The Large-N Limit Of Matrix Models And AdS/CFT

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

Random matrix models have found numerous applications in both Theoretical Physics and Mathematics. In the gauge-gravity duality, for example, the dynamics of the half-BPS sector can be fully described by the holomorphic sector of a single complex matrix model.

In this thesis, we study the large-N limit of multi-matrix models at strong-coupling. In particular, we explore the significance of rescaling the matrix fields. In order to investigate this, we consider the matrix quantum mechanics of a single Hermitian system with a quartic interaction. We "compactify" this system on a circle and compute the first-order perturbation theory correction to the ground-state energy. The exact ground-state energy is obtained using the Das-Jevicki-Sakita Collective Field Theory approach.

We then discuss the multi-matrix model that results from the compactification of the Higgs sector of $\mathcal{N} = 4$ SYM on S^4 (or $T \times S^3$). For the radial subsector, the saddle-point equations are solved exactly and hence the radial density of eigenvalues for an arbitrary number of even Hermitian matrices is obtained. The single complex matrix model is parametrized in terms of the matrix valued polar coordinates and the first-order perturbation theory density of eigenstates is obtained. We make use of the Harish-Chandra-Itzykson-Zuber (HCIZ) formula to write down the exact saddle-point equations.

We then give a complementary approach - based on the Dyson-Schwinger (loop) equations formalism - to the saddle-point method. We reproduce the results obtained for the radial (single matrix) subsector. The two-matrix integral does not close on the original set of variables and thus we map the system onto an auxiliary Penner-type two matrix model. In the absence of a logarithmic potential we derive a radial hemispherical density of eigenvalues. The system is regulated with a logarithm potential, and the Dobroliubov-Makeenko-Semenoff (DMS) loop equations yield an equation of third degree that is satisfied by the generating function. This equation is solved at strong coupling and, accordingly, we obtain the radial density of eigenvalues.

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Chapter 1

Introduction

1.1 The Large-N Limit

The discovery of the Higgs boson is a huge triumph for the Standard Model of Particle Physics. Nevertheless, notwithstanding all its major successes, it is generally accepted that the Standard Model cannot be the ultimate description of the world that we inhabit.

A clear failure of the Standard Model is its inability to provide a theoretical framework that includes all four of the fundamental forces occurring in nature. Even though the Standard Model is valuable in the description of three of the fundamental forces - namely the electromagnetic force, the strong force and the weak force - it conspicuously neglects the earliest of all known forces *i.e.* the gravitational force.

The unification of the principles of general relativity and quantum mechanics has been a very difficult endeavour. At present, a consistent theory of quantum gravity is still lacking. Nevertheless, string theory is acknowledged to be one of the leading candidates for a theory of quantum gravity. The underlying idea behind string theory is relatively simple and consists of regarding all elementary particles - for example, the electron, quarks etc. - not as point-like objects, but as oscillations or specific modes of one dimensional objects that are called strings.

The theory of strings, however, was never intended to be a "Theory Of Everything". In fact, string theory was proposed as a theory to describe strong interactions (hadrons). In the 1960's countless strongly interacting particles (resonances) were discovered. Moreover, it was observed that these resonances tended to display almost linear Regge behaviour *i.e.* they satisfied the relation

$$M^2 = \frac{J}{\alpha'}. \tag{1.1}$$

Here, J and M are the spin and mass (respectively) of the resonance and α' is the Regge slope.

It was argued that Regge behaviour and other properties of the strongly interacting particles could be interpreted in terms of relativistic strings.

The correct description for strong interactions turned out to be given by a Yang-Mills theory with $SU(3)$ gauge group *i.e.* Quantum Chromodynamics[1] (QCD). As is well-known, Quantum Chromodynamics has a running coupling constant. At very high energies (large distances) the theory is effectively free *i.e.* the quarks are weakly interacting. More precisely, Quantum Chromodynamics displays asymptotic freedom. In contrast, for low-energies (large distances) the coupling constant becomes large and the quarks are confined. Accordingly, at low-energies, we are unable to make use of perturbation theory.

In order to evade the difficulty of working with a strongly coupled gauge theory - more precisely, to provide a non-perturbative formulation of Quantum Chromodynamics - 't Hooft suggested that we should consider a general theory with N colours [1] - rather than the phenomenologically correct case with only three colours. (More precisely, we

[1] There were various reasons that tended to disqualify string theory (the dual resonance models) from being accepted as the standard theory of hadronic physics. For example, string theory seemed to require additional dimensions for it to be consistent. In addition, there was the presence of a spin two particle - and also a tachyon - in the spectrum which had nothing to do with the hadrons.

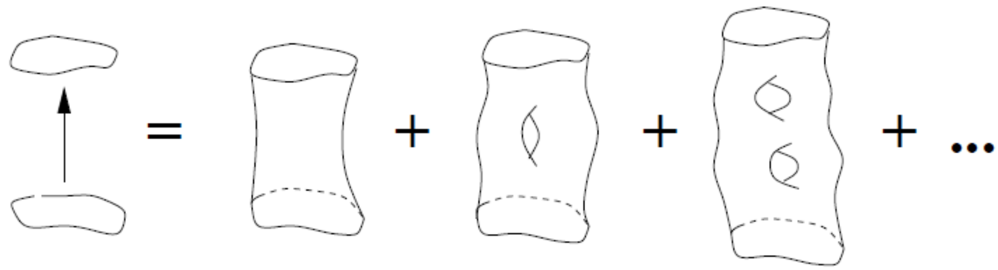

Figure 1.1: The string genus expansion.

consider a theory with gauge group $SU(N)$.)

't Hooft noticed that in the limit as $N \to \infty$, with $\lambda = g_{YM}^2 N$ fixed - this is known as the 't Hooft large-N limit - the Feynman diagrams are rearranged in terms of the genus of the surface upon which they can be drawn. That is, we have a double-expansion of the form [1]

$$F = -\ln Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda). \tag{1.2}$$

In fact, a similar expansion also occurs in string theory:

$$Z = \sum_{g=0}^{\infty} g_s^{2g-2} Z_g, \tag{1.3}$$

where g_s is the string coupling constant.

It is this similarity that led 't Hooft to make the inference that in the large-N limit gauge theories are dual to some theory of closed strings.

The gauge-string theory duality does clarify why the old dual resonance models (string theory) were partially successful in describing phenomena that are correctly described by a gauge theory. Moreover, the duality provides us with another approach to tackle strongly coupled gauge theories. Thus, if we can find the dual string theory it might then be possible to work with the particular strongly coupled gauge theory - of course, the assumption - or hope - is that the dual string theory is analytically simpler than the original gauge theory.

Unfortunately, even with all the simplifications that occur in the large-N limit, we still have not been able to completely solve large-N QCD. The problem being that we have been unable to find the dual string for $(3+1)$-dimensional QCD - for two dimensional QCD some progress has been made [2].

1.2 The AdS/CFT Correspondence

The gauge-string duality was suggested with the hope of helping us to understand the low-energy dynamics of QCD. However, the best known example of the gauge-string duality doesn't involve QCD, but rather involves a superconformal gauge theory. This example was conjectured in 1997 by Maldacena and is known as the AdS/CFT correspondence - the conjecture is also known as the gauge/gravity duality or holography [3, 4, 5] - for reviews, see [6, 7].

The correspondence has found numerous applications. Some of these include the computation of scattering amplitudes at strong-coupling [8], holographic superconductivity [9], hydrodynamics (specifically the computation of a bound for the viscosity to entropy bound [10] which is (miraculously) close to the value for the (strongly coupled) Quark Gluon Plasma). At the moment, the most powerful application to QCD is given by the construction due to Sakai and Sugimoto [11] - for a holistic view of the applications of holography, see [12].

In this section we introduce the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. We begin by looking at the argument leading up to the conjecture by Maldacena. This will be followed by discussing how the parameters on the two sides of the correspondence are related. Finally, we look at the planar limit of the Maldacena conjecture.

1.2.1 Derivation Of The Correspondence

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence is the conjectured equivalence of type IIB string theory on $AdS_5 \times S^5$ and maximally super symmetric four-dimensional $\mathcal{N} = 4$ Yang-Mills theory.

$\mathcal{N} = 4$ super-Yang Mills

$\mathcal{N} = 4$ SYM is a gauge theory, and thus is similar to Quantum Chromodynamics. The thing that distinguishes $\mathcal{N} = 4$ SYM from most gauge theories - including Quantum Chromodynamics - is that it is superconformal *i.e.* it is both supersymmetric and also conformal. In fact, most theories are conformal at the classical level. However, what is special about $\mathcal{N} = 4$ SYM is that this conformal symmetry is present even at the quantum level *i.e.* the conformal symmetry is not anomalous. This is expressed compactly by the vanishing of the beta-function.

The field content of $\mathcal{N} = 4$ contains the gauge field A_μ ($\mu = 1, \ldots, 4$), Weyl fermions λ_α^A, $\lambda_{\dot{\alpha}}^A$ ($\alpha, \dot{\alpha} = 1, 2$; $A = 1, \ldots, 4$) and six scalars Φ_i ($i = 1, \cdots, 6$) [13]. All the fields are in the adjoint representation of $SU(N)$. In fact, from the field content it is possible to already see that the beta-function vanishes at one loop [13]:

$$\begin{aligned}\beta_1 (g_{YM}) &= -\frac{g_{YM}^3}{16\pi^2} \left(\frac{11}{3} N - \frac{1}{6} \sum_{i=1}^{6} C_i - \frac{1}{3} \sum_{j=1}^{8} \tilde{C}_i \right) \\ &= -\frac{g_{YM}^3}{16\pi^2} \left(\frac{11}{3} N - \frac{1}{6} \times 6N - \frac{1}{3} \times 8N \right) = 0 \end{aligned} \tag{1.4}$$

Here, C_i ($i = 1, \ldots, 6$) are the Casimirs for the scalar fields of $\mathcal{N} = 4$ SYM, and similarly $\tilde{C}_j$ ($j = 1, \ldots, 8$) are the Casimirs for the fermions [13]. To arrive at the final result we made use of the fact that for fields in the adjoint representation the Casimir is equal to N.

The Lagrangian for $\mathcal{N} = 4$ SYM can be arrived at by reducing ten-dimensional $\mathcal{N} = 1$ SYM to four dimensions [14].The resulting Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4g_{YM}^2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu} + 2\sum_i D_\mu \Phi_i D^\mu \Phi^i - \sum_{i,j}\left[\Phi_i,\, \Phi_j\right]^2 + fermions\right\} \tag{1.5}$$

The conjecture was arrived at by considering a stack on N D3-branes in type IIB string theory. The stack of D3-branes allows for two different descriptions. That is, we can consider the stack of D3-branes in terms of open strings or in terms of closed string description. (It is precisely because of this this reason that the correspondence is sometimes referred to as an open/closed string duality .)

Let us begin by concentrating on the open string description. In terms of open strings, a Dirichlet p (Dp)-brane is simply a $(p+1)$-dimensional hypersurface where open strings end. In particular, an open string that is attached between two Dp-branes will satisfy Neumann boundary conditions for those coordinates along the brane [15]:

$$\partial_\sigma X^\mu \mid_{\sigma=0} = \partial_\sigma X^\mu \mid_{\sigma=\pi} = 0, \qquad \mu = 0, 1, \cdots, p. \tag{1.6}$$

In contrast, the coordinates that are transverse to the Dp-brane satisfy Dirichlet boundary conditions:

$$\delta X^{\mu}\mid_{\sigma=0}=\delta X^{\mu}\mid_{\sigma=\pi}=0,\qquad \mu=p+1,\cdots,9. \tag{1.7}$$

In addition, the Dp-brane has a world-volume given by [16]

$$\begin{aligned} S &= S_{DBI}+S_{CS} \\ &= -T_p\int d^{p+1}\xi e^{-\Phi}\left(\det\left(\mathcal{P}\left(G_{\mu\nu}+B_{\mu\nu}\right)+2\pi\alpha' F_{\mu\nu}\right)\right)^{1/2} \\ &+\mu_p\int\left(\sum_q \mathcal{P}C_q\right)\wedge e^{\mathcal{P}[B]+2\pi\alpha' F}, \end{aligned} \tag{1.8}$$

where S_{DBI} is the Dirac-Born-Infeld action and S_{CS} is the Chern-Simons action; $F_{\mu\nu}$ is the field strength, C_q is the R-R potential, Φ is the dilaton and $\mathcal{P}\left(G_{\mu\nu}\right)$ denotes the pullback of the metric onto the worldsheet (similarly for the B-field). Finally, μ_p is the R-R charge and the tension is[2]

$$T_{Dp}=\frac{1}{(2\pi)^p\, g_s l_s^{p+1}}. \tag{1.9}$$

Now, let us consider a stack of N D-branes.The action for the stack of D-branes at low-energies[3] can be written as [6]

$$S=S_{bulk}+S_{brane}+S_{int}. \tag{1.10}$$

A few comment are in order. Firstly, S_{bulk} is simply the classical ten-dimensional supergravity action with higher derivative terms [6, 16] - at low-energies the higher derivative

[2]Here, l_s is the fundamental string length scale.

[3]By low-energies we mean that the energy scales that we are considering are smaller than the effective mass string scale $\frac{1}{l_s}$. (In particular, we consider the so-called Maldacena limit *i.e.* $\alpha'\to 0$, $r\to 0$ with $\frac{r}{\alpha'}$ fixed. For the definition of the parameter r, see (1.14) below.) In the low-energy limit, only the massless string states are excited. As explained below, the closed string excitations give rise to ten-dimensional SUGRA while the massless open string excitations are those of a gauge theory [6].

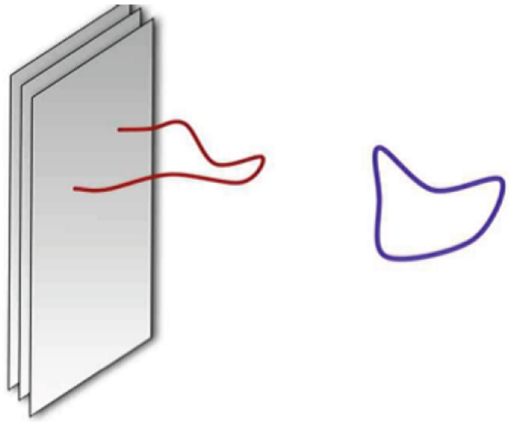

Figure 1.2: A stack of D-branes with an open string attached and a closed string moving in the bulk.

terms to not contribute and consequently S_{bulk} is the action for classical supergravity. More precisely, at low-energies we have [6]

$$S_{bulk} = \frac{1}{2\kappa^2}\int \sqrt{g}\mathscr{R} \sim \int \left(\partial h\right)^2 + \kappa\left(\partial h\right)^2 h + \cdots \tag{1.11}$$

Here $g = \eta + h$ and $\kappa \to g_s\alpha'^2$.

In addition, S_{brane} is the action along the brane. For the stack of $D3$-branes at low-energies, S_{brane} reduces to the action for D=4 $\mathcal{N} = 4$ SYM with $SU\left(N\right)$ gauge group [17]. Finally, S_{int} is the interaction between the bulk and brane modes. In the low-energy limit the bulk modes decouple from the brane modes . In other words, we have that $S_{int} = 0$. Thus, the total action for the stack of $D3$-branes is

$$S = S^{D=4}_{\mathcal{N}=4\,SYM} + S^{10D}_{IIB\,SUGRA}. \tag{1.12}$$

There is, of course, another way to define the Dp-branes. In this approach Dp-branes are identified with the classical p-branes $i.e.$ Dp-branes are regarded as classical solutions to the supergravity equations of motion. The black p-brane solution is given by [15]

$$
\begin{aligned}
ds^2 &= H_p^{-1/2} dx \cdot dx + H_p^{1/2} \left(dr^2 + r^2 d\Omega_{8-p}^2 \right) \\
e^{\Phi} &= H_p^{\frac{3-p}{4}} \\
C_{p+1} &= \left(H_p^{-1} - 1 \right) \wedge dx^0 \wedge \cdots \wedge dx^p .
\end{aligned}
\tag{1.13}
$$

Here, $dx \cdot dx$ refers to the metric along the p-brane [15]; Φ and C_{p+1} are the dilaton and R-R $(p+1)$-form respectively. Moreover, the harmonic function is [15, 16]

$$
H_p(r) = 1 + \left(\frac{R}{r} \right)^{7-p}, \qquad R^{7-p} = \left(2\sqrt{\pi} \right)^{5-p} \Gamma \left(\frac{7-p}{2} \right) g_s N \alpha'^{7-p}. \tag{1.14}
$$

In the case when $p = 3$, we have

$$
ds^2 = \left(1 + \frac{R^4}{r^4} \right)^{-1/2} dx \cdot dx + \left(1 + \frac{R^4}{r^4} \right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2 \right). \tag{1.15}
$$

As expected, in the limit as $r \to \infty$ the metric in (1.15) reduces to flat Minkowski space. Moreover, in the near horizon limit (*i.e.* $r \ll R$) the metric becomes

$$
ds^2 = \frac{r^2}{R^2} dx \cdot dx + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2. \tag{1.16}
$$

We define $z = \frac{R^2}{r}$ and rewrite (1.16) as

$$
ds^2 = \frac{R^2}{z^2} \left(dx \cdot dx + dz^2 \right). \tag{1.17}
$$

This is the metric for $AdS_5 \times S^5$. The radius of AdS_5 (and also the five-sphere) is given by

$$
R = \left(4\pi g_s N \alpha'^2 \right)^{1/4}. \tag{1.18}
$$

Thus, in the closed string description the action can be written as

$$S = S_{IIB}^{AdS_5 \times S^5} + S_{IIB\,SUGRA}^{10D}. \tag{1.19}$$

From (1.12) and (1.19) it is possible to make the inference that type IIB string theory on $AdS_5 \times S^5$ is equivalent to maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory. This is, of course, the strong form of the conjecture. The weak form of the AdS/CFT correspondence conjectures that type IIB strings are dual to $\mathcal{N} = 4$ SYM only in the 't Hooft large-N limit. (There is an even weaker version that states that the type IIB string theory is dual to $\mathcal{N} = 4$ SYM only in the large coupling planar limit *i.e.* strongly coupled $\mathcal{N} = 4$ SYM is dual to classical supergravity [16].)

As expected, the correspondence is holographic. More precisely, the AdS$_{d+1}$/CFT$_d$ conjectures that the bulk theory in AdS_{d+1} is dual to a CFT living on the boundary. Hence the AdS/CFT correspondence agrees with a central idea that any theory of quantum gravity has to be holographic. This idea (*i.e.* the holographic principle [18]) stems from the observation that for a blackhole with surface area A the Bekenstein-Hawking entropy is given by

$$S_{BH} = \frac{A}{4}. \tag{1.20}$$

Naively, this means that the degrees of freedom scale as the surface area of a region and do not (as is usually the case) depend on the volume of the region of interest. This leads us to expect that it is possible to describe the physics of a region in terms of some theory that live on the boundary of the region.

1.2.2 The AdS/CFT Dictionary

The strong version of the correspondence conjectures that the two theories (*i.e.* type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM) are physically equivalent for any value of N and for all values of the coupling constants. It is thus reasonable to expect that any given phenomenon on one side of the duality should always have a corresponding occurrence in the dual theory. Therefore, it should be possible to map CFT observables to the corresponding quantities in the dual string theory.

Firstly, we need to ensure that the symmetries of the bulk theory match the symmetries of the boundary theory. The maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory is a super-conformal theory and its has the symmetry group $PSU\,(2, 2 \mid 4)$ [13, 15]. The superconformal group $PSU\,(2, 2 \mid 4)$ has the bosonic subgroup $SU\,(2, 2) \times SU\,(4) \sim SO\,(4, 2) \times SO\,(6)$. Moreover, $SO\,(4, 2)$ is the four-dimensional conformal group and $SO\,(6)$ is the $\mathcal{R}$-symmetry group that is associated with the six scalars of $\mathcal{N} = 4$ SYM. The subgroup $SO\,(4, 2)$ precisely matches the isometry of $AdS_5 \times S^5$.

Next, we need to discuss how the parameters of the two theories are related. Firstly, the rank of the gauge group is related to the flux of the five-form [15]:

$$N = \int_{S^5} F_5. \tag{1.21}$$

In addition, the coupling constants are related as follows [15, 16]:

$$g_{YM}^2 = g_s. \tag{1.22}$$

As a result, the radius of the AdS_5 and S^5 is

$$R = \left(4\pi g_{YM}^2 N \alpha'^2\right)^{1/4}. \tag{1.23}$$

This means that the AdS/CFT correspondence is a weak/strong duality. More precisely, when the gauge theory is strongly coupled the dual string theory is weakly curved. As a result, when one side of the duality is difficult to analyse the dual side is relatively manageable. Accordingly, this makes the duality both extremely difficult to prove but at the same time it makes the duality to be a powerful tool to tackle strongly coupled theories.

In any CFT the typical objects to consider are the correlation functions. That is, we are interested in expectation values of the form [16]:

$$< \mathcal{O}\left(x_1\right) \mathcal{O}\left(x_2\right) \cdots \mathcal{O}\left(x_n\right) > .$$

As usual, to compute the correlation functions we begin by first introducing the generating function:

$$Z\left[\phi_0\right] = \left\langle e^{\int \phi_0 \mathcal{O}} \right\rangle_{CFT}, \tag{1.24}$$

where ϕ_0 is the source. In particular, the correlation functions can be obtained by taking derivatives of the generating function and finally setting the source to zero. In other words,

$$< \mathcal{O}\left(x_1\right) \mathcal{O}\left(x_2\right) \cdots \mathcal{O}\left(x_n\right) >= \frac{\delta}{\delta\phi_0\left(x_1\right)} \cdots \frac{\delta}{\delta\phi_0\left(x_n\right)} Z\left[\phi_0\right] |_{\phi_0=0} . \tag{1.25}$$

The precise form of the AdS/CFT correspondence is encoded in the GKPW rule [4, 5]:

$$Z_{string}\left[\phi_0\right] = \left\langle exp \int \mathcal{O}\phi_0 \right\rangle_{CFT} . \tag{1.26}$$

In other words, the generating function of the CFT is identified with the string partition function. Furthermore, the source ϕ_0 is identified with the boundary value of some field in the bulk [4, 5]:

$$\phi_0(x) = \lim_{z \to 0} z^{\Delta_-} \phi(z, x), \tag{1.27}$$

where Δ_- is the conformal scaling dimension of some operator and is given by[4]

$$\Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + m^2 R^2}. \tag{1.28}$$

1.2.3 The PP-wave/SYM Duality

Unlike the analogous case in flat spacetime, string theory on the curved $AdS_5 \times S^5$ background has proven to be rather difficult to solve completely. Hence, in the early days of the AdS/CFT correspondence most of the analysis was limited to the weakly curved $AdS_5 \times S^5$ background - this corresponds to studying the AdS/CFT correspondence in the supergravity approximation. Obviously it was of extreme importance to move beyond this supergravity approximation.

Progress in moving beyond the supergravity approximation was achieved by Berenstein *et al.* [19] - for reviews, see [20, 21, 22, 23]. The simple idea was to take a particular example of a Penrose limit. (In the Penrose limit any spacetime can be reduced into a plane-wave background [24].) The advantageous thing about the plane-wave background is that (in contrast to $AdS_5 \times S^5$) it is possible to solve for the string spectrum exactly.

In global coordinates the metric for $AdS_5 \times S^5$ is

$$ds^2 = R^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \cos^2 \psi d\theta^2 + d\theta^2 + \sin^2 \psi d\tilde{\Omega}_3^2 \right). \tag{1.29}$$

[4] In general, the scaling dimension is given by $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}$, where m is the mass of the bulk field.

Anti-de Sitter Space

Basically, anti-de Sitter space (AdS_{p+2}) is the maximally symmetric solution of Einstein's field equation with a negative constant curvature. It is best visualized as a $(p+2)$-dimensional hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = R^2, \tag{1.30}$$

The hyperboloid is embedded in a flat $(p+3)$-dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2. \tag{1.31}$$

It is not difficult to see that the hyperboloid condition can be satisfied by the following parametrization:

$$\begin{aligned} X_0 &= R\cosh\rho\sin\tau, \qquad X_{p+2} = R\cosh\rho\cos\tau \\ X_i &= R\sinh\rho\,\omega_i \ (i=1,\ldots,p+1), \qquad \sum_i \omega_i^2 = 1 \end{aligned} \tag{1.32}$$

In terms of this parametrization, one can show that the induced metric of AdS_{p+2} is

$$ds^2 = R^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_p^2\right). \tag{1.33}$$

Here, $d\Omega_p^2$ is the p-dimensional line element for the p-sphere and is defined inductively by [12]

$$d\Omega_p^2 = d\theta_p^2 + \sin^2\theta_p d\Omega_{p-1}^2, \tag{1.34}$$

with $d\Omega_1 = d\theta_1$.

The coordinates (ρ, τ, Ω_p) are called global coordinates - since by taking $\rho \geq 0, \tau \in [0, 2\pi)$ it is possible to cover the entire hyperboloid. In contrast, we also have the Poincaré coordinates $(u, t, \mathbf{x})$ which only cover half of the hyperboloid. In terms of the Poincaré coordinates the metric (which we have already come across) is

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2 \left(-dt^2 + d\mathbf{x}^2 \right) \right). \tag{1.35}$$

Let us consider the motion of a particle moving along the ψ direction (we set $\rho = 0$ and $\theta = 0$). We begin by introducing light cone coordinates: $\tilde{x}^{\pm} = \frac{1}{2}(t + \psi)$. We rescale the coordinates as follows [19]:

$$x^+ = \tilde{x}^+, \quad x^- = R^2 \tilde{x}^-, \quad \rho = \frac{r}{R} \quad , \theta = \frac{y}{R}. \tag{1.36}$$

Finally, in the limit as $R \to \infty$ we find that $AdS_5 \times S^5$ metric reduces to the pp-wave background:

$$ds^2 = -4dx^+dx^- - \left(r^2 + y^2\right) \left(dx^+\right)^2 + dr^2 + dy^2. \tag{1.37}$$

The natural question to ask is what does the Penrose limit that we have just performed correspond to in the dual gauge theory. To answer this question we need to consider how the energy $E = i\partial_t$ and the angular momentum scale.[5] It is straightforward to show that [19]

[5] Here, the energy E (the angular momentum J) is associated with the conformal dimension Δ ($\mathcal{R}$ charge) of some operator [20].

$$2p^- = -p^+ = i\partial_{\tilde{x}^+} = i\left(\partial_t + \partial_\psi\right) = \Delta - J$$
$$2p^+ = -p_- = \frac{\tilde{p}_-}{R^2} = \frac{1}{R^2} i\partial_{\tilde{x}^-} = \frac{1}{R^2} i\left(\partial_t - \partial_\psi\right) = \frac{1}{R^2}\left(\Delta + J\right)$$

In order for p^+ to remain finite as we take in $R \to \infty$ in the Penrose limit, it will be necessary to have $\Delta \approx J \backsim R^2$. By the AdS/CFT dictionary $R^4 = 4\pi g_{YM}^2 N$, and hence we need to take the limit [21]

$$N \to \infty, \qquad J \sim \sqrt{N}, \qquad g_{YM} \text{ fixed.} \tag{1.38}$$

Put differently, we are considering the double-scaling limit:

$$N, J \to \infty, \qquad \lambda' = \frac{g_{YM}^2 N}{J^2} \text{ and } g_2 = \frac{J^2}{N} \text{ fixed.} \tag{1.39}$$

Here, λ' is the effective loop-counting parameter and g_2 is the effective genus-counting parameter. The doubling-scaling limit in (1.39) is called the BMN limit and is a special case of the 't Hooft large-N limit. In addition, the operators with large $\mathcal{R}$-charge are referred to as BMN operators - this includes both the BPS and non-BPS operators [20].

The Green-Schwarz action is

$$S = \frac{1}{2\pi\alpha'} \int dt \int_0^{2\pi\alpha p^+} d\sigma \left(\frac{1}{2}\dot{z}^2 - \frac{1}{2}z'^2 - \mu^2 z^2 + fermions\right), \tag{1.40}$$

The resulting equations of motions are trivial to solve. Therefore, (as already mentioned) it is possible to determine the string spectrum in the plane-wave background.

The light-cone Hamiltonian is [19]

$$\mathcal{H}_{l.c} = \sum_{n=-\infty}^{\infty} N_n \sqrt{\mu^2 + \left(\frac{n}{\alpha' p^+}\right)^2}, \tag{1.41}$$

where N_n is the occupation number. As is customary, the ground state is defined by [22]:

$$a_n^{\dagger i} \left|0; p^+\right\rangle = 0. \tag{1.42}$$

The ground state is mapped to the single trace operator with $\Delta - J = 0$. Such a unique operator is $\mathrm{Tr} Z^J$ [19]. Here $Z = \Phi_1 + i\Phi_2$ and Φ_I $(I = 1, \ldots, 6)$ are the six scalars of $\mathcal{N} = 4$ SYM. Thus,

$$\mathrm{Tr} Z^J \longleftrightarrow \left|0; p^+\right\rangle. \tag{1.43}$$

Now, we consider the excited states. It is obvious that on the string side the first excited states can be obtained by acting on the ground state with the bosonic creation operator a^i or the fermionic creation operator S^i [19]. On the dual CFT side, the first excited state corresponds to inserting "impurities" with $\Delta - J = 1$ in the operator $\mathrm{Tr} Z^J$. The "impurities" with $\Delta - J = 1$ are the four Higgs scalars that are not rotated by J *i.e.*Φ_i $(i = 1, \ldots, 4)$, and $D_i Z$ $(i = 1, \ldots, 4)$ [20, 22]. Hence,

$$\mathrm{Tr}\left(\Phi_i Z^J\right) \longleftrightarrow a_0^i \left|0; p^+\right\rangle \; i = 1, \ldots, 4 \tag{1.44}$$

$$\mathrm{Tr}\left(D_i Z Z^J\right) \longleftrightarrow a_0^i \left|0; p^+\right\rangle \; i = 5, \ldots, 8. \tag{1.45}$$

The mapping that we have been discussing is valid in the supergravity approximation - more precisely, we have limited our discussion to BPS operators [20]. Naively, in order to move beyond the BPS operators we need to insert "impurities" and also introduce a phase that we will sum over [22]. For example,

$$\sum_{l=0}^{J} e^{\frac{2\pi i n l}{J}} \mathrm{Tr}\left(Z^l \Phi_i Z^{J-l}\right) \longleftrightarrow a_n^i \left|0; p^+\right\rangle, \tag{1.46}$$

or more generally [20],

$$\sum_{l_1,\ldots,l_m=0}^{J} e^{\frac{2\pi i}{J}(n_1 l_1+\cdots+n_m l_m)} \mathrm{Tr}\left(\cdots Z\Phi_i Z \cdots Z\Phi_i Z \cdots\right) \longleftrightarrow a_{n_1}^i \cdots a_{n_m}^i \left|0; p^+\right\rangle. \tag{1.47}$$

The BMN operators are the eigenstates of the dilatation operator. The dilatation operator is a useful tool to obtain the anomalous dimension of operators.[6] In particular, the scaling dimensions of a generic operator are the eigenvalues of the dilatation operator[7] *i.e.*

$$\mathfrak{D}\mathcal{O}_\alpha = \Delta_\alpha \mathcal{O}_\alpha. \tag{1.48}$$

where Δ_α is the scaling dimension. Moreover, as a result of quantum corrections the scaling dimension reads

$$\Delta_\alpha = \left(\Delta_0\right)_\alpha + \gamma, \tag{1.49}$$

where $\left(\Delta_0\right)_\alpha$ is the classical scaling dimension - this can be obtained by trivial power-counting - and γ is the anomalous dimension.

[6] For examples of how the anomalous dimensions of certain BMN operators can be computed see [27, 26].

[7] The scaling dimensions can be obtained directly from the two-point function. Recall that due to the conformal symmetry, the two-point function is restricted to be of the form: $\left\langle \mathcal{O}_\alpha(x)\, \mathcal{O}_\beta(0)\right\rangle = \frac{\delta_{\alpha\beta}}{|x|^{2\Delta_\alpha}}$, where Δ_α is the conformal scaling dimension. However, this is not always the best way of obtaining the scaling dimensions and indeed this is one reason for introducing the dilatation operator.

The dilatation operator has a perturbation expansion of the form [28]:

$$\mathfrak{D} = \sum_{k=0}^{\infty} \left(\frac{g_{YM}^2}{16\pi^2} \right)^k \mathfrak{D}_{2k}. \tag{1.50}$$

In particular, $\mathfrak{D}_0$ is the classical dilatation operator and the one-loop Dilatation operator is [28]

$$\mathfrak{D}_2 = - : \mathrm{Tr}\left[\Phi_m, \Phi_n\right] \left[\check{\Phi}_m, \check{\Phi}_n\right] : -\frac{1}{2} : \mathrm{Tr}\left[\Phi_m, \check{\Phi}_n\right] \left[\Phi_m, \check{\Phi}_n\right] : \tag{1.51}$$

where

$$\check{\Phi}_m = \frac{\delta}{\delta \Phi_m}, \tag{1.52}$$

and - as is customary - the normal ordering is denoted by ::.

In particular, for the BMN theory the one-loop contribution is [28]

$$\mathfrak{D}_2 = -2 : \mathrm{Tr}\left[\Phi_m, \Phi_n\right] \left[\check{\Phi}_m, \check{\Phi}_n\right] : \tag{1.53}$$

In [30], it was shown that the one-loop mixing matrix for $\mathcal{N} = 4$ SYM can identified with the Hamiltonian for an integrable $SO(6)$ spin chain.

Let us now consider the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{R} \times S^3$. Firstly, $\mathbb{R} \times S^3$ - the coordinates for $\mathbb{R} \times S^3$ can be written as (t, θ, ϕ, ψ) - is conformally equivalent to $\mathbb{R}^4$ [29]. Indeed, we have

$$ds^2 = dr^2 + r^2 d\Omega_3^2 = e^{2\tau} \left(d\tau^2 + d\Omega_3^2 \right) \tag{1.54}$$

where $\tau = \ln r$ [29]. Thus, the generator of scale transformations (*i.e.* $r \to cr$) is mapped to the generator of time translations on $\mathbb{R} \times S^3$ [29].

In fact, the argument can be formalized and we find that the dilation operator for $\mathcal{N} = 4$ SYM can be interpreted as the Hamiltonian of a matrix model. The matrix model Hamiltonian reads [25, 30, 50, 60, 31]

$$H = -\frac{g_{YM}^2}{16\pi^2}\mathrm{Tr}\,[Y, Z]\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right]. \tag{1.55}$$

where $Z = \Phi_1 + i\Phi_2$, $Y = \Phi_5 + i\Phi_6$ and the two-point functions are[8] [25]:

$$\left\langle Z_{ij}^{\dagger} Z_{kl}\right\rangle_{free} = \frac{1}{8\pi^2}\delta_{il}\delta_{jk} = \left\langle Y_{ij}^{\dagger} Y_{kl}\right\rangle_{free}. \tag{1.56}$$

Interestingly, the dilatation operator - at least to the first few orders in perturbation theory - agrees completely with the results obtained using a plane-wave matrix model [50].

Thus, we can already see the major role that matrix models play in the gauge-gravity correspondence. Accordingly, in the next section we explain precisely what is a matrix model and review some of the matrix-model technology that we will make use of in this work.

1.3 Random Matrix Models

Random matrices were introduced in Physics by Wigner [32]. The initial motivation was to provide a framework that could describe the spectra of heavy nuclei - the key observation was that the energy levels of a heavy nucleus were similar to the statistics of the eigenvalues for a large Hermitian matrix . Subsequently, they have found numerous applications in both Mathematics and Physics.[9]

[8]The spatial part of the two-point function is fixed by conformal invariance.
[9]For a detailed review of the applications, see [33]

In this section we give a brief review of the one-matrix model and give a some examples that should illustrate the power and utility of random matrix models in Physics.

1.3.1 One-Matrix Model

Random matrix model are simple examples of zero-dimensional theories. The field is given by an $N \times N$ matrix - usually the matrix is chosen to be Hermitian. Moreover, the partition function (path-integral) is

$$Z = \int [dM] \, e^{-N \mathrm{Tr} V(M)}. \tag{1.57}$$

Here, the measure is

$$[dM] = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re} M_{ij} d\mathrm{Im} M_{ij}. \tag{1.58}$$

It is clear that the above matrix model has an overall $U(N)$ symmetry: $M \to UMU^{\dagger}$. Accordingly, it is possible to perform a change of variable from the N^2 matrix elements to the more convenient set of N eigenvalues - we will denote the eigenvalues by λ_i. In terms of the eigenvalues, the measure can be written as [34]

$$\begin{aligned}
[dM] &= \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \, DU \\
&= \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) \, DU,
\end{aligned} \tag{1.59}$$

where $\Delta(\lambda)$ is the Vandermonde determinant and DU is the measure for the unitary group. As a result, the partition function becomes

$$
\begin{aligned}
Z &\sim \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda)\, e^{-N\sum_{i=1}^{N} V(\lambda_i)} \\
&= \int \prod_{i=1}^{N} d\lambda_i e^{-S_{eff}},
\end{aligned} \tag{1.60}
$$

where

$$
S_{eff} = N\sum_{i=1}^{N} V(\lambda_i) - \sum_{i<j} \ln(\lambda_i - \lambda_j)^2 \tag{1.61}
$$

In the large-N limit we can perform the integral by using the steepest descent approach. The saddle-point equations are

$$
V'(\lambda_i) = \frac{2}{N}\sum_{i\neq j} \frac{1}{\lambda_i - \lambda_j}. \tag{1.62}
$$

Next, we introduce the eigenvalue density

$$
\phi(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i), \tag{1.63}
$$

and write the saddle-point equation as

$$
V'(\lambda) = 2 \fint \frac{d\lambda' \phi(\lambda')}{\lambda - \lambda'}. \tag{1.64}
$$

It is possible - in fact, in this thesis we use a different approach to solving the above singular integral equation - to explicitly solve (1.64) for the eigenvalue density [34]. Assuming a “one-cut” solution[10], we have

$$
\phi(z) = -\frac{1}{2\pi i}\left(\omega(z+i\epsilon) - \omega(z-i\epsilon)\right), \tag{1.65}
$$

[10]The generating function - *i.e.* $\omega(z)$ below - is assumed to have a cut in the interval (a, b).

where

$$\omega\left(z\right)=\frac{1}{2}\int_{a}^{b}\frac{d\lambda}{2\pi i}\frac{V'\left(\lambda\right)}{z-\lambda}\left(\frac{\left(z-a\right)\left(z-b\right)}{\left(\lambda-a\right)\left(\lambda-b\right)}\right)^{1/2}. \tag{1.66}$$

1.3.2 Applications Of Random Matrix Models

Matrix models provide us with an interesting "laboratory" in which to study countless phenomena in Physics. In this section we will consider some examples that illustrate the importance of matrix models - it should be stressed that most of the applications mentioned here are not connected with the work done in this thesis.

1.3.2.1 M-theory

As is well-known, there are actually five consistent superstring theories. These are type IIA, type IIB, type I, Heterotic $SO\left(32\right)$ and Heterotic $E_8\times E_8$.

During the second super string revolution, all of the five string theories were shown to be related to one another by a web of dualities [35]. It was further pointed out that all the super strings are part of some eleven-dimensional theory that was dubbed M-theory. Furthermore, the low-energy limit of M-theory is given by 11D SUGRA and it is known that compactification of M-theory leads to strongly coupled type IIA string theory - in fact, the relation between the compactification of 11D SUGRA and type IIA strings was one of the key results that led to the discovery of M-theory.

Even at present, M-theory is still an enigma. However, it has been suggested that the large-N limit of a supersymmetric matrix quantum mechanics in the infinite momentum frame captures all the dynamics of M-theory [36]. The BFSS Lagrangian can be obtained by reducing ten-dimensional $\mathcal{N}=1$ SYM to $(0+1)$ dimensions and is given by

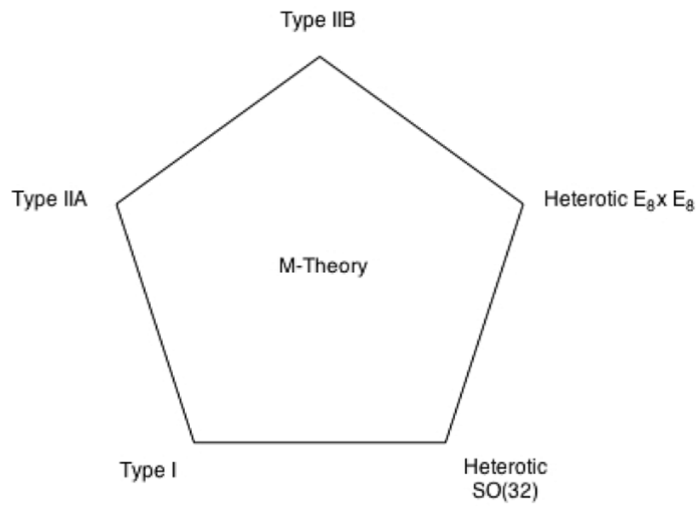

Figure 1.3: Cartoon diagram showing the web of string dualities linking the various superstrings to each other and their relations to M-theory.

$$\mathcal{L} = \mathrm{Tr}\left[\frac{1}{2}D_tX^iD_tX_i - \frac{1}{4}\left[X_i,\, X_j\right]^2 + \Theta^T D_t\Theta + \Theta^T\gamma_i\left[X^i,\, \Gamma^i\Theta\right]\right], \tag{1.67}$$

where X^i $(i = 1, \ldots, 9)$ are $N \times N$ matrices and Θ are 16 Grassmann matrices.

Reduction of ten-dimensional $\mathcal{N} = 1$ SYM to $(0+0)$ dimensions leads to the IKKT matrix model [37]. The action for the IKKT matrix model is given by

$$S = -\alpha\left(\mathrm{Tr}\left[X_\mu,\, X_\nu\right]^2 + \mathrm{Tr}\Psi\Gamma^\mu\left[X_\mu,\, \Psi\right]\right) + \beta\mathrm{Tr}1. \tag{1.68}$$

1.3.2.2 The Simplest Gauge/String Duality

The fact that we still do not have a formal proof of the AdS/CFT correspondence, together with the fact that it is a weak/strong duality, means that we have not fully grasped the underlying workings of this extremely important conjecture. As such it is desirable to have certain simple examples that can lead to a better understanding of the correspondence.

Recently, Gopakumar [38] has suggested a candidate for what might possibly be the simplest example of the gauge-string theory duality. The proposal is that the Hermitian matrix model is dual to the A-model topological string on $\mathbb{P}^1$. It is hoped that this duality will shed some light on the AdS/CFT correspondence.

1.3.2.3 ABJ(M) Matrix Model

ABJM theory [39, 40] is $\mathcal{N}$=6 superconformal Chern-Simons-matter theory in three dimensions. The gauge group is $U(N)_k \times U(N)_{-k}$, where k is the Chern-Simons level. (In fact, ABJM theory describes a stack of N $M2$-branes.)

The AdS_4/CFT_3 conjectures that ABJM theory is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Furthermore, since M-theory is the strong-coupling limit of type IIA, we also have that for $N \gg k^5$ ABJM is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ [39].

The fact that we do not have a complete understanding of M-theory together with the fact that ABJM is slightly more complicated than the analogous $\mathcal{N} = 4$ SYM makes the AdS$_4$/CFT$_3$ conjecture both fascinating and also difficult to work with.

Nevertheless, some results concerning the conjecture - for example, the $N^{3/2}$ scaling behavior of multiple $M2$-branes - have been reproduced using the fact that that ABJM on $\mathbb{R} \times S^3$ reduces to a matrix model with partition function given by [43]

$$Z_{ABJM} = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{d\mu_i d\nu_j}{2\pi} \frac{\prod_{i<j} \sinh^2\left(\frac{\mu_i-\mu_j}{2}\right) \sinh^2\left(\frac{\nu_i-\nu_j}{2}\right)}{\prod_{i,j} \cosh^2\left(\frac{\mu_i-\nu_j}{2}\right)} e^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 - \sum_j \nu_j^2\right)}. \tag{1.69}$$

Similarly, the partition function for the ABJ matrix model is [42][11]

[11] Recall that ABJ theory [41] is $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory with $U(N_1) \times U(N_2)$gauge group and Chern-Simons levels k and $-k$.

$$Z_{ABJ}(N_1,N_2) = \frac{(-1)^{\frac{1}{2}N_1(N_1-1)+\frac{1}{2}N_2(N_2-1)}}{N_1!N_2!}\int \frac{d^N\mu}{(2\pi)^{N_1}}\frac{d^N\nu}{(2\pi)^{N_2}}$$
$$\left(\frac{\prod_{i<k} 2\sinh\frac{\mu_i-\mu_j}{2}\prod_{a<b} 2\sinh\frac{\mu_i-\mu_j}{2}}{\prod_{i,a} 2\cosh\frac{\mu_i-\nu_a}{2}}\right) e^{\frac{ik}{4\pi}\left(\sum_i \mu_i^2-\sum_a \nu_a^2\right)}. \tag{1.70}$$

1.3.2.4 Other Examples

Supersymmetric gauge theories have always been a fascinating field of study. In such theories a special role is played by the superpotential. In a remarkable set of papers [44], Dijkgraaf and Vafa proposed that the superpotentials for a wide class of $\mathcal{N}=1$ super-Yang-Mills theories can be computed by using random matrix models.

Another example of the occurrence of matrix models is given by a conjecture concerning the computation of circular Wilson loops [45]. The conjecture is that the computation of the circular Wilson loops reduces to computing expectation values of a Gaussian matrix model. The conjecture that finally proven by Pestun [46].

As a result of the work done by Vafa and Dijkgraaf [47], matrix models also feature prominently in the AGT correspondence [48] - the AGT correspondence relates the Nekrasov partition function of a four-dimensional $\mathcal{N}=2$ $SU(2)$ gauge theory to the conformal blocks of a two-dimensional CFT *i.e.* Liouville theory.

String theory is consistent only in ten dimensions (or twenty six spacetime dimensions in the case of the bosonic string). Nevertheless, it is still possible to study string theories that are not at critical dimension. An example is two-dimensional string theory. Here, the sum over the metric can be mapped to a lattice theory of discretized surfaces [49] which in turn is mapped to the single Hermitian matrix model with a cubic potential. That is, the partition function of the matrix model is given by

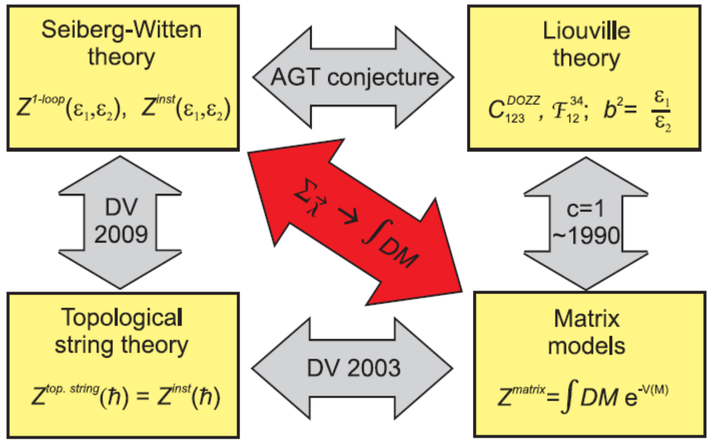

Figure 1.4: Schematic diagram (including the seminal work done by Dijkgraaf and Vafa) indicating how matrix models are related to the AGT correspondence.

$$Z = \int [dM]\, e^{N\left[-\frac{1}{2}\mathrm{Tr}M^2 + g\mathrm{Tr}M^3\right]}. \tag{1.71}$$

1.4 Half-BPS Sector

The high degree of supersymmetry enjoyed by $\mathcal{N} = 4$ SYM implies that the theory has some simplifications. Hence, it is not surprising that $\mathcal{N} = 4$ SYM has certain operators that are protected from receiving quantum corrections. From the context of the AdS/CFT correspondence, such operators are of great importance. In particular, since the correspondence is a weak/strong duality, protected operators provide a powerful way to "probe" or check the validity of the conjecture. This follows since it is possible to perform calculations with the protected operators at weak coupling and check if they agree with corresponding observables in the dual string at weak coupling - it is clear that we can do this as the weakly coupled gauge computation is obviously identical with the strongly coupled gauge theory result.

BPS states

The extended susy algebra can be written as [15]

$$\left\{Q_{\alpha}^{I}, Q_{\beta}^{\dagger J}\right\} = 2M\delta^{IJ}\delta_{\alpha\beta} + 2i\Gamma^{0}_{\alpha\beta}Z^{IJ}. \tag{1.72}$$

Here, Z^{IJ} $(I, J = 1, \ldots, \mathcal{N})$ is the central charge matrix and the Q_{α}^{I} $(I = 1, \ldots, \mathcal{N})$ are the supercharges. Moreover, since the central charge is antisymmetric, it can be brought in block-diagonal form. That is, the central charge can be written as

$$Z^{IJ} = \begin{pmatrix} 0 & Z_1 & 0 & 0 & \\ -Z_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & Z_2 & \\ 0 & 0 & -Z_2 & 0 & \\ & \vdots & & & \ddots \end{pmatrix} \tag{1.73}$$

where $|Z_1| \geq |Z_2| \geq \ldots \geq 0$. The supersymmetric algebra implies that for massive particles, we have [15]

$$M \geq |Z_1|\,. \tag{1.74}$$

This is called the Bogomolnyi-Prasad-Sommerfield (BPS) bound. States that saturate the bound - that is, states for which $M = |Z_1|$ - are called BPS states. In addition, the BPS states belong to a short representation of the supersymmetric algebra. (The shortened representation is a consequence of the zeroes that occur in the susy algebra when the BPS bound is saturated.) To show that this is the case, it is simpler to work in the the rest frame. Accordingly, the four-momentum is $P^{\mu} = (M, 0, 0, 0)$ and the susy algebra takes the form[12] [7]

$$\left\{Q^I_{\alpha\pm}, Q^{\dagger J}_{\beta\pm}\right\} \sim \delta_{\alpha\beta}\delta^{IJ}\left(M \pm Z_I\right). \tag{1.75}$$

This implies that when a bound is saturated, one of the supercharges automatically vanishes - and hence the shortening of the multiplet representation.

Also, depending on the number of central charges that are equal to the mass M - alternatively, the number of broken supersymmetries - we speak of half-BPS or quarter-BPS and so forth.

In addition to the BPS gravity states discussed earlier, another important class of BPS is given by the half-BPS operators of $\mathcal{N} = 4$ SYM.

Since the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory has six scalar fields, it follows that we can group the scalars as

$$Z = \Phi_1 + i\Phi_2,\ Y = \Phi_3 + i\Phi_4,\ X = \Phi_5 + i\Phi_6. \tag{1.76}$$

The half-BPS operators of $\mathcal{N} = 4$ SYM are constructed by taking single trace combinations formed by one of the complex scalars. (Usually the matrix Z is chosen.) In particular, the half BPS operators[13] are given by the chiral primary operators.[14]:

$$\prod_{i=1}^{N} \mathrm{Tr}\left(Z^{n_i}\right). \tag{1.77}$$

where the n_i $(i = 1, \ldots, N)$ are integers with $n_1 \geq n_2 \cdots \geq n_N$.

The duals of these operators are the Kaluza-Klein giants, sphere or AdS giant gravitons. One of the principal idea that emerged from a study of the duals of the chiral primaries of

[13] For a clear introduction to the half-BPS sector of AdS/CFT, see [51]

[14] A chiral primary operator is defined by: $[K_\mu, \mathcal{O}] = 0$, where K_μ is the generator for the special conformal transformations.

$\mathcal{N} = 4$ SYM is that the half-BPS sector can be completely described in terms of the holomorphic sector of a complex matrix model [52, 53, 54] (more precisely, the holomorphic sector of a complex matrix quantum mechanics) in a harmonic potential. This complex matrix model arises from compactification of $\mathcal{N} = 4$ SYM on $R \times S^3$. In addition, the partition function is given by [54]

$$Z = \int \left[dZ\left(t\right)dZ^{\dagger}\left(t\right)\right] \exp\left(i \int dt^{\mathrm{Tr}\left(\dot{Z}(t)\dot{Z}^{\dagger}(t) - Z(t)Z^{\dagger}(t)\right)}\right). \tag{1.78}$$

Here, the measure for the complex matrix model is given by

$$\left[dZ\left(t\right)dZ^{\dagger}\left(t\right)\right] = \prod_{i,j} d\mathrm{Re}Z\left(t\right)_{ij} d\mathrm{Im}Z\left(t\right)_{ij} \tag{1.79}$$

The harmonic potential arises due to the coupling of matter to the positive curvature of the S^3 .

Remarkably, the complex matrix model description is equivalent to a system of free fermions. In the dual string theory side a corresponding free fermion picture also emerges [55, 54].

1.5 Outline

In this thesis, matrix models are used to study the gauge/gravity duality. In particular, we consider the 't Hooft large-N limit - more specifically, the strong-coupling regime - of matrix models with potentials of the form:[15]

$$V = \frac{1}{g_{YM}^2} V'. \tag{1.80}$$

[15]The potential V' is independent of the coupling g_{YM}.

The motivation for this comes from the fact that - as is well-known - the fields in SYM theories can be rescaled so that the action appears multiplied by $\frac{1}{g_{YM}^2}$, as explicitly shown in (1.5). In this formulation, the AdS/CFT identification $g_s = g_{YM}^2$ is manifest.[16]

Hence, we expect the multi-matrix model to "simulate" the AdS/CFT behaviour relating the 't Hooft coupling to the radius of the AdS space. Certainly, we find that the "size" of the eigenvalue density distribution increases with $\lambda = g_{YM}^2 N$.

This thesis is organized as follows. In the next chapter we give a review of the Collective Field Theory Hamiltonian method. In particular, we apply the formalism to a single matrix model with a potential of the form given in (2.48). In this case, and at first sight, it would seem that the theory becomes free at strong-coupling. In order to elucidate this, we "compactify" the Hermitian matrix model on the circle S^1 and carry out first-order pertubation theory. In addition, we use the Collective Field Theory method to obtain the exact ground-state properties of the system. In particular, we compute the ground-state energy in the strong and weakly coupled regimes and compare the exact results with the results that were obtained through pertubation theory.

Chapter 3 discusses the two-matrix integral with a Yang-Mills potential. We review the solution of this model in an approach where one of the matrices is diagonalized and the second matrix is integrated out [66, 67]. This enables us to find an effective action and, consequently, the saddle-point equations. We conclude the chapter by discussing how a radial density distribution can be obtained following the approach by Berenstein *et al.* which is based on commuting matrices at strong-coupling [71].

In Chapter 4 we consider the multi-matrix integral that results from compactification of the Higgs sector of $\mathcal{N} = 4$ SYM on S^4 (or $T \times S^3$).[17] The action is that of three complex matrices (or six Hermitian matrices $X_I (I = 1, \ldots, 6)$) interacting via a Yang-Mills interaction.

[16] Naively, the relation follows from comparing (1.5) and (1.8).

[17] Compactification of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ leads to the $D = 1$ matrix integral *i.e.* a matrix quantum mechanics.

From dimensional analysis, it follows that

$$[X_I] = [\lambda]^{1/4} . \tag{1.81}$$

Hence, this system provides an excellent setting where one may attempt to understand the dynamics of the "AdS/CFT-like" relationship between the radial coordinate and the coupling, provided such a radial coordinate can be identified.

It turns out that for an even number of Hermitian matrices, a matrix valued radial subsector of the system has been identified [75]. We review how to obtain the Jacobian that results from the change of variables from the original Hermitian matrices to the eigenvalues of the radial matrix. We solve the resulting saddle-point equations and obtain the end-points and the radial density of eigenvalues for an arbitrary number of matrices.

Moreover, it turns out that for two Hermitian matrices we can parametrize the system by using matrix valued polar coordinates. This allows us to study the angular degrees of freedom. In particular, we perform first-order perturbation and solve the resulting saddle-point equations. (In fact, the exact result for the integral involving the angular degrees of freedom is given by the Harish-Chandra-Itzykson-Zuber (HCIZ) formula [77]. However, the exact result involves a determinant and is thus not particularly useful.[18])

In Chapter 5 we discuss the Dyson-Schwinger (loop equations) approach to the large-N limit [56, 57]. The loop equations that we consider allow us to write the algebraic equation that is satisfied by the generating function. In particular, we apply the loop equations method to obtain the generating function in the the radial sector. The auxiliary two matrix model is then introduced. For the "pure" quadratic potential - *i.e.* a potential that does not involve the logarithmic term - we are able to reproduce the results given in [66, 67, 71]. In particular, we obtain the radial eigenvalue distribution which

[18]The resulting saddle-point equations lead to a singular eigenvalue distribution.

is identical to the hemisphere distribution that was obtained by Berenstein *et al* [71]. However, the strong-coupling expansion is given in terms of both the 't Hooft coupling and a mass parameter ω. Accordingly, we consider the Dobroliubov-Makeenko-Semenoff (DMS) loop equations[78] with the logarithm term included and obtain a cubic equation that is satisfied by the generating function. The resulting cubic equation is solved at strong-coupling and the eigenvalue density is determined.

In Chapter 6 we give a summary of the key results that were obtained in this dissertation.

In Appendix A, we derive the character expansion, which was used in Chapter 2, for $\mathrm{Tr}U^n$. In Appendix B, we discuss the elliptic integrals and also give a sketch as to how the Hoppe two-matrix integral can be parametrized in terms of the elliptic integrals.

Chapter 2

A Single Matrix Hermitian System

In this chapter we consider the large-N limit of a single Hermitian system. We discuss a generic way of reformulating a given theory in terms of the invariants of that particular theory. The method that we look at is the Collective Field Theory formalism [58].[1] This method was originally introduced by Jevicki and Sakita - the initial motivation was to generalize the Bohm-Pines theory of plasma oscillations. In particular, we apply the formalism to a single Hermitian system with a quartic interaction. The system appears to be free in the strong-coupling limit and we obtain the first-order correction to the ground-state energy. This is then followed by an exact analysis - using the Collective Field Theory approach - of the ground-state energy. Finally, we note that the strong-coupling expansion is elusive as each term in the expansion contributes to the same order in λ.

2.1 The Collective Field Theory Hamiltonian

Let us consider some generic theory with a Hamiltonian given by [59]

[1] The power of reformulating a given theory in terms of the invariants is indeed self-evident. In fact, one of the earliest applications of the Collective Field Theory method was in elucidating attempts to reformulate the whole of Quantum Chromodynamics in terms of the gauge invariant Wilson loops [56, 57].

$$
\begin{aligned}
H &= K + V \\
&= -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial q_i^2} + V\left(q_1, \cdots, q_N\right).
\end{aligned} \tag{2.1}
$$

Our goal is to perform a change of variables from the coordinates q_i, $(i = 1, 2, \ldots, N)$ to a set of invariant quantities that we denote by ϕ_C. A straightforward application of the chain rule yields

$$
\begin{aligned}
K &= -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial q_i^2} \\
&= -\frac{1}{2}\left(\sum_{i=1}^{N}\frac{\partial^2\phi_C}{\partial q_i^2}\right)\frac{\partial}{\partial\phi_C} - \frac{1}{2}\left(\sum_{i=1}^{N}\frac{\partial\phi_C}{\partial q_i}\frac{\partial\phi_{C'}}{\partial q_i}\right)\frac{\partial^2}{\partial\phi_C\partial\phi_{C'}} \\
&= -\frac{1}{2}\omega\left(C\right)\partial_C - \frac{1}{2}\Omega\left(C, C'\right)\partial_C\partial_{C'},
\end{aligned} \tag{2.2}
$$

where we have defined the following quantities:

$$
\omega\left(C\right) = \sum_{i=1}^{N}\frac{\partial^2\phi_C}{\partial q_i^2} \tag{2.3}
$$

$$
\Omega\left(C, C'\right) = \sum_{i=1}^{N}\frac{\partial\phi_C}{\partial q_i}\frac{\partial\phi_{C'}}{\partial q_i}. \tag{2.4}
$$

In general, we have

$$
\begin{aligned}
\omega\left(C\right) &= \sum\phi_{C'}\phi_{C''} \\
\Omega\left(C, C'\right) &= \sum\phi_{C+C'}
\end{aligned}
$$

That is, $\omega(C)$ splits loops ("words") while $\Omega(C, C')$ is responsible for joining the loops ("words"). Accordingly, $\omega(C)$ is sometimes referred to as the "splitting" operator, and likewise, $\Omega(C, C')$ is called the "joining" operator.

The change of variables that we have just performed introduces a nontrivial Jacobian:

$$\int [dq]\, \Psi_1^*(q)\, \Psi_2(q) = \int [d\phi]\, J[\phi]\, \Psi_1^*(\phi)\, \Psi_2(\phi) \tag{2.5}$$

As a result of the non-trivial Jacobian the Hamiltonian is no longer explicitly Hermitian. In order for us to make the Hamiltonian explicitly Hermitian it is necessary to perform a similarity transformation:

$$\partial_C \to J^{1/2} \partial_C J^{-1/2} = \partial_C - \frac{1}{2} \partial_C \ln J. \tag{2.6}$$

The transformed kinetic term is

$$\begin{aligned} K &= -\frac{1}{2}\omega(C)\left(\partial_C - \frac{1}{2}\partial_C \ln J\right) - \frac{1}{2}\Omega(C, C')\left(\partial_C - \frac{1}{2}\partial_C \ln J\right)\left(\partial_{C'} - \frac{1}{2}\partial_{C'} \ln J\right) \\ &= -\frac{1}{2}\omega(C)\,\partial_C + \frac{1}{4}\omega(C)\,\partial_C \ln J - \frac{1}{2}\partial_C\left(\Omega(C, C')\,\partial_{C'}\right) + \frac{1}{2}\partial_C\left(\Omega(C, C')\right)\partial_{C'} \\ &+ \frac{1}{4}\Omega(C, C')\,\partial_C \partial_{C'} \ln J + \frac{1}{2}\Omega(C, C')\left(\partial_{C'} \ln J\right)\partial_C - \frac{1}{8}\Omega(C, C')\left(\partial_C \ln J\right)\left(\partial_{C'} \ln J\right) \\ &= \left(-\frac{1}{2}\omega(C) + \frac{1}{2}\partial_{C'}\left(\Omega(C', C)\right) + \frac{1}{2}\Omega(C, C')\left(\partial_{C'} \ln J\right)\right)\partial_C + \frac{1}{4}\omega(C)\,\partial_C \ln J \\ &- \frac{1}{2}\partial_C\left(\Omega(C, C')\,\partial_{C'}\right) + \frac{1}{4}\Omega(C, C')\,\partial_C \partial_{C'} \ln J - \frac{1}{8}\Omega(C, C')\left(\partial_C \ln J\right)\left(\partial_{C'} \ln J\right). \end{aligned} \tag{2.7}$$

The requirement that the final Hamiltonian should be explicitly Hermitian implies that the coefficient multiplying the derivative ∂_C in (2.7) is zero. That is,

$$0 = -\omega(C) + \partial_{C'}\Omega(C', C) + \Omega(C, C')\,\partial_C \ln J, \tag{2.8}$$

or equivalently:

$$\partial_C \ln J = \Omega^{-1}(C,C')\,\omega(C') - \Omega^{-1}(C,C'')\,\partial_{C'}\Omega(C',C''), \tag{2.9}$$

which is a differential equation satisfied by the non-trivial Jacobian that results from performing the change of variables to the collective fields.

Substituting the equation for the Jacobian (*i.e.* (2.9)) in (2.7), one obtains

$$\begin{aligned}
K &= \frac{1}{4}\omega(C)\,\partial_C \ln J - \frac{1}{2}\partial_C\left(\Omega(C,C')\,\partial_{C'}\right) \\
&+ \frac{1}{4}\Omega(C,C')\,\partial_C\partial_{C'}\ln J - \frac{1}{8}\Omega(C,C')\left(\partial_C \ln J\right)\left(\partial_{C'}\ln J\right) \\
&= \frac{1}{4}\omega(C)\,\Omega^{-1}(C,C')\,\omega(C') - \frac{1}{2}\partial_C\left(\Omega(C,C')\,\partial_{C'}\right) \\
&- \frac{1}{8}\omega(C')\,\Omega^{-1}(C',C)\,\omega(C) + \triangle H \\
&= -\frac{1}{2}\partial_C\left(\Omega(C,C')\,\partial_{C'}\right) + \frac{1}{8}\omega(C)\,\Omega^{-1}(C,C')\,\omega(C') + \triangle H,
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\Delta H &= -\frac{1}{4}\partial_C\omega(C) + \frac{1}{4}\left(\partial_{C''}\Omega(C'',C')\right)\Omega^{-1}(C',C)\,\omega(C) \\
&+ \frac{1}{8}\left(\partial_{C''}\Omega(C'',C)\right)\Omega^{-1}(C,C')\left(\partial_{C'}\Omega(C',C''')\right).
\end{aligned}$$

In the large-N limit the leading contribution to the Hamiltonian is given by [60]

$$H = -\frac{1}{2}\partial_C\Omega(C,C')\,\partial_{C'} + \frac{1}{8}\omega(C)\,\Omega(C,C')\,\omega(C'). \tag{2.11}$$

A simple way to see that this is indeed the case is to show the N dependence in the

Hamiltonian explicitly. This can be done by performing the following rescaling:

$$\begin{aligned}\phi_C \to \sqrt{N}\phi_C,\ \partial_C \to \frac{1}{\sqrt{N}}\partial_C \\ \omega\left(C\right) \to \sqrt{N}\omega\left(C\right),\ \Omega\left(C,C'\right) \to \frac{1}{N}\Omega\left(C,C'\right).\end{aligned} \tag{2.12}$$

Now, let us consider the single matrix model. In this instance, a set of invariant observables is given by [58, 59]

$$\phi_k = \mathrm{Tr}\left(e^{ikM}\right). \tag{2.13}$$

However, it is more convenient to work with the Fourier transform of these invariants. Indeed, for the Fourier transform one obtains

$$\begin{aligned}\phi\left(x\right) &= \int \frac{dk}{2\pi} e^{-ikx}\phi_k \\ &= \sum_{i=1}^{N} \delta\left(x-\lambda_i\right)\end{aligned}$$

which is the eigenvalue density. Moreover, λ_i $(i = 1, \ldots, N)$ are the eigenvalues of the matrix M.

By using the definition of the "joining" operator, one obtains

$$\begin{aligned}\Omega_{kk'} &= \sum_{i,j} \frac{\partial \phi_k}{\partial M_{ij}} \frac{\partial \phi_{k'}}{\partial M_{ji}} \\ &= \sum_{i,j} \left(ike^{ikM}\right)_{ij} \left(ik'e^{ik'M}\right)_{ji} \\ &= -kk'\mathrm{Tr}\left(e^{i(k+k')M}\right) \end{aligned} \tag{2.14}$$

Similarly, the "splitting" operator is [59]

$$\begin{aligned}\omega_k &= \sum_{i,j} \frac{\partial^2 \phi_k}{\partial M_{ij} \partial M_{ji}} \\ &= \sum_{i,j} \frac{\partial}{\partial M_{ij}} \left(ike^{ikM}\right)_{ji} \\ &= -k \sum_{i,j} \int_0^k dk' \left(e^{ik'M}\right)_{ii} \left(e^{i(k-k')M}\right)_{jj} \\ &= -k \int_0^k dk' \phi_k \phi_{k-k'}. \end{aligned} \tag{2.15}$$

In obtaining $\omega(C)$ and $\Omega(C, C')$ we have made use of the following identities:

$$\frac{\partial}{\partial M_{ij}} \left(\mathrm{Tr} f(M)\right) = \left(f(M)\right)_{ji} \tag{2.16}$$

$$\frac{\partial}{\partial M_{ij}} \left(e^{At}\right)_{ab} = \int_0^t d\tau \left(e^{A\tau}\right)_{ai} \left(e^{A(t-\tau)}\right)_{jb} \tag{2.17}$$

The Fourier transforms of (2.14) and (2.15) are

$$\Omega_{xx'} = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ikx} e^{-ik'x} \Omega_{kk'}$$

$$= \partial_x \partial_{x'} \left(\phi\left(x\right) \delta\left(x - x'\right) \right), \tag{2.18}$$

and

$$\omega_x = \int \frac{dk}{2\pi} e^{-ikx} \omega_k$$

$$= -2\partial_x \left(\phi\left(x\right) \fint \frac{dz\phi\left(z\right)}{x - z} \right) \tag{2.19}$$

The Collective Field Theory Hamiltonian at large-N is

$$\begin{aligned} H &= -\frac{1}{2}\partial_k \Omega_{kk'} \partial_{k'} + \frac{1}{8}\omega_k \Omega_{kk'} \omega_{k'} \\ &= -\frac{1}{2}\int dx \int dx' \left(\partial_x \Omega_{xx'} \partial_{x'} + \frac{1}{8}\omega_x \Omega_{xx'} \omega_{x'} \right) + \int dx \phi\left(x\right) \left(v\left(x\right) - \mu\right) \\ &= -\frac{1}{2}\int dx \partial_x \frac{\partial}{\partial \phi\left(x\right)} \phi\left(x\right) \partial_{x'} \frac{\partial}{\partial \phi\left(x'\right)} + \int dx \left(\frac{\pi^2}{6}\phi^3\left(x\right) + \phi\left(x\right)\left(v\left(x\right) - \mu\right) \right). \end{aligned} \tag{2.20}$$

Here we have made use of the identity [63]

$$\int dx \phi\left(x\right) \left(\fint dy \frac{\phi\left(y\right)}{x - y} \right)^2 = \frac{\pi^2}{3} \int dx \phi^3\left(x\right). \tag{2.21}$$

The chemical potential μ has been introduced to ensure that the constraint

$$\int dx \phi\left(x\right) = N, \tag{2.22}$$

is automatically satisfied. To explicitly show the N dependence, we rescale as follows[2]

$$\begin{aligned}
x &\to \sqrt{N}x \\
\phi(x) &\to \sqrt{N}\phi(x) \\
-i\frac{\partial}{\partial\phi} \equiv \Pi &\to -\frac{i}{N}\frac{\partial}{\partial\phi} \\
\mu &\to N\mu
\end{aligned} \tag{2.23}$$

The Hamiltonian can be written as

$$\begin{aligned}
H_{eff} &= \frac{1}{2N^2}\int dx\partial_x\Pi(x)\,\phi(x)\,\partial_x\Pi(x) + N^2\left(\frac{\pi^2}{6}\int dx\phi^3(x) + \int dx\phi(x)\,(v(x)-\mu)\right) \\
&= \frac{1}{2N^2}\int dx\partial_x\Pi\phi(x)\,\partial_x\Pi + N^2V_{eff}.
\end{aligned} \tag{2.24}$$

The background density is determined by the stationary condition: $0 = \frac{\delta V_{eff}}{\delta\phi(x)}$. It is not difficult to see that the background density is

$$\phi_0(x) = \frac{1}{\pi}\sqrt{2(\mu - v(x))}. \tag{2.25}$$

The kinetic part is responsible for the fluctuations, which can be introduced by simply shifting the background density *i.e.*

$$\psi(x) = \phi_0(x) + \frac{1}{\sqrt{\pi}N}\partial_x\eta, \quad \partial_x\Pi(x) = -\sqrt{\pi}NP(x) \tag{2.26}$$

2.2 The Single Matrix Example

The Lagrangian for the matrix quantum mechanics that we will consider is

[2] In addition, we require that $v(x) \to Nv(x)$.

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{2}\dot{M}^2 - V\left(M\right)\right), \tag{2.27}$$

where the potential is given by

$$\mathrm{Tr}V\left(M\right) = \frac{\omega^2}{2}\mathrm{Tr}M^2 + g^2\mathrm{Tr}M^4. \tag{2.28}$$

As motivated earlier, we rescale the "field" M as $M \to \frac{M}{g}$. Then,

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{2g^2}\dot{M}^2 - \frac{\omega^2}{2g^2}\mathrm{Tr}M^2 + \frac{1}{g^2}\mathrm{Tr}M^4\right). \tag{2.29}$$

The corresponding Hamiltonian, in terms of the 't Hooft variables, is

$$H = -\frac{\lambda}{2N}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + N\left(\frac{\omega^2}{2\lambda}\mathrm{Tr}M^2 + \frac{1}{\lambda}\mathrm{Tr}M^4\right). \tag{2.30}$$

We take this Hamiltonian as our starting point. In particular, we are interested in the strong-coupling limit - this is the limit when the mass term tends to zero. In this limit the Hamiltonian takes the form

$$H = -\frac{\lambda}{2N}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + \frac{N}{\lambda}\mathrm{Tr}M^4. \tag{2.31}$$

For strong-coupling it would seem that the theory is free, and it should be possible to carry out perturbation theory. Accordingly, we need to find the eigenstates of the "free" Hamiltonian.

To study the eigenstates of the "free" Hamiltonian, we first "compactify" the single Hermitian matrix model on the circle S^1. More precisely, we write

$$U = e^{\frac{iM}{L}}, \tag{2.32}$$

where U is obviously a unitary matrix.

The eigenvalues of a unitary matrix can be written as $\lambda_i = e^{i\sigma_i}$ $(i = 1, \ldots, N)$, where $\sigma_i \in [-\pi, \pi]$ - hence the terminology that we are "compactifying" the Hermitian matrix on the circle. Moreover, it is straightforward to see that after "compactification" the Hamiltonian becomes[3]

$$
\begin{aligned}
H &= \frac{\lambda}{2NL^2} \mathrm{Tr} \left(U \frac{\partial}{\partial U} \right)^2 + \frac{NL^4}{\lambda} \sum_{i=1}^{N} \sigma_i^4 \\
&= \frac{\lambda}{2NL^2} \mathrm{Tr} \left(t^\alpha U \frac{\partial}{\partial U} \right) \mathrm{Tr} \left(t^\alpha U \frac{\partial}{\partial U} \right) + \frac{NL^4}{\lambda} \sum_{i=1}^{N} \sigma_i^4 \\
&= \frac{\lambda}{2NL^2} \mathrm{Tr} \left(t^\alpha U \frac{\partial}{\partial U} \right) \mathrm{Tr} \left(t^\alpha U \frac{\partial}{\partial U} \right) + \frac{NL^4}{\lambda} \int_{-\pi}^{\pi} d\sigma \phi\left(\sigma\right) \sigma^4. \qquad (2.33)
\end{aligned}
$$

Here, t^α are the generators of the $U\left(N\right)$ group and thus satisfy the following identities:

$$
t^\alpha_{ij} t^\alpha_{ab} = \frac{1}{2} \delta_{ib} \delta_{ja} \qquad (2.34)
$$

$$
\mathrm{Tr}\left(t^\alpha t^\beta\right) = \frac{1}{2} \delta_{\alpha\beta} \qquad (2.35)
$$

In the next subsection we discuss the first-order perturbation correction to the ground-state energy. This will be followed by an exact analysis of the ground-state energy using the Collective Field Theory approach.

2.2.1 The First-Order Correction

As noted earlier, in the strong-coupling limit the theory reduces to a free theory. Thus, we can write the Schrödinger equation as

[3] The first and second line are equivalent since: $\mathrm{Tr}\left(U \frac{\partial}{\partial U}\right)^2 = \sum_{i,j} \sum_{a,b} t^\alpha_{ij} t^\alpha_{ab} \left(U \frac{\partial}{\partial U}\right)_{ji} \left(U \frac{\partial}{\partial U}\right)_{ba}$

$$H\psi(U) = (H_0 + V)\,\psi(U) = E\psi(U)\,, \tag{2.36}$$

where the unperturbed ("free") Hamiltonian is given by

$$\begin{aligned} H_0 &= \frac{\lambda}{2NL^2}\mathrm{Tr}\left(t^\alpha U\frac{\partial}{\partial U}\right)\mathrm{Tr}\left(t^\alpha U\frac{\partial}{\partial U}\right) \\ &= \frac{\lambda}{2NL^2}E^\alpha E^\alpha. \end{aligned} \tag{2.37}$$

First, let us see what the unperturbed Hamiltonian yields when it acts on some of the first invariants[4] of the $U(N)$ group. It is trivial to see that

$$\frac{\lambda}{2NL^2}E^\alpha E^\alpha\left(\mathrm{Tr}U^0\right) = 0. \tag{2.38}$$

Next, we let the unperturbed Hamiltonian act on $\mathrm{Tr}U$ and we obtain

$$\begin{aligned} \frac{\lambda}{2NL^2}E^\alpha E^\alpha \mathrm{Tr}U &= \frac{\lambda}{2NL^2}\mathrm{Tr}\left(t^\alpha U\frac{\partial}{\partial U}\right)\sum_{i,j}\left(t^\alpha U\right)_{ij}\frac{\partial \mathrm{Tr}U}{\partial U_{ji}} \\ &= \frac{\lambda}{2NL^2}\mathrm{Tr}\left(t^\alpha U\frac{\partial}{\partial U}\right)\mathrm{Tr}\left(t^\alpha U\right) \\ &= \frac{\lambda}{2NL^2}\sum_{i,j}\left(t^\alpha U\right)_{ij}\frac{\partial \mathrm{Tr}\left(t^\alpha U\right)}{\partial U_{ji}} \\ &= \frac{\lambda}{4L^2}\mathrm{Tr}U. \end{aligned} \tag{2.39}$$

Similarly,

$$\frac{\lambda}{2NL^2}E^\alpha E^\alpha \mathrm{Tr}U^\dagger = \frac{\lambda}{4L^2}\mathrm{Tr}U^\dagger. \tag{2.40}$$

[4]The invariants of the group $U(N)$ are given by $W_n = \mathrm{Tr}U^n$.

To summarize,

$$
\begin{aligned}
\frac{\lambda}{2NL^2}E^{\alpha}E^{\alpha}\left(\mathrm{Tr}U^0\right) &= 0 \\
\frac{\lambda}{2NL^2}E^{\alpha}E^{\alpha}\left(\mathrm{Tr}U\right) &= \frac{\lambda}{4L^2}\mathrm{Tr}U \\
\frac{\lambda}{2NL^2}E^{\alpha}E^{\alpha}\left(\mathrm{Tr}U^{\dagger}\right) &= \frac{\lambda}{4L^2}\mathrm{Tr}U^{\dagger}
\end{aligned}
\tag{2.41}
$$

That is, the first invariants are eigenstates of the unperturbed Hamiltonian. However, this is not the best way to characterize the eigenstates of the unperturbed Hamiltonian. The most convenient way of writing down the eigenstates is in terms of the characters of the $U(N)$ group - this follows from the trivial observation that any eigenstate is necessarily invariant under the transformation: $U \to VUV^{\dagger}$, where $V \in U(N)$. Accordingly,

$$
\psi(U) = \sum_R a_R \chi_R(U), \tag{2.42}
$$

where R labels some particular representation of $U(N)$. More precisely, the irreducible representations of $U(N)$ can be labeled by N integers $\{n_1, \ldots, n_N\}$. Here, the integer n_k can be identified with the number of boxes in the k^{th} row of a Young tableau.

The first few characters are [63]

$$
\begin{aligned}
\chi_{\yng(1)} &= \mathrm{Tr}U \\
\chi_{\yng(2)} &= \frac{1}{2}\left[(\mathrm{Tr}U)^2 + \mathrm{Tr}U^2\right] \\
\chi_{\yng(1,1)} &= \frac{1}{2}\left[(\mathrm{Tr}U)^2 - \mathrm{Tr}U^2\right] \\
\chi_{\yng(3)} &= \frac{1}{6}\left[(\mathrm{Tr}U)^3 + 3\left(\mathrm{Tr}U\right)\left(\mathrm{Tr}U^2\right) + 2\mathrm{Tr}U^3\right] \\
\chi_{\yng(1,1,1)} &= \frac{1}{6}\left[(\mathrm{Tr}U)^3 - 3\left(\mathrm{Tr}U\right)\left(\mathrm{Tr}U^2\right) + 2\mathrm{Tr}U^3\right] \\
\chi_{\yng(2,1)} &= \frac{1}{2}\left[(\mathrm{Tr}U)^3 - \mathrm{Tr}U^3\right] \\
\chi_{\yng(4)} &= \frac{1}{24}\left[(\mathrm{Tr}U)^4 + 6\mathrm{Tr}U^4 + 3\left(\mathrm{Tr}U^2\right)^2 + 6\left(\mathrm{Tr}U^2\right)\left(\mathrm{Tr}U\right)^2 + 8\left(\mathrm{Tr}U^3\right)\left(\mathrm{Tr}U\right)\right] \\
\chi_{\yng(3,1)} &= \frac{1}{8}\left[(\mathrm{Tr}U)^4 + 6\mathrm{Tr}U^4 + 3\left(\mathrm{Tr}U^2\right)^2 + 6\left(\mathrm{Tr}U^2\right)\left(\mathrm{Tr}U\right)^2 + 8\left(\mathrm{Tr}U^3\right)\left(\mathrm{Tr}U\right)\right] \\
\chi_{\yng(2,2)} &= \frac{1}{12}\left[(\mathrm{Tr}U)^4 - 4\left(\mathrm{Tr}U^3\right)\left(\mathrm{Tr}U\right) + 3\left(\mathrm{Tr}U^2\right)^2\right] \\
\chi_{\yng(2,1,1)} &= \frac{1}{8}\left[(\mathrm{Tr}U)^4 + 2\mathrm{Tr}U^4 - \left(\mathrm{Tr}U^2\right)^2 - 2\left(\mathrm{Tr}U^2\right)\right] \\
\chi_{\yng(1,1,1,1)} &= \frac{1}{24}\left[(\mathrm{Tr}U)^4 - 6\mathrm{Tr}U^4 + 3\left(\mathrm{Tr}U^2\right)^2 - 6\left(\mathrm{Tr}U^2\right)\left(\mathrm{Tr}U\right)^2 + 8\left(\mathrm{Tr}U^3\right)\right]
\end{aligned}
$$

As is well-known, the characters are orthonormal *i.e.*

$$
\int DU \chi_R\left(U^\dagger\right)\chi_{R'}\left(U\right) = \delta_{RR'}. \tag{2.43}
$$

Here, DU is the Haar measure for the unitary group. Using the orthonormality of the characters, it follows that

$$
a_R = \int DU \psi\left(U\right)\chi_R\left(U^\dagger\right). \tag{2.44}
$$

We also have

$$E^\alpha E^\alpha U = t^\alpha t^\alpha U = C_2 U, \tag{2.45}$$

where C_2 is the second Casimir. Hence [61, 64],

$$H_0 \chi_R(U) = C_2^R \chi_R(U). \tag{2.46}$$

Indeed, it can be shown that

$$C_2^{\{n_1,\ldots,n_N\}} = \sum_{j=1}^{N} \frac{1}{2}\left[n_j^2 + n_j(N+1-2j)\right] \tag{2.47}$$

The potential can be written as

$$\begin{aligned} L^4 \mathrm{Tr}\chi^4 &= L^4 \sum_{i=1}^{N} \sigma_i^4 \\ &= L^4 \left[\frac{N\pi^4}{5} + \sum_{i=1}^{N}\sum_{n=1}^{\infty} 8(-1)^n \left(\frac{\pi^2}{n^2} - \frac{6}{n^4}\right)\cos(n\sigma_i)\right] \\ &= L^4 \left[\frac{N\pi^4}{5} + \sum_{i=1}^{N}\sum_{n=1}^{\infty} 4(-1)^n \left(\frac{\pi^2}{n^2} - \frac{6}{n^4}\right)\left\{\mathrm{Tr}U^n + \mathrm{Tr}U^{\dagger n}\right\}\right] \end{aligned} \tag{2.48}$$

Moreover, we can show that[5] [61, 64]

$$\mathrm{Tr}U^n = \begin{cases} \chi_{\{n\}}(U) + \sum_{i=1}^{n-1}(-1)^i \chi_{\{n-i,1^i\}} & \text{for } 1 \le n \le N \\ \chi_{\{n\}}(U) + \sum_{i=1}^{N-1}(-1)^i \chi_{\{n-i,1^i\}} & \text{for } n \ge N \end{cases} \tag{2.49}$$

[5] Here, $\chi_{\{n-i,1^i\}} \equiv \chi_{\left\{n-i,\underbrace{1,1,\ldots,1}_{i},0,\ldots,0\right\}}$.

Using (2.48), (2.49) and the orthonormality of the characters, we get

$$
\begin{aligned}
E_0^{(1)} &= \left\langle 0 \left| \frac{NL^4}{\lambda} \mathrm{Tr}\chi^4 \right| 0 \right\rangle , \\
&= \frac{N^2}{\lambda} \frac{L^4 \pi^4}{5} .
\end{aligned}
\tag{2.50}
$$

It is straightforward to obtain the dependence on L and λ for the second-order correction. One obtains[6]

$$
E_0^{(2)} \sim \left(\frac{L^4}{\lambda} \right)^2 \frac{1}{\left(\frac{\lambda}{L^2} \right)} \sim \frac{L^{10}}{\lambda^3} . \tag{2.51}
$$

The system can be solved exactly, and in the next subsection we discuss how one can use the Collective Field Theory approach to determine the exact ground-state energy.

2.2.2 The Exact Solution

For the unitary group, the invariants are given by

$$
W_n = \mathrm{Tr}\left(U^n \right) . \tag{2.52}
$$

In terms of the invariants we can write

[6]The second-order correction is $E_0^{(2)} = \sum_{R \neq \{0,0,\ldots\}} \frac{|\langle 0|V|R\rangle|^2}{-E_R^{(0)}}$.

$$
\begin{aligned}
H &= \frac{\lambda}{2NL^2}\mathrm{Tr}\left(t^\alpha U\frac{\partial}{\partial U}\right)\mathrm{Tr}\left(t^\alpha U\frac{\partial}{\partial U}\right)+\frac{NL^4}{\lambda}\int_{-\pi}^{\pi}d\sigma\phi\left(\sigma\right)\sigma^4\\
&= \frac{\lambda}{2NL^2}E^\alpha E^\alpha+\frac{NL^4}{\lambda}\int_{-\pi}^{\pi}d\sigma\phi\left(\sigma\right)\sigma^4\\
&= \frac{\lambda}{2NL^2}\left(\frac{1}{2}\Omega_{nn'}\frac{\partial}{\partial W_n}\frac{\partial}{\partial W_{n'}}+\frac{1}{2}\omega_n\frac{\partial}{\partial W_n}\right)+\frac{NL^4}{\lambda}\int_{-\pi}^{\pi}d\sigma\phi\left(\sigma\right)\sigma^4,
\end{aligned}
\tag{2.53}
$$

where [61, 62, 63]

$$
\Omega_{nn'} = 2E^\alpha W_n E^\alpha W_{n'} = nn'W_{n-n'} \tag{2.54}
$$

$$
\omega_n = 2E^\alpha E^\alpha W_n = N\left|n\right|W_n+\left|n\right|\sum_{n'=\epsilon(n)}^{n-\epsilon(n)}W_nW_{n-n'} \tag{2.55}
$$

The Collective Field Theory formalism leads to the following effective Hamiltonian [61]:

$$
\begin{aligned}
H_{eff} &= \frac{\lambda}{2NL^2}\left\{\frac{1}{2}\int d\sigma\partial_\sigma\Pi\phi\partial_\sigma\Pi+\frac{\pi^2}{6}\int d\sigma\phi^3\left(\sigma\right)-\frac{1}{24}\left(\int d\sigma\phi\left(\sigma\right)\right)^3\right\}\\
&\quad+\frac{NL^4}{\lambda}\int d\sigma\sigma^4\phi\left(\sigma\right)+\mu_l\left(N-\int d\sigma\phi\left(\sigma\right)\right).
\end{aligned}
$$

We rescale as follows:

$$
\begin{aligned}
\sigma &\to \sigma\\
\phi\left(\sigma\right) &\to N\phi\left(\sigma\right)\\
\mu_l &\to N\mu_l
\end{aligned}
\tag{2.56}
$$

Thus, the effective Hamiltonian is

$$H_{eff} = \frac{\lambda}{2L^2}\left\{\frac{1}{2N^2}\int d\sigma\partial_\sigma\Pi\phi\partial_\sigma\Pi + \frac{N^2\pi^2}{6}\int d\sigma\phi^3(\sigma) - \frac{N^2}{24}\left(\int d\sigma\phi(\sigma)\right)^3\right\}$$
$$+\frac{N^2L^4}{\lambda}\int d\sigma\sigma^4\phi(\sigma) + N^2\mu_l\left(1-\int d\sigma\phi(\sigma)\right).$$

The background density, which minimizes the effective potential, is

$$\phi_0(\sigma) = \frac{2L}{\pi\sqrt{\lambda}}\left(\mu - \frac{L^4\sigma^4}{\lambda}\right)^{1/2}, \tag{2.57}$$

where

$$\mu = \mu_l + \frac{\lambda}{16L^2}. \tag{2.58}$$

In addition, the ground-state energy is

$$\begin{aligned}
E_0 &= V_{eff}(\phi_0) \\
&= \left[\frac{\lambda N^2\pi^2}{12L^2}\int d\sigma\phi_0^3(\sigma) - \frac{N^2\lambda}{48L^2}\left(\int d\sigma\phi_o(\sigma)\right)^3 + \frac{N^2L^4}{\lambda}\int d\sigma\sigma^4\phi(\sigma)\right] \\
&= N^2\left[\mu - \frac{\lambda\pi^2}{6L^2}\int d\sigma\phi_0^3(\sigma) - \frac{\lambda}{48L^2}\right]
\end{aligned}$$

The turning-points of the density in (2.57) are given by $\sigma_\pm = (\mu\lambda)^{1/4}\frac{1}{L}$. However, $|\sigma_\pm| \leq \pi$. Therefore, this system has two phases separated by a third-order phase transition [65].

For $|\sigma_\pm| < \pi$ - this corresponds to the weakly-coupled regime - the normalization condition for the density yields

$$
\begin{aligned}
1 &= \int_{\sigma_-}^{\sigma_+} \frac{2L}{\pi\sqrt{\lambda}} \left(\mu - \frac{L^4\sigma^4}{\lambda}\right)^{1/2} d\sigma \\
&= \frac{2\mu^{3/4}}{\pi\lambda^{1/4}} \left[\int_{-1}^{1} \left(1-u^4\right)^{1/2} du\right].
\end{aligned} \tag{2.59}
$$

Thus, the chemical potential is

$$
\mu = \left(\frac{\pi}{2\omega_0}\right)^{4/3} \lambda^{1/3}, \tag{2.60}
$$

where

$$
\omega_0 = \int_{-1}^{1} \left(1-u^4\right)^{1/2} du. \tag{2.61}
$$

The ground-state energy is[7]

$$
\begin{aligned}
E_0 &= N^2 \left[\mu - \frac{1}{6L^2}\lambda\pi \int_{\sigma_-}^{\sigma_+} d\sigma \phi_0^3\left(\sigma\right) - \frac{\lambda}{48NL^2}\right] \\
&= N^2 \left[\frac{3}{7}\mu - \frac{L^2\lambda}{48}\right] \\
&= N^2 \left[\frac{3}{7}\left(\frac{\pi}{2\omega_0}\right)^{4/3} \lambda^{1/3} - \frac{\lambda}{48L^2}\right].
\end{aligned} \tag{2.62}
$$

For large λ *i.e.* $\left(\mu\lambda\right)^{1/4}\frac{1}{L} > \pi$, the turning-points are $|\sigma_\pm| = \pi$. Therefore,

[7] Using elementary integration by parts, one obtains:

$$
\int_{\sigma_-}^{\sigma_+} d\sigma \phi_0^3\left(\sigma\right) = \left(\frac{2L}{\pi\sqrt{\lambda}}\right)^3 \left[\left(\mu - \frac{L^4\sigma^4}{\lambda}\right)^{3/2} \sigma \Bigg|_{\sigma_-}^{\sigma_+} + 6\int_{\sigma_-}^{\sigma_+} d\sigma \left(\mu - \frac{L^4\sigma^4}{\lambda}\right)^{1/2} \frac{L^4\sigma^4}{\lambda}\right].
$$

After some trivial manipulations, we obtain the result appearing in the second line of (2.62).

$$
\begin{aligned}
1 &= \int_{-\pi}^{\pi} d\sigma \phi_0 (\sigma) \\
&= \frac{4L}{\pi \lambda^{1/2}} \left[\int_0^{\pi} d\sigma \left(\mu - \frac{L^4 \sigma^4}{\lambda} \right)^{1/2} \right] \\
&= \frac{4L\mu^{1/2}}{\lambda^{1/2}} \left[1 - \frac{1}{10\mu} \left(\frac{L^4 \pi^4}{\lambda} \right) - \frac{1}{72} \frac{L^8 \pi^8}{\mu^2 \lambda^2} + \cdots \right].
\end{aligned} \tag{2.63}
$$

Thus, the chemical potential satisfies the following equation:

$$
\frac{\lambda^{1/2}}{4L} = \mu^{1/2} - \frac{1}{10\mu^{1/2}} \left(\frac{L^4 \pi^4}{\lambda} \right) - \frac{1}{72} \frac{L^8 \pi^8}{\mu^{3/2} \lambda^2} + \cdots \tag{2.64}
$$

This implies that

$$
\mu = \frac{\lambda}{16L^2} + \frac{1}{5} \frac{L^4 \pi^4}{\lambda} + \frac{64}{225} \frac{L^{10} \pi^8}{\lambda^3} + \cdots . \tag{2.65}
$$

For large λ, the ground-state energy is

$$
\begin{aligned}
E_0 &= N^2 \left[\mu - \frac{1}{6L^2} \pi^2 \lambda \int_{-\pi}^{\pi} d\sigma \phi_0^3 (\sigma) - \frac{\lambda}{48L^2} \right] \\
&= N^2 \left[\mu - \frac{8}{3} \frac{L}{\pi \lambda^{1/2}} \int_0^{\pi} d\sigma \left(\mu - \frac{L^4 \sigma^4}{\lambda} \right)^{3/2} - \frac{\lambda}{48L^2} \right].
\end{aligned} \tag{2.66}
$$

Using (2.65), we obtain

$$
\begin{aligned}
E_0 &= N^2\left[\mu - \frac{8}{3}\frac{L\mu^{3/2}}{\lambda^{1/2}}\left(1 - \frac{3}{25}\frac{L^4\pi^4}{\mu\lambda} + \frac{1}{24}\frac{L^8\pi^8}{(\mu\lambda)^2}\right) - \frac{\lambda}{48L^2}\right] \\
&= N^2\left[\frac{\lambda}{48L^2} + \frac{1}{5}\frac{L^4\pi^4}{\lambda} - \frac{64}{225}\frac{L^{10}\pi^{10}}{\lambda^3} + \mathcal{O}\left(\frac{1}{\lambda^5}\right) - \frac{\lambda}{48L^2}\right] \\
&= \frac{1}{5}\frac{N^2L^4\pi^4}{\lambda} - \frac{64}{225}\frac{L^{10}\pi^{10}}{\lambda^3} + \cdots .
\end{aligned}
\tag{2.67}
$$

This is the same result that we obtained earlier using first-order perturbation theory and confirms the second-order perturbation theory estimate.

However, our initial aim was to investigate if the strong-coupling limit of (2.31) was given by a free theory. The system described by (2.31) has only one parameter λ, and by dimensional analysis

$$E_0 = a\lambda^{1/3} \tag{2.68}$$

where a is some arbitrary constant.

Comparing E_0 - as given in (2.68) - with (2.67), or by dimensional analysis, it follows that

$$L \sim \lambda^{1/3}. \tag{2.69}$$

So, even though the expansion (2.67) is a strong-coupling expansion, we find that each case is of order $\lambda^{1/3}$.

It may be possible to consider cases where the system that we have been considering allows L to be dependent on other parameters (such as the radius of compactification or even l_s), in which case the strong-coupling expansion (2.67) is of physical relevance. However, this is beyond the scope of this dissertation.

2.3 Summary

In the context of a single matrix (Hamiltonian) model, we studied,, in this chapter, the effect of the rescaling that brings the potential to the form as given in (1.80). The strong-coupling limit of this Hamiltonian would seem to suggest that the theory is free.

However, in order to obtain the free Hamiltonian, a parameter L has to be introduced. The strong-coupling expansion was then obtained, both using perturbation theory and the exact solution of the theory, using Collective Field theory techniques.

In order to use these results to understand the original Hermitian matrix model, which only depends on a single dimensionful parameter λ at strong-coupling, one is required to identify $L \sim \lambda^{1/3}$. But, as a result, each term in the perturbative expansion contributes to the same order in λ. Therefore, the expectation that the strong-coupling limit of the Hamiltonian considered in this section (*i.e.* (2.31)) is free, turns out to be a naive one.

Chapter 3

The Two-Matrix Model

In this chapter we review the large-N limit of the integral of two matrices coupled via a Yang-Mills interaction. This two-matrix model can be associated with two of the six Higgs scalars of $\mathcal{N} = 4$ SYM. Nevertheless, this particular two-matrix model was introduced and solved by Hoppe [66] years before the AdS/CFT correspondence was put forward. The matrix model reappeared again in a different context in [67]. (Naively, at strong-coupling this particular matrix model can be interpreted as the bosonic part of the IKKT matrix model.) This matrix model also occurs in the theory of knots [68] and a similar matrix model was used to test the Dijkgraaf-Vafa correspondence[1] [69].

In this chapter we review how the matrix model is solved using a standard method whereby we diagonalize on of the matrices and integrate out the second matrix. We then also review a simpler approach due to [71] in the context of commuting matrices, from which a radial density of eigenvalues can be obtained.

[1]The superpotential in [69] is of the form $W(\Phi) = \Phi\left[\Phi^{+}, \Phi^{-}\right] + \omega\Phi^{+}\Phi^{-} + \sum_{p=2}^{N} g_p \Phi^p$. For $g_p = 0$, $p \geq 3$, we can integrate out Φ and obtain a matrix model similar to the one we will consider in this chapter.

3.1 Saddle-Point Method

The action for the two-matrix model is

$$
\begin{aligned}
S &= \frac{\omega^2}{2g_{YM}^2}\mathrm{Tr}\left(X_1^2+X_2^2\right)-\frac{1}{g_{YM}^2}\mathrm{Tr}\left[X_1,\,X_2\right]^2\\
&= \frac{N\omega^2}{2\lambda}\mathrm{Tr}\left(X_1^2+X_2^2\right)-\frac{N}{\lambda}\mathrm{Tr}\left[X_1,\,X_2\right]^2 . \qquad (3.1)
\end{aligned}
$$

It is convenient to rescale the Hermitian matrices as follows: $X_1 \to \lambda^{1/4}X_1$, $X_2 \to \lambda^{1/4}X_2$. Thus,

$$
S = \frac{N\omega^2}{2\lambda^{1/2}}\mathrm{Tr}\left(X_1^2+X_2^2\right)-N\mathrm{Tr}\left[X_1,\,X_2\right]^2 . \qquad (3.2)
$$

Moreover, the above matrix model has an overall U(N) symmetry, and thus we can diagonalize one of the matrices *i.e.* we can write $X_1 = V\Lambda V^\dagger$, where $\Lambda = diag\left(\lambda_1,\ldots,\lambda_N\right)$. Accordingly, the action can be written as

$$
\begin{aligned}
S &= \frac{N\omega^2}{2\lambda^{1/2}}\mathrm{Tr}\left(X_1^2+X_2^2\right)-N\mathrm{Tr}\left[X_1,\,X_2\right]^2\\
&= \frac{N\omega^2}{2\lambda^{1/2}}\sum_{i=1}^{N}\lambda_i^2+\frac{N\omega^2}{2\lambda^{1/2}}\sum_{i,j}\left(X_2\right)_{ij}\left(X_2\right)_{ji}-2N\mathrm{Tr}\left(X_1X_2X_1X_2-X_1^2X_2^2\right)\\
&= \frac{N\omega^2}{2\lambda^{1/2}}\sum_{i=1}^{N}\lambda_i^2+\frac{N\omega^2}{2\lambda^{1/2}}\sum_{i,j}\left(X_2\right)_{ij}\left(X_2\right)_{ji}+N\sum_{i,j}\left(\lambda_i-\lambda_j\right)^2\left(X_2\right)_{ij}\left(X_2\right)_{ji}\\
&= \frac{N\omega^2}{2\lambda^{1/2}}\sum_{i=1}^{N}\lambda_i^2+\sum_{i,j}\left(\frac{N\omega^2}{2\lambda^{1/2}}+N\left(\lambda_i-\lambda_j\right)^2\right)\left(X_2^2\right)_{ij} \qquad (3.3)
\end{aligned}
$$

This action is quadratic in $\left(X_2\right)_{ij}$, and hence

$$
\begin{aligned}
Z &= \int [dX_1] \int [dX_2]\, e^{-S} \\
&= \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} \left(\lambda_i - \lambda_j\right)^2 e^{-\frac{N\omega^2}{2\lambda^{1/2}} \sum_{i=1}^{N} \lambda_i^2} \int [dX_2]\, e^{-\sum_{i,j}\left(\frac{N\omega^2}{2\lambda^{1/2}} + N(\lambda_i - \lambda_j)^2\right)\left(X_2^2\right)_{ij}} \\
&= C \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} \frac{\left(\lambda_i - \lambda_j\right)^2}{\frac{N\omega^2}{2\lambda^{1/2}} + N\left(\lambda_i - \lambda_j\right)^2} e^{-\frac{N\omega^2}{2\lambda^{1/2}} \sum_{i=1}^{N} \lambda_i^2} \\
&= C \int \prod_{i=1}^{N} d\lambda_i e^{-S_{eff}},
\end{aligned}
\tag{3.4}
$$

where

$$
S_{eff} = \frac{N\omega^2}{2\lambda^{1/2}} \sum_{i=1}^{N} \lambda_i^2 + \frac{1}{2} \sum_{i \neq j} \ln \frac{\left(\frac{N\omega^2}{2\lambda^{1/2}} + N\left(\lambda_i - \lambda_j\right)^2\right)}{\left(\lambda_i - \lambda_j\right)^2}. \tag{3.5}
$$

The saddle-point equations easily follow from varying the effective action w.r.t. one of the eigenvalues. Indeed, it is simple to show that

$$
\frac{N\omega^2}{\lambda^{1/2}} \lambda_i = \sum_{i \neq j} \frac{\frac{\omega^2}{\lambda^{1/2}}}{\left(\frac{\omega^2}{2\lambda^{1/2}} + \left(\lambda_i - \lambda_j\right)^2\right)\left(\lambda_i - \lambda_j\right)}, \tag{3.6}
$$

or after trivial simplifications

$$
N\lambda_i = \sum_{i \neq j} \frac{1}{\left(\frac{\omega^2}{2\lambda^{1/2}} + \left(\lambda_i - \lambda_j\right)^2\right)\left(\lambda_i - \lambda_j\right)}. \tag{3.7}
$$

As is customary, we can introduce the eigenvalue density:

$$
\phi\left(x\right) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \lambda_i\right) \tag{3.8}
$$

and the effective action - up to constant terms - can be written as

$$S_{eff} = \frac{N^2\omega^2}{2\lambda^{1/2}} \int dx \phi(x) x^2 + \frac{N^2}{2} \int dx \phi(x) ⨍ dy \phi(y) \ln \frac{\left(\frac{\omega^2}{2\lambda^{1/2}} + (x-y)^2\right)}{(x-y)^2}. \tag{3.9}$$

The saddle-point equations - in terms of the eigenvalue density - are

$$\begin{aligned} x &= ⨍ dy \frac{\phi(y)}{\left(\frac{\omega^2}{2\lambda^{1/2}} + (x-y)^2\right)(x-y)} \\ &= \frac{\lambda^{1/2}}{\omega^2}\left(2 ⨍ \frac{dy\phi(y)}{x-y} - ⨍ \frac{dy\phi(y)}{x-y-\frac{i\omega}{\sqrt{2}\lambda^{1/4}}} - ⨍ \frac{dy\phi(y)}{x-y+\frac{i\omega}{\sqrt{2}\lambda^{1/4}}}\right). \end{aligned} \tag{3.10}$$

It is straightforward to see that we can express the above saddle-point equations in terms of the generating function - which we denote by $W(z)$.[2]

Therefore,

$$\frac{\omega^2 x}{\lambda^{1/2}} = W(x+i\epsilon) + W(x-i\epsilon) - W\left(x - \frac{i\omega}{\sqrt{2}\lambda^{1/4}}\right) - W\left(x + \frac{i\omega}{\sqrt{2}\lambda^{1/4}}\right). \tag{3.11}$$

The saddle-point equations can be drastically simplified if we define the function [66, 67]:

$$G(z) = \zeta = \frac{\omega}{\sqrt{2}\lambda^{1/4}} z^2 + i\left(W\left(z + \frac{i\omega}{2\sqrt{2}\lambda^{1/4}}\right) - W\left(z - \frac{i\omega}{2\sqrt{2}\lambda^{1/4}}\right)\right). \tag{3.12}$$

In terms of the function $G(z)$ the saddle-point equations take the form

[2] Recall that the generating function is defined as

$$W(z) = \int_{-a}^{a} \frac{dz'\phi(z')}{z-z'}.$$

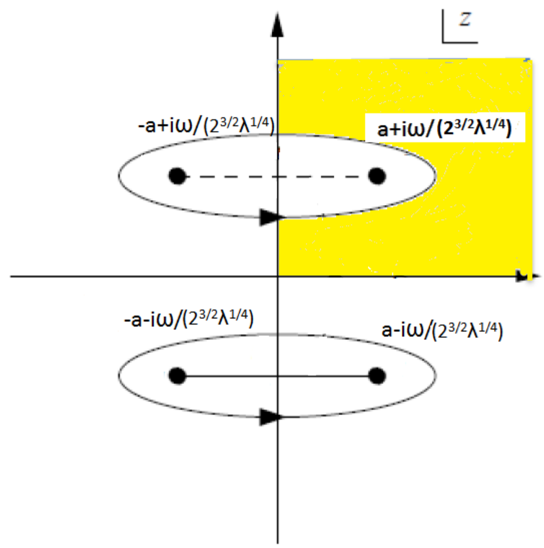

Figure 3.1: The cut-structure of the function $G(z)$; the "coloured" region is mapped to the positive imaginary ζ-plane.

$$G\left(x-\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}\right)=G\left(x+\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}\right). \tag{3.13}$$

A few remarks are in order. Firstly, the function $G(z)$ - except for the two cuts at $\left(-a-\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}, a-\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}\right)$ and $\left(-a+\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}, a+\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}\right)$ - is analytic on the whole complex plane - see Figure (3.1). In addition, the function $G(z)$ maps the "coloured" region in Figure (3.1) to the positive imaginary ζ-plane. Accordingly, by the Schwarz-Christoffel mapping theorem, we have

$$z=f(\zeta)=A\int_{x_1}^{\zeta}\frac{dt\,(t-x_3)}{\sqrt{(t-x_1)(t-x_2)(t-x_4)}} \tag{3.14}$$

The vertices of the polygon in the z-plane are mapped to the corresponding points on the

z	ζ
∞	∞
0	x_1
$\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}$	x_2
$a+\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}$	x_3
$\frac{i\omega}{2\sqrt{2}\lambda^{1/4}}$	x_4
$i\infty$	$-\infty$

Table 3.1: The vertices of the polygon in the z-plane and the corresponding points in the real axis in the ζ-plane.

real ζ-plane; we choose to call the corresponding points on the real ζ-plane $x_1, \ldots, x_4$, see Table (3.1).

From the definition of the function $G(z)$ (*i.e.* (3.12)), it is not difficult to see that for large $|z|$ one has

$$G(z) = \zeta = \frac{\omega}{\sqrt{2}\lambda^{1/4}} z^2 + \frac{\omega}{\sqrt{2}\lambda^{1/4} z^2} + \mathcal{O}\left(\frac{1}{z^4}\right). \tag{3.15}$$

Thus,

$$\begin{aligned} z &= \left(\frac{\sqrt{2}}{\omega}\lambda^{1/4}\zeta - \frac{\omega}{\sqrt{2}\lambda^{1/4}\zeta}\right)^{1/2} \\ &= \left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/2} \zeta^{1/2} - \frac{1}{2\zeta^{3/2}}\left(\frac{\omega}{\sqrt{2}\lambda^{1/4}}\right)^{3/2}. \end{aligned} \tag{3.16}$$

However, $G(z)$ is also given by (3.14) and it is trivial to show that

$$z = 2A\left(\zeta^{1/2} + a_0 + a_1\zeta^{-1/2} + a_2\zeta^{-3/2}\right). \tag{3.17}$$

Comparing the two expressions in (3.16) and (3.17), we can conclude that

$$2A = \left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/2} \tag{3.18}$$

$$a_0 = 0 \tag{3.19}$$

$$x_1 + x_2 + x_4 = 2x_3 \tag{3.20}$$

$$x_1^2 + x_2^2 + x_4^2 - 2x_3^2 = 6\frac{\omega}{\sqrt{2}\lambda^{1/4}} \tag{3.21}$$

Note also that

$$\frac{i\omega}{2\sqrt{2}\lambda^{1/4}} = \int_{x_1}^{x_2} \frac{dt\,(t-x_3)}{\sqrt{(t-x_1)\,(t-x_2)\,(t-x_4)}} = \int_{x_1}^{x_4} \frac{dt\,(t-x_3)}{\sqrt{(t-x_1)\,(t-x_2)\,(t-x_4)}}. \tag{3.22}$$

Therefore,

$$0 = \int_{x_2}^{x_4} \frac{dt\,(t-x_3)}{\sqrt{(t-x_1)\,(t-x_2)\,(t-x_4)}}. \tag{3.23}$$

Performing the above integral leads to the condition:[3]

$$\begin{aligned} 0 &= (2x_3 - 2x_2 + 2x_2 - 2x_1)\,\mathbf{K}\left(\frac{x_2 - x_4}{x_2 - x_1}\right) + 2\,(x_2 - x_1)\,\mathbf{E}\left(\frac{x_2 - x_4}{x_2 - x_1}\right) \\ &= (x_2 + x_4 - x_1)\,\mathbf{K}\,(m) + 2\,(x_1 - x_4)\,\mathbf{E}\,(m)\,, \end{aligned} \tag{3.24}$$

where $\mathbf{K}\,(m)$ ($\mathbf{E}\,(m)$) denotes the complete elliptic integral of the first (second) kind. Also,

[3] Alternatively, the integral can be obtained from [70].

$$m = \frac{x_2 - x_4}{x_4 - x_1}. \tag{3.25}$$

The second condition is

$$\frac{1}{2} = A \int_{x_2}^{x_1} \frac{dt\,(t - x_3)}{\sqrt{(t - x_1)(t - x_2)(t - x_4)}}. \tag{3.26}$$

This condition can be expressed as

$$-\frac{(x_1 + x_2 - x_4)}{\sqrt{x_4 - x_1}} \mathbf{K}(m - 1) + 2\sqrt{x_4 - x_1} \mathbf{E}(m - 1) = \sqrt{\frac{1}{\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/2}}}. \tag{3.27}$$

From (3.24) it follows that

$$\frac{x_2}{x_1 - x_4} \equiv \lambda_2 = 1 - 2\frac{\mathbf{E}(m)}{\mathbf{K}(m)}. \tag{3.28}$$

We can make use of (3.28) to further simplify the second condition:

$$\begin{aligned}
\sqrt{\frac{1}{\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/2} (x_4 - x_1)^{1/2}}} &= -\left(2 - 2\frac{\mathbf{E}(m)}{\mathbf{K}(m)}\right) \mathbf{K}(m - 1) + 2\mathbf{E}(m - 1) \\
&= \frac{2}{\mathbf{K}(m)} \left(\mathbf{E}(m)\mathbf{K}(m - 1) + \mathbf{K}(m)\mathbf{E}(m - 1) - \mathbf{K}(m)\mathbf{K}(m - 1)\right) \\
&= \frac{\pi}{\mathbf{K}(m)}.
\end{aligned} \tag{3.29}$$

where we have made use of the identity:

$$\mathbf{E}(m)\mathbf{K}(m - 1) + \mathbf{K}(m)\mathbf{E}(m - 1) - \mathbf{K}(m)\mathbf{K}(m - 1) = \frac{\pi}{2}. \tag{3.30}$$

After some manipulations we end up with the results:

$$\frac{\sqrt{2}\lambda^{1/4}}{\omega} = \frac{\mathbf{K}^4}{12\pi^4}\left(-3\lambda_2^2 - 2\lambda_2 + 4m\lambda_2 + 1\right). \tag{3.31}$$

$$\omega_2 = \left\langle \mathrm{Tr}X_1^2 \right\rangle = \frac{1}{12} + \frac{\mathbf{K}^4}{5\pi^2}\left(\frac{4m\lambda_2\left(1-m\right) + \left(5\lambda_2^2 - 1\right)\left(2m - 1 - \lambda_2\right)}{4m\lambda_2 + 1 - 3\lambda_2^2 - 2\lambda_2}\right) \tag{3.32}$$

At strong-coupling (this corresponds to taking $m \to 1$) we find [67]:

$$\omega_2 = \left\langle \mathrm{Tr}X_1^2 \right\rangle = \frac{\left(12\pi\right)^{2/3}}{20}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{2/3} - \frac{3}{\left(12\pi\right)^{2/3}}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/3} + \cdots \tag{3.33}$$

$$F = -N^2\left(\frac{3\left(12\pi\right)^{2/3}}{40}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{-2/3} - \frac{9}{5\left(12\pi\right)^{2/3}}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{-5/6} + \cdots\right) \tag{3.34}$$

while for the weakly coupled regime ($m \to 0$) [67]:

$$\omega_2 = \left\langle \mathrm{Tr}X_1^2 \right\rangle = \frac{1}{2}\frac{\sqrt{2}\lambda^{1/4}}{\omega} - \frac{1}{2}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^2 + \cdots \tag{3.35}$$

$$F = N^2\left(\frac{1}{2}\ln\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right) - \frac{1}{2}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right) + \cdots\right). \tag{3.36}$$

Here, X_1 has been rescaled - see the comment appearing below (3.1).

3.2 Commuting Matrices

In[71], Berenstein *et al.* suggested a much simpler way of arriving at the strong-coupling results that we found in the previous section - *i.e.* (3.33) and (3.34) - as well as the eigenvalue density at strong-coupling. It is difficult to find the density by using the approach that we considered in the previous section. For commuting matrices, a radial density corresponding to a hemisphere distribution has been arrived at. Recently, the same analysis has been carried out for all couplings [72].

Let us begin by re-looking at the saddle-point equations at strong-coupling:

$$
\begin{aligned}
x &= \fint dy \frac{\phi\left(y\right)}{\left(\frac{\omega^2}{2\lambda^{1/2}} + \left(x-y\right)^2\right)\left(x-y\right)} \\
&= \frac{\sqrt{2}\lambda^{1/4}\pi}{\omega} \fint dy \phi\left(y\right) \delta'\left(x-y\right) \\
&= -\frac{\sqrt{2}\lambda^{1/4}\pi}{\omega} \phi'\left(x\right),
\end{aligned}
\tag{3.37}
$$

where

$$
\delta\left(x-y\right) = \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{\left(x-y\right)^2 + \epsilon^2}
\tag{3.38}
$$

The eigenvalue density is given by

$$
\phi\left(x\right) = \frac{\omega}{\sqrt{2}\lambda^{1/4}} \frac{1}{2\pi} \left(R^2 - x^2\right)
\tag{3.39}
$$

where

$$
R = \left(\frac{3\pi}{2} \frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/3},
\tag{3.40}
$$

Once we have determined the eigenvalue density it is not difficult to show that

$$
\begin{aligned}
\left\langle \mathrm{Tr} X_1^2 \right\rangle &= \frac{R^2}{5} \\
&= \frac{1}{5}\left(\frac{3\pi}{2}\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{1/3} \\
&= \frac{(12\pi)^{2/3}}{20}\left(\frac{\sqrt{2}\lambda^{1/4}}{\omega}\right)^{2/3}.
\end{aligned}
\tag{3.41}
$$

This is in perfect agreement with the result from the previous section - namely (3.33).

In an approach where the two matrices X_1 and X_2 commute, Berenstein also observed that the spectral density that we have derived can be obtained from the hemisphere distribution:

$$
\phi(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \phi_h(x,\, y)\, dy, \tag{3.42}
$$

where

$$
\phi_h(x,y) \sim \begin{cases} \frac{\omega}{\sqrt{2}\lambda^{1/4}\pi^2}\sqrt{R^2-x^2-y^2} & x^2+y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases} \tag{3.43}
$$

3.3 Summary

The aim of this chapter was to give a review of the solution to the two matrix integral with a Yang-Mills interaction. The effective action was obtained by diagonalizing one of the matrices and integrating out the remaining matrix. In this approach, the saddle-point equations could be written down succinctly in terms of the a single function $G(z)$. Using the analytic properties of this function, we found that the function $G(z)$ maps the interior of a polygon in the z-plane to the positive imaginary ζ-plane. Hence, it is possible to make use of the Schwarz-Christoffel mapping theorem to determine the inverse function of $G(z)$. We were able to obtain both the strong and weak coupling expansions for the free energy and the moment ω_2. Moreover, we found that the strong-coupling expansion depends on a mass parameter ω i.e. a strong-coupling expansion dependent only on λ is not possible.

In addition, we briefly reviewed how to obtain the strong-coupling result using the approach by Berenstein *et al* [71]. Here, we discussed how, for commuting matrices, a radial (hemispherical) density of eigenvalues emerges.

Chapter 4

Higgs Sector

In this chapter we generalize the two-matrix model that we looked at in the previous chapter. More precisely, we look at the multi-matrix model (specifically we study an ensemble with $2m$ Hermitian matrices *i.e.* m complex matrices) with the matrices coupled via a Yang-Mills coupling. This multi-matrix model arises naturally when we fully compactify the Higgs sector of $\mathcal{N} = 4$ SYM on S^4 (or $T \times S^3$). This system contains a subsector that can be identified with a matrix valued radial coordinate. We begin by determining the Jacobian that results when we change to the eigenvalues of this, positive definite, radial matrix coordinate. This will allow us to write the partition function in terms of an effective action. We solve the resulting saddle-point equations in this radial subsector and determine both the end-points and the radial eigenvalue distribution. These results are new [73]. As explicit matrix valued curvilinear coordinate parametrizations are currently not available, except for two Hermitian matrices [74, 75], we then specialize to the case of two Hermitian matrices. We use the perturbative expansion - we consider only the first-order correction - of the logarithm of the integral over the unitary group. An attempt at the next order in perturbation indicates that an exact solution is required. Although this can be written in a closed form, it is not useful to determine the radial density of eigenvalues. We tackle this problem in the following chapter.

4.1 The Jacobian

Full compactification of the Higgs sector of $\mathcal{N} = 4$ SYM leads to the us to consider the following matrix model ensemble

$$Z = \int [dX_I] \, e^{-S}, \tag{4.1}$$

where the action is given by

$$S = \frac{\omega^2}{2g_{YM}^2} \sum_{I=1}^{6} \mathrm{Tr} X_I^2 - \frac{1}{g_{YM}^2} \sum_{I,J} \mathrm{Tr} \left[X_I, \, X_J\right]^2. \tag{4.2}$$

Let us generalize this to an ensemble of $2m$ Hermitian matrices *i.e.* let us consider the matrices X_I $(I = 1, \ldots, 2m)$. Then, it is possible to pair the Hermitian matrices as follows:

$$Z_1 = X_1 + iX_2, \; Z_2 = X_3 + iX_4, \, etc. \tag{4.3}$$

In terms of the complex matrices, we can write the partition as

$$Z = \int \prod_A \prod_{ij} dZ_{A\,ij} dZ^\dagger_{A\,ij} e^{-S}. \tag{4.4}$$

We will be interested in a generic action that will only depend on a positive definite Hermitian matrix

$$\sum_A Z_A Z_A^\dagger. \tag{4.5}$$

The eigenvalues of this matrix will be denoted by $\rho_i = r_i^2$.

Note that from dimensional analysis,[1]

$$[r_i] = \left[g_{YM}^2\right]^{1/4} = L. \tag{4.6}$$

Our objective in this section is to determine the Jacobian that results from changing to the radial eigenvalues that we have just introduced. More precisely, we have

$$\int \prod_A \prod_{ij} dZ_{A_{ij}} dZ_{A_{ij}} = \int \prod_i d\rho_i \mathcal{J}\left(\rho_i\right) d\left[Angular\right]. \tag{4.7}$$

An advantage of the Collective Field Theory formalism is that it allows us to determine the Jacobian that results from performing a generic change of variables. Indeed, in "ρ-space" the Jacobian satisfies the equation:

$$\int d\rho' \Omega_{\rho\rho'} \frac{\partial \ln J\left(\Phi\right)}{\partial \Phi\left(\rho'\right)} + \int d\rho' \frac{\partial \Omega_{\rho\rho'}}{\partial \Phi\left(\rho'\right)} = -\omega_\rho. \tag{4.8}$$

For the matrix ensemble that we are interested in the invariants are

$$\Phi_k = \mathrm{Tr}\left(e^{ikZ_B Z_B^\dagger}\right), \tag{4.9}$$

and the Fourier transform is

$$\begin{aligned}\int \frac{dk}{2\pi} e^{-ik\rho} \mathrm{Tr}\left(e^{ikZZ^\dagger}\right) &= \sum_i \int \frac{dk}{2\pi} e^{-ik\rho} e^{ikr_i^2} \\ &= \sum_i \delta\left(\rho - r_i^2\right) = \Phi\left(\rho\right). \end{aligned} \tag{4.10}$$

It is straightforward to obtain the "joining" operator, and the result is

[1] We extrapolate the $d = 0$ result from the general scalar dimension $[\phi] = \frac{d-2}{2}$.

$$\Omega_{\rho\rho'} = \partial_\rho \partial_{\rho'} \left(\Phi\left(\rho\right) \delta\left(\rho - \rho'\right)\right). \tag{4.11}$$

For the "splitting" operator, one obtains [75]:

$$\begin{aligned}
\omega_k &= \sum_{i,j} \frac{\partial^2 \Phi_k}{\partial Z_{A\,ij} \partial Z^\dagger_{A\,ji}} \\
&= \sum_{i,j} \frac{\partial}{\partial Z_{A\,ij}} \left(ikZ_A e^{ikZ_B Z_B^\dagger} \right)_{ji} \\
&= ikmN\Phi_k - k \int_0^k dk' \Phi_k \mathrm{Tr} \left(ZZ^\dagger e^{i(k-k')ZZ^\dagger} \right).
\end{aligned} \tag{4.12}$$

Taking the Fourier transform of (4.12), we obtain [75]

$$\omega_\rho = -\partial_\rho \left[\rho \Phi\left(\rho\right) \left(2 \fint \frac{d\rho' \Phi\left(\rho'\right)}{\rho - \rho'} + \frac{N\left(m-1\right)}{\rho} \right) \right]. \tag{4.13}$$

Moreover, we note that

$$\int d\rho' \frac{\partial \Omega_{\rho\rho'}}{\partial \Phi\left(\rho'\right)} = 0. \tag{4.14}$$

Thus, equation (4.8) yields

$$\partial_\rho \frac{\partial}{\partial \Phi\left(\rho\right)} \ln \mathcal{J} = 2 \fint \frac{d\rho' \Phi\left(\rho'\right)}{\rho - \rho'} + \frac{N\left(m-1\right)}{\rho}. \tag{4.15}$$

This is easily solved, and we obtain [75]

$$\ln \mathcal{J} = \int d\rho \fint d\rho' \Phi\left(\rho\right) \Phi\left(\rho'\right) \ln\left|\rho - \rho'\right| + N\left(m-1\right) \int d\rho \Phi\left(\rho\right) \ln \rho, \tag{4.16}$$

or

$$\mathcal{J}\left(\rho_i\right) = C_m \prod_i \rho_i^{m-1} \prod_{i<j} \rho_i^{m-1} \rho_j^{m-1} \left(\rho_i - \rho_j\right)^2 .$$

$$= D_m \prod_i r_i^{2m-2} \prod_{i<j} r_i^{2m-2} r_j^{2m-2} \left(r_i^2 - r_j^2\right)^2 . \tag{4.17}$$

4.2 The Radial Sector

The action for the radial sector of (4.2) is given by

$$S_R \equiv \frac{\omega^2}{2g_{YM}^2} \sum_{A=1}^{m} \mathrm{Tr} Z_A^\dagger Z_A + \frac{1}{g_{YM}^2} \sum_{A=1}^{m} \mathrm{Tr} Z_A^\dagger Z_A. \tag{4.18}$$

Accordingly, the partition function can be written as

$$Z = \int \prod_i d\rho_i \mathcal{J}\left(\rho_i\right) e^{-S_R}$$

$$= \int \prod_i d\rho_i e^{-S_{eff}}. \tag{4.19}$$

where the effective action is

$$S_{eff} = \frac{N\omega^2}{2\lambda} \sum_i \rho_i + \frac{N}{2\lambda} \sum_i \rho_i^2 - N\left(m-1\right) \sum_i \ln \rho_i - \sum_{i<j} \ln\left(\rho_i - \rho_j\right)^2 . \tag{4.20}$$

We see that - in addition to the usual repulsion of the eigenvalues that arises from the Vandermonde determinant - we also have a log term that only vanishes only when $m = 1$, and that repels the eigenvalues away from $\rho = 0$.

In the large-N limit, the partition function can be evaluated using the steepest descent method. The saddle-point equations can be obtained by varying the effective action w.r.t.

a single eigenvalue and are

$$2\fint_0^{\infty} \frac{d\rho' \Phi\left(\rho'\right)}{\rho - \rho'} = \frac{\omega^2}{2\lambda} + \frac{\rho}{\lambda} - \frac{(m-1)}{\rho}, \tag{4.21}$$

where $\Phi\left(\rho'\right)$ is the eigenvalue density.

As noted earlier, at strong-coupling (by simple dimensional analysis) we can already infer from (4.21) that

$$R \sim \sqrt{\rho} \sim \lambda^{1/4}. \tag{4.22}$$

To solve (4.21), we begin by introducing the generating function:

$$G\left(z\right) = \int_{z_-}^{z_+} \frac{dz' \Phi\left(z'\right)}{z - z'}, \tag{4.23}$$

where $z_+ > z_- > 0$.

The ansatz is

$$G\left(z\right) = \frac{\omega^2}{4\lambda} + \frac{z}{2\lambda} - \frac{(m-1)}{2z} - \frac{(a_0 + cz)}{2z}\sqrt{\left(z - z_-\right)\left(z - z_+\right)}. \tag{4.24}$$

The function $G\left(z\right)$ has no pole, and hence we need to impose the condition [76]:

$$a_0\sqrt{z_- z_+} = \left(m - 1\right). \tag{4.25}$$

Moreover, the generating function also has the asymptotic behaviour: $G\left(z\right) \sim \frac{1}{z}$. For $|z| \to \infty$, we obtain

$$
\begin{aligned}
G(z) &= \frac{\omega^2}{4\lambda} + \frac{z}{2\lambda} - \frac{(m-1)}{2z} - \frac{(a_0 + cz)}{2z}\sqrt{(z-z_-)(z-z_+)} \\
&= \frac{\omega^2}{4\lambda} + \frac{z}{2\lambda} - \frac{(m-1)}{2z} - \frac{(a_0 + cz)}{2}\left(1 - \frac{z_- + z_+}{2z} - \frac{(z_- - z_+)^2}{8z^2} + \cdots\right) \\
&= \frac{\omega^2}{4\lambda} + \frac{z}{2\lambda} - \frac{(m-1)}{2z} - \frac{(a_0 + cz)}{2}\left(1 - \frac{s}{2z} - \frac{\Delta^2}{8z^2} + \cdots\right).
\end{aligned}
\tag{4.26}
$$

where we have defined:

$$
s = z_- + z_+, \qquad \Delta = z_- - z_+. \tag{4.27}
$$

In the strong-coupling limit, $\frac{\omega^2}{\lambda^{1/2}} \to 0$, and using the fact that $G(z) \sim \frac{1}{z}$, leads to

$$
1 = -\frac{(m-1)}{2} + \frac{\Delta^2}{16\lambda} + \frac{a_0 s}{4} \tag{4.28}
$$

$$
a_0 = \frac{s}{2\lambda}, \qquad c = \frac{1}{\lambda}. \tag{4.29}
$$

In terms of the variables s and Δ, the "no-pole" condition can be written as

$$
\frac{s^2}{16\lambda^2}\left(s^2 - \Delta^2\right) = (m-1)^2. \tag{4.30}
$$

This, together with (4.28), implies that

$$
s^4 - \frac{8\lambda(m+1)s^2}{3} - \frac{16\lambda^2(m-1)^2}{3} = 0. \tag{4.31}
$$

This is easily solved and one obtains

$$s^2 = \frac{4}{3}(m+1)\lambda\left(1+\sqrt{1+3\left(\frac{m-1}{m+1}\right)^2}\right) \tag{4.32}$$

$$\Delta^2 = s^2 - \frac{16\lambda^2(m-1)^2}{s^2}. \tag{4.33}$$

4.3 The Extended Solution

In order to provide a unified description for both the single complex matrix model ($m = 1$) and the case when we have more than two complex matrices ($m \geq 2$), it is necessary to extend the domain of definition of the eigenvalue density to the real line.[2]

First, we define the eigenvalue density in "r-space" as

$$2r\Phi(\rho) = 2r\Phi\left(r^2\right) \equiv \phi(r) \equiv \phi(-r), \qquad r > 0. \tag{4.34}$$

Since $\rho = r^2$, it is not difficult to show that

$$⨍_0^\infty \frac{d\rho'\Phi(\rho')}{\rho-\rho'} = \frac{1}{2r}⨍_{-\infty}^\infty \frac{dr'\phi(r')}{r-r'} \tag{4.35}$$

Thus, the saddle-point equations at strong-coupling - *i.e.* (4.21) - can be written as

$$⨍_{-\infty}^\infty \frac{dr'\phi(r')}{r-r'} = \frac{1}{\lambda}r^3 - \frac{(m-1)}{r}. \tag{4.36}$$

For $m = 1$, (4.36) yields

$$⨍_{-\infty}^\infty \frac{dr'\phi(r')}{r-r'} = \frac{1}{\lambda}r^3. \tag{4.37}$$

[2]Roughly,we wish to work in "r-space" instead of the "ρ space" considered in the last section.

The solution to this singular integral equation is given by the standard one-cut solution associated with a quartic potential.

For the case when we have more than two complex matrices ($m \geq 2$), the solution to (4.36) is given by the two-cut ansatz [76]:[3]

$$G(z) = \frac{1}{\lambda} r^3 - \frac{(m-1)}{r} - \frac{a_0 + cz}{z} \sqrt{(z^2 - r_-^2)(z^2 - r_+^2)}. \tag{4.38}$$

The "no-pole" condition together with the behaviour of $G(z)$ as $|z| \to \infty$ leads to

$$\begin{aligned} a_0 = \frac{s}{2\lambda}, \qquad c = \frac{1}{\lambda}, \\ 2 = -(m-1) + \frac{\Delta^2}{8\lambda} + \frac{a_0 s}{4}. \end{aligned} \tag{4.39}$$

These are the same conditions that we had previously and, obviously, upon solving we obtain (4.32) and (4.33), provided that $z_\pm = r_\pm^2$.

In addition, it turns out that (4.32) and (4.33) extend to the $m = 1$ case provided that $z_- = 0$.

The densities are extracted from the generating function,[4] and we find

$$\pi \Phi(\rho) = \frac{1}{2\lambda\rho} \left(\rho + \frac{1}{2}(\rho_+ + \rho_-) \right) \sqrt{(\rho - \rho_-)(\rho_+ - \rho)}, \qquad \rho_- \leq \rho \leq \rho_+ \tag{4.41}$$

and

[3] The cuts are in the intervals $[-r_-, -r_+]$ and $[r_-, r_+]$, where $r_+ > r_- > 0$.

[4] More precisely, the eigenvalue density is given by

$$\mathrm{Im}\,(G(x \pm i\epsilon)) = \mp \pi \phi(x). \tag{4.40}$$

$$\pi\phi\left(\rho\right) = \frac{1}{2\lambda r}\left(r^2 + \frac{1}{2}\left(r_+^2 + r_-^2\right)\right)\sqrt{\left(r^2 - r_-^2\right)\left(r_+^2 - r^2\right)}, \qquad r_-^2 \le r \le r_+^2. \tag{4.42}$$

From (4.32) and (4.33), it follows that

$$r_\pm = \left[\frac{2}{\sqrt{3}} \times \left(1 + \frac{\sqrt{7}}{2}\right)^{1/2} \pm \left(\frac{4}{3} \times \left(1 + \frac{\sqrt{7}}{2}\right) - \frac{3}{1 + \frac{\sqrt{7}}{2}}\right)^{1/2}\right]^{1/2} \lambda^{1/4}, \quad m = 3$$

$$r_\pm = \left[\left(1 + \frac{2}{\sqrt{3}}\right)^{1/2} \pm \left(1 + \frac{2}{\sqrt{3}} - \frac{1}{1 + \frac{2}{\sqrt{3}}}\right)^{1/2}\right]^{1/2} \lambda^{1/4}, \quad m = 2$$

$$r_+ = \frac{2}{3^{1/4}}\lambda^{1/4}, \quad m = 1.$$

4.4 The Angular Degrees Of Freedom

Let us consider the case when we only have one complex matrix - this corresponds to the two-matrix model that we discussed in the previous chapter. In this case it is possible to introduce matrix valued polar coordinates. In particular, we write:

$$Z = X_1 + iX_2 = RU, \qquad Z^\dagger = U^\dagger R. \tag{4.43}$$

where R is an $N \times N$ Hermitian matrix and U is unitary. In terms of the matrix valued polar coordinates, the action can be written as

$$
\begin{aligned}
S &= \frac{N\omega^2}{2\lambda}\mathrm{Tr}\left(ZZ^\dagger\right) + \frac{N}{4\lambda}\mathrm{Tr}\left[Z,\, Z^\dagger\right]^2 \\
&= \frac{N\omega^2}{2\lambda}\mathrm{Tr}R^2 + \frac{N}{2\lambda}\mathrm{Tr}R^4 - \frac{N}{2\lambda}\mathrm{Tr}\left(R^2 U R^2 U^\dagger\right) \\
&= \frac{N\omega^2}{2\lambda}\sum_i \rho_i + \frac{N}{2\lambda}\sum_i \rho_i^2 - \frac{N}{2\lambda}\mathrm{Tr}\left(\rho U \rho U^\dagger\right). \qquad (4.44)
\end{aligned}
$$

where $\rho_i = r_i^2$, and r_i $(i = 1, \ldots, N)$ are the eigenvalues of the matrix R. Hence, the partition function is

$$
Z = \int \prod_i d\rho_i \Delta^2(\rho)\, e^{-\left(\frac{N\omega^2}{2\lambda}\sum_i \rho_i + \frac{N}{2\lambda}\sum_i \rho_i^2\right)} \int DU e^{\frac{N}{2\lambda}\mathrm{Tr}\left(\rho U \rho U^\dagger\right)}. \qquad (4.45)
$$

The integration over the unitary group can be performed by making use of the Harish-Chandra-Itzykson-Zuber (HCIZ) integral formula [77]. Let A and B be arbitrary matrices with eigenvalues a_i and b_i $(i = 1, \ldots, N)$, then the HCIZ formula is

$$
I(A, B; \beta) \equiv \int DU e^{AUBU^\dagger} = c\frac{\det\left(e^{\beta a_i b_j}\right)}{\Delta(a)\Delta(b)} \qquad (4.46)
$$

where $\Delta(a)$ (similarly for $\Delta(b)$) is the Vandermonde determinant. Accordingly, the partition function becomes

$$
\begin{aligned}
Z &\sim \int \prod_i d\rho_i \Delta^2(\rho)\, e^{-\left(\frac{N\omega^2}{2\lambda}\sum_i \rho_i + \frac{N}{2\lambda}\sum_i \rho_i^2\right)} \frac{\det\left(e^{\frac{N}{2}\rho_i\rho_j}\right)}{\Delta(\rho)\Delta(\rho)} \\
&= \int \prod_i d\rho_i e^{-\left(\frac{N\omega^2}{2\lambda}\sum_i \rho_i + \frac{N}{2\lambda}\sum_i \rho_i^2\right) + \ln\det\left(e^{\frac{N}{2\lambda}\rho_i\rho_j}\right)}. \qquad (4.47)
\end{aligned}
$$

The logarithm of the HCIZ integral also has a perturbation expansion. In particular, we have

$$X(A,B;\beta) \equiv \lim_{N\to\infty} \frac{1}{N^2} \ln\left[I\left(\sqrt{N}A,\, \sqrt{N}B;\beta\right)\right]$$
$$= \sum_{k=1}^{\infty} \frac{\beta^k}{k} X_k(A,B;\beta)\,, \tag{4.48}$$

where the $X_k(A,B;\beta)$ is a symmetric function and homogeneous of degree k. Using the above perturbation expansion - to first order - we can write the effective action as[5]

$$S_{eff} = \frac{N\omega^2}{2\lambda}\sum_i \rho_i + \frac{N}{2\lambda}\sum_i \rho_i^2 - \sum_{i<j} \ln(\rho_i - \rho_j)^2 - \frac{1}{2\lambda}\left(\sum_i \rho_i\right)^2. \tag{4.49}$$

This leads us to the saddle-point equations:

$$2\sum_{i\neq j} \frac{1}{\rho_i - \rho_j} = \frac{N\omega^2}{2\lambda} + \frac{N}{\lambda}\rho_i - \frac{1}{\lambda}\omega_2, \tag{4.50}$$

or equivalently

$$2\int_0^{\infty} \frac{d\rho' \Phi(\rho')}{\rho - \rho'} = \frac{N\omega^2}{2\lambda} + \frac{N}{\lambda}\rho - \frac{1}{\lambda}\omega_2. \tag{4.51}$$

Here,

$$\omega_2 = \int_0^{\infty} d\rho \Phi(\rho)\, \rho. \tag{4.52}$$

The eigenvalue density at strong-coupling ($i.e.$ $\frac{\omega^2}{\lambda^{1/2}} \to 0$) is given by

$$\Phi(\rho) = \frac{1}{2\pi\lambda}\sqrt{\rho(\rho_+ - \rho)}, \tag{4.53}$$

[5]Specifically, the reason for using the perturbation expansion is to determine if the eigenvalue density is still non-vanishing only within a hypersphere.

where

$$\rho_+ = 4\lambda^{1/2}. \tag{4.54}$$

For the extended solution, the saddle-point equations are

$$\fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{r-r'} = \frac{N}{\lambda}r^3 - \frac{r}{\lambda}\omega_2. \tag{4.55}$$

For the density, we obtain

$$\pi\phi(r) = \frac{r^2}{\lambda}\sqrt{r_+^2 - r^2}, \tag{4.56}$$

where $r_+ = 2\lambda^{1/4}$, in perfect agreement with (4.53).[6]

The analysis that we have just performed, unfortunately, breaks down for second-order perturbation theory. In particular, we find that the radial eigenvalue density is confined in a hypersphere with an imaginary radius *i.e.* the end-points of the cut are imaginary. This is a result of the fact that we have insisted in taking the limit $\frac{\omega^2}{\lambda^{1/2}} \to 0$, in which case the system has only λ as a parameter. Indeed, we saw in chapter 3 that in the presence of a non-vanishing "mass term" ω^2, the strong-coupling expansion is actually an expansion in $\frac{\lambda^{1/4}}{\omega}$.

These issues are further elucidated by a naive attempt at a solution of the exact saddle-point equation when $\frac{\omega^2}{\lambda^{1/2}} \to 0$. In this case we start by defining a matrix M with the following matrix elements:

$$M_{ij} = e^{\frac{N}{2\lambda}\rho_i\rho_j}. \tag{4.57}$$

[6] Recall that $2r\Phi(r^2) \equiv \phi(r)$.

Moreover, we have

$$
\begin{aligned}
\frac{\partial}{\partial \rho_{ij}} \ln \det M &= \frac{\partial}{\partial \rho_{ij}} \mathrm{Tr}\left(\ln M\right) \\
&= \frac{N}{\lambda} \sum_{k=1}^{N} M_{jk}^{-1} \rho_k M_{ki}
\end{aligned}
\tag{4.58}
$$

Using (4.58), it is straightforward to show that (at strong-coupling) the saddle-point equations are

$$
\sum_{k=1}^{N} M_{jk}^{-1} \rho_k M_{kj} = \rho_j . \tag{4.59}
$$

The saddle-point equations can be solved for small matrices. Indeed, it is clear that the solution to the saddle-point equations is

$$
\rho_1 = \rho_2 \cdots = \rho_N = Const, \tag{4.60}
$$

or in terms of the eigenvalue density

$$
\Phi\left(\rho\right) \sim \delta\left(\rho - Const.\right). \tag{4.61}
$$

The question is then whether the system with $\frac{\omega^2}{\lambda^{1/2}} = 0$ can be regularized in a way that involves the only dimensionful parameter of the theory, namely λ. A natural suggestion is to consider the effective action in the radial sector of the system with more than two matrices, where a logarithmic term is present. So, in the next chapter we will regulate the two matrix system as

$$
S = -\frac{N\omega^2}{2\lambda} \mathrm{Tr} A - \frac{N}{2\lambda} \mathrm{Tr} A^2 + N\epsilon \mathrm{Tr} \ln A \equiv N \mathrm{Tr} V\left(A\right). \tag{4.62}
$$

and attempt to find the large-N radial density of eigenvalues.

4.5 Summary

The aim of this chapter was to investigate the strong-coupling limit of an ensemble of m complex (or equivalently, $2m$ Hermitian) matrices. We started by reviewing how to make use of the Collective Field Theory technique in order to determine the Jacobian. After obtaining the Jacobian, we were able to write down the effective action in the radial subsector. For an arbitrary number of even Hermitian matrices, the radial density of eigenvalues was then determined. In the case when we have a single complex matrix, we could parametrize our system using matrix valued polar coordinates. The first-order effective action was then written down and the resulting saddle-point equations solved. The density was seen to be non-vanishing in a hypersphere of radius $r_+ = 2\lambda^{1/4}$. However, the higher-order perturbation theory breaks down. It was thus necessary to attempt to solve the exact saddle point equations. Unfortunately, the radial eigenvalue was singular and we have to find another way of regularizing the system. One such method of regularizing the system will be discussed in the next chapter.

Chapter 5

Loop Equations

A closed expression for the Harish-Chandra-Itzykson-Zuber integral formula in terms of the eigenvalues is in general not available. However, we would like to obtain the density of radial eigenvalues which corresponds to the large-N saddle-point configuration of (4.47). Remarkably, methods to obtain the saddle-point density have been developed in the context of the so-called "induced QCD" [80, 81, 78]. Here, in particular, we will make use of the Dobroliubov-Makeenko-Semenoff (DMS) approach, based on the Dyson-Schwinger (loop) equations [78, 79]. (In fact, Dyson-Schwinger (loop) equations - as noted in Chapter 2 - have played a major role in large-N QCD.) We revisit the radial (single matrix) sector and reproduce the previously obtained radial density of eigenvalues using the Dyson-Schwinger equations. We then identify an auxiliary Penner-type two matrix model that maps to the two matrix integral in terms of matrix valued polar coordinates, and apply the DMS approach to obtain the density of eigenvalues.

5.1 Loop Equations In The Radial Sector

Basically the Dyson-Schwinger equations express the trivial fact that - with an appropriate choice of boundary conditions - the integral of a total derivative automatically vanishes. (In particular, in QFT the Dyson-Schwinger equations are used to find relations between the various n-point (Green's) functions.[1])

In this section we wish to revisit the radial sector. We use a set of loop equations in order to obtain an algebraic equation satisfied by the generating function.

The action - in the radial sector - is given by[2]

$$S=-\frac{N\omega^2}{2\lambda}\mathrm{Tr}A-\frac{N}{2\lambda}\mathrm{Tr}A^2+N\epsilon\mathrm{Tr}\ln A\equiv N\mathrm{Tr}V\left(A\right). \tag{5.1}$$

The basic loop equations in the radial sector follow from the identity:

$$0=\frac{1}{N^2}\int\left[dA\right]\frac{\partial}{\partial A_{ij}}e^S\left(\frac{1}{z-A}\right)_{ij}.$$

Hence,

$$\frac{1}{N}\left\langle\mathrm{Tr}\frac{V'\left(A\right)}{z-A}\right\rangle=G^2\left(z\right). \tag{5.2}$$

Here, $G\left(z\right)$ is the generating function.

Now, the L.H.S. of (5.2) yields

[1] As a simple example, consider the generating functional $Z\left[J\right]$; the Dyson-Schwinger equations follow from the identity: $0=\int\mathcal{D}\phi\frac{\delta}{\delta\phi}Z\left[J\right]=\int\mathcal{D}\phi\left[e^{-S+J\cdot\phi}\right]\left(-\frac{\delta S}{\delta\phi}+J\right)=\left\langle\left(-\frac{\delta S}{\delta\phi}+J\right)\right\rangle$.

[2] Our notation has changed: A is ρ and the path-integral weight is e^S.

$$
\begin{aligned}
\frac{1}{N}\left\langle \mathrm{Tr}\frac{V'(A)}{z-A}\right\rangle &= \frac{\omega^2}{2\lambda N}\left\langle \mathrm{Tr}\frac{1}{z-A}\right\rangle + \frac{1}{\lambda N}\left\langle \mathrm{Tr}A\frac{1}{z-A}\right\rangle + \frac{\epsilon}{N}\left\langle \mathrm{Tr}\frac{1}{\mathrm{A}}\frac{1}{z-A}\right\rangle . \\
&= \frac{\omega^2}{2\lambda N}\left\langle \mathrm{Tr}\frac{1}{z-A}\right\rangle - \frac{1}{\lambda} + \frac{z}{\lambda N}\left\langle \mathrm{Tr}\frac{1}{z-A}\right\rangle + \frac{\epsilon}{zN}\left\langle \mathrm{Tr}\frac{1}{A}\right\rangle \\
&+ \frac{\epsilon}{zN}\left\langle \mathrm{Tr}\frac{1}{z-A}\right\rangle \\
&= \left(\frac{\omega^2}{2\lambda}+\frac{z}{\lambda}+\frac{\epsilon}{z}\right)G(z) - \frac{1}{\lambda} - \frac{\epsilon}{zN}\left\langle \mathrm{Tr}\frac{1}{A}\right\rangle . \qquad (5.3)
\end{aligned}
$$

Hence, we can write (5.2) as[3]

$$
\left(\frac{\omega^2}{2\lambda}+\frac{1}{\lambda}z+\frac{\epsilon}{z}\right)G(z) - \frac{1}{\lambda} - \frac{\epsilon\omega_{-1}}{z} = G^2(z). \qquad (5.4)
$$

This equation is simple to solve, and we obtain

$$
G(z) = \frac{1}{2}\left(\frac{z}{\lambda}-\frac{\epsilon}{z}\right) - \frac{1}{2}\sqrt{\left(\frac{z}{\lambda}-\frac{\epsilon}{z}\right)^2 - 4\left(\frac{1}{\lambda}+\frac{\epsilon\omega_{-1}}{z}\right)}. \qquad (5.5)
$$

We could have re-expressed ω_{-1} in terms of ω_1. Since the $\frac{1}{z}$ term on the L.H.S. of (5.4) has to vanish, one obtains

$$
\epsilon\omega_{-1} = \frac{\omega^2}{2\lambda} + \frac{1}{\lambda}\omega_1. \qquad (5.6)
$$

Now, assuming a "one-cut" solution,[4] *i.e.* we can write the generating function as

$$
G(z) = \frac{1}{2}\left(\frac{z}{\lambda}-\frac{\epsilon}{z}\right) - \frac{\left(a_0+\frac{z}{\lambda}\right)}{2z}\sqrt{(z-z_-)(z-z_+)}. \qquad (5.7)
$$

[3] $\omega_{-1} \equiv \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{A}\right\rangle$.

[4] This is a reasonable assumption as we can fix the "moment" ω_{-1} in such a manner as to ensure that the root of the term inside the square-root coalesce, that is, the square-root becomes $\frac{M(z)}{z}\sqrt{(z-z_-)(z-z_+)}$. Here, $M(z)$ is a polynomial of the first degree.

This implies that the quartic polynomial inside (5.5) must have two coincident roots. Alternatively, (5.5) can be expanded for large $|z|$, and when "matched" with (5.7), one obtains (4.28) and (4.29).

5.2 The Auxiliary Two-Matrix Model

Let us return to the two-matrix model that we have been looking at in the previous chapters. The action for the matrix model is

$$S_\rho = -NV_\rho(\rho) + N\beta \mathrm{Tr}\rho U\rho U^\dagger. \tag{5.8}$$

Here,

$$V_\rho(\rho) = \frac{N\omega^2}{2\lambda}\mathrm{Tr}\rho + \frac{N}{2\lambda}\mathrm{Tr}\rho^2 - N\epsilon\mathrm{Tr}\ln\rho, \tag{5.9}$$

and we have added a logarithmic potential similar to the potential for a large number of complex matrices.

Next, let us consider the identity:

$$0 = \int [d\rho]\, DU \frac{d}{d\rho_{ij}}\left[e^S\left(\frac{1}{z_1-\rho}U\frac{1}{z_2-\rho}U^\dagger\right)_{ij}\right]. \tag{5.10}$$

After some trivial manipulations, this can be written as

$$\left\langle \frac{d}{d\rho_{ij}}\left(\frac{1}{z_1-\rho}U\frac{1}{z_2-\rho}U^\dagger\right)_{ij}\right\rangle = \left\langle -\frac{\partial S_\rho}{\partial\rho_{ij}}\left(\frac{1}{z_1-\rho}U\frac{1}{z_2-\rho}U^\dagger\right)_{ij}\right\rangle. \tag{5.11}$$

Thus,

$$G(z_1)\frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z_1-A}U\frac{1}{z_2-B}U^{\dagger}\right\rangle = \left\langle \frac{\partial V_\rho}{\partial \rho_{ij}}\left(\frac{1}{z_1-\rho}U\frac{1}{z_2-\rho}U^{\dagger}\right)_{ij}\right\rangle - \frac{\beta}{N}\left\langle \mathrm{Tr}\left(\rho U^{\dagger}\frac{1}{z_1-\rho}U\frac{1}{z_2-\rho}\right)\right\rangle$$
$$-\frac{\beta}{N}\left\langle \mathrm{Tr}\left(U^{\dagger}\rho U\frac{1}{z_1-\rho}U\frac{1}{z_2-\rho}U^{\dagger}\right)\right\rangle. \tag{5.12}$$

The last term above does not close on the initial set of correlators, which only involves one U and $U^{\dagger}$. So we find it convenient to map this problem to an auxiliary two matrix problem using the saddle-point equations.

Let us consider a two-matrix model with the following action:

$$S = -N\mathrm{Tr}V(A) - N\mathrm{Tr}V(B) + N\mathrm{Tr}AUBU^{\dagger} \tag{5.13}$$

In terms of the eigenvalues (the eigenvalues for the matrix A are denoted by a_i, similarly the eigenvalues of B are b_i), the partition function is[5]

$$\begin{aligned} Z &= \int dA\int dB\int DU e^{-N\mathrm{Tr}V(A)-N\mathrm{Tr}V(B)+N\mathrm{Tr}AUBU^{\dagger}} \\ &= \int\prod_i da_i\int\prod_i db_i\Delta^2(a)\Delta^2(b)e^{-N\sum_i V(a_i)-N\sum_i V(b_i)+\ln I(a,b)} \\ &= \int\prod_i da_i\int\prod_i db_i e^{-N\sum_i V(a_i)-N\sum_i V(b_i)+\ln\Delta^2(a)+\ln\Delta^2(b)+\ln I(a,b)} \end{aligned} \tag{5.14}$$

The saddle-point equations follow trivially from varying the "effective-action" - which can be read off from (5.14) - w.r.t. the eigenvalues a_i and similarly for the eigenvalues b_i. The saddle-point equations are[6]

[5] $I(a,b)$ has been defined in (4.46).

[6] In the notation that follows, $\langle\rangle$ refers to the integrals over the matrix U.

$$0 = -NV'(a_i) + \frac{\partial}{\partial a_i}\ln\Delta^2(a) + N\left\langle UbU^{\dagger}\right\rangle \tag{5.15}$$

$$0 = -NV'(b_i) + \frac{\partial}{\partial b_i}\ln\Delta^2(b) + N\left\langle U^{\dagger}aU\right\rangle \tag{5.16}$$

By adding (5.15) and (5.16), we obtain

$$0 = -NV'(a_i) + \frac{\partial}{\partial a_i}\ln\Delta^2(a) + \frac{N}{2}\left\langle UaU^{\dagger} + U^{\dagger}aU\right\rangle. \tag{5.17}$$

Here we have made the obvious assumption that at the saddle-point we can set $A = B$, and also that the eigenvalue densities of the $N \times N$ matrices A and B are the same.

Now, let us return to the partition function in "ρ-space". In this case the partition function is

$$Z = \int \prod_i d\rho_i \Delta^2(\rho)\, e^{-NV_\rho(\rho) + \beta N \mathrm{Tr}\rho U \rho U^{\dagger}}. \tag{5.18}$$

For the saddle-point equations, we get

$$0 = -NV'_\rho(\rho) + \frac{\partial}{\partial \rho_i}\ln\Delta^2(\rho) + \beta N\left\langle U\rho U^{\dagger} + U^{\dagger}\rho U\right\rangle. \tag{5.19}$$

Next, comparing (5.17) with (5.19) leads to the conclusion that we can make use of the action in (5.13) provided that

$$2\beta = 1 \qquad V'_\rho(\rho) = V'(A). \tag{5.20}$$

The consequence of the analysis that we have just carried out is that in order to tackle the case where we have the two matrices being identical we can consider the same problem in terms of two non-identical matrices provided that the conditions in (5.20) are satisfied. Thus we need to consider the system with the action

$$S = -\frac{N\omega^2}{2\lambda^{1/2}}\mathrm{Tr}A - \frac{N}{2}\mathrm{Tr}A^2 + N\epsilon\mathrm{Tr}\ln A + N\mathrm{Tr}AUBU^\dagger \\ -\frac{N\omega^2}{2\lambda^{1/2}}\mathrm{Tr}B - \frac{N}{2}\mathrm{Tr}B^2 + N\epsilon\mathrm{Tr}\ln B. \tag{5.21}$$

5.3 DMS Loop Equations

We have already mentioned in the introduction that even with all the dramatic simplifications that occur in the large-N limit, $(3+1)$-dimensional QCD has still not been solved. One of the directions taken some decades ago was to consider a simplified lattice version of QCD that was believed could somehow "induce" the real - continuum - QCD [80]. Unfortunately, most of these hopes were unfounded. In particular, the Kazakov-Migdal model has a "hidden" $\mathbb{Z}_N$ symmetry [82] that disqualified it from giving a true description of QCD.[7] Nevertheless, the techniques that were introduced in studying induced QCD are actually relevant to the two-matrix model.[8]

The DMS loop equations are formulated in terms of two analytic functions. The first function is nothing but the generating function (resolvent)

$$G(z_1) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z_1 - A}\right\rangle. \tag{5.22}$$

In addition, we also define the following function:

[7] The "hidden" $\mathbb{Z}_N$ symmetry leads to the result that the expectation values of the Wilson loops vanish.

[8] The action of interest is given by $S_{KM} = N\mathrm{Tr}\left(\sum_{x\in\mathscr{H}} V(\Phi(x)) + \sum_{x,y\in\mathscr{H}} \Phi(x)U(x,y)\Phi(y)U^\dagger(x,y)\right)$, where $\mathscr{H}$ is some hyper-dimensional cubic lattice. Accordingly, the saddle-point equations are: $2\frac{D}{N}\frac{\partial}{\partial\phi_i}\ln I(\phi,\chi)\,|_{\chi=\phi}= V'(\phi_i) - \sum_{i<j}\frac{1}{\phi_i-\phi_j}$, where $I(\phi,\chi)$ is the HCIZ integral.

$$\Phi\left(z_1, z_2\right) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z_1 - A}U\frac{1}{z_2 - B}U^{\dagger}\right\rangle . \tag{5.23}$$

Note that $\Phi\left(z_1, z_2\right) = \Phi\left(z_2, z_1\right)$, as the Haar measure is invariant under the transformation $U \to U^{\dagger}$.

The DMS loop equations follow from the identity:

$$0 = \frac{1}{N^2}\left[\int [dA]\, DU \frac{\partial}{\partial A_{ij}} e^{S}\left(\frac{1}{z_1 - A}U\frac{1}{z_2 - B}U^{\dagger}\right)_{ij}\right], \tag{5.24}$$

where the action is given by

$$S = N\mathrm{Tr}AUBU^{\dagger} - N\mathrm{Tr}V'\left(A\right). \tag{5.25}$$

From (5.24) it follows that

$$\begin{aligned}
0 &= \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z_1 - A}\frac{1}{N}\mathrm{Tr}\frac{1}{z_1 - B}U\frac{1}{z_2 - B}U^{\dagger}\right\rangle + \frac{1}{N^2}\left\langle \frac{\partial S}{\partial A_{ij}}\left(\frac{1}{z_1 - A}U\frac{1}{z_2 - B}U^{\dagger}\right)_{ij}\right\rangle \\
&= G\left(z_1\right)\Phi\left(z_1,\, z_2\right) - \frac{1}{N}\left\langle \mathrm{Tr}\left(V'\left(A\right)\frac{1}{z_1 - A}U\frac{1}{z_2 - B}U^{\dagger}\right)\right\rangle \\
&+ \frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z_1 - A}U\frac{1}{z_2 - B}U^{\dagger}UBU^{\dagger}\right)\right\rangle .
\end{aligned} \tag{5.26}$$

In order to arrive at the final result we have made use of the fact that in the large-N limit correlation functions factorize. Also, we have

$$\begin{aligned}
\left\langle \mathrm{Tr}\left(UBU^{\dagger}\frac{1}{z_1 - A}U\frac{1}{z_2 - B}U^{\dagger}\right)\right\rangle &= \left\langle \mathrm{Tr}\left(\left(B - z_2 + z_2\right)U^{\dagger}\frac{1}{z_1 - A}U\frac{1}{z_2 - B}\right)\right\rangle \\
&= -G\left(z_1\right) + z_2\Phi\left(z_1, z_2\right).
\end{aligned} \tag{5.27}$$

Hence, we can write (5.26) as

$$\frac{1}{N}\left\langle \mathrm{Tr}\left(V'\left(A\right)\frac{1}{z_1-A}U\frac{1}{z_2-B}U^{\dagger}\right)\right\rangle = G\left(z_1\right)\Phi\left(z_1,\, z_2\right)-G\left(z_1\right)+z_2\Phi\left(z_1,\, z_2\right). \quad (5.28)$$

This is the principal equation that we will make extensive use of in this chapter and is known as the DMS loop equation. Also, it is useful to study the behaviour of this equation for large values of, say, $|z_1|$.

Firstly, let us consider the large $|z_1|$ behaviour of the two analytic functions that we introduced above. The large $|z_1|$ behaviour of the generating function $G\left(z_1\right)$ is well-known - and indeed very simple to derive. Indeed for large $|z_1|$, we have

$$G\left(z_1\right) = \frac{1}{z_1}+\sum_{k=1}^{\infty}\frac{\omega_k}{z_1^{k+1}} \quad (5.29)$$

$$\Phi\left(z_1, z_2\right) = \frac{G\left(z_2\right)}{z_1}+\sum_{k=1}^{\infty}\frac{\mathcal{G}_k\left(z_2\right)}{z_1^{k+1}}. \quad (5.30)$$

where

$$\mathcal{G}_k\left(z\right) = \frac{1}{N}\left\langle \mathrm{Tr}\left(A^kU\frac{1}{z-B}U^{\dagger}\right)\right\rangle = \frac{1}{N}\left\langle \mathrm{Tr}\left(B^kU\frac{1}{z-A}U^{\dagger}\right)\right\rangle. \quad (5.31)$$

Now, let us consider a generic potential whose derivative is of the form

$$V'\left(\omega\right) = \sum_{m\geq -1}L_m\omega^m. \quad (5.32)$$

Then, using (5.29) and (5.30), we can write the DMS loop equation for large $|z_1|$ as

$$\frac{1}{z_1}\left(\sum_{m\geq -1} L_m\mathcal{G}_m\left(z_2\right)\right)+\sum_{k=1}^{\infty}\frac{1}{z_1^{k+1}}\left(\sum_{m\geq -1} L_m\mathcal{G}_{m+k}\left(z_2\right)\right)=\frac{1}{z_1}\left(z_2 G\left(z_2\right)-1\right)$$
$$+\sum_{k=1}^{\infty}\frac{1}{z_1^{k+1}}\left\{-\omega_k+\sum_{k'=0}^{k-1}\omega_{k'}\mathcal{G}_{k-1-k'}\left(z_2\right)+z_2\mathcal{G}_k\left(z_2\right)\right\},$$

where

$$\mathcal{G}_0\left(z_2\right)\equiv G\left(z_2\right). \tag{5.33}$$

Accordingly, we have

$$\sum_{m\geq -1} L_m\mathcal{G}_m\left(z_2\right)=z_2 G\left(z_2\right)-1 \tag{5.34}$$

$$\sum_{m\geq -1} L_m\mathcal{G}_{m+k}\left(z_2\right)=-\omega_k+\sum_{k'=0}^{k-1}\omega_{k'}\mathcal{G}_{k-1-k'}\left(z_2\right)+z_2\mathcal{G}_k\left(z_2\right),\ k\geq 1. \tag{5.35}$$

Finally, by expanding the DMS equations in terms of $\frac{1}{z_2}$, one obtains

$$\frac{1}{z_2}\frac{1}{N}\left\langle\mathrm{Tr}\left(V'\left(A\right)\frac{1}{z_1-A}\right)\right\rangle+\sum_{k=1}^{\infty}\frac{1}{Nz_2^{k+1}}\left\langle\mathrm{Tr}\left(V'\left(A\right)\frac{1}{z_1-A}UB^kU^{\dagger}\right)\right\rangle$$
$$=\frac{1}{z_2}\left[G^2\left(z_1\right)+\mathcal{G}_1\left(z_1\right)\right]+\sum_{k=1}^{\infty}\frac{1}{z_2^{k+1}}\left(\mathcal{G}_{k+1}\left(z_1\right)+G\left(z_1\right)\mathcal{G}_k\left(z_1\right)\right). \tag{5.36}$$

5.4 The Penner Potential

The matrix model with a logarithmic potential was introduced by Penner [83] - and, accordingly, it is referred to as a Penner matrix model. The Penner matrix model was initially suggested as a way of arriving at a result due to Zagier and Harer [84] - more precisely, the free energy of the Penner matrix model is a generating function of the virtual Euler characteristic which was computed by Harer and Zagier. Harer and Zagier showed that the virtual Euler characteristic for the moduli space of a Riemann surface of genus g with n punctures - we denote the moduli space by $\mathcal{M}_{g,n}$ - is

$$\chi_V\left(\mathcal{M}_{g,n}\right) = \frac{(n+2g-3)\,(2g-1)}{n!\,(2g)!}B_{2g}. \tag{5.37}$$

Here, B_{2g} are the Bernoulli numbers. Moreover,

$$F = \ln Z_{Penner} = \sum_{g,n} \chi_V\left(\left(\mathcal{M}_{g,n}\right)\right) N^{2-2g} \tag{5.38}$$

where

$$Z_{Penner} = \int [dM]\, e^{-N\mathrm{Tr}[-\ln(1-M)-M]}. \tag{5.39}$$

In this section we consider a Penner-type matrix model. More precisely, we consider a potential whose derivative is of the form[9]

$$V'\left(\omega\right) = \frac{L_{-1}}{\omega} + L_0 + L_1\omega. \tag{5.40}$$

In terms of our previous parameters, we have

$$L_{-1} = -\epsilon, \qquad L_0 = \frac{\omega^2}{2\lambda^{1/2}}, \qquad L_1 = 1. \tag{5.41}$$

[9]This is nothing but the derivative of the auxiliary matrix model that we considered in the previous section.

Let us begin by evaluating the L.H.S. of (5.28). It is simple to see that

$$\begin{aligned}
\frac{1}{N}\left\langle \mathrm{Tr}\left(V'(A)\frac{1}{z_1-A}U\frac{1}{z_2-B}U^\dagger\right)\right\rangle &= \frac{L_{-1}}{N}\left\langle \mathrm{Tr}\left(\frac{1}{A}\frac{1}{z_1-A}U\frac{1}{z_2-B}U^\dagger\right)\right\rangle \\
&+\frac{L_0}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z_1-A}U\frac{1}{z_2-B}U^\dagger\right)\right\rangle \\
&+\frac{L_1}{N}\left\langle \mathrm{Tr}\left(A\frac{1}{z_1-A}U\frac{1}{z_2-B}U^\dagger\right)\right\rangle \\
&=\frac{L_{-1}}{z_1}\mathcal{G}_{-1}(z_2)+\frac{L_{-1}}{z_1}\Phi(z_1,\, z_2)+L_0\Phi(z_1,\, z_2) \\
&+L_1 z_1\Phi(z_1,\, z_2)-L_1 G(z_2) \\
&=\frac{L_{-1}}{z_1}\mathcal{G}_{-1}(z_2)+V'(z_1)\,\Phi(z_1,\, z_2)-L_1 G(z_2)\,.
\end{aligned} \tag{5.42}$$

Accordingly, the DMS equations can be written as

$$\begin{aligned}
V'(z_1)\,\Phi(z_1,\, z_2)+\frac{L_{-1}}{z_1}\mathcal{G}_{-1}(z_2)-L_1 G(z_2) &= G(z_1)\,\Phi(z_1,\, z_2) \\
&-G(z_1)+z_2\Phi(z_1,\, z_2)\,.
\end{aligned} \tag{5.43}$$

Therefore,

$$\Phi(z_1,\, z_2)=\frac{G(z_1)+\frac{L_{-1}}{z_1}\mathcal{G}_{-1}(z_2)-L_1 G(z_2)}{G(z_1)+z_2-V'(z_1)}. \tag{5.44}$$

From (5.36), we have

$$\frac{1}{N}\left\langle \mathrm{Tr}\left(V'(A)\frac{1}{z_1-A}\right)\right\rangle = G^2(z_1)+\mathcal{G}_1(z_1)\,. \tag{5.45}$$

Since

$$
\begin{aligned}
\frac{1}{N}\left\langle \mathrm{Tr}\left(V'(A)\frac{1}{z_1-A}\right)\right\rangle &= \frac{L_{-1}}{N}\left\langle \mathrm{Tr}\left(\frac{1}{A}\frac{1}{z_1-A}\right)\right\rangle + \frac{L_0}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z_1-A}\right)\right\rangle \\
&+ \frac{L_1}{N}\left\langle \mathrm{Tr}\left(A\frac{1}{z_1-A}\right)\right\rangle \\
&= \frac{L_{-1}}{Nz_1}\left\langle \mathrm{Tr}\frac{1}{A}\right\rangle + \frac{L_{-1}}{Nz_1}\left\langle \mathrm{Tr}\frac{1}{z_1-A}\right\rangle + \frac{L_0}{N}\left\langle \mathrm{Tr}\frac{1}{z_1-A}\right\rangle \\
&+ \frac{L_1 z_1}{N}\left\langle \mathrm{Tr}\frac{1}{z_1-A}\right\rangle - L_1 \\
&= V'(z_1)\,G(z_1) + \frac{L_{-1}}{z_1}\left\langle \mathrm{Tr}\frac{1}{A}\right\rangle - L_1. \qquad (5.46)
\end{aligned}
$$

It follows that we can write (5.45) as

$$
V'(z_1)\,G(z_1) + \frac{L_{-1}}{z_1}\omega_{-1} - L_1 = G^2(z_1) + \mathcal{G}_1(z_1)\,, \qquad (5.47)
$$

or equivalently

$$
\mathcal{G}_1(z) = V'(z)\,G(z) - L_1 + \frac{L_{-1}}{z}\omega_{-1} - G^2(z)\,. \qquad (5.48)
$$

Using (5.34) and (5.48), we have

$$
\begin{aligned}
L_{-1}\mathcal{G}_{-1}(z_2) &= (z_2 - L_0)\,G(z_2) - 1 - L_1\mathcal{G}_1(z_2) \\
&= (z_2 - L_0)\,G(z_2) - 1 - L_1\left[V'(z_2)\,G(z_2) - L_1 + \frac{L_{-1}}{z_2}\omega_{-1} - G^2(z_2)\right].
\end{aligned}
\qquad (5.49)
$$

Substituting this into (5.44), we obtain

$$\Phi(z_1,z_2) = \frac{G(z_1) + \frac{L_{-1}}{z_1}\mathcal{G}_{-1}(z_2) - L_1 G(z_2)}{G(z_1) + z_2 - V'(z_1)}$$
$$= \frac{1}{z_1 G(z_1) + z_1 z_2 - z_1 V'(z_1)}\Bigg[z_1 G(z_1) - z_1 L_1 G(z_2)$$
$$+ (z_2 - L_0) G(z_2) - 1 - L_1\left(V'(z_2) G(z_2) - L_1 + \frac{L_{-1}}{z_2}\omega_{-1} - G^2(z_2)\right)\Bigg].$$

The function $\Phi(z_1,z_2)$ is symmetric in its arguments *i.e.* $\Phi(z_1,z_2) = \Phi(z_2,z_1)$. This is sufficient to determine $G(z)$. Accordingly, we have

$$\frac{1}{z_1 G(z_1) + z_1 z_2 - z_1 V'(z_1)}\Bigg[z_1 G(z_1) - z_1 L_1 G(z_2) + (z_2 - L_0) G(z_2)$$
$$- 1 - L_1\left(V'(z_2) G(z_2) - L_1 + \frac{L_{-1}}{z_2}\omega_{-1} - G^2(z_2)\right)\Bigg]$$
$$= \frac{1}{z_2 G(z_2) + z_1 z_2 - z_2 V'(z_2)}\Bigg[z_2 G(z_2) - z_2 L_1 G(z_1) + (z_1 - L_0) G(z_1)$$
$$- 1 - L_1\left(V'(z_1) G(z_1) - L_1 + \frac{L_{-1}}{z_1}\omega_{-1} - G^2(z_1)\right)\Bigg]. \tag{5.50}$$

After some tedious algebra, we obtain

$$\left[L_1 G^2(z_2) + (z_2 - L_0 - L_1 V'(z_2)) G(z_2) + \left(L_1^2 - 1 - \frac{L_1 L_{-1}}{z_2}\omega_{-1}\right)\right]\left[z_2 G(z_2) - z_2 V'(z_2)\right]$$
$$+ L_1 L_{-1}\omega_{-1} z_2 + z_2 L_{-1} G(z_2) = \Bigg[L_1 G^2(z_1) + (z_1 - L_0 - L_1 V'(z_1)) G(z_1)$$
$$+ \left(L_1^2 - 1 - \frac{L_1 L_{-1}}{z_1}\omega_{-1}\right)\Bigg]\left[z_1 G(z_1) - z_1 V'(z_1)\right]$$
$$+ L_1 L_{-1}\omega_{-1} z_1 + z_1 L_{-1} G(z_1) = C.$$

This can written in canonical form as

$$
\begin{aligned}
\frac{C}{z_1} &= L_1 G^3(z_1) + \left[z_1 - L_0 - 2L_1 V'(z_1)\right] G^2(z_1) \\
&+ \left[\left(L_1^2 - 1 - \frac{L_1 L_{-1}}{z_1}\omega_{-1}\right) - V'(z_1)\left(z - L_0 - L_1 V'(z_1)\right) + L_{-1}\right] G(z_1) \\
&+ L_1 L_{-1}\omega_{-1} - \left(L_1^2 - 1 - \frac{L_1 L_{-1}}{z_1}\omega_{-1}\right) V'(z_1). \qquad (5.51)
\end{aligned}
$$

The constant C can be determined by considering the large $|z_1|$ behaviour of (5.51). From (5.29) and (5.51), it follows that

$$
\begin{aligned}
\frac{C}{z_1} &= \left[-\omega_1 L_1\left(1 - L_1^2\right) + L_0 L_1 (1 + L_1) + L_1 L_{-1}\omega_{-1}(1 + L_1)\right] \\
&+ \frac{1}{z_1}\Big[\omega_1 L_0 (1 + L_1)(2L_1 - 1) - \omega_2 L_1\left(1 - L_1^2\right) \\
&+ L_0^2\left(1 + L_1^2\right) + L_1^2 L_{-1} - L_1^2 + L_0 L_1 L_{-1}\omega_{-1}\Big] + \cdots \qquad (5.52)
\end{aligned}
$$

Firstly, we need to check that the term independent of z_1 on the R.H.S. of (5.52) vanishes. In order to show that the term that is independent of z_1 vanishes, it is clear that we need an identity that expresses ω_{-1} in terms of the moment ω_1. For large values of $|z_1|$, (5.48) yields

$$
L_{-1}\omega_{-1} = (1 - L_1)\,\omega_1 - L_0. \qquad (5.53)
$$

Accordingly,

$$
\begin{aligned}
-\omega_1 L_1\left(1 - L_1^2\right) + L_0 L_1 (1 + L_1) + L_1^2 L_{-1}\omega_{-1} &= -\omega_1 L_1\left(1 - L_1^2\right) + L_0 L_1 (1 + L_1) \\
&+ L_1 (1 + L_1)\left[(1 - L_1)\,\omega_1 - L_0\right] = 0. \qquad (5.54)
\end{aligned}
$$

Furthermore,

$$\begin{aligned}C &= L_1\left(L_1^2-1\right)\omega_2 + L_0\left(1+L_1\right)\left(2L_1-1\right)\omega_1\\&\quad + L_0L_1L_{-1}\omega_{-1} + L_{-1}\left(1+L_1^2\right)\\&\quad - L_1^2 + L_0^2\left(1+L_1\right).\end{aligned} \tag{5.55}$$

We therefore arrive - for a generic potential of the form given in (5.40) - at the following cubic equation for $G(z)$:

$$\begin{aligned}&\frac{1}{z}\times\Bigg[L_1\left(L_1^2-1\right)\omega_2 + L_0\left(1+L_1\right)\left(2L_1-1\right)\omega_1\\&+L_0L_1L_{-1}\omega_{-1} + L_{-1}\left(1+L_1^2\right) - L_1^2 + L_0^2\left(1+L_1\right)\Bigg]\\&= L_1G^3\left(z_1\right) + \left[z_1 - L_0 - 2L_1V'\left(z_1\right)\right]G^2\left(z_1\right)\\&+\left[\left(L_1^2 - 1 - \frac{L_1L_{-1}}{z_1}\omega_{-1}\right) - V'\left(z_1\right)\left(z - L_0 - L_1V'\left(z_1\right)\right) + L_{-1}\right]G\left(z_1\right)\\&+L_1L_{-1}\omega_{-1} - \left(L_1^2 - 1 - \frac{L_1L_{-1}}{z_1}\omega_{-1}\right)V'\left(z_1\right).\end{aligned} \tag{5.56}$$

5.5 The Hoppe Integral In Radial Matrix Coordinates

We first wish to discuss the two matrix integral integral in matrix polar coordinates. From (5.21), this corresponds to

$$L_{-1} = -\epsilon = 0, \qquad L_0 = \frac{\omega^2}{2\lambda^{1/2}}, \qquad L_1 = 1. \tag{5.57}$$

The simplest way to obtain the resolvent $G(z)$ is to observe that from (5.53), the condition $L_1 = 1$ implies that $L_{-1}\omega_{-1} = -L_0$, and the hence (5.49) yields

$$G^2(z) - 2L_0 G(z) + \frac{2L_0}{z} = 0. \tag{5.58}$$

The standard solution, extending to the negative values of z on the real axis is

$$G(z) = L_0 - \sqrt{\frac{L_0^2}{4} - \frac{2L_0}{z}} \tag{5.59}$$

However, the solution that we seek can only have a non-zero density on the positive real axis, and thus we seek a solution to the resolvent matching the asymptotically the form

$$G(z) = \frac{\alpha}{2} - \frac{\epsilon}{z} - \frac{\alpha}{2z}\sqrt{(z - z_-)(z - z_+)}, \tag{5.60}$$

in the limit as $\epsilon \to 0$. Then,

$$\epsilon = \frac{\alpha}{2}\sqrt{z_- z_+}, \qquad \frac{\alpha}{4}(z_+ + z_-) = 1 + \epsilon. \tag{5.61}$$

As $\epsilon \to 0$, $z_- = 0$, and one obtains

$$\pi\Phi(\rho) = \frac{\alpha}{2\rho^{1/2}}\sqrt{\frac{4}{\alpha} - \rho}, \qquad 0 \leq \rho \leq \frac{4}{\alpha}. \tag{5.62}$$

Correspondingly,

$$\pi\phi(r) = \alpha\sqrt{\frac{4}{\alpha} - r^2}, \qquad -\frac{2}{\alpha^{1/2}} \leq r \leq \frac{2}{\alpha^{1/2}} \tag{5.63}$$

With $\alpha = 2L_0 = \frac{\omega^2}{\lambda^{1/2}}$. This radial density is of the form given in (3.43). There were no approximations in our approach.

5.6 Strong-Coupling Limit

We now consider the case when L_0 and L_{-1} are small in comparison to L_1, and also fix L_1 *i.e.* we set $L_1 = 1$. The reason for doing this is that we wish to have only one dimensionful parameter - namely 't Hooft's coupling - in our theory. We regularize this system with a Penner-type potential.

Firstly, when $L_0 = 0$ and $L_1 = 1$ it is straightforward to show that we can write (5.56) as

$$\frac{2L_{-1} - 1}{z} = G^3(z) - G^2(z)\left[z + \frac{2L_{-1}}{z}\right] + G(z)\left[2L_{-1} + \frac{L_{-1}^2}{z^2}\right],$$

where we have made use of (5.55) and the identity in (5.53). Moreover, the identity in (5.53) reduces to

$$L_{-1}\omega_{-1} = 0. \tag{5.64}$$

As long as $L_{-1} \neq 0$, this condition only allows for an even solution. However, if $L_{-1} = 0$, this need not be the case. With $L_{-1} = 0$, one obtains

$$0 = G^3(z) - zG^2(z) + \frac{1}{z}. \tag{5.65}$$

It is natural to ask how is it possible to write the generating function - which in this case is the root of a cubic equation - as a "one-cut" solution *i.e.*

$$G(x \pm i\epsilon) \sim V'(x) \mp i\pi M(x)\sqrt{(x-a)(x-b)}, \tag{5.66}$$

where $M(x)$ is some polynomial whose degree, obviously, depends on the degree of the potential $V(x)$. For the single matrix case it was clear how to write the generating function as a "one-cut" solution. In the case when we have an equation of third order it might not be clear how this can be achieved. However, if we schematically write the root (generating function) of the cubic equation as

$$G(z) = (z - z_1)(z - z_2)(z - z_3), \tag{5.67}$$

we can consider the equation satisfied by, say, the second and third roots *i.e.*

$$0 = (z - z_2)(z - z_3)$$

Indeed, it is trivial to show that

$$z_\pm = \frac{(z_2 + z_3)}{2} \pm \frac{\sqrt{(z_2 + z_3)^2 - 4z_2 z_3}}{2}. \tag{5.68}$$

Let us now return to the cubic equation given in (5.65). Firstly, before we formally solve for the roots, it is worthwhile to roughly estimate the behaviour of the roots for both large and small values of $|z|$. Let us begin with the case when $|z| \to \infty$. In this case, the leading contribution to the cubic equation is[10]

$$0 = -zG^2(z) + \frac{1}{z}. \tag{5.69}$$

[10]Recall the trivial fact that for large $|z|$ the generating function has the behaviour $G(z) \sim \frac{1}{z}$, and thus $G^3(z)$ can be ignored.

Accordingly,

$$G\left(z\right)=\pm\frac{1}{z}. \tag{5.70}$$

In addition, assuming that one of the roots has the asymptotic behaviour of the form $G\left(z\right)\sim z^{n}$, one obtains that $n=1$ and hence

$$G_{3}\left(z\right)\sim z. \tag{5.71}$$

Similarly, we now consider the limit $\left|z\right|\rightarrow 0$; and it is not difficult not difficult to see that the leading contribution to (5.65) is

$$G^{3}\left(z\right)=-\frac{1}{z} \tag{5.72}$$

and hence the roots are of the form[11]

$$\begin{aligned}
G_{1}\left(z\right) &\sim -\frac{1}{z^{1/3}}\\
G_{2}\left(z\right) &\sim \frac{1}{2z^{1/3}}+\frac{i\sqrt{3}}{2\left|z\right|^{1/3}}\\
G_{3}\left(z\right) &\sim \frac{1}{2z^{1/3}}-\frac{i\sqrt{3}}{2\left|z\right|^{1/3}}.
\end{aligned}$$

We now consider the formal solution to the cubic equation in (5.65). Firstly, let us assume we have the following generic cubic equation:

$$0=z^{3}+a_{2}z^{2}+a_{1}z+a_{0}. \tag{5.73}$$

[11] The behaviour of the roots - in particular $G_{2}\left(z\right)$ and $G_{3}\left(z\right)$ - is already promising as we expect that as we approach the cut the generating function must be of the form: $G\left(x\pm i\epsilon\right)=\fint\frac{dy\phi(y)}{x-y}\mp i\pi\phi\left(x\right)$. In other words, the roots should have some imaginary part.

The roots to this cubic equation are

$$z_1 = -\frac{a_2}{3} + (s_1 + s_2) \tag{5.74}$$

$$z_2 = -\frac{a_2}{3} - \frac{1}{2}(s_1 - s_2) + \frac{i\sqrt{3}}{2}(s_1 + s_2) \tag{5.75}$$

$$z_3 = -\frac{a_2}{3} - \frac{1}{2}(s_1 - s_2) + \frac{i\sqrt{3}}{2}(s_1 + s_2). \tag{5.76}$$

Here,

$$s_1 = \sqrt[3]{r + (q^3 + r^2)^{1/2}},\ s_2 = \sqrt[3]{r - (q^3 + r^2)^{1/2}}, \tag{5.77}$$

where

$$q = \frac{1}{3}a_1 - \frac{1}{9}a_2^2;\ r = \frac{1}{6}(a_1 a_2 - 3a_0) - \frac{1}{27}a_2^3. \tag{5.78}$$

We now consider the cubic equation given in (5.65); and in this case, one obtains

$$q^3 + r^2 = \frac{1}{4z^2} - \frac{z^2}{27}. \tag{5.79}$$

For $|z| \to 0$ the discriminant is positive and hence we have one real root and a pair of complex conjugate roots. Similarly, for $|z| \to \infty$ we find that the discriminant is negative and consequently all three roots are real.

Using (5.79), we can write the generating function - that is, the root of (5.65) - as

$$
\begin{aligned}
G(z) \equiv G_3(z) = \frac{z}{3} + \frac{1}{2}\Bigg[& -\left(-\frac{1}{2z} + \frac{z^3}{27} - \frac{\sqrt{\frac{1}{4} - \frac{z^4}{27}}}{|z|}\right)^{1/3} - \left(-\frac{1}{2z} + \frac{z^3}{27}\right. \\
& \left. - \frac{\sqrt{\frac{1}{4} - \frac{z^4}{27}}}{|z|}\right)^{1/3}\Bigg] - \frac{i\sqrt{3}}{2}\Bigg[-\left(-\frac{1}{2z} + \frac{z^3}{27} - \frac{\sqrt{\frac{1}{4} - \frac{z^4}{27}}}{|z|}\right)^{1/3} \\
& + \left(-\frac{1}{2z} + \frac{z^3}{27} - \frac{\sqrt{\frac{1}{4} - \frac{z^4}{27}}}{|z|}\right)^{1/3}\Bigg].
\end{aligned} \tag{5.80}
$$

Accordingly, the eigenvalue density is

$$
\begin{aligned}
\Phi(\rho) &= \frac{\sqrt{3}}{2\pi}(s_1 - s_2) \\
&= \frac{\sqrt{3}}{2\pi}\Bigg[-\left(-\frac{1}{2\rho} + \frac{\rho^3}{27} - \frac{\sqrt{\frac{1}{4} - \frac{\rho^4}{27}}}{|\rho|}\right)^{1/3} \\
&\quad + \left(-\frac{1}{2\rho} + \frac{\rho^3}{27} - \frac{\sqrt{\frac{1}{4} - \frac{\rho^4}{27}}}{|\rho|}\right)^{1/3}\Bigg].
\end{aligned} \tag{5.81}
$$

Moreover, the eigenvalue density has a finite support which is given by the roots of the equation

$$
0 = \frac{1}{4} - \frac{\rho^4}{27}. \tag{5.82}
$$

This is easily solved, and we obtain[12]

$$
\rho_\pm = \pm\frac{3^{3/4}}{\sqrt{2}}. \tag{5.83}
$$

Moreover, in "ρ-space" the density seems to "blow-up" close to the origin - see figure

[12] We assume that the cut is on the real axis, and hence we ignore the other two imaginary solutions to the quartic equation.

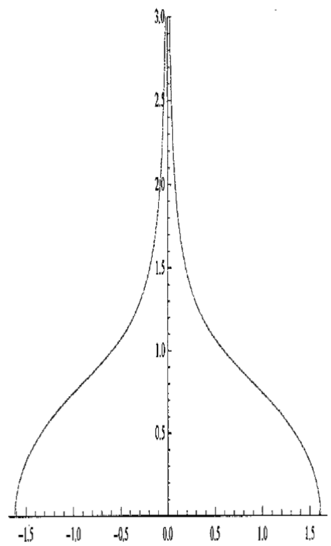

Figure 5.1: A plot of the eigenvalue density in "ρ-space".

(5.6). However, this is not a huge difficulty and can be resolved easily by working with the density in "r-space". In particular, close to the origin the density in "r-space" has the behaviour:

$$\phi(r) = 2r\Phi\left(r^2\right) \sim \frac{2r}{r^{2/3}} = 2r^{1/3}. \tag{5.84}$$

Finally, we note that the eigenvalue density "opens-up" in a non-physical region - obviously this can be seen from the end-points *i.e.* $\rho_{\pm} = \pm\frac{3^{3/4}}{\sqrt{2}}$. Naturally, we could attempt to restrict ("truncate") the eigenvalue density in such a manner that it is only defined in the physical region. That is, we could define - indeed, this is possible as we have been solving the equations "point-wise" - the eigenvalue density as a piece-wise function.

5.7 Summary

The main objective of this chapter was to provide an alternate approach to the saddle-point equations that we have considered in previous chapters. This alternate approach is given by the Dyson-Schwinger (loop) equations approach. We began by revisiting the radial subsector. In particular, we used a set of loop equations that enabled us to write down the algebraic equation satisfied by the generating function. Subsequently, the equation was then solved and indeed we obtained the same results that we had found previously. For the two-matrix integral, written in terms of matrix valued polar coordinates, we saw that the loop equations do not close on the original set of variables. It was then necessary to map this system onto a Penner-type two-matrix model. In the case when we do not have a logarithm potential, we were able to reproduce the radial hemispherical distribution [71]. Finally, the resulting cubic equation satisfied by the generating function was solved at strong-coupling. The density of eigenvalues was then obtained. However, we found that the density opens up in a non-physical domain.

Chapter 6

Conclusions And Outlook

The half-BPS sector of the gauge-gravity duality is described in terms of a single complex matrix model in a harmonic oscillator potential. It has always been clear that in order to move beyond the half-BPS sector, one has to consider much more complicated multi-matrix integrals [85]. In this dissertation, we investigated the strong-coupling large-N limit of such multi-matrix models.[1] One of the main objectives of the work done in this dissertation was to inquire into the significance of rescaling the matrix fields in such a manner as to bring the potential into the form given in (1.80).

In order to investigate the significance of such a rescaling, we started by looking at the dynamics of a simple single Hermitian matrix system. In the strong-coupling limit, the system appeared to be free. To see if this was indeed the case, we "compactified" the Hermitian system on the circle S^1. We were then able to write down the eigenstates of the free Hamiltonian. Accordingly, the first-order correction to the ground-state energy was then obtained and an estimate for the second-order correction was made. Using the Collective Field Theory, we were able to obtain the exact ground-state energy - both in the strong and weak coupling regimes. However, this approach introduced a length parameter L into the theory. We found that all the terms in the strong-coupling limit

[1] In fact, this limit is of great importance in the study of the emergence of gravity [51, 86, 87].

contributed to the same order in λ and thus the strong-coupling limit was not particularly useful.

In Chapter 3, we reviewed the solution to the Hoppe two-matrix integral. The solution involved using the "gauge-symmetry" to diagonalize one of the matrices. The other matrix was subsequently integrated out, and an effective action written down. The resulting saddle-point equations were written down in terms of a single analytic function $G(z)$. By using the properties of this function - more specifically, the fact that the function $G(z)$ maps the "coloured" region in Figure to the positive imaginary ζ-plane - we were able to obtain an expression for the inverse of $G(z)$.[2] The free energy and the moment ω_2 were then computed. Unfortunately, we found that the strong-coupling expansion was given in terms of both the 't Hooft coupling and a mass parameter ω. This was undesirable as in the strong-coupling limit we anticipated that the theory should depend on only one dimensionful parameter.

We then discussed a generalization of the Hoppe two-matrix integral. In particular, we considered an ensemble of $2m$ Hermitian matrices. This multi-matrix model results from a full compactification of the Higgs sector of $\mathcal{N}=4$ SYM on S^4. After reviewing how to use the Collective Field Theory approach in order to determine the Jacobian, we obtained the effective action for the radial subsector. The resulting saddle-point equations were then obtained and the radial density of eigenvalues was obtained. This density was seen to be non-vanishing within a hyperannulus. For the single complex matrix, we were able to parametrize the system using matrix valued polar coordinates. Here, we found that the density, at least to first-order in perturbation theory, was non-vanishing in a hypersphere with radius given by $r_+ = 2\lambda^{1/2}$. However, the higher-order results were senseless. As a result, we attempted to make use of the Harish-Chandra-Itzykson-Zuber (HCIZ) formula to obtain the exact radial density of eigenvalues. The exact saddle-point equations, at strong-coupling, yielded a singular density.

[2]The inverse of $G(z)$ maps the positive imaginary ζ-plane to the "coloured" region in Figure 1.

In Chapter 5, we gave a complementary approach to the saddle-point method. This approach was based on the Dyson-Schwinger (loop) equations formalism. In particular, the approach allows us to write an algebraic equation satisfied by the generating function. We revisited the radial subsector. In particular, using a set of loop equations we were able to obtain a quadratic equation that was satisfied by the generating function. Indeed, we were able to obtain the results that were obtained using the saddle-point method. For the single complex matrix model, we find that the loop equations do not close. As a result, we find it convenient to make the system onto a two-matrix Penner-type matrix model. Moreover, this allowed us to regulate the theory with a Penner-type potential similar to the one that we have for larger number of even Hermitian matrices. Indeed, in the absence of the Penner-type potential we were able to reproduce a radial hemispherical density of eigenvalues, and by extension the strong-coupling results obtained in [66, 67]. Using the DMS loop equations, we obtained an equation, of third degree, satisfied by the generating. This equation was solved in the strong-coupling limit and a radial density of eigenvalues was then obtained. We saw that the resulting radial density of eigenvalues "opened-up" in a non-physical domain.

Recently, the strong-coupling limit of the Hoppe two-matrix integral has been studied using the renormalization group [88]. This represents a fresh new approach to solving the two-matrix model and it might be worthwhile to try and apply some of these techniques to the problems that were considered in this dissertation.

As noted in the introduction, ABJM theory on the three-sphere (using the localization technique) can be reduced to a matrix integral. Moreover, this matrix integral can be written as the grand partition function of an ideal Fermi gas [89]. (A similar analysis has been carried out for the ABJ matrix model in [90].) The starting point of this description is the observation that we can write the grand partition function as Fredholm determinant:

$$\Xi(z) = \det(1 + z\hat{\rho}) \tag{6.1}$$

where $\hat{\rho}$ is defined via

$$\rho(x_1, x_2) \equiv \langle x_1 | \hat{\rho} | x_2 \rangle = \frac{1}{2\pi k} \frac{1}{(2\cosh x_1)^{1/2} (2\cosh x_2)^{1/2} 2\cosh\left(\frac{x_1 - x_2}{2}\right)}. \tag{6.2}$$

Remarkably, the grand partition function of the Hoppe two-matrix integral can also be written as a Fredholm determinant [67]. Naturally, one possible future direction that we might choose to consider is to try and give a similar Fermi gas picture for the Hoppe two-matrix integral. In fact, such a result has already been anticipated in [91]. Here, it was observed that by using the HCIZ formula, the partition for the two-matrix integral could be interpreted as describing N fermions in an inverse oscillator potential.

For more than two Hermitian matrices, there is currently no matrix valued curvilinear coordinate parametrization that is available. Hence, it might be interesting to see how we can introduce the angular degrees of freedom in such cases. In addition, we would like to exhaustively study the multi-matrix quantum mechanics of the radial sector. Already, we have already identified, in the collective fields description, the emergence of an innate metric.

Appendix A

The Character Expansion For $\mathrm{Tr}U^n$

In this appendix we wish to give the argument that led to the character expansion given in (2.49).

Firstly, we can expand the invariants $\mathrm{Tr}U^n$ as

$$\begin{aligned}\mathrm{Tr}U^n &= \sum_R a_R \chi_R(U) \\ &= \sum_{\{n_1,\ldots,n_N\}} a_{\{n_1,\ldots,n_N\}} \chi_{\{n_1,\ldots,n_N\}}(U) \, .\end{aligned} \tag{A.1}$$

where we have made use of the fact that the irreducible representations of $U(N)$ can be labeled by a set of integers $\{n_1, \ldots, n_N\}$, with $n_1 \geq n_2 \cdots \geq n_N$. The characters are orthonormal and hence

$$a_{\{n_1,\ldots,n_N\}} = \int DU \chi^*_{\{n_1,\ldots,n_N\}}(U) \, \mathrm{Tr}U^n. \tag{A.2}$$

The Haar measure is, by definition, invariant under both left and right transformations. Accordingly, it is possible to diagonalize the matrix U *i.e.* we can write $U = T\Lambda T^\dagger$, where $\Lambda = diag\left(e^{i\phi_1}, \ldots, e^{i\phi_N}\right)$. The integral over T is trivial and hence we can express the Haar measure in terms of the eigenvalues $e^{i\phi_k}$, with $\phi_k \in [-\pi, \pi]$.

In the Weyl parametrization, the Haar measure is

$$DU = \prod_{k=1}^{N} \frac{d\phi_k}{2\pi} \Delta\left(\phi\right) \Delta^*\left(\phi\right) \tag{A.3}$$

where

$$\Delta\left(\phi\right) = \frac{1}{\sqrt{N!}} \det\left(e^{i(N-j)\phi_k}\right)$$

Accordingly,

$$\begin{aligned} a_{\{n_1,\ldots,n_N\}} &= \int DU \chi^*_{\{n_1,\ldots,n_N\}}\left(U\right) \mathrm{Tr}U^n \\ &= \int \prod_{k'=1}^{N} \frac{d\phi_k}{2\pi} \Delta\left(\phi\right) \Delta^*\left(\phi\right) \chi^*_{\{n_1,\ldots,n_N\}}\left(U\right) \mathrm{Tr}U^n \\ &= \int \prod_{k'=1}^{N} \frac{d\phi_k}{2\pi} \det\left(e^{i(N-j)\phi_k}\right) \det\left(e^{-i(N-j+n_j)\phi_k}\right) \left(\sum_{k=1}^{N} e^{in\phi_k}\right) \end{aligned} \tag{A.4}$$

where we have used

$$\Delta^*\left(\phi\right) \chi^*_{\{n_1,\ldots,n_N\}}\left(U^\dagger\right) = \frac{1}{\sqrt{N!}} \det\left(e^{-i(N-j+n_j)\phi_k}\right). \tag{A.5}$$

Expanding the determinants in (A.4), one obtains

$$
\begin{aligned}
a_{\{n_1,\ldots,n_N\}} &= \frac{1}{N!}\epsilon_{i_1,\ldots i_N}\epsilon_{j_1,\ldots j_N}\sum_{k=1}^{N}\int\prod_{k=1}^{N}\frac{d\phi_k}{2\pi}\left(e^{i\left(i_{k'}-j_{k'}-n_{i_{k'}}\right)\phi_k}\right)\left(e^{in\phi_k}\right) \\
&= \frac{1}{N!}\epsilon_{i_1,\ldots i_N}\epsilon_{j_1,\ldots j_N}\sum_{k=1}^{N}\left[\prod_{k\neq k'}\delta\left(i_{k'}-j_{k'}-n_{i_{k'}}\right)\delta\left(i_k-j_k-n_{i_k}+n\right)\right] \qquad \text{(A.6)}
\end{aligned}
$$

Let us fix the integers $\{i_1,\ldots,i_N\}$ and also the integer k. Then, it is clear that we need to determine a set of integers $\{j_1,\ldots,j_N\}$ that will satisfy the following conditions:

$$
j_{k'} = i_{k'} - n_{i_{k'}}, j_k = i_k - n_{i_k} + n \qquad \text{(A.7)}
$$

$$
1 \leq j_{k'} \leq N, j_{k'} = 2,\ldots,N. \qquad \text{(A.8)}
$$

Equivalently, we have

$$
\begin{aligned}
n_{i_{k'}} &< i_{k'} && \text{(A.9)} \\
n_{i_{k'}} &\geq N - i_{k'} && \text{(A.10)} \\
n_{i_k} &< i_k - n && \text{(A.11)} \\
n_{i_k} &\geq N - n - i_k && \text{(A.12)}
\end{aligned}
$$

Firstly, let us consider the case when $i_k \neq 1$. Suppose that $i_{k_0} = 1$. It is straightforward to see that (A.11) implies that $n_{k_0} < 1$ and hence $n_i \leq 0$ $(i = 1,\ldots,N)$.

For $n \geq N-1$, (A.11) yields:

$$
0 \geq i_k - 1 \geq 1
$$

Thus, the conditions given implicitly by the delta functions in (A.6) cannot be satisfied.

Similarly, for $1 \leq n \leq N-1$ with $i_k = N$, (A.10) leads to the result that $1 \geq n_N \geq n \geq 0$. Again, we see that the conditions cannot be satisfied.

However, when $1 \leq n \leq N-1$ and $i_k \neq N$, we find that $0 \geq n_N \geq 0$. This implies that $n_i = 0$ $(i = 1, \ldots, N)$. Hence, because of the properties of the we can write (A.6) as

$$a_{\{n_1,\ldots,n_N\}} \sim \frac{1}{N!} \epsilon_{i_1,\ldots i_N} \sum_{\substack{k=1 \\ i_k \neq 1 \\ i_k \leq N-n}} \epsilon_{i_1,\ldots,i_k+n,i_{k+1},\ldots i_n} \tag{A.13}$$

Now, let $j = n + i_k$. Since $n \neq 0$, it follows that $j \neq i_k$, and thus there is some k' that satisfies $i_{k'} = j$. Hence, by the properties of the ϵ symbol, the coefficients $a_{\{n_1,\ldots,n_N\}}$ automatically vanish for $i_k \neq 1$.

Next, we look at case when $i_k = 1$. Here, the conditions (A.10) and (A.12) can be written as

$$1 \geq n_j \geq j - N \tag{A.14}$$

$$n \geq n_1 \geq 1 + N - j \tag{A.15}$$

This implies that for $j = N$, we have $1 \geq n_N \geq 0$. Thus,

$$n \leq N : n \geq n_1 \geq 1 + n - N; \; 0 \leq n_j \leq 1 \; j = 2, 3, \ldots, N$$

$$n \geq N : n \geq n_1 \geq 1; \qquad 0 \leq n_j \leq 1 \; j = 2, 3, \ldots, N$$

and we obtain

$$n \leq N: \quad a_{\{n_1,\dots,n_N\}} = \begin{cases} \epsilon_{1+n-n_1,2-n_2\dots,N-n_N} & 1 \leq n_1 \leq n;\ 0 \leq n_j \leq 1,\ j = 2,3,\dots,N \\ 0 & \text{otherwise} \end{cases} \tag{A.16}$$

$$n \geq N: \quad a_{\{n_1,\dots,n_N\}} = \begin{cases} \epsilon_{1+n-n_1,2-n_2\dots,N-n_N} & 1+n-N \leq n_1 \leq n;\ 0 \leq n_j \leq 1,\ j = 2,3,\dots,N \\ 0 & \text{otherwise} \end{cases} \tag{A.17}$$

After some manipulations, (A.16) and (A.17) can be written as

$$n \leq N: \quad a_{\{n_1,\dots,n_N\}} = \begin{cases} (-1)^{n-n_1},\ \sum_{n_i=1}^{N} & 1 = n,\ \text{if } 1 \leq n_1 \leq n;\ 0 \leq n_j \leq 1,\ j = 2,3,\dots,N \\ 0 & \text{otherwise} \end{cases}$$

$$n \geq N: \quad a_{\{n_1,\dots,n_N\}} = \begin{cases} (-1)^{n-N_1} & \sum_{n_i=1}^{N} 1 = n,\ \text{if } n_1 \geq 1+n-N,\ n \geq;\ 0 \leq n_j \leq 1,\ j = 2,3,\dots,N \\ 0 & \text{otherwise} \end{cases}$$

This yields the trace expansion that we used in Chapter 2.

Appendix B

The Elliptic Integrals

The elliptic integral of the first kind is defined by

$$F(\varphi, k) = u(\sin\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin\vartheta}} = \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{B.1}$$

Here, k is the modulus[1]- we also have the complementary modulus which is defined as: $k' = \sqrt{1-k^2}$ - and φ is called the modular angle. In fact, the elliptic integrals are the inverses of the Jacobian elliptic functions.[2] For example,

$$u = F(\mathrm{sn}(u,k), k). \tag{B.2}$$

Similarly, the elliptic integral of the second kind is defined in terms of an integral:

$$E(k, \varphi) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2\vartheta}\, d\vartheta = \int_0^{\sin\varphi} \frac{\sqrt{(1-k^2t^2)}dt}{\sqrt{(1-t^2)}}. \tag{B.3}$$

The elliptic function of the third kind is

[1] The notation is slightly different to the one that was used in Chapter 3. In particular, the parameter m used in Chapter 3 is given by $m = k^2$.

[2] The most well-known Jacobian elliptic functions are the sine amplitude elliptic function $\mathrm{sn}(u,k)$, cosine amplitude elliptic function $\mathrm{cn}(u,k)$ and the delta amplitude elliptic function $\mathrm{dn}(u,k)$.

$$\Pi\left(\varphi,\alpha^2,k\right) = \int_0^{\varphi} \frac{1}{1-\alpha^2\sin^2\vartheta}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \tag{B.4}$$

For $\varphi = \pi$, the elliptic functions are said to complete. For example, the complete elliptic integral of the first kind is defined as

$$\mathbf{K}\left(k\right) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \int_0^1 \frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}, \tag{B.5}$$

and the complete elliptic integral of the second kind is

$$\mathbf{E}\left(k\right) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\vartheta}d\vartheta = \int_0^1 \frac{\sqrt{(1-k^2t^2)}dt}{\sqrt{(1-t^2)}}. \tag{B.6}$$

Moreover, the elliptic integrals satisfy the so-called Legendre relation *i.e.*

$$\mathbf{E}\left(k\right)\mathbf{K}\left(k'\right) + \mathbf{E}\left(k'\right)\mathbf{K}\left(k\right) - \mathbf{K}\left(k\right)\mathbf{K}\left(k'\right) = \frac{\pi}{2} \tag{B.7}$$

In fact, the complete elliptic integrals can be written in terms of the hypergeometric function. In particular, we have [70, 92]

$$\begin{aligned}
\mathbf{K}\left(k\right) &= \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,k^2\right)\\
\mathbf{E}\left(k\right) &= \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2},1,k^2\right)
\end{aligned}$$

Therefore,

$$
\mathbf{K}(k) = \begin{cases} \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{8}\right)^2 \frac{k^4}{3} + \cdots\right] & k \to 0 \\ \ln\frac{k'}{4} + \left(\frac{1}{2}\right)^2 \left(\ln\frac{k'}{4} - \frac{2}{1\cdot 2}\right) k'^2 + \cdots & k' = \sqrt{1-k^2} \to 0 \end{cases}
$$

$$
\mathbf{E}(k) = \begin{cases} \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{3}{8}\right)^2 \frac{k^4}{3} + \cdots\right] & k \to 0 \\ \ln\frac{k'}{4} + \left(\frac{1}{2}\right)^2 \left(\ln\frac{k'}{4} - \frac{1}{2}\right) k'^2 + \cdots & k' = \sqrt{1-k^2} \to 0 \end{cases}
$$

The solution of the two-matrix integral required us to introduce the inverse mapping function $z(\zeta)$. It turns out that this function - and hence the solution of the two-matrix integral - can be parametrized in terms of the elliptic integrals.

From Chapter 3, the function $z(\zeta)$ is defined as

$$
z(\zeta) = A \int_{x_1}^{\zeta} \frac{dt\,(t-x_3)}{\sqrt{(t-x_1)(t-x_2)(t-x_4)}} \tag{B.8}
$$

The integral in (B.8) yields[3]

$$
\begin{aligned}
z(u) &= \frac{2A}{\sqrt{x_4 - x_1}}\left[F(\varphi,k)(x_2-x_3) + \frac{x_1(x_1-x_2)}{x_2}\Pi(\varphi,1,k)\right] \\
&= \frac{2A}{\sqrt{x_4 - x_1}}\left[(x_2-x_3)\,u + \frac{x_1(x_1-x_2)}{x_2}\int_0^u \frac{du'}{1-\mathrm{sn}^2 u'}\right] \\
&= \frac{2A}{\sqrt{x_4 - x_1}}\left[\left(x_2 - x_3 + \frac{x_1(x_1-x_2)}{x_2}\right) u + \frac{x_1(x_1-x_2)}{x_2}\left(-\mathbf{E}(u) + \mathrm{dn}u\mathrm{tn}u\right)\right].
\end{aligned}
$$

Here,

[3]Here, we will assume that $\zeta > x_1 > x_2 > x_3 > x_4$.

$$k = \sqrt{\frac{x_2 - x_4}{x_1 - x_4}} \tag{B.9}$$

$$\varphi = \sin^{-1}\left(\sqrt{\frac{\zeta - x_1}{\zeta - x_2}}\right) \tag{B.10}$$

From (B.10), it follows that

$$\zeta\left(u\right) = \frac{x_1 - x_2\,\mathrm{sn}^2 u}{1 - k^2 \mathrm{sn}^2 u}. \tag{B.11}$$

In the case when $\alpha \neq 1$, the function $z\left(u\right)$ can be expressed in terms of the Jacobi function $H_1\left(u\right)$ and the Heuman's Lambda function [93].

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