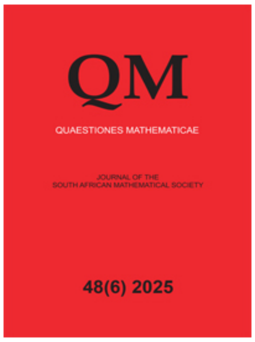

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ON THE ANALYSIS AND INTEGRABILITY OF THE TIME-FRACTIONAL STOCHASTIC POTENTIAL-KDV EQUATION

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Abstract. This study investigates the invariance properties of the time-fractional stochastic potential-KdV (FSP-KdV) equation, intending to enhance our understanding of the dynamics associated with nonlinear photon and optical soliton propagation. The application of the Lie group analysis method enables the derivation of vector fields and symmetry reductions for the equation. Additionally, employing power series theory, the paper systematically constructs explicit power series solutions, offering a detailed derivation. The resulting wave propagation patterns of these solutions are depicted along the x-axis at various temporal instances. Finally, leveraging a new conservation theorem, the study formulates two distinct conservation laws for the equation, presenting comprehensive and detailed derivations for each.

Mathematics Subject Classification (2020): 35A09, 35C06.
Key words: Lie symmetry analysis, time-fractional stochastic potential-KdV equation, similarity reductions, optical soliton, explicit solutions, conservation laws.

1. Introduction. Recent progress in fractional differential equations (FRDEs) [27, 29, 13, 30, 4, 12] has been catalyzed by emerging applications across diverse

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fields, including viscoelasticity [33, 17, 16], mathematical biology [32, 28, 14, 35], electrochemistry [24, 22, 26, 23], physics [34, 36, 5, 38], and fluid mechanics [37, 21]. The Lie theory of symmetry groups stands as a well-established, systematic, and efficient methodology for tackling differential equations. In their work [6], Gazizov used the Lie symmetry method to systematically analyze symmetries in FRDEs, introducing prolongation formulae for fractional derivatives (FDs). This method enabled the Lie symmetry analysis of several time-fractional equations utilizing the Riemann-Liouville (RL) derivative [31, 9, 7].

The well-established Noether theorem establishes a fundamental relationship between symmetries and conservation laws within the context of differential equations, specifically when these equations conform to the Euler-Lagrange formalism [18]. In reference [15], fractional extensions of the Noether operators were initially introduced, leading to the derivation of conservation laws for equations governing time-fractional subdiffusion and diffusion-wave phenomena. These conservation laws were obtained through the application of a new theorem on conservation laws proposed by Ibragimov [10].

In the domain of plasmas and electrical circuits, the stochastic potential-KdV equation emerges as a nonlinear model with implications for predicting the propagation of nonlinear photons and optical solitons. This mathematical formulation, expressed as

$$\Phi_t + \alpha\Phi_x + \beta(\Phi_x)^2 + \gamma\Phi_{xxx} = 0, \tag{1}$$

incorporates significant coefficients, with γ signifying dispersion, β denoting nonlinearity, and α representing a stochastic parameter. Commonly known as the potential-KdV equation, it finds applications in the study of water waves, particularly when the stochastic factor is absent (α= 0). Originating from the work of Alhami *et al.*, Equation (2) facilitates the derivation of explicit solutions such as lumps, breathers, and multi-soliton types using methodologies like the simplified Hirota method and the Cole-Hopf transformation [3]. Further exploration of closed-form solutions in a fractional context is presented in [2]. The Lie symmetry analysis of this equation is addressed in [1]. In this context, we consider the time fractional form given by

$$D_t^\rho\Phi + \alpha\Phi_x + \beta(\Phi_x)^2 + \gamma\Phi_{xxx} = 0, \tag{2}$$

introducing fractional components to enhance the illustrative capacity for physical densities in comparison to integer order partial differential equations (PDEs), which may potentially overlook certain information. In Equation (2), $D_t^\rho\Phi$ denotes the RL fractional derivative operator, characterized by its definition as follows

$$D_t^\rho\Phi(x,t) = \begin{cases} \frac{1}{\Gamma(m-\rho)}\frac{\partial^m}{\partial t^m}\int\limits_0^t (t-\vartheta)^{m-\rho-1}\Phi(x,\vartheta)d\vartheta, & m-1<\rho<m, \\ \frac{\partial^m\Phi}{\partial t^m}, & \rho = m \in \mathbb{N}, \end{cases} \tag{3}$$

where the ($0 < \rho \leq 1$) parameter characterizes the order of the fractional derivative, exerting a significant influence on the properties of this equation. Here, t represents time, and x corresponds to the spatial coordinate.

The present study contributes significantly through various sections. Section 2 elucidates foundational definitions and the general algorithm of Lie point symmetries. Subsequently, in Section 3, this algorithm is applied to the governing model. The resulting Lie symmetries are then harnessed in Section 3 for the one-dimensional optimal system, leading to further utilization in similarity reductions. In Section 4, the application of power series theory yields exact explicit structures for the considered model, as outlined in Equation (2). A thorough convergent analysis of these results is conducted. Section 5 focuses on describing local conservation laws using the Ibragimov method. The paper concludes in Section 7 with summarizing remarks, and potential avenues for future research are suggested.

2. Some definitions and general description of the Lie symmetry method. Throughout this section, we will bear in mind several introductory definitions that are utilized throughout our article

DEFINITION 1. The RL fractional derivative (ordinary) is defined as [8]

$$D^{\rho}g(t) = \begin{cases} \frac{d^m}{dt^m} I^{m-\rho} g(t), & m-1<\rho<m, \\ \frac{d^m h}{dt^m}, & \rho = m \in \mathbb{N}, \end{cases} \tag{4}$$

where $I^{\rho}h(t)$ is integration of order ρ of RL given by

$$I^{\rho}g(t) = \frac{1}{(\rho-1)!}\int_0^t (t-\varpi)^{\rho-1} g(\varpi)d\varpi, \qquad \rho>0$$

$$I^{\rho}g(t) = g(t) \qquad \rho = 0.$$

By using a result, we can write $(\rho-1)! = \Gamma(\rho)$.

DEFINITION 2. Given a one-parameter Lie group of transformation, we express

$$\bar{x} = F(x;\varepsilon), \tag{5}$$

as a Taylor series in the parameter ε around $\varepsilon = 0$. Utilizing the condition $x = F(x;\varepsilon)|_{\varepsilon=0}$, we derive what is termed the infinitesimal transformations of the Lie group of transformation $\bar{x} = F(x;\varepsilon)$:

$$\bar{x} = x + \varepsilon\zeta(x) + O\left(\varepsilon^2\right),$$

where

$$\zeta(x) = \left.\frac{\partial \bar{x}}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{6}$$

The components of the vector $\zeta(x) = (\zeta_1(x), \zeta_2(x), \ldots, \zeta_n(x))$ are referred to as the infinitesimals of (5).

DEFINITION 3. The operator is defined as

$$\mathbf{U} = \sum_{i=1}^{n} \zeta_i(x)\frac{\partial}{\partial x_i}, \tag{7}$$

is recognized as the infinitesimal generator (operator) associated with the one-parameter Lie group of transformations (5). Here, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\zeta(x) = (\zeta_1(x), \zeta_2(x), \ldots, \zeta_n(x))$ represent the infinitesimals of (5).

DEFINITION 4. A Lie algebra $\mathbf{L}$ is characterized as a vector space over a field $\mathbf{F}$ endowed with a prescribed bilinear commutation law (known as the commutator), which adheres to certain properties;
1. Closure: For $A, B \in \mathbf{L}$ it follows that $[A, B] \in \mathbf{L}$.
2. Bilinearity:

$$[A, \alpha_1 B + \alpha_2 C] = \alpha_1 [A, B] + \alpha_2 [A, C], \quad \alpha_1, \alpha_2 \in \mathbf{F}, \quad A, B, C \in \mathbf{L}.$$

3. Skew-symmetry:

$$[A, B] = -[B, A].$$

4. Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

2.1. General description of Lie symmetry method for time fractional non-linear PDEs. Here, we describe the main concept of the Lie symmetry method [8] by introducing the following general (1+1)-dimensional fractional PDEs

$$\partial_t^\rho \Phi(x,t) = G(t, x, \Phi, \Phi_x, \Phi_{xx}, \cdots), \qquad (0 < \rho \leq 1). \tag{8}$$

One parameter symmetry group of transformations is as follows

$$\begin{aligned} t^\star &= t + \varepsilon \vartheta^{(t)}(x,t,\Phi) + O(\varepsilon^2), \\ x^\star &= x + \varepsilon \vartheta^{(x)}(x,t,\Phi) + O(\varepsilon^2), \\ \Phi^\star &= \Phi + \varepsilon \Theta(x,t,\Phi) + O(\varepsilon^2), \end{aligned} \tag{9}$$

where $\varepsilon < 1$ is the Lie group parameter and from Def 3 its associated Lie algebra has the following form

$$\mathbf{U} = \vartheta^{(t)} \frac{\partial}{\partial t} + \vartheta^{(x)} \frac{\partial}{\partial x} + \Theta \frac{\partial}{\partial \Phi} \cdot \tag{10}$$

Similarly,

$$\begin{aligned} \frac{\partial^\rho \Phi^\star}{\partial t^{\star\rho}} &= \frac{\partial^\rho \Phi}{\partial t^\rho} + \varepsilon \Theta_\rho^0(x,t,\Phi) + O(\varepsilon^2), \\ \frac{\partial \Phi^\star}{\partial x^\star} &= \frac{\partial \Phi}{\partial x} + \varepsilon \Theta^x(x,t,\Phi) + O(\varepsilon^2), \\ \frac{\partial^2 \Phi^\star}{\partial x^{\star 2}} &= \frac{\partial^2 \Phi}{\partial x^2} + \varepsilon \Theta^{xx}(x,t,\Phi) + O(\varepsilon^2), \\ \frac{\partial^3 \Phi^\star}{\partial x^{\star 3}} &= \frac{\partial^3 \Phi}{\partial x^3} + \varepsilon \Theta^{xxx}(x,t,\Phi) + O(\varepsilon^2). \\ &\ \ \vdots \end{aligned} \tag{11}$$

The Lie algebra associated to (8) is prolongation of (10) written as

$$\mathbf{U}^{\rho,n} = \mathbf{U} + \Theta^0_\rho \frac{\partial}{\partial t^\rho \Phi} + \Theta^x \frac{\partial}{\partial \Phi_x} + \Theta^{xx} \frac{\partial}{\partial \Phi_{xx}} + \Theta^{xxx} \frac{\partial}{\partial \Phi_{xxx}} + \dots, \tag{12}$$

where n is the order of Equation (8) and

$$\begin{aligned} \Theta^x =& \mathcal{D}_x(\Theta) - \Theta_t \mathcal{D}_x(\vartheta^{(t)}) - \Theta_x \mathcal{D}_x(\vartheta^{(x)}), \\ \Theta^{xx} =& \mathcal{D}_x(\Theta^x) - \Theta_{tx} \mathcal{D}_x(\vartheta^{(t)}) - \Theta_{xx} D_x(\vartheta^{(x)}), \\ \Theta^{xxx} =& \mathcal{D}_x(\Theta^{xx}) - \Theta_{txx} \mathcal{D}_x(\vartheta^{(t)}) - \Theta_{xxx} \mathcal{D}_x(\vartheta^{(x)}), \end{aligned} \tag{13}$$

where the total differential operator $\mathcal{D}_j$ is defined as

$$\mathcal{D}_j = \frac{\partial}{\partial x^j} + \Phi_j \frac{\partial}{\partial \Phi} + \Phi_{jl} \frac{\partial}{\partial \Phi_l} + ..., \qquad j, l = 1, 2, 3. \tag{14}$$

Also, operator defined by (12) is a point symmetry of (8) iff,

$$\mathbf{U}^{\rho,n}(\Delta)|_{\Delta=0} = 0, \tag{15}$$

where

$$\Delta := \partial_t^\rho \Phi(x,t) - G(x, t, \Phi, \Phi_x, \Phi_{xx}, \cdots).$$

The invariance condition

$$\vartheta^{(t)}(x, t, \Phi)|_{t=0} = 0, \tag{16}$$

is necessary to the transformation (9).
Also, extended infinitesimals of ρ order in explicit form is given as

$$\begin{aligned} \Theta^0_\rho =& \frac{\partial^\rho \Theta}{\partial t^\rho} + \left(\Theta_\Phi - \rho \mathcal{D}_t(\vartheta^{(t)})\right) \frac{\partial^\rho \Phi}{\partial t^\rho} - \Phi \frac{\partial^\rho \Theta_\Phi}{\partial t^\rho} \\ &+ \sum_{m=1}^{\infty} \left[\binom{\rho}{m} \frac{\partial^\rho \Theta_\Phi}{\partial t^\rho} - \binom{\rho}{m+1} \mathcal{D}_t^{n+1}(\vartheta^{(t)}) \right] \mathcal{D}_t^{\rho-m}(\Phi) \\ &- \sum_{m=1}^{\infty} \binom{\rho}{m} \mathcal{D}_t^m(\vartheta^{(x)}) \mathcal{D}_t^{\rho-m}(\Phi_x) + \upsilon, \end{aligned} \tag{17}$$

where

$$\binom{\rho}{m} = \frac{(-1)^{m-1} \rho \Gamma(m-\rho)}{\Gamma(1-\rho)\Gamma(m+1)},$$

and

$$\upsilon = \sum_{m=2}^{\infty} \sum_{l=2}^{m} \sum_{s=2}^{l} \sum_{q=0}^{s-1} \binom{\rho}{m} \binom{m}{l} \binom{s}{q} \frac{1}{s!} \frac{t^{m-\rho}}{\Gamma(m+1-\rho)} (-1)^q \Phi^q \frac{\partial^l}{\partial t^l} [\Phi^{s-q}] \frac{\partial^{m-l+s}}{\partial t^{m-l} \partial \Phi^s}. \tag{18}$$

DEFINITION 5. The $\Phi = \phi(x, t)$ is called an invariant solution of Equation (8) associated to the symmetry generator (10) if
1- $\Phi = \phi(x, t)$ is the invariant surface of Equation (13), i-e.,

$$\mathbf{U}\phi = 0 \Leftrightarrow \left(\vartheta^{(t)} \frac{\partial}{\partial t} + \vartheta^{(x)} \frac{\partial}{\partial x} + \Theta \frac{\partial}{\partial \Phi}\right)(\phi) = 0,$$

2- $\Phi = \phi(x, t)$ adheres to Equation (8).

3. Symmetry analysis of the FSP-KdV Equation (2). Assume that Equation (2) remains invariant upon transformation Equation (9) and Equation (11), we get

$$D_t^{\rho}\tilde{\Phi} + \alpha\tilde{\Phi}_x + \beta(\tilde{\Phi}_x)^2 + \gamma(\tilde{\Phi}_{xxx}), \tag{19}$$

such that $\tilde{\Phi} = \phi(x,t)$ adheres to Equation (2), then applying prolongation Equation (12) to above equation. We have the following (reduced) invariant equation:

$$\Theta^0_{\rho} + \alpha\Theta^x + 2\beta\Phi_x\Theta^x + +\gamma\Theta^{xxx} = 0. \tag{20}$$

By substituting the values of Θ^x, Θ^{xxx} and Θ^0_{ρ} in above equation and setting all the monomials of derivatives of Φ equal to zero, we obtained following determining system

$$\begin{aligned}
&\partial_t^{\rho}\Theta + \alpha\Theta_x + \gamma\Theta_{xxx} = 0,\\
&\binom{\rho}{m}\partial_t^{\rho}\Theta - \binom{\rho}{m+1}D_t^{m+1}(\vartheta^{(t)}) = 0, \quad m = 1,2,3,...\\
&\Theta_{\Phi\Phi} = \vartheta^{(t)}_{\Phi} = \vartheta^{(x)}_{\Phi} = \vartheta^{(t)}_{x} = \vartheta^{(x)}_{t} = 0,\\
&\alpha\rho\vartheta^{(t)}_t - \alpha\vartheta^{(x)}_x + 2\beta\Theta_x + 3\gamma\Theta_{xx\Phi} = 0\\
&\alpha\rho\vartheta^{(t)}_t - 2\vartheta^{(x)}_x + \Theta_{\Phi} = 0\\
&\Theta_{x\Phi} - \vartheta^{(x)}_{xx} = 0\\
&\rho\vartheta^{(t)}_t - 3\vartheta^{(x)}_x = 0.
\end{aligned} \tag{21}$$

The solution of the above system yields

$$\vartheta^{(x)} = a_1 x + a_2, \qquad \vartheta^{(t)} = \frac{3}{\rho}a_1 t, \qquad \Theta = -a_1\Phi - \frac{\alpha a_1}{\beta}x + f(t),$$

where a_1 and a_2 are constants. We have infinite-dimensional symmetry algebra for the FSP-KdV Equation (2) given by

$$\mathbf{U_1} = \frac{\partial}{\partial x}, \quad \mathbf{U_2} = x\frac{\partial}{\partial x} + \frac{3}{\rho}t\frac{\partial}{\partial t} - (\Theta + \frac{\alpha x}{\beta})\frac{\partial}{\partial\Theta}, \quad \mathbf{U}_{\infty} = f(x,t)\frac{\partial}{\partial\Theta}\cdot \tag{22}$$

To make this finite-dimensional we take $f(t) = 1$, and the obtained symmetry algebra is

$$\mathbf{U_1} = \frac{\partial}{\partial x}, \quad \mathbf{U_2} = x\frac{\partial}{\partial x} + \frac{3}{\rho}t\frac{\partial}{\partial t} - (\Theta + \frac{\alpha x}{\beta})\frac{\partial}{\partial\Theta}, \quad \mathbf{U_3} = \frac{\partial}{\partial\Theta}\cdot \tag{23}$$

Theorem 1. *The vector fields* $\mathbf{U}_i$ $(i = 1,2,3)$ *for the FSP-KdV Equation (2) constitute a three-dimensional Lie algebra.*

Proof 1. As indicated in Table 1, an anti-symmetrical pattern is noticeable, and zero diagonal elements are evident. The determination of structure constants is easily accomplished by examining the commutator table, and the Jacobi identity verification is straightforward. □

4. Optimal system and its use for symmetry reductions. The concept of an optimal system of subalgebras within a given Lie algebra, aimed at producing fundamentally distinct invariant solutions, was likely first introduced by Ovsyannikov [20]. Subsequently, Ibragimov [11] and Olver [19] further developed this idea. The process of identifying the optimal system entails examining sets of equivalent classes of one-dimensional subalgebras and analyzing their behavior under the adjoint representation.

The adjoint action representation can be formulated as:

$$\mathrm{Adj}(\exp(\varepsilon \mathbf{U}_i).\mathbf{U}_j) = \mathbf{U}_j - \varepsilon[\mathbf{U}_i, \mathbf{U}_j] + \frac{\varepsilon^2}{2!}[\mathbf{U}_i, [\mathbf{U}_i, \mathbf{U}_j]] - \cdots, \tag{24}$$

where the symbol ε denotes a real number, and $[\mathbf{U}_i, \mathbf{U}_j]$ denotes the Lie product, which is defined as follows

$$[\mathbf{U}_i, \mathbf{U}_i] = \mathbf{U}_i\mathbf{U}_j - \mathbf{U}_j\mathbf{U}_i. \tag{25}$$

The commutator Table for the Lie algebra (23) is given in Table 1.

Optimal system.

In this scenario, the algebra is three-dimensional, characterized by non-zero commutators

$$[\mathbf{U}_1, \mathbf{U}_3] = \mathbf{U}_1 - \frac{\alpha \mathbf{U}_3}{\beta}, \; [\mathbf{U}_2, \mathbf{U}_3] = \mathbf{U}_3.$$

Table 2 illustrates the adjoint table, aiding in the computation of the optimal system of one-dimensional subalgebras.

Table 1: Commutator Table

$[.,.]$	$\mathbf{U}_1$	$\mathbf{U}_3$	$\mathbf{U}_3$
$\mathbf{U}_1$	0	$\mathbf{U}_1 - \frac{\alpha \mathbf{U}_3}{\beta}$	0
$\mathbf{U}_2$	$-\mathbf{U}_1 + \frac{\alpha \mathbf{U}_3}{\beta}$	0	$\mathbf{U}_3$
$\mathbf{U}_3$	0	0	$-\mathbf{U}_3$

Table 2: Adjoint Table

$Adj(e^{\varepsilon})$	$\mathbf{U}_1$	$\mathbf{U}_2$	$\mathbf{U}_3$
$\mathbf{U}_1$	$\mathbf{U}_1$	$-\varepsilon\mathbf{U}_1 + \mathbf{U}_2 + \frac{\alpha\varepsilon\mathbf{U}_3}{\beta}$	$\mathbf{U}_3$
$\mathbf{U}_2$	$e^{\varepsilon}\mathbf{U}_1 + \frac{1}{2}\frac{\alpha(e^{-\varepsilon}-e^{\varepsilon})\mathbf{U}_3}{\beta}$	$\mathbf{U}_2$	$e^{-\varepsilon}\mathbf{U}_3$
$\mathbf{U}_3$	$\mathbf{U}_1$	$\mathbf{U}_3\varepsilon + \mathbf{U}_2$	$\mathbf{U}_3$

Let us consider an arbitrary element $\mathcal{S}$ belonging to the symmetry algebra L_3, defined as

$$\mathcal{S} = \mu_1\mathbf{U}_1 + \mu_2\mathbf{U}_2 + \mu_3\mathbf{U}_3 \tag{26}$$

Case-I. For the case $\mu_3 \neq 0, \; \mu_2 = 0, \mu_1 \neq 0$, we have

$$\mathcal{S} = \mu_1\mathbf{U}_1 + \mu_3\mathbf{U}_3. \tag{27}$$

Possible actions in this case are $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$. The adjoint action by $\mathbf{U}_2$ results as

$$\mathcal{S}' = \mathrm{Adj}(e^{\varepsilon}\mathbf{U}_2)\mathcal{S} = \mathbf{U}_1 + \frac{1}{\mu_1}e^{-2\varepsilon}[1 - \frac{\alpha}{2\beta}(e^{-2\varepsilon} - 1)]\mathbf{U}_3. \tag{28}$$

For $\varepsilon = -\frac{1}{2}\ln\left(\frac{2\beta}{\alpha} + 1\right)$, we get the element

$$\mathcal{S}_1 = \langle \mathbf{U}_1 \rangle.$$

Case-II. For the case $\mu_3 \neq 0,\ \mu_2 \neq 0, \mu_1 = 0$, we have

$$\mathcal{S} = \mu_2\mathbf{U}_2 + \mu_3\mathbf{U}_3. \tag{29}$$

Possible actions in this case are $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$. The adjoint action by $\mathbf{U}_3$ results as

$$\mathcal{S}' = \mathrm{Adj}(e^{\varepsilon}\mathbf{U}_3)\mathcal{S} = \mathbf{U}_2 + [\varepsilon + \frac{\mu_3}{\mu_2}]\mathbf{U}_3. \tag{30}$$

For $\varepsilon = -\frac{\mu_3}{\mu_2}$, we get the element

$$\mathcal{S}_2 = \langle \mathbf{U}_2 \rangle.$$

Case-III. For the case $\mu_3 \neq 0,\ \mu_1 = 0,\ \mu_2 = 0$,

$$\mathcal{S} = \mu_3\mathbf{U}_3. \tag{31}$$

Without any adjoint action, we get following the non-similar class

$$\mathcal{S}_3 = \langle \mathbf{U}_3 \rangle.$$

In the remaining cases, we get the aforementioned results. Therefore, the optimal system can be represented by a set of symmetry subalgebras described as follows.

$$\begin{aligned} \mathcal{S}_1 &= \langle \mathbf{U}_1 \rangle, \\ \mathcal{S}_2 &= \langle \mathbf{U}_2 \rangle, \\ \mathcal{S}_3 &= \langle \mathbf{U}_3 \rangle. \end{aligned} \tag{32}$$

Reduction by subalgebra $\mathcal{S}_1 = \langle \mathbf{U}_1 \rangle$.
For symmetry $\mathbf{U}_1$, one can write

$$\frac{dt}{0} = \frac{dx}{1} = \frac{d\Phi}{0}.$$

This gives the following similarity transformation

$$\varkappa = t, \quad \Phi(x,t) = h(\varkappa). \tag{33}$$

Putting these values in Equation (2), we get

$$D_t^{\rho}\Phi = 0,$$

which gives

$$\Phi(x,t) = k_1 t^{\rho-1}, \quad k_1 > 0. \tag{34}$$

Reduction by subalgebra $\mathcal{S}_2 = \langle \mathbf{U_2} \rangle$.
In particular, for the symmetry $\mathbf{U_2}$, we also get

$$\frac{\rho dt}{3t} = \frac{dx}{x} = \frac{d\Phi}{-(\Phi + \frac{\alpha x}{\beta})},$$

and this gives similarity transformation and similarity variables

$$\varkappa = xt^{-\frac{\rho}{3}}, \quad \Phi(x,t) = -\frac{\alpha x}{\beta} + t^{-\frac{\rho}{3}} G(\varkappa), \tag{35}$$

where G is a function of independent variable $\varkappa$. Inserting Equation (35) in Equation (2), we have a special non-linear FODE.

Now, we have an important theorem given below

THEOREM 2. *The FSP-KdV Equation (2) reduces to a nonlinear fractional ODE by using the Equation (35) given by*

$$\left(\mathcal{P}_{\frac{3}{\rho}}^{1-\frac{4\rho}{3},\rho} G\right)(\varkappa) - \alpha t^{\frac{2\rho}{3}} G_\varkappa + \beta (G_\varkappa)^2 - \gamma G_{\varkappa\varkappa\varkappa} = 0, \tag{36}$$

with the Erdélyi-Kober (EK) fractional differential operator defined by

$$\left(\mathcal{P}_\Upsilon^{\tau,\rho} G\right)(\varkappa) = \prod_{j=0}^{m-1} \left(\tau + j - \frac{1}{\Upsilon}\varkappa\frac{d}{d\varkappa}\right)\left(\mathcal{K}_\Upsilon^{\tau+\rho,m-\rho} G\right)(\varkappa), \tag{37}$$

$$m = \begin{cases} [\rho]+1, & \rho \notin \mathbb{N}, \\ \rho, & \rho \in \mathbb{N}, \end{cases} \tag{38}$$

such that

$$\left(\mathcal{K}_\Upsilon^{\tau,\rho} G\right)(\varkappa) = \begin{cases} \frac{1}{\Gamma(\rho)} \int\limits_1^\infty (u-1)^{\rho-1} u^{-(\tau+\rho)} G(\varkappa u^{\frac{1}{\Upsilon}}) du, & \rho > 0 \\ G(\varkappa), & \rho = 0, \end{cases} \tag{39}$$

is the EK fractional integral operator [8].

Proof 2. Suppose that $m-1 < \rho < m$, $m = 1, 2, 3, ...$ and order(m)=order(p) where $p \in \mathbb{R}$. By taking RL definition into account in Equation (33), one reaches

$$\frac{\partial^\rho \Phi}{\partial t^\rho} = \frac{\partial^q}{\partial t^q}\left[\frac{1}{\Gamma(q-\rho)} \int\limits_0^t (t-\varpi)^{q-\rho-1} \varpi^{\frac{-\rho}{3}} G\left(x\varpi^{\frac{-\rho}{3}}\right) d\varpi\right]. \tag{40}$$

Substitute $r = t/\varpi \Rightarrow d\varpi = -(t/r^2)dr$, then Equation (40) can be written as:

$$\frac{\partial^\rho \Phi}{\partial t^\rho} = \frac{\partial^q}{\partial t^q}\left[t^{q-\frac{4\rho}{3}} \frac{1}{\Gamma(q-\rho)} \int\limits_1^\infty (r-1)^{q-\rho-1} r^{-(1-\frac{\rho}{3}+q-\rho)} G\left(\varkappa r^{\frac{\rho}{3}}\right) dr\right], \tag{41}$$

by using the EK fractional integration Equation (39) in Equation (41), we have:

$$\frac{\partial^\rho \Phi}{\partial t^\rho} = \frac{\partial^q}{\partial t^q}\left[t^{q-\frac{4\rho}{3}}\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right]. \tag{42}$$

Now simplifying the right hand side of Equation (42), one can get

$$\begin{aligned}\frac{\partial^\rho \Phi}{\partial t^\rho} &= \frac{\partial^{q-1}}{\partial t^{q-1}}\left[\frac{\partial}{\partial t}\left(t^{q-\frac{4\rho}{3}}\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},n-\rho}G\right)(\varkappa)\right)\right],\\ &= \frac{\partial^{q-1}}{\partial t^{q-1}}\left[t^{q-1-\frac{4\rho}{3}}\left(q-\frac{4\rho}{3}-\frac{\rho}{3}\varkappa\frac{d}{d\varkappa}\right)\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right],\end{aligned} \tag{43}$$

repeat order $q-1$ times yields

$$\begin{aligned}&\frac{\partial^p}{\partial t^p}\left[t^{q-\frac{4\rho}{3}}\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right]\\ &\quad= \frac{\partial^{q-1}}{\partial t^{q-1}}\left[\frac{\partial}{\partial t}\left(t^{q-\frac{4\rho}{3}}\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right)\right],\\ &\quad= \frac{\partial^{q-1}}{\partial t^{q-1}}\left[t^{q-1-\frac{4\rho}{3}}\left(q-\frac{4\rho}{3}-\frac{\rho}{3}\varkappa\frac{d}{d\varkappa}\right)\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right],\\ &\quad\;.\\ &\quad\;.\\ &\quad= t^{-\frac{4\rho}{3}}\prod_{j=0}^{m-1}\left[\left(1-\frac{4\rho}{3}+j-\frac{\rho}{3}\varkappa\frac{d}{d\varkappa}\right)\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right],\end{aligned}$$

applying the EK fractional derivative Equation (37), we get

$$\frac{\partial^q}{\partial t^q}\left[t^{q-\frac{4\rho}{3}}\left(\mathcal{K}_{\frac{3}{\rho}}^{1-\frac{\rho}{3},q-\rho}G\right)(\varkappa)\right] = t^{-\frac{4\rho}{3}}\left(\mathcal{P}_{\frac{3}{\rho}}^{1-\frac{4\rho}{3},\rho}G\right)(\varkappa). \tag{44}$$

Putting Equation (44) in to Equation (42), we get

$$\frac{\partial^\rho \Phi}{\partial t^\rho} = t^{-\frac{4\rho}{3}}\left(\mathcal{P}_{\frac{3}{\rho}}^{1-\frac{4\rho}{3},\rho}G\right)(\varkappa), \tag{45}$$

thus, Equation (2) is converted into the following fractional ODE

$$\left(\mathcal{P}_{\frac{3}{\rho}}^{1-\frac{4\rho}{3},\rho}G\right)(\varkappa) - \alpha t^{\frac{2\rho}{3}}G_\varkappa + \beta(G_\varkappa)^2 - \gamma G_{\varkappa\varkappa\varkappa} = 0. \tag{46}$$

This completes the theorem. □

5. Series solutions for the FSP-KdV Equation (2). In this section, we aim to deduce explicit solutions for the FSP-KdV Equation (2). To achieve this goal, we introduce the following transformation

$$\Phi(x,t) = \varrho(\nu), \qquad \nu = rx - \frac{\kappa t^\rho}{\Gamma(1+\rho)}, \tag{47}$$

where r and κ are constants. By substituting Equation (47) in Equation (2), we obtain a nonlinear ODE

$$(\alpha r-\kappa)\varrho'+\beta r^2(\varrho')^2+\gamma r^3\varrho'''=0. \tag{48}$$

Assume that

$$\varrho\prime(\nu)=\mathcal{V}(\nu). \tag{49}$$

Now, Equation (48) becomes

$$(\alpha r-\kappa)\mathcal{V}+\beta r^2(\mathcal{V})^2+\gamma r^3\mathcal{V}''=0. \tag{50}$$

Suppose that solution of Equation (50) is

$$\mathcal{V}(\nu)=\sum_{s=0}^{\infty}c_s\nu^s, \tag{51}$$

where c_s are coefficients of the above series to be evaluated later. Now, putting Equation (51) into Equation (50), we have the following relation

$$\begin{aligned}(\alpha r-\kappa)\sum_{s=0}^{\infty}c_s\nu^s+\beta r^2\sum_{s=0}^{\infty}\sum_{u=0}^{s}c_uc_{s-u}\nu^s+\\ \gamma r^3\sum_{s=0}^{\infty}(s+1)(s+2)c_{s+2}\nu^s=0,\end{aligned} \tag{52}$$

from Equation (52), comparing coefficients for $s=0$, one reaches

$$c_2=\frac{-(\alpha r-\kappa)}{2\gamma r^3}c_0-\frac{\beta}{2\gamma r}c_0^2, \tag{53}$$

where c_0 is an arbitrary constant also $r\neq 0$. Generally, for $s\geq 1$, we can get

$$c_{s+2}=\frac{-1}{\gamma r^3(s+1)(s+2)}\left[(\alpha r-\kappa)c_s+\beta r^2\sum_{u=0}^{s}c_uc_{s-u}\right], \tag{54}$$

where from Equation (53) and Equation (54), we can evaluate any coefficient c_{s+2} for $s\geq 1$ of Equation (51), in which the r, β, γ and c_1 are real constants.
The power series solution for Equation (50) can be written as

$$\mathcal{V}(\nu)=c_0+c_1\nu+\sum_{s=2}^{\infty}c_s\nu^s. \tag{55}$$

Hence, by following above expression the explicit profile of Equation (2) can be evaluated easily.

$$\Phi(x,t) = \int \mathcal{V}(\nu)d\nu = \int c_0 d\nu + \int c_1 \nu d\nu + \int \sum_{s=2}^{\infty} c_s \nu^s d\nu,$$

$$\Phi(x,t) = c_0\nu + c_1\frac{\nu^2}{2} + \sum_{s=2}^{\infty} c_s \frac{\nu^{s+1}}{s+1} = c_0\nu + c_1\frac{\nu^2}{2} + c_2\frac{\nu^3}{3} + \sum_{s=1}^{\infty} c_{s+2}\frac{\nu^{s+3}}{s+3},$$

$$= c_0\left(rx - \frac{\kappa t^\rho}{\Gamma(1+\rho)}\right) + \frac{c_1}{2}\left(rx - \frac{\kappa t^\rho}{\Gamma(1+\rho)}\right)^2 - \frac{1}{3}\left(\frac{\alpha r-\kappa}{2\gamma r^3}c_0 + \frac{\beta}{2\gamma r}c_0^2\right) \quad (56)$$

$$\left(rx - \frac{\kappa t^\rho}{\Gamma(1+\rho)}\right)^3 + \sum_{s=1}^{\infty} \frac{-1}{\gamma r^3(s+1)(s+2)(s+3)}\Bigg[(\alpha r-\kappa)c_s +$$

$$\beta r^2 \sum_{u=0}^{s} c_u c_{s-u}\Bigg]\left(rx - \frac{\kappa t^\rho}{\Gamma(1+\rho)}\right)^{s+3}.$$

5.1. The physical explanation of the explicit solution for FSP-KdV Equation (2). For the sake of analysis of properties of power series solutions, we plot different dimensional graphics of solution (56) shown in Figures (1-5) by taking appropriate values for parameters.

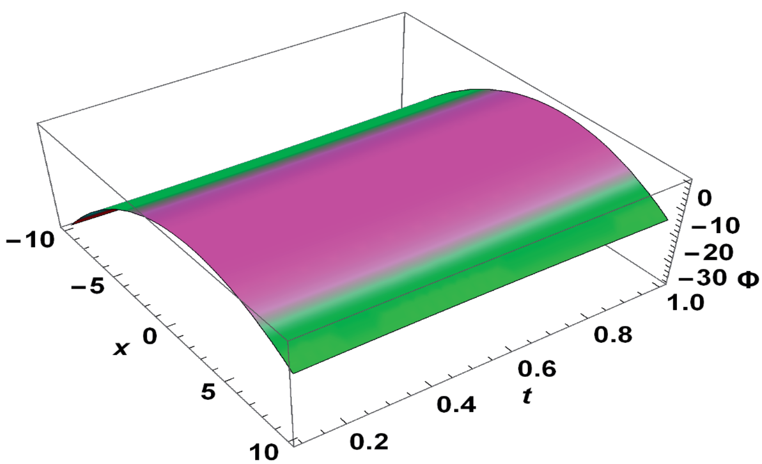

Figure 1: The influence of ρ on solitary wave profile (56) of Equation (2) by fixing the value of time $t = 1$ with constants $c_0 = 1.8$, $r = 0.3$, $c_1 = 5.55$, $\kappa = 0.4$, $\alpha = \beta = 1$ and $\gamma = 1$.

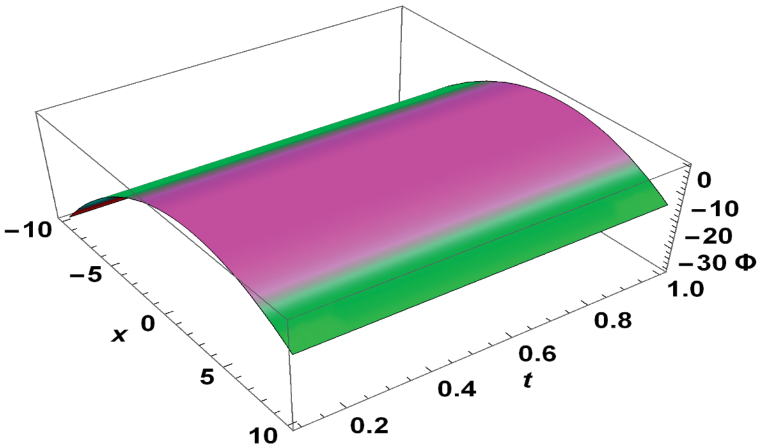

Figure 2: Solitary wave solution (56) of Equation (2) when $\rho = 0.2$, $m = 4$ and same parameter values.

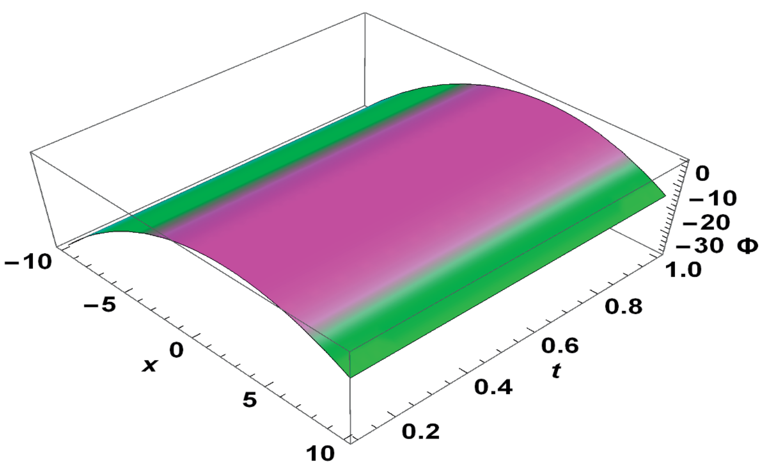

Figure 3: Solitary wave solution (56) of Equation (2) when $\rho = 0.5$, $m = 4$ and same parameter values.

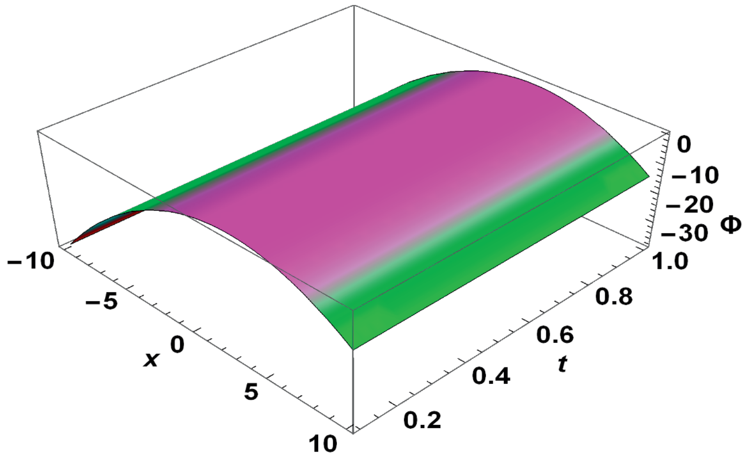

Figure 4: Solitary wave profile (56) of Equation (2) when $\rho = 0.9$, $m = 4$ and same parameter values.

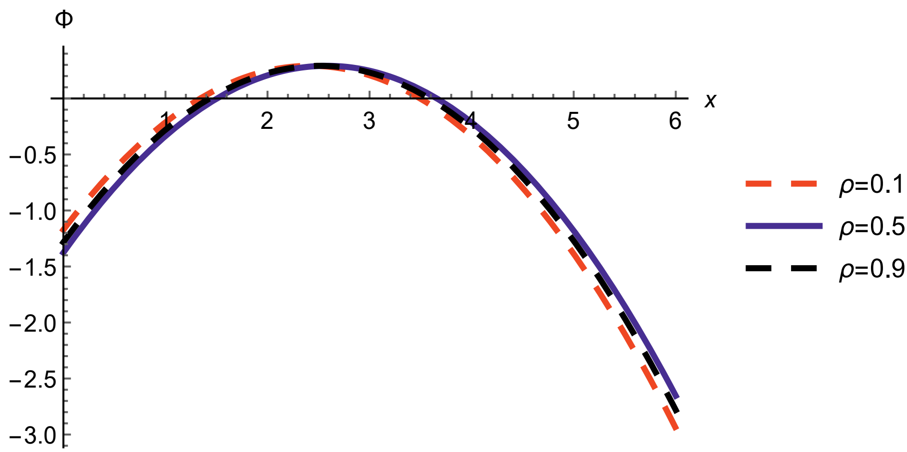

Figure 5: The corresponding $2D$ plot at $t = 0.5$ of Fig. 1 reveals that by increasing the value of ρ, the amplitude of the wave decreases which means that the solitary wave profile looks like a dynamical system.

5.2. Convergence analysis of the explicit structures. In this analysis, we examine how the explicit solution (51), with coefficients provided by Equation (53) and Equation (54), converges. Equation (54) takes the following form

$$|c_{s+2}| \leq K\left[|c_s| + \sum_{u=0}^{s} |c_u||c_{s-u}|\right], \tag{57}$$

where $K = \max\{\frac{\alpha r-\kappa}{\gamma r^3}, \frac{\beta}{\gamma r}\}$. Now, consider another power series

$$\mathcal{W}(\nu) = \sum_{s=0}^{\infty} w_s \nu^s, \tag{58}$$

with $w_i = |c_i|$, $i = 0$ so, we one can write

$$w_{s+2} = K\left[|w_s| + \sum_{u=0}^{s} |w_u||w_{s-u}|\right], \tag{59}$$

where $s = 0, 1, 2, \cdots$ and it is clear that

$$|c_{s+2}| \leq w_{s+2} \quad \Rightarrow \quad |c_s| \leq w_s,$$

the series given by Equation (58) is a majorant series of Equation (51). Now, we have to show that series $\mathcal{W} = \mathcal{W}(\nu)$ whose radius of convergence is positive. Then, Equation (58) takes the following form

$$\begin{aligned} \mathcal{W}(\nu) &= w_0 + w_1\nu + w_2\nu^2 + \sum_{3}^{\infty} w_s \nu^s, \\ &= w_0 + w_1\nu + w_2\nu^2 + w_3\nu^3 + K\sum_{s=1}^{\infty} w_s \nu^{s+2} + K\sum_{s=1}^{\infty}\sum_{u=0}^{s} w_u w_{s-u} \nu^{s+2}. \end{aligned} \tag{60}$$

Next, we will prove that Equation (58) possesses a positive radius of convergence. Now, considering an implicit functional system concerning ν, we get

$$\delta(\nu, \mathcal{W}) = \mathcal{W} - w_0 - w_1\nu - K[\mathcal{W} - w_0]\nu^2 - K[\mathcal{W}^2 - w_0^2]\nu^2, \tag{61}$$

since δ is an analytic function in the neighborhood of $(0, w_0)$ where $\delta(0, w_0) = 0$ and $\frac{\partial \delta}{\partial W}(0, w_0) \neq 0$.

THEOREM 3. ([25]) *Suppose that f be a $\Xi-$mapping of an open set $A \subset \mathbb{P}^{m+n}$ into $\mathbb{P}^n$, such that $h(a, b) = 0$ for some point $(a, b) \in E$. Put $B = h'(a, b)$ and suppose that B_x is invertible. Then $\exists$ an open sets $X \subset \mathbb{P}^{m+n}$ and $V \subset \mathbb{P}^m$ with $(a, b) \in X$ and $b \in V$ which have the following properties:*
(i) To each $y \in V$, corresponds a unique x such that $(x, y) \in X$ and $h(x, y) = 0$,
(ii) If this x is defined to be $f(y)$, then

$$\begin{gathered} f(b) = a, \\ h(f(y), y) = 0 \quad (y \in V), \\ f'(b) = -(B_x)^{-1}B_y, \end{gathered}$$

where f is a Ξ-mapping of V into $\mathbb{P}^n$.
(iii) The function f is implicitly given by (ii).

One can see that $\mathbb{P} = \mathbb{P}(\xi)$ is analytic in the neighborhood of point $(0, w_0)$ and has a positive radius. It has cleared from the above discussion that the power series Equation (51) converges in the neighborhood of the point $(0, w_0)$.

6. Local conservation laws for FSP-KdV Equation (2). In this section, we calculate conservation laws for FSP-KdV Equation (2) based on formal Lagrangian and Lie symmetries. A conservation law or conserved flux satisfies the condition

$$\left[D_t(\hat{\mathcal{T}}^t) + D_x(\hat{\mathcal{T}}^x)\right]_{Equation(2)} = 0, \tag{62}$$

where $\hat{\mathcal{T}}^t$ and $\hat{\mathcal{T}}^x$ are conserved quantities. Based on the results given in [6, 10, 8], Equation (2) has the formal Lagrangian of the following form

$$\mathcal{L} = \omega(x,t)[D_t^\rho \Phi + \alpha\Phi_x + \beta(\Phi_x)^2 + \gamma\Phi_{xxx}], \tag{63}$$

where introduce $\omega(x,t)$ as new dependent variable. By considering Equation (63), we can write action integral of the form

$$\int_0^t \int_\Omega \mathcal{L}(x,t,\Phi,\omega,D_t^\rho\Phi,\Phi_x,\Phi_{xxx})dxdt. \tag{64}$$

Now, the Euler-Lagrange operator is of the form

$$\frac{\delta}{\delta\Phi} = \frac{\partial}{\partial\Phi} + (D_t^\rho)^* \frac{\partial}{\partial D_t^\rho \Phi} - D_x\frac{\partial}{\partial\Phi_x} - D_x^3\frac{\partial}{\partial\Phi_{xxx}}, \tag{65}$$

where $(D_t^\rho)^*$ represents adjoint operator of (D_t^ρ) and we can define the adjoint equation for as

$$\frac{\delta\mathcal{L}}{\delta\Phi} = 0. \tag{66}$$

Adjoint operator $(D_t^\rho)^*$ for RL has following form

$$(D_t^\rho)^* = (-1)^m I_T^{m-\rho}(D_t^m) = {}_t^C D_T^\rho, \tag{67}$$

where $I_T^{m-\rho}$ has following form

$$I_T^{m-\rho}h(x,t) = \frac{1}{\Gamma(m-\rho)}\int_t^\tau \frac{h(x,\tau)}{(\tau-t)^{\rho-m+1}}d\tau. \tag{68}$$

Now, taking dependent variable Φ and independent variables t, x and y we get

$$\hat{\mathbf{U}} + \mathcal{D}_t(\vartheta^t)\hat{I} + \mathcal{D}_x(\vartheta^x)\hat{I} = W\frac{\delta}{\delta\Phi} + \mathcal{D}_t(\hat{\mathcal{T}}^t) + \mathcal{D}_x(\hat{\mathcal{T}}^x), \tag{69}$$

where $\frac{\delta}{\delta\Phi}$ is EL-operator and $\hat{I}$ is known to be an identity operator. So $\hat{\mathbf{U}}$ takes the form

$$\hat{\mathbf{U}} = \vartheta^t\frac{\partial}{\partial t} + \vartheta^x\frac{\partial}{\partial x} + \Phi_\rho^0\frac{\partial}{\partial D_t^\rho\Phi} + \Phi^x\frac{\partial}{\partial\Phi_x} + \Phi^{xxx}\frac{\partial}{\partial\Phi_{xxx}}, \tag{70}$$

and W is the characteristic function and its value is $W = \Phi - \vartheta^t\Phi_t - \vartheta^x\Phi_x$. For $\mathbf{U}_i$, the function W_i follows

$$W_1 = -\Phi_x, \qquad W_2 = -(\Phi + \frac{\alpha x}{\beta} + x\Phi_x + \frac{3}{\rho}t\Phi_t), \qquad W_3 = 1. \tag{71}$$

By considering RL-fractional derivative, the density component $\hat{\mathcal{T}}^t$ of conserved quantity is given by

$$\hat{\mathcal{T}}^t = \vartheta^t \mathcal{L} + \sum_{s=0}^{m-1} (-1)^s_0 D_t^{\rho-1-s}(W_\iota) D_t^s \frac{\partial \mathcal{L}}{\partial_0 D_t^\rho \Phi} - (-1)^m J\left(W_\iota, D_t^n \frac{\partial \mathcal{L}}{\partial_0 D_t^\rho \Phi}\right), \quad (72)$$

where $J(.)$ is defined by

$$J(f,g) = \frac{1}{\Gamma(m-\rho)} \int_0^t \int_t^\tau \frac{f(\tau,x)g(\kappa,x)}{(\kappa-\tau)^{\rho-m+1}} d\kappa d\tau, \quad (73)$$

and the flux components $\hat{\mathcal{T}}^x$ for independent variables x defined as

$$\begin{aligned}\hat{\mathcal{T}}^x =& \vartheta^x \mathcal{L} + W_\iota \left[\frac{\partial \mathcal{L}}{\partial \Phi_j^\iota} - D_j(\frac{\partial \mathcal{L}}{\partial \Phi_{jk}^\iota}) + D_j D_k(\frac{\partial \mathcal{L}}{\partial \Phi_{jkl}^\iota}) - ...\right] \\ &+ D_j(W_\iota)\left[\frac{\partial \mathcal{L}}{\partial \Phi_{jk}^\iota} - D_k(\frac{\partial \mathcal{L}}{\partial \Phi_{jkl}^\iota}) + ...\right] \\ &+ D_j D_k(W_\iota)\left[\frac{\partial \mathcal{L}}{\partial \Phi_{jkl}^\iota} - ...\right] + ...,\end{aligned} \quad (74)$$

where $j,k,l = 1,2$ and $\iota = 1,2,3$.

(Case: 1) For $\mathbf{U}_1$, we have $W_1 = -\Phi_x$ and $\vartheta^x = 1$. Substituting these values in Equation (72-74), we find

$$\begin{aligned}\hat{\mathcal{T}}^t &= \omega D_t^{\rho-1}(-\Phi_x) + J(-\Phi_x, \omega_t), \\ \hat{\mathcal{T}}^x &= \omega[D_t^\rho \Phi - \beta(\Phi_x)^2 - \gamma\Phi_{xx} + \gamma\Phi_{xxx}] - \gamma\Phi_x\omega_{xx} + \gamma\Phi_{xx}\omega_x.\end{aligned}$$

(Case: 2) For $\mathbf{U}_2$, we have $W_2 = -(\Phi + \frac{\alpha x}{\beta} + x\Phi_x + \frac{3}{\rho}t\Phi_t)$, $\vartheta^t = \frac{3}{\rho}t$ and $\vartheta^x = x$. Substituting these values in Equation (72-74), we get:

$$\begin{aligned}\hat{\mathcal{T}}^t =& \frac{3}{\rho}t\omega\mathcal{L} - \omega D_t^{\rho-1}(\Phi + \frac{\alpha x}{\beta} + x\Phi_x + \frac{3}{\rho}t\Phi_t) + J(-(\Phi + \frac{\alpha x}{\beta} + x\Phi_x + \frac{3}{\rho}t\Phi_t), \omega_t), \\ \hat{\mathcal{T}}^x =& \omega[xD_t^\rho\Phi - \beta(\Phi_x)^2 - \alpha\Phi - \frac{\alpha^2 x}{\beta} - 2\alpha x\Phi_x - \frac{3\alpha}{\rho}t\Phi_t - \frac{6\beta}{\rho}t\Phi_x\Phi_t - 3\gamma\Phi_{xx} \\ &- \gamma\Phi_{xxx} + \gamma x\Phi_{xxx} - \frac{3\gamma}{\rho}t\Phi_{xxt}] - 2\beta\Phi\Phi_x + \frac{\gamma\alpha}{\beta}\Phi_x - \gamma\Phi\omega_{xx} - \frac{\alpha\gamma}{\beta}x\omega_{xx} \\ &- \gamma x\Phi_x\omega_{xx} - \frac{3\gamma}{\rho}t\Phi_t\omega_{xx} + 2\gamma\Phi_x\omega_x + \gamma x\Phi_{xx}\omega_x + \frac{3\gamma}{\rho}t\Phi_{xt}\omega_x.\end{aligned}$$

(Case: 3) For $\mathbf{U}_3$, we have $W_3 = 1$. Substituting these values in Equation (72-74), we have

$$\begin{aligned}\hat{\mathcal{T}}^t &= \omega D_t^{\rho-1}(1) + J(1, \omega_t), \\ \hat{\mathcal{T}}^x &= \omega(\alpha + 2\beta\Phi_x) + \gamma\omega_{xx}.\end{aligned}$$

7. Discussion and conclusions. In this investigation, we apply classical Lie symmetry group analysis to FRDEs. Specifically, we utilize the fractional Lie symmetries method to explore the fractional symmetry of the FSP-KdV equation described by Equation (2) with RL derivative. Initially, we identify Lie point symmetries, which serve as fundamental elements for the symmetry algebra and are utilized to derive a system of one-dimensional subalgebras. This optimal system was then used to perform symmetry reductions directly and also employed the Erdélyi-Kober fractional differential operator. Subsequently, by leveraging power series theory, we rigorously derived explicit power series solutions for the equation. The physical interpretation of these solutions, as depicted in Figures 1-5, revealed that as the value of ρ increases, the amplitude of the wave decreases, confirming that the solitary wave profile resembles a dynamical system. The properties of power series solutions were further discussed with the aid of the implicit function theorem. Furthermore, we establish conservation laws using the RL operator for the first time in this study.

The results obtained serve as benchmarks for accuracy testing and comparison of numerical results. However, the exploration of symmetry properties in FRDEs is in its early stages, warranting further investigation. For instance, our analysis currently involves only two independent variables (x,t) and one dependent variable (Φ). Extending this analysis to time FDEs with more independent and dependent variables raises questions about deriving nonlocal results. Addressing these issues requires further research to advance our understanding of the symmetry properties of FDEs.

Authors Contribution. A.H.K. and F.D.Z. have given the main idea and was implemented by A.H. and N.Z.

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