Stabilizability preserving quotients for non-linear systems

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I would like to dedicate this thesis firstly to my mother, the person who instilled in me a love for knowledge. Without your sacrifice and courage this would not have been possible, this thesis is for you. To my brothers Tafadzwanashe and Saul I thank you for being by my side through out this journey, you were a constant source of inspiration when the journey got hard.

And Keilah Makanakaishe Chingozha this is for you.

To my father and my sister Linnet I wish you could be here to witness this.

Abstract

The design of feedback stabilizing controllers is an essential component of control engineering theory and practice. A large part of the modern literature in control theory is devoted to coming up with new methods of designing feedback stabilizing controllers. This sustained interest in answering the question "*how to design a feedback stabilizing controller*" has not been accompanied by equal interest in the more existential and fundamental problem of "*when is a system stabilizable by feedback*". As such the theory of control Lyapunov functions still remains the most general framework that characterizes stabilizability as a system property. This approach however simply replaces one elusive and difficult concept (i.e stabilizability) with an equally difficult concept, the existence of control Lyapunov functions. In this thesis we analyse control system stabilizability from the perspective of control system quotients which are generalized control system reductions, the focus being on the propagation of the stabilizability property from the lower order quotient system to the original system.

For the case where the quotient system is a linear controllable system we prove that propagation of the stabilizability property to the original system is possible if the *zero dynamics* of the original system are stable. A novel way of constructing the *zero dynamics* which does not involve the solution of a system of partial differential equations is devised. More generally for analytic non-linear systems given a stabilizable quotient system, we develop a new method of constructing a control Lyapunov function for the original system, this construction involves the solution of a system of partial differential equations. By studying the integrability conditions of this associated system of partial differential equations we are able to characterize obstructions to

our proposed method of constructing control Lyapunov functions in terms of the structure of the original control system.

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Chapter 1

Introduction

The concept of control system stabilizability plays a central role in control engineering theory and practice, James Clerk Maxwell's paper "On Governors" arguably the first academic paper on control theory was on the stability analysis of the steam engine centrifugal speed governor [1][2]. For modern control engineering, designing of stabilizing feedback controllers is a quintessential task. It comes as no surprise that a large part of the control theory literature is devoted to developing methods for stabilizing feedback controller design [3]. The success of these stabilizing feedback design methods belies some fundamental problems with regards to characterization of stabilizability as a control system property [4]. In contrast to the development of the characterization of the equally important property of controllability which is geometric in flavour, structural and intrinsic, the most general framework for studying stabilizability is the theory of control Lyapunov functions. Lyapunov function theory essentially replaces what should be an intrinsic system property with an external structure (i.e Lyapunov functions) [5] for which there exists no general prescription of the form of the candidate Lyapunov function based on the structure of the system. Therefore instead of asking whether a system is stabilizable this question is replaced with the equally challenging question of whether a system possesses a Lyapunov function. For lower order systems it is possible to proceed heuristically in the construction of Lyapunov functions however as the dimension increases constructing Lyapunov functions becomes more of an art than a science.

To circumvent the "dimensionality curse" and to interrogate the interplay of system structure and stabilizability a hierarchical approach to studying system stabilizability is taken in this work. The hierarchical structure is established using quotients, where quotienting of control systems is a method of reducing the order of a control system by aggregating the state space via some equivalence relation. Typically, quotienting is done by submersion maps on the state space, i.e state dependent transformations. However, the approach taken in this research will look at more general state and control input dependent transformations. The quotienting is required to preserve stabilizability such that given a control system and its lower order quotient, the original control system is stabilizable if and only if its lower order quotient is stabilizable. Existence of such stabilizability preserving quotients therefore provides a theoretical basis for hierarchical control design methods and hierarchical construction of control Lyapunov functions.

1.1 Thesis Contribution

Before briefly summarising the main contributions of this work we will state the research question.

Problem Statement 1. *Consider the two control affine systems Σ and $\tilde{\Sigma}$,*

$$\Sigma \ : \ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \ \mathbf{x} \in \mathbb{R}^m, \mathbf{u} \in \mathbb{R}^r. \tag{1.1}$$

$$\tilde{\Sigma} \ : \ \dot{\mathbf{y}} = \tilde{\mathbf{f}}(\mathbf{y}) + \tilde{\mathbf{g}}(\mathbf{y})\mathbf{v}, \ \mathbf{y} \in \mathbb{R}^n, \mathbf{v} \in \mathbb{R}^s. \tag{1.2}$$

Assume that there exists a quotient map which is a pair of maps (ϕ, ψ) that effect a state and control input transformation such that trajectories of Σ are mapped to trajectories of $\tilde{\Sigma}$.

$$\phi : \mathbb{R}^m \mapsto \mathbb{R}^n, m > n. \tag{1.3}$$

$$\psi : \mathbb{R}^m \times \mathbb{R}^r \mapsto \mathbb{R}^n \times \mathbb{R}^s. \tag{1.4}$$

If $\tilde{\Sigma}$ is stabilizable i.e there exists a feedback $\mathbf{v} = \alpha(\mathbf{y})$ and a control Lyapunov function $\tilde{V}(\mathbf{y})$, under what conditions is Σ stabilizable.

In this thesis we tackle two variants of the above problem, the first case is when the quotient system $\tilde{\Sigma}$ is a controllable linear system and there is no reduction of the input dimension, the second case is when both Σ and $\tilde{\Sigma}$ are analytic control systems. The geometric approach to control theory is taken in the thesis and we make extensive use of ideas from differential geometry. Pivotal to the constructions developed in this work is the theory of Ehresmann connections, jet bundle theory and the geometric theory of partial differential equations. The following contributions are made in this work. Note that parts of these contributions are also contained in the published work [6] and in the paper under review with preprints available [7].

1. Let $\phi : \mathbb{R}^m \mapsto \mathbb{R}^n$ be a smooth map, a novel construction is developed which pulls-back the control system $\tilde{\Sigma}$ defined on $\mathbb{R}^n$ onto $\mathbb{R}^m$. The new pulled-back control system on $\mathbb{R}^m$ denoted $\phi^*(\tilde{\Sigma})$ is such that the map ϕ carries trajectories of $\phi^*(\tilde{\Sigma})$ to trajectories of $\tilde{\Sigma}$.

2. For the case where $\tilde{\Sigma}$ is a linear controllable system we show that if the zero dynamics of Σ are stable then Σ will be stabilizable. More importantly we develop a novel method of constructing the zero dynamics of Σ which does not involve explicitly solving systems of partial differential equations.

3. For the general case we transform the above problem into an associated system of under-determined partial differential equations. If a function $V : \mathbb{R}^m \mapsto \mathbb{R}$ is a solution to this system of partial differential equations then the modified function $\phi^*(\tilde{V}) + V$ (note $\phi^*(\tilde{V})$ is the pull-back of $\tilde{V}$) is a control Lyapunov function for Σ. Using the geometric theory of partial differential equations we develop integrability conditions which when satisfied allow for the iterative solution of the function V. These integrability conditions can be viewed as obstructions to the stabilization scheme that we propose.

1.2 Structure of the thesis

This thesis is organised as follows.

- **Chapter 2** presents the required mathematical machinery and notation that is used in developing our results. The theory of Ehresmann connections and jet bundle theory is presented in this chapter, we closely follow the presentation and notation of [8] and the reader unfamiliar with these concepts is advised to consult this text. Another important theoretical construction presented in this chapter is the geometric theory of partial differential equations. Since this theory is somewhat "non-standard" our presentation follows a tutorial style and at times technical precision is sacrificed for clarity. For a complete coverage of these ideas the interested reader is directed to the following sources [9][10][11][12][13].

- **Chapter 3** presents the fundamental concepts in geometric control theory, the fibre bundle representation of control systems, systems trajectories and quotient maps of control systems are defined within the geometric control theory framework. This chapter also provides a survey of quotienting/reduction based approaches to the problem of characterizing control system stabilizability and controllability. Special focus is put on the two feedback design techniques, centre manifold theory method and the immersion and invariance methods which are instances where quotienting is implicitly used in designing stabilizing feedback controllers.

- **Chapter 4** contains the main results of this thesis. Section 4.2 contains the specialised result where $\tilde{\Sigma}$ (as defined in the problem statement above) is assumed to be a linear and controllable system. Section 4.3.1 contains the general main theorem and its proof, the chapter ends with an example.

- **Chapter 5** contains the concluding remarks and tentative ideas on possible extensions to the work reported in this document.

Chapter 2

Mathematical preliminaries

Summary

This chapter presents the mathematical machinery and notation that will be used in the rest of this thesis. The aim of this chapter is to provide the reader with just those aspects of differential geometry which are required to develop the theory of Ehresmann connections and the geometric theory of partial differential equations. As such for the sake of brevity and clarity some theorems are quoted without proof, for a fuller picture the interested reader is directed to the texts referenced in this chapter.

2.1 Notation

Following standard convention, $\mathbb{N}$ will denote the set of all natural numbers, $\mathbb{R}$ denotes the set of real numbers. Consider the k-fold Cartesian product $\overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{k}$ which will be denoted as $\mathbb{R}^k$, elements of $\mathbb{R}^k$ are k-tuples of real numbers represented as $(x^1, \cdots, x^k)$.

Consider $f : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}$, f is said to be continuously differentiable k times at $x \in U$ if f's partial derivatives at x of order less than or equal to k exist and are continuous. If f is k times continuously differentiable on every $x \in U$ then f is a function of differentiability class C^k, if $k = \infty(C^\infty)$ then f is said to be a smooth function. The map $g : U \subseteq \mathbb{R}^m \mapsto V \subseteq \mathbb{R}^n$ is of class C^k if each of its scalar components is class C^k. Let $n = m$, the map $g : U \mapsto V$ is a diffeomorphism if g is bijective and if both g and g^{-1} are C^∞. The sets U and V are said to be diffeomorphic if there exists a diffeomorphism that maps U to V.

2.2 Differentiable manifolds

An m-dimensional differentiable manifold can be intuitively understood as a space which locally looks like $\mathbb{R}^m$. "Locally looks like" is here used to mean homeomorphic to $\mathbb{R}^m$. Coordinate charts which are defined below formalise this idea.

Definition 2.2.1. *[14] Let M be a set. A **coordinate chart** is a pair (U, ψ) where $U \subset M$ and $\psi : U \mapsto \mathbb{R}^m$ is a continuous bijection from U to some open subset in $\mathbb{R}^m$.*

For some coordinate chart (U, ψ) the map ψ is called the local coordinate map. Let $\psi = (\psi^1, \cdots, \psi^m)$ the functions $\psi^i : U \mapsto \mathbb{R}$ are termed the coordinate functions and for any point $p \in U$, $(\psi^1(p), \cdots, \psi^m(p))$ are the local coordinates of p. A differentiable manifold structure is constructed by patching together coordinate charts such that they cover the space M. Additionally we require these charts to be compatible, a family of such compatible charts is called an atlas.

Definition 2.2.2. *[14] A C^k **atlas** on M is a family of charts $\mathcal{A} = \{(U_i, \psi_i) | i \in I, I \subset \mathbb{N}\}$ such that*

- *$M = \bigcup_{\forall i \in I} U_i$*
- *For any two charts (U_i, ψ_i) and (U_j, ψ_j) in $\mathcal{A}$ with $U_i \cap U_j \neq \emptyset$ and an overlap map defined as $\psi_{ij} = \psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \mapsto \psi_j(U_i \cap U_j)$. The overlap map is a diffeomorphism of class C^k, charts that satisfy this condition are said to be compatible. See figure (2.2).*

The idea of a differentiable manifold can now be formally defined as follows.

Definition 2.2.3. *[15] A $\mathbf{C^k}$ **differentiable manifold of dimension $\boldsymbol{m}$** is a pair $(M, \mathcal{A})$, where M is a set and $\mathcal{A}$ is a C^k maximal atlas. In the sequel the atlas $\mathcal{A}$ will be omitted and only M will be referred to as a manifold*

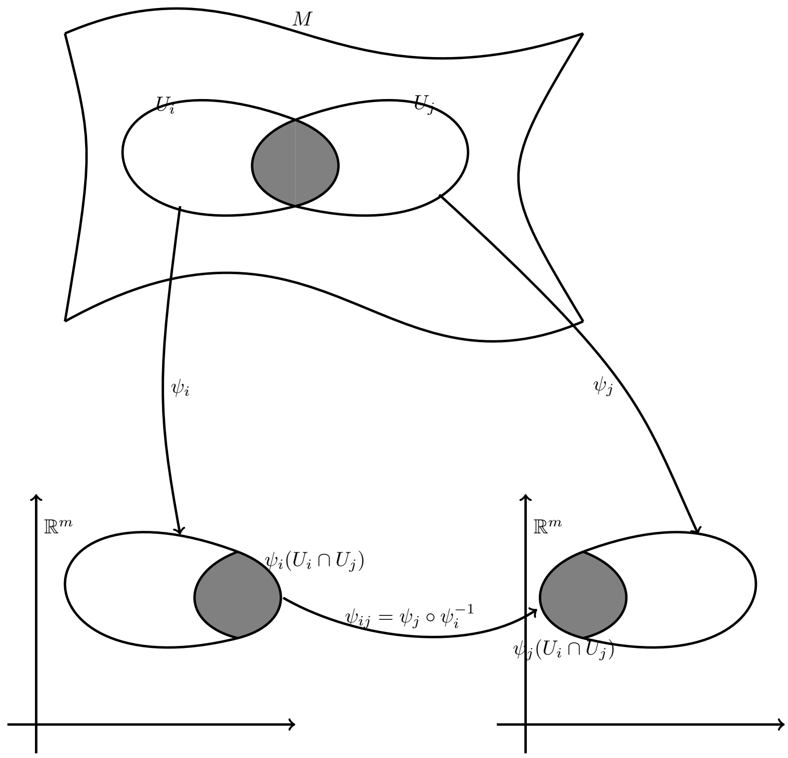

Figure 2.1: Compatible coordinate charts

2.2.1 Maps between manifolds

Definition 2.2.4. *Let M and N be C^k manifolds of dimensions m and n respectively. The map $f : M \mapsto N$ is of* ***class*** *C^r $(0 \leq r \geq k)$, if for each $x \in M$ and charts (U, ψ), (V, ϕ) of M and N respectively such that $x \in U$ and $f(x) \in V$, the local representative of f, $f_{\psi\phi} = \phi \circ f \circ \psi^{-1} : U \subset \mathbb{R}^m \mapsto V \subset \mathbb{R}^n$, is of class C^r.*

For $r = \infty$ we shall simply refer to the map as smooth. If the map $f : M \mapsto N$ is a smooth bijection with the inverse map f^{-1} also being smooth then f is a **diffeomorphism**. When a map between manifolds M and N is a diffeomorphism the manifolds are called **diffeomorphic**, this shall be denoted as $M \simeq N$. Being "diffeomorphic" is an equivalence relation as such two manifolds are essentially indistinguishable if they are diffeomorphic.

Definition 2.2.5. *For* (U, ψ) *and* (V, ϕ) *charts of* M *and* N *respectively and* $p \in U$*. The **rank** of the map* $f : M \mapsto N$ *at* p*, where* $f(p) \in V$ *is the rank of the Jacobian matrix of* $f_{\psi\phi}$*(the local representative of* f*) at* $\psi(p)$*. Let* $(x^1, \cdots x^m)$ *and* $(y^1, \cdots y^n)$ *be the coordinates induced by the charts* (U, ψ) *and* (V, ϕ) *respectively. The Jacobian(**J**) of* f *at* p *is given by*

$$\mathbf{J} = \begin{pmatrix} \frac{\partial \tilde{f}^1}{\partial x^1} & \cdots & \frac{\partial \tilde{f}^1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}^n}{\partial x^1} & \cdots & \frac{\partial \tilde{f}^n}{\partial x^m} \end{pmatrix}$$

where $\tilde{f} = \phi \circ f \circ \psi^{-1}$.

The rank of a map is a coordinate invariant property that does not depend on the coordinate charts used in the above definition. If the rank of f is constant for all $p \in M$ then f is referred to as a **constant rank** map and the point p will be omitted in referring to the rank of f. Using the notion of rank, smooth maps can be categorised as follows.

Definition 2.2.6. *Let* $f : M \mapsto N$ *be a smooth constant rank map.*

1. f *is a **submersion** if rank* f *= dim* N.
2. f *is an **immersion** if rank* f *= dim* M.
3. *If* f *is an immersion that is injective(1-to-1) and a homeomorphism onto its image then* f *is called a **smooth embedding**.*

The simplest example of a submersion is the projection map $\pi : \mathbb{R}^{n+k} \mapsto \mathbb{R}^n$ that projects onto the first n coordinates. For an embedding the simplest example is the inclusion map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$. An example of an immersion which will be encountered in this text numerous times is a smooth trajectory/curve in a manifold $\gamma : I \subset \mathbb{R} \mapsto M$ such that $\gamma\prime(t) \neq 0$ for all $t \in I$. This brief discussion of rank will be concluded by stating a theorem that provides a canonical representation of constant rank theorems.

Theorem 2.2.1. *[16] Let* $f : M \mapsto N$ *be a smooth constant rank map and suppose dim* M *=* m*, dim* N *=* n *and rank* f *=* k *at every point of* M*. If* $p \in M$*, then* *there exists coordinate charts* (U, ψ) *and* (V, ϕ) *such that* $\psi(p) = (0, \cdots, 0)$ *and* $\phi(f(p)) = (0, \cdots, 0)$ *and* $f_{\psi\phi} = \phi \circ f \circ \psi^{-1}$ *is given by*

$$f_{\psi\phi}(x^1, \cdots, x^m) = (x^1, \cdots, x^k, 0, \cdots, 0). \tag{2.1}$$

2.2.1.1 Pull-back of functions

Consider the smooth real valued function defined on some manifold N, $h : N \mapsto \mathbb{R}$. The set of all such functions will be denoted $C^\infty(N; \mathbb{R})$. Let there be a smooth mapping from some manifold M to N, $f : M \mapsto N$. This induces a smooth real valued function on M called the pull-back of h by f denoted $f^*h = h \circ f : M \mapsto \mathbb{R}$. Thus the map f induces a mapping $f^* : C^\infty(N; \mathbb{R}) \mapsto C^\infty(M; \mathbb{R})$.

2.2.2 Submanifolds

Submanifolds are subsets of a manifold which can be equipped with a differentiable structure and are manifolds in their own right. More precisely submanifolds are defined as follows.

Definition 2.2.7. *[17] A k-dimensional* ***submanifold*** *of a manifold* M *is a subset* $W \subset M$ *such that for every* $p \in W$ *there exists a local chart* (U, ψ) *containing* p *such that,*

$$\psi(U \cap W) = \psi(U) \cap \mathbb{R}^k. \tag{2.2}$$

W *is a manifold with a differentiable structure generated by the atlas*

$$\{(U \cap W, \psi|_{(U \cap W)}) | \psi(U \cap W) = \psi(U) \cap \mathbb{R}^k\} \tag{2.3}$$

A chart such that for $p \in W$, $\psi(p) = (x^1, \cdots, x^k, 0, \cdots, 0)$ *is said to be adapted to* W.

In practice submanifolds are presented as either level sets or images of maps. The following theorems will prove useful in this regard.

Theorem 2.2.2. *[16]* ***Submersion theorem****. Consider a smooth constant rank map* $f : M \mapsto N$*, with rank(f) = dim(N). For* $q \in N$*, the level set* $f^{-1}(q) = \{p \in M | f(p) = q\}$ *is a closed submanifold of* M *with dimension = dim(M) - dim(N).*

As a simple application of the submersion theorem consider the map $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x^1, x^2) = (x^1)^2 + (x^2)^2 - 1$. The set $f^{-1}(0)$ defines the unit circle as a 1 dimensional submanifold of $\mathbb{R}^2$. Another useful theorem for constructing and identifying submanifolds is the embedding theorem.

Theorem 2.2.3. *[16] **Embedding theorem**. Let $f : M \mapsto N$ be an embedding then $f(M)$ is a submanifold of N.*

2.2.2.1 Quotient manifolds

Before stating the definition of a quotient manifold the notion of an equivalence relation is first developed.

Definition 2.2.8. *[14] Let M be a differentiable manifold. An **equivalence relation** R on M is a binary relation that possesses the following properties,*

1. *for $u \in M$, uRu;*
2. *for $u, v \in M$, uRv iff vRu; and*
3. *for $u, v, w \in M$ uRv and vRw implies uRw.*

For $u \in M$ the set of all points in M that are R-equivalent to u is called the equivalence class of u denoted $[u] = \{v \in M | uRv\}$. The set of all such equivalence classes is the quotient space denoted M/R. There exists a canonical projection map $\pi : M \mapsto M/R, \pi(u) = [u]$. Having sketched the idea of equivalence relations, quotient manifolds can be defined as follows.

Definition 2.2.9. *[14] The equivalence relation R defined on a differentiable manifold M is called regular if the quotient space M/R can be equipped with a manifold structure such that the canonical projection $\pi : M \mapsto M/R$ is a submersion. If R is a regular equivalence relation then M/R is a **quotient submanifold** of M by R.*

As an example consider the equivalence relation defined on $\mathbb{R}$, xRy if $x - y = n(2\pi), n \in \mathbb{Z}$. The quotient manifold $\mathbb{R}/R$ is the circle S^1.

2.3 Tangent bundle

The tangent bundle of a manifold is constructed by attaching to each point of the manifold a vector space of tangent vectors. Several formulations of these tangent vector spaces exist in literature the most common being,

1. **Derivation approach** A tangent vector is defined as a linear operator which acts on smooth functions as derivations [18].

2. **Curve approach** A tangent vector is defined as an equivalence set of curves passing through a point on a manifold [14]. This approach will be used below.

2.3.1 Tangent space

Let M be a manifold with a chart (U, ψ) and $p \in U$. Consider two curves c_1 and c_2, for which there exists some interval $I \subset \mathbb{R}, 0 \in I$ such that $c_i : I \mapsto U, i = 1, 2$. The curves c_1 and c_2 are said to be **tangent at** p if both curves pass through p ($c_1(0) = c_2(0) = p$) and also if $(\psi \circ c_1)\prime(0) = (\psi \circ c_2)\prime(0)$($\prime$ differentiation with respect to curve parameterization). In defining the tangency property reference is made to a particular coordinate chart, however it can be shown that tangency is a chart invariant property. It can also be shown that tangency is an equivalence relation on the set of all curves passing through a particular point on the manifold.

Definition 2.3.1. *[17] For $p \in M$, the tangent space at p denoted as T_pM is defined as the space of equivalence classes $[c]$ of curves passing through p. Where*

$$[c] = \{c_i \,|c_i(0) = c(0) = p, (\psi \circ c_i)\prime(0) = (\psi \circ c)\prime(0)\} \tag{2.4}$$

The tangent space T_pM can be equipped with a linear structure as follows;

$$[c_1]+[c_2] = \{c\,|\,c(0) = c_1(0) = c_2(0), (\psi\circ c)\prime(0) = (\psi\circ c_1)\prime(0)+(\psi\circ c_2)\prime(0)\} \tag{2.5}$$

$$\lambda[c_1] = \{c \mid c(0) = c_1(0), (\psi \circ c)\prime(0) = \lambda(\psi \circ c_1)\prime(0)\} \tag{2.6}$$

The mapping $\upsilon_\psi : T_pM \mapsto \mathbb{R}^m, \upsilon_\psi([c]) = (\psi \circ c)\prime(0)$ can be easily shown to be an isomorphism thus proving that $\dim(T_pM) = \dim(M)$.

Let $e_1, \dots, e_m$ be the standard basis vectors of $\mathbb{R}^m$. Define the following curves in $\mathbb{R}^m$.

$$a_i : I \mapsto \mathbb{R}^m, \quad a_i(t) = \psi(p) + te_i. \tag{2.7}$$

These curves can be mapped back to $U \subset M$ by the action of ψ^{-1} (which by definition is a local diffeomorphism) $\psi^{-1} \circ a_i(t) : I \mapsto M$. The tangent vectors corresponding to these curves are defined as follows:

$$\frac{\partial}{\partial x^i} = [\psi^{-1} \circ a_i]. \tag{2.8}$$

By construction $\frac{\partial}{\partial x^i} \in T_pM$ and since $\{e_1, \dots, e_m\}$ is the standard basis of $\mathbb{R}^m$ the vectors $\frac{\partial}{\partial x^i}$ form a basis for T_pM. Elements of the tangent space T_pM will henceforth be represented as

$$X_p \in T_pM = X_p^i \frac{\partial}{\partial x^i}, \quad i = 1, \dots m, \tag{2.9}$$

where Einstein's summation convention has been employed.

2.3.1.1 Push-forward of vectors

Definition 2.3.2. *[17] Given a smooth map $f : M \mapsto N$, the* ***tangent map*** *or* ***differential*** *or* ***push-forward*** *of f at $p \in M$ denoted Tf_p, df_p and f_* respectively is given by,*

$$df_p = Tf_p = f_* : T_pM \mapsto T_{f(p)}N, \quad f_*([c]) = [f \circ c]. \tag{2.10}$$

Let (U, ψ) be a coordinate chart of M containing p and (V, ϕ) a coordinate chart of N containing $f(p)$ such that $p = (x^1, \dots, x^m)$ and $f(p) = (y^1, \dots, y^n)$. The coordinate representative of the curve $c(t)$ in the chart (U, ψ) is given by $\psi \circ c(t) = (x_c^1(t), \cdots, x_c^m(t))$. The corresponding vector $[c]$ in coordinates becomes $[\psi \circ c] = \dot{x}_c^i(0)\frac{\partial}{\partial x^i}$. Applying the above definition of the push-forward produces

$f_*([c]) = [f \circ c] = [(\phi \circ f \circ \psi^{-1}) \circ (\psi \circ c)] = [f_{\psi\phi} \circ \psi \circ c];$

$$f_*([c]) = \begin{bmatrix} \frac{\partial f^1_{\psi\phi}}{\partial x^1} & \dots & \frac{\partial f^1_{\psi\phi}}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n_{\psi\phi}}{\partial x^1} & \dots & \frac{\partial f^n_{\psi\phi}}{\partial x^m} \end{bmatrix} \begin{bmatrix} \dot{x}^1_c(0) \\ \vdots \\ \dot{x}^m_c(0) \end{bmatrix}. \tag{2.11}$$

In coordinates the push-forward map is just the Jacobian matrix.

2.3.2 Definition of the tangent bundle

Definition 2.3.3. *[17] The* ***tangent bundle*** *of a manifold M is the disjoint union of all tangent spaces at all points of M.*

$$TM = \bigcup_{p \in M} T_p M. \tag{2.12}$$

The tangent bundle TM is a differentiable manifold of dimension $2m$ where $m = dim(M)$.

TM is equipped with a canonical projection map $\pi_M : TM \mapsto M$ defined as follows,

$$\forall (p, X_p) \in TM, \quad \pi_M(p, X_p) = p. \tag{2.13}$$

Consider a smooth mapping $f : M \mapsto N$, the map f induces a map on the tangent bundles of the respective manifolds denoted $Tf : TM \mapsto TN$ and defined as follows.

$$Tf(p, X_p) = (f(p), Tf_p(X_p)). \tag{2.14}$$

Tf is such that the following diagram commutes.

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

2.4 Co-tangent bundle

The co-tangent bundle is the dual space to the tangent bundle and it is constructed by attaching to each point of the manifold the dual space to the tangent space. The dual space to the tangent space is the co-tangent space and it is defined in the following section.

2.4.1 Co-tangent space

Definition 2.4.1. *[17] Let M be a smooth m-dimensional manifold with $p \in M$ and T_pM the tangent space at p. The **co-tangent space** T_p^*M is the space of linear functions on the tangent space i.e $\omega \in T_p^*M, \omega : T_pM \mapsto \mathbb{R}$. Elements of T_p^*M are called covectors. If T_pM has the basis vectors $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\}$ then there exists covectors $\omega^1, \cdots, \omega^m$ such that*

$$\omega^i \left(\frac{\partial}{\partial x^j} \right) = \delta^{ij} \tag{2.15}$$

where δ^{ij} is the Kronecker delta. The covectors $\{\omega^1, \cdots, \omega^m\}$ form a basis for T^*M and are denoted $\{dx^1, \cdots, dx^m\}$.

2.4.1.1 Pull-back of co-vectors

The smooth map $f : M \mapsto N$ induces a homomorphism, $f^* : T^*_{f(p)}N \mapsto T_p^*M$ called the pull-back of f at p. Let $X_p \in T_pM$ and $\omega \in T^*_{f(p)}N$ then the pull-back of f is defined as,

$$f^*(\omega)(X_p) = \omega(f_*(X_p)). \tag{2.16}$$

2.4.2 Definition of co-tangent bundle

Definition 2.4.2. *[17] The **co-tangent bundle** of a manifold T^*M is the $2m$ dimensional manifold constructed by taking the disjoint union of all co-tangent spaces of all points of M.*

$$T^*M = \bigcup_{p \in M} T_p^*M. \tag{2.17}$$

The co-tangent bundle T^*M is equipped with a canonical projection denoted π_M^* and defined as,

$$\pi_M^* : T^*M \mapsto M, \quad \pi_M^*(p, \omega_p) = p. \tag{2.18}$$

2.5 Vector fields

Definition 2.5.1. *[17] Let M be a smooth manifold. A* ***vector field*** *on M is a mapping $X : M \mapsto TM$ such that $\pi_M \circ X = id_M$(i.e X is a section of TM). The set of all C^r vector fields on M is denoted by $\mathfrak{X}^r(M)$, the set of all smooth vector fields shall be simply denoted by $\mathfrak{X}(M)$ for simplicity of presentation.*

A vector field assigns to every point $p \in M$ an element of T_pM. In some coordinate chart (U, x^i) we have $X = X^i \frac{\partial}{\partial x^i}$, the X^i's are the local components of the vector field. If $X \in \mathfrak{X}^r$ then the local components X^i are C^r functions on M.

Definition 2.5.2. *[17] Let $X \in \mathfrak{X}^r(M)$, an* ***integral curve*** *of X is a curve $c : I \subset \mathbb{R} \mapsto M$ such that $\dot{c}(t) = X(c(t))$.*

For some coordinate chart of M let $(X^1, \ldots, X^m)$ and $(c^1, \ldots, c^m)$ be local representatives of X and c respectively. The above definition implies that c is a solution to the system of differential equations given by

$$\frac{dc^1(t)}{dt} = X^1(c^1(t), \ldots, c^m(t)) \tag{2.19}$$

$$\vdots \qquad \vdots \tag{2.20}$$

$$\frac{dc^m(t)}{dt} = X^m(c^1(t), \ldots, c^m(t)). \tag{2.21}$$

If the integral curves(i.e solutions of the above differential equations) exist for all time $I = (-\infty, +\infty)$ then the vector field is said to be **complete**.

2.5.1 ϕ-related vector fields

Definition 2.5.3. *[14] Let $\phi : M \mapsto N$ be a smooth mapping of manifolds. The vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called* ***ϕ-related*** *denoted $X \sim_\phi Y$ if $T\phi \circ X = Y \circ \phi$.*

Let c_X denote the integral curve of $X \in \mathfrak{X}(M)$ and assume that there exists a vector field $Y \in \mathfrak{X}(N)$ such that $X \sim_\phi Y$. Consider the curve $\phi \circ c_X : I \subset \mathbb{R} \mapsto N$ then

$$\frac{d(\phi \circ c_X)(t)}{dt} = T\phi \left(\frac{dc_X(t)}{dt} \right) = T\phi(X(c_X(t))) = Y((\phi \circ c_X)(t)). \tag{2.22}$$

Thus $(\phi \circ c_X)(t)$ is an integral curve of Y. Denote the integral curve of Y by c_Y. For $p \in M$, $q \in N$ such that $q = \phi(p)$ if $c_X(0) = p$ and $c_Y(0) = q$ then by the uniqueness property of integral curves we have $c_Y(t) = (\phi \circ c_X)(t)$.

2.5.2 Lie bracket

A vector X_p acts on a function f to give $X_p(f)$ which is the directional derivative of f in the direction given by X_p at p. Extending this construction to vector-fields, let X be a vector field on M and f a function on M. X acts on f to give another function $X(f)$ which in coordinates is denoted $X^i \frac{\partial f}{\partial x^i}$.

Consider the bilinear map $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$ defined as follows [15]:

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad X, Y \in \mathfrak{X}(M). \tag{2.23}$$

The bilinear map defined above is called the **Lie bracket** and it obeys the chain rule of differentiation(i.e $[X, Y](fg) = f[X, Y](g) + g[X, Y](f)$). Thus $[X, Y]$ is a well defined vector-field. Some useful properties of the Lie bracket are here listed [15].

1. Bilinear : For $a_1, a_2 \in \mathbb{R}, [a_1X_1 + a_2X_2, Y] = a_1[X_1, Y] + a_2[X_2, Y]$ and $[X, a_1Y_1 + a_2Y_2] = a_1[X, Y_1] + a_2[X, Y_2]$.

2. Antisymmetric : $[X, Y] = -[Y, X]$.

3. Jacobi identity : $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

The vector space $\mathfrak{X}(M)$ equipped with the Lie bracket is an $\mathbb{R}$-algebra.

2.6 Distributions and Frobenius' theorem

Distributions generalize the idea of a vector field to higher dimensions. A vector field assigns an element of the tangent space to every point on a manifold, distributions assign a sub-space of the tangent space to each point on a manifold. More precisely a distribution is defined as follows.

Definition 2.6.1. *[19] Let M be a C^r-manifold.*

1. *A **distribution** D is an assignment to each point $p \in M$ of a subspace D_p of T_pM.*

2. *A distribution is C^r if it is a C^r embedded sub-manifold of TM.*

3. *A C^r distribution is **regular** if it is a C^r subbundle of TM.*

For a smooth and regular distribution D there always exists a finite set of vector fields $\{X_1 \cdots X_k\}$ such that for every point $p \in M$, $D_p = \text{span}\{X_1(p) \cdots X_k(p)\}$. Such a distribution D is said to be locally spanned(finitely generated) by the vector fields $X_1 \cdots X_k$.

A property of distributions that will prove useful in the sequel is involutivity. A distribution D is **involutive** if for any two vector fields X and Y taking values in D(i.e. $X(p), Y(p) \in D_p$) then the vector field given by the Lie bracket of X and Y also takes values in D($[X, Y](p) \in D_p$). If the distribution D is finitely generated by the set of vector fields $\{X_1 \cdots X_k\}$ then checking for involutivity amounts to checking if the following condition holds[19].

$$[X_i, X_j](p) = c_{ij}^l(p) X_l(p), \quad i, j, l \in 1, \cdots, k \tag{2.24}$$

Where $c_{ij}^l(p)$ are C^r functions.

2.6.1 Integral submanifolds

Integral submanifolds are to distributions what integral curves are to vector fields. Let D be a regular distribution on some manifold M, a **local integral manifold** of D is an immersed submanifold S such that $T_pS \subset D_p$ for all $p \in S$. If $T_pS = D_p$ for all $p \in S$ then the integral submanifold is said to be **maximal**.

For a smooth regular 1-dimensional distribution there always exists a maximal integral submanifold. In this case the integral submanifold corresponds to the integral curve of the vector field which spans the 1-dimensional distribution. Existence of maximal integral submanifolds is then guaranteed by the uniqueness and existence theorem of solutions of ordinary differential equations.

It is important to note that maximal integral manifolds do not always exist. A distribution is said to be **integrable** if it admits integral manifolds. Frobenius' theorem stated below provides an easy way to test if a distribution is integrable.

Theorem 2.6.1. *[19] A regular smooth distribution is integrable if and only if it is involutive.*

2.7 Tensors

Tensors are multilinear maps that are defined on finite copies of a vector space and its dual space. For manifolds the vector space and dual space are the tangent space and co-tangent space.

Definition 2.7.1. *[17] Let M be a smooth m-dimensional manifold, a* ***tensor of type*** (r, s) *at $p \in M$ is a real valued $(r + s)$-multilinear map defined on the Cartesian product of r copies of the T_p^*M and s copies of T_pM. The set of all (r, s) tensors at p is denoted $T_s^r(T_pM)$. For some $t \in T_s^r(T_pM)$,*

$$t : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{r-copies} \times \underbrace{T_pM \times \cdots \times T_pM}_{s-copies} \mapsto \mathbb{R} \tag{2.25}$$

Tangent vectors are thus type $(1, 0)$ tensors whereas covectors are type $(0, 1)$ tensors. Multiplication and addition of tensors of the same type follows intuitively from the multiplication and addition operation on $\mathbb{R}$, this makes the space

$T_s^r(T_pM)$ into a vector space. It is possible to multiply tensors of different types via the tensor product operator which is denoted by the symbol $\otimes$. Let t_1 be a type (r,s) tensor and t_2 be a type (p,q) tensor, the tensor product of t_1 and t_2 denoted $t_1 \otimes t_2$ is a $(r+p, s+q)$ tensor defined as follows.

$$
\begin{aligned}
&(t_1 \otimes t_2)(\omega_1, \cdots, \omega_r, \omega_{r+1}, \cdots, \omega_{r+p}, X_1, \cdots, X_s, X_{s+1}, \cdots, X_{s+q}) \\
&\qquad = t_1(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s) t_2(\omega_{r+1}, \cdots, \omega_{r+p}, X_{s+1}, \cdots, X_{s+q}).
\end{aligned}
$$

where $\omega_i \in T_p^*M$ and $X_j \in T_pM$ for $i = \{1, \cdots r+p\}$ and $j = \{1, \cdots, s+q\}$.

2.7.1 Tensor fields

Following the approach taken in defining vector/covector fields the vector bundle of (r,s) tensors on the manifold M is defined in the same way. The vector bundle of (r,s) tensors denoted $T_s^r(M)$ is constructed by taking the union of (r,s) tensor spaces on all points on M.

$$
\begin{aligned}
T_s^r(M) &= \{(p, t_p) | p \in M, t_p \in T_s^r(T_pM)\} &(2.26)\\
&= \bigcup_{p \in M} \left(p \times T_s^r(T_pM)\right). &(2.27)
\end{aligned}
$$

The tensor bundle is a smooth differentiable manifold. Note that $T_0^1 M = TM$ the tangent bundle and $T_1^0 M = T^*M$ the co-tangent bundle.

Definition 2.7.2. *[14] Let M be a smooth manifold. A* ***tensor field of type (r,s)*** *on M is the mapping $t : M \mapsto T_s^r M, t(p) = (p, t_p)$. Where $t_p \in T_s^r(T_pM)$.*

Operations on tensors discussed in the previous section extend naturally to tensor fields, if for some coordinate chart $p = (x^1, \cdots, x^m)$ this coordinate chart thus induces the following basis for $T_s^r(T_pM)$,

$$
\left\{ \frac{\partial}{\partial x^{i_1}} \otimes, \cdots, \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes, \cdots, \otimes dx^{j_s} | i_1, \cdots, i_r, j_1, \cdots, j_s = 1, \cdots, m \right\}. \tag{2.28}
$$

In this coordinate chart the tensor field t is defined by the component functions $t^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}(x^1,\cdots,x^m)$.

2.7.1.1 Pull-back of tensors by diffeomorphisms

Let M and N be smooth manifolds with $\Phi : M \mapsto N$ being a diffeomorphism. Consider the (r,s) tensor field on N denoted by T. The pull-back of T by Φ, $\Phi^*(T) \in T^r_s M$ is defined as

$$\Phi^*(T)(\Omega^1,\cdots,\Omega^r,X_1,\cdots,X_s) = T((\Phi^{-1})^*\Omega^1,\cdots,(\Phi^{-1})^*\Omega^r,\Phi_*X_1,\cdots,\Phi_*X_s).$$

Where $\Omega^i \in T^*M$ and $X_j \in TM$ for $i = \{1,\cdots,r\}$ and $j = \{1,\cdots,s\}$.

2.7.2 Lie derivative

In section 2.3 it was mentioned that tangent vectors can be defined as derivation operators that act on functions. The Lie derivative extends the derivation action of vector fields to general tensor fields.

Definition 2.7.3. *[15] Let X be a smooth vector field defined on a smooth manifold M with the flow $F_t : M \mapsto M$. Given a smooth (r,s) tensor field τ on M, the **Lie derivative** of τ with respect to X denoted $\mathcal{L}_X\tau$ is defined as*

$$(\mathcal{L}_X\tau)(p) = \frac{d}{dt}\,|_{t=0}\,(F_t^*\tau)(p) = \lim_{t\to 0}\frac{F_t^*(\tau(F_t(p))) - \tau(p)}{t}, p \in M. \tag{2.29}$$

Evaluating the Lie derivative for tensors of rank $(0,0)$ i.e functions on M recovers the familiar directional derivative ($\mathcal{L}_X f = X^i\frac{\partial f}{\partial x^i}$). For $\tau = Y$ a vector field on M, $\mathcal{L}_X Y = [X,Y]$.

2.8 Fibre bundles

Fibre bundles generalize the familiar notion of product spaces. Formally a fibre bundle is defined as follows.

Definition 2.8.1. *[20] A **fibre bundle** is a 4-tuple (B, M, π, F) where*

1. *B is smooth manifold called the **total space**,*
2. *M is a smooth manifold called the **base space**,*
3. *$\pi : B \mapsto M$ is a surjective map called the **projection**,*
4. *F is a space called the **typical fibre**,*
5. *Let $\{V_j\}$ be a family of open sets covering M with $j \in J \subset \mathbb{N}$. For each $j \in J$ there exists a homeomorphism $\phi_j : V_j \times F \mapsto \pi^{-1}(V_j)$.*

For brevity at times the projection π will be used to identify the fibre bundle (B, M, π, F). For the general fibre bundle (B, M, π, F) let $\dim(M) = m$, $\dim(B) = m + n$ and (U, ψ) be a coordinate chart of B such that $\psi : U \subset B \mapsto \mathbb{R}^{m+n}$. (U, ψ) is called an **adapted coordinate chart** if for $p, p' \in U$ and $\pi(p) = \pi(p')$ then $pr_1(\psi(p)) = pr_1(\psi(p'))$, where pr_1 is the projection to $\mathbb{R}^m$ [8]. With the adapted coordinate chart the first m coordinates of points that lie on the same fibre are equal.

It is common to categorise fibre bundles by the structure carried by the typical fibre. For vector bundles the typical fibre has a vector space structure, for affine bundles the typical fibre is an affine space while for principal bundles the typical fibre has a Lie group structure. The tangent bundle, co-tangent bundle and tensor bundle are all examples of vector bundles.

Definition 2.8.2. *[8] A map $\phi : M \mapsto B$ is called a **section** of π if $\pi \circ \phi = id_M$. The set of all sections will be denoted $\Gamma(\pi)$.*

Tensor fields can be defined as sections of the tensor bundle.

2.8.1 Bundle morphisms

Bundle morphisms are maps between fibre bundles that preserve the fibre bundle structure. Preserving the fibre bundle structure means that for any two points of the total space that lie on the same fibre, their image must also lie on the same fibre.

Definition 2.8.3. *[8] If (B, M, π, F) and (E, N, ρ, H) are fibre bundles then a* ***bundle morphism*** *is a pair of maps $(f, \bar{f})$ where $f : B \mapsto E$, $\bar{f} : M \mapsto N$ and $\rho \circ f = \bar{f} \circ \pi$.*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\pi"] & E \arrow[d, "\rho"] \\ M \arrow[r, "\bar{f}"] & N \end{tikzcd}$$

An example of a bundle morphism is the tangent map $(Tf, f) : (TM, M, \tau_M, \mathbb{R}^m) \mapsto (TN, N, \tau_N, \mathbb{R}^n)$.

2.8.2 Pull-back of Bundles

Pulling back fibre bundles is a construction that makes it possible to construct new bundles from old ones.

Definition 2.8.4. *[8] Consider a fibre bundle (E, N, ρ, H) and a map $f : M \mapsto N$. The pull-back of ρ by f is the bundle $(f^*(E), M, f^*(\rho), H)$, where*

$$f^*(E) = \{(p, q) \in M \times E \mid f(p) = \rho(q)\} \tag{2.30}$$

$$f^*(\rho)(p, q) = p. \tag{2.31}$$

The pull-back bundle $f^*(\rho)$ has the same typical fibre as ρ in fact the pull-back bundle is constructed by attaching copies of the fibres of ρ to points in M.

2.8.3 Ehresmann connections

Consider the fibre bundle (B, M, π, F) let $(\mathcal{U}, x^i)$ be some chart of M which induces an adapted coordinate chart $(\pi^{-1}(\mathcal{U}), x^i, u^\alpha)$ for B. For some $q \in \pi^{-1}(\mathcal{U})$ there is a canonical tangent sub-space $V_q\pi \subset T_qB = \ker T_q\pi$ which will be referred to as the **vertical sub-space**. The disjointed union of these vertical sub-spaces defines the vertical sub-bundle of π.

Definition 2.8.5. *[8] If (B, M, π, F) is a fibre bundle, then the* ***vertical sub-bundle*** *$V\pi$ is a vector sub-bundle of the tangent bundle TB defined as,*

$$V\pi = \{X \in TB | T\pi(X) = \boldsymbol{0}\} \tag{2.32}$$

An Ehresmann connection represents a non-canonical way to specify a sub-space complementary to the canonical vertical sub-space.

Definition 2.8.6. *[8] A* ***connection*** *on the fibre bundle (B, M, π, F) can be equivalently defined as*

1. *a smooth distribution distribution $H\pi \subset TB$ called the horizontal sub-bundle such that $TB = V\pi \oplus H\pi$,*
2. *a smooth vector bundle homomorphism $K : TB \mapsto TB$ for which $K(TB) = V\pi$ and $K \circ K = K$. This is equivalent to the above definition since it can be shown that $H\pi = kerK$.*

The connection allows one to relate vectors on the base manifold M to vectors on the total space B in the following way. Consider $q \in B$ and $p \in M$ such that $\pi(q) = p$, the restriction of the tangent map $T\pi$ to the horizontal sub-space $H_q\pi$ is an isomorphism $T\pi : H_q\pi \mapsto T_pM$. The inverse to this isomorphism is called the horizontal lift $\mathrm{Hor}_q : T_{\pi(q)}M \mapsto H_q\pi$, this map is uniquely defined by the connection and provides a way of "*lifting*" vectors from TM to TB.

Consider the adapted coordinate chart $(\mathcal{W}, (x^i, u^\alpha))$ with $q \in \mathcal{W}$. In these coordinates the vertical sub-space takes the simple form $V\pi = \mathrm{span}\{\frac{\partial}{\partial u^\alpha}\}$. The vector bundle homomorphism K can be viewed as a $V\pi$ valued one-form on B, in the adapted coordinate system this gives

$$K = \left(du^\alpha - \Gamma_i^\alpha(x^i, u^\alpha)dx^i\right) \otimes \frac{\partial}{\partial u^\alpha}. \tag{2.33}$$

The functions $\Gamma_i^\alpha(x^i, u^\alpha)$ uniquely define the connection. In these coordinates the horizontal sub-bundle takes the following form,

$$H\pi = \mathrm{span}\left\{\frac{\partial}{\partial x^i} + \Gamma_i^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}\right\} \tag{2.34}$$

The horizontal lift map can be also be viewed as a $H\pi$ valued one-form on M which is written as

$$\mathrm{Hor}_q = dx^i \otimes \left(\frac{\partial}{\partial x^i} + \Gamma_i^\alpha(x^i, u^\alpha) \frac{\partial}{\partial u^\alpha} \right). \tag{2.35}$$

This can also be written conveniently in matrix form as shown below.

$$\mathrm{Hor}_q = \begin{array}{c} \\ 1 \\ \vdots \\ m \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccc} 1 & \quad & m \end{array} \\ \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \Gamma_1^1 & \cdots & \Gamma_m^1 \\ \vdots & \ddots & \vdots \\ \Gamma_1^n & \cdots & \Gamma_m^n \end{bmatrix} \end{array} \tag{2.36}$$

Where $\dim(M) = m, \dim(B) = m + n$.

An important property of a connection is its curvature defined below.

Definition 2.8.7. *[8] The curvature tensor of the connection is the $(1,2)$-tensor $R : \mathfrak{X}(B) \times \mathfrak{X}(B) \mapsto \mathfrak{X}(B)$ defined by*

$$R(X,Y) = K\left([(X - K(X)), (Y - K(Y))]\right). \quad X, Y \in \mathfrak{X}(B). \tag{2.37}$$

The coordinate expression of the curvature tensor R is

$$R = \frac{1}{2}\left(\frac{\partial \Gamma_{i_2}^\alpha}{\partial x^{i_1}} + \Gamma_{i_1}^{\alpha_1} \frac{\partial \Gamma_{i_2}^\alpha}{\partial u^{\alpha_1}} - \frac{\partial \Gamma_{i_1}^\alpha}{\partial x^{i_2}} - \Gamma_{i_2}^{\alpha_1} \frac{\partial \Gamma_{i_1}^\alpha}{\partial u^{\alpha_1}} \right) dx^{i_1} \wedge dx^{i_2} \otimes \frac{\partial}{\partial u^\alpha}. \tag{2.38}$$

2.9 Jet bundles

In section 2.3.1 tangent vectors were defined as equivalence classes of curves in a manifold up to the first derivative. Jet bundles build on this idea by replacing a manifold with a fibre bundle, curves in a manifold with sections on the bundle, equivalence of curves up to first derivative with equivalence of sections up to the

k^{th} derivative. The jet bundle formalism provides a geometric way of describing partial differential equations and will play a central role in the main result of this work. This presentation of the theory of jet bundles follows closely the approach of D.J.Saunders in [8], the interested reader can consult this source for a more detailed coverage of these ideas.

Let (B, M, π, F) be a fibre bundle, $p \in M$. The local sections $\phi, \psi \in \Gamma_p(\pi)$ are said to be locally k-equivalent at the point p if $\phi(p) = \psi(p)$ and if their derivatives up to the k^{th} order are equal. If (x^i, u^α) is some adapted coordinate system in some neighbourhood of $\phi(p)$ then ϕ and ψ are said to be k-equivalent if

$$\frac{\partial^j \phi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_j}} = \frac{\partial^j \psi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_j}}. \tag{2.39}$$

Where $j \in (1, \cdots, k)$, $i_1 \leq \cdots \leq i_j$, $i_1, \cdots, i_j \in (1, \cdots, dim(M) = m)$, $\alpha \in (1, \cdots, dim(F) = n)$. The k-equivalence set at p containing ϕ is called the k-**jet** of ϕ and is denoted $j^k_p\phi$.

Definition 2.9.1. *[8] The k^{th}-**jet manifold** of (B, M, π, F) is the set of all k-jets and is denoted $J^k\pi$*

$$J^k\pi = \bigcup_{p \in M} \{j^k_p\phi \mid \phi \in \Gamma_p(\pi)\} \tag{2.40}$$

*The k^{th}-jet bundle is equipped with maps π_k and $\pi_{k,0}$ called the **source** and **target projections** respectively, these maps are defined as follows*

$$\begin{aligned} \pi_k \quad : \quad & J^k\pi \mapsto M \\ & j^k_p\phi \mapsto p \end{aligned}$$

and

$$\begin{aligned} \pi_{k,0} \quad : \quad & J_k\pi \mapsto B \\ & j^k_p\phi \mapsto \phi(p) \end{aligned}$$

If the bundle π has the adapted coordinates (x^i, u^α) is some open set $W \subset B$, then the k^{th} jet bundle $J^k\pi$ has the induced coordinates $(x^i, u^\alpha, u^\alpha_j)$ where $j =$

$(1, \cdots, k)$. Consider $j^k_p\phi \in J^k\pi$ in these induced coordinates we have, $x^i(j^k_p\phi) = x^i(p), u^\alpha(j^k_p\phi) = u^\alpha(\phi(p))$ and

$$u^\alpha_j(j^k_p\phi) = \frac{\partial^j\phi^\alpha}{\partial x^{i_1}\cdots\partial x^{i_j}}.$$

2.9.1 Operations on jets

Consider the k-jet $j^k_p\phi$, let $1 \leq l < k$ the l-jet projection is the map denoted $\pi_{k,l}$ which takes $j^k_p\phi$ to a l-jet. The l-jet projection is defined as

$$\begin{aligned} \pi_{k,l} \quad : \quad & J^k\pi \mapsto J^l\pi \\ & j^k_p\phi \mapsto j^l_p\phi. \end{aligned}$$

The jet projection operation essentially involves forgetting all the derivatives of order greater than l. Let ϕ be a local section of π, the k^{th} prolongation of ϕ is the map $j^k\phi : M \mapsto J^k\pi$, $j^k\phi(p) = j^k_p\phi$. The prolongation operation can be extended to fibre bundle morphism in the following way.

Definition 2.9.2. *[8] Let (B, M, π, F) and (E, N, ρ, H) be fibre bundles and let $F = (f, \tilde{f})$ be a bundle morphism where f is a diffeomorphism. The k^{th}* ***prolongation*** *of F is the map $j^kF : J^k\pi \mapsto J^k\rho$ defined by*

$$j^kF(j^k_p\phi) = j^k_{\tilde{f}(p)}(\tilde{f}\circ\phi\circ f^{-1}). \tag{2.41}$$

2.9.2 Manifold structure of jet bundles

It has been shown that $J^k\pi$ is a manifold in its own right. Additionally $J^k\pi$ can be equipped with a fibre bundle structure.

Corollary 2.9.1. *[8] The functions $\pi_{k,l} : J^k\pi \mapsto J^l\pi$ (where $1 \leq l \geq k$), $\pi_{k.0} : J^k\pi \mapsto B$ and $\pi_k : J^k\pi \mapsto M$ are smooth surjective submersions.*

Thus $J^k\pi$ can be viewed as the total space over the base manifolds $J^l\pi$, B and M. $\pi_{k,k-1} : J^k\pi \mapsto J^{k-1}\pi$ is not just a bare fibre bundle but is actually an affine bundle. The bundle $\pi_{k,k-1}$ is an affine bundle modelled on the vector bundle

$\pi_{k-1}^*(S^kT^*M) \otimes \pi_{k-1,0}^*(V\pi)$ where S^kTM is the k-symmetric tensor bundle and $V\pi$ is the vertical bundle.

2.10 Geometric partial differential equations

This section presents the geometric theory of partial differential equations as developed by Goldschmidt [9],[10]. The results presented here play a pivotal role in the main contribution of this thesis, for a more in-depth coverage of the material consulting the papers of Goldschmidt [9][10] is highly recommended.

Definition 2.10.1. *[10] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles. A* ***partial differential equation*** *of order k is a fibred embedded sub-manifold $R_k \subset J^k\pi$. Additionally there always exists a fibre bundle morphism $\Phi : J^k\pi \mapsto \tilde{\pi}$ such that $R_k = ker\,\Phi$.*

To see how the above definition gives the common differential equation representation we give an example.

Example 2.10.1. *Consider the smooth manifolds $M = \mathbb{R}^3$ with coordinates (x, y, z), $B = \mathbb{R}^4$ with coordinates (x, y, z, f) , $E = \mathbb{R}^6$ with coordinates (x, y, z, w^1, w^2, w^3) from which the fibre bundles $(B, M, \pi, \mathbb{R})$ and $(E, M, \tilde{\pi}, \mathbb{R})$ are constructed. Note π and $\tilde{\pi}$ are the projections onto the first three terms. The first order jet manifold on π will have coordinates $\{x, y, z, f, f_x, f_y, f_z\}$. Define the fibre bundle morphism $\Phi : J^1\pi \mapsto \tilde{\pi}$ as follows,*

$$\begin{aligned}\Phi : (x, y, z, f, f_x, f_y, f_z) \;\; \mapsto \;\; & (x, y, z, w^1 = f_x - X^1(x, y, z), w^2 - X^2(x, y, z), \\ & w^3 - X^3(x, y, z))\end{aligned} \tag{2.42}$$

where $X^i(x, y, z)$ are smooth functions. The kernel of Φ will define the following partial differential equation

$$\frac{\partial f}{\partial x} = X^1(x,y,z) \tag{2.43}$$
$$\frac{\partial f}{\partial y} = X^2(x,y,z) \tag{2.44}$$
$$\frac{\partial f}{\partial z} = X^3(x,y,z) \tag{2.45}$$

The solution of a partial differential equation is defined next.

Definition 2.10.2. *[8] Let (B, M, π, F) be a fibre bundle and let $R_k \subset J^k\pi$ be a k^{th}-order partial differential equation. A local **solution** of R_k is a local section $\phi : \mathfrak{U} \subset M \mapsto B$ such that $j^k_p\phi \in R_k$ for every $p \in \mathfrak{U}$.*

2.10.1 Prolongation of partial differential equations

Prolonging a partial differential equation refers to differentiating a differential equation to generate a higher order equation. Within the geometric setting prolongation is defined as follows.

Definition 2.10.3. *[8] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th} order differential equation $R_k \subset J_k\pi$ defined by the fibre bundle morphism $\Phi : J^k\pi \mapsto E$. The l^{th} prolongation of R_k is the $(k+l)^th$-order differential equation $R_{k+l} \subset J^{k+l}\pi$ defined by the fibre bundle morphism $\rho_l(\Phi) : J^{k+l}\pi \mapsto J^l\tilde{\pi}, \rho_l(\Phi)(j^{k+l}_p\phi) = j^l_p(\Phi(j^k_p\phi))$. Where $p \in M$ and ϕ is a section of π.*

Example 2.10.2. *Continuing from example (2.10.1), let $(x, y, z, f, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{xz}, f_{yy}, f_{yz}, f_{zz})$ be the coordinates of $J^2\pi$ and $(x, y, z, w^1, w^2, w^3, w^1_{xyz}, w^2_{xyz}, w^3_{xyz})$ be the coordinates of $J_1\tilde{\pi}$ with $w^i_{xyz} = [w^i_x, w^i_y, w^i_z]$. For the first prolongation $\rho_1(\Phi)(j^2_{(x,y,z)}f) \mapsto (x, y, z, w^1, w^2, w^3, w^1_{xyz}, w^2_{xyz}, w^3_{xyz})$, we have*

$$\begin{array}{lll} w^1_x = f_{xx} - X^1_{,x} & w^1_y = f_{xy} - X^1_{,y} & w^1_z = f_{xz} - X^1_{,z} \\ w^2_x = f_{xy} - X^2_{,x} & w^2_y = f_{yy} - X^2_{,y} & w^2_z = f_{yz} - X^2_{,z} \\ w^3_x = f_{xz} - X^3_{,x} & w^3_y = f_{yz} - X^3_{,y} & w^3_z = f_{zz} - X^3_{,z}. \end{array}$$

Where $X^i_{,x} = \frac{\partial X^i}{\partial x}, X^i_{,y} = \frac{\partial X^i}{\partial y}, X^i_{,z} = \frac{\partial X^i}{\partial z}$.

2.10.2 Symbol of partial differential equations

The symbol of the differential equations encodes information about the highest order elements in the linearization of the differential equation[10]. Before stating the definition recall that $\pi_{k,k-1} : J^k\pi \mapsto J^{k-1}\pi$ is an affine fibre bundle that is modelled on the vector bundle over $J^{k-1}\pi$ with total space $\pi^*_{k-1}\left(S^kT^*M\right) \otimes \pi^*_{k-1,0}\left(V\pi\right)$. There exists a canonical inclusion map

$$\epsilon_k : \pi^*_k\left(S^kT^*M\right) \otimes \pi^*_{k,0}\left(V\pi\right) \mapsto V\pi_k. \tag{2.46}$$

Assuming (x^i, u^α) are adapted coordinates for some chart of π the coordinate expression for ϵ_k is

$$\epsilon : \xi^\alpha_{[i_1\cdots i_k]}dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial u^\alpha} \mapsto \xi^\alpha_{i_1\cdots i_k}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_k}}. \tag{2.47}$$

Definition 2.10.4. *[10] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th}-order differential equation $R_k \subset J^k\pi$ defined by the fibre bundle morphism $\Phi : J^k\pi \mapsto E$. The **symbol** of R_k denoted $\sigma(\Phi)$ is the vector bundle morphism $\sigma(\Phi) = V\Phi \circ \epsilon_k : \pi^*_k\left(S^kT^*M\right) \otimes \pi^*_{k,0}\left(V\pi\right) \mapsto V\tilde{\pi}$. Where $V\Phi$ is the restriction of the tangent map $T\Phi$ to $V\pi_k$. Let $ker\, \rho(\sigma(\Phi)) = G_k$, at times we refer to G_k as the symbol of R_k.*

Example 2.10.3. *Continuing with our example we have the following local coordinates $T^*M = span\{dx, dy, dz\}$, $V\tilde{\pi} = span\{\frac{\partial}{\partial w^1}, \frac{\partial}{\partial w^2}, \frac{\partial}{\partial w^3}\}$, $V\pi_1 = span\{\frac{\partial}{\partial f}, \frac{\partial}{\partial f_x}, \frac{\partial}{\partial f_y}, \frac{\partial}{\partial f_z}\}$. We identify $T^*M \otimes V\pi$ with T^*M. The inclusion map ϵ_1 becomes,*

$$\epsilon_1 : \xi_x dx + \xi_y dy + \xi_z dz \mapsto \xi_x\frac{\partial}{\partial f_x} + \xi_y\frac{\partial}{\partial f_y} + \xi_z\frac{\partial}{\partial f_z}. \tag{2.48}$$

Applying $T\Phi|_{V\pi_1}$ we get,

$$V\Phi \circ \epsilon_1 : \xi_x dx + \xi_y dy + \xi_z dz \mapsto \xi_x\frac{\partial}{\partial w^1} + \xi_y\frac{\partial}{\partial w^2} + \xi_z\frac{\partial}{\partial w^3} \tag{2.49}$$

which is just the identity map, therefore we have $G_1 = \{\mathbf{0}\}$.

2.10.2.1 Prolongation of symbol of partial differential equations

The prolongation of the symbol of a partial differential equation is defined as follows.

Definition 2.10.5. *[21] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th}-order differential equation $R_k \subset J^k\pi$ defined by the fibre bundle morphism $\Phi : J^k\pi \mapsto E$. For $p_k \in R_k$ the l^{th}-prolongation of the symbol $\sigma(\Phi)|_{p_k}$ is the map*

$$\rho_l(\sigma(\Phi)|_{p_k}) : S^{k+l}T^*_{\pi_k(p_k)}M \otimes V_{\pi_{k,0}(p_k)}\pi \mapsto S^l T^*_{\pi_k(p_k)}M \otimes V_{\Phi(p_k)}\tilde{\pi}, \tag{2.50}$$

defined by $(id_{S^l T^*_{\pi_k(p_k)}M} \otimes \sigma(\Phi)|_{p_k}) \circ (\Delta_{k,l} \otimes id_{V\pi})$. Where $\Delta_{k,l} : S^{k+l}T^*_{\pi_k(p_k)}M \mapsto S^l T^*_{\pi_k(p_k)}M \otimes S^k T^*_{\pi_k(p_k)}M$ is the natural inclusion. Let $ker(\rho_l(\sigma(\Phi)|_{p_k}) = G_{k+l}$, at times we refer to G_{k+l} as the l^{th} prolongation of the symbol of R_k.

Example 2.10.4. *Continuing the example we define the following spaces*

$$\begin{aligned}
T^*M \otimes V\tilde{\pi} &= span\{\frac{\partial}{\partial w^1} \otimes dx, \frac{\partial}{\partial w^1} \otimes dy, \frac{\partial}{\partial w^1} \otimes dz, \frac{\partial}{\partial w^2} \otimes dx, \frac{\partial}{\partial w^2} \otimes dy, \\
&\quad \frac{\partial}{\partial w^2} \otimes dz, \frac{\partial}{\partial w^3} \otimes dx, \frac{\partial}{\partial w^3} \otimes dy, \frac{\partial}{\partial w^3} \otimes dz\}, \\
S^2T^*M \otimes V\pi &\cong S^2T^*M = span\{dx \otimes dx, dy \otimes dy, dz \otimes dz, \\
&\quad dx \otimes dy + dy \otimes dx = dx \odot dy, dx \otimes dz + dz \otimes dx = dx \odot dz, \\
&\quad dy \otimes dz + dz \otimes dy = dy \odot dz\}.
\end{aligned} \tag{2.51}$$

The first prolongation of the PDE symbol is then the map $\rho_1(\sigma(\Phi))$ defined below.

$$\xi_{xx}dx\otimes dx+\xi_{yy}dy\otimes dy+\xi_{zz}dz\otimes dz+\xi_{xy}dx\odot dy+\xi_{xz}dx\odot dz+\xi_{yz}dy\odot dz$$

$$\Big\downarrow{\scriptstyle \rho_1(\sigma(\Phi))}$$

$$\xi_{xx}\tfrac{\partial}{\partial w^1}\otimes dx+\xi_{yy}\tfrac{\partial}{\partial w^2}\otimes dy+\xi_{zz}\tfrac{\partial}{\partial w^3}\otimes dz+\xi_{xy}\left(\tfrac{\partial}{\partial w^2}\otimes dx+\tfrac{\partial}{\partial w^1}\otimes dy\right)$$

$$+\xi_{xz}\left(\tfrac{\partial}{\partial w^3}\otimes dx+\tfrac{\partial}{\partial w^1}\otimes dz\right)+\xi_{yz}\left(\tfrac{\partial}{\partial w^3}\otimes dy+\tfrac{\partial}{\partial w^2}\otimes dz\right).$$

This is a trivial inclusion map therefore $G_{1+1}=\{\mathbf{0}\}$.

2.10.3 Formal solutions of partial differential equations

The notion of a formal solution formalizes the idea of approximating the solution of a partial differential equation by a finite order Taylor series.

Definition 2.10.6. *[11] Let* (B, M, π, F) *and* $(E, M, \tilde{\pi}, G)$ *be fibre bundles and consider the* k^{th}*-order differential equation* $R_k \subset J^k\pi$ *defined by the fibre bundle morphism* $\Phi : J^k\pi \mapsto E$. *A* ***local formal solution*** *of order* k *is a local section of* R_k *i.e* $\phi_k \in \Gamma(\pi_k|_{\mathcal{U}})$, $\phi_k : \mathcal{U} \subset M \mapsto R_k \subset J^k\pi$.

The process of constructing the Taylor series solution of a differential equation can only be successful if a formal solution of order k can be prolonged to a formal solution of higher order. This quality of being able to iteratively construct Taylor series solutions is the essence of the concept of formal integrability defined below.

Definition 2.10.7. *[9] Let* (B, M, π, F) *and* $(E, M, \tilde{\pi}, G)$ *be fibre bundles and consider the* k^{th}*-order differential equation* $R_k \subset J^k\pi$ *defined by the fibre bundle morphism* $\Phi : J^k\pi \mapsto E$. *The partial differential equation* R_k *is* ***formally integrable*** *if* R_{k+l} *is a fibred submanifold and if the maps* $\pi_{k+l,k} : R_{k+l} \mapsto R_k$ *are epimorphisms for* $l \in \mathbb{Z}_{>0}$.

From the above definition the property of being formally integrable is untestable as it involves checking the surjectivity of an infinite number of maps. The central result of the geometric theory of partial differential equations developed in [10] provides testable conditions for formal integrability. Before stating this theorem some prerequisite definitions and theorems are required.

Definition 2.10.8. *[11] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th}-order differential equation $R_k \subset J^k\pi$ defined by the fibre bundle morphism $\Phi : J^k\pi \mapsto E$. Let G_k be the symbol of R_k, for $p_k \in R_k$ the basis $\{e^1, \ldots, e^m\}$ of $T^*_{\pi_k(p_k)}M$ is called **quasi-regular** if*

$$dim(G_{k+1}|_{p_{k+1}}) = dim(G_k|p_k) + \sum_{j=1}^{m-1} dim(G_{k,j}|_{\pi_k(p)}). \tag{2.52}$$

Where $G_{k,j}|_{\pi_k(p)}$ is given by

$$G_{k,j}|_{\pi_k(p)} = G_k|_{p_k} \cap S^k\Sigma_j|_{\pi_k(p_k)}, \quad \Sigma_j = span\{e^{j+1}, \ldots, e^m\}. \tag{2.53}$$

Example 2.10.5. *Continuing with our example, since $G_1 = \{\mathbf{0}\}$ and $G_{1+1} = \{\mathbf{0}\}$ G_1 trivially satisfies the quasi-regularity condition.*

Theorem 2.10.1. *[22] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th}-order differential equation $R_k \subset J^k\pi$ defined by the fibre bundle morphism $\Phi : J^k\pi \mapsto E$. If there exists a quasi-regular basis for $T^*_{\pi_k(p_k)}M$ where $p_k \in R_k$, then the symbol G_k is said to be involutive.*

We can now state the central theorem that allows the development of integrability conditions for partial differential equations.

Theorem 2.10.2. *[10] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th}-order differential equation $R_k \subset J^k\pi$ defined by the fibre bundle morphism $\Phi : J^k\pi \mapsto E$. If*

1. *R_{k+1} is a fibred submanifold of $J^{k+1}\pi$*

2. *$\pi_{k+1,k} : R_{k+1} \mapsto R_k$ is surjective*

3. *G_k is involutive*

then R_k is formally integrable.

This theorem only requires prolonging the partial differential equation once and testing if the prolonged differential equation projects onto the original partial differential equation. Requiring $\pi_{k+1,k}$ to be surjective can be shown to be equivalent to requiring the zeroing of the so-called curvature map $\kappa : R_k \subset J^k\pi \mapsto S^lT^*M \otimes V\tilde{\pi}/\mathrm{Im}\rho_1(\sigma(\Phi))$ [10][11]. Let $p_k \in R_k$ and $p_{k+1} \in J^{k+1}\pi$ be such that $\pi_{k+1,k}(p_{k+1}) = p_k$ the curvature map is defined as

$$\kappa(p_k) = \tau\left(\rho_1(\Phi)(p_{k+1}) - j^1\Phi(p)\right), \tag{2.54}$$

$$\tau : S^lT^*M \otimes V\tilde{\pi} \mapsto S^lT^*M \otimes V\tilde{\pi}/\mathrm{Im}\rho_1(\sigma(\Phi)). \tag{2.55}$$

Where τ is the canonical projection map onto $S^lT^*M \otimes V\tilde{\pi}/\mathrm{Im}\rho_1(\sigma(\Phi))$.

Example 2.10.6. *Continuing with the example we want to construct the curvature map. Let $p_1 \in R_1 \subset J^1\pi, p_1 = (x, y, z, \tilde{f}, \tilde{f}_x, \tilde{f}_y, \tilde{f}_z)$ since $\Phi(p_1) = 0$ this means $\tilde{f}_x - X^1(x, y, z) = \tilde{f}_y - X^2(x, y, z) = \tilde{f}_y - X^3(x, y, z) = 0$. The point $p_2 \in J^2\pi$ projects to p_1 therefore its coordinates have the general form,*

$$p_2 = (x, y, z, f, \tilde{f}_x, \tilde{f}_y, \tilde{f}_z, s_{xx}, s_{xy}, s_{xz}, s_{yy}, s_{yz}, s_{zz})$$

where $s_{i_1i_2}, i = (x, y, x)$ are arbitrary. Taking the first jet prolongation of $\Phi(p_1)$, $j^1\Phi(p_1) \in J^1\tilde{\pi}$,

$$\begin{aligned} j^1\Phi(p_1) = {} & (x, y, z, \tilde{f}_x - X^1(x, y, z), \tilde{f}_y - X^2(x, y, z), \tilde{f}_z - X^3(x, y, z), \\ & \tilde{f}_{x,x} - X^1_{,x}, \tilde{f}_{x,y} - X^1_{,y}, \tilde{f}_{x,z} - X^1_{,z}, \tilde{f}_{y,x} - X^2_{,x}, \tilde{f}_{y,y} - X^2_{,y}, \tilde{f}_{y,z} - X^2_{,z}, \\ & \tilde{f}_{z,x} - X^3_{,x}, \tilde{f}_{z,y} - X^3_{,y}, \tilde{f}_{z,z} - X^3_{,z}). \end{aligned} \tag{2.56}$$

Note $\tilde{f}_{i_1,i_2} = \frac{\partial \tilde{f}_{i_1}}{\partial i_2}, i_1, i_2 = \{x, y, z\}$. Evaluating the first prolongation of Φ at the point $p_2 \in J^2\pi$ we get the following,

$$
\begin{aligned}
\rho_1(\Phi)(p_2) \;&=\; (x,y,z,\tilde f_x - X^1(x,y,z), \tilde f_y - X^2(x,y,z), \tilde f_z - X^3(x,y,z), \\
&\quad \tilde s_{xx} - X^1_{,x}, \tilde s_{xy} - X^1_{,y}, \tilde s_{xz} - X^1_{,z}, \tilde s_{xy} - X^2_{,x}, \tilde s_{yy} - X^2_{,y}, \tilde s_{yz} - X^2_{,z}, \\
&\quad \tilde s_{xz} - X^3_{,x}, \tilde s_{yz} - X^3_{,y}, \tilde s_{zz} - X^3_{,z}). \qquad (2.57)
\end{aligned}
$$

Calculating $\rho_1(\Phi)(p_2) - j_1(\Phi(p_1)) \in T^*M \otimes V\tilde\pi$ *which is an element of a nine dimensional vector space,*

$$
\begin{aligned}
\rho_1(\Phi)(p_2) - j_1(\Phi(p_1)) \;&=\; [(s_{xx} - \tilde f_{x,x}), (s_{xy} - \tilde f_{x,y}), (s_{xz} - \tilde f_{x,z}), \\
&\quad (s_{xy} - \tilde f_{y,x}), (s_{yy} - \tilde f_{y,y}), (s_{yz} - \tilde f_{y,z}), \\
&\quad (s_{xz} - \tilde f_{z,x}), (s_{yz} - \tilde f_{z,y}), (s_{zz} - \tilde f_{z,z})] \qquad (2.58)
\end{aligned}
$$

The projection map $\tau : T^*M \otimes V\tilde\pi \mapsto T^*M \otimes V\tilde\pi / Im(\rho_1(\sigma(\Phi)))$ *is defined by the following matrix,*

$$
\tau = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \qquad (2.59)
$$

The curvature map then becomes,

$$
\kappa(p_1) = \tau\left(\rho_1(\Phi)(p_2) - j_1(\Phi(p_1))\right) = \begin{bmatrix} \tilde f_{y,x} - \tilde f_{x,y} \\ \tilde f_{z,x} - \tilde f_{x,z} \\ \tilde f_{z,y} - \tilde f_{y,z} \end{bmatrix} = \begin{bmatrix} X^2_x - X^1_y \\ X^3_x - X^1_z \\ X^3_y - X^2_z \end{bmatrix} \qquad (2.60)
$$

This differential equation is formally integrable if the following condition is met.

$$
\begin{bmatrix} X^2_x - X^1_y \\ X^3_x - X^1_z \\ X^3_y - X^2_z \end{bmatrix} = 0. \qquad (2.61)
$$

This is just the familiar requirement for the curl of the vector field $X =$

$[X^1(x,y,z), X^2(x,y,z), X^3(x,y,z)]$ *to be zero.*

In general formal integrability does not guarantee existence of solutions however for analytic partial differential equations formal integrability is equivalent to the existence of analytic solutions.

Theorem 2.10.3. *[10] Let (B, M, π, F) and $(E, M, \tilde{\pi}, G)$ be fibre bundles and consider the k^{th}-order analytic partial differential equation $R_k \subset J^k\pi$ defined by the analytic fibre bundle morphism $\Phi : J^k\pi \mapsto E$. If R_k is formally integrable, given $p_{k+l} \in R_{k+l}$ with $\pi_{k+l}(p_{k+l}) = p$, there exists an analytic solution $\phi \in \Gamma(\pi)$ of R_k on some neighbourhood of p such that $j^{k+l}\phi(p) = p_{k+l}$.*

Chapter 3

Control Theory

Summary

This chapter presents concepts of geometric control theory with special emphasis on control system representations. Additionally a review of the state of the art in control theory literature with respect to the problem of stabilizability preserving quotients is presented.

3.1 Introduction

A brief survey of the literature on control theory will reveal a diverse selection of paradigms to problem solving. It is therefore appropriate from the outset to state the paradigm that is adopted in this work. The approach taken in this work is greatly influenced by the ideas of J.C.Willems dubbed "The behavioural approach to control" [23],[24]. In the behavioural approach a control system is defined as the collection of time trajectories of the system variables the so-called behaviours. It should be noted here that "system variables" encompasses inputs, outputs and states, in the behavioural approach all these variables are looked at the same. The quintessential feature of the behavioural approach is that it puts systems trajectories at the centre of control theory, thus a distinction is made between a control system and its representation. With this view concepts such as controllability and observability are elevated from being mere representation dependent concepts [25] to more fundamental structural properties of the control system. This focus on fundamental aspects of control systems allows for the development of general results which are applicable to systems irrespective of the chosen representation. In this thesis the complete machinery of the behavioural approach will not be used, rather the idea of the centrality of system trajectories in control theory will be extensively used. The approach taken in this work is more in line with the "spirit" of the behavioural approach, not so much the letter of it.

Another overarching idea that shapes the approach taken in this work is the appeal to differential geometric techniques in both the phrasing and solution of the research question. The application of differential geometric tools in control theory has yielded significant results in the solution of such problems as controllability of non-linear control systems [26], feedback linearization [27], energy based control [28] e.t.c. One of the main advantages of the differential geometric approach is that it allows for the development of co-ordinate invariant results which means that results can be derived which are not dependent on the chosen control system representation.

3.2 Control system modelling

Modelling of control systems typically involves defining control inputs, system states, systems outputs and connecting these variables using differential-algebraic equations. This gives the following familiar representation of a general non-linear system

$$\dot{x} = f(x, u) \tag{3.1}$$

$$y = g(x) \tag{3.2}$$

where u, x, y are input, state and output variables respectively. In as much as the above control system representation is popular it does prove limiting with regards to general and global analysis. A case in point is when analysing mechanical systems with rotational degrees of freedom, written in the form of equations (3.1) and (3.2). It is possible that the system might appear to be globally continuously stabilizable in this form however it is known that there are topological obstructions to continuous stabilization of systems with rotational degrees of freedom [29]. The fact that equations (3.1) and (3.2) assume a Euclidean state space can thus hide certain topological/geometric properties inherent in the actual system. Additionally the above representation is inherently local and hence does not easily lend itself to coordinate invariant analysis [4].

To overcome the limitations stated above a number of control system models exist within the geometric control theory literature which use different geometric structures such as, sets of vector fields, exterior differential systems, distributions and fibre bundles [26][30][31][32]. Not much depends on the choice of model used but it should be noted that since these models are based on different geometric structures they lend themselves to different analysis techniques. In this thesis the fibre bundle control model will be used.

Definition 3.2.1. *A* ***control system*** *is a 5-tuple* $\Sigma = (B, M, \pi_M, U, F)$ *where the 4-tuple* (B, M, π_M, U) *is a fibre bundle and a smooth map* $F : B \mapsto TM$ *such that the following diagram commutes.*

$$\begin{tikzcd} B \arrow[dd, "\pi_M"'] \arrow[dr, "F"] & \\ & TM \arrow[dl, "\pi_{TM}"] \\ M & \end{tikzcd}$$

π_M *and* π_{TM} *are the canonical projections of the fibre bundle and the tangent bundle respectively.*

The base manifold M models the state space of the control system and the typical fibre models the control input space. Locally the total space looks like a product space of the state and control input space however globally the topology can change drastically, allowing the model to accommodate instances where the control input space depends on the state space in a non-trivial way. By choosing fibre respecting coordinates for B the usual representation of the control system as a set of differential equations can be easily recovered. Within this framework of control system representation trajectories of a control system are defined as follows [33].

Definition 3.2.2. *Let $\Sigma = (B, M, \pi_M, U, F)$ be a control system, the smooth curve $\gamma^M(t) : \mathbb{R} \mapsto M$ is a trajectory of Σ if there exists a curve $\gamma^B(t) : \mathbb{R} \mapsto B$ such that:*

1. $\pi_M \circ \gamma^B(t) = \gamma^M(t)$
2. $\frac{d}{dt}\gamma^M(t) = F \circ \gamma^B(t)$

3.3 Control system equivalence

Consider two control systems $\Sigma = (B, M, \pi_M, U, F)$ and $\tilde{\Sigma} = (\tilde{B}, N, \pi_N, V, G)$, let Σ_{traj} and $\tilde{\Sigma}_{\text{traj}}$ be the set of trajectories of the control systems respectively. Σ and $\tilde{\Sigma}$ are said to be equivalent if and only if the sets Σ_{traj} and $\tilde{\Sigma}_{\text{traj}}$ can be put in one-to-one correspondence[34].

Proposition 3.3.1. *The control systems Σ and $\tilde{\Sigma}$ are said to be equivalent if there exists a bundle isomorphism $\Phi = (\phi, \psi)$ such that the following diagram commutes.*

$$\begin{tikzcd}
B \arrow[rrr, "\psi"] \arrow[dr, "F"] \arrow[dd, "\pi_M"'] & & & \tilde{B} \arrow[dl, "G"'] \arrow[dd, "\pi_N"] \\
& TM \arrow[r, "T\phi"] \arrow[dl, "\pi_{TM}"] & TN \arrow[dr, "\pi_{TN}"] & \\
M \arrow[rrr, "\phi"] & & & N
\end{tikzcd}$$

Proof. Let $\gamma^M(t)$ be a trajectory of Σ by definition there exists a curve $\gamma^B(t)$ such that $\pi_M \circ \gamma^B(t) = \gamma^M(t)$. From the fibre preserving property,

$$\begin{aligned}
\pi_N \circ \psi \circ \gamma^B(t) &= \phi \circ \pi_M \circ \gamma^B(t) \\
\pi_N \circ (\psi \circ \gamma^B)(t) &= \phi \circ \gamma^M(t)
\end{aligned}$$

This proves the first part of the trajectory definition. For the second part differentiate the curve $\phi \circ \gamma^M(t)$.

$$\begin{aligned}\frac{d}{dt}(\phi \circ \gamma^M(t)) &= T\phi \circ \frac{d}{dt}(\gamma^M(t)) \\ &= T\phi \circ F \circ \gamma^B(t) \\ &= G \circ \psi \circ \gamma^B(t) \\ &= G \circ (\psi \circ \gamma^B)(t)\end{aligned}$$

Therefore $\phi \circ \gamma^M(t)$ is a trajectory of $\tilde{\Sigma}$. Since Φ is a bundle isomorphism there exists a smooth inverse bundle morphism $\Phi^{-1} : \pi_N \mapsto \pi_M$. By following the exact same steps as above it can be shown that if $\gamma^N(t)$ is a trajectory of $\tilde{\Sigma}$ then $\phi^{-1}(\gamma^N(t))$ is a trajectory of Σ. □

The question of determining if two control systems are equivalent is one that has been studied extensively by control theoreticians. In its most general form the equivalence problem of two control systems essentially involves solving an underdetermined system of partial differential equations (i.e. $T\phi \circ F = G \circ \psi$). As with most things in control theory the equivalence problem was first solved for linear systems in which it was shown that if a control system is controllable then it is equivalent(via linear feedback transformation) to the Brunovsky canonical form [35]. The next step towards a general solution of the equivalence problem involved looking at the equivalence of a general non-linear system to a controllable linear system. This line of research yielded the much celebrated feedback linearization techniques [36]. For the case of general equivalence of non-linear systems fruitful results have been achieved by restricting equivalence to particular classes of nonlinear systems e.g equivalence to triangular systems [37], equivalence to passive systems [38], equivalence to feed-forward forms [39].

The results sketched out in the above paragraph have been developed by the application of various methodologies, some of those methods will be discussed here. The mathematician Elie Cartan developed a method for solving the equivalence problem for geometric objects aptly called "Cartan's moving frames method" [40]. Gardener [41] introduced the use of Cartan methods in developing solutions to the equivalence problem for control systems [42]. Cartan's method makes use of co-frames(basis of one-forms) and as such it works best if the control system is represented as an exterior differential system. Another method that has been widely applied in deriving equivalence results is the approach developed by Kang et al [43] [44], this method is based on Poincaré's theory of normal forms [45]. In this approach one considers the action of the Taylor series approximation of the feedback transformation term by term on the series approximation of the control system. Term wise analysis of the transformation action produces a set of algebraic equations(homological equations) which have to be solved to prove equivalence. It is important to note that since this method uses Taylor series approximations convergence of the normal forms has to be proved, unfortunately the convergence of normal forms still remains an open problem [46]. This idea of iteratively solving the equivalence problem will play a key role in the results developed in this work.

3.4 Quotients of control systems

The notion of control system quotients formalises the idea of abstracting/reducing a control system, the quotient control system is a lower order approximation of the original control system where some of the information in the original system has been factored out.

Definition 3.4.1. *[33] Consider the control systems $\Sigma_M = (B, M, \pi_M, U, F)$ and $\tilde{\Sigma} = (\tilde{B}, N, \pi_N, V, G)$ where dim$(M) >$ dim(N), $\tilde{\Sigma}$ is a* ***quotient control system*** *of Σ if there exists a fibre bundle morphism $\Phi = (\phi, \psi) : \pi_M \mapsto \pi_N$ which satisfies the following conditions,*

1. *The maps $\phi : M \mapsto N$ and $\psi : B \mapsto \tilde{B}$ are surjective submersions i.e Φ is a bundle epimorphism.*
2. *$\Phi = (\phi, \psi)$ is such that the following diagram commutes.*

$$
\begin{tikzcd}
B \arrow[rrr, "\psi"] \arrow[dr, "F"] \arrow[dd, "\pi_M"'] & & & \tilde{B} \arrow[dl, "G"'] \arrow[dd, "\pi_N"] \\
& TM \arrow[r, "T\phi"] \arrow[dl, "\pi_{TM}"] & TN \arrow[dr, "\pi_{TN}"] & \\
M \arrow[rrr, "\phi"] & & & N
\end{tikzcd}
$$

$\tilde{\Sigma}$ being a quotient of Σ implies that $\Phi = (\phi, \psi)$ carries trajectories of Σ to trajectories of $\tilde{\Sigma}$, since Φ is an epimorphism there is a many-to-one relation between Σ_{traj} and $\tilde{\Sigma}_{traj}$. It should be noted that only the existence not the uniqueness of a suitable bundle morphism is required. It is possible given a control system and its quotient to have multiple fibre bundle morphisms that effect the quotienting. Tabauda et al [33] define a notion of quotient control system that requires uniqueness of the fibre bundle morphism, this requires placing additional conditions in the definition. In this work uniqueness of the bundle morphism does not play a critical role hence it is omitted in the definition above.

An interesting property of quotients of control systems proved in Tabauda et al [33] is the surprising fact that for any control system Σ existence of a quotient control system is guaranteed under really mild conditions. The theorem is stated here with out proof.

Theorem 3.4.1. *[33] Consider the control system $\Sigma = (B, M, \pi_M, U, F)$ and $\phi : M \mapsto N$ a surjective submersion, if $T\phi \circ F : B \mapsto TN$ has constant rank and connected fibres then there exists,*

1. *a control system $\tilde{\Sigma} = (\tilde{B}, N, \pi_N, V, G)$,*

2. *a fibre preserving lift $\psi : B \mapsto \tilde{B}$ of ϕ such that $\tilde{\Sigma}$ is a quotient control system of Σ with fibre bundle morphism (ϕ, ψ).*

Consider the case where the control system Σ has a constant rank control distribution and a non-vanishing drift vector field (i.e. F is a constant rank map) then the above theorem guarentees the existence of a control system on N which is a quotient of Σ.

With the requisite mathematical constructions having been presented it becomes easier to see that most of the reduction techniques employed in control theory are actually instances of quotients. Symmetry based reduction can be viewed as quotienting where the map ϕ is the projection to the quotient manifold generated by the Lie group action on the state space manifold [47]. The same goes for system decomposition via controlled invariance distributions, the state-space quotienting map ϕ corresponds to the projection map from the state space manifold to its quotient sub-manifold generated by factoring out the integral sub-manifolds of the controlled invariant distribution [48].

3.5 Control system property preserving quotients

The analysis of control systems and synthesis of controllers becomes increasingly difficult as the dimension of the control system gets bigger. This fact has been the fundamental motivation in the study of reduction and decomposition techniques for non-linear control systems. Of course for the analysis of the lower order reduction to be of any use to the analysis of the original system the reduction process should be such that the property of interest is preserved.

3.5.1 Controllability preserving quotients of control systems

The various notions of controllability capture the idea of how much the trajectory of the system can be influenced by the control input. Controllability plays a fundamental role in control theory, most control theoretic analysis and design methods have controllability as a necessary condition. Consider as examples for linear systems the existence of a linear stabilizing feedback is equivalent to the system possessing stable uncontrollable modes [49], for most motion planning algorithms controllability is a necessary condition [50], [19], [51]. Given the ubiquity of controllability it comes as no surprise that the problem of controllability preserving quotients/reductions has been extensively studied in various flavours.

One of the earliest studies of this problem is the work of Martin et al [52], this work focusses on systems evolving on principal fibre bundles. The authors proved that if a system modelled on a principal fibre bundle is accessible then the system will be controllable if and only if the projection of the system to the base manifold is controllable. This result implies that for an accessible control system on a principal bundle to test for controllability it suffices to verify the property on the lower dimension projection of the system.

In Grasse [53] the question of controllability preserving reductions is answered via the novel concept of "liftability".

Definition 3.5.1. *[53] Consider two C^1 control systems $\Sigma = (M \times \mathbb{R}^p, M, \pi_M, \mathbb{R}^p, F)$ and $\tilde{\Sigma} = (N \times \mathbb{R}^q, N, \pi_N, \mathbb{R}^q, G)$ additionally assume that there exists a C^1 map $\Phi : M \mapsto N$. $\tilde{\Sigma}$ is said to be liftable to Σ if there exists a map $l : \mathbb{R}^q \mapsto \mathbb{R}^p$ such that for every $\omega \in \mathbb{R}^q$, the vector fields $F(\cdot, l(\omega))$ is Φ-related to $G(\cdot, \omega)$.*

If $\tilde{\Sigma}$ is liftable to Σ, Grasse [53] proved that controllability of $\tilde{\Sigma}$ is a sufficient condition for the controllability of Σ. This approach however has a major disadvantage in that it does not provide a method to construct the map l for general control systems thus limiting the applicability of this approach.

The idea of controllability preserving quotients is more explicitly used albeit under different terminology in the work of Grasse and Ho [54],[55] who make use of the language of simulation relations and also in Pappas et al [56] who use the term control system abstractions. A simulation relation between two systems is a state space equivalence relation that is compatible with the systems' dynamics, while an abstraction as defined in [56] is a state space transformation that maps trajectories into trajectories. These ideas are actually equivalent if the equivalence relation of a simulation is associated with the graph of an abstraction map. That being said Grasse et al [54] use the language of simulations to develop conditions for which simulation relations propagate controllability properties for control systems with disturbance inputs, more precisely if Σ is controllable and if there is a simulation relation with another system $\tilde{\Sigma}$ (i.e $\tilde{\Sigma}$ simulates Σ) under what conditions is $\tilde{\Sigma}$ controllable. However the reverse question(if $\tilde{\Sigma}$ is controllable and simulates Σ is Σ controllable?) is the one that is of practical interest in most instances since $\tilde{\Sigma}$ can be of lower dimension. It is this direction of inquiry that is pursued in the work of Pappas et al[56]. Given a control system Σ defined on a manifold M and an abstraction map $\Phi : M \mapsto N$, a method of constructing a control system on N which is Φ-related to Σ is developed in [56]. Furthermore they prove that if the kernel of the abstraction map is contained in the Lie algebra generated by

Σ's control bundle then the control system on N is a consistent abstraction of Σ. Consistency here implies that the abstracted control system is controllable if and only if Σ is controllable. These results are used in [57] to develop a hierarchical controllability algorithm for linear control systems.

3.5.2 Stabilizability preserving quotients of control systems

Before defining stabilizability a working definition of stability is needed.

Definition 3.5.2. *[19] Let X be a vector field on M and $c : I \subset \mathbb{R} \mapsto M$ the integral curve of X. The equilibrium point p_0 of X is called*

1. ***Lyapunov stable*** *if for any neighbourhood $\mathcal{U}$ of p_0, there exists a neighbourhood $\mathcal{W}$ such that if $p\prime \in \mathcal{W}$ and $c(0) = p\prime$ then $c(t) \in \mathcal{U}$ for all time.*

2. ***locally asymptotically stable*** *if it is Lyapunov stable and $c(t)$ converges to p_0 for all initial conditions in some neighbourhood $\mathcal{U}$ of p_0.*

Stabilizability is then defined as follows.

Definition 3.5.3. *The control system $\Sigma = (B, M, \pi_M, U, F)$ with equilibrium point $p_0 \in M$ is locally* ***stabilizable*** *if there exists an at least C^1 local bundle section of π_M defined on some neighbourhood $\mathcal{U}$ of p_0, $\alpha : \mathcal{U} \subset M \mapsto B$ such that the vector field $F \circ \alpha(p)$ has p_0 as a locally asymptotically stable equilibrium point.*

The question of stabilizability is one that has been completely answered for linear systems. A linear system is stabilizable by linear feedback if its uncontrollable modes are stable, thus for linear systems complete controllability is equivalent to stabilizability [49]. Leveraging the equivalence of controllability and stabilizability and their work on controllability preserving abstractions, Pappas et al [58] developed a method of constructing stabilizability preserving abstractions for linear systems. Their results fully characterize stabilizability preserving abstraction for linear control systems. For non-linear systems however the question is far from being conclusively answered, the major reason being that there is no simple relationship between stabilizability and controllability. This makes it impossible to follow the approach taken by Pappas et al [58] for linear systems to leverage off the vast results of controllability preserving quotients since controllability is not a necessary condition for the existence of a smooth stabilizing feedback control [59].

Despite the lack of explicit results on the problem of stabilizability preserving quotients for non-linear systems, the idea is implicitly invoked in many stabilizing feedback controller design methods. Literature on stabilizability of non-linear systems mostly takes one of two approaches, linearization based techniques or control Lyapunov techniques. In the following subsections we present centre manifold theory and immersion and invariance theory as exemplars of these techniques where quotients play a key role.

3.5.2.1 Centre manifold theory methods

It is a well known linearization result that if the linearization of a nonlinear system does not have uncontrollable modes in the open right hand plane then the original non-linear system can be locally stabilized by linear feedback [60], if there are uncontrollable modes in the open right hand plane then the original system cannot be stabilized. This leaves unanswered the case where the uncontrollable modes lie on the imaginary axis. This case was studied for single input affine non-linear systems by Aeyels [61] using the centre manifold theory, a brief sketch of the result is presented below to show how this method implicitly involves a quotienting of the stabilizability problem.

Consider the C^∞ single input affine nonlinear system

$$\dot{x} = f(x) + Bu, \quad x \in \mathbb{R}^n, \tag{3.3}$$

with $f(0) = 0$. After appropriate linear state and feedback transformation assume that the linearization of the system can be written in the following form,

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + O(2). \tag{3.4}$$

Where $x = (x^1, x^2)$ is a partition of the state-space and (A_{11}, B_1) is a controllable pair and A_{22} has pure imaginary eigenvalues. Controllability of the pair (A_{11}, B_1)

allows for arbitrary pole placement for the closed loop matrix $(A_{11} + B_1 K)$. Applying the feedback transformation $u = Kx_1 + v$, diagonalizing the feedback transformed system into the Jordan normal form and repartitioning the state space ($x = (x^1, x^2, x^3)$) gives,

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_1 \end{bmatrix} v + \begin{bmatrix} f_1(x^1, x^1, x^3) \\ f_2(x^1, x^2, x^3) \\ f_3(x^1, x^2, x^3) \end{bmatrix}. \tag{3.5}$$

Where the matrices A_{22} has eigenvalues in the open left hand plane, A_{11} has critical eigenvalues and the pair (A_{33}, B_1) is controllable. Effectively the original stabilization problem for the system (3.3) has been transformed to finding a stabilizing feedback $v(x^1, x^2, x^3)$ for (3.5). The centre manifold is defined as follows [62].

Definition 3.5.4. *Let E_c be the eigenspace of the matrix A_{11}, the submanifold manifold $W_c \subset \mathbb{R}^n$ is called the centre manifold for (3.5) if W_c is locally invariant and tangent to the eigenspace E_c. By the implicit function theorem,*

$$W_c = \{(x^1, x^2, x^3) | (x^2, x^3) = h(x^1), h(0) = Dh(0) = 0\}, with \quad h = (h_1, h_2)$$

where h is some smooth map.

By construction the dynamics of (x^2, x^3) are linearly stable therefore the stability of 3.5 can be determined by studying the projected dynamics onto the centre manifold. The projected centre dynamics are given by,

$$\dot{x}^1 = A_{11}x^1 + f_1(x^1, h_1(x^1), h_2(x^1)). \tag{3.6}$$

This shows that the centre manifold approach is effectively a reduction based technique, the original system (3.3) will be stabilizable if and only if the lower dimensional system (3.6) is stable. To define the centre manifold the map h must be determined, this is not an easy task as it involves solving the system of partial differential equations shown below [61].

$$\begin{aligned}\frac{\partial h_1}{\partial x^1}\left[A_{11}x^1 + f_1(x^1, h_1(x^1), h_2(x^1))\right] &= A_{22}h_1(x_1) + f_2(x^1, h_1(x_1), h_2(x_1)) \\ \frac{\partial h_2}{\partial x^1}\left[A_{11}x^1 + f_1(x^1, h_1(x^1), h_2(x^1))\right] &= A_{33}h_2(x^1) + f_3(x^1, h_1(x^1), h_2(x^1))\end{aligned}$$

3.5.2.2 Immersion and Invariance method

The immersion and invariance method for nonlinear systems initially appeared in the work of Astolfi et al[63]. A brief account of this method is presented here as an example of a control Lyapunov function based solution to the stabilizability problem which implicitly employs system reduction. Below we state without proof the theorem central to the immersion and invariance method, for the proof see [63].

Theorem 3.5.1. *[63] Consider the control affine system,*

$$\dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{3.7}$$

with equilibrium point x_0. Assume that there exists

1. *Target dynamics given by the dynamical system,*

$$\dot{\xi} = \alpha(\xi), \xi \in \mathbb{R}^p, p < n \tag{3.8}$$

   with an asymptotically stable equilibrium point ξ_0.

2. *An immersion $\pi : \mathbb{R}^p \mapsto \mathbb{R}^n$, $\pi(\xi_0) = x_0$ and a map $c : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that*

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi). \tag{3.9}$$

3. *A map $\phi : \mathbb{R}^n \mapsto \mathbb{R}^{n-p}$ such that the image of the immersion π can be expressed as a level set of ϕ.*

$$\{x \in \mathbb{R}^n | \phi(x) = 0\} = \{x \in \mathbb{R}^n | \exists \xi \in \mathbb{R}^p, \pi(\xi) = x\}, \tag{3.10}$$

4. All the trajectories of the system

$$\begin{aligned}\dot{z} &= \frac{\partial \phi}{\partial x}\left[f(x) + g(x)\psi(x,z)\right] \quad (3.11)\\ \dot{x} &= f(x) + g(x)\psi(x,z) \quad (3.12)\end{aligned}$$

are bounded and satisfy $\lim_{x\to\infty} z(t) = 0$.

Then x_0 *is a globally asymptotically stable equilibrium point of the closed loop system*

$$\dot{x} = f(x) + g(x)\psi(x, \phi(x)). \quad (3.13)$$

The target dynamics (3.8) are assumed to be given *a priori* however in practice it's usually the case that one quotients the original dynamics via symmetry, invariant distributions e.t.c and then designs a stabilizing controller for the quotient system to produce the target dynamics. Thus the immersion and invariance approach can be viewed as providing a method to design a stabilizing controller for the original higher order system if the lower order quotient system is stabilizable. As with the center manifold approach finding the map π involves solving the partial differential equation (3.9) with c being a free parameter, this is a non-trivial task and solutions are not guaranteed.

The center manifold approach and the immersion and invariance approach are typical of the general trend within control theory regarding quotienting/reduction based study of system stabilizability. Most methods for designing stabilizing controllers tacitly involve the interplay of quotient maps and stabilizability, examples include methods such as backstepping control and sliding mode control. However the general question of stabilizability preserving quotients has not been addressed in its own right(with the exception of the linear case[58]). This work aims at contributing to the field of control theory by explicitly addressing the question of stabilizability preserving quotients for general nonlinear systems.

Chapter 4

Main Results

Summary

This is the central chapter of this thesis. In this chapter we develop and prove the main result of this work.

4.1 Introduction

Before presenting the main results of this work we restate the research question using the differential geometric and control theoretic tools that have been developed in the previous chapters.

4.1.1 Problem statement

Problem Statement 2. *Consider the two control affine systems* $\Sigma = (B, M, \pi_M, U, F)$ *and* $\tilde{\Sigma} = (E, N, \pi_N, V, G)$. *Assume that*

1. *dim(M)* $\geq$ *dim(N)*
2. $\tilde{\Sigma}$ *is a quotient control system of* Σ, *i.e there exists a fibre bundle epimorphism* $\Phi = (\phi, \psi)$ *such that the following diagram commutes.*

$$
\begin{tikzcd}
B \arrow[rrr, "\psi"] \arrow[dd, "\pi_M"'] \arrow[dr, "F"] & & & E \arrow[dd, "\pi_N"] \arrow[dl, "G"'] \\
& TM \arrow[r, "T\phi"] \arrow[dl, "\pi_{TM}"] & TN \arrow[dr, "\pi_{TN}"] & \\
M \arrow[rrr, "\phi"] & & & N
\end{tikzcd}
$$

3. *$\tilde{\Sigma}$ is stabilizable about some equilibrium $q_0 \in N$, i.e there exists a local control Lyapunov function $\tilde{V} : N \mapsto \mathbb{R}$ and a local section of π_N, $\alpha \in \Gamma(\pi_N)$ such that the vector field $G \circ \alpha(q)$ is locally asymptotically stable about q_0.*

Under what conditions is Σ stabilizable about some equilibrium point $p \in \phi^{-1}(q_0)$.

It is important to highlight that this question does not focus on the issue of stabilizability propagation via quotients. The question of stabilizability propagation via quotients looks at the possibility of constructing quotienting maps in such a way that if the original system was (un)stabilizable then the quotient system would also be (un)stabilizable. This work looks only at the converse direction of this question since this is of practical importance for hierarchical stabilizing controller design.

By stability we specifically refer to Lyapunov stability: as such, we identify stabilizability with the existence of a control Lyapunov function. The problem statement above can be re-interpreted as saying "under what conditions does Σ possess a control Lyapunov function given the assumed data". A solution is constructed by devising a new method of constructing a Lyapunov function for Σ using the assumed data.

Before presenting the main result an auxiliary result for the case where $\tilde{\Sigma}$ is a linear controllable system is presented. This auxiliary result ties some of the ideas developed in this work with the established results in feedback linearization theory. It should be noted that this auxiliary result has been published in [6].

4.2 The case for linear controllable quotient

In this section we provide a solution to the following variation of the research problem.

Problem Statement 3. *Consider the two control affine systems* $\Sigma = (B, M, \pi_M, U, F)$ *and* $\tilde{\Sigma} = (E, N, \pi_N, V, G)$*. Assume that*

1. *dim(M) > dim(N).*
2. *dim(U) = dim(V).*
3. $\tilde{\Sigma}$ *is a linear controllable system.*

Under what conditions is Σ ***locally stabilizable about the equilibrium point*** p***, where*** p ***gets mapped to the origin****.*

From the above assumptions $\tilde{\Sigma}$ is strictly of a lower dimension than Σ, both control systems have the same number of control inputs and since $\tilde{\Sigma}$ is linear and controllable it is also linearly stabilizable. Before stating the solution we define the novel concept of a horizontally lifted control system which will be used in the solution.

Definition 4.2.1. *Consider the control affine system* $\tilde{\Sigma} = (E, N, \pi_N, V, G)$*, the fibre bundle* (B, M, π_M, U) *and the fibre bundle morphism* $\Phi = (\phi, \psi) : \pi_M \mapsto \pi_N$*. The* ***horizontally lifted system*** $\tilde{\Sigma}^H$ *is the control system on* π_M *induced by* $\tilde{\Sigma}$ *defined as* $\tilde{\Sigma}^H = (B, M, \pi_M, U, G^H)$ *with*

$$G^H(p\prime) = Hor_p \circ G \circ (\phi, \psi) \circ p\prime. \tag{4.1}$$

where Hor_p *is the horizontal lift map defined by the Ehresmann connection on the fibre bundle* $\phi : M \mapsto N$ *and* $p \in M, p\prime \in B, \pi_M(p\prime) = p$*.*

The following proposition shows the relation between the trajectories of $\tilde{\Sigma}^H$ and $\tilde{\Sigma}$.

Proposition 4.2.1. *If $\gamma^M(t) \in M$ is a trajectory of the horizontally lifted system $\tilde{\Sigma}^H$, then the curve $\phi \circ \gamma^M(t) \in N$ is a trajectory of $\tilde{\Sigma}$.*

Proof. Since $\gamma^M(t) \in M$ is a trajectory of $\tilde{\Sigma}^H$ by definition there exists a curve $\gamma^B(t) \in B$ such that,

$$\frac{d}{dt}\gamma^M(t) = G^H(\gamma^B(t)) = \mathrm{Hor}_{\gamma^M(t)} \circ G \circ (\phi, \psi) \circ \gamma^B(t). \tag{4.2}$$

Consider the curve $\phi \circ \gamma^M(t) \in N$, by the bundle morphism property $\pi_N \circ (\psi, \phi) \circ \gamma^B(t) = \phi \circ \pi_M \circ \gamma^B(t) = \phi \circ \gamma^M(t)$. Now differentiating $\phi \circ \gamma^M(t)$,

$$\begin{aligned}
\frac{d}{dt}\phi \circ \gamma^M(t) &= T_{\gamma^M(t)}\phi \circ \frac{d}{dt}\gamma^M(t) && (4.3)\\
&= T_{\gamma^M(t)}\phi \circ G^H(\gamma^B(t)) && (4.4)\\
&= T_{\gamma^M(t)}\phi \circ \mathrm{Hor}_{\gamma^M(t)} \circ G \circ (\phi, \psi) \circ \gamma^B(t) && (4.5)\\
&= G \circ (\phi, \psi) \circ \gamma^B(t). && (4.6)
\end{aligned}$$

The last equality holds since $T\phi$ and Hor are inverses by definition. Therefore $\phi \circ \gamma^M(t)$ is a trajectory of $\tilde{\Sigma}$. □

We can now state the auxiliary result.

Theorem 4.2.1. *Assume the linear feedback $\nu : N \mapsto E$ stabilizes the control system $\tilde{\Sigma}$ defined in the problem statement 3. The system Σ is stabilizable if the zero dynamics autonomous system (Z) defined below is stable.*

$$Z(p) = (F - G^H) \circ \psi^{-1} \circ \nu \circ \phi(p) \tag{4.7}$$

Proof. Since only the local case is considered assume that there are charts with adapted coordinates $(\mathbf{x}^1, \mathbf{x}^2, u)$, $(\mathbf{y}, v)$ for π_M and π_N respectively such that the systems have the following forms

$$\begin{aligned}
\Sigma : \dot{\mathbf{x}}^1 &= f_0^1(\mathbf{x}^1, \mathbf{x}^2) + f_1^1(\mathbf{x}^1, \mathbf{x}^2)u \\
\dot{\mathbf{x}}^2 &= f_0^2(\mathbf{x}^1, \mathbf{x}^2) + f_1^2(\mathbf{x}^1, \mathbf{x}^2)u && (4.8)
\end{aligned}$$

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}v \tag{4.9}$$

and the bundle morphism has the following form

$$\begin{aligned} \phi(\mathbf{x}^1, \mathbf{x}^2) &\mapsto (\mathbf{y} = \mathbf{x}^1) && (4.10) \\ \psi(\mathbf{x}^1, \mathbf{x}^2, u) &\mapsto (v = \alpha(\mathbf{x}^1, \mathbf{x}^2) + \beta(\mathbf{x}^1, \mathbf{x}^2)u) && (4.11) \end{aligned}$$

For the fibre bundle $\phi : M \mapsto N$ the vertical bundle($V\phi$) and horizontal bundle($H\phi$) take the form,

$$\begin{aligned} V\phi &= \operatorname{span}\left\{\frac{\partial}{\partial \mathbf{x}^2}\right\} && (4.12) \\ H\phi &= \operatorname{span}\left\{\frac{\partial}{\partial \mathbf{x}^1} + \Psi(\mathbf{x}^1, \mathbf{x}^2)\frac{\partial}{\partial \mathbf{x}^2}\right\}. && (4.13) \end{aligned}$$

where $\Psi(\mathbf{x}^1, \mathbf{x}^2)$ is a matrix $(m - n) \times n$ matrix, $m = \dim(M), n = \dim(N)$. The distribution $H\phi$ defines the vector valued connection one-form given by

$$K = \left(d\mathbf{x}^2 - \Psi(x^1, x^2)d\mathbf{x}^1\right) \otimes \frac{\partial}{\partial \mathbf{x}^2}. \tag{4.14}$$

Since $\tilde{\Sigma}$ is a quotient of Σ under the action of the bundle morphism (ϕ, ψ) by definition this implies that $G \circ (\phi, \psi) = T\phi \circ F$. This gives the following equality,

$$f_0^1(\mathbf{x}^1, \mathbf{x}^2) + f_1^1(\mathbf{x}^1, \mathbf{x}^2)u = \mathbf{A}\mathbf{x}^1 + \mathbf{B}\alpha(\mathbf{x}^1, \mathbf{x}^2) + \mathbf{B}\beta(\mathbf{x}^1, \mathbf{x}^2)u. \tag{4.15}$$

Applying the connection vector valued one-form K, the control system Σ can be split into its vertical(Σ^V) and horizontal components(Σ^H).

$$\Sigma^V = K(\Sigma) = \begin{bmatrix} 0 \\ f_0^2(\mathbf{x}^1, \mathbf{x}^2) + f_1^2(\mathbf{x}^1, \mathbf{x}^2)u - \Psi(\mathbf{x}^1, \mathbf{x}^2)(f_0^1(\mathbf{x}^1, \mathbf{x}^2) + f_1^1(\mathbf{x}^1, \mathbf{x}^2)u) \end{bmatrix} \tag{4.16}$$

$$\Sigma^H = \Sigma - K(\Sigma) = \begin{bmatrix} f_0^1(\mathbf{x}^1, \mathbf{x}^1) + f_1^1(\mathbf{x}^1, \mathbf{x}^2)u \\ \Psi(\mathbf{x}^1, \mathbf{x}^2)(f_0^1(\mathbf{x}^1, \mathbf{x}^1) f_1^1(\mathbf{x}^1, \mathbf{x}^2)u) \end{bmatrix} \tag{4.17}$$

If $\tilde{\Sigma}$ is stabilizable by the linear feedback $v = -\mathbf{K}\mathbf{y}$ this induces the feedback u^* for Σ given by

$$u^*(\mathbf{x}^1, \mathbf{x}^2) = \psi^{-1} \circ v \circ \phi(\mathbf{x}^1, \mathbf{x}^2) = \frac{-\mathbf{K}\mathbf{x}^1 - \alpha(\mathbf{x}^1, \mathbf{x}^2)}{\beta(\mathbf{x}^1, \mathbf{x}^2)}. \tag{4.18}$$

Applying the feedback u^* to the horizontal component of Σ and making use of the equality condition from equation (4.15) ,

$$\Sigma_{CL}^H = \begin{bmatrix} \mathbf{A}\mathbf{x}^1 - \mathbf{B}\mathbf{K}\mathbf{x}^1 \\ \Psi(\mathbf{x}^1, \mathbf{x}^2)(\mathbf{A}\mathbf{x}^1 - \mathbf{B}\mathbf{K}\mathbf{x}^1) \end{bmatrix} \tag{4.19}$$

The closed loop dynamics Σ_{CL}^H are actually just the horizontally lifted dynamics of the closed loop dynamics of $\tilde{\Sigma}$, as such the dynamics Σ_{CL}^H have the submanifold defined by the level set $\phi(\mathbf{x}^1, \mathbf{x}^2) = 0$ as an asymptotically stable submanifold. Additionally once on this submanifold the vector field $\Sigma_{CL}^H = \mathbf{0}$. Focussing attention on the vertical component of Σ it is evident that the same submanifold($\phi(\mathbf{x}^1, \mathbf{x}^2) = \mathbf{0}$) is an invariant submanifold of Σ^V. Thus the stabilizability of the control system turns on the (in)stability of the closed loop dynamics of Σ^V under the action of the feedback u^*. The closed loop dynamics of the vertical component of Σ are

$$\Sigma_{CL}^V = \begin{bmatrix} 0 \\ f_0^2(\mathbf{x}^1, \mathbf{x}^2) + f_1^2(\mathbf{x}^1, \mathbf{x}^2) \left[\frac{-\mathbf{K}\mathbf{x}^1 - \alpha(\mathbf{x}^1, \mathbf{x}^2)}{\beta(\mathbf{x}^1, \mathbf{x}^2)} \right] - \Psi(\mathbf{x}^1, \mathbf{x}^2) \left(\mathbf{A}\mathbf{x}^1 - \mathbf{B}\mathbf{K}\mathbf{x}^1 \right) \end{bmatrix} \tag{4.20}$$

.

The dynamics described by Σ_{CL}^V represent the closed loop dynamics of system Σ on the asymptotically stable submanifold $\phi(\mathbf{x}^1, \mathbf{x}^2) = \mathbf{0}$. Evaluating the zero dynamics as defined in the proposition shows that $Z(\mathbf{x}^1, \mathbf{x}^2) = \Sigma_{CL}^V$.

□

The fact that stabilizability of a partially feedback linearizable system depends on the (in)stability of the zero dynamics is a standard result in geometric control theory [27]. In the standard approach determining the zero dynamics essentially involves solving a set of partial differential equations [27]. Our method of constructing the zero dynamics does not have this drawback and hence presents an easier way of calculating zero dynamics.

Example 4.2.1. *Consider the SISO system* $\Sigma = (B = \mathbb{R}^4, M = \mathbb{R}^3, \pi_M, \mathbb{R}, F)$, *with the following coordinates for the manifolds involved* $B = (x^1, x^2, x^3, u)$, $M = (x^1, x^2.x^3)$ *and the map* F *defined as*

$$F(x^1, x^2, x^3, u) = \begin{bmatrix} -x^1 \\ x^1x^2 \\ x^2 \end{bmatrix} + \begin{bmatrix} e^{x^2} \\ 1 \\ 0 \end{bmatrix} u \tag{4.21}$$

The control system $\tilde{\Sigma} = (\tilde{B} = \mathbb{R}^3, N = \mathbb{R}^2, \pi_N, \mathbb{R}, G)$, *with the following coordinates* $\tilde{B} = (z^1, z^2, v)$, $N = (z^1, z^2)$ *and the map* G *defined as*

$$G(z^1, z^2, v) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \tag{4.22}$$

is a quotient of Σ *under the action of the fibre bundle morphism* $\Phi = (\phi, \psi)$ *defined as*

$$\phi \ : \ (x^1, x^2, x^3) \mapsto (z^1 = x^3, z^2 = x^2) \tag{4.23}$$
$$\psi \ : \ (x^1, x^2, x^3, u) \mapsto (z^1 = x^3, z^2 = x^2, v = u + x^1x^2) \tag{4.24}$$

To calculate the zero dynamics of Σ *define a connection one-form* K *for the fibre bundle* $\phi : M \mapsto N$,

$$K = \left(dx^1 - \Gamma^1_2(x^1, x^2, x^3)dx^2 - \Gamma^1_3(x^1, x^2, x^3)dx^3\right) \otimes \frac{\partial}{\partial x^1}. \tag{4.25}$$

This induces a horizontal lift map $Hor_{\mathbf{x}} : \mathbb{R}^2 \mapsto \mathbb{R}^3$ *which in matrix form is*

$$Hor_{(x^1,x^2,x^3)} = \begin{bmatrix} \Gamma^1_3(x^1, x^2, x^3) & \Gamma^1_2(x^1, x^2, x^3) \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.26}$$

.

Evaluating the zero dynamics $Z(x^1, x^2, x^3)$ gives

$$Z(x^1,x^2,x^3) = -x^1 - x^2\Gamma^1_3(x^1,x^2,x^3) - x^1x^2\Gamma^1_2(x^1,x^2,x^3) - u(\Gamma^1_2(x^1,x^2,x^3) - e^{x^2}). \tag{4.27}$$

In as much as any choice of the Γ's will do, a careful selection of these coefficients can actually make the stability study of zero dynamics easier. If we choose $\Gamma^1_2(x^1,x^2,x^3) = e^{x^2}$ and $\Gamma^1_3(x^1,x^2,x^3) = -x^1e^{x^2}$ the zero dynamics become the trivially stable dynamics

$$Z(x^1,x^2,x^3) = -x^1. \tag{4.28}$$

4.3 The general case

For the general case the problem is re-stated with all the relevant assumptions for convenience.

Problem Statement 4. *Consider two control affine systems $\Sigma = (B, M, \pi_M, U, F)$ and $\tilde{\Sigma} = (\tilde{B}, N, \pi_N, V, G)$. Assume that*

1. *dim(M) > dim(N),*
2. *the control distribution defined by $F \circ \pi_M^{-1}(p) \subset T_pM$ is constant rank and smooth,*
3. *$\tilde{\Sigma}$ is a quotient control system of Σ under the action of the smooth fibre bundle morphism $\Phi = (\phi, \psi)$*
4. *the fibre bundle morphism Φ in adapted coordinates (x^i, u^j) and (y^q, v^k) for π_M and π_N respectively has the following form*

$$\phi \;:\; (x^i) \mapsto (y^q = x^q) \tag{4.29}$$

$$\psi \;:\; (x^i, u^j) \mapsto (y^q = x^q, v^k = \varphi^k(x^i) + u^j\beta^k_j(x^i)). \tag{4.30}$$

where $i = 1, \cdots, m$, $j = 1, \cdots, r$, $q = 1, \cdots, n$, $k = 1, \cdots, s$ and the functions $\varphi^k(x^i)$, $\beta^k_j(x^i)$ are all smooth.

5. *$\tilde{\Sigma}$ is stabilizable i.e there exists a smooth section α of π_N and a positive definite function $\tilde{V} : N \mapsto \mathbb{R}$ such that the closed loop dynamics $G \circ \alpha(q), q \in N$ are asymptotically stable in the Lyapunov sense about the equilibrium point q_0.*

***Under what conditions is Σ locally stabilizable about the equilibrium point** $p_0 \in \phi^{-1}(q_0)$?*

4.3.1 Main theorem

Let $\Sigma = (B, M, \pi_M, U, F)$ and $\tilde{\Sigma} = (\tilde{B}, N, \pi_N, V, G)$ be control affine systems. Assume there are coordinate charts $\mathcal{O} \subset B$, $\mathcal{Q} \subset \tilde{B}$ such that there are coordinates $(x^1, \cdots, x^m, u^1, \cdots, u^r)$ and $(y^1, \cdots, y^n, v^1, \cdots, v^s)$ for $\mathcal{O} \subset B$ and $\mathcal{Q} \subset \tilde{B}$ respectively such that $q_0 = \mathbf{0}$ and $p_0 = \mathbf{0}$. The control systems Σ and $\tilde{\Sigma}$ can be written in the following form,

$$\Sigma \quad : \quad \begin{bmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^m \end{bmatrix} = \begin{bmatrix} f_0^1(\mathbf{x}) \\ \vdots \\ f_0^m(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} f_1^1(\mathbf{x}) \\ \vdots \\ f_1^m(\mathbf{x}) \end{bmatrix} u^1 + \cdots + \begin{bmatrix} f_r^1(\mathbf{x}) \\ \vdots \\ f_r^m(\mathbf{x}) \end{bmatrix} u^r. \tag{4.31}$$

$$\tilde{\Sigma} \quad : \quad \begin{bmatrix} \dot{y}^1 \\ \vdots \\ \dot{y}^n \end{bmatrix} = \begin{bmatrix} g_0^1(\mathbf{y}) \\ \vdots \\ g_0^n(\mathbf{y}) \end{bmatrix} + \begin{bmatrix} g_1^1(\mathbf{y}) \\ \vdots \\ g_1^n(\mathbf{y}) \end{bmatrix} v^1 + \cdots + \begin{bmatrix} g_s^1(\mathbf{y}) \\ \vdots \\ g_s^n(\mathbf{y}) \end{bmatrix} v^s. \tag{4.32}$$

Where $\mathbf{x} = (x^1, \cdots, x^m)$ and $\mathbf{y} = (y^1, \cdots, y^n)$. As stated in the problem statement (4) it is assumed that $\tilde{\Sigma}$ is stabilizable about the origin, i.e there exists feedback controls $v^1 = \alpha^1(\mathbf{y}), \cdots, v^s = \alpha^s(\mathbf{y})$ and a control Lyapunov function $\tilde{V} : N \mapsto \mathbb{R}$ such that

$$\begin{aligned} W(\mathbf{y}) \quad = \quad & \frac{\partial \tilde{V}}{\partial y^1} \left[g_0^1(\mathbf{y}) + g_1^1(\mathbf{y})\alpha^1(\mathbf{y}) + \cdots + g_s^1(\mathbf{y})\alpha^s(\mathbf{y}) \right] + \\ \cdots \quad & + \frac{\partial \tilde{V}}{\partial y^n} \left[g_0^n(\mathbf{y}) + g_s^1(\mathbf{y})\alpha^1(\mathbf{y}) + \cdots + g_s^n(\mathbf{y})\alpha^s(\mathbf{y}) \right], \end{aligned} \tag{4.33}$$

where $W(\mathbf{y})$ is a negative definite function. From the control system Σ's control vector fields construct the constant rank smooth distribution C defined by,

$$C = \text{span}\left\{ f_1^i(\mathbf{x})\frac{\partial}{\partial x^i}, \cdots, f_r^i(\mathbf{x})\frac{\partial}{\partial x^i} \right\}. \tag{4.34}$$

Since C is constant rank and smooth it is possible to construct a complementary constant rank and smooth distribution D such that $TM = C \oplus D$. The canonical projection onto the distribution D will be denoted $P_D : TM \mapsto D$. The projection map P_D can be represented as a vector valued one-form. Let $D = \text{span}\{e_1, \cdots, e_{m-r}\}$ then $P_D = P^a_{D,i}(\mathbf{x})dx^i \otimes e_a$ for $a = 1, \cdots, m-r$. In matrix form P_D is represented as a $(m-r) \times m$ matrix where the co-efficient $P^a_{D,i}$ corresponds to the (a, i) matrix element.

The fibre bundle $\phi : M \mapsto N$ can be equipped with a connection defined below.

Proposition 4.3.1. *The fibre bundle defined by the surjective submersion $\phi : M \mapsto N$ can be equipped with an Ehresmann connection which in coordinates $(x^1, \cdots, x^m)$ and $(y^1, \cdots, y^n)$ for M and N respectively has the following equivalent representations.*

1. *The canonical vertical bundle of the fibre bundle denoted VM has the form $VM = span\{\frac{\partial}{\partial x^{n+1}}, \cdots, \frac{\partial}{\partial x^m}\}$. A connection defined as a complementary subspace to the canonical vertical bundle will be denoted HM and has the following form*

$$\begin{aligned} HM &= span\{\frac{\partial}{\partial x^1} + \Gamma_1^{n+1}(\mathbf{x})\frac{\partial}{\partial x^{n+1}} + \cdots + \Gamma_1^m(\mathbf{x})\frac{\partial}{\partial x^m}, \\ \cdots &, \frac{\partial}{\partial x^n} + \Gamma_n^{n+1}\frac{\partial}{\partial x^{n+1}} + \cdots + \Gamma_n^m(\mathbf{x})\frac{\partial}{\partial x^m}\}. \end{aligned} \tag{4.35}$$

2. *As a $V\pi$-valued one-form the connection has the following form,*

$$\begin{aligned} K &= \left(dx^{n+1} - \Gamma_1^{n+1}(\mathbf{x})dx^1 - \cdots - \Gamma_n^{n+1}(\mathbf{x})dx^n\right) \otimes \frac{\partial}{\partial x^{n+1}} + \cdots \\ &+ \left(dx^m - \Gamma_1^m(\mathbf{x})dx^1 - \cdots - \Gamma_n^m(\mathbf{x})dx^n\right) \otimes \frac{\partial}{\partial x^m}. \end{aligned} \tag{4.36}$$

3. *The connection defines the horizontal lift map as a $H\pi$-valued one-form*

denoted $Hor_{\mathbf{x}}$,

$$\begin{aligned} Hor_{\mathbf{x}} &= dy^1 \otimes \left(\frac{\partial}{\partial x^1} + \Gamma_1^{n+1}(\mathbf{x}) \frac{\partial}{\partial x^{n+1}} + \cdots + \Gamma_1^m(\mathbf{x}) \frac{\partial}{\partial x^m} \right) + \cdots \\ &+ dy^n \otimes \left(\frac{\partial}{\partial x^n} + \Gamma_n^{n+1}(\mathbf{x}) \frac{\partial}{\partial x^{n+1}} + \cdots + \Gamma_n^m(\mathbf{x}) \frac{\partial}{\partial x^m} \right). \end{aligned} \tag{4.37}$$

The main theorem which answers the research problem as stated above in problem statement (4) can now be stated below.

Theorem 4.3.1. *Consider the control affine system $\Sigma = (B, M, \pi_M, U, F)$ and assume that this system has a stabilizable quotient system $\tilde{\Sigma}$ which is also a control affine system. Given the assumptions stated in the problem statement (4). The control system Σ is stabilizable if the following conditions are met.*

1. *The control systems Σ and $\tilde{\Sigma}$ are analytic.*

2. *The fibre bundle $\phi : M \mapsto N$ is equipped with a flat connection.*

3.

$$A_{i_1}^{a_1,a_2} = \sum_{i=1}^{m} \left[P_{D,i}^{a_1}(\mathbf{x}) \frac{\partial}{\partial x^i} \left(P_{D,i_1}^{a_2}(\mathbf{x}) \right) - P_{D,i}^{a_2}(\mathbf{x}) \frac{\partial}{\partial x^i} \left(P_{D,i_1}^{a_1}(\mathbf{x}) \right) \right] = 0. \tag{4.38}$$

4. *For $X = Hor_{\mathbf{x}} \circ G \circ \alpha(\phi(\mathbf{x})) - \mathbf{d}\phi^*(\tilde{V})(\mathbf{x}) - f_0(\mathbf{x})$,*

$$B^{a_1,a_2} = \sum_{i=1}^{m} \sum_{i_1=1}^{m} \left[P_{D,i}^{a_2}(\mathbf{x}) P_{D,i_1}^{a_1}(\mathbf{x}) - P_{D,i}^{a_1}(\mathbf{x}) P_{D,i_1}^{a_2}(\mathbf{x}) \right] \frac{\partial}{\partial x^i} (X^{i_1})(\mathbf{x}) = 0. \tag{4.39}$$

4.3.2 Proof

Before proceeding with the technical content of the proof a sketch of the main ideas guiding the proof will be presented. Since stabilizability implies the existence of a control Lyapunov function [5], proving stabilizability of Σ is equivalent to proving the existence of a control Lyapunov function for Σ. The theorem quoted above therefore provides conditions under which it is possible to construct a control Lyapunov function for Σ from the assumed data. The proof contained proceeds in a constructive manner showing how such a control Lyapunov function can be constructed. Below is a list of the steps that are followed in the proof.

1. Postulate a form of the control Lyapunov function of Σ which will be denoted $V^*(\mathbf{x}) = \phi^*\tilde{V}(\mathbf{x}) + V(\mathbf{x})$. where $V(\mathbf{x})$ is an unknown modification term which needs to determined and $\phi^*\tilde{V}$ is the pull-back of $\tilde{V}$.

2. Construct target dynamics Σ^{target}, which is a vector field that is asymptotically stable in the Lyapunov sense with V^* being a Lyapunov function for the vector field Σ^{target}.

3. Set up a partial differential in V such that if a solution exists then there exists a feedback controller that makes the closed loop dynamics of Σ equal to the target dynamics Σ^{target}.

4. Using the geometric theory of partial differential equations, extract the integrability conditions for the partial differential equation. The conditions stated in the theorem above will turn out to be the integrability conditions for this partial differential equation.

4.3.2.1 Target dynamics

Proposition 4.3.2. *Consider the affine control system* $\tilde{\Sigma} = (\tilde{B}, N, \pi_N, V, G)$ *which in some adapted chart* $\mathcal{Q} \subset \tilde{B}$ *has the form given in equation (4.32) and is stabilizable. There exists a local section* $\alpha : \pi_N^{-1}(\mathcal{Q}) \subset N \mapsto \mathcal{Q} \subset \tilde{B}$ *and a local control Lyapunov function* $\tilde{V} : \pi_N^{-1}(\mathcal{Q}) \subset N \mapsto \mathbb{R}, \tilde{V}(\mathbf{0}) = 0$ *such that the vector field* $G \circ \alpha(\mathbf{y})$ *is asymptotically stable about the origin. Let* $\phi : \mathcal{O} \subset M \mapsto \mathcal{Q} \subset N$ *be a smooth surjective submersion with* $\phi(\mathbf{0}) = \mathbf{0}$. *Assume there exists a function* $V : \mathcal{O} \subset M \mapsto \mathbb{R}, V(\mathbf{0}) = 0$ *such that*

1. $\phi^*\tilde{V}(\mathbf{x}) + V(\mathbf{x})$ *is positive definite,*

2. *the differential of* V *annihilates horizontal vector fields,* $\mathbf{d}V \in \mathbf{ann}(HM)$.

Then the vector field Σ^{target} defined below is locally asymptotically stable about the origin.

$$\Sigma^{target} = Hor_{\mathbf{x}} \circ G \circ \alpha \circ \phi(x) - \Delta^{\sharp} \circ \mathbf{d}\phi^* \tilde{V}(\mathbf{x}) - \Delta^{\sharp} \circ \mathbf{d}V(\mathbf{x}). \tag{4.40}$$

Where $\Delta^{\sharp}$ is a bundle isomorphism $\Delta^{\sharp} : T^*M \mapsto TM$, $\Delta^{\sharp}(x^i, \omega_i dx^i) \mapsto (x^i, \omega_{i_1} \delta^{i_1,i} \frac{\partial}{\partial x^i})$. Note $\delta^{i_1,i}$ is the Kronecker delta symbol.

Proof. To prove that Σ^{target} is asymptotically stable at the origin consider the coordinate form of Σ^{target}.

$$\Sigma^{target} = \begin{bmatrix} \tilde{g}^1(\mathbf{x}^1) - \frac{\partial \tilde{V}}{\partial x^1}(\mathbf{x}^1) - \frac{\partial V}{\partial x^1}(\mathbf{x}^1, \mathbf{x}^2) \\ \vdots \\ \tilde{g}^n(\mathbf{x}^1) - \frac{\partial \tilde{V}}{\partial x^n}(\mathbf{x}^1) - \frac{\partial V}{\partial x^n}(\mathbf{x}^1, \mathbf{x}^2) \\ \Gamma_1^{n+1}(\mathbf{x}^1, \mathbf{x}^2)\tilde{g}^1(\mathbf{x}^1) + \cdots + \Gamma_n^{n+1}(\mathbf{x}^1, \mathbf{x}^2)\tilde{g}^n(\mathbf{x}^1) - \frac{\partial V}{\partial x^{n+1}}(\mathbf{x}^1, \mathbf{x}^2) \\ \vdots \\ \Gamma_1^{m}(\mathbf{x}^1, \mathbf{x}^1)\tilde{g}^1(\mathbf{x}^1) + \cdots + \Gamma_n^{m}(\mathbf{x}^1, \mathbf{x}^2)\tilde{g}^n(\mathbf{x}^1) - \frac{\partial V}{\partial x^{m}}(\mathbf{x}^1, \mathbf{x}^2) \end{bmatrix} \tag{4.41}$$

where $\mathbf{x}^1 = (x^1, \cdots, x^n)$, $\mathbf{x}^2 = (x^{n+1}, \cdots, x^m)$ and $\tilde{g}^q(\mathbf{x}^1) = g_0^q(\mathbf{x}^1) + \alpha^1(\mathbf{x}^1)g_1^q(\mathbf{x}^1) + \ldots \alpha^s(\mathbf{x}^1)g_s^q(\mathbf{x}^1)$. The assumption that $\mathbf{d}V \in \mathbf{ann}(HM)$ implies that V satisfies the following system of partial differential equations.

$$\begin{aligned} \frac{\partial V}{\partial x^1}(\mathbf{x}) + \Gamma_1^{n+1}(\mathbf{x})\frac{\partial V}{\partial x^{n+1}}(\mathbf{x}) + \cdots + \Gamma_1^{m}(\mathbf{x})\frac{\partial V}{\partial x^{m}}(\mathbf{x}) &= 0 &(4.42)\\ \vdots \quad &\vdots \quad \vdots \\ \frac{\partial V}{\partial x^n}(\mathbf{x}) + \Gamma_n^{n+1}(\mathbf{x})\frac{\partial V}{\partial x^{n+1}}(\mathbf{x}) + \cdots + \Gamma_n^{m}(\mathbf{x})\frac{\partial V}{\partial x^{m}}(\mathbf{x}) &= 0. &(4.43) \end{aligned}$$

For asymptotic stability of the target dynamics Σ^{target} consider the time derivative of $\phi^*\tilde{V}(\mathbf{x}) + V(\mathbf{x})$ along the trajectories of Σ^{target}.

$$
\begin{aligned}
\frac{d}{dt}\left(\phi^*\tilde{V}(\mathbf{x}) + V(\mathbf{x})\right) &= \left(\frac{\partial \tilde{V}}{\partial x^1} + \frac{\partial V}{\partial x^1}\right)\left(\tilde{g}^1 - \frac{\partial \tilde{V}}{\partial x^1} - \frac{\partial V}{\partial x^1}\right) + \cdots \\
&+ \left(\frac{\partial \tilde{V}}{\partial x^n} + \frac{\partial V}{\partial x^n}\right)\left(\tilde{g}^n - \frac{\partial \tilde{V}}{\partial x^n} - \frac{\partial V}{\partial x^n}\right) \\
&+ \frac{\partial V}{\partial x^{n+1}}\left(\Gamma_1^{n+1}\tilde{g}^1 + \cdots \Gamma_n^{n+1}\tilde{g}^n - \frac{\partial V}{\partial x^{n+1}}\right) + \cdots \\
&+ \frac{\partial V}{\partial x^m}\left(\Gamma_1^m\tilde{g}^1 + \cdots \Gamma_n^m\tilde{g}^n - \frac{\partial V}{\partial x^m}\right) \qquad (4.44) \\
&= W(\mathbf{x}^1) - \left(\frac{\partial \tilde{V}}{\partial x^1}(\mathbf{x}^1) + \frac{\partial V}{\partial x^1}(\mathbf{x}^1, \mathbf{x}^2)\right)^2 - \cdots \\
&- \left(\frac{\partial \tilde{V}}{\partial x^n}(\mathbf{x}^1) + \frac{\partial V}{\partial x^n}(\mathbf{x}^1, \mathbf{x}^2)\right)^2 \\
&- \left(\frac{\partial V}{\partial x^{n+1}}(\mathbf{x}^1, \mathbf{x}^2)\right)^2 - \cdots - \left(\frac{\partial V}{\partial x^m}(\mathbf{x}^1, \mathbf{x}^2)\right)^2 \qquad (4.45)
\end{aligned}
$$

The time derivative of $\phi^*\tilde{V} + V$ along the trajectories of Σ^{target} is therefore negative definite and by Lyapunov's second method the dynamics of Σ^{target} are asymptotically stable about the origin.

□

4.3.2.2 Setting up partial differential equations

Having defined the target dynamics Σ^{target} the next step is to define a condition (in the form of P.D.Es) for the equivalence of the target dynamics and the closed loop dynamics of Σ denoted Σ^{CL}. Existence of a feedback control that makes Σ equal to Σ^{target} is equivalent to requiring $\Sigma^{target}(\mathbf{x}^1, \mathbf{x}^2) - f_0(\mathbf{x}^1, \mathbf{x}^2) \in C$. Recall that $\ker P_D = C$ the requirement can be rewritten in the following form,

$$P_D \circ (\Sigma^{target}(\mathbf{x}^1, \mathbf{x}^2) - f_0(\mathbf{x}^1, \mathbf{x}^2)) = \mathbf{0}. \qquad (4.46)$$

Equation (4.46) above is a partial differential equation with V being the dependent variable, if a function $V(\mathbf{x}^1, \mathbf{x}^2)$ satisfies this equation and the requirement

$\mathbf{d}V \in \mathbf{ann}(VM)$. This implies that its possible to find a feedback control that makes $\Sigma^{target} = \Sigma^{CL}$ and additionally $\phi^*\tilde{V} + V$ is a control Lyapunov function for Σ.

To transform these partial differential equation requirements into the language of geometric differential equations we will need the following geometric objects.

1. The fibre bundle $(M \otimes \mathbb{R}, M, \pi, \mathbb{R})$ where π is the natural projection. A chart $\mathcal{U} \subset M$ with coordinates x^i for $i = 1, \cdots, m$ induces the adapted coordinates (x^i, V) in the open set $\pi^{-1}(\mathcal{U})$.
2. The first jet bundle $J^1\pi$ has the induced coordinates (x^i, V, V_i).
3. A bundle morphism $\Phi_d : J^1\pi \mapsto T^*M$, $\Phi_d(x^i, V, V_i) \mapsto (x^i, V_i dx^i)$.
4. A bundle isomorphism $\Delta^b : T^*M \mapsto TM$, $\Delta^b(x^i, \omega_i dx^i) \mapsto (x^i, \omega_{i_1}\delta^{i_1,i}\frac{\partial}{\partial x^i})$. Δ^b is the inverse of $\Delta^\sharp$.
5. Recall for a connection equipped fibre bundle $\phi : M \mapsto N$, $TM = VM \oplus HM$ where VM is the canonical vertical bundle and HM is the horizontal bundle defined by the connection. Since VM and HM are regular distributions their annihilators provide a splitting of the cotangent bundle $T^*M = \mathbf{ann}(VM) \oplus \mathbf{ann}(HM)$.
6. A projection $P_{VM} : T^*M \mapsto \mathbf{ann}(VM)$ such that $\ker(P_{VM}) = \mathbf{ann}(HM)$. P_{VM} is a $(1,1)$-tensor and has the coordinate expression $P_{VM}(\mathbf{x}) = P^i_{VM,q}(\mathbf{x})$ $\frac{\partial}{\partial x^i} \otimes dx^q$, where $\mathbf{ann}(VM) = \text{span}\{dx^1, \cdots, dx^n\}$ and $q = 1, \cdots, n$. In matrix form P_{VM} is a $m \times n$ matrix where the co-efficient $P^i_{VM,q}$ corresponds to the (i,q) entry in the matrix.

Proposition 4.3.3. *Consider the fibred submanifold $\mathcal{R} \subset J^1\pi$ defined as follows,*

$$\mathcal{R} = \{(x^i, V, V_i) | P_D \circ (X(x^i) - \Delta^\sharp \circ \Phi_d(x^i, V, V_i)) = P_{VM} \circ \Phi_d(x^i, V, V_i) = \mathbf{0}\}. \quad (4.47)$$

$$X \in \mathfrak{X}(M) = Hor_{x^i} \circ G \circ \alpha \circ \phi(x^i) - \Delta^\sharp \circ \mathbf{d}\phi^*\tilde{V}(x^i) - f_0(x^i). \quad (4.48)$$

$\mathcal{R}$ *is the geometric representation of the system of partial differential equations (4.46), (4.42).*

For the associated fibre bundle morphism to the differential equation $\mathcal{R}$, consider the fibre bundle $(M \otimes (D \oplus \mathbf{ann}(VM)), M, \tilde{\pi}, \mathbb{R}^{m+n-r})$. The local adapted coordinates for $\tilde{\pi}$ will be denoted $(x^i, \mathcal{X}^a, \omega_q)$. Let $\Psi : J^1\pi \mapsto \tilde{\pi}$ be a fibre bundle morphism defined below.

$$\Psi(x^i, V, V_i) \mapsto (x^i, P^a_{D,i}\delta^{i,i_1}V_{i_1} - P^p_{D,i}X^i, P^i_{VM,q}V_i). \tag{4.49}$$

The fibred submanifold $\mathcal{R}$ is the zero level set of Ψ.

Proposition 4.3.4. *Consider the jet bundles $J^2\pi$ and $J^1\tilde{\pi}$ with the induced adapted local coordinates $(x^i, V, V_i, V_{[i,i_1]}), V_{[i,i_1]} = V_{[i_1,i]}$ and $(x^i, \mathcal{X}^a, \omega_q, \mathcal{X}^a_i, \omega_{qi})$ respectively. The first prolongation of the differential equation $\mathcal{R}$ is defined as the kernel of the prolonged morphism $\rho_1(\Psi) : J^2\pi \mapsto J^1\tilde{\pi}$,*

$$\begin{aligned}
\rho_1(\Psi)(x^i, V, V_i, V_{[i,i_1]}) &= (x^i, P^a_{D,i}\delta^{i,i_1}V_{i_1} - P^a_{D,i}X^i, P^i_{VM,q}V_i, P^a_{D,i}\delta^{i,i_1}V_{[i,i_1]} \\
&+ \frac{\partial}{\partial x^i}\left(P^a_{D,i_1}\right)\delta^{i,i_1}V_i - \frac{\partial}{\partial x^i}(P^a_{D,i_1})X^{i_1} - P^a_{D,i_1}\frac{\partial}{\partial x^i}(X^{i_1}) \\
&, \frac{\partial}{\partial x^{i_1}}(P^{i_1}_{VM,q})V_i + P^{i_1}_{VM,q}V[i, i_1])
\end{aligned} \tag{4.50}$$

4.3.2.3 Symbol of a partial differential equation

The symbol of the partial differential equation $\mathcal{R}$ is the vector bundle morphism $\sigma(\Psi) : V\Psi \circ \epsilon_1 : \pi_1^*(T^*M) \otimes \pi_{1,0}^*(V\pi) \mapsto V\tilde{\pi}$. Identify $T^*M \otimes V\pi$ with T^*M and $V\tilde{\pi} \equiv D \oplus \mathbf{ann}(VM)$ can be identified with $\mathbb{R}^{m-r} \oplus \mathbb{R}^n$ with local coordinates $(\mathcal{X}^a, \omega_q)$. In local coordinates the inclusion map $\epsilon_1 : T^*M \mapsto V\pi_0^1$ has the form

$$\epsilon_1(\omega_i dx^i) \mapsto \left(\mathbf{0}\frac{\partial}{\partial V} + \omega_{i_1}\delta^{i_1,i}\frac{\partial}{\partial V_i}\right). \tag{4.51}$$

The symbol vector bundle morphism $\sigma(\Psi)$ then becomes

$$\sigma(\Psi)(\omega_i dx^i) \mapsto (P^a_{D,i}\delta^{i,i_1}\omega_{i_1}, P^i_{VM,q}V_i). \tag{4.52}$$

Recall that alternatively the symbol of $\mathcal{R}$ can be defined as a subbundle $G_1 \subset T^*M \otimes V\pi$ where $G_1 = \ker \sigma(\Psi)$. Therefore G_1 is defined as

$$G_1 = \ker \sigma(\Psi) = \Delta^b(C) \cap \mathbf{ann}(HM), \tag{4.53}$$

To prolong the symbol $\sigma(\Psi)$ consider the following identifications. $S^2T^*M \otimes V\pi$ is identified with S^2T^*M (can be viewed as the set of symmetric $m \times m$ matrices) and $T^*M \otimes V\tilde{\pi} = T^*M \otimes (D \oplus \mathbf{ann}(VM))$ is identified with $(T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM))$. The inclusion $S^2T^*M \hookrightarrow T^*M \otimes T^*M$ in coordinates is defined as

$$\Omega_{[i_1,i_2]}dx^{i_1} \otimes dx^{i_2} \in S^2T^*M \hookrightarrow (\omega^1_{i_1}dx^{i_1}) \otimes (\omega^2_{i_2}dx^{i_2}) \in T^*M \otimes T^*M | \Omega_{[i_1,i_2]} = \omega^1_{i_1}\omega^2_{i_2}. \tag{4.54}$$

The first prolongation of the symbol is the vector bundle morphism $\rho_1(\sigma(\Psi)) = id_{T^*M} \otimes \sigma(\Psi) : S^2T^*M \mapsto (T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM))$ defined as

$$\rho_1(\sigma(\Psi))(\Omega_{[i_1,i_2]}dx^{i_1} \otimes dx^{i_2}) \mapsto (\Omega_{[i,i_1]}\delta^{i_1,i_2}P^a_{D,i_2}dx^i \otimes e_a, \Omega_{[i,i_1]}P^{i_1}_{VM,q}dx^i \otimes dx^q). \tag{4.55}$$

In matrix representation the bundle morphism $\rho_1(\sigma(\Psi))$ has the form,

$$\rho_1(\sigma(\Psi))(\Omega) \mapsto (P_D\Omega^T, \Omega P_{VM}), \Omega \in S^2T^*M. \tag{4.56}$$

Proposition 4.3.5. *The kernel and co-kernel of the vector bundle morphism $\rho_1(\sigma(\Psi))$ are the sub-bundles defined as*

$$\begin{aligned} G_{1+1} &= \mathit{ker}\, \rho_1(\sigma(\Psi)) = S^2(\Delta^b(C)) \cap S^2\mathbf{ann}(HM) && (4.57)\\ \textit{co-ker}\, \rho_1(\sigma(\Psi)) &= (T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM))/\mathit{Im}\, \rho_1(\sigma(\Psi)) \\ &\equiv \wedge^2(\Delta^b(D)) \oplus \wedge^2(\mathbf{ann}(VM)). && (4.58) \end{aligned}$$

Proof. Consider the splitting of the map $\rho_1(\sigma(\Psi)) = (\rho^a, \rho^b)$ with the component maps defined as,

$$\begin{aligned}
\rho^a &: S^2T^*M \hookrightarrow T^*M\otimes T^*M \mapsto T^*M\otimes D \equiv T^*M\otimes \Delta^b(D) && (4.59)\\
\rho^a(\Omega) &= (id_{T^*M}\otimes \Delta^b\circ P_D\circ \Delta^\sharp)(i(\Omega)) && (4.60)\\
\rho^b &: S^2T^*M \hookrightarrow T^*M\otimes T^*M \mapsto T*M\otimes \mathbf{ann}(VM) && (4.61)\\
\rho^b(\Omega) &= (id_{T^*M}\otimes P_{VM})(i(\Omega)). && (4.62)
\end{aligned}$$

The kernel and co-kernel of $\rho_1(\sigma(\Psi))$ can be expressed in terms of the kernels and co-kernels of the maps ρ^a and ρ^b,

$$\begin{aligned}
\ker\rho_1(\sigma(\Psi)) &= \ker\rho^a \cap \ker\rho^b. && (4.63)\\
\text{co-ker}\,\rho_1(\sigma(\Psi)) &= \text{co-ker}\,\rho^a \cup \text{co-ker}\,\rho^b. && (4.64)
\end{aligned}$$

To characterise the kernel and co-kernel of ρ^a consider the splitting $T^*M = \Delta^b(C)\oplus\Delta^b(D)$. This induces the following splitting of the vector bundles S^2T^*M, $T^*M\otimes T^*M$ and $T^*M\otimes\Delta^b(D)$.

$$\begin{aligned}
S^2T^*M &= S^2(\Delta^b(C))\oplus S^2(\Delta^b(D))\oplus(\Delta^b(C)\otimes\Delta^b(D)). && (4.65)\\
T^*M\otimes T^*M &= (\Delta^b(C)\otimes\Delta^b(C))\oplus(\Delta^b(C)\otimes\Delta^b(D))\oplus(\Delta^b(D)\otimes\Delta^b(C))\\
&\oplus (\Delta^b(D)\otimes\Delta^b(D))\\
&= S^2(\Delta^b(C))\oplus\wedge^2(\Delta^b(C))\oplus(\Delta^b(C)\otimes\Delta^b(D))\oplus(\Delta^b(D)\otimes\Delta^b(C))\\
&\oplus S^2(\Delta^b(D))\oplus\wedge^2(\Delta^b(D)). && (4.66)\\
T^*M\otimes\Delta^b(D) &= (\Delta^b(C)\oplus\Delta^b(D))\otimes\Delta^b(D)\\
&= S^2(\Delta^b(D))\oplus\wedge^2(\Delta^b(D))\oplus(\Delta^b(C)\otimes\Delta^b(D)) && (4.67)
\end{aligned}$$

Under this splitting $i(S^2T^*M)\subset T^*M\otimes T^*M$ has the following form

$$\begin{aligned}
i(S^2T^*M) &= S^2(\Delta^b(C))\oplus\mathbf{0}_{\wedge^2(\Delta^b(C))}\oplus(\Delta^b(C)\otimes\Delta^b(D))\oplus\mathbf{0}_{(\Delta^b(D)\otimes\Delta^b(C))}\\
&\oplus \mathbf{0}_{\wedge^2(\Delta^b(D))}\oplus S^2(\Delta^b(D)). && (4.68)
\end{aligned}$$

Since $id_{T^*M}\otimes(\Delta^b\circ P_D\circ\Delta^\sharp)$ is a full rank map the kernel and co-kernel of ρ^a can be identified as

$$\ker\rho^a = S^2(\Delta^b(C)) \tag{4.69}$$

$$\text{co-}\ker\rho^a = \wedge^2(\Delta^b(D)) \tag{4.70}$$

Following the same procedure for ρ^b gives

$$\ker\rho^b = S^2(\mathbf{ann}(HM)) \tag{4.71}$$

$$\text{co-}\ker\rho^b = \wedge^2(\mathbf{ann}(VM)) \tag{4.72}$$

Evaluating expressions (4.63) and (4.64) using the above expressions of the kernel and co-kernel of ρ^a and ρ^b then proves the proposition. □

Proposition 4.3.6. *The symbol G_1 of $\mathcal{R}$ is involutive.*

To prove this proposition consider the following lemma.

Lemma 4.3.1. *Let V^* be a m-dimensional covector space. Consider the subspaces $E^*, F^* \subset V^*$ of dimension r and s respectively. If $G_1 = E^* \cup F^*$ and $G_{1+1} = S^2E^* \cap S^2F^*$, there exists a quasi-regular basis for V^*.*

Proof. Consider the case where $dim(E^*) > dim(F^*)$ and $E^* \cap F^* \neq \mathbf{0}$, there exists a basis $V^* = \text{span}\{v^1, \cdots, v^m\}$ such that E^* and F^* have the following form.

$$E^* = \text{span}\{v^1, \cdots, v^s, v^{s+1}, \cdots, v^r\} \tag{4.73}$$

$$F^* = \text{span}\{v^1, \cdots, v^s\} \tag{4.74}$$

The spaces $E^* \cap F^*$, $S^2(E^*)$ and $S^2(F^*)$ will then have the form,

$$E^* \cap F^* = \text{span}\{v^1, \cdots, v^s\} \tag{4.75}$$

$$S^2(E^*) = \text{span}\{v^{i_1} \otimes v^{i_2}\}, i_1 \geq i_2.\ i_1, i_2 = 1, \cdots, r \tag{4.76}$$

$$S^2(F^*) = \text{span}\{v^{j_1} \otimes v^{j_2}\}, j_1 \geq j_2.\ j_1, j_2 = 1, \cdots, s \tag{4.77}$$

$$S^2(E^*) \cap S^2(F^*) = \text{span}\{v^{j_1} \otimes v^{j_2}\}, j_1 \geq j_2.\ j_1, j_2 = 1, \cdots, s \tag{4.78}$$

Therefore $\dim(E^* \cap F^*) = s$ and $\dim((S^2(E^*)) \cap S^2(F^*)) = \frac{s(s+1)}{2}$. Let $\Sigma_k = \mathrm{span}\{v^{k+1}, \cdots, v^m\}$ then we have,

$$\begin{aligned} \dim((E^* \cap F^*) \cap \Sigma_{k_1}) &= s - k_1, \text{ for } k_1 = 1, \cdots, s-1 && (4.79) \\ \dim((E^* \cap F^*) \cap \Sigma_{k_1}) &= 0, \text{otherwise.} && (4.80) \end{aligned}$$

From which we evaluate the following,

$$\dim(E^* \cap F^*) + \sum_{k=1}^{m-1} \dim((E^* \cap F^*) \cap \Sigma_k) = \frac{s(s+1)}{2}. \tag{4.81}$$

Proving that the chosen basis is indeed quasi-regular. Following the same procedure for the cases where $dim(E^*) > dim(F^*)$ and $dim(E^*) = dim(F^*)$ proves the lemma. $\square$

Applying the above lemma to the case where $V^* = T^*_{\mathbf{x}}M, E^* = \Delta^b(C(\mathbf{x})), F^* = \mathbf{ann}(H_{\mathbf{x}}M)$ and Theorem (2.10.1) [22] proves the proposition.

4.3.2.4 Curvature map

Let $p \in \mathcal{R} \subset J^1\pi$ with co-ordinates $p = (x^i, V, V_i)$. Any point $q \in J^2\pi$ that projects onto p will have coordinates of the form $q = (x^i, V, V_i, \tilde{V}_{[i_1,i_2]})$. The curvature map $\kappa : \mathcal{R} \mapsto \text{co-ker}\rho_1(\sigma(\Psi))$ is calculated as

$$\kappa(p) = \tau\left(\rho_1(\Psi)(q) - j^1\Psi(p)\right). \tag{4.82}$$

Let $(x^i, \mathcal{X}^a, \omega_q, \mathcal{X}^a_i, \omega^i_q)$ be local coordinates for $J^1\tilde{\pi}$. By definition $\rho_1(\Psi)(q) \in J^1\tilde{\pi}$ projects onto $\Psi(p)$ and therefore $\rho_1(\Psi)(q)$ will differ from $j^1\Psi(p)$ in the $\mathcal{X}^p_i$ and ω^i_q co-ordinates only. Furthermore since the fibre bundle $\tilde{\pi}^1_0$ has an affine structure modelled on $T^*M \otimes V\tilde{\pi} \equiv (T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM))$, $\rho_1(\Psi)(q) - j^1\Psi(p)$ can be taken to be an element of $(T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM))$. Applying equation (4.49) and proposition (4.3.4) gives,

$$\rho_1(\Psi)(q) - j_1\Psi(p) = \Bigg[\sum_{i_1=1}^{m} P^a_{D,i_1}\left(\tilde{V}_{[i,i_1]} - \frac{\partial}{\partial x^i}(V_{i_1})\right) dx^i \otimes e_a \ ,$$

$$\sum_{i_1=1}^{m} P^{i_1}_{VM,q}\left(\tilde{V}_{[i,i_1]} - \frac{\partial}{\partial x^i}(V_{i_1})\right) dx^i \otimes dx^q.\Bigg] \quad (4.83)$$

Recall that it is shown in proposition (4.3.5) that co-ker$\rho_1(\sigma(\Psi)) = \wedge^2(\Delta^b(D)) \oplus \wedge^2(\mathbf{ann}(VM))$. Make the following identifications,

1. $S^2T^*_{\mathbf{x}}M$ is the space of symmetric $m \times m$ matrices.
2. $T^*_{\mathbf{x}}M \otimes D(\mathbf{x})$ is the space of $(m-r) \times m$ matrices.
3. $T^*_{\mathbf{x}}M \otimes \mathbf{ann}(V_{\mathbf{x}}M)$ is the space of $m \times n$ matrices.
4. $\wedge^2(\Delta^b(D(\mathbf{x})))$ is the space of skew-symmetric $(m-r) \times (m-r)$ matrices.
5. $\wedge^2(\mathbf{ann}(V_{\mathbf{x}}M))$ is the space of skew-symmetric $n \times n$ matrices.

With these identifications the prolonged symbol map $\rho_1(\sigma(\Psi)) : S^2T^*M \mapsto (T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM))$ becomes $\rho_1(\sigma(\Psi))(\Omega) = (P_D\Omega^T, \Omega P_{VM})$. The map $\tau : (T^*M \otimes D) \oplus (T^*M \otimes \mathbf{ann}(VM)) \mapsto \wedge^2(\Delta^b(D)) \oplus \wedge^2(\mathbf{ann}(VM))$ becomes $\tau(A, B) = (AP_D^T - P_DA^T, B^TP_{VM} - P_{VM}^TB)$.

Evaluating the curvature map $\kappa(p) = (G(p), H(p)), G(p) \in \wedge^2(\Delta^b(D(\mathbf{x})))$, $H(p) \in \wedge^2(\mathbf{ann}(V_{\mathbf{x}})M))$,

$$G^{a_1,a_2}(p) = \sum_{i=1}^{m}\sum_{i_1=1}^{m}\left(P^{a_1}_{D,i_1}P^{a_2}_{D,i} - P^{a_1}_{D,i}P^{a_2}_{D,i_1}\right)\frac{\partial}{\partial x^i}(V_{i_1}). \quad (4.84)$$

$$H_{q_1,q_2}(p) = \sum_{i_1=1}^{m}\sum_{i=1}^{m}\left(P^{i}_{VM,q_1}P^{i_1}_{VM,q_2} - P^{i}_{VM,q_2}P^{i_1}_{VM,q_1}\right)\frac{\partial}{\partial x^i}(V_{i_1}). \quad (4.85)$$

Since $p \in \mathcal{R}$ i.e Ψ evaluates to zero at p, V_{i_1} satisfies the following equations.

$$\begin{aligned} \sum_{i_1=1}^{m} \left(P^{a}_{D,i_1} V_{i_1} - P^{a}_{D,i_1} X^{i_1} \right) &= 0. && (4.86) \\ \sum_{i_1=1}^{m} P^{i_1}_{VM,q} V_{i_1} &= 0. && (4.87) \end{aligned}$$

Differentiating these equations with respect to x^i allows for the elimination of $\frac{\partial}{\partial x^i}(V_{i_1})$ in equations (4.84) and (4.85). Differentiating equation (4.86) with respect to x^i yields,

$$\sum_{i_1=1}^{m} \left(\frac{\partial}{\partial x^i} \left(P^{a}_{D,i_1} \right) \left(V_{i_1} - X^{i_1} \right) + P^{a}_{D,i_1} \frac{\partial}{\partial x^i}(V_{i_1}) - P^{a}_{D,i_1} \frac{\partial}{\partial x^i}(X^{i_1}) \right) = 0. \quad (4.88)$$

Substituting into (4.84) gives,

$$\begin{aligned} G^{a_1,a_2}(p) &= \sum_{i=1}^{m} \sum_{i_1=1}^{m} \left(\left[P^{a_1}_{D,i} \frac{\partial}{\partial x^i}(P^{a_2}_{D,i_1}) - P^{a_2}_{D,i} \frac{\partial}{\partial x^i}(P^{a_1}_{D,i_1}) \right] (V_{i_1} - X^{i_1}) \right. \\ &+ \left. \left[P^{a_2}_{D,i} P^{a_1}_{D,i_1} - P^{a_1}_{D,i} P^{a_2}_{D,i_1} \right] \frac{\partial}{\partial x^i}(X^{i_1}) \right). && (4.89) \end{aligned}$$

Recall for the partial differential equation $\mathcal{R}$ to be integrable the curvature map must be a zero map. $G^{a_1,a_2}(p)$ is a zero map if the following conditions are satisfied.

$$\begin{aligned} \sum_{i=1}^{m} \left[P^{a_1}_{D,i} \frac{\partial}{\partial x^i}(P^{a_2}_{D,i_1}) - P^{a_2}_{D,i} \frac{\partial}{\partial x^i}(P^{a_1}_{D,i_1}) \right] &= 0 && (4.90) \\ \sum_{i=1}^{m} \sum_{i_1=1}^{m} \left[P^{a_2}_{D,i} P^{a_1}_{D,i_1} - P^{a_1}_{D,i} P^{a_2}_{D,i_1} \right] \frac{\partial}{\partial x^i}(X^{i_1}) &= 0. && (4.91) \end{aligned}$$

Equations (4.90) and (4.91) correspond to conditions 3 and 4 Theorem (4.3.1).

Differentiating equation (4.87) with respect to x^i gives,

$$\sum_{i_1=1}^{m} \left[\frac{\partial}{\partial x^i}(P^{i_1}_{VM,q_2}) V_{i_1} + P^{i_1}_{VM,q_2} \frac{\partial}{\partial x^i}(V_{i_1}) \right] = 0. \quad (4.92)$$

Substituting this into equation (4.85) gives

$$H_{q_1,q_2}(p) = \sum_{i_1=1}^{m}\left(\sum_{i=1}^{m}\left[P^{i}_{VM,q_2}\frac{\partial}{\partial x^i}(P^{i_1}_{VM,q_1}) - P^{i}_{VM,q_1}\frac{\partial}{\partial x^i}(P^{i_1}_{VM,q_2})\right]\right)V_{i_1}. \tag{4.93}$$

Requiring $H_{q_1,q_2}(p)$ to be a zero map imposes the following condition

$$\sum_{i=1}^{m}\left[P^{i}_{VM,q_2}\frac{\partial}{\partial x^i}(P^{i_1}_{VM,q_1}) - P^{i}_{VM,q_1}\frac{\partial}{\partial x^i}(P^{i_1}_{VM,q_2})\right] = 0. \tag{4.94}$$

From proposition (4.3.1) the projection map $P_{VM} : T^*M \mapsto \mathbf{ann}(VM)$ in matrix form is,

$$P_{VM} = \begin{array}{c} \\ 1 \\ \vdots \\ n \\ n+1 \\ \vdots \\ m \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & & n \end{array} \\ \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \Gamma^{n+1}_{1} & \cdots & \Gamma^{n+1}_{n} \\ \vdots & \ddots & \vdots \\ \Gamma^{m}_{1} & \cdots & \Gamma^{m}_{n} \end{bmatrix} \end{array}. \tag{4.95}$$

The component $P^{i}_{VM,q}$ corresponds to the matrix element in the i^{th}-row and q^{th} column. Substituting this into equation (4.94), the integrability condition becomes

$$\frac{\partial}{\partial x^{q_2}}\left(\Gamma^{l}_{q_1}\right) - \frac{\partial}{\partial x^{q_1}}\left(\Gamma^{l}_{q_2}\right) + \sum_{i_1=n+1}^{m}\left[\Gamma^{l_1}_{q_2}\frac{\partial}{\partial x^{l_1}}\left(\Gamma^{l}_{q_1}\right) - \Gamma^{l_1}_{q_1}\frac{\partial}{\partial x^{l_1}}\left(\Gamma^{l}_{q_2}\right)\right] = 0 \tag{4.96}$$

where $l, l_1 = n+1, \cdots, m$. This corresponds to the components of the curvature form of the connection being zero thus proving the flatness requirement in the theorem.

Remarks. The conditions stated in the main theorem (4.3.1) are nothing but just the integrability conditions for the system of partial differential equations defined in proposition (4.3.3). If these conditions are met, the function $V(\mathbf{x}^1, \mathbf{x}^2)$ can be constructed iteratively by solving the system of partial differential equations

(4.3.3) via the Taylor series method. Having constructed $V(\mathbf{x}^1, \mathbf{x}^2)$ the stabilizing feedback controller can be found by solving the following under-determined system of algebraic equations, where $u_1, \cdots, u_r$ are the unknowns.

$$\begin{bmatrix} f_1^1(\mathbf{x}) & \cdots & f_r^1(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ f_1^m(\mathbf{x}) & \cdots & f_r^m(\mathbf{x}) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = \begin{bmatrix} \tilde{g}^1(\mathbf{x}^1) - \frac{\partial \tilde{V}}{\partial x^1}(\mathbf{x}^1) - \frac{\partial V}{\partial x^1}(\mathbf{x}) \\ \vdots \\ \tilde{g}^n(\mathbf{x}^1) - \frac{\partial \tilde{V}}{\partial x^n}(\mathbf{x}^1) - \frac{\partial V}{\partial x^n}(\mathbf{x}) \\ \Gamma_1^{n+1}(\mathbf{x})\tilde{g}^1(\mathbf{x}^1) + \cdots + \Gamma_n^{n+1}(\mathbf{x})\tilde{g}^n(\mathbf{x}^1) - \frac{\partial V}{\partial x^{n+1}} \\ \vdots \\ \Gamma_1^m(\mathbf{x})\tilde{g}^1(\mathbf{x}^1) + \cdots + \Gamma_n^m(\mathbf{x})\tilde{g}^n(\mathbf{x}^1) - \frac{\partial V}{\partial x^m} \end{bmatrix} \tag{4.97}$$

4.3.3 Example

Example 4.3.1. *Consider the control systems $\Sigma = (\mathbb{R}^2 \times \mathbb{R}, \pi, \mathbb{R}^2, \mathbb{R}, F)$ and $\tilde{\Sigma} = (\mathbb{R} \times \mathbb{R}, \tilde{\pi}, \mathbb{R}, \mathbb{R}, G)$ with the following local representations.*

$$\Sigma \ : \ \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} f(x^1, x^2) \\ u \end{bmatrix}. \tag{4.98}$$

$$\tilde{\Sigma} \ : \ \dot{y} = v. \tag{4.99}$$

The system $\tilde{\Sigma}$ is a quotient control system of Σ under the action of the bundle morphism (ϕ, ψ) defined as

$$\phi \ : \ \mathbb{R}^2 \mapsto \mathbb{R} \tag{4.100}$$

$$(x^1, x^2) \mapsto (y = x^1) \tag{4.101}$$

$$\psi \ : \ \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R} \tag{4.102}$$

$$(x^1, x^2, u) \mapsto (y = x^1, v = f(x^1, x^2)). \tag{4.103}$$

The fibre bundle $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ can be equipped with a connection defined by the *horizontal distribution denoted* H*,*

$$H = span\left\{\frac{\partial}{\partial x^1} + \Gamma(x^1, x^2)\frac{\partial}{\partial x^2}\right\}. \tag{4.104}$$

The connection H *is flat since all one-dimensional connections are flat. The one-dimensional control distribution* $C = span\{\frac{\partial}{\partial x^2}\}$ *induces the trivial projection map* $P_D : \mathbb{R} \mapsto \mathbb{R}, P_D = P^1_{D,1}dx^1 \otimes \frac{\partial}{\partial x^1} + P^2_{D,1}dx^2 \otimes \frac{\partial}{\partial x^1}$ *with components* $P^1_{D,1} = 1$ *and* $P^2_{D,1} = 1$*. The map* P_D *and the connection defined by* H *trivially satisfies the conditions of theorem 4.3.1, therefore its possible to construct the Lyapunov function of* Σ *iteratively.*

To see how the Lyapunov function of Σ *can be constructed let* $\alpha(y)$ *be the stabilizing feedback of* $\tilde{\Sigma}$ *with the control Lyapunov function* $\tilde{V}(y)$*. Consider the system of partial differential equations that is to be solved.*

$$\begin{aligned} \alpha(x^1) - f(x^1, x^2) - \frac{\partial \tilde{V}}{\partial x^1} - \frac{\partial V}{\partial x^1} &= 0 && (4.105) \\ \frac{\partial V}{\partial x^1} + \Gamma(x^1, x^2)\frac{\partial V}{\partial x^2} &= 0. && (4.106) \end{aligned}$$

The Taylor series of all the functions involved in the partial differential equations are given below.

$$\begin{aligned} V(x^1, x^2) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} V_{mn}(x^1)^m(x^2)^n,\ V_{mn} = \frac{\partial^{m+n}V}{\partial(x^1)^m\partial(x^2)^n},\ V_{00} = 0. \\ \alpha(x^1) &= \sum_{i=0}^{\infty} \alpha_1 x_1^i,\ \alpha_i = \frac{\partial^i \alpha}{\partial(x^1)^i},\ \alpha_0 = 0. \\ f(x^1, x^2) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f_{jk}(x^1)^j(x^2)^k,\ f_{jk} = \frac{\partial^{j+k}f}{\partial(x^1)^j(x^2)^k},\ f_{00} = 0 \\ \tilde{V}(x^1) &= \sum_{l=0}^{\infty} \tilde{V}_l(x^1)^l,\ \tilde{V}_l = \frac{\partial^l \tilde{V}_l}{\partial(x^1)^l},\ \tilde{V}_0 = 0. \\ \Gamma(x_1, x^2) &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \Gamma_{pq}(x^1)^p(x^2)^q,\ \Gamma_{pq} = \frac{\partial^{p+q}\Gamma}{\partial(x^1)^p\partial(x^2)^q}. \end{aligned}$$

Substituting into the partial differential equations and solving for V_{mn} and Γ_{pq} up to second order terms gives

$$V_{10} = -\tilde{V}_1. \tag{4.107}$$

$$V_{01} = \frac{\tilde{V}}{\Gamma_{00}}. \tag{4.108}$$

$$V_{11} = -f_{01}. \tag{4.109}$$

$$V_{20} = \frac{1}{2}\alpha_1 - \tilde{V}_2 - \frac{1}{2}f_{10}. \tag{4.110}$$

$$V_{02} = \frac{(\Gamma_{00}f_{01} - \tilde{V}_1\Gamma_{01})}{2\Gamma_{00}}. \tag{4.111}$$

The control Lyapunov function for Σ up to second order becomes,

$$\phi^*\tilde{V} + V = \frac{1}{2}(\alpha_1 - f_{10})(x^1)^2 + \frac{\tilde{V}_1}{\Gamma_{00}}x^2 - f_{01}x^1x^2 + \frac{(\Gamma_{00}f_{01} - \tilde{V}_1\Gamma_{01})}{2\Gamma_{00}}(x^2)^2. \tag{4.112}$$

Remarks. From the example above a couple of points are worth noting. The choice of the stabilizing feedback of $\tilde{\Sigma}$ affects the kind of solution that one can get. For the second order solution above to be a positive definite function it is necessary for $(\alpha_1 > f_{10})$, if this condition is not met it becomes necessary to construct higher order series approximations. The connection component plays the role of a free parameter that can be chosen arbitrarily in this case, however it should again be noted that for this second order approximation the connection needs to be chosen such that the co-efficient of the $(x^2)^2$ term is positive.

Chapter 5

Conclusions and future work

5.1 Concluding remarks

This thesis presents a solution to the problem of determining the stabilizability properties of a control system given that its quotient is known to be stabilizable. The problem addressed in this thesis is central to any hierarchical stabilizing feedback design method and it has been shown that the interplay between quotienting and stabilizability is implicitly assumed in some of the most successful hierarchical stabilizing feedback design techniques. Our solution to the problem of stabilizability preserving quotients is two pronged in nature. Firstly, we develop a solution for the case where the quotient control system is assumed to be a linear controllable system. Using a novel construction dubbed"horizontally lifting" a control system, it is shown that the original system is stabilizable if it possesses stable zero dynamics. Additionally we provide a new method of characterizing these zero dynamics which does not involve solving a system of under-determined partial differential equations.

The second part of the solution is for the case where both the system and its quotient are assumed to be an analytic control affine systems. Proceeding in a constructive manner we develop a method of designing a candidate control Lyapunov function for the original system from the control Lyapunov function of the quotient system. Determining the modification term in the candidate Lyapunov function involves solving a system of under-determined partial differential equations. Using the geometric theory of partial differential equations we develop integrability conditions for this system of differential equations. These integrability conditions can be understood as encoding the system structural obstructions to the candidate Lyapunov function construction scheme that we propose.

5.2 Future work

The work presented in this thesis is by no means exhaustive and the questions listed below provide possible extensions and avenues of enquiry that build on what has been developed in this thesis.

- For a system Σ and its quotient $\tilde{\Sigma}$ that satisfy the integrability conditions of the main theorem, it is possible to construct the modification/cross-term component of the candidate Lyapunov function of Σ by a power series solution of the associated system of partial differential equations. A useful and very practical extension of this result would be to devise an algorithm for solving the system of algebraic equations that come up in this power series solution in an efficient way.

- Again of practical interest are questions to do with the robustness of the construction developed in this work. More specifically, let $\mathcal{R}_1 \subset J^2\pi$ be the prolonged partial differential equation defined in Section 4.3.2.2 and let $\mathcal{H} = \{j^2\varphi | \varphi \in \Gamma(\pi), \text{Hessian of } \varphi > 0\}$, i.e $\mathcal{H}$ is the set of positive definite functions. If a function $\varphi \in \Gamma(\pi)$ is positive definite then its second prolongation $j^2\varphi \in \mathcal{H}$. For robustness we are asking if the set $\mathcal{R}_1 \cap \mathcal{H}$ is an open set in $J^2\pi$.

- Another variation of the preceeding line of inquiry is to ask what influence the choice of stabilizing feedback and control Lyapunov function for the quotient control system have on the size of the set of feasible solutions for the system of partial differential equations.

- For a complete theory of stabilizability preserving quotients we need to understand how the property of stabilizability is propagated. The task here is to devise a method of constructing a quotient map (i.e fibre morphism

$(\phi, \psi))$ such that the projected system will be stabilizable if the original system is stabilizable.

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