union*	*Seemingly having correct thinking about the union, explaining that elements in the intersection are included but only written once.*
105	**Helen**: You use it once?	*Transactive question-request for clarification*		
106	**Kenny**: Ja, you use it once, but it can be both in *A* and *B*.	*Transactive response-clarification*	*L1b: clarifying the meanings of intersection and union*	*Seemingly having correct thinking about the union, explaining that elements in the intersection are included but only written once.*
107	**T**: Any other ideas, remarks?	*Transactive pompt- request*		

		for clarification		
108	**Christine**: Can I try another example?	*Contribution to idea*	*L1c: illustrating the notions of intersection and union with examples*	*Students realizing that using examples can bring much more clarity.*
109	**T**: Ja			
110	**Christine**: *[goes to the board]* I say you've got *A* here and the numbers 1, 2, 3, 4, right? *[draws a circle with 1, 2, 3, 4 inside the circle and labels it A.* Then *B*. It's 5, 3, 4, 7 *[draws another circle, B, with 5, 3, 4, 7 inside the circle]* Right? Then the intersection is going to go like this. *[Draws 2 intersecting circles – with 1, 2 in the left circle, 5, 7 in the right circle and 3, 4 in the intersection.]* They've got like 2 common numbers – 3 and 4 here *[points to the first two circles]* and 3 and 4 here. Then if it was a union then it would be something like this. *[Draws 2 intersecting circles with 1, 2, 3, 4 in the left circle and 5, 7, 8, 9 in the right circle with nothing in the intersection]* They don't have anything in common. That's what I'm thinking. You get it?	*Contribution to proof-writing and clarification of mathematical objects; giving pictorial examples*	*L1c: illustrating the notions of intersection and union with examples* *L1bx: incorrect explanation and description of the union.*	*This example is more detailed than the last given by Gary as it shows elements of sets and thus clarifies which elements are included in the intersection and union. Christine gives two examples, the first example shows two sets that have two elements in their intersection and she depicts the intersection correctly. Then in order to depict the union of two sets, she chooses another two sets such that these two sets have no elements in their intersection. She depicts the union of these sets correctly but since the example for the union is engineered such that the two sets do not have any elements in their intersection, it gives the*

				impression that when considering the union of two sets there cannot be an intersection. This is emphasized by Christine saying: "They don't have anything in common".
111	**Maria**: Mmm			
112	**T**: Any other ideas?	*Transactive prompt- request for more clarification*		
113	**Maria**: So for union they don't have anything in common?	*Transactive question- request for clarification*	***L1b**: seeking clarity for the term union*	*Maria questioning to make sure that she has understood correctly. One of the drawbacks of students learning from each other in the EZPD is that when a wrong idea or contribution is made, it is also spread to the other participants unless there are more knowledgeable peers / lecturers/ tutors who correct this thinking.*
114	**Joseph**: No, for union it's just a combination of all the numbers.	*Transactive response- clarification*	*L1b: the term union still being clarified in own words*	*Joseph seems to agree with Christine and Gary's explanations and exhibits complex thinking.*
115	**Bonnie**: OK			
116	**Joseph**: But in those where there's 3 and 4 you don't have to repeat	*Transactive response- clarification*		
117	**Edgar**: Another thing… Another thing is in	*Contribution to proof-*	*L1c: illustrating the concepts*	*Edgar bringing in another*

	the Venn diagram *[goes to the board]* I think if we have elements *A* like, let's take 2, 3, 4, 6. Ja, if we've got these ones *[writes: A = {2, 3, 4, 6}]* And *B* we've got 0, 1, 2, 3 and then 4 *[writes: B = {0, 1, 2, 3, 4}]* For intersection, *A* intersecting *B*, what we write is a, we look at the elements which are the same from both *A* and *B* which is 2, 3,	*writing and clarification of mathematical objects; giving pictorial examples*	*of intersection and union with examples*	*example for further clarity. This time the sets A and B do have elements in their intersection.*
118	**S**: and 4	*Contribution to proof*		
119	**Edgar**: 2, 3, 4 and nothing *[writes $A \cap B$ = {2, 3, 4}]* *(laughter)* So *A* union *B* we include all these elements starting, we start from zero, even those that are not contained in here *[writes: $A \cup B$ = {0, 1, 2, 3, 4, 6}]*	*Contribution to proof-writing and clarification of Mathematical objects; giving pictorial examples*	*L1c: illustrating the notions of intersection and union with examples*	*Edgar correctly identifies both the intersection and union of two sets indicating concept level thinking about the notions of intersection and union. Edgar's participation in the EZPD is hopefully shedding light on the notions of intersection and union and guiding other students.*
120	**T**: Ok, I like that	*Facilitative- confirming and encouraging*		
121	**Edgar**: 0, 1, 2, 3, 4 and 6		*L1c: illustrating the notions of intersection and union with examples*	
122	**Edgar**: That's what I think			
123	**T**: OK, so in that Venn diagram where you showed the union what was wrong with that	*Facilitative- highlighting learning/misconceptions*		

	one? You see, in the intersection you had 2, 3 and 4 but in the union you still have the 2, 3 and 4 don't you?			

Sub-episode 2.7: Maria's third attempt at the proof of $A \subseteq B \Rightarrow A \cap B = A$, discussion and clarification of method of proof of equality

The lecturer tries to re-direct discussion towards proof construction strategy:

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
124	**T**: Alright, where are we trying to go here? We still don't have a very clear picture of that. We are assuming that A is a subset of B, we want to show that A intersection B equals A, which means what? What are we trying to show? We are trying to show two sets are equal. What do we have to do to show that two sets are equal?	*Transactive prompt- request for strategy, reflection, clarification*		
125	**Joseph**: Prove the left then prove the right. So you use	*Transactive response- explanation*	*L2cx: incorrect/confused description of methodology of proof of equality of two sets*	*Joseph's description of the proof methodology to prove equality of sets seems to be at complex level, as he seems to think of the proof of equality of sets as proving the left and right hand of an equation.*

126	**T**: What does that mean?	*Transactive prompt- request for clarification*		
127	**Joseph**: It means if the two are equal, you find that if the left is true then the right must be true.	*Transactive response- explanation*	*L2cx: incorrect description of methodology of proof of equality of two sets*	*Joseph's description of the proof methodology to prove equality of sets seems to be at complex level, thinking of the proof of equality of sets as proving the left and right hand of an equation.*
128	**T**: What do you mean left and right?	*Transactive prompt- request for clarification*		
129	**Joseph**: Say *B* intersection *A* is equals to *A*, then you let *x* to be in *A*, then you should show that *x* is also in the set of *A* intersection *B*.	*Transactive response- explanation*	*L2c: correctly identifies methodology of proof of first component of equality of two sets*	*Joseph's actual method for proving equality of sets seems to be correct and does in fact prove equality. Thus his thinking about the method of proof of equality could be pseudoconceptual as he is able to apply the method of proof correctly but is unable to explain the thinking behind it correctly, showing incomplete knowledge of the mathematical object.*
130	**T**: Does that mean you're happy with that?	*Transactive prompt- request for reflection*		
131	**Joseph**: Now in that case we said let *x* be an element of *A*. Now since you assume that *A*	*Transactive response- explanation*	*L2a: selection of correct opening statement for proof*	*Joseph's method to prove equality seems to be*

	is an element of *B*, it means that *A* is in *x* and, I mean *x* is in *A* and *x* is in *B*. Then once you say…		*L1ax: incorrect use of mathematical language (using element of instead of subset of)*	*correct, while his explanations are incorrect seeming to indicate pseudoconceptual thinking.*
132	**T**: OK, I don't want to go too far. That's good. In brackets put down what we are trying to show. Next to *A* intersection *B* equals *A* what are we trying to show in the brackets? Just write it down for me.	*Transactive prompt- request for clarification*		
133	**Maria**: *[is at the board]* What?			
134	**T**: Next to 'We show that $A \cap B = A$' What are we trying to show?	*Transactive prompt- request for clarification*		
135	**Maria**: Here we're trying to show that *A* equals *[Next to* We show that $A \cap B = A$ *writes* ((a) $\Rightarrow$ (b)]	*Transactive response- explanation*	***L2cx**: unclear about how to prove equality of two sets and whole proof framework*	*Maria doesn't seem to have picked up on Joseph's explanation. She is still unclear about what is actually being done in the proof. Although she correctly explains the definition of equality of sets, she is unable to use that definition to get the proof methodology.*
136	**T**: Anybody can help her there? What did you mean by left and right? What's…	*Transactive prompt- request for clarification*		
137	**Joseph**: *[comes up to the board and erases* (a) $\Rightarrow$ (b) *and replaces it with: if* $x \in A$ *then* $x \in (A \cap B)$ *]*	*Transactive response- explanation*	*L2c:tries to clarify method of proof of equality of two sets*	*Joseph's method seems to be correct but he is not able to use the required*

	If x belongs to *A* then we can write *A* intersection *B*			*logic and reasoning perhaps indicating that he may still be exhibiting complex level thinking.*
138	**T**: In other words, what are you trying to do?	*Transactive prompt- request for strategy*		
139	*[Joseph stands back and looks at the board]* **T**: In other words, what are you trying to do? Go back to the definition guys, when are two sets equal? …When are two sets equal? What does the definition of equality say?	*Transactive prompt- request for strategy, reference to definition*		*Lecturer's scaffolding and reminding the students of the definition of equality of two sets seems to bring more clarity.*
140	**Edgar**: When every element in the other one is also contained in the other one.	*Transactive response- explanation*	*L1b: provides a definition of equality in his own words*	*Edgar's description of equality of sets is correct perhaps indicating concept level thinking.*
141	**T**: Ok. Which means?	*Transactive prompt- clarification*		
142	**Edgar**: Which means that…			
143	**T**: How do we show that two sets are equal?	*Transactive prompt- clarification and strategy*		
144	**Edgar**: We need to prove that it is, this is true… When, when… Let's take a set *A* and a set *B*. We need to prove that every element in *A* is contained in *B*. And also every element in *B* is also contained in *A*.	*Transactive response- explanation*	*L1b: provides a definition of equality in his own words H1a: correctly explains the main idea or approach used to prove equality of two sets in own words.*	*Edgar's description of equality of two sets seems to be correct perhaps indicating that his understanding is at concept level.*
145	**T**: And what do we call that?	*Transactive prompt- clarification and strategy*		
146	**Edgar**: Um…			
147	**T**: Yes?			
148	**Helen**: Oh, I think we try to, to prove that if	*Transactive response-*	*L2c: correctly identifies proof*	*The lecturer, Edgar and*

	we have a, a subset *A* and a subset *B* we try to show that *A* is a subset of *B* and *B* is a subset of *A*.	*explanation*	*method to use to prove equality*	*Helen are participating in the ZPD to clarify notion of equality and the method of proof. Helen translates what Edgar proposed as the approach to be used to prove equality of two sets into a method of proof.*
149	**T**: Good. Write that definition down for us please. That *A* equals *B*. It's very, very important and everybody is missing it here, you know. It's a fundamental definition. *A* equals *B*… You can write it right at the top there. Ja, at the top, even at the top - you're nice and tall so you can reach *[all laugh]*	*Facilitative- highlighting learning, reference to definition*		
150	**Helen**: *[comes to the board and writes: A=B when* $A \subseteq B$ *&* $B \subseteq A$ *]*	*Contribution to proof writing*	*L2c: correctly identifies proof method to use to prove equality*	
151	**T**: Beautiful. Very nice. That's what I want. Does everybody remember that definition?	*Facilitative- highlighting learning, providing encouragement, reference to definition*		
152	**S** (chorus): Yes			
153	**T**: OK. Good.			
154	**T**: Now we are trying to show that *A* intersection *B* equals *A*. What do you think we're trying to show?	*Transactive prompt- request for clarification and strategy*		
155	**Gary**: That *A* is a subset of *A* intersection *B* and *A* intersection *B* is a subset of *A*.	*Transactive response- explanation*	*L2c: correctly identifies method to be followed in the*	*Gary applies methodology of equality to this*

			proof	*particular proof, exhibiting good understanding of the definition of the equality of sets and its application, thus perhaps exhibiting concept level thinking of proof method of equality of sets.*
156	**T**: Good. *[to Maria]* So write that down please. Rub out what was in the brackets and just write what you are trying to do	*Transactive prompt- request for strategy*		
157	**Maria**: Here *[rubs out: let x∈A and if x∈A then x∈ (A∩B)]*			
158	**T**: Rub out all the brackets. Even above that, ja. *[Maria erases a lot that was written on the board]* And the other things			
159	**T**: Now even the, even the ones she has written… And carry on			
160	**Edgar**: Even this one.			
161	**T**: Ah, that's it. And… Carry on			
162	**Maria**: *[After We show that A∩B = A Maria writes : A∩B ⊆A & A⊆A∩B}]*	*Transactive argument- writing on board*	***L2c**: strategy and method for proof of equality*	*Maria showing some development in her understanding of proof methodology*

Sub-episode 2.8: Completion of proof of $A \cap B \subseteq A$

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
163	**T**: OK, good. So now we have a good idea of what we are trying to do. It's always important to get an idea of what you are trying to do. Before you start anything please guys don't start anything without knowing what you are going to do. It's no use, huh?	*Facilitative- highlighting learning, encouraging*		
164	**S**: Yes, that's true.		*L2c: students show understanding of importance of proof methods and strategy*	
165	**T**: OK, go ahead. Now you can go on.			
166	**Maria**: *[erases: $A \cap B = A$ that was on the board]* So we will show that A intersection B is a subset of A. *[writes: $A \cap B \subseteq A$]*	*Transactive argument- explaining as writing on board*	***L2a**: identifying first steps in proof of equality of sets and* ***L2c**: following correct proof method of equality of sets.*	*Maria is following the correct proof method for showing equality.*
167	**T**: OK	*Facilitative- confirming*		
168	**Maria**: So…			
169	**T**: Is that what you are trying to show?	*Transactive prompt- request for clarification*		
170	**Maria**: Ja			

171	**T**: OK, so just write To show. You can't just write the statement.	*Directive- immediate feedback*		
172	**Maria**: *[Writes: to show before $A \cap B \subseteq A$]*		***L2a:*** *selecting statements that add to the logic of the proof construction.*	
173	**T**: because that implies that that is true. So we don't know that that is true yet – we are trying to show that. OK, go ahead.	*Directive- immediate feedback*		
174	**Maria**: So let *x* be an element of *A* *[writes: let $x \in A$]* This would imply that… OK *[underneath writes: $\Rightarrow$]* Let *x* be an element of *A* intersection *B* *[next to let $x \in A$ writes $\cap B$, thus we have let $x \in A \cap B$]*	*Transactive argument- explaining as writing on board*	***L2a****: starts the proof incorrectly but realizes the mistake and corrects it herself* ***L2c****:seems to have an understanding of proof method*	*Maria starts the proof of showing that one set is a subset of another correctly, and seems to have concept level understanding of this proof method.*
175	**T**: That's good. Everybody happy?	*Facilitative- confirming, encouraging*		
176	**S**: Yes			
177	**Maria**: So this would imply that *x* is an element of *A* intersection *B* which is a subset of *A* *[writes: $\Rightarrow x \in A \cap B \subseteq A$]*	*Transactive argument- explaining as writing on board*	***L2bx:*** *selecting the statement to be proved as an assumption* ***L3ax****: assertion made without any basis* ***L1ax****: incorrect use of mathematical symbols*	*After the first correct step, Maria writes the desired deduction that would lead to the correct conclusion without any justification showing that she still has not realized that deductions cannot be made without the necessary reasoning and justification.*
178	**T**: Is that true? *[Maria looks at T]*	*Transactive prompt- request for critique, justification,*		*The lecturer is trying to get Maria to realize that*

	Is it true? Every step of the way you must be sure that it is true. Is that true?	*Directive- introducing the notion that students must be sure of the truth of statements they write*		*she cannot make deductions without justification as she has often been doing in the past.*
179	**S**: No			
180	**Helen**: Not yet, because we're trying to prove that.	*Contribution to proof- point out error and explaining*	*L3b: clarifies and explains that every statement needs to have justification and reasoning behind it.*	*Helen shows good understanding of the logical reasoning and justification process, contributing to the EZPD.*
181	**S**: No, we're trying to prove.	*Transactive response- to agree with correction*	*L3b: clarifies and explains that every statement needs to have justification and reasoning behind it.*	*Other students also contributing in the EZPD.*
182	**Joseph**: I would say x is an element of A and x is an element of B.	*Contribution to proof- further an idea*	*L3a: makes a correct deduction from previous statement*	*Joseph making a correct deduction from the previous statement showing good understanding of the reasoning and justification process.*
183	**T**: That's what we were trying to prove.	*Facilitative- re-voicing previous student's ideas*		
184	**Maria**: x is an element of B. I just? Hmm?			
185	**Joseph**: I just said x is an element of A and x is an element of B	*Contribution to proof- repeating an idea*	*L3a: makes a correct deduction from previous statement*	*Joseph is guiding Maria to recognize and follow correct reasoning and logic.*
186	**Maria**: This will imply that x is an element of A and x is an element of B.	*Transactive argument- writing on board and*	***L3a**: makes a correct deduction*	*Maria is still bringing in irrelevant assumptions*

	[writes: $\Rightarrow x\in A$ and $x\in B$]	*explaining*	***L1a**: Mathematical language, terms and symbols used correctly –spoken and written.*	*and non- useful deductions indicating lack of strategic knowledge. She does not seem to be following the correct proof method seeming to revert to proving equality of both sides (obtaining $x\in B$) perhaps indicating complex thinking about proof method of showing one set is a subset of another. She seems to have totally lost sight of what she is trying to prove.*
187	And if *x* is an element of *A* and an element of *B* it will mean that x is a, *A* is a subset of *B*. That's what I think, because I say that if *A* is a subset of *B* *[writes: $A\subseteq B$]*		***L2bx**: brings in the assumption ($A\subseteq B$) without having any use or purpose for this assumption.*	
	it means that *x* is an element of *A* *[writes: $x \in A$]* We should also imply that *x* is an element of *B* *[writes: $\Rightarrow x\in B$]* *[Now on board: to show $A\cap B \subseteq A$* *let $x\in A\cap B$* *$\Rightarrow x\in A$ and $x\in B$* *$A\subseteq B$* *$x \in A$* *$\Rightarrow x\in B$]*		***L2bx**: selecting non-useful deductions* ***L2cx**: Following incorrect method of proof.*	
188	**T**: Go back to what you are trying to show	*Facilitative- attempt to re-structure proof writing*		
189	**Maria**: Ja			
190	**T**: What are you trying to show?	*Transactive prompt- request for strategy*		
191	**Joseph**: It will imply that *x* is an element of *A*…	*Contribution to proof*	*L3a: makes correct deduction from previous statement*	*Joseph's understanding of the proof method seems to be at concept level. He also shows good strategic*

				knowledge seeming to know how to make useful and correct deductions.
192	**T**: Why?	*Transactive prompt- request for elaboration*		
193	**Joseph**: Because it's an element of *A* and *B* and we want to show that *A* intersection *B* is a subset of *A*.	*Transactive response- to elaborate*	*L3c: identifies the reason why the deduction is made*	*Joseph's understanding of the proof method of subset and equality seems to be at concept level. Also displaying good strategic knowledge.*
194	**T**: You got it! What are we trying to show?	*Facilitative- highlighting learning, repeating*		
195	**S**: That *A* intersect *B* is a subset of *A*	*Transactive response- to answer*	*L2c: identifies what is needed to be shown*	*Other students participating in the EZPD are also showing understanding of the proof methodology.*
196	**T**: So so far we've had that *x* is an element of *A*	*Facilitative- highlighting learning, repeating*		
197	**Maria**: And *B*	*Contribution to proof*	***L2bx****: perhaps not realizing why the focus is on x being an element of A*	*Although she has identified what needs to be shown, she has not yet learned how to extract and manipulate the given information to get to this- perhaps indicating lack of strategic knowledge.*
198	**T**: And *B*. So is it an element of *A*?	*Facilitative- re-voicing Transactive prompt- request*		

		for reflection		
199	**S**: Yes			
200	**T**:*[nods]* For sure. It's both. It's both an element of *A* and an element of *B*. So we can make the conclusion that *x* is an element of *A*, as you were saying, that *x* is an element of *A*. Is that right?	*Facilitative- confirming* *Transactive prompt- request for reflection*		
200	**Maria**: *[under* $\Rightarrow x \in A$ *and* $x \in B$ *writes:* $\Rightarrow x \in A$ *]*	*Writing on board*	***L3a****: writes the correct deduction*	
201	**Edgar**: I don't understand what *x* is there for.	*Transactive request- clarification*	*L2cx: does not recognize why assertion is made and how it is arrived at*	*Edgar also seemingly exhibits a lack of strategic knowledge-Edgar's question seems to go unnoticed and unanswered*
202	**Maria**: *A* intersection *B* is a subset of *A*. So *A* intersection *B*…	*Transactive argument- explanation*	***L3a****: new assertion made correctly*	*Maria seems to recognize why the conclusion can be made from previous statement*
203	**Joseph**: is a subset of *A*.	*Contribution to proof*	*L3a: new assertion made correctly*	*Joseph and other students offer their guidance and help to Maria every step of the way making for a very effective EZPD in which the student is supported and nurtured.*
204	**T**: Has she concluded? OK	*Transactive prompt- request for justification*		
205	**Joseph**: Thus…			
206	**Maria**: *[writes: Thus* $A \cap B \subseteq A$ *]*	*Writing conclusion*	***L3c****: correct conclusion made*	*Maria makes the correct conclusion, indicating*

				growing understanding of the proof method of showing one set is a subset of another.
207	**Frank**: Conclude again that *A* intersect *B* equal to….	*Contribution to proof*		*Frank seems to think that since* $A \cap B$ *is a subset of A, they have now been shown to be equal. He either does not have an understanding of the proof method of equality or perhaps he still associates the double implication with the implication indicating complex level thinking.*
208	**Maria**: Hmm?			
209	**Frank**: At the bottom there you must conclude again that therefore *A* intersect *B* is equals to *A*.	*Contribution to proof*	*L3ax: assertion made without justification*	*Perhaps Frank is still looking at implication as a double implication indicating complex thinking.*
210	**Maria**: OK		***L3bx****: acceptance (unquestioning) of assertion without justification*	*Maria seems to accept Frank's contribution without questioning.*
211	**T**: Is that right?	*Transactive prompt-critique and justification*		
212	**S**: No			
213	**Maria**: We haven't proved yet	*Transactive response- to clarify*	***L3b****: clarifies that assertion can't be made without*	*Maria is strengthened in the EZPD created by*

			justification	*peers and lecturer to be firm in her insistence that every statement needs to be justified.*
214	**Edgar**: You must show the other side, the other side.	*Contribution to proof*	*L3b: clarifies that assertion can't be made without justification*	*Edgar offers his complex/pseudoconcept level thinking of the method of proof of equality of two sets*
215	**Helen**: Yes, I think you have to say that because we concluded that x is an element of A because x is an element of A and B. Therefore A is a subset of A intersection B.	*Contribution to proof*	*L3ax: assertion made without justification*	*Helen also thinks that the reverse assertion can be made- perhaps associating the implication with double implication which may indicate complex thinking.*
216	**Joseph**: If you had used the double implication sign then we could have concluded…	*Contribution to proof*	*L3b: recognizes that the assertion can't be made* *L2c: clarifies that it could have been done with double implication*	*Joseph recognizing the root of Helen and Frank's misunderstanding.*
217	**T**: Can we? Can we use the double implication sign? Remember we can only use it if we are sure that we can go both ways.	*Transactive prompt- request for critique and justification*		
218	**S**: We can	*Transactive response- to answer*	*L2cx: students think that double implication could have been used in proof*	
219	**Maria**: So I say Thus A is a subset of A intersection B.	*Transactive argument- saying while writing on*	***L3ax**: assertion made without justification*	*Although she correctly realized that this had not*

	[writes: Thus $A \subseteq A \cap B$ *]* And then again I can…	*board*		*been proved here, Maria lacks the conviction and follows other students' suggestions.*
220	**S**: No	*Contribution to proof*	*L3b: recognizes that the assertion can't be made*	
221	**T**: Is that right?	*Transactive prompt- request for critique and justification*		
222	**S**: That one is not ?	*Contribution to proof*	*L3b: recognizes that the assertion can't be made*	
223	**Christine**: You can't conclude in that manner	*Contribution to proof*	*L3b: recognizes that the assertion can't be made*	*Christine firmly corrects Maria's error in concluding* $A \subseteq A \cap B$ *indicating good (concept level) understanding of the proof method and the notion of implication.*
224	**S**: No	*Contribution to proof*	*L3b: recognizes that the assertion can't be made*	
225	**Joseph**: The one we have just done, we must complete them. That's what we are trying to show.	*Contribution to proof*	*L2c: clarifying proof method and logical thought*	*Joseph also seems to show good understanding of the proof method.*
226	**T**: Ok, …Ok. Why don't you two go up and tell me why. I'm sure she is still got a few questions because she wrote *A* is a subset of *A* intersection *B*. Why can she not write that, but why must she write this? *[to Christine]* I think you said, you said that she must write this. Can you tell us why? Clarify for her because she is still confused.	*Facilitative- trying to re-structure discussion and proof writing*		

227	**Christine**: Oh, she can't conclude by saying that *A* is a subset of *A* intersection *B* because she didn't prove that part. She proved the part that *A* intersection *B* is a subset of *A*.	*Transactive response- to clarify and elaborate*	*L3b: clarifies and elaborates why assertion can't be made and why the previous one can*	*Christine's conviction about what has been proved and what has not been proved indicates good understanding of method of proof of equality and may indicate concept level thinking.*
228	**T**: Why? Tell us how… Take us through those steps. Go up and take us through those steps. How did she start? Why is it now that we can make that conclusion, please?	*Transactive prompt- request for clarification and elaboration*		
229	**Christine**: *[goes up to the board and points to the statements]* Ok, we tried to show that *A* intersection *B* is a subset of *A* and then we let *x* be an element of *A* intersection *B*. This means that *x* is in both *A* and *B* and this *[points to* $A \cap B$*]* is an intersection of *A*. So *x* is an element of *A* and *x* is an element of *B* *[points to* $x \in A$ *and* $x \in B$*]* This means that since *x* is in both *A* and *B*, then *x* is also going to be in *A*, which you are trying to prove. Then we conclude by saying that *A* intersection *B* is a subset of *A*. *[points to Thus* $A \cap B \subseteq A$*]* So she was supposed to write this part *[draws a line under* $A \cap B \subseteq A$ *]*	*Transactive argument- explaining and giving justification as writing on board*	*L1a,b: correct use of terms, explaining terms using own words, identifying implications of terms and statements,* *L2a,b,c: identifies proof framework and what needs to be shown and how to go about it,* *L3b,c: Recognizes why assertions can be made from previous statements and identifies basis for conclusion*	*The unfamiliar terms such as 'implies' and 'intersection' are used with ease and described and interpreted correctly. She clearly identifies what needs to be shown and the connection between that and the statements made indicating concept level thinking for method of proof. She also identifies the basis for each deduction and the conclusion and demonstrates that each deduction should be made with the necessary*

				justification.
230	**T**: Do you see why you cannot write the other one yet?	*Facilitative- highlighting learning*		
231	**Maria**: Yes. from this side to this side.	*Transactive response- to agree*	***L2c**: gaining understanding in proof method of equality*	*Maria seems to be gaining knowledge in the method of proof of equality of two sets and progressing to pseudoconceptual thinking.*
232	**T**: OK, great, thank you. Will that always be true? *A* intersection *B*, will that always be a subset of *B*? Any ideas?	*Facilitative- encouraging Transactive prompt- request for reflection*		
233	**S**: Yes, it will	*Transactive response- to agree*	*L1b: seems to identify implications of an intersection*	
234	**T**: It will always be. Actually we didn't even need to go and prove that because you could have just said that *A* intersection *B* is a subset of *A* always. Because that intersection, right, the intersection of two sets… Just draw that venn diagram again – draw the intersection again, of *A* and *B*	*Directive- immediate feedback, Transactive prompt- request for pictorial example*		
235	**Maria**: *[draws the two intersecting circles, marks them A and B and draws lines in the intersection]*			
236	**T**: Do you see that that intersection is always going to be contained in *A*?	*Facilitative- highlighting learning*		
237	**S**: Yes			
238	**T**: It will always be contained also in?	*Facilitative- highlighting learning*		

239	**S**: *B*	*Transactive response- give an answer*		
240	**T**: So *A* intersection *B* – write that down please – *A* intersection *B* is always going to be a subset of *A* and it will always be a subset of *B*.	*Facilitative- highlighting learning*		
241	**Maria**: *[writes:* $A \cap B \subseteq A$ *&* $A \cap B \subseteq B$ *]*	*Writing on board*	***L3a**: identifying identities or assertions that will always be true*	
242	**Joseph**: Always	*Contribution*	*L3a: identifies that it will always be true*	
243	**Maria**: *[after* $A \cap B \subseteq A$ *&* $A \cap B \subseteq B$ *adds always]*	*Writing on board*	***L3a**: identifying identities or assertions that will always be true*	

Sub-episode 2.9: Maria's attempt at the proof of $A \subseteq A \cap B$ to complete the proof of $A = A \cap B$ *in* a) $\Longrightarrow$ b)

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
244	**T**: Good. That's good. Alright. So now we can go ahead.	*Facilitative- encouraging*		
245	**Maria**: So that means I now show that *A* is a subset of *A* intersection *B*?	*Transactive argument- continuation of proof and proposal of plan/ strategy*	***H1a**: identifies the main idea or approach to be used*	*Maria identifies the correct plan for the first time indicating developing knowledge of the method of proof of equality.*

246	**T**: Right. Write that down please.	*Facilitative- confirming*		
247	**Maria**: *[next to to show $A \cap B \subseteq A$ writes: to show $A \subseteq A \cap B$]* Again let… OK, can I ask you a question?	*Start of new plan, Transactive question-request for clarification*	***L2a**: selects correct statement for beginning the next section of proof*	*Maria identifies the correct plan for the first time indicating developing knowledge of the method of proof of equality.*
248	**S**: Yes			
249	**Maria**: So there *[points to*: *to show $A \subseteq A \cap B$]* I'm coming to show that *A* is a subset of *A* intersection *B* - do I have to start with this side *[points to the first A]* or this side *[points to the $A \cap B$]*	*Transactive question-request for clarification and strategy*	***L2ax,cx**: cannot identify how to start with proof of showing that one set is a subset of another* ***H2ax**: unable to transfer method just used in previous proof*	*Methodology of how to prove that one set is a subset of another is not understood. Maria has been unable to transfer method of $A \cap B \subseteq A$ to do proof of $A \subseteq A \cap B$ and views subset proof as having two sides. Thus Maria may still have complex level understanding of this proof method. This shows how students doing proof construction on their own would battle as at every turn there are myriads of possible places where students can get stuck.*
250	**Helen**: Start with *A*. Left	*Transactive response-clarification*	*L2c: clarifies how to start proof*	*Helen clarifies where to start in every-day language but does not elaborate on the reason*

				why this is done.
251	**Maria**: Mmm?	*Transactive question-request for clarification*		
252	**Helen**: Start with the *A*.	*Transactive response-clarification*	*L2c: clarifies how to start proof in the proof framework*	*Helen clarifies where to start in every-day language but does not elaborate on the reason why this is done*
253	**Maria**: With *A*?	*Transactive question-request for clarification*	***L2cx**: seeking clarification on method of proving one set is a subset of another*	*Although Maria is told how to start she is obviously still unclear-perhaps needing further reasons behind the method of proof.*
254	**H**: Mmm			
255	**Maria**: OK			
256	**Christine**: Because if you start with the right you are going to have…	*Transactive response-clarification*		
257	**Maria**: So will *A* take me back here?	*Transactive question-request for clarification*	***L2cx**: seeking clarification on method of proving one set is a subset of another*	*Maria is not convinced that she'll be able to prove what is needed- as a result of being unsure of proof methodology.*
258	**Helen**: That's what we want to show. That's what we want to show.	*Transactive response-clarification*	*L2c: clarifying what needs to be shown in the proof framework*	
259	**Maria**: Hmm?	*Transactive question-*		*Maria still unclear and*

		request for clarification		*not sure of the proof method.*
260	**S**: That's what we want to show.	*Transactive response- clarification*	*L2c: clarifying what needs to be shown*	
261	**Maria**: Ok. So I let *x* be an element of *A*. If *x* is an element of *A* and *A* is a subset of *A* intersection *B* *[writes: let x∈A]* Hmm?	*Transactive argument- explaining as writing on board*	***L2a**: Beginning the proof correctly* ***L1a**: writing opening statement correctly* ***L2bx:** selecting the statement to be proved as an assumption* ***L3ax**: makes the assumption A is a subset of A intersection B without justification; this is in fact what she needs to prove* ***H2ax**: unable to transfer ideas just used in proof of* $A \cap B \subseteq A$ *to do proof of* $A \subseteq A \cap B$	*Maria's logical reasoning processes and grasp of the proof method seems to be at complex level. She has not been able to grasp and to transfer the idea that she cannot use the statement that she needs to prove as an assumption.*
262	**T**: It's not! That's what you're trying to show… that's what you're trying to show… So please don't get confused with what you are trying to show, you cannot assume that. But what have you assumed, what have you got?	*Directive- providing immediate feedback* *Facilitative- highlighting learning* *Transactive prompt- request for reflection*		
263	**S**: *(Some comments) A* is a subset of *B*.	*Transactive response- to answer*	*L2a:recognizing statement assumed at beginning of proof*	
264	**T**: What have you assumed?	*Facilitative- highlighting learning*		

265	**Maria**: *A* is a subset of *B*	*Transactive response- to answer*	***L2a****:recognizing statement assumed at beginning of proof*	*Maria benefitting from the EZPD. Another important fact that students often lose sight of and forget are assumptions that they have made and making use of what has been assumed.*
266	**T**: Ah… And	*Transactive prompt- request for elaboration*		
267	**Maria**: This would imply that x is an element of *B* since *A* is a subset of *B*. *[writes:* $\Rightarrow x \in B$ *(sin* $A \subseteq B$*)]*	*Transactive argument- explaining and writing on board*	***L3a****: correct deduction on the basis of A being a subset of B.*	*Making a correct deduction after being guided to be aware of the statement that is assumed.*
	Therefore I conclude that *A* intersection *B* is equals to *B*?		***L3cx****: makes a conclusion A intersection B is equal to B, without any justification* ***L2cx****: doesn't follow proof framework and method*	*Although at the outset she correctly identified what needed to be done, it seems that she has lost sight of this now- she seems to want to show something else. She has again fallen back into the habit of making assertions without justification.*
268	**T**: Not yet. Any ideas?	*Transactive prompt- request for reflection and strategy*		
269	**Joseph & Christine & Helen**: This again implies that *x* is an element of *A* and *x* is an element of *B*	*Contribution to proof- correct deduction from previous statement*	*L3a, L3c: correctly identify the deduction that can follow from previous statement and show good understanding of the underlying reason that*	*Joseph, Chrisitine and Helen and probably a few others recognize the steps needed in this proof method and that each step*

			this is done	*needs to be justified and they're creating the EZPD for Maria to develop.*
270	**Maria**: *[writes: $\Rightarrow x \in A$ and $x \in B$]*	*Transactive argument-writing on board*	***L3a**: helped by others to make correct deduction*	
271	**Joseph**: Thus	*Contribution to proof*		
272	**Maria**: *[writes: $\Rightarrow A \cap B$ Thus $A \subseteq A \cap B$]*	*Transactive argument-writing on board*	***L1ax**: use of term " x element of" forgotten* ***L3a, L3c**: Maria now makes the correct deduction and conclusion from previous statement*	*Maria developing her understanding of the proof method – seems to be suddenly seeing the light with the help of among others Joseph, Christine and Helen.*
273	**Christine**: It implies that *A*?	*Taking on role of more knowing other*	*L1a: guiding Maria to see the error of omitting "x element of"*	*Christine is also mindful of the correct use of mathematical terms, symbols and signs.*
274	**Maria**: Hmm? What?			
275	**Christine**: Why did you write *A* intersection *B*?	*Taking on role of more knowing other*	*L1a: guiding Maria to see the error of omitting "x element of"*	*Christine is also mindful of and guides Maria by prompting her on the correct use of mathematical terms, symbols and signs.*
276	**Maria**: OK, because it's what I'm trying to show you.	*Transactive response- to reason and explain*	***L2ax**:does not identify the statement as a deduction from the previous statement but rather as the goal that she wants to reach*	

277	**S**: This one?			
278	**S**: Implies that *A* intersection *B*	*Taking on role of more knowing other*		
279	**Maria**: Oh, this one? *[points to $\Rightarrow A \cap B$]*	*Transactive response-reflection*		
280	**Helen**: I think she wanted to say that it implies that *x* is an element of *A* intersection *B*.	*Taking on role of more knowing other*	*L1a: guiding Maria to see the error of omitting "x element of"*	
281	**Maria**: Ja, Ja	*Transactive response-to agree*		
282	**T**: Don't forget that.	*Facilitative- highlighting learning*		
283	**Maria**: *[erases: $\Rightarrow A\cap B$ and replaces it with $\Rightarrow x\in A\cap B$]*	*Transactive argument-correcting mistake*	***L1a**: correcting mathematical language, symbols, signs*	
284	**T**: What made you just write *A* intersection *B*?	*Facilitative- highlighting learning*		
285	**Maria**: I just jumped a step - this one *(underlines the $\Rightarrow x\in A\cap B$)*	*Transactive response- to explain*	***L1ax**: refers to omitting "x element of" as jumping a step*	*This mistake (which is commonly observed) leads us to question whether she really understands the thinking behind the proof and that we're looking at elements of sets. Might have something to do with conception of proving an implication as two sides of an equality- as Maria said "it's what I'm trying to show you"*
286	**T**: Did you jump the *x* element of? Did you get excited? You found *A* intersection *B*…	*Facilitative- highlighting learning*		

	(laughter) You jumped to that… OK, good. Is everybody happy so far?			
287	**S**: Yes			
288	**Frank**: Then what about 1 and 2?	*Contribution to proof*	*L2c: Frank associates parts 1) and 2) with the proof framework and method of proof of equality of two sets.*	*Frank seems to be attached to the form and structure of the proof. He seems to have a good grasp now of the proof methodology for showing equality of sets.*
289	**Maria**: Then I name this 1 and 2. *[labels Thus $A \cap B \subseteq A$ (1) and Thus $A \subseteq A \cap B$ (2)]* Then 1 and 2 imply that *[writes: Thus (1) and (2) imply that]* *A* intersection *B* is equals to and *A* is a subset of *B* which implies that…	*Transactive argument-explaining as writing on board*	***L2a**: selecting phrases that help clarify the proof construction process and the purpose of components in the proof and the proof method.* ***L2c:** showing good understanding of the proof method of equality of sets.*	*Maria perhaps a little unclear about where in the proving process of ($A \subseteq B \Rightarrow A \cap B = A$) she is.*
290	**S**: Hmm?	*Transactive question-clarification*	*L3b: peers questioning the deduction made (spoken)*	
291	**Maria**: OK, from this 1 and 2 *[points to Thus (1) and (2) imply that]* I conclude that *A* intersection *B* is equal to *A*.	*Transactive argument-explaining as writing on board*	***L3a**: makes the correct deduction from previous statements* ***L2c:** showing good understanding of the proof method of equality of sets.*	*Maria explains why she can now make the conclusion indicating that she's made progress in understanding the proof method of equality of two sets.*
292	**Joseph**: Since the two are… I'm saying if, since the two implies that *A* intersect *B* implies, I mean is a subset of *A* and also *A* is a subset of *A* intersection *B*, then that you	*Contribution to proof-elaboration on reasons behind the conclusion*	*L3a,c: recognizes and gives justification on why assertions and conclusions can be made*	*Joseph confirms and elaborates why the conclusion can be made. Students learning from*

	can say the two are equal.		*L2c: showing good understanding of the proof method of equality of sets.*	*each other in the EZPD.*
293	**Maria**: *[After Thus (1) and (2) imply that continues writing $A \cap B = A$]*	*Transactive argument- concluding as writing on board*	***L1a**: correctly interprets the correct spoken deduction into written mathematical language* ***L2c:** showing good understanding of the proof method of equality of sets.*	

Sub-episode 2.10: Taken up by lecturer's explanation of the notion of equivalence

In sub-episode 2.10 Maria asks whether she should now prove *b)* $\Longrightarrow$ *a)* and it is explained by the lecturer that this is necessary to prove equivalence. The transcript from lines 296 to 309 has been omitted as it shows the lecturer explaining that this component of the proof needs to be done to show equivalence. The proof construction follows in sub-episode 2.11.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
294	**T**: Any questions, comments? Are we happy? Are we all together? Good. OK.	*Transactive prompt- request for reflection*		
295	**Maria**: So here do I also have to prove this? *[points to $A \subseteq B \Rightarrow A \cap B = A$]* or do I just conclude that a) implies b)?	*Transactive question- clarification*	***L2c**: Maria realizes that the proof method is not yet complete*	*Maria is developing a much clearer picture of the method of the proof of an implication and also realizing that she now*

				needs to prove the converse.

Sub-episode 2.11: proof of b) $\Longrightarrow$ a) *or* $A \cap B = A \Longrightarrow A \subseteq B$

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
310	**Maria**: Assume that *A* intersection *B* is equal to *A* *[writes: Assume* $A \cap B = A$ *]* I want to show that *A* is a subset of *B* *[writes: we show that* $A \subseteq B$ *]*	*Start of new component of proof* *Proposal of new plan or strategy*	***L2a, L2c**: identifies what is assumed and what needs to be shown and the proof methodology in implication proof,* ***L1a**: mathematical language, terms and symbols are used correctly (spoken and written).* ***H2a**: uses proof method (implication) and ideas grappled with in previous proof correctly*	*Maria is visibly more confident about the proof framework and method now and uses the pronoun "I" when telling the group what needs to be shown indicating confidence in her own understanding and ability.* *She seems to have developed concept level understanding of proof method of an implication.*
	I let *x* be an element of *A* *[writes: Let* $x \in A$ *]* And if *x* is an element of *A*…		***L2c:**identifies correct proof method of showing one set is a subset of another* ***L2a**: selects correct statement to begin the proof* ***L1a**: correct use of mathematical terms, symbols and signs*	*Maria seems to have developed concept level understanding of proof method of showing one set is a subset of another.*

311	**Frank**: You know what? When you say x is an element of A you are not letting b) to c)	*Contribution to proof- point of confusion*	*L2cx: hasn't recognized the need to prove b) implies a) and would like to move on to prove b) implies c)*	*Frank doesn't seem to have realized that in order to prove equivalence, one needs to show the reverse implication.*
312	**Maria**: Mmm?			
313	**Frank**: That b) implies c)	*Contribution to proof- point of confusion*	*L2cx: hasn't recognized the need to prove b) implies a) and would like to move on to prove b) implies c)*	
314	**Maria**: We are proving that b) implies a)	*Transactive response- explanation*	***H1a**: explanation of proof framework and plan*	*Maria firmly explains what she is proving now and is not swayed by Frank's suggestion.*
315	**Frank**: You are doing like that?	*Transactive question- request for clarification*		
316	**Maria**: Yes	*Confirmation*		
317	**Frank**: Again? *(laughter)*	*Transactive question- request for clarification*	*H1ax: hasn't recognized the need to prove b) implies a)*	
318	**S**: Not again, for the first time	*Transactive response- explanation*	*H1a: explanation of proof method*	
319	**Maria**: It's the first time we prove it.	*Transactive response- explanation*	***H1a**: explanation of main idea*	*Maria is confident in her defense of the correct proof method.*
320	**Frank**: For the first time?	*Transactive question- request for clarification*		
321	**S**: It's a double implication. There's more than one proof.	*Transactive response- explanation*	*L2c: explanation of double implication proof*	
322	**Gary**: For the first time we proved that a)	*Transactive response-*	*L2c:explanation of double*	*Gary showing good*

	implies b) right? So if a) implies b) we must also prove that b) implies a). So right now we've just shown that a) implies b). We must go back to show that b) implies a). Ok.	*explanation*	*implication proof*	*understanding (perhaps concept level) of how to show equivalence of two statements.*
323	**Frank**: Ok			
324	**Maria**: OK. And this, from this it would imply that *x* is an element of *B* because here it says that *A* intersection *B* is equal to *A*. *[writes: $\Rightarrow x \in B$]*	*Transactive argument- reasoning and explaining as writing on board*	***L3ax**:makes a deduction that doesn't follow simply from the previous statement* ***H2b**: realizes from previous proof that assumptions made at beginning must be used in order to reach conclusion*	*It's not clear whether Maria makes the deduction from logical reasoning or if she is just guessing as she knows what the conclusion should be. She seems to have realized that she does need to use assumptions made at the outset to reach the conclusion.*
325	**T**: Break it down into simple steps for us…	*Transactive prompt- request for clarification and explanation*		
326	**Maria**: *[completes the statement she was writing: $\Rightarrow x \in B$ (since $A \cap B = A$)]*	*Transactive argument- writing on board*	***L3ax**:makes a deduction that doesn't follow simply from the previous statement* ***H2b**: realizes from previous proof that assumptions made at beginning must be used in order to reach conclusion*	
327	**T**: Are we clear? I think you missed a step.	*Transactive prompt- request for clarification and explanation*		

328	**Frank**: Since *A* is a subset of *B*	*Contribution to proof- point of confusion*	*L2bx: wants to use as an assumption what needs to be shown in the proof*	
329	**Maria**: Hmm?			
330	**Frank**: Since *A* is a subset of *B* we want to show that. We want to show that.	*Contribution to proof- point of confusion*	*L2bx: wants to use as an assumption what needs to be shown in the proof*	*Frank realizes that A is a subset of B is what is needed to be shown but still suggests using it as an assumption.*
331	**T**: And is it clear for all of us, is it? Is it?	*Transactive prompt- request for reflection*		*T is still addressing Maria- not really taking in Frank's suggestion*
332	**Christine**: No it's not	*Contribution to proof*	*L3b: recognizes that the assertion has been made without justification*	
333	**T**: OK, just go back and think about how to make that a bit more clear.	*Transactive prompt- request for clarification*		
334	**Maria**: Like?	*Transactive question- request for clarification*		
335	**T**: What do we know? What are our assumptions?	*Facilitative- structuring proof writing, highlighting attention to assumptions*		
336	**Maria**: You know that *A* intersection *B* is equal to *A*.	*Transactive response- to answer*	***L2b**: correctly identifies assumptions*	*Shows understanding that when an assumption is made it means that "we know this"*
337	**T**: OK, now you have let *x* to be?	*Transactive prompt- request for reflection and clarification*		

338	**Maria**: an element of *A*.	*Transactive response- to answer*		
339	**T**: What does that mean then?	*Transactive prompt- request for logical reasoning*		
340	**Maria**: Like I've let *x* be, to be contained in *A*.	*Transactive response- to explain*		*Correctly explaining what is meant when x is an element of A*
341	**T**: Mmm			
342	**Maria**: And since I've already assumed that *A* intersection *B* is equal to *A* it means that *x* is also contained in *B*.	*Transactive response- to explain*	***L3ax**: deduction doesn't follow directly from previous statement*	*It's possible that Maria is making the connection intuitively.*
343	**T**: Ok, that's not clear. You have written *x* is an element of *A*	*Transactive prompt- request for logical reasoning*		
344	**S**: Right	*Confirmation*		
345	**T**: And *A* is…?	*Transactive prompt- request for logical reasoning*		*T is prompting Maria to recognize the strategy she needs to use. This strategic knowledge is often what's lacking.*
346	**Maria**: Is a subset of	*Transactive response- to answer*	***L2bx**: doesn't identify correct assumption*	*Maria does not identify the useful or appropriate step to be used here.*
347	**T**: *A* is…?	*Transactive prompt- request for logical reasoning*		
348	**S**: equals to	*Transactive response- to answer*	*L2a: identifies correct assumption*	*Other students helping Maria in the EZPD*
349	**Maria**: is equal to *A* intersection *B*	*Transactive response- to answer*	***L2a**: identifies correct assumption*	*Maria is helped to realize how the assumption can be used to get the desired result.*

350	**T**: So what is *x* an element of?	*Transactive prompt- request for reasoning*		
351	**S**: *x* is an element of *A* intersect *B*	*Transactive response- to answer*	*L3a: able to make the correct deduction from previous statements*	*Other students participating in the EZPD with the lecturer to help Maria with logical reasoning and justification*
352	**T**: Right. Write that down.	*Facilitative- confirming*		
353	**Edgar**: *x* is an element of *A* intersect *B*.	*Contribution to proof- echoing what has been learnt*	*L3a: repeating the correct deduction made from previous statements*	*Creating the EZPD with the teacher to help Maria with logical reasoning and justification.*
354	**Maria**: *[erases: $\Rightarrow x\in B$ (since $A\cap B = A$)]*	*Transactive argument- writing on board*		
355	**Edgar**: And… that since *x*, since *x* is an element in *A*	*Contribution to proof- echoing what has been learnt*	*L3a: clarifying reasoning behind the deduction to be made*	*Interesting to note the degree of excitement and enthusiasm Edgar shows.*
356	**Maria**: *[writes: $\Rightarrow x\in A\cap B$ (since $A = A\cap B$)]*	T*ransactive argument- writing on board*	***L3a:*** *correctly writing correct deduction from previous statement and providing the justification.*	*Maria realizing the importance of justification in the proof process*
357	**T**: OK, good. Carry on	*Facilitative- confirming student's proof construction actions*		
358	**Maria**: *[erases and $x\in$ from the statement let $x\in A$ and $x\in$]*		***L2a****: Maria is also correcting what she wrote previously and making sure what she's written makes sense and adds to the logic.*	*Maria is being helped to realize the importance of using appropriate mathematical language/ symbols in the proof process.*

359	**T**: Is everybody clear and happy? You've brought in an assumption that *A* equals *A* intersection *B*, go ahead	*Transactive prompt- request for reflection on the logical reasoning,* *Facilitative- highlighting learning*		
360	**Maria**: Then this would imply that *x* is contained in *A* and also contained in *B* *[writes: $\Rightarrow x \in A$ and $x \in B$]*	*Transactive argument- reasoning and explaining as writing on board*	***L1a***:*correctly uses newly met terms, symbols and signs (spoken and written)* ***L3a***: *able to identify implications of previous statement and make correct deductions*	*Maria seems to have developed her understanding of the proving process considerably and is making great strides.*
361	**T**: Yes.	*Facilitative- confirming*		
362	**Maria**: It's like this it means that *x* is contained in *B*. *[writes: $\Rightarrow x \in$ B]*	*Transactive argument- reasoning and explaining as writing on board*	***L2b***:*able to identify useful implications of previous statement* ***L3a:*** *makes a correct deduction* ***L1a***: *able to speak and write the correct terms/ mathematical terminology*	*Maria is beginning to develop her understanding of the useful implications that will allow her to reach the correct conclusion- making gains in strategic knowledge (category L2b)*
	And this means that if *x* is contained in *A* and in *B* it means that *A* is a subset of *B* *[writes: Thus $A \subseteq B$]*		***L3c***: *makes a correct deduction from previous statement and recognizes the basis for a conclusion* ***L2c:*** *confirms correct understanding and application of the method of proof of showing one set is a subset of another.* ***L1a***: *able to write the correct*	*Maria is still under the impression that showing that "x is contained in A and in B", will mean that A is a subset of B. This may indicate pseudoconceptual thinking of the proof method of showing that one set is a subset of another as she is*

			terms/ mathematical terminology	*able to construct the proof correctly but her explanation for the proof is not quite correct.*
	So now because I've proven this side to be equal to this side *[pointing to* $A \cap B = A$ *and Thus* $A \subseteq B$*]* so I conclude that a) implies b) and b) implies a).		***H1a****: able to see the big picture, recognize the components of the proof* ***L2c:*** *confirming correct understanding and application of implication proof method.* ***L3c:*** *makes correct conclusions with necessary justification*	*Maria seems to have achieved concept level thinking of double implication proof methodology but still sometimes referring to it as equality.*
363	**T**: Can you do that, guys? You'll be happy?	*Transactive prompt- request for reflection and critique*		
364	**S**: Yes	*Confirmation*		
365	**T**: OK, good	*Facilitative- confirming*		
366	**Maria**: *[writes: (a)* $\Leftrightarrow$ *(b)]*	*Transactive argument- writing on board*	***L3c****: concludes the proof correctly* ***L1a****: writing mathematical terms, symbols correctly*	*Maria seems to have achieved concept level thinking of double implication proof methodology.*
367	**T**: Very nice, thank you.	*Facilitative- confirming and encouraging*		
	Other students then come up to continue with other parts of the proof			

Session 2: Episode 3

At the second session which took place one week after the first session, six of the participating students; Edgar, Christine, Joseph, Gary, Maria and Frank were present. After a brief revision of the notion of the Cartesian product of sets, Edgar goes to the board to attempt the proof of the proposition: $(A \cup B) \times C = (A \times C) \cup (B \times C)$. A successful proof construction of this proposition requires knowledge of the proof method of proving equality of sets as well as knowledge of the precise definitions of union, intersection and the Cartesian product and the ability to use these definitions in the logical reasoning and justification of each step in the proof.

Sub-episode 3.1: Proof of $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$

	Speech and actions	**Student and teacher interactions**	**Proof comprehension**	**Interpretation according to T.F. and general comments on the proof construction process**
1	**Edgar**: *[comes to the board]* Ladies and gentlemen, I will just try to solve in a way I think in my mind. Um, since we are having this what we need to do to prove the whole of this *[points to $(A \cup B) \times C$]* is a subset of this one *[points to $(A \times C) \cup (B \times C)$]* and the whole of this *[points to $(A \times C) \cup (B \times C)$]* is a subset of this one *[points to $(A \cup B) \times C$]* So I think I should write, I'll write, I'll write this down.	*Proposal of a new plan or strategy*	***H1a***: *correctly describing overview of the whole proof method* ***L2c***: *method of proof of equality is correctly described.* ***L1a***: *mathematical language, symbols and signs are used correctly – spoken and written.* ***L2a***: *selecting appropriate statements and phrases that add logic to the proof process.* ***H2a***: *is able to transfer methods used in the previous proofs to*	*Edgar confidently and correctly begins the proof and describes the plan of action to be taken. He seems to have concept level thinking of the method of proof of equality of $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and correctly breaks the proof up into the two components: $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ and $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$*

	[writes: proof: We show that $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ *and* $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$*]* Are you able to see?		*this proof*	
2	**S**: Ja			
3	**Edgar**: OK, that's what I need to show. So to prove that, um... Firstly we let these Cartesian points; *x* and *y* be an element of *A* union *B* brackets… *[writes: let* $(x, y) \in (A \cup B) \times C$ *]*	*Putting plan into action*	***L2a****: selecting appropriate statement to begin the proof.* ***L2c:*** *method of proof of showing one set is a subset of another correctly begun.* ***L1a****: selecting correct mathematical language, symbols and signs – spoken and written*	*Method of proof of showing one set is a subset of another seems to be at concept level.*
	This will imply that *x* and *y* are both elements of *A* union *B* and *x, y* an element of, and both of them are an element of *C*. *[writes:* $\Rightarrow (x, y) \in (A \cup B)$ *and* $(x, y) \in C$ *]* Yes?	*Making an incorrect deduction*	***L3ax****: incorrect deduction made as a result of :* ***L1ax****:incorrect understanding of the Cartesian product of sets*	*Edgar's understanding of Cartesian product of two sets and the notion of an ordered pair seems to be at complex level as he seems to associate the notions of the Cartesian product of sets and the intersection with the word 'and'.*
4	**Gary**: According to your statement there	*Transactive question-request for clarification, acting as more knowing other*		
5	**Edgar:** Yes?			
6	**Gary**: you are saying we must show that *A* union *B*, um	*Transactive questions-requests for clarification, acting as more knowing*		

		other		
7	**Edgar**: *A* union *B* and *C* is a subset of this	*Transactive response- to clarify*		
8	**Gary**: Ja.			
9	**Edgar**: Mmm			
10	**Gary:** But then right now we must show that, we must first prove that *A* union *B* multiplied by *C* is a subset of that, right?	*Transactive questions- requests for clarification*	*L1ax: referring to the Cartesian product as "multiplied"*	*Gary is trying to guide Edgar by using prompts to help him to reflect on his actions in the proof construction.*
11	**Edgar**: Yes			
12	**Gary**: So why didn't you write the statement that we are proving first?	*Contribution to an idea*	*L2a: selecting statements that will add to the logic of the proof construction*	*Students are encouraged to write and distinguish which component they're starting with if there are several components in the proof to add clarity and Gary is pointing this out.*
13	**Edgar**: Oh, you mean writing the statement?			
14	**Gary**: Ja			
15	**Edgar:** Whole of this?			
16	**Gary**: Ja			
17	**Edgar**: No problem. I thought that because I've already written it here *[points to* $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ *]* I'll solve it and then afterwards I'm	*Transactive response- to explain one's thinking*	***H1a****: further elaboration of the overall approach.* **L2a:** *addition of statement suggested by Gary to add clarity to the proof construction.*	*Edgar's explanation shows that he might have thought the statement suggested by Gary to be superfluous and clearly explains the two components*

	going to write only that one to show that that one is the subset of that one *[points to the board]* Then we, then I do the proof finally. But I can do that for you. Ok, let this one *[points to:* let $(x, y) \in (A \cup B) \times C$. *Above this writes: now to show* $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ *]* Ok			*of the proof and his plan of action.*
18	**Gary**: Another thing. You say let *x* and *y* be an element of that, right?	*Transactive question- request for reflection. Acting as more knowing other*		*Gary acting as more knowing peer and guiding Edgar in the EZPD using prompts that cause him to reflect on his thinking.*
19	**Edgar**: Yes			
20	**Gary:** Then after that you say it implies that *x, y* is an element of *A* union *B*?	*Transactive question- request for reflection. Acting as more knowing other*	*L3b: questioning when assertions have been made without any basis*	*Gary acting as more knowing peer and guiding Edgar in the EZPD using prompts that cause him to reflect on his thinking.*
21	**Edgar**: Yes			
22	**Gary**: Why do you say that?	*Transactive question- request for reflection. Acting as more knowing other*	*L3b: questioning when assertions have been made without any basis*	*Gary is acting as more knowing peer and guiding Edgar in the EZPD using prompts that cause him to reflect on his thinking.*
23	**Edgar**: It's an element of…. Oh, agiri if you look at this is an element of the	*Transactive response- to explain one's thinking*		*Edgar's explanation again seems to confirm that his*

	whole of this, isn't it? *[points to $(A \cup B) \times C$]* So now, um, this, it implies that this one *[points to (x, y)]* is an element of this one *[points to $(A \cup B)$]* and again is an element of C – both of them x and y.		***L3ax**: incorrect deduction made as a result of :* ***L1ax**:incorrect understanding of the Cartesian product of sets*	*understanding of the notions of the Cartesian product and intersection are associated with the word 'and' seeming to indicate complex thinking about the terms Cartesian product and intersection*
24	**Gary**: Let's go back to our actual definition.	*Reference to definition. Acting as more knowing other*	*L1b: making reference to the definition.*	*Gary acting as more knowing peer and making reference to the definition- reminding others of the importance of the definition.*
25	**Edgar**: The definition, ja.			
26	**Gary**: It says x is, x comes from A	*explanation of definition*	*L1b: explaining the definition in his own words*	*Gary taking Edgar through the definition in his own words. His explanation shows that he may have concept level understanding of the Cartesian product.*
27	**Edgar:** Mmm			
28	**Gary**: And y will come from B	*explanation of definition*	*L1b: explaining the definition in his own words*	*Gary taking Edgar through the definition in his own words. His explanation may show that he has concept level understanding of the Cartesian product.*
29	**Edgar:** Mmm			
30	**Gary**: while working with Cartesian products, right?			*Gary reminds Edgar all the time that we're talking about the Cartesian product.*

31	**Edgar**: Mmm			
32	**Gary**: So right now we're working with Cartesian products you tell us that *A* union *B*, it means that *x* must come from *A* union *B*.	*Application of definition*	*L1b: application of knowledge of definition to this specific problem*	*Gary now helps Edgar to understand how to apply the definition to specific cases. His explanation and application may show that he has concept level understanding of the Cartesian product.*
33	**Edgar**: *x* must come from *x* union *B*?	*Transactive response-reflection*		*Edgar reflecting on the application of the definition*
34	**Gary**: *x* should come from *A* union *B*. And then *y* comes from *C*.	*Transactive response-explanation*	*L1b: application of knowledge of definition to this specific problem*	*Gary now helping Edgar to understand how to apply the definition to specific cases. His explanation and application may show that he has concept level understanding of the Cartesian product.*
35	**Edgar:** Before, before you do that I think, I think according to my understanding I don't know, according to my understanding I think I have to… This one *[points to: let* $(x, y) \in (A \cup B) \times C$ *]* If the… this one *[points to* (x, y) *]* is an element of both these *[underlines* $(A \cup B) \times C$ *]* isn't it?	*Transactive response-explanation, questioning*	***L1ax***:*incorrect understanding of the Cartesian product of sets*	*It's clear that Edgar is associating the Cartesian product with the notion of intersection which may indicate complex thinking. This may be because both definitions contain the word "and" and the notion of the ordered pair is not properly understood.*

36	**Frank**: If you write it like that you must state that if, and only if	*Transactive response- point of confusion*	*L2ax: selecting inappropriate statements that do not make sense or add to the logic of proof construction*	*Frank also shows confused thinking of the notion of the Cartesian product.*
37	**Edgar:** Yes?			
38	**Frank**: you are stating like this. If you write it like this			
39	**Joseph**: You can say it's an element of both $A \cup B$ and C if we are talking of an intersection.	*Contribution to an idea- stating the root of the confusion*	*L1b: identifying the cause of the problem as association with another sign or symbol whose properties he clarifies*	*Joseph very cleverly identifies the cause of the confusion in Edgar's reasoning as being Edgar's association of the Cartesian product with the intersection. This indicates that he may have concept level thinking about both the notions of Cartesian product and the intersection.*
40	**Edgar**: If we are talking of an intersection?	*Transactive response- to reflect on one's thinking*		*Edgar is being guided in the ZPD to really reflect on his actions.*
41	**Joseph:** Ja. And if that cross wasn't there A union B intersection C	*Transactive response- to elaborate*	*L1b: explaining what might have been the cause of Edgar's mistake, that of seeing the Cartesian product as an intersection.*	*Joseph continues explaining how the association of the Cartesian product with the intersection is causing the problem.*
42	**Edgar**: Ok	*confirmation*		
43	**Joseph**: So in this case we are talking of cross product.	*Transactive response- to elaborate*		*Joseph reinforcing the fact that these are two different mathematical objects.*
44	**Edgar:** Yes	*confirmation*		
45	**Joseph**: It means the element x belongs	*Transactive response- to*	*L2b: correctly applying*	*Joseph interprets the*

	to the set that is before the cross	*elaborate*	*knowledge of the definition to a specific case*	*definition of the Cartesian product very simply to convey more understanding.*
46	**Edgar**: Yes	*confirmation*		
47	**Joseph**: And then y belongs to the set that is after the cross. So in this case he is right in saying x is an element of A union B.	*Transactive response- to elaborate*	*L2b: correctly applying knowledge of the definition to a specific case*	*Joseph interprets the definition of the Cartesian product very simply to convey more understanding.*
48	**Edgar**: OK. I was confused. Thanks a lot for that *[erases : $\Rightarrow (x, y) \in (A \cup B)$ and $(x, y) \in C$]* OK *[writes: $\Rightarrow x \in (A \cup B)$ and $y \in C$]* Is that what you are saying?	*Moment of realization* *Continuation of plan or strategy- correct deduction from previous statement*	***L1a**: using newly met terms, symbols correctly (spoken and written)* ***L3a**: making the correct deduction from the previous statement*	*Edgar seems to have understood the notion now and makes the correct deductions.*
	So this implies that x is an element of A or x is an element of B and y an element of C. OK *[writes: $\Rightarrow x \in A$ or $x \in B$ and $y \in C$]*	*correct deduction from previous statement*	***L3a**: making the correct deduction from the previous statement* ***L1a**: using newly met terms, symbols correctly (spoken and written)*	
49	**Edgar**: So… So now we know that this *[points to: $\Rightarrow x \in A$ or $x \in B$ and $y \in C$]* because there's an 'or' between that means x can be an element of A and y an element of C or x an element of B and y an element of C. It's fine like that? *[writes: $\Rightarrow x \in A$ and $y \in C$ or $x \in B$ and $y \in C$]*	*Continuation of plan or strategy* *Talking out loud while writing- reasoning*	***L1a**: correctly using newly met terms, symbols and signs and* ***L1b:**explaining the meaning of terms and symbols and the reasoning used to reach the next deduction* ***L2a**: selects correct phrases/statements that add to the logic of the proof*	*Edgar makes the next deduction correctly too indicating good understanding of the Cartesian product and its elements, ordered pairs.*

			construction. ***L3a**: making a correct deduction from previous statement.*	
	So this implies that *x* is an element of *A* cross *C* or *x* an element of *x* cross *[writes: ⇒x∈ (A×C) or x∈ (x ×]*		***L3ax**: incorrect deduction from previous statement* ***L1ax**: incorrect selection and use of newly met terms; Cartesian product and ordered pairs, also a slight writing error*	*Edgar's next deduction is incorrect showing that he has not quite achieved concept level thinking of the notions of Cartesian product and ordered pairs.*
50	**S**: *B*	*Contribution to proof*		
51	**Edgar**: Is it *x*? *[erases x and writes B*] cross *C* *[So the statement is written: ⇒x∈ (A×C) or x∈ (B×C)]* Thus	*Correction of mistake*	***L1ax**: corrects writing error but use of newly met terms still incorrect* ***L3ax**: repeating the incorrect deduction from previous statement*	*Edgar has not yet realized his mistake*
52	**Gary**: This one	*Transactive question-request for reflection*		
53	**Edgar**: Sorry?			
54	**Gary**: Can you explain that last statement?	*Transactive question-request for reflection* *Acting as more knowing other*	*L3b: questioning the incorrect deduction made by Edgar*	*Gary is again acting as the more knowing other and guiding Edgar using prompts that promote reflection and justification to discover his mistake.*
55	**Edgar:** This one? *[points to: ⇒x∈ (AxC) or x∈ (BxC)]*			
56	**Gary**: Ja			
57	**Edgar**: That one. OK. Um, we know	*Moment of realization*	***L1a**: realizing the incorrect*	*Gary has successfully been*

	that x represents a cross, so *x* is an element of *A* and this *y* *[points to: $y \in C$]* OK…*x* is an element of *A* and *y* is an element of *C*, so it's *x, y*. It's *x, y*. *[Erases: $\Rightarrow x \in (A\times C)$ or $x \in (B\times C)$]* Ja, it's *x, y*. OK. Sorry, it's *x, y*		*usage of terms and symbols and correcting this by:* ***L1b**: reflecting on and explaining the meaning of terms and symbols in own words.*	*able to guide Edgar to recognize his mistake and to go on to correct it, using prompts requesting reflection and justification. Edgar's quick realization of the mistake and his explanation indicate that his understanding of these notions is evolving towards concept level understanding.*
	[writes: $\Rightarrow (x, y) \in (A\times C)$ or $(x, y) \in (B\times C)$] Is it fine now?		***L3a**: making the correct deduction*	
58	**S**: Ja			
59	**Edgar**: Thus we can conclude that this one is a subset of that *[points to: now to show $(A\cup B) \times C \subseteq (A\times C) \cup (B\times C)$]* *[writes: Thus $(A\cup B) \times C \subseteq (A\times C) \cup (B\times C)$]*	*conclusion of plan or strategy*	***L3ax**: missing one deduction that needs to come before the conclusion* ***L3c**: making the correct conclusion and explaining the reasoning behind it.* ***L2c:** proof method of showing one set is a subset of another correct*	*Edgar's correct conclusion confirms that his understanding of the proof method for proving that one set is a subset of another seems to be at concept level.*
	And then I'll do the other one. So now I've done this one, and now I have to do this one. *[points to: $(A\times C) \cup (B\times C) \subseteq (A\cup B) \times C$]*	*Start of plan/ strategy*	***H1a**: explains the main idea behind the next portion of proof*	
60	**Joseph**: You're sure that's it?	*Transactive question-request for reflection*		
61	**Edgar:** Yes			
62	**Joseph**: You should show that the Cartesian product also belongs to *A*	*Contribution to an idea-acting as more knowing*	*L2a: suggesting selection of statement that adds to the logic*	*Edgar had missed this deduction before making the*

	cross *C* and *B* cross *C* just from the last statement.	*other*	*of the proof construction before the conclusion* *L3a: pointing out the deduction that was missed before the conclusion*	*conclusion and Joseph is pointing this out. Joseph's understanding of the both the proof method and the logical steps in the proof construction seem to be at concept level.*
63	**Edgar**: Before this one?			
64	**Joseph**: Mmm			
65	**Edgar**: The Cartesian product belong to *[points to:* $\Rightarrow (x, y) \in (A\times C)$ *or* $(x, y) \in (B\times C)$ *]* *A*, you mean I put a union here?	*Transactive question- request for clarification*		
66	**Joseph**: Ja			
67	**Edgar**: Ok. Is it a must?	*Transactive question- request for justification*	***L2ax***: *questioning addition of a statement that would add to the logic of the proof construction.* ***L1ax:*** *his question shows incomplete understanding of the symbol union.*	*This may indicate incomplete understanding of the logical relationship as well as the use and meaning of mathematical symbols and terminology, in this case the union.*
68	**Joseph**: I think so	*Confirmation*		
69	**Edgar**: Is it a must or?	*Transactive question- request for justification*	***L2ax, L1ax***: *questioning addition of a statement that would add to the logic of the proof construction.*	
70	**Joseph**: Ja, even that one should have *A* is a subset of	*Transactive response- to explain*	*L1c: illustrating with examples*	*Here Joseph might be illustrating with examples from previous proofs*
71	**Edgar**: Mmm			
72	**Joseph:** So if you had that in the first place to say *A* is a subset of *B*	*Transactive response- to explain*	*L1c: illustrating with examples*	*Here Joseph might be illustrating with examples*

				from previous proofs
73	**Edgar**: Ok, what do the others think? Is it a must or	*Transactive question- request for justification*		*Edgar is still not convinced about the necessity of the statement that Joseph has suggested.*
74	**S**: Mmm			
75	**Edgar**: Is it?	*Transactive question- request for justification*		
76	**S**: Yes *[laughter]*			
77	**Edgar**: *[he erases: Thus* $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ *]* It's just that if it was a computer I was going to *[shows how he would move it down]* move it down.	*Humour- laughter and jokes*		
	After much laughter Edgar continues			
78	*[* $(x, y) \in (A \times C)$ *or* $(x, y) \in (B \times C) \Rightarrow (x, y) \in (A \times C) \cup (x, y$ *]*		***L1ax****: incorrect use of mathematical symbol*	*Edgar seems to think that the symbol of union can be substituted directly for "or" indicating complex level thinking as Edgar is associating the symbol of the union with the word 'or' found in the definition.*
79	**S**: Just write *B* cross *C*	*Correction of mistake*	*L1a: correcting the use of mathematical symbol/sign*	*Other students participating in the ZPD realize this mistake.*
80	**Edgar**: I must just write…?	*Transactive question- request for explanation*		
81	**S**: *B* cross *C*	*Correction of mistake*	*L1a: correcting the use of mathematical symbol/sign*	
82	**Edgar**: Ok, *B* cross…	*Talking out loud while*	***L3a****: correct deduction*	*Edgar is guided to make the*

	[erases the second x, y and writes $(B\times C)$ *]* *[Thus he has written:* $\Rightarrow (x, y) \in (A\times C) \cup (B\times C)$ *]*	*writing- saying what is being written*	***L1a**: correct use of mathematical terms, symbols and signs.*	*correct deduction but makes a writing error when writing the conclusion.*
	[talks while he writes:] And that A union B cross… is a subset of… *[writes: Thus* $(A\cup B) \times C \subseteq (A\times B) \cup (B\times C)$ *]*		***L1ax**: typing error in writing the conclusion resulting in:* ***L3cx**: incorrect conclusion*	
83	**S**: A question	*Correction of mistake*		
84	**Edgar:** *[erases the B in* $(B\times C)$ *]* Is it A cross C?	*Transactive question-request for clarification*		
85	**S**: No, the first brackets	*Correction of mistake*		
86	**Edgar**: Here? *[points to* $(A\times B)$ *]*	*Transactive question-request for clarification*		
87	**S**: Ja			
88	**Edgar**: *[erases the B in* $(A\times B)$ *and replaces it with a C. He also puts the B back into* $(B\times C)$*. Thus the statement reads:* *Thus* $(A\cup B) \times C \subseteq (A\times C) \cup (B\times C)$*]*	*Correction of mistake* *Completion of first part of proof*	***L3c**: correct conclusion made* ***L1a**: mathematical terms and symbols used correctly*	*Edgar makes the correct conclusion and completes the first component of the proof*

Sub-episode 3.2: Proof of $(A\times C) \cup (B\times C) \subseteq (A\cup B) \times C$

In the next sub-episode Edgar attempts the next component of the proof: $(A\times C) \cup (B\times C) \subseteq (A\cup B) \times C$.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments

				on the proof construction process
88	**Edgar:** [Continuing the next part:] So the other one says [*writes: to show* $(A\times C)\cup(B\times C)\subseteq(A\cup B)\times C$	*Start of second part of proof* *Proposal of a new plan*	***L2a***: *selection of appropriate statements that add to the logic of the proof construction process*	*Edgar identifies the next component of the proof and selects the appropriate statement to begin the proof which may indicate concept level understanding of the proof methods involved.*
	let $(x, y)\in(A\times C)\cup(B\times C)$		***L2a***: *selecting appropriate statements to start next portion of the proof* ***L2c***: *proof method seems to be well understood and followed* ***L1a***: *correct mathematical terminology*	
	$\Rightarrow x\in(A\times C)$ *or* $y\in(B\times C)$		***L3ax,***: *incorrect deduction made from previous statement* ***L1ax***: *incorrect use of mathematical symbols and signs* ***H2ax:*** *does not transfer ideas from the first component of the proof to the next component such as the Cartesian product and the ordered pair*	*Edgar is unable to transfer the knowledge of Cartesian product and the ordered pair to the second component of the proof. He shows an inclination to follow the form rather than the thinking and reasoning behind the notions just used in the first part of the proof. This may indicate complex thinking about the notion of the Cartesian product as he seems to associate its application with the example in the previous sub-episode.*
	$\Rightarrow x\in A$ *and* $x\in C$ *or* $y\in B$ *and* $y\in C$]	*Transactive question-*	***L3ax***: *incorrect deduction made*	*Edgar seems to again*

	So it's fine?	*request for reflection*	*from previous statement* **L1ax:** *incorrect use of mathematical symbol*	*interpret the '×' as the word 'and' indicating complex thinking about this mathematical object. He seems to easily slip back into misuse of the Cartesian product indicating that he still has not achieved concept level thinking.*
89	**S**: Yes	*Confirmation*		*Other students seem to have similar complex thinking about the notion of Cartesian product.*
90	**Edgar:** I want to show that… *x*… Oh *x*…	*Continuation of plan*		
91	**Gary**: Oh this statement after letting *x*, *y* be an element of *A* cross *C* union *B* cross *C* ah, can you clarify?	*Transactive question- request for clarification*	*L3b: Gary questioning and probing wrong deductions made without any basis.*	*Gary is acting as more knowing other and guiding Edgar by prompting him to reflect on his actions.*
92	**Edgar**: Which one?			
93	**Gary**: The first statement, *x*…. after that	*Transactive question- request for clarification*	*L3b: Gary questioning and probing wrong deductions made without any basis*	
94	**Edgar**: After the left			
95	**Gary**: Ja. We have made			
96	**Edgar**: Oh this is a union and this is *x*, *y*.	*Transactive response- to clarify*		
97	**Gary**: We have *A* cross *C* meaning…	*Transactive question- request for reflection*	*L1b: questioning meaning of the Cartesian product*	*Gary is encouraging Edgar to reflect on the meaning of the mathematical symbols and not just to use them without*

				thinking.
98	**Edgar**: There's no cross here, it's a union, it's an "or". That means that this can be this, or	*Transactive response- to clarify*		*Edgar showing that he does not seem to be aware of the cross, and is just paying attention to the union symbol. Perhaps this also indicates that students need to develop their sense of accuracy when writing mathematical statements and deductions. Often students are not aware that every written symbol and sign has a meaning and a consequence.*
99	**Gary**: Ja			
100	**Edgar**: This can be that, ja. Thanks. That's a mistake I've been making on the right, yes. Let me *[erases:* $\Rightarrow x\in (A\times C)$ *or* $y\in (B\times C)$ *and* $\Rightarrow x\in A$ *and* $x\in C$ *or* $y\in B$ *and* $y\in C$ *and writes:* $\Rightarrow x\in A$ *and* $y\in C$ *or* $x\in B$ *and* $y\in C$	*Moment of realization*	***L1a**: correct use of symbols, terms and mathematical language* ***L3a**: correct deduction made*	*Edgar realizes his mistake after Gary's questioning and seems to understand where he went wrong but he seems to keep making the same mistakes – new mathematical notions are difficult to be absorbed immediately and it takes time to achieve full understanding and to learn how to use the new terms and symbols correctly.*
	$\Rightarrow x\in A$ *or* $x\in B$ *and* $y\in C$		***L3a**: correct deduction made* ***L1a**: correct use of symbols, terms and mathematical language*	

	⇒ $x \in (A \cup B) \times C$]		***L3ax**: incorrect deduction made as a result of:* ***L1ax**: incorrect use of Cartesian product*	*Edgar seems to have forgotten about the second element in the ordered pair. Also he seems to view the Cartesian product symbol '×' to be interchangeable with the word 'and'.*
101	**Gary**: *y* must be an element of *C*	*Contribution to an idea – point out misconceptions*	*L1a: correction of use of the Cartesian product and ordered pair*	*Gary reminds him of his omission.*
102	**Edgar**: Pardon?			
103	**Gary**: an element of *C*	*Contribution to an idea – point out misconceptions*	*L1a: correction of use of the Cartesian product and ordered pair*	*Gary reminds him of his omission*
104	**Edgar:** *[erases the C and puts $y \in C$. Thus the statement now reads: ⇒ $x \in (A \cup B) \times y \in C$.]*	*Talking out loud while writing- reasoning, explaining concepts*	***L3ax:** incorrect deduction as a result of:* ***L1ax**: incorrect use of mathematical symbol (Cartesian product)*	*It is clear that in Edgar's view the Cartesian product symbol '×' is interchangeable with the word 'and'.*
	⇒ $(x, y) \in (A \cup B) \times C$		***L3a**: correct deduction* ***L1a**: correct use of mathematical symbols and signs.*	*Correct deduction and conclusion made showing definite improvement in Edgar's understanding.*
	Thus $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$		***L3c:**correct conclusion made* ***L1a**: correct use of mathematical symbols and signs*	
	Thus *[points to let $(x, y) \in (A \times C) \cup (B \times C)$]* This one, to show that *[continues after Thus $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$]*		***L2a**: use of appropriate statements, phrases that help make sense of the proof*	*Edgar shows good understanding of the method of proof of equality of sets explaining how the two components of the proof result*

	Let me call the (1), this is (1) *[writes (1) next to Thus* $(A\cup B)\times C\subseteq(A\times C)\cup(B\times C)$ *]* This is (2) *[writes (2) next to Thus* $(A\times C)\cup(B\times C)\subseteq(A\cup B)\times C$ *]*			*in the correct conclusion of equality, perhaps showing concept level thinking about method of proof of equality.*
	[After Thus $(A\cup B)\times C\subseteq(A\times C)\cup(B\times C)$ *he writes:* *From (1) and (2) :* $(A\cup B)\times C=(A\times C)\cup(B\times C)$ *]*	*conclusion of plan*	***L3c**: correct conclusion drawn on the basis of correct reasoning* ***L1a**: correct use of mathematical terms, symbols and signs*	
105	**Joseph:** I've got a question	*Transactive question*		
106	**Edgar**: Yes			
107	**Joseph**: Can you say suppose you have *x* is an element of *A* intersect *B*	*Transactive question- request for clarification*	*L1c: illustrating a concept with examples*	*Joseph is finding the root of the problem of an incorrect deduction and incorrect use of mathematical symbols and signs in first part of line 104, and bringing in other similar examples to illustrate the error more clearly.*
108	**Edgar**: For example?			
109	**Joseph**: For example, ja. From there can you say *x* is an element of *A*, write the intersection sign *x* is an element of *B*?	*Giving examples- narrative*	*L1c: illustrating a concept with examples*	*Joseph is finding the root of the problem of an incorrect deduction and incorrect use of mathematical symbols and signs in first part of line 104, and bringing in other similar examples to illustrate the error more clearly.*

110	**Edgar**: You come and…			
111	**Joseph**: [goes to the board] Suppose you have *x* is an element of *A* intersect *B* *[writes: $x \in (A \cap B)$]* You say it says to us that *x* is an element of *A* intersect *x* is an element of *B* *[writes: $\Rightarrow x \in A \cap x \in B$]* Because I think this intersection *[points to: $x \in (A \cap B)$]* tells us that we are thinking of one set	*Giving examples- narrative*	*L1c: illustrating a mathematical object with examples* *H3a: illustrates inferences from mathematical symbols with examples*	*Joseph illustrates the misconception with several other examples showing a good understanding and grasp of these notions (perhaps showing concept level understanding). By doing this he easily and very ably transfers his knowledge to his peers.*
112	**Edgar:** Ok			
113	**Joseph:** And can we say that? Can we move from there to there? *[points to $x \in (A \cap B) \quad \Rightarrow x \in A \cap x \in B$]* Why I'm asking this, I see this here *[underlines: $\Rightarrow x \in (A \cup B) \times \; y \in C$]* So I'm happy that we came across this because I'm also getting confused. Can we say this? *[points to $x \in (A \cap B) \; \Rightarrow x \in A \cap x \in B$]* Or even can we say *x* being an element of *A* union *B* implies that *x* is an element of *A* or *x* is an element of *B*? *[writes: $x \in (A \cup B)$ $\Rightarrow x \in A \cup x \in B$]* Can we say?	*Transactive question- request for clarification* *Giving examples- narrative and written*	*L1c: illustrating a mathematical object with examples* *H3a: illustrates inferences from mathematical symbols with examples*	*Joseph illustrates the misconception with several other examples showing a good understanding and grasp of these notions (perhaps showing concept level understanding). By doing this he easily and very ably helps and contributes to his peers' development.*
	The lecturer confirms the error and the correction is made, that is $\Rightarrow x \in$			

	$(A \cup B) \times \ y \in C$ *is changed to* $\Rightarrow x \in (A \cup B)$ *and* $y \in C$			

Session 2: Episode 4

After a few words of encouragement and praise to all the students from the lecturer for all the improvement that has been observed, the next proposition put up on the board for the students to attempt is: ***Proposition:*** $\boldsymbol{(A \cap B) \times C = (A \times C) \cap (B \times C)}$. A successful proof construction of this proposition requires knowledge of the proof method of showing set equality as well as the precise definitions of subset, intersection and the Cartesian product and the ability to use these definitions in the logical reasoning and justification of each step in the proof. Maria volunteers to come to the board to do this proof.

Sub-episode 4.1: Proof of $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$

	Speech and actions	**Student and teacher utterances**	**Proof comprehension**	**Interpretation according to T.F. and general comments on the proof construction process**
1	**Maria:** I think if we want to show that this is equal to this. *[points to the proposition]*	*Proposal of a new plan or strategy*	***H1a****:describes the main idea or over-arching approach used*	
	We also want to show that this *[points to* $(A \cap B) \times C$ *]* is a subset of this *[points to* $(A \times C) \cap (B \times C)$ *]* and this *[points to* $(A \times C) \cap (B \times C)$ *]* is a subset of this		***H1b****: breaks down the proof into components* ***H2a****: uses proof methods that she previously grappled with*	*Maria has successfully mastered the proof methodology of showing equality of sets- proof methodology seems to be at concept level*

	[points to (A∩B) × C]			
	So now we want to show that *A* union *B* times *C* is a subset of *[writes: We want to show that (A∪B) × C ⊆]*	*Start of plan*	***L2a**: selects appropriate opening statement adding to the logic of the proof process* *L1ax: makes a typing error and refers to Cartesian product as 'times'*	
2	**Gary:** *A* intersect *B*	*Correction of mistake*	*L1a: correction of error*	
3	**Maria:** Hmm?			
4	**Gary:** *A* intersect *B*	*Correction of mistake*	*L1a: correction of error*	
5	**Maria:** *[erases the ∪ sign and replaces it with ∩ then continues]* Subset of *A* times *C* intersect *B* times *C* and *[writes: ⊆(A×C) ∩(B ×) and*	*Talking out loud while writing- saying what is being written*	***L1a**: correction of error* *L1ax: makes a writing error and refers to Cartesian product as 'times'*	*Maria refers to the Cartesian product as 'times' but is able to use it correctly and sensibly, thus showing that she can work with the mathematical object but perhaps is just reluctant to use the longer name of the symbol*
6	**S:** Times *C*		*L1a: correction of error*	
7	**Maria:** Oh, thanks. We want to show that *[adds C so it reads: (A∩B) × C ⊆ (A×C) ∩(B×C) and (A×C) ∩(B×C) ⊆ (A∩B) × C]* And *A* times *C* intersect *B* times *C* is also a subset of *A* intersect *B* times *C*.		***H1a**:describes the main idea or over-arching approach used* ***H1b**: breaks down the proof into components* ***L1a**: correctly using newly met terms, symbols and signs* ***L2c**: identifying correct proof method* *L1ax: refers to Cartesian*	*proof methodology of showing equality of sets seems to be at concept level*

			product as 'times'	
	So now we first by proving this. *[points to $(A\cap B) \times C \subseteq (A\times C) \cap (B\times C)$]* *[writes: Now to show $(A\cap B) \times C \subseteq (A\times C) \cap (B\times C)$]*		***L2a****: selecting appropriate statement to start the proof plan* ***L1a****: correctly using newly met terms, symbols and signs*	*Maria has correctly broken down the proof construction process into components and now systematically starts the proof of the first component. This shows a huge improvement from the first session and a big jump in reasoning ability and systematic thinking.*
	Now to show that *A* intersect *B* times *C* is a subset of *A* times *C* intersect *B* times *C*. So we introduce… OK we start by writing: *x* and *y* being the element of this. *[points to $(A\cap B) \times C$]* *[writes: let $(x, y) \in (A\cap B) \times C$]* Let *x, y* be an element of *A* intersect *B* times *C*.		***L2c****: selects correct proof method* ***H2a****: uses proof methods that she previously grappled with* ***L1a****: correctly using newly met terms, symbols and signs* ***L2a****: selecting correct statement to start the proof construction* *L1ax: referring to Cartesian product as 'times'*	*Maria also shows her mastery of the proof of showing that one set is a subset of another- this proof methodology also seems to be at concept level. She also seems to have concept level understanding of the Cartesian product but we'll see below that this might be pseudoconceptual.*
	If this *[points to x]* is an element of this *[points to $(A\cap B)$]* it will imply that this *[points to x and $(A\cap B)$]* and *y* is an element of *C*. *[points to y and C]* *[writes while she talks]*… intersect *B* and *y* is an element of *C* *[writes: $\Rightarrow x\in (A\cap B)$ and $y\in C$]*	*Talking out loud while writing- reasoning*	***L3a****: makes correct deductions from previous statement* ***L1a****: correctly using newly met terms, symbols and signs* ***L2b:*** *selecting appropriate aspects of the definition of Cartesian product*	*Maria is justifying every statement she makes- she has now realized that she can't make deductions or bring in assertions without justification. This is seen throughout the proof to have become a well- established habit.* *Maria's explanations show*

	And if this *[points to x]* is an element of this *[points to (A∩B)]* it means that *x* is an element of *A* and again *x* is an element of *A* and *x* is an element of *B* and *y* is an element of *C*. *[writes: ⇒x∈A and x∈B and y∈C]*	*Talking out loud while writing- reasoning*	***L3a**: makes correct deductions from previous statement* ***L1a**: correctly using newly met terms, symbols and signs* ***L2b:** selecting appropriate aspects of the definition of intersection*	*that she is now very happy with the methodology of how to prove that one set is a subset of another and seems to want to share this knowledge with others.*
	So here we've got *[points to y∈C]* y being an element of *C* you can introduce it to both *A* and *B* seeing that *[writes while she talks]* *x* is an element of *A* and *y* is an element of *C*. Also *x* is an element of *B* and *y* is an element of *C*. *[writes: ⇒x∈A and y∈C and x∈B and y∈C]*	*Talking out loud while writing- reasoning*	***L3a**: makes correct deductions from previous statement* ***L1a**: correctly using newly met terms, symbols and signs*	
	So when grouping this you can say that *[writes while she talks]* *x* and *y* are elements of *A* and *C* and *x* and *y* again are elements of *B* and *C*. *[writes: ⇒(x, y) ∈ (A×C) and (x, y) ∈ (B×C)]*	*Talking out loud while writing- reasoning*	***L3a**: makes correct deductions from previous statement* ***L1a**: correctly using newly met terms, symbols and signs* ***L2b:** selecting appropriate aspects of the definition of Cartesian product*	
	And also again between these you can see that *x* and *y* are an element of *A* and *C* is… *[pointing to the statement just above]* we know that this is an intersection	*Talking out loud while writing- reasoning*	***L3a**: makes correct deductions from previous statement* ***L1a**: correctly using newly met terms, symbols and signs* *L1ax: referring to Cartesian*	

	because… we introduce the intersection sign and *B* times *C*. *[writes:* $\Rightarrow (x, y) \in (A\times C) \cap (B\times C)$ *]*		*product as 'times'* ***L2b:*** *selecting appropriate aspects of the definition of intersection*	
	So we started here with *A* intersection *B* *[points to* $(A\cap B)\times C$ *in the statement:* $\Rightarrow x\in (A\cap B)$ *and* $y\in C$ *]* with *x* and *y* being elements of *A* intersection *B* multiplied by *C* *[points to: let* $(x, y) \in (A\cap B) \times C$ *]* and here we ended up with this *[points to:* $\Rightarrow (x, y) \in (A\times C) \cap (B\times C)$ *]* And this implies that this *[points to: let* $(x, y) \in (A\cap B) \times C$ *]* is a subset of this *[points to:* $\Rightarrow (x, y) \in (A\times C) \cap (B\times C)$ *]* We start with this *[points to: let* $(x, y) \in (A\cap B) \times C$ *]* and end with this. *[points to:* $\Rightarrow (x, y) \in (A\times C) \cap (B\times C)$ *]*	*Talking out loud while writing- reasoning*	*L1ax: referring to Cartesian product as 'multiplied'* ***L3c:*** *making the correct conclusion based on the necessary justification*	

Sub-episode 4.2: Discussion of the association of the Cartesian product with the intersection

	Speech and actions	**Student and teacher utterances**	**Proof comprehension**	**Interpretation according to T.F. and general comments on the proof construction**

				process
8	**Christine:** Can I ask something?	*Transactive question-request for explanation*		
9	**Maria**: Ja			
10	**Christine**: Because 'and' means intersection can we say, in the bracket say *A* intersection *C*. Can you say that?	*Transactive question-request for explanation and clarification*	*L1bx: questioning use and meaning of terms, symbols and signs*	*Christine seems to be associating both the intersection and Cartesian product with the word 'and'. She now wants to know if she can substitute an intersection sign for the symbol denoting the Cartesian product since in her mind they both mean 'and'. She seems to be exhibiting complex level thinking.*
11	**Maria:** Hmm?			
12	**Christine:** That cross stands for an intersection, right? Can we put intersections in the bracket?	*Transactive question-request for explanation and clarification*	*L1bx: questioning use and meaning of terms, symbols and signs*	*Christine seems to be associating both the intersection and Cartesian product with the word 'and'. She now wants to know if she can substitute an intersection sign for the symbol denoting the Cartesian product since in her mind they both mean 'and'. She seems to be exhibiting complex level thinking.*
13	**S:** intersections?			

14	**T:** It's a good question. Can you…? What she means is can you put *x*, being an element of *A* intersection *C*? Is that what you meant?	*Facilitative- re-voicing*		
15	**Christine:** *(nods)* Mmm			
16	**T:** OK. And why can we not? Why do we know that we cannot do that – maybe that's a better way to ask the question.	*Transactive prompts- request for clarification*		
17	**Student:** I think you can…	*Transactive response- to clarify*	*L1bx: not able to identify the correct meaning and use of terms, symbols and signs*	
18	**Maria:** I think that here because we're speaking of a multiplication…. *[points to: Proposition: $(A \cap B) \times C = (A \times C) \cap (B \times C)$]* here we started with a multiplication sign and we want to prove that you see this side here *[points to: $(A \cap B) \times C$]* we've got an intersection and here we've got a multiplication sign. And here we've got *[points to: $(A \times C) \cap (B \times C)$]* two multiplication signs. So if we prove this *[points to the lower part of the board]* we must prove this also looking at this side that what this side contains *[points to: $(A \times C) \cap (B \times C)$]*	*Transactive response- to clarify*	***L1bx**: not able to identify the correct meaning and use of terms, symbols and signs*	*Maria seems to exhibit pseudoconceptual thinking about the Cartesian product as she is able to use, interpret and apply the notion of the Cartesian product correctly in the proof construction but does not seem to be able to explain the reasoning and logic behind the notion correctly.*

19	**Christine**: So, if I were to put intersections there, then are you changing the question somehow? You know what I'm saying?	*Transactive question- request for clarification*	*L1bx: questioning use and meaning of terms, symbols and signs*	*Christine is still confused about her association of the Cartesian product and the intersection.*
20	**Maria:** Ja, I think, because…			
21	**T:** We're not quite there yet, any other, any other input? Anyone else want to say? Yes?	*Transactive prompt- request for elaboration*		
22	**Joseph:** I think in terms of the intersection it is when you say like one variable, suppose *x* is in both sets *A* and *B*. Now when you have the crosses where you have two variables – *x* is in *A* and *y* is in *C*. So we've got there, we have *x, y* – *x* is the set of, I mean is an element of the set before the cross. And *y* is an element of the set after the cross. When you see a cross we actually speak of two variables.	*Contribution to an idea – point out misconceptions and state the root of the confusion*	*L1b: describing and explaining the meanings of terms and symbols used*	*Joseph's understanding of the notions of intersection and Cartesian product of sets seems to be at concept level and his very able and thorough explanation is helping his peers develop in the EZPD created.*
23	**T:** Is that clear? So that's the important thing. Look over there, we should just put the definition up of *A* intersection *B*, *x* element of *A* intersection *B*. It's the same variable *x* in *A* and *B*, whereas here we have *x* element of *A*, and *y* element of *C*. So then you know that we are talking of the Cartesian product, you're not talking of an intersection. OK? Alright?	*Didactive – reference to definition* *Facilitative- re-voicing and confirming others' ideas to highlight misconceptions and learning about these*		
24	**Joseph:** And again before the last thing	*Contribution to an idea –*	*L1b: describing and explaining*	*Joseph seems to display true*

	there you said you have *x, y* being the element of *A* and *x, y*. You can treat *x, y* as one variable and then the intersection comes when the *x, y* is contained in both *A* cross *C* and *B* cross *C*.	*point out misconceptions and state the root of the confusion*	*the meanings of terms and symbols used*	*concept level understanding of Cartesian product and ordered pairs.*

Sub-episode 4.3: Conclusion of the first component of the proof and attempt at proof construction of $(A\times C) \cap (B\times C) \subseteq (A\cap B) \times C$

Maria then goes on to complete the proof in sub-episode 4.3.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
25	**Maria:** Can I go on? *[says while she writes]* So *A* intersection *B* multiplied by *C* is a subset of *A* multiplied by *C* intersection *B* multiplied by *C [writes: So* $(A\cap B) \times C \subseteq (A\times C) \cap (B\times C)$ *]*	*conclusion of plan*	***L3c****: making the right conclusion and providing all the justification necessary* ***L1a****: selecting correct mathematical terms, symbols and signs* *L1ax: referring to Cartesian product as 'multiplied'*	*Maria seems to exhibit true concept thinking about the proof of showing a set is a subset of another- bringing proof to conclusion*
26	I've now already proved this one so let's come this side. *[says while she writes]* I want to show that *A* multiplied by *C*	*Talking out loud while writing- saying what is being written and reasoning*	***L2a****: selecting appropriate phrases and statements to start the next component of proof* ***L1a****: selecting the correct terms,*	*Maria still refers to this 'side' when talking about proof of equality of sets but is able to do the proof perfectly.*

	intersection *B* multiplied by *C*. This is a subset of *A* intersection *B* multiplied by *C*. *[writes: to show* $(A\times C) \cap (B\times C) \subseteq (A\cap B) \times C$ *]*Again we introduce the Cartesian – what do you call the *x* and *y*?	*Start of plan*	*signs and symbols* *L1ax: referring to Cartesian product as multiplied*	
27	**Student:** Cartesian product	*Transactive response- to give an answer*	*L1ax: selecting the incorrect mathematical language (should be an ordered pair)*	
28	**Maria:** the Cartesian product. OK *[writes: x and then erases it and writes: let* $(x, y) \in (A\times C) \cap (B\times C)$ *says while she writes]* Let *x* and *y* to be the element of this *A* times *C* intersection *B* times *C*.	*Talking out loud while writing- saying what is being written and reasoning*	***L2a**: selecting correct opening statement for proof of next component* ***L1a**: selecting correct mathematical terms, symbols and signs* *L1ax: referring to Cartesian product as times*	
	Then if you have this *[points to (x, y)]* are an element of this *[points to A×C]* We know that this *[points to (x, y)]* can be an element of this *[points to A×C]* and again an element of this *[points to B×C]* because you've got the intersection sign. Such that *x, y* are an element of *A* times	*Talking out loud while writing- saying what is being written and reasoning*	***L1b:** explaining the notion of intersection in own words* ***L3a**: correct deduction made from previous statements* ***L2b:** selecting appropriate aspects of the definition of*	*Maria is able to explain and unpack exactly what the notions of intersection and Cartesian product mean and seems very able to use these newly met terms with ease and correctly.*

	C and again x, y are element of B times C. *[writes:* $\Rightarrow (x, y) \in (A\times C)$ *and* $(x, y) \in (B\times C)$ *]*		*intersection* ***L1a**: selecting correct terms, symbols and signs* *L1ax: referring to Cartesian product as times*	
	So now if these are elements of $A\times C$, it means *[points to* $(x, y) \in (A\times C)$ *]* x is an element of A and y is an element of C. And again x is an element of B and y is an element of C. *[writes:* $\Rightarrow x\in A$ *and* $y\in C$ *and* $x\in B$ *and* $y\in C$ *]*	*Talking out loud while writing- saying what is being written and reasoning*	***L3a**: correct deduction made from previous statements* ***L1a**: selecting correct terms, symbols and signs* ***L2b:** selecting appropriate aspects of the definition of Cartesian product*	*Maria gives detailed justification of each deduction and conclusion made. Her correct deductions indicate that she now has a great appreciation of the need for justification and logical reasoning in the proof. They also indicate that she may now have concept level knowledge of the Cartesian product.*
	So because we know that we have the same thing, you can just write it once. We say that x is an element of A and x is an element of B. Also y is an element of C. *[writes:* $\Rightarrow x\in A$ *and* $x\in B$ *and* $y\in C$ *]*	*Talking out loud while writing- saying what is being written and reasoning*	***L3a**: correct deduction made from previous statements* ***L1a**: selecting correct terms, symbols and signs*	
	And what does this mean? Because you have got x being both an element of A and B you can write this to be, you can combine it and say x is an element of A intersection B and y is an element of C. *[writes:* $\Rightarrow x\in (A\cap B)$ *and* $y\in C$ *]*	*Talking out loud while writing- saying what is being written and reasoning*	***L3a**: correct deduction made from previous statements* ***L1a**: selecting correct terms, symbols and signs* ***L2b:** selecting appropriate aspects of the definitions of Cartesian product and intersection*	
	And from the definition there *[points to the top of the board]* it says that if x is an element of this and	*Reference to definition and explanation of definition*	***L1b:** referring to the definition of the Cartesian product and explaining in own words*	*Maria's reference to and use of the definition of the Cartesian product shows that*

	this *[points to $(A \cap B)$ and $y \in C$]* it means that the Cartesian products *x* and *y* are elements of *A* intersection *B* multiplied by *C*. *[writes: $\Rightarrow (x, y) \in (A \cap B) \times C$]*		***L3a**: correct deduction made from previous statements* ***L1a**: selecting correct terms, symbols and signs* ***L2b:** selecting appropriate aspects of the definition of Cartesian product* *L1ax: referring to Cartesian product as 'multiply' and to the ordered pair as Cartesian product*	*she is now understanding the definition and is not afraid to use and apply it, which is another huge step in the right direction and may be indicative of concept level knowledge of these mathematical objects.*
	So here we started with this *[points to: let $(x, y) \in (A \times C) \cap (B \times C)$]* and end with this *[then points to $\Rightarrow (x, y) \in (A \cap B) \times C$]* It means that this *[points to: let $(x, y) \in (A \times C) \cap (B \times C)$]* is a subset of this *[points to: $\Rightarrow (x, y) \in (A \cap B) \times C$]* So *A* times *C* intersection *B* times *C* is a subset of *A* intersection *B* times *C*. *[writes: So $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$]*	*Talking out loud while writing- saying what is being written and reasoning* *Conclusion of second component*	***L3c**: making the correct conclusion with all the necessary justification* ***L1a**: selecting correct terms, symbols and signs* ***L2c**: showing good understanding of proof framework and methodology* *L1ax: referring to Cartesian product as times*	*Maria's thorough explanation of why she can now make the conclusion of the second component of the proof and the final conclusion of the proof, seem to confirm concept level understanding of the proof methodologies.*
	This will be (1) *[writes* (1) *next to So $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$]* And this it will be (2) *[writes* (2) *next to So $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$]*		***L2a**: selecting appropriate statements that add to the logic of proof construction* ***L2c**: showing good understanding of proof framework and methodology* ***H1d**: recognizes the logical relationship between the*	

			components of the proof	
	So from this *[points to* (1)] if this *[points to: $(A \cap B) \times C$]* is a subset of this *[points to $(A\times C) \cap (B\times C)$]* And this *[points to $(A\times C) \cap (B\times C)$]* is a subset of this *[points to $(A \cap B) \times C$]* It implies that this *[points to*: $(A \cap B) \times C$ *]* is equal to this *[points to $(A\times C) \cap (B\times C)$]*		***H1c***:*identifies the purpose of the two components of the proof* ***L2c***: *showing good understanding of proof framework and methodology*	
	So we can conclude and say that *A* intersection *B* times *C* is equal to *A* times *C* intersection *B* times *C* *[writes: Thus : $(A \cap B) \times C = (A\times C) \cap (B\times C)$]* *[returns to her seat]*	*Conclusion of proof*	*L1ax: referring to Cartesian product as times* ***L2c***: *showing good understanding of proof framework and methodology* ***L1a***: *selecting correct terms, symbols and signs* ***L3c***:*making the correct conclusion with all necessary justification and successfully ending the proof construction*	

Session 2: Episode 5

The next new mathematical object that is covered in the second session is that of power sets. Briefly the power set of the set A is the set of all possible subsets of A. The notion of the power set has been covered previously in lectures and after a very brief revision of its definition, the next proposition is put up on the board: *Proposition:* $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$. The proof of this proposition involves the proof of the two implications $A \subseteq B \Rightarrow P(A) \subseteq P(B)$ and $P(A) \subseteq P(B) \Rightarrow A \subseteq B$ and requires knowledge of the proof methods of the double implication, implication and of how to show that one set is a subset of another set, as well as the knowledge of the precise definitions of subset and power set. Frank volunteers to come up and do this proof.

Sub-episode 5.1: Proof of first component of the proof, $A \subseteq B \Rightarrow P(A) \subseteq P(B)$

	Speech and actions	Student and teacher utterances	Proof comprehension and construction	Interpretation according to T.F. and general comments on the proof construction process
1	**Frank:** *[comes to the board]* *[writes: (a)* $A \subseteq B \Rightarrow P(A) \subseteq P(B)$		***H1b****: breaks down the proof into components* ***L2a:*** *selection of statements that add to the logic of the proof construction process* ***L2c****:selection of correct proof methodology for double implication proof* ***L1a****: correct use of mathematical terms, symbols and signs*	*Proof method for proof of a double implication and an implication seems to be well understood and seem to be at concept level.*

	Assume $A \subseteq B$ we show that $P(A) \subseteq P(B)$		***L2a**: selection of correct opening statement that adds to the logic of proof of the first component of an implication proof* ***L2c**:selection of correct proof methodology for implication proof* ***L1a**: correct use of mathematical terms, symbols and signs*	
	let $X \in P(A)$		***L2a**: correct opening statement for proof* ***L1a**: correct use of mathematical terms, symbols and signs* ***L2c:** selection of correct proof methodology for subset proof*	*Frank's correct use of capital X as an element of the power set of A, and his correct deduction that this translates to X being a subset of A seems to indicate that his understanding of this newly met notion, the power set is at pseudoconcept or concept level. Similarly Frank translates X being a subset of B to X being an element of the power set of B again indicating pseudoconcept or concept level understanding of the notion of power sets. Frank's understanding of methodology of subset proof also seems to be at concept level.*
	$\Rightarrow X \subseteq A$		***L3a**: correct deduction made from previous statement* ***L1a**: correct use of mathematical terms, symbols and signs*	
	$\Rightarrow X \subseteq B$ (since $A \subseteq B$)		***L3a**: correct deduction made from previous statement* ***L1a**: correct use of mathematical terms, symbols and signs*	
	$\Rightarrow X \in P(B)$		***L3a**: correct deduction made from previous statement* ***L1a**: correct use of*	

			mathematical terms, symbols and signs	*Each deduction is justified and his grasp of logical reasoning appears sound.*
	thus $P(A) \subseteq P(B)$ *]*		***L3c**: correct conclusion made* ***L2c:** correct method of proof for subset proof* ***L1a**: correct use of mathematical terms, symbols and signs*	*Frank's understanding of methodology of subset proof seems to be at concept level*
2	**T:** Good. Can you tell us what you were doing?	*Transactive prompt: request for explanation*		
3	**Frank:** You see what we must do here is to… You see that here we have the double implication *[points to the board]* So we must start with the implication. The… that we must show that *A* is a subset of *B* will lead us to *P*, of the power set of *A* is a subset of the power set of *B*. So we assume that *A* is a subset of *B*, then we must show that *P*, power set of *A* is a subset of power set of *B*. So let *X* be an element of power set of *A*. So when we say *X* is an element of power set of *A* we must use the capital letter *X*, not a small letter *x* *[writes: X x]*	*Transactive response: explanation and clarification*	***L1b**: explaining the meaning of terms or symbols* ***L2c**: clarifying the method of proof*	*Frank's explanation of the method of proof of an implication confirms that his understanding of this proof method seems to be at concept level. He is able to use all the correct terminology connected with the proof method such as 'assume'.*
4	**Student:** Why?	*Transactive question – request for clarification and explanation*		

5	**Frank:** Because it is a power set. Then this implies that X may be a subset of A *[points to the board]* and then…	*Transactive response: explanation and clarification*	***L3a**: explanation of deduction made*	*When questioned about why an element of the power set should be in capital letters, he does not really expand on this perhaps indicating incomplete understanding. Thus perhaps his understanding of the power set is pseudoconceptual.*
6	**T:** Is it 'may be' or it definitely is?	*Transactive prompt-clarification*		
7	**Frank:** It is definitely. So this implies that X may be a subset of B since A is a subset of B. Then X may be an element of power set of B.	*Transactive response: explanation and clarification*	***L3a:** explanation of deduction made*	*His explanations of why deductions were made indicate good understanding of the notions of subset and the power set and his ability to use these mathematical objects correctly in the proof construction .*
8	**Gary:** So in your case X is an element?	*Transactive question – request for clarification and explanation*	*L1b: questioning the meaning of power sets and their elements*	*Gary's questioning serves the benefit of clarifying the notion of the power set further.*
9	**Frank:** Yes, it's an element… in, of the power set, but it's a subset of…	*Transactive response: explanation and clarification*		*Frank is now explaining the notion of the power set and what it means to be an element of a power set showing good understanding of the notion.*
10	**Gary:** You create then your capital letter X – where is it?	*Transactive question – request for clarification and explanation*	*L1b: questioning the meaning of power sets and their elements*	*Gary's questioning serves the benefit of clarifying the notion of the power set further.*

11	**Frank:** Oh, my capital letter *X*?			
12	**Gary:** Ja			
13	**Frank:** This one?			
14	**Gary:** What does it represent?	*Transactive question – request for clarification and explanation*	*L1b: questioning the meaning of power sets and their elements*	*Gary's questioning serves the benefit of clarifying the notion of the power set further.*
15	**Frank:** When you, when you… when you are talking in terms of power set we want to make a variable to be an element of a power set, we must make it in a capital letter. You understand? You must not make it a…	*Transactive response: explanation and clarification*	***L1a**: correctly describing elements of the power set*	*Frank is saying that elements of a power set must be denoted by a capital letter but not really explaining **why** an element of a power set must be denoted by a capital letter.*
16	**Joseph:** You are saying *X* is a subset of *A*?	*Transactive question – request for clarification and explanation*	*L1b: questioning the meaning of power sets and their elements*	*Joseph does not seem to be sure of the newly met notion of the power set*
17	**Frank:** Yes	*Transactive response- to answer*	***L1b**: correctly identifying the elements of the power set of A as subsets of A*	
18	**Joseph:** Meaning that when you talk of power sets we say it consists of sets, … a power set of *A* which means it consists of all possible sets of	*Transactive argument- thinking and reasoning aloud*	*L1b: exploring the definition and meaning of power sets*	*Joseph is developing his understanding in the EZPD created*
19	**Frank and Edgar:** *A*			
20	**Frank:** yes			
21	**Joseph:** Now you can't say *X* is in itself is an element of… you must say *X* is a subset since a power set consists of sets.	*Contribution to an idea- point of confusion*	*L1bx: suggesting incorrect usage of 'subset of' instead of 'element of' as a result of incomplete understanding of the notion of power sets*	*Joseph is developing his understanding of the notion of the power set, having elements that are sets, and thinks that since X is a subset of A, it should be a subset of*

				the power set of A, which may indicate complex thinking.
22	**Frank:** Oh, you want me to say that *X* is a subset of this? *[points to let* $X \in P(A) \Rightarrow X \subseteq A$ *]* Oh, here we must change this element to a subset?	*Transactive question – request for clarification*	***L1b****: questioning Edgar's suggestion to use 'subset of' instead of 'element of'*	*Frank's original statement is correct but because of other students' uncertainty, he is beginning to question this. This could indicate pseudoconceptual thinking.*
23	**Joseph:** Of course! Power sets consist of subsets.	*Transactive argument-thinking aloud*	*L1b: exploring the definition and meaning of power sets*	*Joseph seems to be developing his grasp of power sets in the EZPD created*
24	**Maria:** I think since a power set consists of sets and then I can say that *X* is an element of the power set of *A* it means that *X* is contained in the power set of *A*, not *X* being a subset of the power set of *A*, I think.	*Contribution to an idea-explanation of terms*	*L1b: explaining about the power set and its elements*	*Maria seems to have the correct idea about power sets and their elements.*

Sub-episode 5.2: Discussion around the notion of the power set using examples to illustrate

The discussion about power sets continues in this sub-episode with Joseph once again trying to clarify mathematical objects using examples showing that this has become a well- established habit for him.

	Speech and actions	**Student and teacher utterances**	**Proof comprehension**	**Interpretation according to T.F. and general comments on the proof construction process**
25	**Frank:** So what is the argument?			

26	**Joseph:** This *X*, suppose we write, we write all the possible… the sets of the power set. Say you should *[goes to the board]* You say *X* is a subset of the power set of *A*. Now suppose I am writing them down… *[writes: P(A) = {A* *or* *= {{* am I going to say *A* or am I going to say, *[changes the A to X, so it reads: P(A) = {X,… or = {{X},…]* am I going to say this and all the subsets? Which one between this and this *[points to: P(A) = {X and: = {{X},….]* is true?	*Contribution to an idea-examples*	*L1ax: incorrect (spoken) use of the newly met term referring to X as a subset of the power set whereas it is an element.* *L1c: Joseph illustrating the notion of the power set and its elements with an example*	*Using examples to illustrate and clarify mathematical objects has become an established habit, and Joseph is turning to this example to clarify the notion of the power set and its elements for himself and other students.*
27	**Frank:** *[goes to the board]* The power set contains the empty set. *[puts a Ø in the set P(A) = {X]*	*Contribution to the example*	***L1b**: selects correct depiction of the power set and correctly identifies other elements in the power set.*	*Frank identifies the correct depiction of the power set and the elements contained in them, confirming that he has good understanding of the power set.*
28	**Joseph:** Ja, those are the other sets.			
29	**Edgar:** The most probable one is the second one with the set here because we want, *[pointing to {{X}, ….]*the most probable one is this one		*L1bx: incorrect explanation of the elements of a power set.*	*Edgar has an incorrect idea about the power set and its elements, and thinks that sets in the power set should be inside curly brackets.*

30	**Joseph:** *[points to* $P(A) = \{ \emptyset, X, \quad$ *]* The second one? *[points to:* $= \{\{X\},\ldots.$ *]* And if I write it like this?		*L1c: Joseph illustrating the notion of the power set and its elements with an example*	*Joseph is still trying to clarify any misunderstandings with an example*
31	**Frank:** Is that an X?			
32	**Joseph:** You see? Can you see X?			
33	**T:** But what does that capital X represent?	*Transactive prompt-clarification*		
34	**Maria:** That X, representing the set of X, like when you say A is equal to this set…		*L1b: trying to describe the elements in the power set*	*Maria is trying to clarify that X is a set.*
35	**Edgar:** And I also think that that X means that that X is contained in the power set of A. That X is contained inside there. Like up here you wrote that set X, so inside it's a power set. X is contained inside.	*Contribution to an idea*	*L1bx: trying to describe the elements in the power set*	*Edgar seems to confirm his belief that sets in the power set must be enclosed in curly brackets.*
36	**Joseph:** OK, sure, you must be careful there, be careful of writing the power set and also the set so that's contained? né? *[points to the board]*	*Contribution to an idea*	*L1b: trying to describe the elements in the power set*	*Joseph seems to be getting some clarity about the elements of the power set.*
37	**Maria:** Ja			
	Frank: So X represents subsets…	*Transactive question-clarification*	***L1b**: trying to describe the elements in the power set*	*Frank seems to be gaining certainty about the elements of a power set being subsets.*
38	**Joseph:** Ja, ja, no, you are right, it's OK		*L1b: realizing that Frank had been correct in proof construction, about the elements of a power set.*	*Joseph seems to be getting some clarity about the elements of the power set.*
39	**T:** It's good that you clarify. So that	*Didactive- explanation of*		

	capital *X* is a set, is a set and it's one of the elements of the power set.	*the power set and its elements*		
40	**Student:** Ok. It is a set?	*Transactive question-clarification*		
41	**T:** It is a set, ja. So, if you like I mean just, just to clarify I'll draw a venn diagram. I find venn diagrams always help me. *[goes to the board and draws a large circle, A]* Say we've got the set *A* there, right, and we've got, in this case we've got *A* is this set where it's got the two elements 1 and 2 *[writes:* {1, 2} *and draws two dots in the circle and labels them 1 and 2]* 1 and this is 2. A has got just those two elements. Then in the power set of *A* what are you going to have? All the subsets of *A*, right? So what are some of the subsets of *A*? 1 is a subset. Let's call this the set *X*, let's call *Y* the set containing 2, it's also a subset. And *A* is a subset. *[writes:* $A = \{1, 2\}$, $X=\{1\}$, $Y=\{2\} \subseteq A$ $X \subseteq A$, $Y \subseteq A$ *]* So here we are going to have *X*. *X* is a set. It's the set containing the element 1 and *Y* is the set containing the element 2. *[writes:* $X = \{1\}$, $X \subseteq A$	*Didactive- illustrating the power set and its elements with an example*		

	Y = {2}, Y ⊆ A again] What else is in here?			
42	**Student:** The set	*Transactive response- to answer*		
43	**T:** The set itself *[writes: P(A) = and draws another Venn diagram showing the power set of A containing the sets X, Y, A and Ø }* And what else? The empty set. So it's got these 4 elements in this power set.	*Didactive- illustrating the power set and its elements with an example*		

Sub-episode 5.3: Frank attempts the proof $P(A) \subseteq P(B) \Rightarrow A \subseteq B$ completed with Joseph's help

In sub-episode 5.3 Frank continues with the proof of $P(A) \subseteq P(B) \Rightarrow A \subseteq B$.

	Speech and actions	Student and teacher utterances	Proof comprehension	Interpretation according to T.F. and general comments on the proof construction process
44	**Frank:** And we go back *[writes: $\Leftarrow P(A) \subseteq P(B)$]*	*Continuation of proof*	***L2a**: abbreviated but correct opening statement adding sense to the proof construction*	*Referring to the reverse implication as "going back".*
45	**Frank:** *[writes: Assume $P(A) \subseteq P(B)$ we show that $A \subseteq B$]*	*Start of second component of proof*	***L2a**: introducing statements that add to the logic of the proof construction* ***L2c**:selection of correct proof methodology* ***L1a**: correctly using newly met terms, symbols and signs*	*Methodology of implication proof well understood and seemingly at concept level.*

46	**T:** Is it alright? Are you clear?	*Transactive prompt- request for reflection*		
47	**Frank:** *[writes: let {x} ∈ A]*		***L1ax****: incorrect use of terms, symbols and signs* ***L2c****:selection of correct proof methodology*	*Frank's action of wanting to choose an element of A to start the proof is correct but the set {x} is a subset not an element of the set A. Methodology of subset proof confirmed to be at concept level.*
48	**T:** Now, think about that. What does everybody say about that? *[silence]* We want to show… What do we want to show?	*Transactive prompt- request for reflection and strategy*		
49	**Frank:** *A* is a subset of *B*	*Transactive response- to answer*	***L2b****: identifying what needs to be shown*	*This confirms that Frank's understanding of the implication proof method is correct and may be at concept level.*
50	**T:** So then you have to pick any element	*Directive- providing information on proof construction*		
51	**Frank:** Ja			
52	**T:** in where?	*Transactive prompt- request for reflection*		
53	**Frank:** *A*	*Transactive response- to answer*	***L2c****: identifying how to go about the proof construction method*	*This confirms that Frank's understanding of the subset proof method is correct and may be at concept level.*
54	**T:** In *A*, right? And *A* is just a set	*Directive- providing information on proof*		

		construction		
55	**Frank:** Ok. *[erases the brackets so it now reads: let $x \in A$]*		***L2a**: correctly identifying the statement to start the next component*	*Frank realizes his error quickly when reminded that A is simply a set.*
	[writes: $\Rightarrow x \in B$ (since $P(A) \subseteq P(B)$)]		***L3ax**: making a deduction that doesn't follow simply from the previous statement* ***L2bx:** selecting an inappropriate statement to continue the proof*	*Frank is stuck and does not know where to go after the first step lacking strategic knowledge. He makes the desired deduction which would lead to the right conclusion, but without the correct justification. This is perhaps confirmation of Frank's grasp of the notion of the power set being at pseudoconcept level, as he struggles to use and apply the definition of the power set to go on to the next step.*
56	**T:** Do we agree with that?	*Transactive prompt-reflection*		
57	**T:** So you wanted to show that *A* is a subset of *B*. You've taken an element in *A* and then you immediately go to say that element is in *B*. Since…	*Directive- providing immediate feedback on proof construction*		
58	**Student:** Is *x* not in power set *B*?	*Contribution to proof construction*	*L3ax: suggesting a deduction that doesn't follow from previous statement*	*Students seem to associate the set B with the power set of B seemingly revealing complex level thinking. Thus they seem to think that if x is in B it will be in the power set of B.*

59	**T:** Since what? Does it follow immediately?	*Transactive prompt- request for justification and reflection*		
60	**Student:** No, it does not follow immediately	*Transactive response- to answer*	*L3a: realizing that this deduction is not justified.*	
61	**Gary:** I was thinking; x being an element of A, right? Ah, since x is an element of A, what it means that…	*Contribution to proof construction*		*Gary is developing understanding through interaction with peers in the EZPD.*
62	**Joseph:** {x} is an element of the power set…	*Contribution to proof construction*	*L3a: making a correct deduction from previous statement*	*Joseph seemingly showing good strategic knowledge, being able to determine what the next appropriate and correct deduction should be.*
63	**Gary: subset** {x} can be an element of power set of A, subset {x} is an element of the power set of A.	*Contribution to proof construction*	*L3a: making a correct deduction from previous statement*	*Gary seemingly showing good strategic knowledge, being able to determine what the next appropriate and correct deduction should be.*
64	**T:** OK, do you want to write that down? Maybe… Draw the Venn diagram of that set A that I put up, that example. Ja, and see what… Do you remember how we got the power set? We had the elements 1 and 2	*Transactive prompt- request for examples to clarify the mathematical object*		*The lecturer feels the need for further clarification and asks for a repeat of the previous example.*
65	**Frank:** *[draws a Venn diagram and writes Ø inside]*		***L1c**: illustrating mathematical objects with examples*	*Students having difficulty generating examples that will be useful in clarifying mathematical objects, in this case Frank draws the power set of A without drawing the*

				set A.
66	**T:** No, first draw *A*, the set of *A*	*Transactive prompt- request for examples to clarify the mathematical object*		
67	**Frank:** Oh, *A*, here? *[labels the diagram A]*		***L1c**: illustrating mathematical objects with examples*	
68	**T:** What did *A* have in it? 1 and 2.	*Transactive prompt- request for examples to clarify the mathematical object*		
69	**Frank:** 1 and 2			
70	**T:** Right			
71	**Frank:** *[erases Ø, and writes 1, 2 in Venn diagram labelled A]* 1 and 2	*Contribution to proof- using pictorial examples*	***L1c**: illustrating mathematical objects with examples*	
72	**T:** Uh huh. Now the power set is…	*Transactive prompt- request for examples to clarify the mathematical object*		
73	**Frank:** *[writes: P(A) and draws a Venn diagram with Ø, {1}, {2}, {1, 2}]*	*Contribution to proof- using pictorial examples*	***L1c**: illustrating mathematical objects with examples*	
74	**T:** Right. Does that give you a clue?	*Transactive prompt- reflection on the example for inspiration*		
75	**Frank:** *x* is in *A*.	*Transactive response- to answer*		
76	**T:** *x* is in *A*. So *x* is, it can be the 1 or the 2 in this case. Do you want to go up and show us?	*Transactive prompt- request for examples to clarify the mathematical object*		
77	**Joseph:** *[goes to the board and says as he writes]* Subset {*x*} is an element of power set *A*… is an element of the power set *B* since, and this is an element of *B* since this	*Contribution to proof construction*	*L3a: making a correct deduction L1a: using correct mathematical terms, symbols, signs*	*Joseph maybe showing concept level understanding of the power set and its elements and is able to make correct deductions by*

	[writes: $\Rightarrow \{x\} \in P(A)$			*applying this notioncorrectly. He also shows good strategic knowledge being able to make appropriate and correct steps and deductions.*
	$\Rightarrow \{x\} \in P(B)$ (since $P(A) \subseteq P(B)$)		*L3a: making a correct deduction* *L1a: using correct mathematical terms, symbols, signs*	
	$\Rightarrow x \in B$		*L3a: making a correct deduction* *L1a: using correct mathematical terms, symbols, signs*	
	Thus $A \subseteq B$]		*L3c: making the correct conclusion of the second component of the proof* *L1a: using correct mathematical terms, symbols, signs*	
78	**Frank:** But at the beginning I was trying to show that the set was…	*Transactive question-clarification*	***L1ax**: still believing that his original incorrect use of terms, symbols and signs was correct*	*Frank seems to think that he was on the right track when he took the action in line 52, where he had:* *"let $\{x\} \in A$"*
79	**Student:** No you can't say that a set is an element of a set.	*Transactive response-clarification*	*L1a: identifying correct use of terms, symbols and signs correctly*	*Other students participating and showing understanding.*
80	**T:** Look at *A*…	*Transactive prompt-reflection*		
81	**Joseph:** The set, it's like saying this is an *x* *[in the circle labeled A, he erases 1and 2, and replaces this with x and y]* so that we say *x*, we say *y*. And then this *x* that's in here it can be considered as a subset so we say *x* *[draws the Venn diagram P(A) and writes {x}, {y}, {x, y} and Ø and erases*	*Contribution to proof construction- using examples*	*L1c: illustrating mathematical objects with examples*	*Joseph alters the example drawn by Frank on the board by replacing the elements 1 and 2 by the general variables x and y to better illustrate the relation between elements of a set and elements of the power set, perhaps showing concept level understanding of these*

	the Ø, {1}, {2}, {1,2}] and the subset will be $\{x\}$, $\{y\}$ and $\{x, y\}$ which is the set itself . And this one is just the same like we had a subset.			*mathematical objects. Joseph has developed this understanding in the EZPD created by all participants.*
82	**T:** Good. Is it clear guys?	*Transactive prompt-request for reflection*		
83	**Students:** Ja			
84	**T:** Are you sure? From, from the set containing x being an element of power set of B can you go through x being an element of B? Is that correct? Does everybody agree with that?	*Transactive prompt- request for reflection and critique*		
85	**Students:** Yes. It's not complete. You must conclude and stand up	*Contribution to proof construction*		
86	**Frank:** *[writes (1) next to $A \subseteq B \Rightarrow P(A) \subseteq P(B)$ and (2) next to $P(A) \subseteq P(B) \Rightarrow A \subseteq B$ and then writes: From (1), (2) Therefore $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$]*		***L3c**: correct conclusion of the whole proof*	*Proof methods of implication and double implication seem to be confirmed to be at concept level as proof is concluded correctly with all the necessary justification.*