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TRIBONACCI NUMBERS THAT ARE PRODUCTS OF TWO FIBONACCI NUMBERS

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ABSTRACT. Let T_m be the mth Tribonacci number and F_n be the nth Fibonacci number. In this paper, we solve the Diophantine equation

$$T_m = F_n F_k$$

in positive integer unknowns m, n, and k.

1. INTRODUCTION

The Tribonacci sequence $\{T_m\}_{m\geq 0}$ is given by $T_0 = 0$, $T_1 = T_2 = 1$, and the recurrence

$$T_{m+3} = T_{m+2} + T_{m+1} + T_m \quad \text{for all } m \geq 0.$$

Its characteristic polynomial is $X^3 - X^2 - X - 1 = (X - \alpha)(X - \beta)(X - \bar{\beta})$, where

$$\alpha = \frac{1 + r_1 + r_2}{3},\ \beta = \frac{2 - (r_1 + r_2) + i\sqrt{3}(r_1 - r_2)}{6},$$

with

$$r_1 = \sqrt[3]{19 + 3\sqrt{33}} \qquad \text{and} \qquad r_2 = \sqrt[3]{19 - 3\sqrt{33}}.$$

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for } n \geq 0.$$

Its characteristic polynomial is $X^2 - X - 1 = (X - \gamma)(X - \delta)$, where

$$\gamma = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \delta = \frac{1 - \sqrt{5}}{2}.$$

In this paper, we study the Diophantine equation

$$T_m = F_n F_k \tag{1}$$

in positive unknowns m, n, and k.

Theorem 1. *The only nonzero Tribonacci numbers that are products of two Fibonacci numbers are*

$$1,\ 2,\ 4,\ 13, \text{ and } 24.$$

Our method of proof involves the application of Baker's lower bounds for nonzero linear forms in logarithms of algebraic numbers, and the Baker-Davenport reduction procedure. Computations are done with the help of a computer program in Mathematica.

2. Preliminary Results

2.1. **Results on Tribonacci and Fibonacci Numbers.** Here, we recall some well-known formulas and inequalities on Tribonacci and Fibonacci sequences. Binet's formula for the Tribonacci sequence is

$$T_m = a\alpha^m + b\beta^m + \bar{b}\bar{\beta}^m \quad \text{for all } m \geq 0,$$

where

$$a = \frac{5\alpha^2 - 3\alpha - 4}{22}, \quad \text{and} \quad b = \frac{5\beta^2 - 3\beta - 4}{22}.$$

The minimal polynomial of a over integers is $44X^3 - 2X - 1$, with zeros a, b, and $\bar{b}$ with $\max\{|a|, |b|, |\bar{b}|\} < 1$. We have the numerical estimates

$$\begin{aligned} 1.83 < \alpha < 1.84; \\ 0.73 < |\beta| = \alpha^{-1/2} < 0.74; \\ 0.33 < |a| < 0.34; \\ 0.25 < |b| < 0.27. \end{aligned} \tag{2}$$

For $m \geq 1$, denoting $e(m) := T_m - a\alpha^m$, we have

$$|e(m)| < \alpha^{-\frac{m}{2}}. \tag{3}$$

Furthermore,

$$\alpha^{m-2} \leq T_m \leq \alpha^{m-1} \qquad \text{for all} \quad m \geq 1. \tag{4}$$

Binet's formula for the Fibonacci sequence is

$$F_n = \frac{\gamma^n - \delta^n}{\sqrt{5}} \quad \text{for all} \quad n \geq 0.$$

One has $\gamma\delta = -1$. Furthermore, for $n \geq 1$, one has

$$\gamma^{n-2} \leq F_n \leq \gamma^{n-1}. \tag{5}$$

For a number field $\mathbb{K}$, let $d_{\mathbb{K}}$ be its degree over $\mathbb{Q}$, also usually denoted by $[\mathbb{K} : \mathbb{Q}]$. Because $d_{\mathbb{Q}(\alpha)} = 3$ and $d_{\mathbb{Q}(\gamma)} = 2$, it follows that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\gamma)$. Further, $\mathbb{Q}(\alpha) = \mathbb{Q}(a)$. The numbers α, γ, and a are positive and belong to the real field $\mathbb{K} = \mathbb{Q}(\alpha, \gamma)$ of degree $d_{\mathbb{K}} = 6$.

2.2. **Auxiliary Results on Linear Forms in Logarithms of Algebraic Numbers.** In this subsection, we point out some useful results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. Let $\eta \neq 0$ be an algebraic number of degree d and let

$$a_0(X - \eta^{(1)}) \cdots (X - \eta^{(d)}) \in \mathbb{Z}[X]$$

be the minimal polynomial over $\mathbb{Z}$ of $\eta = \eta^{(1)}$ with positive leading coefficient $a_0 \geq 1$. Then, the absolute logarithmic Weil height is defined by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \max\{0, \log|\eta^{(i)}|\}\right).$$

The height has the following basic properties. For algebraic numbers η, γ, and $s \in \mathbb{Z}$, we have

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{6}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{7}$$

$$h(\eta^s) = |s|\, h(\eta) \quad (s \in \mathbb{Z}). \tag{8}$$

In the case that η is a rational number, say $\eta = p/q \in \mathbb{Q}$ with coprime integers p, $q \geq 1$, we have $h(p/q) = \max\{\log|p|, \log q\}$.

Let $\mathbb{K}$ be a real number field, $\eta_1, \ldots, \eta_t \in \mathbb{K}$ and $b_1, \ldots, b_t \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_t|\}$ and put

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1.$$

Let A_1, ..., A_t be real numbers with

$$A_i \geq \max\{d_{\mathbb{K}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, t.$$

With these basic notations, we have the following version of a lower bound for a nonzero linear form in logarithms from Matveev [6].

Theorem 2. [3, Theorem 9.4] *Assume that $\Lambda \neq 0$. Then,*

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_{\mathbb{K}}^2 \cdot (1 + \log d_{\mathbb{K}}) \cdot (1 + \log B) \cdot A_1 \cdots A_t.$$

We also need the following lemma.

Lemma 1. [5, Lemma 7] *If $l \geq 1$, $H > \left(4l^2\right)^l$, and $H > L/(\log L)^l$, then*

$$L < 2^l H (\log H)^l.$$

Applying these results to some appropriate linear forms in logarithms resulting from equation (1), we end up with a large upper bound on $\max\{k, m, n\}$, which we need to reduce. For that, we use the following result of Dujella and Pethő [4], which is a variant of the reduction method from Baker and Davenport [1].

Lemma 2. *Let M be a positive integer, p/q be a convergent of the continued fraction expansion of the irrational number τ such that $q > 6M$, and A, B, and μ be some real numbers with $A > 0$ and $B > 1$. If*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\| > 0,$$

then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. The Proof of the Main Result

Let (m, n, k) be a solution of Diophantine equation (1). We suppose $2 \leq n \leq k$ and $m \geq 2$ because $F_1 = F_2 = T_1 = T_2 = 1$. We assume that $n \geq 3$, because for $n = 2$, we get $T_m = F_k$ and the only common terms of the Fibonacci and Tribonacci sequence are 1, $2 = F_3 = T_3$, and $13 = F_7 = T_6$ (see, for example, Theorem 1 in [2] for a more general result). From now on, we assume $k \geq n \geq 3$ and $m \geq 4$.

From (4) and (5), we have

$$\alpha^{m-2} < T_m = F_n F_k < \gamma^{n+k-2} \quad \text{and} \quad \gamma^{n+k-4} < \alpha^{m-1}.$$

This implies that

$$\frac{\log\gamma}{\log\alpha}(n+k) - 2.2 < m < \frac{\log\gamma}{\log\alpha}(n+k) + 0.5. \tag{9}$$

Furthermore, from equation (1), we have

$$\begin{aligned}\left|\frac{5a\alpha^m}{\gamma^{n+k}}-1\right| &= \left|-5e(m)\gamma^{-(n+k)}-(-1)^n\delta^{2n}-(-1)^k\delta^{2k}+(-1)^{n+k}\delta^{2(n+k)}\right| \\ &< 5|e(m)|\gamma^{-(n+k)}+|\delta|^{2n}+|\delta|^{2k}+|\delta|^{2(n+k)} \\ &< 5\alpha^{\frac{-m}{2}}\gamma^{-(n+k)}+\gamma^{-2n}+\gamma^{-2k}+\gamma^{-2(n+k)} \\ &< (5\alpha^{-1}+3)\gamma^{-2n},\end{aligned}$$

because $m \geq 4$ and $k \geq n$. We thus obtain

$$\left|\frac{5a\alpha^m}{\gamma^{n+k}}-1\right| < 5.74\gamma^{-2n}. \tag{10}$$

Let $\Lambda_1 := 5a\alpha^m\gamma^{-n-k}-1$. We have $\Lambda_1 \neq 0$. To see this, if $\Lambda_1 = 0$, then

$$5a\cdot\alpha^m\cdot 2^{n+k} = (1+\sqrt{5})^{n+k} \in \mathbb{Q}(\sqrt{5}),$$

which is false ($a\alpha^m$ is a real number larger than 1 that has two complex conjugates of absolute values smaller than 1, which cannot be in the normal real field $\mathbb{Q}(\sqrt{5})$).

We can now apply Theorem 2 to Λ_1 with $t := 3$ and

$$(\eta_1,\, b_1) := (\alpha,\, m), \quad (\eta_2,\, b_2) := (\gamma,\, -n-k), \quad \text{and} \quad (\eta_3,\, b_3) := (5a, 1).$$

We compute upper bounds on the absolute logarithmic Weil height of each of the above algebraic numbers and we get

$$\begin{aligned}h(\eta_1) &= h(\alpha) < 0.204, \\ h(\eta_2) &= h(\gamma) < 0.241, \\ h(\eta_3) &= h(5a) \leq \log 5 + h(a) < 2.871.\end{aligned} \tag{11}$$

We have $d_{\mathbb{K}} = 6$ and we can choose

$$A_1 = 1.23, \quad A_2 = 1.45, \quad A_3 = 17.23, \quad \text{and} \quad B = n+k.$$

We then get

$$\log|\Lambda_1| > -8.85\cdot 10^{14}\log(n+k).$$

In the above, we used that $1+\log(n+k) < 2\log(n+k)$ because $n+k \geq 4$. Combining the above bound with inequality (10), we obtain

$$2n\log\gamma < 8.85\cdot 10^{14}\log(n+k)+\log 5.74 < 8.85\cdot 10^{14}\log(n+k)+1.748,$$

so

$$n\log\gamma < 4.43\cdot 10^{14}\log(n+k). \tag{12}$$

We rewrite equation (1) as

$$a\alpha^m + e(m) = F_n\left(\frac{\gamma^k-\delta^k}{\sqrt{5}}\right).$$

Because $n \geq 3$, we have

$$\begin{aligned}\left|\frac{a\alpha^m}{F_n} - \frac{\gamma^k}{\sqrt{5}}\right| &< \frac{|e(m)|}{F_n} + \frac{|\delta|^k}{\sqrt{5}} \\ \left|\frac{\sqrt{5}a\alpha^m}{F_n\gamma^k} - 1\right| &< \frac{\sqrt{5}}{\gamma^k}\left(\frac{1}{F_n} + \frac{1}{\sqrt{5}\gamma^k}\right) \\ &< \left(\frac{\sqrt{5}}{2} + \frac{1}{\gamma^3}\right)\gamma^{-k},\end{aligned}$$

because $k \geq 3$ and $F_n \geq 2$. Thus, putting

$$\Lambda_2 := (\sqrt{5}a/F_n)\alpha^m\gamma^{-k} - 1,$$

we obtain

$$|\Lambda_2| < 1.4\gamma^{-k}. \tag{13}$$

Of course, we have $\Lambda_2 \neq 0$ again because $a\alpha^m \notin \mathbb{Q}(\sqrt{5})$. So, we can apply Theorem 2 again to Λ_2 with $t := 3$ and

$$(\eta_1, b_1) := (\alpha, m), \quad (\eta_2, b_2) := (\gamma, -k), \quad \text{and} \quad (\eta_3, b_3) := \left(\frac{\sqrt{5}a}{F_n}, 1\right).$$

We have

$$h\left(\frac{\sqrt{5}a}{F_n}\right) \leq h(\sqrt{5}) + h(a) + \log(F_n) < h(a) + \log(\sqrt{5}) + \log\left(\gamma^{n-1}\right) < 2.5n\log\gamma,$$

where the last inequality above holds because $n \geq 3$. We again have $d_{\mathbb{K}} = 6$ and we can choose

$$A_1 = 1.23, \quad A_2 = 1.45, \quad \text{and} \quad A_3 = 15n\log\gamma,\ B = n + k$$

and get

$$\log|\Lambda_2| > -7.7 \cdot 10^{14} n\log\gamma\log(n+k).$$

Combining this with inequality (13), we obtain

$$k\log\gamma < 7.7 \cdot 10^{14} \cdot n\log\gamma\log(n+k) + \log(1.35)$$

leading to

$$k\log\gamma < 7.71 \cdot 10^{14}(n\log\gamma)\log(n+k). \tag{14}$$

Therefore, considering the upper bound of $n\log\gamma$ from inequality (12), we get

$$k < \left(\frac{(7.71 \cdot 10^{14}) \cdot (4.43 \cdot 10^{14})}{\log\gamma}\right)\log^2(n+k) < 7.1 \cdot 10^{29}\log^2(2k).$$

or

$$2k < 14.2 \cdot 10^{30}\log(2k)^2.$$

Applying Lemma 1 with $l = 2$, $L = 2k$, and $H = 14.2 \cdot 10^{30}$, we obtain

$$k < 1.37 \cdot 10^{34}. \tag{15}$$

We next reduce the above bound by applying Lemma 2. Recall that for a positive real x, if $|x-1| < 0.5$, then $|\log x| < 1.5\,|x-1|$ (see [7, Lemma 4]). Hence, we have from inequality (10) (note that because $n \geq 3$ the right side of inequality (10) is indeed smaller than 0.5),

$$0 < |m\log\alpha - (n+k)\log\gamma + \log(5a)| < 8.61\gamma^{-2n},$$

which implies that

$$0 < \left| m\frac{\log \alpha}{\log \gamma} - (n+k) + \frac{\log(5a)}{\log \gamma} \right| < 17.893\gamma^{-2n}. \tag{16}$$

Note that α and γ are multiplicatively independent. Indeed, if $\alpha^x = \gamma^y$ for integers x and y, then this common value lives in $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\gamma) = \mathbb{Q}$ because $\mathbb{Q}(\alpha)$ has degree 3 and $\mathbb{Q}(\gamma)$ has degree 2. Thus, $\alpha^x = \gamma^y$ is a rational unit so it is ± 1, and this is possible only when $x = y = 0$. Thus, $\log \alpha / \log \gamma$ is an irrational. From inequalities (9) and (15), we have

$$m < \frac{\log \gamma}{\log \alpha} \cdot (2 \cdot k) + 0.5 < \frac{\log(0.5 + 0.5\sqrt{5})}{\log 1.83} \cdot (2 \cdot 13700 + 0.5) \cdot 10^{30};$$

i.e.,

$$m < 2.22 \cdot 10^{34}. \tag{17}$$

We apply Lemma 2 with $w := 2n$,

$$\tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := \frac{\log(5a)}{\log \gamma}, \quad A := 17.893, \quad B := \gamma, \quad \text{and} \quad M := 2.22 \times 10^{34}.$$

With the help of Mathematica, we find that the 71st convergent of τ is

$$\frac{p_{71}}{q_{71}} = \frac{4525445232205414399824110390796611 13}{3573641069135323348796366297377338 70}.$$

Its denominator satisfies $q_{71} > 6M$ and $\varepsilon > 0.247647 > 0$. Thus, inequality (16) has no solution for

$$2n \geq \frac{\log(17.893 q_{71}/\varepsilon)}{\log \gamma} > 179.01,$$

which implies that

$$n \leq 89.$$

Substituting this upper bound for n into inequality (14), we obtain

$$k < 6.9 \cdot 10^{16} \log(n+k) \leq 6.9 \cdot 10^{16} \log(2k).$$

We again apply Lemma 1 with $l = 1$, $H = 13.8 \cdot 10^{16}$, and $L = 2k$ and get

$$k < 5.49 \cdot 10^{18}.$$

From here, because $n \leq k$ and from inequality (9), we have

$$m < 8.75 \cdot 10^{18}.$$

We consider Λ_2 and we get from inequality (13)

$$0 < \left| m \log \alpha - k \log \gamma + \log\left(\frac{\sqrt{5}a}{F_n}\right) \right| < 3\gamma^{-k}.$$

This implies that

$$0 < \left| m\frac{\log \alpha}{\log \gamma} - k + \frac{\log\left(\sqrt{5}a/F_n\right)}{\log \gamma} \right| < 7\gamma^{-k}. \tag{18}$$

We then apply Lemma 2 with $w := k$,

$$\tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := \frac{\log\left(\sqrt{5}a/F_n\right)}{\log \gamma}, \quad A := 7, \quad B := \gamma, \quad M := 8.75 \cdot 10^{18}.$$

With the help of Mathematica, we find that the 41st convergent of τ is

$$\frac{p_{41}}{q_{41}} = \frac{237161759629456603958}{1872812421214946661 47}.$$

It satisfies $q_{41} > 6M$ and $\varepsilon > 0.010633 > 0$ for all $n \in [3, 89]$. Hence, inequality (18) has no solution for

$$k \geq \frac{\log(7q_{41}/\varepsilon)}{\log \gamma} > 110.4.$$

Thus, we obtain $k \leq 110$ and consequently $m \leq 175$ by inequality (9). We now check for the solutions of equation (1) for $3 \leq n \leq 89$, $n \leq k \leq 110$, and $4 \leq m \leq 175$. This is done quickly with a small program in Mathematica and we get

$$(m, k, n) \in \{(4,\ 3,\ 3),\ (7,\ 6,\ 4)\}.$$

This yields the additional Tribonacci numbers $T_4 = 4 = F_3^2$ and $T_7 = 24 = F_4 \cdot F_6$, which are products of two Fibonacci numbers from the statement of Theorem 1. This finishes the proof.

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