Extensions of the First Borel-Cantelli Lemma in Riesz Spaces

Nyasha Takunda Mushambi
Student Number: 549147
Date: February 9, 2018

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the degree of Master of Science.

University of the Witwatersrand
School of Mathematics
Johannesburg
South Africa

Abstract

The classical First and Second Borel-Cantelli theorems as well as the Kolmogorov Zero-One Laws have been extended to the abstract setting of Riesz spaces by Wen-Chi Kuo, Coenraad C. A. Labuschagne and Bruce A. Watson. This dissertation aims to extend upon the work of these authors. In particular, an extension of the Barndorff-Nielsen Zero-One Law to the Riesz space setting.

Declaration

I declare that the contents of this dissertation are original except where due references have been made. It has not been submitted before for any degree to any other institution.

(Signature)

Acknowledgments

I would like to convey my sincerest appreciation to Professor Bruce Watson and Doctor Bertin Zinsou for their guidance throughout the writing of this dissertation. Furthermore, I would like to thank my friends and family for their continual support and encouragement.

Contents

Chapter 1

Introduction

We consider the Borel-Cantelli Lemma, a result in probability theory, which can be stated as follows:

Let $(A_n)_{n\geq 1}$ *be a sequence of events in the probability space* $(\Omega, \mathcal{A}, P)$.

(a) *If* $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then* $P(\limsup_{n\to\infty} A_n) = 0$.

(b) *If the events* A_n *are independent and* $\sum_{n=1}^{\infty} P(A_n) = \infty$, *then* $P(\limsup_{n\to\infty} A_n) = 1$.

The results (a) and (b) are formally called the First and Second Borel-Cantelli lemmas respectively. They were obtained by Émile Borel, see [3, 4], and Francesco Paolo Cantelli, see [5]. These theorems are important for deducing results about almost sure convergences. In particular they are crucial in proving the Strong laws of large numbers and the laws of iterated logarithms given below, see [7]:

1. *Let* $X_1, X_2, \ldots$ *be independent and identically distributed random variables with* $\mathbb{E}[X_i] = \mu$ *and* $\mathbb{E}[X_i^4] < \infty$. *If* $S_n = X_1 + \ldots + X_n$ *then* $S_n/n \to \mu$ *almost surely.*

2. *If* $X_1, X_2, \ldots$ *are independent and identically distributed random variables with* $\mathbb{E}[X_i] = \infty$, *then* $P(|X_n| \geq n$ *infinitely often*$) = 1$. *If* $S_n = X_1 + \ldots + X_n$ *then* $P(\lim S_n/n$ *exists* $\in (-\infty, \infty)) = 0$.

The laws of iterated logarithm for Brownian motion:

Let $B_t,\ t \geq 0$ *be a standard Brownian motion, that is a process with a Gaussian distribution with parameters* $(0,\ t)$ *for mean and variance respectively, then*

$$\frac{\limsup\limits_{t \to \infty} B_t}{(2t \log \log t)^{1/2}} = 1$$

almost surely.

Nash stated the following remark, see [13]. Let $\alpha_n = I(A_n)$, the indicator of the event A_n, that is

$$\alpha_n = \begin{cases} 1 & \text{when } A_n \text{ occurs} \\ 0 & \text{when } A_n \text{ fails to occur.} \end{cases}$$

Let $P(A_n | \alpha_1 \alpha_2 \ldots \alpha_{n-1})$ denote the conditional probability of the event A_n, given the outcomes of the previous $n-1$ trials. When $n = 1$, the expression is taken to represent the unconditional probability $P(A_1)$. The Borel criterion, see [4], stated that:

For a sequence of events $(A_n)_{n\geq 1}$, *if*

$0 < p'_n \leq P(A_n|\alpha_1\alpha_2\dots\alpha_{n-1}) \leq p''_n < 1,$ *for every* n, *whatever be* $\alpha_1\alpha_2\dots\alpha_{n-1}$,

then

$$\sum_{n=1}^{\infty} p''_n < \infty \implies P(\limsup A_n) = 0,$$
$$\sum_{n=1}^{\infty} p'_n = \infty \implies P(\limsup A_n) = 1.$$

Cantelli, see [5], proved that the condition $\sum\limits_{n=1}^{\infty} P(A_n) < \infty$ always implies that $P(\limsup A_n) = 0$.

This research focuses on the extension of the First Borel-Cantelli Lemma in Riesz Spaces. In particular, we prove the Barndorff-Nielsen Theorem in Riesz spaces, see Theorem 5.3, which in the case of probability spaces is as follows:

Theorem. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space and* $(A_n)_{n\geq 1}$ *a sequence of events* ($\mathcal{A}$*-measurable*) *such that* $\liminf\limits_{n\to\infty} P(A_n) = 0$ *and* $\sum\limits_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$. *Then,*

$$P(\limsup_{n\to\infty} A_n) = 0 \text{ and } P(A_n) \to 0, \text{ as } n \to \infty.$$

The Balakrishnan-Stepanov Theorem:

Theorem. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space with a sequence of events* $(A_n)_{n\geq 1}$. *If* $P(A_n) \to 0$ *and*

$$\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}^c \cap \ldots \cap A_{n+m-1}^c \cap A_{n+m}) < \infty$$

for some $m \geq 1$, then $P(A_n$ infinitely often$) = 0$.

is also extended to the Riesz space setting. In addition we prove Fatou and Reverse Fatou lemmas in the Riesz space setting as given in Theorem 3.4 and Theorem 3.6.

An application of Riesz spaces processes is given below from [11]:

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space with $\mu(\Omega) = \infty$, and let $(\Omega_i)_{i\in\mathbb{N}}$ be a μ-measurable partition of Ω into sets of finite positive measure. Let $\mathcal{A}_0$ be the sub-μ-algebra of $\mathcal{A}$ generated by $(\Omega_i)_{i\in\mathbb{N}}$. The Riesz space $E = L^\infty(\Omega, \mathcal{A}, \mu)$ and the conditional expectation operator $T = \mathbb{E}[\,\cdot\,|\mathcal{A}_0]$. For $f \in E$ we have

$$Tf(\omega) = \frac{\int_{\Omega_i} f d\mu}{\mu(\Omega_i)}, \text{ for } \omega \in \Omega_i. \tag{1.1}$$

We have that the universal completion, E^u, of E is the space of all $\mathcal{A}$-measurable functions. The T-universal completion of E is the space

$$\mathcal{L}^1(T) = \left\{ f \in E^u \;\middle|\; \int_{\Omega_i} |f| d\mu < \infty \text{ for all } i \in \mathbb{N} \right\},$$

which is characterized by $f|_{\Omega_i} \in L^1(\Omega, \mathcal{A}, \mu)$, for each $i \in \mathbb{N}$. We note that T can be extended to an $\mathcal{L}^1(T)$ conditional expectation operator as per (1.1). We also note that E has weak order unit $e = \mathbf{1}$, the function identically $\mathbf{1}$ on Ω, which again is a weak order unit for $\mathcal{L}^1(T)$, but is not in $L^1(\Omega, \mathcal{A}, \mu)$. Let $\mathcal{C}$ and $\mathcal{D}$ be sub-σ-algebras of $\mathcal{A}$ which contain $\mathcal{A}_0$. The strong mixing coefficient that is, α-mixing coefficient, of $\mathcal{C}$ and $\mathcal{D}$ conditioned on $\mathcal{A}_0$, which in measure theoretic terms could be denoted $\alpha_{\mathcal{A}_0}(\mathcal{C}, \mathcal{D})$, is $\alpha_T(U, V)$. Here U and V are the restrictions to $\mathcal{L}^1(T)$ of the extensions to $\mathcal{L}^1(U)$ and $\mathcal{L}^1(V)$ respectively of the conditional expectation operators U and

V on E conditioning with respect to the σ-algebras $\mathcal{C}$ and $\mathcal{D}$. In this case the operators are

$$U(f) = \sum_{i=1}^{\infty} \mathbb{E}_i[f I_{\Omega_i} | \mathcal{C}],$$
$$V(f) = \sum_{i=1}^{\infty} \mathbb{E}_i[f I_{\Omega_i} | \mathcal{D}],$$

for $f \in \mathcal{L}^1(T)$. Here the conditional expectation $\mathbb{E}_i[f I_{\Omega_i} | \mathcal{C}]$ is the conditional expectation on Ω_i of $f|_{\Omega_i}$ with respect to the probability measure $\mu_i(A) := \frac{\mu(A \cap \Omega_i)}{\mu(\Omega_i)}$ and the σ-algebra $\{C \cap \Omega_i | C \in \mathcal{C}\}$, and similarly for $\mathcal{C}$ replaced by $\mathcal{D}$. We have

$$\begin{aligned}
\alpha_T(U, V) &= \alpha_{\mathcal{A}_0}(\mathcal{C}, \mathcal{D}) \\
&= \sum_{i=1}^{\infty} I_{\Omega_i} \sup \left\{ \left| \frac{\mu(C \cap D \ \cap \Omega_i)}{\mu(\Omega_i)} - \frac{\mu(C \cap \Omega_i)}{\mu(\Omega_i)} \frac{\mu(C \cap \Omega_i)}{\mu(\Omega_i)} \right| \Bigg| \ C \in \mathcal{C}, D \in \mathcal{D} \right\} \\
&= \sum_{i=1}^{\infty} \alpha_i(\mathcal{C}, \mathcal{D}) I_{\Omega_i},
\end{aligned}$$

where $\alpha_i(\mathcal{C}, \mathcal{D})$ is the α-mixing coefficient of σ-algebras $\mathcal{C}$ and $\mathcal{D}$ with respect to the probability measure μ_i.

In Chapter 2 we give definitions, presenting notations and summarise properties of Riesz spaces. We prove a theorem stating that the summation of disjoint elements in a Riesz space is equal to the suprema of the same elements. In Chapter 3 we give a definition of a conditional expectation in a Dedekind complete Riesz space. We prove Fatou Lemma and Reverse Fatou Lemma in the Riesz space setting. In Chapter 4 we prove some preliminary results on sequences of band projections. In Chapter 5 we prove the

Barndorff-Nielsen Theorem as well the Balakrishnan and Stepanov Theorem in the Riesz space setting.

Chapter 2

Preliminaries

2.1 Partially Ordered Sets

We begin this chapter by defining partially ordered sets, which can be found in [16]. If X is a non-empty set and the points of X are denoted by $x, y, \ldots,$ the set of all ordered pairs (x, y) with x and y members of X is called the *Cartesian product* of X and X itself; notation $X \times X$. A *relation* denoted R in X is a non-empty subset of $X \times X$; written xRy if $(x, y) \in X \times X$.

The following definition is taken from [16, page 1].

Definition 2.1.1. *The relation R in X is called a partial ordering,*

(i) *If xRx for every $x \in X$ (R is reflexive),*

(ii) *If xRy and yRz implies xRz (R is transitive),*

(iii) *If xRy and yRx implies $x = y$ (R is anti-symmetric).*

For a partial order relation R we write $x \leq y$ (or, equivalently, $y \geq x$) instead of xRy. If x and y are points of X such that $x \leq y$ or $x \geq y$, we say that x and y are *comparable*. If neither $x \leq y$ nor $x \geq y$ holds then x and y are *incomparable*.

If X is partially ordered and Y is a non-empty subset of X, then the partial ordering of X induces a partial ordering in Y. If $x \in X$ is such that $y \leq x$ for all $y \in Y$, then x is called an *upper bound* of Y. The subset Y is said to be *bounded above.* If x is an upper bound of Y such that $x \leq x'$ for any other upper bound x' of Y, then x is called the *least upper bound* or *supremum* of Y.

If X is partially ordered and Y is a non-empty subset of X and the point $x \in X$ satisfies $x \leq y$ for all $y \in Y$, then x is called a *lower bound* of Y. The subset Y is said to be *bounded below.* If x is a lower bound of Y such that $x \geq x'$ for any other lower bound of Y, then x is called the *greatest lower bound* or *infimum* of Y.

The following definition is taken from [16, page 3].

Definition 2.1.2. *Let X be a partially ordered set. Then*

(i) *The set X is called order complete (or complete) if every non-empty subset of X has a supremum and an infimum.*

(ii) *The set X is called Dedekind complete if every non-empty subset of X that is bounded above (bounded below) has a supremum (infimum).*

(iii) *The set X is called Dedekind σ-complete (or countably Dedekind complete) if every non-empty finite or countable subset of X that is bounded above (bounded below) has a supremum (infimum).*

2.2 Lattices

In this section we present some simple definitions and theorems on lattices. The following definition is taken from [16, page 3] part (iv).

Definition 2.2.1. *Let X be a partially ordered set. X is called a lattice if every subset of X consisting of two points has a supremum and an infimum.*

Let X be a lattice. We denote the supremum of the set consisting of the elements x and y in X by $x \vee y$ and the infimum as $x \wedge y$. By induction it is evident that every finite subset of X has a supremum and an infimum. For $F = \{x_1, x_2, \ldots, x_n\}$ a finite subset of X, the supremum and infimum of F are denoted respectively by $\bigvee_{i=1}^{n} x_i$ and $\bigwedge_{i=1}^{n} x_i$. The following definition is taken from [16, page 5].

Definition 2.2.2. *The lattice X is called distributive if*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

for all x, y, z in X.

The following definition is taken from [16, page 6].

Definition 2.2.3. *The smallest element of a lattice X, when it exists is called the zero element and it is denoted by θ. The largest element of a lattice X, when it exists is called the unit element and it is denoted by e.*

Every finite lattice has a zero element and a unit element. A lattice consists of one element if and only if $\theta = e$. If X is a lattice with θ and e and if the elements x and x' satisfy $x \wedge x' = \theta$ and $x \vee x' = e$, then x' is called a complement of x. Similarly, x is the complement of x'.

The following proposition is taken from [16, page 6].

Proposition 2.2.4. *In a distributive lattice X with θ and e, complements are unique.*

2.3 Riesz Spaces

In this section we present definitions, properties and notations of Riesz spaces.

The following definition is from [16, page 13].

Definition 2.3.1. *The real vector space E is called an ordered vector space if E is partially ordered and for each $f, g, h \in E$ we have,*

(i) *If $f \leq g$, then $f + h \leq g + h$;*

(ii) *If $f \geq 0$, then $\alpha f \geq 0$ for every $\alpha \geq 0$ in $\mathbb{R}$.*

If, in addition, E is a lattice with respect to the partial ordering, then E is called a Riesz space or a vector lattice.

Example 2.3.2. [16, Example 4.2 (1)], The n-dimensional space $\mathbb{R}^n$ with coordinatewise addition and scalar multiplication is a vector space. If the ordering is defined *coordinatewise*, that is, for $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$ we defined $x \leq y$ whenever $x_k \leq y_k$ for $k = 1, \ldots, n$, then $\mathbb{R}^n$ is a Riesz space.

We know that $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ is a vector space. The partial order on $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ almost everywhere, (a.e), is a follows. For $f, g \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, then $f \leq g$ if and only if $f(x) \leq g(x)$ for almost all $x \in \Omega$. We can show that $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ satisfies the three properties of a Riesz space below.

For $f, g, h \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and $\alpha \in \mathbb{R}^+$,

(i) If $f \leq g$ then $f(x) \leq g(x)$ a.e in Ω. It follows that $f(x) + h(x) \leq g(x) + h(x)$ a.e in Ω. Hence, $f \leq g$ implies $f + h \leq g + h$.

(ii) If $f \geq 0$ then it follows that $\alpha f \geq 0$ for all $\alpha \geq 0$.

(iii) We have that $\sup\{f, g\}(x) = \max\{f(x), g(x)\}$ for all x and $\sup\{f, g\} \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$.

The following definition and proposition are taken from [16, page 16].

Definition 2.3.3. *Let E be a Riesz space. The subset $E^+ = \{f : 0 \leq f \in E\}$ is called the positive cone of E and the elements of E^+ are called the positive elements of E.*

Proposition 2.3.4. *Let E be a Riesz space. Then the positive cone E^+ has the following properties:*

(i) *If $f \in E^+,\ g \in E^+$, then $f + g \in E^+$,*

(ii) *If $f \in E^+$, then $\alpha f \in E^+$ for any $0 \leq \alpha \in \mathbb{R}$,*

(iii) *If $f \in E^+$ and $-f \in E^+$, then $f = 0$.*

We mention some further properties of a Riesz space taken from [16, page 16].

Proposition 2.3.5. *Let E be a Riesz space. Then*

(a) *If $f,\ g \in E$ and $f \geq g$, then this is equivalent to $f - g \in E^+$ and to $-g \geq -f$.*

(b) *If $f,\ g \in E$ and $f = f \vee g$, then this is equivalent to $f \geq g$ and to $g = f \wedge g$.*

(c) *If $f,\ g \in E$ and $f \geq g$, then this is equivalent to $\alpha f \geq \alpha g$ for $0 < \alpha \in \mathbb{R}$ and also $\alpha f \leq \alpha g$ for $0 > \alpha \in \mathbb{R}$.*

(d) *If $f,\ g \in E$, then $f \wedge g = -\{(-f) \vee (-g)\}$, similarly $f \vee g = -\{(-f) \wedge (-g)\}$.*

(e) *If $f,\ g,\ h \in E$, then*

$$(f \vee g) + h = (f + h) \vee (g + h),\ (f \wedge g) + h = (f + h) \wedge (g + h).$$

(f) *For* $0 \leq \alpha \in \mathbb{R}$ *and* $f,\ g \in E$ *we have*

$$(\alpha f) \vee (\alpha g) = \alpha(f \vee g),\ (\alpha f) \wedge (\alpha g) = \alpha(f \wedge g).$$

Similarly it follows that, for real $\alpha \leq 0$, *we have*

$$(\alpha f) \vee (\alpha g) = \alpha(f \wedge g),\ (\alpha f) \wedge (\alpha g) = \alpha(f \vee g).$$

(g) *If* $f,\ g,\ h \in E$, *then* $\{(f \vee g) \vee h\} = \{f \vee (g \vee h)\}$, *and similarly for infima.*

The properties of Proposition 2.3.5 can be extended to any finite number of terms.

The following definition is taken from [16, page 17].

Definition 2.3.6. *Let E be a Riesz space. We introduce some notations as follows. For any $f \in E$ we define*

$$f^+ = f \vee 0, \ f^- = (-f) \vee 0 \text{ and } |f| = f \vee (-f).$$

We quote Theorem 5.1 [16, page 17], for later use.

Theorem 2.3.7. *Let E be a Riesz space, then the following holds.*

(i) *For $f \in E$, f^+, f^- are in E^+ and $(-f)^+ = f^-$, $(-f)^- = f^+$. Furthermore $|-f| = |f|$.*

(ii) *For $f \in E$, $f = f^+ - f^-$, $f^+ \wedge f^- = 0$ and $|f| = f^+ + f^-$. Hence $|f| \in E^+$.*

(iii) *For $f \in E$, $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$. Furthermore $-f^- \leq f \leq f^+$ and $|f| = 0$ if and only if $f = 0$.*

(iv) *For $f, g \in E$, $f \leq g$ if and only if $f^+ \leq g^+$ and $f^- \geq g^-$.*

We note Theorem 5.2 [16, page 17].

Theorem 2.3.8. *For f and g in a Riesz space E we have*

$$(f \vee g) + (f \wedge g) = f + g \text{ and}$$
$$(f \vee g) - (f \wedge g) = |\ f - g\ |.$$

Hence

$$f \vee g = \frac{1}{2}(f + g) + \frac{1}{2}|\ f - g\ | \text{ and}$$
$$f \wedge g = \frac{1}{2}(f + g) - \frac{1}{2}|\ f - g\ |.$$

From Theorem 6.1 [16, page 21], we have.

Theorem 2.3.9 (Infinite distributive laws)**.** *Let D be a subset of the Riesz space E possessing a supremum, $f_0 = \sup D = \sup\{f : f \in D\}$ exists. Then, for any $g \in E$, we have*

$$f_0 \wedge g = \sup\{f \wedge g : f \in D\}.$$

Similarly, if $f_1 = \inf D$ exists, then

$$f_1 \vee g = \inf\{f \vee g : f \in D\}.$$

The following distributive corollary is Corollary 6.2 [16, page 21].

Corollary 2.3.10. *Let f, g and $h \in E$, where E is a Riesz space. We have*

$$(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h),$$
$$(f \wedge g) \vee h = (f \vee h) \wedge (g \vee h).$$

From Theorem 6.4 [16, page 21], we state the Riesz Decomposition Property.

Theorem 2.3.11. *Let the elements u, z_1, z_2 in E, where E is a Riesz space, satisfy $u \leq z_1 + z_2$. Then there exist elements u_1, u_2 in E^+ such that $u_1 \leq z_1, u_2 \leq z_2$ and $u = u_1 + u_2$.*

We quote Theorem 6.5 [16, page 22].

Theorem 2.3.12. *Let E be a Riesz space and u, v, w be elements in E^+ and let f, g be arbitrary elements in E. Then*

$$(u + v) \wedge w \leq (u \wedge w) + (v \wedge w), \tag{2.1}$$

$$(f + g) \vee w \leq (f \vee w) + (g \vee w). \tag{2.2}$$

Furthermore, $v \wedge w = 0$ implies

$$(u + v) \wedge w = u \wedge w, \tag{2.3}$$

and if $u \wedge v = 0$, then there is equality in (2.1).

The following definition is taken from [16, page 46].

Definition 2.3.13. *A sequence (f_n) in a Riesz space E is said to be increasing if $f_1 \leq f_2 \leq \ldots$ and it is said to be decreasing if $f_1 \geq f_2 \geq \ldots$. We denote an increasing sequence (f_n) by $f_n \uparrow$ and a decreasing sequence (f_n) by $f_n \downarrow$. If $f_n \uparrow$ and $f = \sup f_n$ exists in E, we write $f_n \uparrow f$ and $f_n \downarrow f$ whenever $f_n \downarrow$ and $f = \inf f_n$ exists in E. We say that f_n converges monotonely to f as n tends to infinity.*

We state Theorem 16.1 [12, page 78].

Theorem 2.3.14. *Let $(f_n)_{n\geq 1}$ be a sequence of elements in the Riesz space E. The sequence is order convergent to $f \in E$ whenever there exists a sequence $p_n \downarrow 0$ in E such that $|f - f_n| \leq p_n, -p_n \leq f - f_n \leq p_n$, holds for all n. This is denoted $f_n \to f$. The following properties hold.*

(i) *If $f_n \to f$ and $f_n \to g$, then $f = g$.*

(ii) *If $f_n \uparrow f$ or $f_n \downarrow f$, $\sup f_n = f$ or $\inf f_n = f$, then $f_n \to f$, $\lim f_n = f$.*

(iii) *If $f_n \uparrow$ or $f_n \downarrow$ and $f_n \to f$, then $f_n \uparrow f$ or $f_n \downarrow f$ respectively.*

(iv) *If $f_n \to f$, $g_n \to g$, and a and b are real numbers, then $af_n + bg_n \to af + bg$, $f_n \vee g_n \to f \vee g$ and $f_n \wedge g_n \to f \wedge g$. In particular, if $f_n \to f$, then $f_n^+ \to f^+$, $f_n^- \to f^-$ and $|f_n| \to |f|$.*

(v) *If $f_n \to f$, then $f_{n_k} \to f$ for every subsequence $(f_{n_k})_{k\geq 1}$ such that $n_1 < n_2 < \dots$.*

We note Definition 15.7 [12, page 73].

Definition 2.3.15. *An indexed subset $\{f_\tau : \tau \in \{\tau\}\}$ of a Riesz space L is called directed upwards if for each pair $\tau_1, \tau_2 \in \{\tau\}$ there exists $\tau_3 \in \{\tau\}$ such that $f_{\tau_3} \geq f_{\tau_1}$ and $f_{\tau_3} \geq f_{\tau_2}$ hold simultaneously, and $\{f_\tau : \tau \in \{\tau\}\}$ is called directed downwards if for each pair $\tau_1, \tau_2 \in \{\tau\}$ there exists $\tau_3 \in \{\tau\}$ such that $f_{\tau_3} \leq f_{\tau_1}$ and $f_{\tau_3} \leq f_{\tau_2}$ hold simultaneously. This is denoted by $f_\tau \uparrow$ or $f_\tau \downarrow$ respectively. If $f_\tau \uparrow$ and $f = \sup f_\tau$ exists in L, we will write $f_\tau \uparrow f$, or $f_\tau \uparrow_\tau f$ if necessary. If $f_\tau \downarrow$ and $f = \inf f_\tau$ exists in L, we will write $f_\tau \downarrow f$.*

The upwards directed sets $\{f_\tau\}$ and $\{g_\tau\}$, indexed by means of the same index set $\{\tau\}$, will be called equidirected if for any pair $\tau_1, \tau_2 \in \{\tau\}$ there exists $\tau_3 \in \{\tau\}$ such that $f_{\tau_3} \geq f_{\tau_1}$ and $f_{\tau_3} \geq f_{\tau_2}$ as well as $g_{\tau_3} \geq g_{\tau_1}$ and $g_{\tau_3} \geq g_{\tau_2}$ hold simultaneously. The upwards directed sets $\{f_\tau\}$ and $\{g_\tau\}$ are equidirected in particular if $f_{\tau_1} \leq f_{\tau_2}$ holds if and only if $g_{\tau_1} \leq g_{\tau_2}$ holds (note, however, this is not a necessary condition for being equidirected). A similar definition holds for downwards directed sets.

The following definition is taken from [1, page 210].

Definition 2.3.16. *A subset Q of a Riesz space is order bounded from above if there is an element, m, that dominates each element of Q, that is, for each $g \in Q$, we have $g \leq m$. A subset that is order bounded from below is defined similarly.*

A subset is order bounded if it is order bounded from above and below.

We note the following definition from [1, page 218].

Definition 2.3.17. *Let E be a Dedekind complete Riesz space. If (f_n) is an order bounded sequence of elements in E, then*

$$\limsup f_n = \inf_n \left(\sup_{m \geq n} f_m \right) = \bigwedge_n \left(\bigvee_{m \geq n} f_m \right),$$
$$\liminf f_n = \sup_n \left(\inf_{m \geq n} f_m \right) = \bigvee_n \left(\bigwedge_{m \geq n} f_m \right).$$

Below we prove the Sandwich Theorem in Riesz spaces.

Theorem 2.3.18 (Sandwich)**.** *Let $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be sequences in the Dedekind complete Riesz space E with the same limit f in E. If $(z_n)_{n\geq 1}$ is a sequence in E such that $f_n \leq z_n \leq g_n$ for all n, then $(z_n)_{n\geq 1}$ converges to f.*

Proof. Let $(p_n)_{n\geq 1}$ be a sequence in E such that $p_n \downarrow 0$. Let $P, Q \in \mathbb{N}$ such that

$$|f_n - f| \leq p_n \text{ for all } n > P,$$
$$|g_n - f| \leq p_n \text{ for all } n > Q.$$

If $N = \max\{P,\ Q\}$, then for all $n > N$

$$-p_n \leq f_n - f \leq z_n - f \leq g_n - f \leq p_n,$$

Hence $|z_n - f| \leq p_n$. □

The Sandwich Theorem is used below to show that an order bounded sequence is convergent if and only if its inferior and superior limits coincide.

Theorem 2.3.19. *Let $(f_n)_{n\geq 1}$ be an order bounded sequence of elements in the Dedekind complete Riesz space E. The sequence converges if and only if*

$$\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = \lim_{n\to\infty} f_n = f.$$

Proof. Suppose that

$$\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = f.$$

Then $g_n \downarrow f$ and $h_n \uparrow f$ as $n \to \infty$ where

$$g_n = \sup_{m \geq n} f_m, \;\; h_n = \inf_{m \geq n} f_m. \tag{2.4}$$

From (2.4), we get $h_n \leq f_n \leq g_n$. Hence the Sandwich Theorem implies that

$$\lim_{n \to \infty} f_n = f.$$

Conversely suppose $(f_n)_{n \geq 1}$ converges to f in E. Then there exists a sequence $p_n \downarrow 0$ such that $|f_n - f| \leq p_n$ for all n, thus

$$|f_m - f| \leq p_n \text{ for all } m \geq n,$$
$$f - p_n \leq f_m \leq f + p_n \text{ for all } m \geq n.$$

It follows that

$$f - p_n \leq \inf_{m \geq n} f_m = h_n,$$
$$g_n = \sup_{m \geq n} f_m \leq f + p_n.$$

Using the fact that $h_n \leq g_n$ we obtain $f - p_n \leq h_n \leq g_n \leq f + p_n$ for all n. Therefore $g_n, h_n \to f$ as $n \to \infty$. Hence

$$\liminf_{n \to \infty} f_n = \limsup_{n \to \infty} f_n = \lim_{n \to \infty} f_n = f. \qquad \square$$

We prove the following theorem relating summation to suprema which will be used later.

Theorem 2.3.20. *Let E be a Dedekind complete Riesz space. If $f_n \in E^+$ with f_n order bounded and $f_n \wedge f_m = 0$ for all $n \neq m$, where $n, m \in \mathbb{N}$, then*

$$\sum_{n \in \mathbb{N}} f_n = \bigvee_{n \in \mathbb{N}} f_n.$$

Proof. Let $f, g \in E^+$ with $f \wedge g = 0$, then from Theorem 2.3.8

$$f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|, \tag{2.5}$$
$$f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|. \tag{2.6}$$

Multiplying (2.6) by 2 we get

$$\begin{aligned} 2(f \wedge g) &= 2\left(\frac{1}{2}(f+g) - \frac{1}{2}|f-g|\right) \\ &= (f+g) - |f-g|. \end{aligned}$$

But $f \wedge g = 0$, so

$$(f+g) = |f-g|. \tag{2.7}$$

Substituting (2.7) into (2.5) gives

$$\begin{aligned} f \vee g &= \frac{1}{2}(f+g) + \frac{1}{2}(f+g) \\ &= f + g. \end{aligned} \tag{2.8}$$

By induction we now show that

$$\bigvee_{n=1}^{N} f_n = \sum_{n=1}^{N} f_n. \tag{2.9}$$

For $N = 1$

$$f_1 = f_1.$$

Suppose that for $N = k$,

$$\bigvee_{n=1}^{k} f_n = \sum_{n=1}^{k} f_n, \tag{2.10}$$

then for $N = k + 1$

$$\bigvee_{n=1}^{k+1} f_n = \left(\bigvee_{n=1}^{k} f_n \right) \vee f_{k+1} = \left(\sum_{n=1}^{k} f_n \right) \vee f_{k+1}. \tag{2.11}$$

Applying (2.8) to (2.11) with $f = \sum_{n=1}^{k} f_n$ and $g = f_{k+1}$ gives,

$$\begin{aligned} \bigvee_{n=1}^{k+1} f_n &= \left(\sum_{n=1}^{k} f_n \right) \vee f_{k+1} \\ &= \sum_{n=1}^{k} f_n + f_{k+1} \\ &= \sum_{n=1}^{k+1} f_n, \end{aligned}$$

which concludes the induction.

As (f_n) is order bounded and $\left(\bigvee_{n=1}^{N} f_n \right)$ is an increasing sequence, then there exits $h \in E^+$ such that

$$\bigvee_{n=1}^{N} f_n \uparrow h, \text{ as } N \to \infty. \tag{2.12}$$

It follows from Theorem 2.3.19

$$\begin{aligned} h &= \lim_{N\to\infty} \bigvee_{n=1}^{N} f_n \\ &= \liminf_{N\to\infty} \bigvee_{n=1}^{N} f_n. \end{aligned}$$

From Definition 2.3.17,

$$h = \bigvee_{k=1}^{\infty} \left(\bigwedge_{N=k}^{\infty} \bigvee_{n=1}^{N} f_n \right). \tag{2.13}$$

As the sequence

$$\left(\bigvee_{n=1}^{N} f_n \right)$$

is increasing, we have

$$\bigwedge_{N=k}^{\infty} \bigvee_{n=1}^{N} f_n = \bigvee_{n=1}^{k} f_n. \tag{2.14}$$

Substituting (2.14) into (2.13) we have,

$$\begin{aligned} h &= \bigvee_{k=1}^{\infty} \bigvee_{n=1}^{k} f_n \\ &= \bigvee_{n=1}^{\infty} f_n. \end{aligned} \tag{2.15}$$

By definition of infinite summation

$$\sum_{n=1}^{N} f_n \uparrow \sum_{n=1}^{\infty} f_n$$

so,

$$\sum_{n=1}^{\infty} f_n = \lim_{N \to \infty} \sum_{n=1}^{N} f_n = \lim_{N \to \infty} \bigvee_{n=1}^{N} f_n = \bigvee_{n=1}^{\infty} f_n = h.$$

It follows that

$$\sum_{n=1}^{\infty} f_n = \bigvee_{n=1}^{\infty} f_n. \qquad \square$$

2.4 Ideals and Bands

We start this section by giving definitions of ideals and bands. Definition 7.1 of [16, page 27] is given below.

Definition 2.4.1. *Let E be a Riesz space. Subsets of E are assumed to inherit the ordering from E.*

(i) *The linear subspace V of E is called a Riesz subspace of E if for all pairs f and g in V, the elements $f \vee g$ and $f \wedge g$ are elements in V ($f \vee g, f \wedge g$ exist in E).*

(ii) *The subset S of E is said to be solid if for each $f \in S$ and each $g \leq f$, we have that $g \in S$.*

(iii) *The subset A of E is called an ideal in E if A is a solid linear subspace of E.*

(iv) *The ideal B in E is called a band if, whenever a subset of B possesses a supremum in E, this supremum is a member of B, that is, if it follows from $D \subset B$ and $f = sup\ D$ that $f \in B$.*

The following definitions are taken from [16, page 30].

Definition 2.4.2. *For any non-empty subset D of the Riesz space E, the intersection of all Riesz subspaces containing D is called the Riesz subspace generated by D. The ideal and the band generated by D are defined similarly.*

Definition 2.4.3. *The ideal A_D generated by D can be described explicitly as follows. If $f_1, \ldots, f_n$ are finite elements of D and $\alpha_1, \ldots, \alpha_n$ are real* *numbers, then* $|\alpha_1 f_1| + \ldots + |\alpha_n f_n| \in A_D$*, and so every* $g \in E$ *satisfying*

$$|g| \leq |\alpha_1 f_1| + \ldots + |\alpha_n f_n|,$$

is an element of A_D*. In the case that* D *has a single element* f *we write* $A_D = A_f$*. The principal band,* A_f*, generated by* f *is given below*

$$A_f = \{g \in E : |g| \leq |\alpha f|, \alpha \in \mathbb{R}\}.$$

We state Definition 21.4 [12, page 111].

Definition 2.4.4. *Let* E *be a Riesz space.*

(i) *The element* $e \in E^+$ *is called a strong order unit in* E *if the principal ideal generated by* e *is the whole space* E*, that is, if for every* $f \in E$ *there exists a postive number* a*, depending upon* f*, such that* $|f| \leq ae$*.*

(ii) *The element* $e \in E^+$ *is called a weak order unit in* E *if the principal band generated by* e *is the whole space* E*.*

We quote Theorem 7.2 [16, page 28].

Theorem 2.4.5. *Let* E *be a Riesz space.*

(i) *Every band in* E *is an ideal and every ideal in* E *is a Riesz subspace. The set* $\{0\}$ *consisting only of the null element is a band in* E*. Similarly,* E *itself is a band.*

(ii) *Every Riesz subspace of a Riesz subspace of* E *is itself a Riesz subspace of* E*.*

(iii) *Every ideal in an ideal in E is itself an ideal in E.*

(iv) *Every band in a band in E is itself a band in E.*

The following definition is taken from [16, page 30].

Definition 2.4.6. *If V_1 and V_2 are subsets of a vector space V, then we define the algebraic sum $V_1 + V_2 = \{f_1 + f_2 : f_1 \in V_1, f_2 \in V_2\}$.*

If V_1 and V_2 are linear subspaces of V, then $V_1 + V_2$ is also a linear subspace. Now if $V_1 \cap V_2 = \{0\}$, then any $f \in V_1 + V_2$ can be written uniquely as $f = f_1 + f_2$ with $f_1 \in V_1$ and $f_2 \in V_2$. The subspace $V_1 + V_2$ is the *direct sum* of V_1 and V_2 and is denoted $V_1 \bigoplus V_2$.

We state Theorem 7.6 [16, page 31].

Theorem 2.4.7. *If A_1 and A_2 are ideals in the Riesz space E, then $A_1 + A_2$ is an ideal in E. It is evident that $A_1^+ + A_2^+$ is contained in $(A_1 + A_2)^+$. The converse holds as well, that is, $(A_1 + A_2)^+ = A_1^+ + A_2^+$. In other words, for any $f \in (A_1 + A_2)^+$ there exist $f_1 \in A_1^+$ and $f_2 \in A_2^+$, not uniquely determined in general, such that $f = f_1 + f_2$. If $A_1 \cap A_2 = \{0\}$, that is, if $A_1 + A_2 = A_1 \bigoplus A_2$, then f_1 and f_2 are uniquely determined. In this case, if f and g are elements of $A_1 \bigoplus A_2$ having decompositions $f = f_1 + f_2$ and $g = g_1 + g_2$, then $f \leq g$ implies $f_1 \leq g_1$ and $f_2 \leq g_2$.*

Remark 2.4.8 The algebraic sum of bands is not always a band.

Example 2.4.9. [16, page 31]. Let $E = C([-1, 1])$, be the space of all real continuous functions on the interval $[-1, 1]$, and let the bands B_1 and B_2 be defined by

$$B_1 = \{f \in E : f = 0 \text{ on } [0, 1]\},$$
$$B_2 = \{f \in E : f = 0 \text{ on } [-1, 0]\}.$$

Then $B_1 + B_2 = B_1 \bigoplus B_2 = \{f \in E : f(0) = 0\}$ is an ideal in E, but not a band. The band generated by $B_1 \bigoplus B_2$ is the entire space $C([-1, 1])$.

The following definition is taken from Chapter 3, Section 8 [16, page 34].

Definition 2.4.10. *Let E be a Riesz space. The elements f and g in E are said to be disjoint if $|f| \wedge |g| = 0$ and we write $f \perp g$. For any non-empty subset D of E the set*

$$D^d = \{f \in E : f \perp g \textit{ for all } g \in D\},$$

is called the disjoint complement of D. The disjoint complement $D^{dd} = (D^d)^d$ of D^d is called the second disjoint complement of D. If D_1 and D_2 are non-empty subsets of E such that $d_1 \perp d_2$ for every $d_1 \in D_1$ and every $d_2 \in D_2$, then D_1 and D_2 are said to be disjoint (notation $D_1 \perp D_2$).

We quote Theorem 8.1 [16, page 34].

Theorem 2.4.11. *Let E be a Riesz space with the elements $f, g, \ldots$.*

(i) *If $f \perp g$ and $|h| \leq |f|$, then $h \perp g$.*

(ii) *If $f \perp g$ and $\alpha \in \mathbb{R}$, then $\alpha f \perp g$.*

(iii) *If $f_1 \perp g$ and $f_2 \perp g$, then $(f_1 + f_2) \perp g$.*

(iv) *$f \perp g$ if and only if $f^+ \perp g$ and $f^- \perp g$.*

(v) *For every $f \in E$ we have $f^+ \perp f^-$.*

(vi) *If D_1 and D_2 are non-empty subsets of E such that $D_1 \perp D_2$, then $D_1 \cap D_2$ is either empty or equal to the set $\{0\}$ containing only the null element.*

(vii) *If D is a subset of E such that $f_0 = \sup D$ exists and $g \perp f$ holds for all $f \in D$, then $g \perp f_0$.*

(viii) *If $\{f_1, \ldots, f_n\}$ is a set of nonzero and mutually disjoint elements, then this is a linearly independent set.*

We note Theorem 8.2 [16, page 36].

Theorem 2.4.12. *Let E be a Riesz space with elements f, g.*

(1) *If $f \perp g$, then $|f+g| = |f-g| = |f| + |g| = ||f| - |g|| = |f| \vee |g|$,*

(2) *If $f \perp g$, then $(f+g)^+ = f^+ + g^+ = f^+ \vee g^+$,*

(3) *If $f \perp g$, then $(f+g)^- = f^- + g^- = f^- \vee g^-$.*

We quote Theorem 8.4 [16, page 36].

Theorem 2.4.13. *Let D be a non-empty subset of the Riesz space E. Then the disjoint complement D^d is a band in E. Furthermore, D is a subset of D^{dd} and $D^d = D^{ddd}$. Finally $D^d \cap D^{dd} = \{0\}$, so the algebraic sum $D^d + D^{dd}$ is a direct sum. In general this direct sum is an ideal, not necessarily a band.*

We quote Theorem 8.5 [16, page 37].

Theorem 2.4.14. (i) *For any ideal A in E, the set A^{dd} is the largest among all ideals B in E having the property that for every $0 \neq f \in B$ there exists an element $g \in A$ such that $0 \neq |g| \leq |f|$.*

(ii) *If A is an ideal in E, then $A^d = \{0\}$, that is, $A^{dd} = E$, if and only if for every $0 \neq f \in E$ there exists an element $g \in A$ such that $0 \neq |g| \leq |f|$.*

2.5 Projections and Dedekind Completeness

We have already seen in Section 2.4 that the algebraic sum of two ideals in the Riesz space E is again an ideal, but the algebraic sum of two bands (even the direct sum) is not necessarily a band.

We note Theorem 11.1 [16, page 55].

Theorem 2.5.1. *If A and B are ideals in the Riesz space E such that $A \bigoplus B = E$, then $B = A^d$ and $A = B^d$, so $A = A^{dd}$ and $B = B^{dd}$. Hence, A and B are bands, each the disjoint complement of the other one.*

Before stating the next definition, we recall some definitions and notations from linear algebra. A mapping T from the vector space V into the vector space W is called a *linear mapping* or *linear transformation* (denoted by $T : V \to W$) if

$$T(\alpha f + \beta g) = \alpha T f + \beta T g,$$

for all f, g in V and all real α, β. Such a mapping is called a *linear operator* or, briefly, an *operator.* If T_1 and T_2 are operators from V into V itself, the operator T_1T_2 is defined by $(T_1T_2)f = T_1(T_2)f$ for all $f \in V$. We write T^2 for TT. If the operator $T : V \to V$ satisfies $T^2 = T$, then T is said to be *idempotent* and T is called a *projection*.

We state Definition 24.4 [12, page 133].

Definition 2.5.2. *Any band A in the Riesz space L, having the property that $A \bigoplus A^d = L$, is called a projection band. In this case, if $f = f_1 + f_2$ is the decomposition of an arbitrary $f \in L$ as the sum of $f_1 \in A$ and $f_2 \in A^d$, then f_1 and f_2 are called the components of f in A and A^d respectively.*

The following theorem is Theorem 11.4 [16, page 57].

Theorem 2.5.3. (i) *The band B in the Riesz space E is a projection band if and only if, for any $u \in E^+$, the element*

$$u_1 = \sup\{v : v \in B, 0 \leq v \leq u\}$$

exists, and u_1 is then the component of u in B. Similarly,

$$u_2 = \sup\{w : w \in B^d, 0 \leq w \leq u\}$$

is then the component of u in B^d. Hence, these suprema satisfy $u = u_1 + u_2$ with $u_1 \in B$ and $u_2 \in B^d$.

(ii) *For any $f \in E$, denote the component of f in the projection band B by P_Bf. The mapping P_B from E into E itself has the following properties:*

(a) *P_B is linear and idempotent, $P_B^2 = P_B$, so P_B is a projection.*

(b) $0 \le P_B u \le u$ *holds for every* $u \in E^+$.

On account of these properties P_B *is called the band projection or order projection on the projection band* B.

(iii) *In the converse direction, if* P *is a projection mapping* E *into* E *itself such that* $0 \le Pu \le u$ *holds for every* $u \in E^+$, *there exists a projection band* B *such that* P *is the band projection on* B.

The following definition is from [16, page 39].

Definition 2.5.4. *The Riesz space* E *is said to be Archimedean if*

$$\inf\{n^{-1}u : n = 1, 2, \ldots\} = 0,$$

holds for every $u \in E^+$.

We quote Theorem 9.1 [16, page 39].

Theorem 2.5.5. *Let* E *be a Riesz space.*

(i) *The space* E *is Archimedean if and only if it is true for every* $u \in E^+$ *that* $\inf\{\epsilon_n u : n = 1, 2, \ldots\} = 0$ *holds for any sequence* $(\epsilon_n)_{n \in \mathbb{N}}$ *of non-negative real numbers satisfying* $\epsilon_n \to 0$.

(ii) *The space* E *is Archimedean if and only if, given* u *and* v *in* E^+ *such that* $0 \le nv \le u$ *for* $n = 1, 2, \ldots$, *it follows that* $v = 0$. *In other words, for any* $v \ge 0, v \ne 0$, *the sequence* $(nv : n = 1, 2, \ldots)$ *is not bounded above.*

(iii) *If E is Archimedean, then every Riesz subspace of E is also Archimedean. In particular, ideals in E and bands in E are Archimedean Riesz spaces on their own.*

We note Theorem 32.1 [16, page 209].

Theorem 2.5.6. *If B_1 and B_2 are projection bands in the Archimedean Riesz space E, then $B_3 = B_1 \cap B_2$ is also a projection band and the corresponding band projections satisfy*

$$P_1P_2 = P_2P_1 = P_3.$$

Furthermore, we have $P_3u = (P_1u) \wedge (P_2u)$ for any $u \in E^+$.

We state Theorem 32.2 [16, page 210].

Theorem 2.5.7. *If B_1 and B_2 are projection bands in the Archimedean Riesz space E with corresponding band projections P_1 and P_2, then the following conditions are equivalent.*

(i) $B_1 \subseteq B_2$.

(ii) $P_1P_2 = P_2P_1 = P_1$.

(iii) $P_1 \leq P_2$.

The following definition is taken from [16, page 61].

Definition 2.5.8. *The Riesz space E is Dedekind complete whenever every non-empty subset D of E that is bounded above has a supremum.*

We note Theorem 12.1 [16, page 62].

Theorem 2.5.9. (i) *The Riesz space E is Dedekind complete if and only if every non-empty subset of E^+ that is upwards directed and bounded above has a supremum.*

(ii) *The Riesz space E is Dedekind σ-complete if and only if every increasing sequence in E^+ that is bounded above has a supremum, in other words, if and only if it follows from $0 \leq u_n \leq v_0$ that there exists an element $u_0 \in E$ such that $0 \leq u_n \uparrow u_0$.*

Remark 2.5.10 Dedekind completeness trivially implies the property to be Archimedean. Indeed, the Riesz space E is Archimedean whenever $0 \leq nv \leq u$, $n = 1, 2, \ldots$, implies $v = 0$. Assume now that E is Dedekind complete and let $0 \leq nv \leq u$ for $n = 1, 2, \ldots$. Then, in view of the Dedekind completeness, $u_0 = \sup_n nv$ exists. Hence $2u_0 = \sup_n 2nv = \sup_n nv = u_0$, so $u_0 = 0$, which implies $v = 0$.

We quote Theorem 12.2 [16, page 62].

Theorem 2.5.11. *Let E be a Dedekind complete Riesz space. Then the following holds.*

(i) *If B_1 and B_2 are disjoint bands in E, then $B_1 \bigoplus B_2$ is a band.*

(ii) *Every band B in E satisfies $B \bigoplus B^d = E$, that is every band B in E is a projection band.*

2.6 Linear Operators on Riesz Spaces

In this section we will present properties of linear operators on Riesz spaces.

Let V and W be ordered vector spaces, not necessarily Riesz spaces. We denote by T the mapping $T : V \to W$ and recall that T is a linear operator (operator for brevity) if

$$T(\alpha f + \beta g) = \alpha T f + \beta T g.$$

for all f, g in V and all real α, β. The set $\mathcal{L}(V, W)$ of all operators from V into W is a vector space if, for T_1, T_2 in $\mathcal{L}(V, W)$ and α, β real or complex, we define $\alpha T_1 + \beta T_2$ by

$$(\alpha T_1 + \beta T_2) f = \alpha T_1 f + \beta T_2 f, \text{ for all } f \in V.$$

If V, W, Z are vector spaces and T_1, T_2 are operators such that $T_1 : W \to Z$ and $T_2 : V \to W$, then the product operator $T_1 T_2 : V \to Z$ is defined by

$$(T_1 T_2) f = T_1 (T_2) f, \text{ for all } f \in V.$$

The following definition is taken from [16, page 122].

Definition 2.6.1. *The operator $T : V \to W$ is called a positive operator if T maps the positive cone of V into the positive cone of W, that is, $f \geq 0$ in V implies $Tf \geq 0$ in W.*

We state Definition 19.1 [16, page 124].

Definition 2.6.2. *The linear operator T from the Riesz space E into the Riesz space F is called a Riesz homomorphism (or lattice homomorphism) if*

$$T(f \vee g) = (Tf) \vee (Tg),$$

holds for all f, g in E.

We quote Theorem 19.2 [16, page 125].

Theorem 2.6.3. *Let $T : E \to F$ be a linear operator from the Riesz space E into the Riesz space F. Then the following conditions for T are equivalent.*

(i) *T is a Riesz homomorphism, that is,*

$$T(f \vee g) = (Tf) \vee (Tg) \text{ for all } f \text{ and } g \text{ in } E.$$

(ii) *For all f and g in E, $T(f \wedge g) = (Tf) \wedge (Tg)$.*

(iii) *For $f,\ g \in E, f \wedge g = 0$ implies $(Tf) \wedge (Tg) = 0$.*

(iv) *For all $f \in E$, $|Tf| = T(|f|)$.*

(v) *For all $f \in E$, $Tf^+ = (Tf)^+$.*

The following definition is taken from [16, page 145].

Definition 2.6.4. *The linear operator T mapping the Riesz space E into the Riesz space F is said to be order continuous if, for any downwards directed set D in E having infimum the null element ($D \downarrow 0$ in E), we have $\inf\{|Tf| : f \in D\} = 0$ in F.*

We quote from [16, page 145].

Theorem 2.6.5. *Let E and F be Riesz spaces.*

(i) *If T is order continuous, then so is αT for any real number α.*

(ii) *If $T \geq 0$* (T is positive), *then T is order continuous if and only if it follows from $D \downarrow 0$ in E that $\{Tf : f \in D\} \downarrow 0$ in F (equivalently, $D \uparrow u$ in E implies $\{Tf : f \in D\} \uparrow Tu$ in F).*

(iii) *If $T \geq 0$ and T is order continuous, and if $0 \leq S \leq T$, then S is an order continuous operator.*

We prove the following Theorem below.

Theorem 2.6.6. *Let E be a Riesz space. If $f \in E$ and T is a positive operator ($T \geq 0$), then $|Tf| \leq T|f|$.*

Proof. We have

$$|f| = f^+ + f^- \text{ and } f = f^+ - f^-, \text{ see Theorem 2.3.7 (ii).}$$

So

$$\begin{aligned} |Tf| &= |T(f^+ - f^-)| \\ &= |Tf^+ - Tf^-| \\ &\leq |Tf^+| + |Tf^-| \\ &= Tf^+ + Tf^- \\ &= T(f^+ + f^-) \\ &= T|f|. \end{aligned}$$

Hence $|Tf| \leq T|f|$. $\square$

Chapter 3

Conditional Expectations in Riesz spaces

In this Chapter we discuss the classical conditional expectation and its extension to the Riesz space setting.

A probability space, $(\Omega, \mathcal{F}, P)$, is a triple where Ω is a set, called the sample space, $\mathcal{F}$ is a σ-algebra of subsets of Ω and P is a positive measure (probability) on $\mathcal{F}$ with $P(\Omega) = 1$. For a probability space $(\Omega, \mathcal{F}, P)$, we denote $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ the collection of all measurable and absolutely integrable real functions. The elements of $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ are referred to as random variables.

For a random variable, f, the expectation of f is given by

$$\mathbb{E}[f] = \int_\Omega f dP.$$

Let Σ a sub σ-algebra of $\mathcal{F}$. The conditional expectation of f with respect to Σ, denoted $\mathbb{E}[f|\Sigma]$, is F where F is Σ-measurable and for each $A \in \Sigma$,

$$\int_A F dP = \int_A f dP.$$

If (f_n) is an almost everywhere increasing sequence in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ with an almost everywhere pointwise limit $f \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, then $\mathbb{E}[f_n|\Sigma]$ is an increasing sequence in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ with almost everywhere pointwise limit $\mathbb{E}[f|\Sigma]$.

The mapping $f \mapsto \mathbb{E}[f|\Sigma]$ has the following important properties which form the motivation for the definition of conditional expectation on Riesz spaces given in [8].

(1) $f \mapsto \mathbb{E}[f|\Sigma]$ is linear;

(2) if $f \geq 0$ then $\mathbb{E}[f|\Sigma] \geq 0$;

(3) $\mathbb{E}[\mathbf{1}|\Sigma] = \mathbf{1}$, where $\mathbf{1}$ is the function with value 1 almost everywhere;

(4) $\mathbb{E}[\mathbb{E}[f|\Sigma]|\Sigma] = \mathbb{E}[f|\Sigma]$, that is, $\mathbb{E}[\cdot|\Sigma]$ is idempotent;

(5) if $f_n \uparrow f$ in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}[f_n|\Sigma] \uparrow \mathbb{E}[f|\Sigma]$ in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$, that is $\mathbb{E}[\cdot|\Sigma]$ is order continuous.

(6) $\mathcal{R}(\mathbb{E}[\cdot|\Sigma]) = \mathcal{L}^1(\Omega, \Sigma, P)$.

From conditions (1)-(5) we can see that a conditional expectation is a linear positive order continuous projection, which maps the constant $\mathbf{1}$ function to the constant $\mathbf{1}$ function. Thus a conditional expectation operator on a Riesz space is defined in [8] as follows.

Definition 3.1. *Let E be a Riesz space with weak order unit. A positive order continuous projection T on E with range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation if $T(e)$ is a weak order unit of E for each weak order unit e of E.*

The following definition is [10, Definition 3.1].

Definition 3.2. *Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$. Let P and Q be band projections on E, we say that P and Q are independent with respect to T if*

$$TPTQ = TPQ = TQTP.$$

From [9] Theorem 3.2 we have the following theorem which gives the interaction of conditional expectations and band projections.

Theorem 3.3. *Let E be a Dedekind complete Riesz space with weak order unit, T a conditional expectation on E and B the band in E generated by $0 \leq g \in R(T)$, with associated band projection P. Then $TP = PT$.*

We quote Theorem 5.3.2 of [14, page 132], below.

Theorem 3.4 (Fatou Lemma)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$, then*

$$\mathbb{E}[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

We now give a generalization of Fatou Lemma to the Riesz space setting.

Theorem 3.5. *Let E be a Dedekind complete Riesz space with conditional expectation operator T with weak order unit $e = Te$. If $(f_n)_{n\geq 1}$ is a bounded sequence in E then*

$$T(\liminf_{n\to\infty} f_n) \leq \liminf_{n\to\infty} Tf_n.$$

Proof. Let $f \in E$, denote the limit inferior of f_n. For every $k \in \mathbb{N}$ define

$$g_k = \inf_{n\geq k} f_n. \tag{3.1}$$

Then the sequence $(g_k)_{k\geq 1}$ is increasing and converges to f. For $k \leq n$, we have $g_k \leq f_n$ so, as T is a positive linear operator,

$$Tg_k \leq Tf_n. \tag{3.2}$$

Taking the infimum over $n \geq k$ in (3.2) we have

$$Tg_k \leq \inf_{n\geq k} Tf_n. \tag{3.3}$$

We know $g_k \uparrow$ with order limit f, so by the order continuity of T we have,

$$\lim_{k\to\infty} Tg_k = Tf. \tag{3.4}$$

Also

$$\inf_{n\geq k} Tf_n \uparrow \liminf_{n\to\infty} Tf_n$$

so taking $k \to \infty$ we have

$$Tf \leq \liminf_{n\to\infty} Tf_n. \qquad \square$$

Corollary 5.3.2, [14, page 132] is as follows.

Theorem 3.6 (Reverse Fatou Lemma)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$. If $X_n \leq Z$ for all $n \geq 1$ where Z is a random variable on the probability space, then*

$$\mathbb{E}[\limsup_{n\to\infty} X_n] \geq \limsup_{n\to\infty} \mathbb{E}[X_n].$$

We prove the generalization of Reverse Fatou Lemma in the Riesz space setting below.

Theorem 3.7. *Let E be a Dedekind complete Riesz space with conditional expectation operator T with weak order unit $e = Te$. If $(f_n)_{n\geq 1} \subset E^+$ and $g \in E^+$ with $f_n \leq g$ for all $n \geq 1$, then*

$$T(\limsup_{n\to\infty} f_n) \geq \limsup_{n\to\infty} Tf_n.$$

Proof. We know that $f_n \leq g$, then $g - f_n \geq 0$, and by Theorem 3.5 we have

$$T\big(\liminf_{n\to\infty}(g - f_n)\big) \leq \liminf_{n\to\infty} T(g - f_n),$$

so

$$Tg + T\big(\liminf_{n\to\infty}(-f_n)\big) \leq Tg + \liminf_{n\to\infty}\big(T(-f_n)\big).$$

Using the relation $-\liminf(-f_n) = \limsup f_n$ we have

$$T\big(-\limsup_{n\to\infty} f_n\big) \leq -\limsup_{n\to\infty} Tf_n.$$

Hence

$$T(\limsup_{n\to\infty} f_n) \geq \limsup_{n\to\infty} Tf_n. \qquad \square$$

Chapter 4

Sequences of Band projections

In this Chapter we prove some preliminary results on sequences of band projections.

In Lemmas 4.1-4.4 we build the results which are required to prove Theorem 4.5, a major step in the extension of the proof of the Barndorff-Nielsen Theorem to Riesz space setting.

Lemma 4.1. *Let $(P_n)_{n\geq 1}$ be a sequence of band projections in a Dedekind complete Riesz space, then*

$$\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}(I - P_n) \geq \limsup_{n\to\infty} P_n(I - P_{n+1}).$$

Proof. We know that

$$\sup_{k\geq n}(I - P_n) \geq I - P_{j+1}, \text{ for } j \geq n - 1.$$

So

$$P_j \cdot \sup_{k\geq n}(I-P_n) \geq P_j(I-P_{j+1}), \text{ for } j \geq n-1.$$

Hence

$$\sup_{j\geq n-1} P_j \cdot \sup_{k\geq n}(I-P_n) \geq \sup_{j\geq n-1} P_j(I-P_{j+1}).$$

Taking the limit as n approaches infinity gives the result. □

Lemma 4.2. *Let E be a Dedekind complete Riesz space with weak order unit e. Let $(P_n)_{n\geq 1}$ be a sequence of band projections on E and*

$$f = \left(\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}(I-P_n) - \limsup_{n\to\infty} P_n(I-P_{n+1})\right)e,$$

then

$$\left(\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}(I-P_n)\right)f = f, \tag{4.1}$$

$$\left(I - \limsup_{n\to\infty} P_n(I-P_{n+1})\right)f = f. \tag{4.2}$$

Proof. By Lemma 4.1, it follows that $f \geq 0$. Let

$$P = \limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}(I-P_n) \text{ and } Q = \limsup_{n\to\infty} P_n(I-P_{n+1})$$

be band projections. Note that $P \geq Q$ implies that $PQ = Q$ by Theorem 2.5.7(ii), (iii). We have

$$P(P-Q) = P^2 - PQ. \tag{4.3}$$

Since band projections are idempotent it follows that (4.3) simplifies to

$$P(P-Q) = P - Q$$

giving (4.1) where $f = (P - Q)e$.

Also $P(I - Q) = P - Q$, so

$$\begin{aligned}(I-Q)(P-Q) &= P - PQ - Q + Q^2, \text{ idempotent property gives} \\ &= P - PQ - Q + Q, \text{ but } PQ = Q \\ &= P - Q \\ &= P(I-Q)\end{aligned}$$

It follows that $(I - Q)(P - Q)e = (P - Q)e = f$ which gives (4.2.) $\square$

Lemma 4.3. *Let E be a Dedekind complete Riesz space with weak order unit e. Let $(P_n)_{n\geq 1}$ be a sequence of band projections on E and*

$$f = \left(\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}(I - P_n) - \limsup_{n\to\infty} P_n(I - P_{n+1})\right)e,$$

then

$$\limsup_{n\to\infty} P_n f = f \text{ and } \limsup_{n\to\infty} (I - P_n) f = f. \tag{4.4}$$

Proof. From Lemma 4.2 we have,

$$f = \left(\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty} (I - P_n)\right) f. \tag{4.5}$$

Applying $\limsup\limits_{n\to\infty} P_n$ to f,

$$\limsup_{n\to\infty} P_n f = \left(\left(\limsup_{n\to\infty} P_n\right)^2 \cdot \limsup_{n\to\infty} (I - P_n)\right) f. \tag{4.6}$$

But by Theorem 2.5.3 (ii) (a),

$$\left(\limsup_{n\to\infty} P_n\right)^2 = \limsup_{n\to\infty} P_n. \tag{4.7}$$

Substituting (4.7) into (4.6), then by (4.5) we have,

$$\begin{aligned}\limsup_{n\to\infty} P_n f &= \left(\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}\ (I-P_n)\right) f,\\ &= f. \end{aligned} \tag{4.8}$$

Similarly, if we apply $\lim\limits_{n\to\infty}\sup\ (I-P_n)$ to (4.5) and using Theorem 2.5.6 we get

$$\begin{aligned}\limsup_{n\to\infty}\ (I-P_n) f &= \left(\left(\limsup_{n\to\infty}\ (I-P_n)\right)^2 \cdot \limsup_{n\to\infty} P_n\right) f\\ &= \left(\limsup_{n\to\infty}\ (I-P_n) \cdot \limsup_{n\to\infty} P_n\right) f\\ &= f. \end{aligned}$$

□

Lemma 4.4. *Let E be a Dedekind complete Riesz space. If $(P_n)_{n\geq 1}$ is a sequence of band projections on E, then*

$$P_n = \prod_{k=n}^{n+m} P_k + \sum_{i=1}^{m} \prod_{k=0}^{i-1} P_{n+k}(I-P_{n+i}). \tag{4.9}$$

Proof. The last term of (4.9) is equivalent to

$$\sum_{i=1}^{m} \prod_{k=0}^{i-1} P_{n+k}(I-P_{n+i}) = \sum_{i=1}^{m} (I-P_{n+i}) \prod_{k=0}^{i-1} P_{n+k}, \tag{4.10}$$

since band projections commute. From the right hand side of (4.10) we get the telescoping series given below

$$\begin{aligned}\sum_{i=1}^{m}(I-P_{n+i})\prod_{k=0}^{i-1}P_{n+k} &= \sum_{i=1}^{m}\left(\prod_{k=0}^{i-1}P_{n+k}-\prod_{k=0}^{i}P_{n+k}\right)\\ &= \prod_{k=0}^{0}P_{n+k}-\prod_{k=0}^{m}P_{n+k}\\ &= P_n-\prod_{k=n}^{n+m}P_k \end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.10) we get

$$\sum_{i=1}^{m}\prod_{k=0}^{i-1}P_{n+k}(I-P_{n+i}) = P_n-\prod_{k=n}^{n+m}P_k \tag{4.12}$$

from which (4.9) follows. □

We now prove the following theorem below.

Theorem 4.5. *Let E be a Dedekind complete Riesz space with weak order unit e. For any sequence of band projections $(P_n)_{n\geq 1}$ on E,*

$$\limsup_{n\to\infty}P_n\cdot\limsup_{n\to\infty}\,(I-P_n) = \limsup_{n\to\infty}P_n(I-P_{n+1}). \tag{4.13}$$

Proof. From the order continuity of band projections the left hand side of (4.13) can be written as

$$\begin{aligned}\limsup_{n\to\infty}P_n\cdot\limsup_{n\to\infty}\,(I-P_n) &= \lim_{n\to\infty}\sup_{k\geq n}P_k\cdot\lim_{m\to\infty}\sup_{r\geq m}\,(I-P_r)\\ &= \lim_{m\to\infty}\left(\lim_{n\to\infty}\sup_{k\geq n}P_k\right)\sup_{r\geq m}\,(I-P_r).\end{aligned}$$

As band projections commute with supremum and infimum we have

$$\begin{aligned}
\lim_{m\to\infty}\left(\lim_{n\to\infty}\sup_{k\geq n} P_k\right)\sup_{r\geq m}(I-P_r) &= \lim_{m\to\infty}\lim_{n\to\infty}\sup_{k\geq n}\left(P_k \sup_{r\geq m}(I-P_r)\right)\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sup_{k\geq n}\sup_{r\geq m} P_k(I-P_r)\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sup_{\substack{k\geq n\\ r\geq m}} P_k(I-P_r).\\
&= \inf_{\substack{m\\ n}}\sup_{\substack{k\geq n\\ r\geq m}} P_k(I-P_r)\\
&\geq \inf_{n}\sup_{\substack{k\geq n\\ r=k+1}} P_k(I-P_r). \qquad (4.14)
\end{aligned}$$

Here (4.14) holds as

$$\{(r,k)|r\geq m, k\geq n\}\supset\big\{(k+1,k)|k\geq \max\{n,m-1\}\big\}.$$

Hence

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\sup_{\substack{k\geq n\\ r\geq m}} P_k(I-P_r) &\geq \inf_{n}\sup_{k\geq n} P_k(I-P_{k+1})\\
&= \lim_{n\to\infty}\sup_{k\geq n} P_k(I-P_{k+1}), \qquad (4.15)
\end{aligned}$$

giving

$$Q=\limsup_{n\to\infty} P_n\cdot\limsup_{n\to\infty}(I-P_n)-\limsup_{n\to\infty} P_n(I-P_{n+1})\geq 0.$$

Let $f=Qe$, then $f=Qe\neq 0$. From Lemma 4.4 we have

$$P_n=\prod_{k=n}^{n+m} P_k+\sum_{i=1}^{m}\left(\prod_{k=0}^{i-1} P_{n+k}(I-P_{n+i})\right). \qquad (4.16)$$

From Theorem 2.5.6

$$\prod_{k=n}^{n+m} P_k = \bigwedge_{k=n}^{n+m} P_k, \tag{4.17}$$

$$\prod_{k=0}^{i-1} P_{n+k}(I - P_{n+i}) = \bigwedge_{k=0}^{i-1} P_{n+k}(I - P_{n+i}). \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.16), we have

$$P_n = \bigwedge_{k=n}^{n+m} P_k + \sum_{i=1}^{m} \left(\bigwedge_{k=0}^{i-1} P_{n+k}(I - P_{n+i}) \right). \tag{4.19}$$

From Theorem 2.3.20,

$$\begin{aligned} \sum_{i=1}^{m} \left(\bigwedge_{k=0}^{i-1} P_{n+k}(I - P_{n+i}) \right) &= \bigvee_{i=1}^{m} \left(\bigwedge_{k=0}^{i-1} P_{n+k}(I - P_{n+i}) \right) \\ &\leq \bigvee_{i=1}^{m} P_{n+i-1}(I - P_{n+i}). \end{aligned} \tag{4.20}$$

Replacing $n + i - 1$ by k on the right hand side of (4.20), we get

$$\bigvee_{i=1}^{m} P_{n+i-1}(I - P_{n+i}) = \bigvee_{k=n}^{n+m-1} P_k(I - P_{k+1}). \tag{4.21}$$

Combining (4.20) and (4.21) gives

$$\sum_{i=1}^{m} \left(\bigwedge_{k=0}^{i-1} P_{n+k}(I - P_{n+i}) \right) \leq \bigvee_{k=n}^{n+m-1} P_k(I - P_{k+1}). \tag{4.22}$$

Combining (4.22) and (4.19), gives

$$P_n \leq \bigwedge_{k=n}^{n+m} P_k + \bigvee_{k=n}^{n+m-1} P_k(I - P_{k+1}) \tag{4.23}$$

Letting $m \to \infty$ in (4.23), yields

$$P_n \leq \bigwedge_{k \geq n} P_k + \bigvee_{k \geq n} P_k(I - P_{k+1}). \tag{4.24}$$

From Lemma 4.2

$$f = \big(I - \limsup_{n \to \infty} P_n(I - P_{n+1})\big)f, \tag{4.25}$$

hence

$$\limsup_{n \to \infty} P_n(I - P_{n+1})f = 0. \tag{4.26}$$

Similarly from Lemma 4.3

$$f = \limsup_{n \to \infty}\, (I - P_n)f,$$

so

$$\liminf_{n \to \infty} P_n f = 0. \tag{4.27}$$

Taking the supremum of (4.24) applied to f for $n \geq m$ gives

$$\begin{aligned}\Big(\bigvee_{n \geq m} P_n\Big)f &\leq \Big(\bigvee_{n \geq m} \bigwedge_{k \geq n} P_k\Big)f + \Big(\bigvee_{n \geq m} \bigvee_{k \geq n} P_k(I - P_{k+1})\Big)f \\ &= \Big(\bigwedge_{k \geq m} P_k\Big)f + \Big(\bigvee_{k \geq m} P_k(I - P_{k+1})\Big)f. \end{aligned} \tag{4.28}$$

But $f = \limsup\limits_{n \to \infty}\, P_n f$ by (4.8). Letting $m \to \infty$ in (4.28) we have

$$f = \limsup_{n \to \infty} P_n f \leq \liminf_{k \to \infty} P_k f + \limsup_{k \to \infty} P_k(I - P_{k+1})f. \tag{4.29}$$

But by (4.26) and (4.27) respectively,

$$\limsup_{k\to\infty} P_k(I-P_{k+1})f = 0,$$
$$\liminf_{k\to\infty} P_k f = 0.$$

Hence $f = Qe = 0$, which contradicts the assumption that $Q \neq 0$. Therefore

$$\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}\ (I-P_n) = \limsup_{n\to\infty} P_n(I-P_{n+1}). \qquad \square$$

We now prove the corollary to Theorem 4.5 below.

Corollary 4.6. *Let E be a Dedekind complete Riesz space with a sequence of band projections $(P_n)_{n\geq 1}$ and P a single band projection. Assume that*

(a) $\liminf_{n\to\infty}\ PP_n = 0,$

(b) $\limsup_{n\to\infty}\ PP_n(I-P_{n+1}) = 0$ *or* $\limsup_{n\to\infty}\ P(P_{n+1}(I-P_n) = 0.$

Then $\limsup_{n\to\infty}\ P_n \leq I - P$.

Proof. From (b) we have

$$\begin{aligned} 0 &= \limsup_{n\to\infty} PP_n(I-P_{n+1}) \\ &= P\limsup_{n\to\infty} P_n(I-P_{n+1}), \end{aligned} \tag{4.30}$$

as band projections commute.

Applying Theorem 4.5 to (4.30) gives

$$0 = P\limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty}(I-P_n). \tag{4.31}$$

As band projections commute with lim sup we have

$$
\begin{aligned}
0 &= \limsup_{k\to\infty} P_k \cdot \limsup_{n\to\infty}(P - PP_n) \\
&= \limsup_{k\to\infty} P_k \cdot \big(P - \liminf_{n\to\infty} PP_n\big) \qquad (4.32)
\end{aligned}
$$

But from (a), $\liminf_{n\to\infty} PP_n = 0$. Hence

$$
\begin{aligned}
0 &= \limsup_{k\to\infty} P_k \cdot \big(P - 0\big) \\
&= \limsup_{k\to\infty} P_k P \\
&= \limsup_{k\to\infty} P_k P. \qquad (4.33)
\end{aligned}
$$

By band commutation we get

$$
\limsup_{n\to\infty} P_n P = P \limsup_{n\to\infty} P_n = 0. \qquad (4.34)
$$

It follows that

$$
\begin{aligned}
\limsup_{n\to\infty} P_n &= \limsup_{n\to\infty} P_n - P \limsup_{n\to\infty} P_n + P \limsup_{n\to\infty} P_n \\
&= (I - P) \limsup_{n\to\infty} P_n + P \limsup_{n\to\infty} P_n \\
&= (I - P) \limsup_{n\to\infty} P_n. \qquad (4.35)
\end{aligned}
$$

We know that

$$
\limsup_{n\to\infty} P_n \leq I. \qquad (4.36)
$$

Substituting (4.36) into (4.35) we have

$$\begin{aligned}\limsup_{n\to\infty} P_n &= (I-P)\limsup_{n\to\infty} P_n \\ &\leq I-P. \qquad\qquad \square \end{aligned} \tag{4.37}$$

The following Lemma will be used in the proof of the Barndorff-Nielsen Theorem.

Lemma 4.7. *Let E be a Dedekind complete Riesz space with weak order unit e. Let $(P_n)_{n\geq 1}$ be a sequence of band projections on E . Assume that*

(a) $\liminf\limits_{n\to\infty} P_n = 0$; *and*

(b) $\limsup\limits_{n\to\infty} P_n(I-P_{n+1}) = 0.$

Then $\limsup\limits_{n\to\infty} P_n = 0.$

Proof. Recall from Theorem 4.5

$$\limsup_{n\to\infty} P_n(I-P_{n+1}) = \limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty} (I-P_n).$$

Applying assumption (b) and the equation above we have

$$\begin{aligned} 0 &= \limsup_{n\to\infty} P_n(I-P_{n+1}) \\ &= \limsup_{n\to\infty} P_n \cdot \limsup_{n\to\infty} (I-P_n) \\ &= \limsup_{n\to\infty} P_n\big(\limsup_{n\to\infty} I + \limsup_{n\to\infty}(-P_n)\big) \\ &= \limsup_{n\to\infty} P_n(I - \liminf_{n\to\infty} P_n). \end{aligned} \tag{4.38}$$

Applying assumption (a) to (4.38) we have that

$$\limsup_{n\to\infty} P_n = 0. \qquad \square$$

Lemma 4.8. *Let E be a Dedekind complete Riesz space with weak order unit e. If $(P_n)_{n\geq 1}$ is a sequence of band projections acting on E then,*

$$\bigvee_{n=k}^{N} P_n = \bigvee_{n=k}^{N} P_n \prod_{j=k}^{n-1}(I - P_j). \tag{4.39}$$

Proof. We show by induction that

$$\bigvee_{n=k}^{N} P_n = \bigvee_{n=k}^{N} P_n \prod_{j=k}^{n-1}(I - P_j). \tag{4.40}$$

For $N = k$, we have the following for the right hand side of (4.40)

$$\bigvee_{n=k}^{k} P_n = P_k, \tag{4.41}$$

similarly for the left hand side we have

$$\bigvee_{n=k}^{k} P_n \prod_{j=k}^{n-1}(I - P_j) = P_k \prod_{j=k}^{k-1}(I - P_j) = P_k, \tag{4.42}$$

since $I = \prod_{j=k}^{k-1}(I - P_j)$.

Suppose that $N = M$

$$\bigvee_{n=k}^{M} P_n = \bigvee_{n=k}^{M} P_n \prod_{j=k}^{n-1}(I - P_j) \tag{4.43}$$

holds, then for $N = M + 1$

$$\begin{aligned}
\bigvee_{n=k}^{M+1} P_n &= \left(\bigvee_{n=k}^{M} P_n\right) \vee P_{M+1} \\
&= \left(\bigvee_{n=k}^{M} P_n\right) \vee \left(P_{M+1}(I - \bigvee_{n=k}^{M} P_n)\right). \qquad (4.44)
\end{aligned}$$

Applying the inductive hypothesis to (4.44) we get

$$\bigvee_{n=k}^{M+1} P_n = \bigvee_{n=k}^{M} P_n \prod_{j=k}^{n-1}(I - P_j) \vee \left(P_{M+1}(I - \bigvee_{n=k}^{M} P_n)\right). \qquad (4.45)$$

But

$$I - \bigvee_{n=k}^{M} P_n = \bigwedge_{n=k}^{M}(I - P_n). \qquad (4.46)$$

Applying Theorem 2.5.6 to the right hand side of (4.46) we get

$$I - \bigvee_{n=k}^{M} P_n = \prod_{n=k}^{M}(I - P_n). \qquad (4.47)$$

Substituting (4.47) into (4.45) gives

$$\begin{aligned}
\bigvee_{n=k}^{M+1} P_n &= \bigvee_{n=k}^{M} P_n \prod_{j=k}^{n-1}(I - P_j) \vee \left(P_{M+1} \prod_{n=k}^{M}(I - P_n)\right) \\
&= \bigvee_{n=k}^{M+1} P_n \prod_{j=k}^{n-1}(I - P_j). \qquad \square
\end{aligned}$$

Chapter 5

Barndorff-Nielsen Theorem in the Riesz space setting

In this Chapter we now prove the main result of the dissertation.

Corollary 4.2 of [10] is a generalization of the First Borel-Cantelli Lemma to Riesz spaces and it is quoted below.

Corollary 5.1. *Let E be a Dedekind complete Riesz space with strictly positive conditional expectation operator T and weak order unit $e = Te$. If*

$$\sum_{j=1}^{\infty} T(P_j e) \in E,$$

where $(P_j)_{j\geq 1}$ is a sequence of band projections in E, then

$$T(\limsup_{j\to\infty} P_j e) = 0.$$

The classical Barndorff-Nielsen Theorem [6, page 51], can be stated as follow.

Theorem 5.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $(A_n)_{n\geq 1}$ is a sequence of events in the probability space $(\Omega, \mathcal{F}, P)$ such that*

$$\liminf_{n\to\infty} P(A_n) = 0 \text{ and } \sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty,$$

then

$$P(\limsup_{n\to\infty} A_n) = 0 \text{ and } P(A_n) \to 0, \text{ as } n \to \infty.$$

The main result of this dissertation is the extension of the Barndorff-Nielsen Theorem to the Riesz space setting.

Theorem 5.3 (Barndorff-Nielsen Theorem in Riesz spaces)**.** *Let $(P_n)_{n\geq 1}$ be a sequence of band projections on a Dedekind complete Riesz space E with strictly positive conditional expectation operator T and weak order unit $e = Te$ such that*

$$\liminf_{n\to\infty} T(P_n e) = 0 \text{ and } \sum_{n=1}^{\infty} T\big(P_n(I - P_{n+1})e\big) \text{ is convergent in } E,$$

then

$$\lim_{n\to\infty} P_n e = 0.$$

Proof. Put

$$Q_n = P_n(I - P_{n+1}), n \geq 1. \tag{5.1}$$

Then Q_n is a band projection and by the Borel-Cantelli Lemma in Riesz spaces, see Corollary 5.1, as $\sum\limits_{n=1}^{\infty} TQ_n e$ converges in E then $T(\limsup\limits_{n\to\infty} Q_n e) = 0$ and

$$\limsup_{n\to\infty} Q_n e = 0, \tag{5.2}$$

as T is strictly positive. By Lemma 3.5,

$$0 \le T(\liminf_{n\to\infty} P_n e) \le \liminf_{n\to\infty} \; T(P_n e) = 0.$$

Now, since T is a strictly positive, we have that

$$\liminf_{n\to\infty} P_n e = 0. \tag{5.3}$$

By Lemma 4.7 and (5.1), (5.2), (5.3) we have that $\limsup\limits_{n\to\infty} \; P_n = 0$. From Theorem 2.3.19 we have that

$$0 = \liminf_{n\to\infty} P_n e = \limsup_{n\to\infty} P_n e \text{ and hence } \liminf_{n\to\infty} P_n e = 0. \qquad \square$$

Theorem 5.3 is a conditional generalization of the classical Barndorff-Nielsen result to a Dedekind complete Riesz space with weak order unit and conditional expectation operator. If $E = \mathcal{L}^1(\Omega, \mathcal{F}, P)$ then E is a Dedekind complete Riesz space. Then $e = \mathbf{1}$ and χ_{A_n} denote the constant 1 function and the indicator function of A_n respectively. The map $P_n f = \chi_{A_n} f$ is a band projection on E and e is a weak order unit for E. Let $(\Omega, \mathcal{F}, P)$ be a probability space, if $\mathcal{A}$ is a sub σ-algebra of $\mathcal{F}$ then $Tf = \mathbb{E}[f|\mathcal{A}]$ is a conditional expectation on the Riesz space setting on E.

Note (a) In Theorem 5.3 it is shown that $\lim\limits_{n\to\infty} P_n = 0$ from which it follows, by the order continuity of T, that $\lim\limits_{n\to\infty} T P_n e = 0$.

(b) Theorem 5.3 yields in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ the approved result that $\limsup\limits_{n\to\infty} A_n$ is the event with probability zero, $P\{x \in \underset{n\to\infty}{A_n}, \text{infinitely often}\} = 0$.

For reference we state the classical Balakrishnan and Stepanov Theorem [6, page 52], below.

Theorem 5.4. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and* (A_n) *be a sequence of events in* $(\Omega, \mathcal{F}, P)$. *If* $P(A_n) \to 0$ *as* $n \to \infty$ *and*

$$\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}^c \cap \ldots \cap A_{n+m-1}^c \cap A_{n+m}) < \infty$$

for some $m \geq 1$, *then* $P(A_n$ *infinitely often*$) = 0$.

We prove our second major result, the generalization of the Balakrishnan and Stepanov Theorem in Riesz spaces.

Theorem 5.5 (Balakrishnan-Stepanov Theorem in Riesz spaces)**.** *Let* E *be a Dedekind complete Riesz space with* T *a strictly positive conditional expectation operator on* E *and weak order unit* $e = Te$. *Let* $(P_n)_{n\geq 1}$ *be a sequence of band projections on* E *with* $T(P_n e) \to 0$ *and*

$$\sum_{n=1}^{\infty} T\left(P_{n+m} \prod_{j=n}^{n+m-1} (I - P_j)e\right) \in E$$

for some $m \geq 1$. *Then* $\limsup_{n\to\infty} P_n = 0$.

Proof. From Definition 2.3.17 we know that

$$\limsup_{n\to\infty} P_n e = \bigwedge_{k\in\mathbb{N}} \bigvee_{n\geq k} P_n e.$$

Hence

$$T\left(\limsup_{n\to\infty} P_n e\right) = T\left(\bigwedge_{k\in\mathbb{N}} \bigvee_{n\geq k} P_n e\right)$$
$$\leq T\left(\bigvee_{n\geq k} P_n e\right). \tag{5.4}$$

Applying e to Lemma 4.8 and letting $N \to \infty$ we get

$$\bigvee_{n\geq k} P_n e = \bigvee_{n\geq k} P_n \prod_{j=k}^{n-1}(I - P_j)e. \tag{5.5}$$

Hence

$$T\left(\limsup_{n\to\infty} P_n e\right) \leq T\left(\bigvee_{n\geq k} P_n e\right) = T\left(\bigvee_{n\geq k} P_n \prod_{j=k}^{n-1}(I - P_j)e\right) = K. \tag{5.6}$$

We show that the sequence

$$\left(\bigvee_{n=k}^{N} P_n \prod_{j=k}^{n-1}(I - P_j)e\right)$$

is disjoint. Let $r, m, k \in \mathbb{N}, r > m \geq k$, by Theorem 2.5.6 we have

$$L = \left(P_r \prod_{j=k}^{r-1}(I - P_j)e\right) \wedge \left(P_m \prod_{j=k}^{m-1}(I - P_j)e\right)$$
$$= \left(P_r \bigwedge_{j=k}^{r-1}(I - P_j)e\right) \wedge \left(P_m \prod_{j=k}^{m-1}(I - P_j)e\right)$$
$$\leq P_r(I - P_m)e \wedge \left(P_m \prod_{j=k}^{m-1}(I - P_j)e\right) \tag{5.7}$$

Applying Theorem 2.5.6 to (5.7) we get

$$\begin{aligned} L &\le P_r(I-P_m)e \wedge \left(P_m \prod_{j=k}^{m-1}(I-P_j)e\right) \\ &\le (I-P_m)P_m e = 0 \end{aligned} \tag{5.8}$$

proving disjointness. Applying Theorem 2.3.20 to the right hand side of (5.5) we get

$$\bigvee_{n\ge k} P_n \prod_{j=k}^{n-1}(I-P_j)e = \sum_{n\ge k} P_n \prod_{j=k}^{n-1}(I-P_j)e. \tag{5.9}$$

Substituting (5.9) into (5.6) and by the order continuity of T we have

$$\begin{aligned} K &= T\left(\sum_{n\ge k} P_n \prod_{j=k}^{n-1}(I-P_j)e\right) \\ &= \sum_{n\ge k} T\left(P_n \prod_{j=k}^{n-1}(I-P_j)e\right). \end{aligned} \tag{5.10}$$

Expanding (5.10) we have

$$\begin{aligned} K &= \sum_{n\ge k} T\left(P_n \prod_{j=k}^{n-1}(I-P_j)e\right) \\ &= \sum_{n=k}^{k+m-1} T\left(P_n \prod_{j=k}^{n-1}(I-P_j)e\right) + \sum_{n=k+m}^{\infty} T\left(P_n \prod_{j=k}^{n-1}(I-P_j)e\right). \end{aligned} \tag{5.11}$$

Changing the index for the second term in the right hand side of (5.11) we have

$$\sum_{n=k+m}^{\infty} T\left(P_n \prod_{j=k}^{n-1}(I-P_j)e\right) = \sum_{n=k}^{\infty} T\left(P_{n+m} \prod_{j=k}^{n+m-1}(I-P_j)e\right). \tag{5.12}$$

Substituting (5.12) into (5.11) we have

$$K = \sum_{n=k}^{k+m-1} T\left(P_n \prod_{j=k}^{n-1}(I - P_j)e\right) + \sum_{n=k}^{\infty} T\left(P_{n+m} \prod_{j=k}^{n+m-1} (I - P_j)e\right). \quad (5.13)$$

Combining (5.13) and (5.6) we have

$$T\left(\limsup_{n\to\infty} P_n e\right) \leq \sum_{n=k}^{k+m-1} T\left(P_n \prod_{j=k}^{n-1}(I - P_j)e\right) + \sum_{n=k}^{\infty} T\left(P_{n+m} \prod_{j=k}^{n+m-1} (I - P_j)e\right). \quad (5.14)$$

We know that

$$\prod_{j=k}^{n-1}(I - P_j)e \leq e \quad (5.15)$$

since products of band projections are band projections and are bounded above by I. Combining (5.15) and (5.14) we have for $n \geq k$,

$$T\left(\limsup_{n\to\infty} P_n e\right) \leq \sum_{n=k}^{k+m-1} T(P_n e) + \sum_{n=k}^{\infty} T\left(P_{n+m} \prod_{j=k}^{n+m-1} (I - P_j)e\right). \quad (5.16)$$

Since $T(P_n e) \to 0$, it follows

$$\sum_{n=k}^{k+m-1} T(P_n e) \to 0 \text{ as } k \to \infty. \quad (5.17)$$

By assumption

$$\sum_{n=1}^{\infty} T\left(P_{n+m} \prod_{j=n}^{n+m-1} (I - P_j)e\right) \in E, \quad (5.18)$$

hence we have

$$\sum_{n=k}^{\infty} T\left(P_{n+m} \prod_{j=k}^{n+m-1} (I - P_j)e\right) \to 0 \text{ as } k \to \infty. \quad (5.19)$$

Letting $k \to \infty$ in (5.16) and applying (5.17) and (5.19) we have

$$T\left(\limsup_{n\to\infty} P_n e\right) = 0.$$

Since T is a strictly positive linear operator it follows that $\limsup_{n\to\infty} P_n = 0$. $\square$

Conclusion

We proved the conditional expectation analogues of the Fatou and Reverse Fatou Lemmas in the Riesz space setting. We also proved an interesting lemma about a sequence of band projections acting on a Dedekind complete Riesz space. From this lemma and the First Borel-Cantelli Lemma in Riesz spaces, we proved, a conditional version of the Barndorff-Nielsen Theorem as well as a conditional version of the Balakrishnan and Stepanov Theorem in Riesz spaces.

Bibliography

[1] C. D. Aliprantis, K. C. Border, *Infinite dimensional analysis: A hitch-hiker's guide.* Springer-Verlag, Berlin (1994).

[2] P. Billingsley, *Probability and Measure.* Wiley, New York (1995).

[3] E. Borel, *Les probabilitiés dénombrables et leurs applications arithmétiques.* Rend. Circ. Mat. Palermo 27, (1909), pp. 247-271.

[4] E. Borel, *Sur un problème de probabilitiés relatif aux fractions continues.* Math. Ann. 77, (1912), pp. 578-587.

[5] F.P. Cantelli, *Sulla probabilitià come limite della frequenza.* Rend. Accad. Lincei Ser.5. 24, (1917), pp. 39-35.

[6] T. K. Chandra, *The Borel-Cantelli Lemma.* Springer, India/New York (2012).

[7] R Durrett, *Probability theory and examples.* Cambridge University Press (2010)

[8] W-C Kuo, C.C.A. Labuschagne, B.A. Watson, *Discrete time stochastic processes in Riesz spaces.* Indag. Math, 15 (2004) pp. 15-40.

[9] W-C Kuo, C.C.A. Labuschagne, B.A. Watson, *Conditional Expectation on Riesz spaces*. J. Math. Anal. Appl, 303 (2005) pp. 509-521.

[10] W-C Kuo, C.C.A. Labuschagne, B.A. Watson, *A Zero-One Law for Riesz spaces and Fuzzy processes*. Beijing, Springer-Verlag, (2005) pp. 393-397.

[11] W-C Kuo, M.J. Rogans, B.A. Watson, *Mixing inequalities in Riesz spaces*. J. Math. Anal. Appl, 456 (2017) pp. 992-1004.

[12] W. A. J. Luxemburg, A. C. Zaanen, *Riesz spaces I*. North Holland, Amsterdam. London (1971).

[13] S.W Nash, *An extension of the Borel-Cantelli lemma*. AMS **25** (1954) pp. 165-167.

[14] S.I. Resnick, *A probability path*. Birkhäuser, (2001).

[15] F. Riesz, *Sur la décomposition des opérations fonctionelles linéaires.* Atti del Congr. Internaz. dei Mat., Bologna, (1928), pp. 143-148.

[16] A.C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*. Springer-Verlag, Berlin/New York, (1997).